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Newton flows for elliptic functions I
Structural stability: characterization & genericity

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ABSTRACT

Newton flows are *dynamical systems* generated by a continuous, desingularized Newton method for mappings from a Euclidean space to itself. We focus on the special case of meromorphic functions on the complex plane. Inspired by the analogy between the rational (complex) and the elliptic (i.e. doubly periodic meromorphic) functions, a theory on the class of so-called *Elliptic Newton flows* is developed. With respect to an appropriate topology on the set of all elliptic functions f of fixed order $r (\geqslant 2)$ we prove: For *almost all* functions f, the corresponding Newton flows are *structurally stable* i.e. topologically invariant under small perturbations of the zeros and poles for f [*genericity*]. They can be described in terms of nondegeneracy-properties of f similar to the rational case [*characterization*].

1. Meromorphic Newton flows

In this section, we briefly explain the concept of meromorphic Newton flow. For details and historical notes, see [1–6]. In the sequel, let f stand for a non-constant meromorphic function on the complex plane. So, $f(z)$ is complex analytic for all z in $\mathbb{C}$ with the possible exception of (countably many) isolated singularities: the *poles* for f.

The (damped) Newton method for finding zeros of f (with starting point z^0) is given by

$$z_{n+1} - z_n = -t_n \frac{f(z_n)}{f'(z_n)}, \ t_n \neq 0, \ n = 0, 1, \ldots, z_0 = z^0. \tag{1}$$

Dividing both sides of (1) by the 'damping factor' t_n and choosing t_n smaller and smaller yields an 'infinitesimal version' of (1), namely

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{-f(z)}{f'(z)}. \tag{2}$$
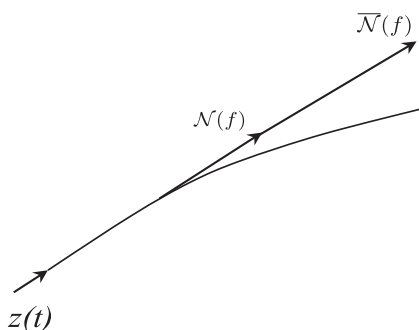
Figure 1. $\mathcal{N}(f)$ vs. $\overline{\mathcal{N}}(f)$.

Conversely, Euler's method applied to (2) gives rise to an iteration of the form (1). A dynamical system of type (2) is denoted by $\mathcal{N}(f)$. For this system, we will interchangeably use the following terminologies: *vector field* [i.e. the expression on its r.h.s.], or *(Newton-)flow* [when we focus on its phase portrait(=family of all maximal trajectories as point sets)].

Obviously, zeros and poles for f are removable singularities for $\frac{f}{f'}$ and turn into *isolated equilibria* for $\mathcal{N}(f)$. Special attention should be paid to those points z where $f(z) \neq 0$ and $f'(z) = 0$. In these (isolated!) so-called *critical points* , the vector field $\mathcal{N}(f)$ is not well defined. We overcome this complication by introducing an additional 'damping factor' $(1 + |f(z)|^4)^{-1}|f'(z)|^2(\geqslant 0)$ and considering a system $\overline{\mathcal{N}}(f)$ of the form

$$\frac{dz}{dt} = -(1 + |f(z)|^4)^{-1}\overline{f'(z)}f(z). \tag{3}$$

Clearly, $\overline{\mathcal{N}}(f)$ may be regarded as another 'infinitesimal version' of Newton's iteration (1). Note that, where both $\mathcal{N}(f)$ and $\overline{\mathcal{N}}(f)$ are well defined, their phase portraits coincide, including the orientations of the trajectories (cf. Figure 1). Moreover, $\overline{\mathcal{N}}(f)$ is a smooth, even real (but not complex) analytic vector field on the *whole* plane. For our aims, it is enough that $\overline{\mathcal{N}}(f)$ is of class C^1. We refer to $\overline{\mathcal{N}}(f)$ as to a *desingularized Newton flow* for f on $\mathbb{C}$.

Integration of (2) yields:

$$f(z(t)) = e^{-t}f(z_0), \; z(0) = z_0, \tag{4}$$

where $z(t)$ denotes the maximal trajectory for $\mathcal{N}(f)$ through a point z_0. So we have

$$\mathcal{N}(f)\text{-trajectories and also those of } \overline{\mathcal{N}}(f) \text{ are made up of lines } \arg f(z) = \text{constant.} \tag{5}$$

It is easily verified that these Newton flows fulfil a *duality property* which will play an important role in the sequel:

$$\mathcal{N}(f) = -\mathcal{N}\left(\frac{1}{f}\right) \text{ and } \overline{\mathcal{N}}(f) = -\overline{\mathcal{N}}\left(\frac{1}{f}\right). \tag{6}$$

As a consequence of (5), (6) and using properties of (multi-)conformal mappings, we picture the local phase portraits of $\mathcal{N}(f)$ and $\overline{\mathcal{N}}(f)$ around their equilibria. See the comments on Figure 2, where $N(f)$, $P(f)$ and $C(f)$ stand for, respectively, the set of zeros, poles and critical points of f.

Comments on Figure 2: Figure 2(a) and (b) In case of a k-fold zero (pole), the Newton flow exhibits an attractor (repellor) and each (principal) value of $\arg f$ appears precisely k times on equally distributed incoming (outgoing) trajectories. Moreover, as for the (positively measured) angle between two different incoming (outgoing) trajectories, they intersect under a non vanishing angle $\frac{\Delta}{k}$, where Δ stands for the difference of the $\arg f$ values on these trajectories, i.e. these equilibria are star nodes. In the sequel we will use: If two incoming (outgoing) trajectories at a *simple* zero (pole) admit the same $\arg f$ value, those trajectories coincide.

Figure 2(c) and (d): In case of a k-fold critical point (i.e. a k-fold zero for f', no zero for f) the Newton flow exhibits a k-fold saddle, the stable (unstable) separatrices being equally distributed around this point. The two unstable (stable) separatrices at a onefold saddle, see Figure 2(c), constitute the 'local' unstable (stable) manifold at this saddle point.

In the sequel, we shall need:

Remark 1.1: Let z_0 be either a *simple* zero, pole or critical point for f. Then z_0 is a hyperbolic[1] equilibrium for $\overline{\mathcal{N}}(f)$. (In case of a zero or critical point for f, this follows by inspection of the linearization of the r.h.s. of $\overline{\mathcal{N}}(f)$: in case of a pole use (6).)

Remark 1.2 (Desingularized meromorphic Newton flows in $\mathbb{R}^2$-setting): If we put $F : (\mathrm{Re}(z), \mathrm{Im}(z))^T (= (x_1, x_2)^T) \mapsto (\mathrm{Re}f(z), \mathrm{Im}f(z))^T$, the desingularized Newton flow $\overline{\mathcal{N}}(f)$ takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1, x_2)^T = -[1 + |F(x_1, x_2)|^4]^{-1} \det (DF(x_1, x_2))(DF)^{-1}(x_1, x_2)F(x_1, x_2) \quad (7)$$
$$= -[1 + |F(x_1, x_2)|^4]^{-1}\tilde{D}F(x_1, x_2)F(x_1, x_2),$$

where $(\,\cdot\,)^T$ stands for transpose, and $\tilde{D}F(\,\cdot\,)$ for the co-factor (adjoint) matrix[2] of the Jacobi matrix $DF(\,\cdot\,)$ of F. (The r.h.s. of (7) vanishes at points corresponding to poles of f)

We end up with the Figures 3 and 4 illustrating the above explanation.

$\circ$: attractor (in $N(f)$) $\bullet$: repellor (in $P(f)$) $+$: 1-fold saddle (in $C(f)$) $*$: 2-fold saddle (in $C(f)$)

(a)   **(b)**   **(c)**   **(d)**
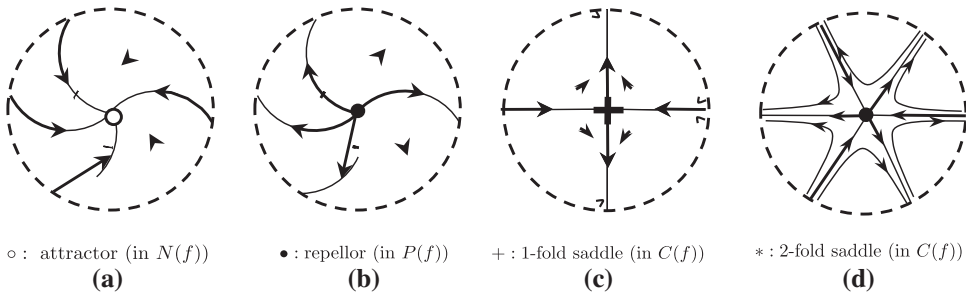
Figure 2. Local phase portraits around equilibria of $\overline{\mathcal{N}}(f)$.
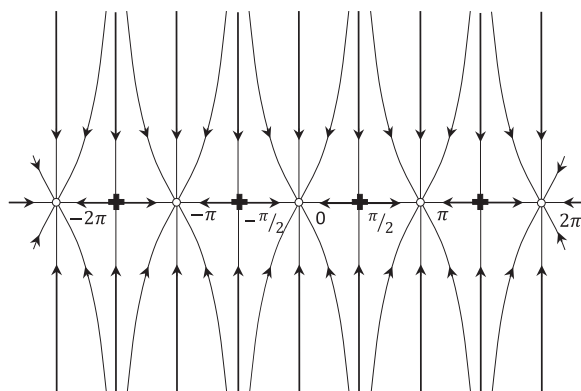
Figure 3. Phaseportrait $\overline{\mathcal{N}}(\sin z)$.
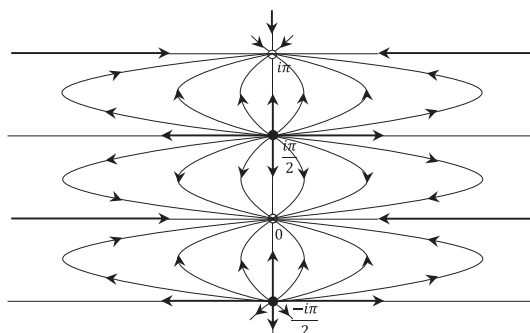
Figure 4. Phaseportrait of $\overline{\mathcal{N}}(\tanh z)$.

2. Rational Newton flows; the purpose of the paper

Here we present some earlier results on meromorphic Newton flows in the special case of rational functions. Throughout this section, let f be a (*non-constant*) rational function. By means of the transformation $w = \frac{1}{z}$, we may regard f as a function on the extended complex plane $\mathbb{C} \cup \{z = \infty\}$. As usual, we identify the latter set with the sphere S^2 (as a Riemann surface) and the set $\mathcal{R}$ of extended functions f with the set of all (non-constant) meromorphic functions on S^2. The transformation $w = \frac{1}{z}$ turns the '*planar rational Newton flow*' $\overline{\mathcal{N}}(f)$, $f \in \mathcal{R}$, into a *smooth* vector field on S^2, denoted $\overline{\overline{\mathcal{N}}}(f)$, cf. [7,8]. In the theory on such vector fields the concept of structural stability plays an important role, see e.g. [9] or [10]. Roughly speaking, structural stability of $\overline{\overline{\mathcal{N}}}(f)$ means 'topological invariance of its phase portrait under sufficiently small perturbations of the problem data'. Here we briefly summarize the results as obtained by Jongen et al., Shub (cf. [4,5,7,11]):

Theorem 2.1 (Structural stability for rational Newton flows): *Let $f \in \mathcal{R}$, then:*

(i) *Characterization:*
   The flow $\overline{\overline{\mathcal{N}}}(f)$, $f \in \mathcal{R}$, is structurally stable iff f fulfils the following conditions:

   • *All finite zeros and poles for f are simple.*

- *All critical points for f, possibly including $z = \infty$, are simple.*
- *No two critical points for f are connected by an $\overline{\overline{\mathcal{N}}}(f)$-trajectory.*

(ii)   *Genericity:*
      For 'almost all' functions f in $\mathcal{R}$, the flows $\overline{\overline{\mathcal{N}}}(f)$ are structurally stable, i.e. the functions f as in (i) constitute an open and dense subset of $\mathcal{R}$ (w.r.t. an appropriate topology on $\mathcal{R}$).

(iii) *Classification:*
      The conjugacy classes of the structurally stable flows $\overline{\overline{\mathcal{N}}}(f)$ can be classified in terms of certain sphere graphs that are generated by the phase portraits of these flows.

(iv)   *Representation:*
      Up to conjugacy for flows and (topological) equivalency for graphs, there is a 1-1-correspondence between the set of all structurally stable flows $\overline{\overline{\mathcal{N}}}(f)$ and the set of all so-called Newton graphs, i.e. cellularly embedded sphere graphs that fulfil some combinatorial (Hall) condition.

The purpose of the present paper and its sequel [12] is to find out whether similar results hold for *elliptic* Newton flows (i.e. meromorphic Newton flows in the case of elliptic functions), be it that, especially in the cases (iii) Classification and (iv) Representation, the proofs are much harder, see [12]. In the present paper, we focus on the first two properties mentioned in Theorem 2.1: characterization and genericity.

Phase portraits of rational Newton flows (even structurally stable) on $\mathbb{C}$ are presented in Figures 5 and 6. The simplest example of a spherical rational Newton flow is the so-called North–South flow, given by $\overline{\overline{\mathcal{N}}}(z^n), n \geqslant 1$, see Figure 7; structurally stable if $n = 1$. Intuitively, it is clear that the phase portraits of $\overline{\overline{\mathcal{N}}}(z^n)$ and $\overline{\overline{\mathcal{N}}}((\frac{z-a}{z-b})^n), a \neq b$, are topologically equivalent (i.e. equal up to conjugacy), see Figures 7 and 8.

One of the first applications of Newton flows was Branin's method for solving non linear problems, see [1,2,4]. It was Smale, see [6], who stressed the importance for complexity theory of classifying Newton graphs on the sphere that determine the desingularized rational Newton flows. This was done for a class of polynomials in [5] and in general in [11]. Also in the elliptic case, the classification of so-called elliptic Newton graphs on the torus, which determine the desingularized elliptic Newton flows has implications for complexity theory, see [12].
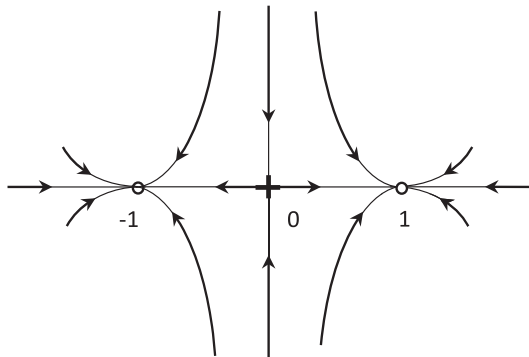
Figure 5. Phaseportrait $\overline{\mathcal{N}}(z^2 - 1)$.

Figure 6. Phaseportrait $\overline{\mathcal{N}}(\frac{1}{z^2-1})$.

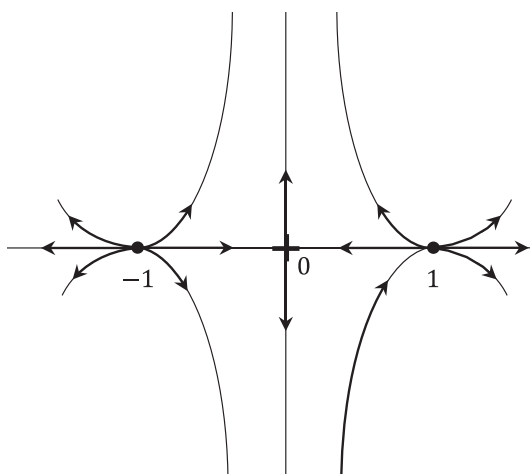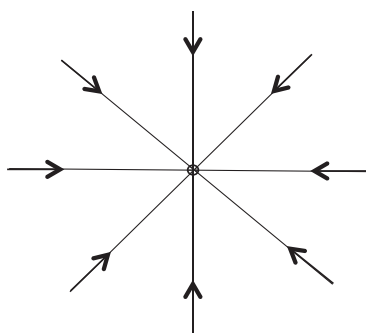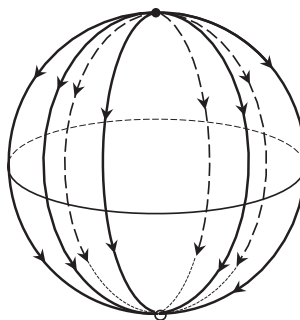$$\overline{\mathcal{N}}(z^n), n \geqslant 1 \qquad \overline{\overline{\mathcal{N}}}(z^n), n \geqslant 1$$

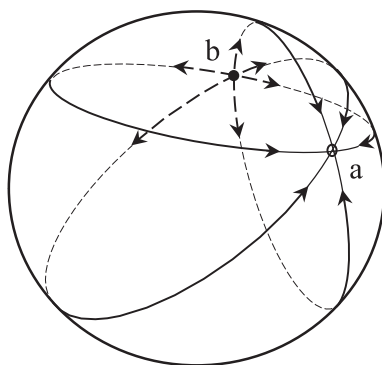**Figure 7.** The planar and spherical North–South flow.

Figure 8. $\overline{\overline{\mathcal{N}}}\left(\left(\frac{z-a}{z-b}\right)^n\right)$.

3. Elliptic Newton flows: definition

Throughout this section, let f be a (non-constant) elliptic, i.e. a meromorphic and doubly periodic function of order r ($2 \leqslant r < \infty$) with (ω_1, ω_2) as a pair of *basic periods*.[3] We always assume that $\mathrm{Im}\frac{\omega_2}{\omega_1} > 0$. The associated period lattice is denoted by Λ, and P_{ω_1, ω_2} stands for the 'half open/half closed' period parallelogram $\{t_1\omega_1 + t_2\omega_2 \mid 0 \leqslant t_1 < 1,\ 0 \leqslant t_2 < 1\}$. On P_{ω_1,ω_2}, the function f has r zeros and r poles (counted by multiplicity).

By Liouville's Theorem, these sets of zeros and poles determine f up to a multiplicative constant $C(\neq 0)$, and thus also the class $[f]$ of all elliptic functions of the form Cf, $C(\neq 0)$.

Consider the Riemann surface $T(\Lambda) = \mathbb{C}/\Lambda$. It can be obtained from P_{ω_1,ω_2} by identifying opposite sides in the boundary of this parallelogram. The planar, desingularized Newton flow $\overline{\mathcal{N}}(f)$ is doubly periodic on $\mathbb{C}$ with respect to the lattice Λ periods (ω_1, ω_2). Hence, this flow may be interpreted as a C^1-(even smooth, but nowhere meromorphic) vector field, say $\overline{\overline{\mathcal{N}}}_\Lambda(f)$, on $T(\Lambda)$; its trajectories correspond to the lines $\arg f(z) = \mathrm{constant}$, cf. (5). It will be treated as such. We refer to $\overline{\overline{\mathcal{N}}}_\Lambda(f)$ as the *(desingularized) elliptic Newton flow* for f on $T(\Lambda)$.

If g is another function in $[f]$, the planar flows $\overline{\mathcal{N}}(g)$ and $\overline{\mathcal{N}}(f)$ have equal phase portraits, as follows by inspection of the expressions of these flows in Section 1; see also Figure 1. Hence, the flows $\overline{\overline{\mathcal{N}}}_\Lambda(f)$ and $\overline{\overline{\mathcal{N}}}_\Lambda(g)$, both defined on $T(\Lambda)$, have equal phase portraits.

Definition 3.1: If f is an elliptic function of order r, then:

(1) The elliptic Newton flow for f, denoted $\overline{\overline{\mathcal{N}}}_\Lambda([f])$, is the collection of all flows $\overline{\overline{\mathcal{N}}}_\Lambda(g)$, for any $g \in [f]$.

(2) The set of all elliptic Newton flows of order r with respect to a given period lattice Λ is denoted $N_r(\Lambda)$.

Regarding flows on $T(\Lambda)$ with the same phase portraits as equal, compare the 'desingularization' step leading from (2) to (3) and see also Figure 1, we may interpret the elliptic Newton flow $\overline{\overline{\mathcal{N}}}_\Lambda([f])$ as a smooth vector field on the compact torus $T(\Lambda)$. Consequently, it is allowed to apply the theory for C^1-vector fields on compact two-dimensional differential manifolds. For example: Since there are no closed orbits by (4), and applying the Poincaré–Bendixson–Schwartz Theorem, cf. [3,4,9], we find:

Lemma 3.1: *The limiting set of any (maximal) trajectory of $\overline{\overline{\mathcal{N}}}_\Lambda([f])$ tends – for increasing t – to either a zero or a critical point for f on $T(\Lambda)$, and – for decreasing t – to either a pole or a critical point for f on $T(\Lambda)$.*

We also have:

Remark 3.1: Hyperbolic equilibria for $\overline{\mathcal{N}}(f)$ correspond to such equilibria for $\overline{\overline{\mathcal{N}}}([f])$.

4. Elliptic Newton flows: representation

Let f be as in Section 3, i.e. an elliptic function of order r ($2 \leqslant r < \infty$) with (ω_1, ω_2), $\mathrm{Im}\frac{\omega_2}{\omega_1} > 0$, an (arbitrary) pair of basic periods generating a period lattice $\Lambda = \Lambda_{\omega_1,\omega_2}$. The set of all such functions is denoted by $E_r(\Lambda)$.

Let the zeros and poles for f on P_{ω_1,ω_2} be $a_1,\ldots,a_r$, resp. $b_1,\ldots,b_r$ (counted by multiplicity). Then we have: (cf. [13])

$$a_i \neq b_j, i,j = 1,\ldots,r \text{ and } a_1 + \cdots + a_r = b_1 + \cdots + b_r \bmod \Lambda. \tag{8}$$

We consider f as a meromorphic function on the quotient space $T(\Lambda) = \mathbb{C}/\Lambda$. The zeros and poles for f on $T(\Lambda)$ are given by, respectively, $[a_1],\ldots,[a_r]$ and $[b_1],\ldots,[b_r]$, where $[\cdot]$ stands for the congruency class mod Λ of a number in $\mathbb{C}$. Apparently, from (8) it follows:

$$[a_i] \neq [b_j], i,j = 1,\ldots,r \text{ and } [a_1] + \cdots + [a_r] = [b_1] + \cdots + [b_r]. \tag{9}$$

Moreover, a parallelogram of the type P_{ω_1,ω_2} contains one representative of each of these classes: the r zeros/poles for f on this parallelogram.

An elliptic Newton flow $\overline{\overline{\mathcal{N}}}_\Lambda([f])(\in N_r(\Lambda))$ corresponds uniquely to the class $[f]$. So we may identify the set $N_r(\Lambda)$ with the set $\{[f] \mid f \in E_r(\Lambda)\}$.

On its turn, the class $[f]$ is uniquely determined (cf. Section 3) by sets of zeros/poles, say $\{a_1,\ldots,a_r\}/\{b_1,\ldots,b_r\}$, both situated in some P_{ω_1,ω_2}. Thus, the sets

$$\{[a_1],\ldots,[a_r]\}, \{[b_1],\ldots,[b_r]\}$$

fulfil the conditions (9). Conversely, we have:

Let two sets $\{[a_1],\ldots,[a_r]\}$, $\{[b_1],\ldots,[b_r]\}$ of classes mod Λ (repetitions permitted!), fulfilling conditions (9), be given. Assume that the representatives, $a_1,\ldots,a_r$ and $b_1,\ldots,b_r$, of these classes are situated in a half open/half closed parallelogram spanned by an (arbitrary) pair of basic periods of Λ. We put

$$b_r' = a_1 + \cdots + a_r - b_1 - \ldots - b_{r-1}, \text{ thus } [b_r'] = [b_r].$$

Consider functions of the form

$$C \frac{\sigma(z-a_1)\ldots\sigma(z-a_r)}{\sigma(z-b_1)\ldots\sigma(z-b_{r-1})\sigma(z-b_r')}, \tag{10}$$

where $C(\neq 0)$ is an arbitrary constant, and σ stands for Weierstrass' sigma function corresponding to Λ (cf. [13]). Since σ is a holomorphic, quasi-periodic function with only simple zeros at the lattice points of Λ, a function given by (10) is elliptic of order r; the zeros and poles are $a_1,\ldots,a_r$ resp. $b_1,\ldots,b_r$ (cf. [13]). Such a function determines precisely one element of $N_r(\Lambda)$. (Note that if we choose $a_1,\ldots,a_r,b_1\ldots b_r$ in any other period parallelogram, we obtain a representative of the same Newton flow, cf. Section 3; moreover, the incidental role of b_r does not affect generality)

Altogether, we have proved:

Lemma 4.1: *The flows in $N_r(\Lambda)$ are given by all ordered pairs $(\{[a_1],\ldots,[a_r]\}, \{[b_1],\ldots,[b_r]\})$ of sets of classes mod Λ that fulfil (9).*

Remark 4.1: Interchanging the roles of $(\{[a_1],\ldots,[a_r]\}$ and $\{[b_1],\ldots,[b_r]\})$ reflects the duality property, cf. (6). In fact, we have $\overline{\overline{\mathcal{N}}}_\Lambda([\frac{1}{f}]) = -\overline{\overline{\mathcal{N}}}_\Lambda([f])$.

On the subset $V_r(\Lambda)$ in $T^r(\Lambda) \times T^r(\Lambda)$ of pairs (c,d), $c = ([c_1],\ldots,[c_r])$, $d = ([d_1],\ldots,[d_r])$, that fulfil condition (9), we define an equivalence relation ($\approx$):

$$(c,d) \approx (c',d') \text{ iff } \{[c_1],\ldots,[c_r]\} = \{[c_1'],\ldots,[c_r']\} \text{ and } \{[d_1],\ldots,[d_r]\} = \{[d_1'],\ldots,[d_r']\}$$

The topology τ_0 on $E_r(\Lambda)$

Clearly, the set $V_r(\Lambda)/\approx$ may be identified with a representation space for $N_r(\Lambda)$ and thus for $\{[f] \mid f \in E_r(\Lambda)\}$. This space can be endowed with a topology which is successively induced by the quotient topology on $T(\Lambda)(= \mathbb{C}/\Lambda)$, the product topology on $T^r(\Lambda) \times T^r(\Lambda)$, the relative topology on $V_r(\Lambda)$ as a subset of $T^r(\Lambda) \times T^r(\Lambda)$ and the quotient topology w.r.t. the relation $\approx$.

Finally, we endow $E_r(\Lambda)$ with the weakest topology, say τ_0, making the mapping

$$E_r(\Lambda) \to N_r(\Lambda) : f \mapsto [f]$$

continuous.

The topology τ_0 on $E_r(\Lambda)$ is induced by the Euclidean topology on $\mathbb{C}$, and is natural in the following sense: Given f in $E_r(\Lambda)$ and $\epsilon > 0$ sufficiently small, a τ_0-neighbourhood $\mathcal{O}$ of f exists such that for any $g \in \mathcal{O}$, the zeros (poles) for g are contained in ε-neighbourhoods of the zeros (poles) for f.

Until now, we dealt with elliptic Newton flows $\overline{\overline{\mathcal{N}}}_\Lambda(f)$ with respect to an arbitrary, but fixed, lattice, namely the lattice Λ for f. Now, we turn over to a different lattice, say Λ^*, i.e. pairs of basic periods for Λ and Λ^* are not necessarily related by a unimodular transformation. Firstly, we treat a simple case: For $\alpha \in \mathbb{C}\setminus\{0\}$, we define $f^\alpha(z) = f(\alpha^{-1}z)$. Thus f^α is an elliptic function, of order r, with basic periods $(\alpha\omega_1, \alpha\omega_2)$ generating the lattice $\Lambda^* = \alpha\Lambda$.

W.r.t. the transformation $f \mapsto f^\alpha$, there holds the following lemma:

Lemma 4.2: *The transformation $z \mapsto w = \alpha z$ induces a homeomorphism from the torus $T(\Lambda)$ onto $T(\alpha\Lambda)$, such that the phase portraits of the flows $\overline{\overline{\mathcal{N}}}_\Lambda([f])$ and $\overline{\overline{\mathcal{N}}}_\Lambda([f^\alpha])$ correspond under this homeomorphism, thereby respecting the orientations of the trajectories.*

Proof: Under the transformation $w = \alpha z$, the flow $\overline{\mathcal{N}}(f)$ given by (3), transforms into:

$$\frac{dw}{dt} = -|\alpha|^2(1 + |f^\alpha(w)|^4)^{-1}\overline{(f^\alpha)'(w)}f^\alpha(w).$$

Since $|\alpha|^2 > 0$, this C^1-flow on $\mathbb{C}$ has the same phase portraits as $\overline{\mathcal{N}}(f^\alpha)$. The assertion follows because the transformation $w = \alpha z$ induces a homeomorphism between the tori $T(\Lambda)$ and $T(\alpha\Lambda)$. $\qquad \square$

In other words: from a topological point of view, the Newton flows $\overline{\overline{\mathcal{N}}}_\Lambda([f]) \in N_r(\Lambda)$ and $\overline{\overline{\mathcal{N}}}_{\alpha\Lambda}([f^\alpha]) \in N_r(\alpha\Lambda)$ may be considered as equal.

More general, we call two Newton flows $\overline{\overline{\mathcal{N}}}_\Lambda([f]) \in N_r(\Lambda)$ and $\overline{\overline{\mathcal{N}}}_{\Lambda^*}([g]) \in N_r(\Lambda^*)$ equivalent ($\sim$) if they attain representatives, say $\overline{\overline{\mathcal{N}}}_\Lambda(f)$, respectively, $\overline{\overline{\mathcal{N}}}_{\Lambda^*}(g)$, and there is a homeomorphism $T(\Lambda) \to T(\Lambda^*)$, induced by the linear (over $\mathbb{R}$) basis transformation $(\omega_1, \omega_2) \mapsto (\omega_1^*, \omega_2^*)$, such that their phase portraits correspond under this homeomorphism, thereby respecting the orientations of the trajectories.

From now on, we choose for (ω_1, ω_2) a pair of so-called *reduced*[4] periods for f, so that the quotient $\tau = \frac{\omega_2}{\omega_1}$ satisfies the conditions:

$$\begin{cases} \operatorname{Im} \tau > 0, |\tau| \geqslant 1, -\frac{1}{2} \leqslant \operatorname{Re} \tau < \frac{1}{2}, \\ \operatorname{Re} \tau \leqslant 0, \quad \text{if } |\tau| = 1, \end{cases} \tag{11}$$

(Such a choice is always possible (cf. [14]). Moreover, τ is unique in the following sense: if (ω_1', ω_2') is another pair of reduced periods for f, such that $\tau' = \frac{\omega_2'}{\omega_1'}$ also satisfies the conditions (11), then $\tau = \tau'$).

Together with Lemma 4.2 this yields:

$\overline{\mathcal{N}}([f])$ is equivalent with $\overline{\mathcal{N}}\left(\left[f^{\frac{1}{\omega_1}}\right]\right)$ in $\Lambda_{1,\tau}$ and $(1, \tau)$ a pair of reduced periods for $f^{\frac{1}{\omega_1}}$.

More generally, we have:

Lemma 4.3: *Let f be – as before – an elliptic function of order r with Λ as period lattice, and let Λ^* be an arbitrary lattice. Then, there exists a function, say f^*, with $f^* \in E_r(\Lambda^*)$, such that $\overline{\mathcal{N}}_\Lambda([f]) \sim \overline{\mathcal{N}}_{\Lambda^*}([f^*])$.*

Proof: Choose (ω_1, ω_2) and (ω_1^*, ω_2^*), $\operatorname{Im} \frac{\omega_2^*}{\omega_1^*} > 0$, as basic periods for Λ resp. Λ^* and let H be a linear basis transformation from (ω_1, ω_2) to (ω_1^*, ω_2^*). The zeros and poles for f are represented by the tuples $\{[a_1], \ldots, [a_r]\}$ resp. $\{[b_1], \ldots, [b_r]\}$ in P_{ω_1, ω_2} that fulfil (8). Under H these tuples turn into tuples $\{[a_1^*], \ldots, [a_r^*]\}$, $\{[b_1^*], \ldots, [b_r^*]\}$ in $P_{\omega_1^*, \omega_2^*}$, satisfying (8) with Λ^* in the role of Λ. The latter pair of tuples determines a function f^* in $E_r(\Lambda^*)$ and thus, by Lemma 4.1, a Newton flow $\overline{\mathcal{N}}_{\Lambda^*}([f^*])$ in $N_r(\Lambda^*)$. Now, the chain rule applied to the $\mathbb{R}^2$-versions (cf. (7)) of $\overline{\mathcal{N}}_\Lambda([f])$ and $\overline{\mathcal{N}}_{\Lambda^*}([f^*])$ yields the assertion. (Use that, by assumption, $\det(H) > 0$). $\square$

Remark 4.2: Note that if $\Lambda = \Lambda^*$ (thus the basic periods for Λ, and Λ^* are related by unimodular transformations), then: $f = f^*$. Also we have: $g \in [f]$ implies $g^* \in [f^*]$.

We summarize the homogeneity results, specific for continuous elliptic Newton flows, as obtained in this and the preceding section (choose $\Lambda^* = \Lambda_{1,\tau} : \tau = i$)

Theorem 4.1 (The canonical form for elliptic Newton flows): *Given an arbitrary elliptic Newton flow, say $\overline{\mathcal{N}}_\Lambda([f])$, on $T(\Lambda)$, there exists a function f^* in $E_r(\Lambda_{1,i})$, of the form (10) with $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ in the parallellogram $P_{1,i}$ and $C = 1$, such that*

$$\overline{\mathcal{N}}_\Lambda([f]) \sim \overline{\mathcal{N}}_{\Lambda_{1,i}}([f^*]).$$

In particular, it is always possible to choose $\Lambda^* = \Lambda_{1,\tau}$, τ as in (11) and to apply the linear transformation $(1, \tau) \mapsto (1, i)$. So that we even may assume that $(1, i)$ is a pair of reduced periods for the corresponding elliptic function on $\Lambda_{1,i}$.

Hence, in the sequel, we suppress – unless strictly necessary – references to: $(1, \tau)$, Λ, class $[\cdot], \ldots$ and write $\Lambda, T, E_r, \overline{\mathcal{N}}(f), \ldots$ instead of $\Lambda_{1,\tau}, T(\Lambda), E_r(\Lambda), \overline{\mathcal{N}}_\Lambda([f]), \ldots$.

We end up by presenting two pictures of Newton flows for sn, where sn stands for a Jacobian function. This is a second-order elliptic function, attaining only simple zeros, poles and critical points, characterized by the basic periods $4K, 2iK'$. So does the phase portrait of its Newton flow. Here K, K' are two parameters defined in terms of the Weierstrass function $\wp$.

It turns out that for the phase portrait there are – up to topological equivalency – only two possibilities, corresponding to the form (rectangular or not) of the parallelogram $P_{1,\tau}$ with $(1, \tau) \in D$ and $\tau = (\frac{2iK'}{4K})$ mod 1. In fact, the crucial distinction between these two cases is whether there occur so-called 'saddle-connections' (i.e. (un)stable manifolds connecting saddles) or not (cf. [15]). Hence, it is sufficient to select for each possibility one suitably chosen example. See Figure 9 [non-rectangular, equiharmonic subcase, given by $\tau = \frac{1}{3}\sqrt{3}\exp(\frac{\pi i}{3})$] and Figure 8 [rectangular subcase given by $\text{Re}\tau = 0$]. For a detailed argumentation, see our previous work [15]; compare also the Remarks 2.14 and 2.15 in the forthcoming [12].

Note that in Figures 9 and 10 the points, labelled by $0, 4K, 2iK'$ and $4K+2iK'$ correspond to the same toroidal zero for sn (denoted by $\circ_1$), whereas both $2K$ and $2K+2iK'$ correspond to the other zero (denoted by $\circ_2$).

Similarly, $2K+iK'$ stands for a pole (denoted by $\bullet_3$) on the torus, the pair $(iK', 4K+iK')$ for the other pole (denoted by $\bullet_4$). The four torodial critical points (denoted by $+_5, \cdots, +_8$) are represented by, respectively, the pairs $(K, K + 2iK')$, $(3K, 3K + 2iK')$ and the points $K + iK'$ and $3K + iK'$; see e.g. [16] or [13].

It is well known that the periods $4K, 2iK'$ are not independent of each other, but related via a parameter $m, 0 < m < 1$, see e.g. [16]. In the situation of Figure 10: if $m \downarrow 0$, then $4K \rightarrow 2\pi, \pm 2iK' \rightarrow \infty$ and the phase portraits of $\overline{\mathcal{N}}(\text{sn})$ turn into that of $\overline{\mathcal{N}}(\sin)$; if
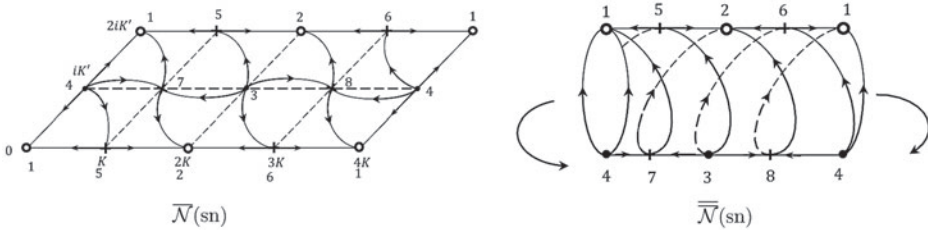

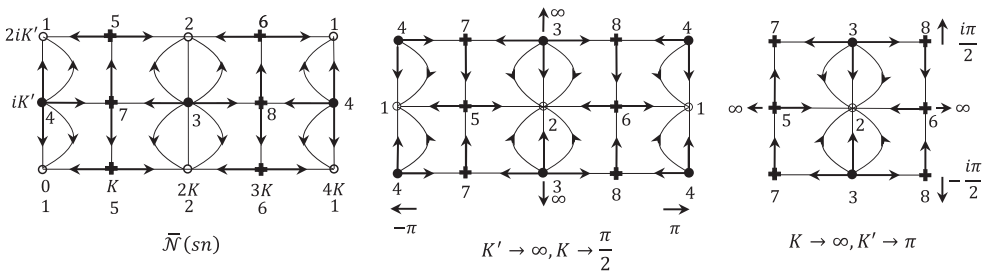**Figure 9.** Newton flows for sn; non-rectangular case; $\tau = \frac{1}{3}\sqrt{3}\exp(\frac{\pi i}{3})$.

Figure 10. Newton flows for sn; rectangular case; Re $\tau = 0$.

$m \uparrow 1$, then $\pm 4K \to \infty, 2iK' \to 2\pi i$ and the phase portraits of $\overline{\mathcal{N}}(\text{sn})$ turn into that of $\overline{\mathcal{N}}(\text{tanh})$; compare also Figures 3 and 4.

In part II (cf. [12]) of our serial on elliptic Newton flows, it has been proved – that up to conjugacy – there is only one second-order structurally stable elliptic Newton flow. So that, in a certain sense, the pictures in Figure 9 represent all examples of possible structurally stable Newton flows of order 2. On the other hand, in the case of order $r = 3$, there are several different possibilities as is explained in part III (cf. [17]) of our serial.

5. Structural stability: characterization and genericity

Adopting the notations introduced in the preceding section, let f be a function in the set $E_r(= E_r(\Lambda))$ and $\overline{\overline{\mathcal{N}}}(f)(= \overline{\overline{\mathcal{N}}}([f])) \in N_r(= N_r(\Lambda))$ its associated Newton flow (as a smooth vector field on the torus $T(= T(\Lambda))$).

By $X(T)$ we mean the set of all C^1-vector fields on T, endowed with the C^1-topology (cf. [18]). We consider the map:

$$\mathcal{F}_\Lambda : E_r \to X(T) : f \mapsto \overline{\overline{\mathcal{N}}}(f)$$

The topology τ_0 on E_r and the C^1-topology on $X(T)$ are matched by:

Lemma 5.1: *The map $\mathcal{F}_\Lambda$ is $\tau_0 - C^1$ continuous.*

Proof: In accordance with Theorem 4.1 and (10), we assume

$$f(z) = \frac{\sigma(z-a_1)\ldots\sigma(z-a_r)}{\sigma(z-b_1)\ldots\sigma(z-b_{r-1})\sigma(z-b'_r)}.$$

Put $p(z) = \sigma(z-a_1)\ldots\sigma(z - a_r)$ and $q(z) = \sigma(z-b_1)\ldots\sigma(z-b_{r-1})\sigma(z-b'_r)$. Then, the planar version $\overline{\mathcal{N}}(f)$ of the flow $\overline{\overline{\mathcal{N}}}(f)$ takes the form: (cf. (3))

$$\frac{dz}{dt} = -(|p(z)|^4 + |q(z)|^4)^{-1}(p(z)\overline{p'(z)}|q(z)|^2 - q(z)\overline{q'(z)}|p(z)|^2) \tag{12}$$

The expression in the r.h.s. is well defined (since $|p(z)|^4 + |q(z)|^4 \neq 0$ for all z) and depends – as function (F) on $\mathbb{R}^2$ – continuously differentiable on $x(=\text{Re } z)$ and $y(=\text{Im } z)$. So does the Jacobi matrix (DF) of F. Analogously, a function $g \in E_r$ chosen τ_0-close to f, gives rise to a system $\overline{\mathcal{N}}(g)$ and a function G with Jacobi matrix DG. Taking into account the very definition of C^1-topology on $X(T)$, the mapping $\mathcal{F}_\Lambda$ is continuous as a consequence of the following observation: If -w.r.t. the topology τ_0- the function g approaches f, i.e. the zeros and poles for g approach those for f, then G and DG approach F, respectively, DF on every compact subset of $\mathbb{R}^2$. □

Next, we make the concept of structural stability for elliptic Newton flows precise.

Definition 5.1: Let f, g be two functions in E_r. Then, the associated Newton flows are called *conjugate*, denoted $\overline{\overline{\mathcal{N}}}(f) \sim \overline{\overline{\mathcal{N}}}(g)$, if there is a homeomorphism from T onto itself, mapping maximal trajectories of $\overline{\overline{\mathcal{N}}}(f)$ onto those of $\overline{\overline{\mathcal{N}}}(g)$, thereby respecting the orientation of these trajectories.

Note that the above definition is compatible with the concept of 'equivalent representations of elliptic Newton flows' as introduced in Section 4; compare also (the comment on) Definition 3.1.

Definition 5.2: The flow $\overline{\overline{\mathcal{N}}}(f)$ in N_r is called τ_0-*structurally stable* if there is a τ_0-neighbourhood $\mathcal{O}$ of f, such that for all $g \in \mathcal{O}$ we have: $\overline{\overline{\mathcal{N}}}(f) \sim \overline{\overline{\mathcal{N}}}(g)$.

The set of all structurally stable Newton flows $\overline{\overline{\mathcal{N}}}(f)$ is denoted $\tilde{N}_r$.

From Lemma 5.1 it follows:

Corollary 5.1: *If $\overline{\overline{\mathcal{N}}}(f)$, as an element of $\mathcal{X}(T)$, is C^1-structurally stable [9], then this flow is also τ_0-structurally stable.*

So, when discussing structural stability in the case of elliptic Newton flows, we may skip the adjectives C^1 and τ_0.

Definition 5.3: The function f in E_r is called *non-degenerate* if:

- All zeros, poles and critical points for f are simple;
- No two critical points for f are connected by an $\overline{\overline{\mathcal{N}}}(f)$-trajectory.

The set of all non degenerate functions in E_r is denoted by $\tilde{E}_r$.

Note: If f is non-degenerate, then $\dfrac{1}{f}$ also, and these functions share their critical points; moreover, the derivative f' is elliptic of order $2r$ (=number, counted by multiplicity, of the poles for f on T). Since all zeros for f are simple, the $2r$ zeros for f' (on T) are the critical points (all simple) for f, and we find that f, as a function on T, has precisely $2r$ different critical points. Compare also the forthcoming Lemma 5.2 (Case $A = B = r$).

The main result of this section is:

Theorem 5.1: *[Characterization and genericity of structural stability]*

(1) $\overline{\overline{\mathcal{N}}}(f)$ *is structurally stable if and only if f in $\tilde{E}_r$.*
(2) *The set $\tilde{E}_r$ is open and dense in E_r.*

Proof: Will be postponed until the end of this section. □

We choose another lattice, say Λ^*. The functions f and g in $E_r(\Lambda)$ determine, respectively, functions f^* and g^* in $E_r(\Lambda^*)$, compare Lemma 4.3. It is easily verified:

- $\overline{\overline{\mathcal{N}}}(f) \sim \overline{\overline{\mathcal{N}}}(g)$ if and only if $\overline{\overline{\mathcal{N}}}(f^*) \sim \overline{\overline{\mathcal{N}}}(g^*)$
- $\overline{\overline{\mathcal{N}}}(f)$ is structurally stable if and only if $\overline{\overline{\mathcal{N}}}(f^*)$ is structurally stable.
- f is non-degenerate if and only if f^* is non-degenerate.

As an intermezzo, we look at (elliptic) Newton flows from a slightly different point of view. This enables us to perform certain calculations inserting more specific properties of elliptic functions.

Steady streams

We consider a *stream* on $\mathbb{C}$ (cf. [19]) with complex potential

$$w(z) = -\log f(z). \tag{13}$$

The *stream lines* are given by the lines $\arg f(z) = $ constant, and the *velocity field* of this stream by $\overline{w'(z)}$. Zeros and poles for f of order n resp. m, are just the *sinks* and *sources*
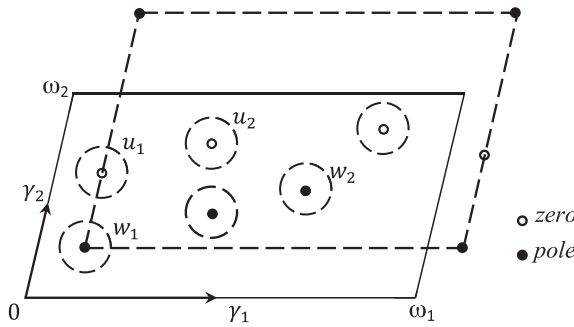
Figure 11. All zeros and poles for f inside P; after shift.

of strength n, resp. m. Moreover, it is easily verified that the so-called *stagnation points* of the steady stream (i.e. the zeros for $w'(z)$) are the critical points of the planar Newton flow $\mathcal{N}(f)$. Altogether, we may conclude that the velocity field of the steady stream given by $w(z)$ and the (desingularized) planar Newton flow $\mathcal{N}(f)$ exhibit equal phase portraits.

From now on, we assume that f has – on a period parallelogram P – the points $(\mathbf{a_1}, \ldots, \mathbf{a_A})$ and $(\mathbf{b_1}, \ldots, \mathbf{b_B})$ as zeros, resp. poles, with multiplicities $n_1, \ldots, n_A$, resp. $m_1, \ldots, m_B$ We even may assume[5] that all these zeros and poles are situated inside P (not on its boundary), cf. Figure 11.

Since f is elliptic of order r, we have:

$$\begin{cases} n_1 + \cdots + n_A = m_1 + \cdots + m_B = r \\ n_1\mathbf{a_1} + \cdots + n_A\mathbf{a_A} = m_1\mathbf{b_1} + \cdots + m_B\mathbf{b_B} \bmod \Lambda, \text{ i.e.} \\ \mathbf{b_B} = \frac{1}{m_B}[n_1\mathbf{a_1} + \cdots + n_A\mathbf{a_A} - m_1\mathbf{b_1} - \cdots - m_{B-1}\mathbf{b_{B-1}} + \lambda^0], \text{ some } \lambda^0 \in \Lambda. \end{cases} \quad (14)$$

Note that there is an explicit formula for λ^0. In fact, we have:

$$\lambda^0 = -\eta(f(\gamma_2))\omega_1 + \eta(f(\gamma_1))\omega_2,$$

where $\omega_1 (= 1)$ and $\omega_2 (= i)$ are reduced periods for Λ, and $\eta(\cdot)$ stands for winding numbers of the curves $f(\gamma_1)$ and $f(\gamma_2)$ (compare Figure 11 and [13]).

The derivative f' is an elliptic function of order $(m_1 + 1) + \cdots + (m_B + 1) = r + B$, and

$$\sharp(\text{crit. points for } f) = r + B - (n_1 - 1) - \cdots - (n_A - 1) = A + B(=: K)$$

By (10) we have:

$$f(z) = \frac{\sigma^{n_1}(z - \mathbf{a_1})\ldots\sigma^{n_A}(z - \mathbf{a_A})}{\sigma^{m_1}(z - \mathbf{b_1})\ldots\sigma^{m_B-1}(z - \mathbf{b_B})\sigma(z - \mathbf{b_B'})}, \text{ with}$$

$$\mathbf{b_B'} = n_1\mathbf{a_1} + \cdots + n_A\mathbf{a_A} - m_1\mathbf{b_1} - \cdots - (m_B - 1)\mathbf{b_B}, \quad (15)$$

$$\text{and thus } \mathbf{b_B'} = \mathbf{b_B} \bmod \Lambda.$$

So, by (13)

$$
w'(z) = - n_1 \frac{\sigma'(z - \mathbf{a_1})}{\sigma(z - \mathbf{a_1})} - \cdots - n_A \frac{\sigma'(z - \mathbf{a_A})}{\sigma(z - \mathbf{a_A})} + m_1 \frac{\sigma'(z - \mathbf{b_1})}{\sigma(z - \mathbf{b_1})} + \cdots
$$

$$
+ (m_B - 1) \frac{\sigma'(z - \mathbf{b_B})}{\sigma(z - \mathbf{b_B})} + \frac{\sigma'(z - \mathbf{b_B'})}{\sigma(z - \mathbf{b_B'})}
$$

$$
= - n_1 \zeta(z - \mathbf{a_1}) \ldots - n_A \zeta(z - \mathbf{a_A}) + m_1 \zeta(z - \mathbf{b_1}) \cdots + (m_B - 1) \zeta(z - \mathbf{b_B}) + \zeta(z - \mathbf{b_B'})
$$

where ζ stands for the Weierstrass zeta function (cf. [13]) associated with the lattice Λ. Since ζ is a quasi-periodic, meromorphic function with only poles (all simple!) in the points of the lattice Λ, the function $w'(z)$ is elliptic of order $K (= A + B)$ with K (simple) poles given by: $\mathbf{a_1}, \ldots, \mathbf{a_A}, \mathbf{b_1}, \ldots, \mathbf{b_B}$, situated in its period parallelogram P. It follows that $w'(z)$ has also K zeros (counted by multiplicity) on P. These zeros correspond with the critical points for $\overline{\overline{\mathcal{N}}}(f)$ on the torus T.

Since $\zeta' = -\wp$, where $\wp$ stands for the (elliptic!) Weierstrass $\wp$-function (cf. [13]) associated with Λ, we find (use also $\mathbf{b_B'} = \mathbf{b_B} \bmod \Lambda$):

$$
w''(z) = n_1 \wp(z - \mathbf{a_1}) \cdots + n_A \wp(z - \mathbf{a_A}) - m_1 \wp(z - \mathbf{b_1}) \cdots - m_B \wp(z - \mathbf{b_B}) \tag{16}
$$

From (14) and (15), it follows: $\mathbf{b_B}$ is determined by $\mathbf{a_1}, \ldots, \mathbf{a_A}, \mathbf{b_1}, \ldots, \mathbf{b_{B-1}}$. In mutually disjoint and suitably small[6] neighbourhoods, say $U_1, \ldots, U_A$, and $W_1, \ldots, W_{B-1}$, of respectively $\mathbf{a_1}, \ldots, \mathbf{a_A}, \mathbf{b_1}, \ldots, \mathbf{b_{B-1}}$ we choose arbitrary points $a_1, \ldots, a_A, b_1, \ldots, b_{B-1}$ and put, with fixed values n_i, m_j:

$$
b_B = \frac{1}{m_B} [n_1 a_1 + \cdots + n_A a_A - m_1 b_1 - \cdots - m_{B-1} b_{B-1} + \lambda^0]. \tag{17}
$$

In this way $a_1, \ldots, a_A, b_1, \ldots, b_{B-1}$ are close to, respectively, $\mathbf{a_1}, \ldots, \mathbf{a_A}, \mathbf{b_1}, \ldots, \mathbf{b_{B-1}}$ and b_B is close to $\mathbf{b_B}$. Finally, we put

$$
b_B' = n_1 a_1 + \cdots + n_A a_A - m_1 b_1 \cdots - m_{B-1} b_{B-1} - (m_B - 1) b_B \quad \text{(close to } \mathbf{b_B'}\text{)}. \tag{18}
$$

Perturbating $\mathbf{a_1}, \ldots, \mathbf{a_A}, \mathbf{b_1}, \ldots, \mathbf{b_{B-1}}$ into respectively $a_1, \ldots, a_A, b_1, \ldots, b_{B-1}$, and putting $\check{a} = (a_1, \ldots, a_A)$, $\check{b} = (b_1, \ldots, b_{B-1})$, we consider functions[7] $\mathbf{f}(z; \check{a}, \check{b})$ on the product space $\mathbb{C} \times U_1 \times \cdots U_A \times W_1 \times \cdots W_{B-1}$ given by: (compare (15))

$$
\mathbf{f}(z; \check{a}, \check{b}) = \frac{\sigma^{n_1}(z - a_1) \cdots \sigma^{n_A}(z - a_A)}{\sigma^{m_1}(z - b_1) \cdots \sigma^{m_B - 1}(z - b_B(\check{a}, \check{b})) \sigma(z - b_B'(\check{a}, \check{b}))}. \tag{19}
$$

Then, for each $(\check{a}, \check{b})$ in $U_1 \times \cdots U_A \times W_1 \times \cdots W_{B-1}$, the function $f|_{\check{a}, \check{b}}(\cdot) = \mathbf{f}(\cdot; \check{a}, \check{b})$ is elliptic in z, of order r. The points $a_1, \ldots, a_A$, resp. $b_1, \ldots, b_B$ are the zeros and poles for $f|_{\check{a}, \check{b}}$ on P (of multiplicity $n_1, \ldots, n_A$, resp. $m_1, \ldots, m_B$). Moreover, $f|_{\check{a}, \check{b}}$ has $K (= A + B)$ critical points on P (counted by multiplicity). Note that the Newton flow $\overline{\overline{\mathcal{N}}}(f|_{\check{a}, \check{b}})$ on T is

represented by the pair $(\check{a}, \check{b})$ in $U_1 \times \cdots U_A \times W_1 \times \cdots W_{B-1}$, i.e. by

$$a_1, \cdots, a_A; \; b_1, \cdots, b_{B-1}, \qquad \text{arbitrarily chosen in suitably small}$$
$$\uparrow \qquad \uparrow \;\; \uparrow \qquad \uparrow \quad \text{neighbourhoods } U_1 \times \cdots U_A \times W_1 \times \cdots W_{B-1}$$
$$1\times \quad 1\times \; 1\times \quad \; 1\times$$

but also by the pair (a, b) in the quotient space $V_r(\Lambda)/\approx$ as introduced in Section 4, i.e. by

$$(([a_1], \cdots, [a_A]), ([b_1], \cdots, [b_B])), \;\; \text{that fulfil condition (9)}$$
$$\uparrow \qquad\quad \uparrow \qquad \uparrow \qquad\quad \uparrow$$
$$n_1\times \qquad n_A\times \; m_1\times \qquad m_B\times$$

In particular,

- If $(\check{a}, \check{b}) = (\check{\mathbf{a}}, \check{\mathbf{b}})$, then $f|_{\check{\mathbf{a}}, \check{\mathbf{b}}}(z) = f(z)$ and thus $\overline{\overline{\mathcal{N}}}(f|_{\check{\mathbf{a}}, \check{\mathbf{b}}}) = \overline{\overline{\mathcal{N}}}(f)$;
- If $A = B = r$ (thus $K = 2r$), i.e. all zeros, poles are simple, then

$$(\check{a}, \check{b}) = (a_1, \ldots, a_r; \; b_1, \ldots, b_{r-1}) \in U_1 \times \cdots U_r \times W_1 \times \cdots W_{r-1}, \text{ and}$$
$$(a, b) = (([a_1], \cdots, [a_r]), ([b_1], \cdots, [b_r])) + \text{ condition (9)}$$

The decomposition of multiple critical points is by far the hardest part in the density proof of Theorem 5.1 and strongly relies on the following lemma. Note that in case of *non-degenerate* functions this lemma is trivial (see the note on Page 13 between Definition 5.3 and Theorem 5.1). But we also we have to cope with *degenerate* functions f. So, we need an analysis of the influence of perturbations of the zeros/poles for such functions on their critical points (being implicitly determined by these zeros/poles). Here is the lemma we need:

Lemma 5.2:   *Let $K(= A + B) > 2$. Then:*
Under suitably chosen – but arbitrarily small– perturbations of the zeros and poles for f, thereby preserving the multiplicities of these zeros and poles, the Newton flow $\overline{\overline{\mathcal{N}}}(f))$ turns into a flow $\overline{\overline{\mathcal{N}}}(f|_{\check{a}, \check{b}})$ with only (K different)onefold saddles.

Proof: We consider $\hat{w}(z; \check{a}, \check{b}) = -\log\mathbf{f}(z; \check{a}, \check{b})$ and write:

$$\frac{\partial \hat{w}}{\partial z} = \hat{w}'(z; \check{a}, \check{b}); \;\; \frac{\partial^2 \hat{w}}{\partial^2 z} = \hat{w}''(z; \check{a}, \check{b}).$$

These are meromorphic functions in each of the variables $z, a_1, \ldots, a_A, b_1, \ldots, b_{B-1}$. Define:

$$\Sigma = \{(z; \check{a}, \check{b}) \mid \hat{w}'(z; \check{a}, \check{b}) = 0\} \qquad\qquad\qquad \text{'critical set'}$$
$$\Sigma_{nd} = \{(z; \check{a}, \check{b}) \mid \hat{w}'(z; \check{a}, \check{b}) = 0, \hat{w}''(z; \check{a}, \check{b}) \neq 0\} \qquad \text{'non-degenerate critical set'}$$
$$\Sigma_d = \{(z; \check{a}, \check{b}) \mid \hat{w}'(z; \check{a}, \check{b}) = 0, \hat{w}''(z; \check{a}, \check{b}) = 0\} \qquad\quad \text{'degenerate critical set'}$$

Since the $a_i, i = 1,\ldots, A$, and $b_j, j = 1,\ldots, B$, are poles for $\hat{w}'(\cdot; \check{a}, \check{b})$ as an elliptic function in z, we have: If $(z; \check{a}, \check{b}) \in \Sigma$, then:

$$\begin{cases} z \neq a_i \bmod \Lambda, z \neq b_j \bmod \Lambda, z \neq b'_B \bmod \Lambda, \text{ and (by construction)} \\ a_{i_1} \neq a_{i_2} \bmod \Lambda, i_1 \neq i_2, i_1, i_2 = 1,\ldots, A; b_{j_1} \neq b_{j_2} \bmod \Lambda, j_1 \neq j_2, j_1, j_2 = 1,\ldots, B-1. \end{cases}$$

The subset $\mathcal{V}$ of elements $(z; \check{a}, \check{b}) \in \mathbb{C} \times U_1 \times \cdots U_A \times W_1 \times \cdots W_{B-1}$ that fulfils these inequalities is open in $\mathbb{C} \times \mathbb{C}^A \times \mathbb{C}^{B-1}$. On this set $\mathcal{V}$ (that contains the critical set Σ), the function $\hat{w}'$ is analytic in each of its variables. (Thus Σ is a closed subset of $\mathcal{V}$). For the partial derivatives of $\hat{w}'$ on $\mathcal{V}$ we find: (use (13), (19), compare also (16) and Footnote 7)

$$\frac{\partial \hat{w}'}{\partial z} : n_1 \wp(z - a_1) + \cdots + n_A \wp(z - a_A) - m_1 \wp(z - b_1) - \cdots - m_B \wp(z - b_B)$$

$$\frac{\partial \hat{w}'}{\partial a_i} : \frac{\partial}{\partial a_i}[-n_1 \zeta(z - a_1) \cdots - n_A \zeta(z - a_A) + m_1 \zeta(z - b_1) \cdots$$

$$+ (m_B - 1)\zeta(z - b_B) + \zeta(z - b'_B)]$$

$$= [\text{ using the formulas (17) and (18)}]$$

$$= \left[-n_i \wp(z - a_i) + (m_B - 1)\left[\frac{n_i}{m_B}\right]\wp(z - b_B) \right.$$

$$\left. + (n_i - (m_B - 1))\left[\frac{n_i}{m_B}\right]\wp(z - b'_B) \right]$$

$$= -n_i(\wp(z - a_i) - \wp(z - b_B)) \qquad (i = 1,\ldots, A)$$

In a similar way:

$$\frac{\partial \hat{w}'}{\partial b_j} : m_j(\wp(z - b_i) - \wp(z - b_B)) \quad (j = 1,\ldots, B - 1)$$

$$[\frac{\partial \hat{w}'}{\partial b_B} = 0 \quad (m_B = 1, 2,\ldots)]$$

By the Addition Theorem of the $\wp$-function [cf. [13]], we have:

$$\frac{\partial \hat{w}'}{\partial a_i} : -\frac{\sigma(a_i - b_B)\sigma(2z - a_i - b_B)}{\sigma^2(z - a_i)\sigma^2(z - b_B)}, i = 1,\ldots, A$$

So, let $(z; \check{a}, \check{b})$ in $\mathcal{V}$, then

$$\frac{\partial \hat{w}'}{\partial a_i}\Big|_{(z;\check{a},\check{b})} = 0, \text{ some } i \in \{1,\ldots, A\} \Leftrightarrow$$

$$\begin{cases} a_i = b_B \bmod \Lambda, & \text{[in contradiction with '}a_i, b_B \text{ different']} \\ \text{or} & \\ 2z = a_i + b_B \bmod \Lambda & \text{[if } 2z = a_{i_1} + b_B \bmod \Lambda, 2z = a_{i_2} + b_B \bmod \Lambda, i_1 \neq i_2, \\ & \text{then } a_{i_1} = a_{i_2}; \text{ in contradiction with '}a_{i_1}, a_{i_2} \text{ different']} \end{cases}$$

From this follows:

If $(z; \check{a}, \check{b}) \in \mathcal{V}$, then at most one of $\frac{\partial \hat{w}'}{\partial a_i}|_{(z;\check{a},\check{b})}, i = 1,\ldots,A$, vanishes. By a similar reasoning, we even may conclude:

$$\begin{cases} \text{At most one of the partial derivatives} \\ \frac{\partial \hat{w}'}{\partial a_i}(z; \check{a}, \check{b}), \frac{\partial \hat{w}'}{\partial b_j}(z; \check{a}, \check{b}), (z; \check{a}, \check{b}) \in \mathcal{V}, i = 1,\ldots,A, j = 1,\ldots,B-1, \\ \text{vanishes, and thus, } \textbf{in case K} > \textbf{2:} \\ \frac{\partial \hat{w}'}{\partial a_i}(z; \check{a}, \check{b}) \neq 0, \frac{\partial \hat{w}'}{\partial b_j}(z; \check{a}, \check{b}) \neq 0, \\ \text{for at least one } i \in \{1,\ldots,A\} \text{ or } j \in \{1,\ldots,B-1.\} \end{cases} \tag{20}$$

The latter conclusion cannot be drawn in case $K = 2$; however, see the forthcoming Remark 5.1. Note that always $K \geqslant 2$.

Under the assumption that $K > 2$: let $\mathbf{z_1}, \mathbf{z_2},\ldots,\mathbf{z_L}$ be the different critical points for $f = (\mathbf{f}(\cdot, \check{\mathbf{a}}, \check{\mathbf{b}}))$ with multiplicities $K_1,\ldots,K_L, K_1 \geqslant \cdots \geqslant K_L \geqslant 1, K_1 + \cdots + K_L = K$. If $(\check{a}, \check{b})$ tends to $(\check{\mathbf{a}}, \check{\mathbf{b}})$, then K_l of the K critical points for $\mathbf{f}(\cdot, \check{a}, \check{b})$(counted by multiplicity) tend to the K_l-fold saddle $\mathbf{z_l}$ for $\overline{\overline{\mathcal{N}}}(f)$). It follows that, if $(\check{a}, \check{b})$ is sufficiently close to $(\check{\mathbf{a}}, \check{\mathbf{b}})$, then K_l critical points for $\mathbf{f}(\cdot, \check{a}, \check{b})$(counted by multiplicity) are situated in, suitably small, disjunct neighbourhoods, say V_l, around $\mathbf{z_l}, l = 1,\ldots,L$. We choose $(\check{a}, \check{b})$ so close to $(\check{\mathbf{a}}, \check{\mathbf{b}})$ that this condition holds. If all the critical points for f, i.e. the saddles of $\overline{\overline{\mathcal{N}}}(f)$) are simple, there is nothing to prove. So, let $K_1 > 1$, thus $\hat{w}''(\mathbf{z_1}; \check{\mathbf{a}}, \check{\mathbf{b}}) = 0$, i.e. $(\mathbf{z_1}; \check{\mathbf{a}}, \check{\mathbf{b}}) \in \Sigma_d \subset \Sigma$. Without loss of generality, we assume (see (20)) that $\frac{\partial \hat{w}'}{\partial a_i}(\mathbf{z_1}; \check{\mathbf{a}}, \check{\mathbf{b}}) \neq 0$. According to the Implicit Function Theorem a local parametrization of Σ around $(\mathbf{z_1}; \check{\mathbf{a}}, \check{\mathbf{b}})$ exists, given by:

$$(z; a_1(z, a_2,\ldots,a_A, b_1,\ldots,b_{B-1}), a_2,\ldots,a_A, b_1,\ldots,b_{B-1}),$$

where $a_1(\mathbf{z_1}, \mathbf{a_2},\ldots,\mathbf{a_A}, \mathbf{b_1},\ldots,\mathbf{b_{B-1}}) = \mathbf{a_1}$. Thus, at $(\mathbf{z_1}; \check{\mathbf{a}}, \check{\mathbf{b}})$ we have:

$$\hat{w}'' + \left[\frac{\partial \hat{w}'}{\partial a_1}\right]\left[\frac{\partial a_1}{\partial z}(z, a_2,\ldots)\right] = 0.$$

Since $\hat{w}''(\mathbf{z_1}; \check{\mathbf{a}}, \check{\mathbf{b}}) = 0$ and $\frac{\partial \hat{w}'}{\partial a_i}(\mathbf{z_1}; \check{\mathbf{a}}, \check{\mathbf{b}}) \neq 0$, it follows that

$$\frac{\partial a_1}{\partial z}(\mathbf{z_1}, \check{\mathbf{a}}, \check{\mathbf{b}}) = 0$$

Note that $a_1(z, a_2,\ldots,a_A, b_1,\ldots,b_{B-1})$, depends complex differentiable on z. So the zeros for $\frac{\partial a_1}{\partial z}(z; a_2,\ldots,a_A, b_1,\ldots,b_{B-1})$ are *isolated*. Thus, on a *reduced* neighbourhood of $(\mathbf{z_1}, \check{\mathbf{a}}, \check{\mathbf{b}})$, say $\hat{U}$, neither $\frac{\partial a_1}{\partial z}(\cdot)$ nor $\frac{\partial \hat{w}'}{\partial a_1}$ vanish. If z tends to $\mathbf{z_1}$, then:

$$(z; a_1(z, \mathbf{a_2},\ldots,\mathbf{a_A}, \mathbf{b_1},\ldots,\mathbf{b_{B-1}}), \mathbf{a_2},\ldots,\mathbf{a_A}, \mathbf{b_1},\ldots,\mathbf{b_{B-1}})$$

tends to $(\mathbf{z_1}, \check{\mathbf{a}}, \check{\mathbf{b}})$ along Σ, and we cross $\hat{U}$, meeting elements $(z, \check{a}, \check{b}) \in \Sigma$, such that

$$\begin{cases} \hat{w}''(z, \check{a}, \check{b}) + \frac{\partial \hat{w}'}{\partial a_1}(z, \check{a}, \check{b})\frac{\partial a_1}{\partial z}(z, \check{a}, \check{b}) = 0 \\ \frac{\partial \hat{w}'}{\partial a_1}(z, \check{a}, \check{b}) \neq 0, \frac{\partial a_1}{\partial z}(z, \check{a}, \check{b}) \neq 0 \end{cases}$$

Thus,

$$\hat{w}''(z, \check{a}, \check{b}) \neq 0.$$

Hence, we have: $(z, \check{a}, \check{b}) \in \Sigma_{nd}$. So, the K_1 critical points for $\mathbf{f}(\cdot; \check{a}, \check{b})$ that approach $\mathbf{z_1}$ via the 'curve'

$$(z; a_1(z, \mathbf{a_2}, \ldots, \mathbf{a_A}, \mathbf{b_1}, \ldots, \mathbf{b_{B-1}}), \mathbf{a_2}, \ldots, \mathbf{a_A}, \mathbf{b_1}, \ldots, \mathbf{b_{B-1}})$$

are all simple, whereas the critical points for $\mathbf{f}(\cdot, \check{a}, \check{b})$ approaching $\mathbf{z_2}, \ldots, \mathbf{z_L}$ are still situated in, respectively $V_2, \ldots, V_L$. If $K_2 > 1$, we repeat the above procedure with respect to $\mathbf{z_2}$, etc. In a finite number of steps, we arrive at a flow $\overline{\overline{\mathcal{N}}}(\mathbf{f}|_{\check{a}, \check{b}})$ with only simple saddles and $(\check{a}, \check{b})$ arbitrary close to $(\check{\mathbf{a}}, \check{\mathbf{b}})$. □

Remark 5.1: The case $A = B = 1$ (i.e. $K = 2$).
If $K = 2$, then the function f has – on T – only one zero and one pole, both of order r; the corresponding flow $\overline{\overline{\mathcal{N}}}(f))$ is referred to as to a *nuclear* Newton flow. In this case, the assertion of Lemma 5.2 is also true. In fact, even a stronger result holds:

'All nuclear Newton flows – of any order r– are conjugate, in particular each of them
has precisely two saddles (simple) and there are no saddle connections'.

Nuclear Newton flows will play an important role in the creation of elliptic Newton flows, but we postpone the discussion on this subject to a sequel of the present paper, see [20].

We end up by presenting the (already announced) proof of Theorem 5.1

Proof of Theorem 5.1:
The 'density part' of Assertion 2: Let $\mathcal{O}$ be an arbitrarily small τ_0-neighbourhood of a function f as in Lemma 5.2. We split up each of the A different zeros and B different poles for f into n_i resp. m_j mutually different points, contained in disjoint neighbourhoods U_i resp. $W_j, i = 1, \ldots, A, j = 1, \ldots, B$ (compare Figure 11) and take into account relation (8). In this way, we obtain $2r$ different points, giving rise to an elliptic function of the form (10), with these points as the r simple zeros /r simple poles in P. We may assume that this function is still situated in $\mathcal{O}$, see the introduction of the topology τ_0 in Section 4. Now, we apply Lemma 5.2 (case $A = B = r, K = 2r$) and find in $\mathcal{O}$ an elliptic function, of order r with only simple zeros, poles and critical points. This function is non-degenerate if the corresponding Newton flow does not exhibit trajectories that connect two of its critical points. If this is the case, none of the straight lines connecting two critical values for our function, passes through the point $0 \in \mathbb{C}$. If not, then adding an arbitrarily small constant $c \in \mathbb{C}$ to f does not affect the position of its critical points, and yields a function – still in[8] $\mathcal{O}$- with only simple zeros and poles. By choosing c suitably, we find a function, renamed f, such that none of the straight lines connecting critical values (for different critical points) contains $0 \in \mathbb{C}$. So, we have: $f \in \tilde{E}_r$, i.e. $\tilde{E}_r$ is dense in E_r.

The 'if part' of Assertion 1: Let $f \in \tilde{E}_r$. Then all equilibria for the flow $\overline{\overline{\mathcal{N}}}(f)$ are hyperbolic (cf. Remarks 1.1 and 3.1). Moreover, there are neither saddle-connections nor closed orbits (compare (4)) and the limiting sets of the trajectories are isolated equilibria (cf. Lemma 3.1). Now, the Baggis–Peixoto Theorem for structurally stable C^1-vector fields on compact two-dimensional manifolds (cf. [9,21]) yields that $\overline{\overline{\mathcal{N}}}(f)$ is C^1-structurally stable, and is by Corollary 5.1 also τ_0-structural stable.

The **'only if part' of Assertion 1**: Suppose that $f \notin \tilde{E}_r$, but $\overline{\overline{\mathcal{N}}}(f)$ in $\tilde{N}_r$. Then there is a τ_0-neighbourhood of f, say $\mathcal{O}$, such that for all $g \in \mathcal{O}$: $\overline{\overline{\mathcal{N}}}(f) \sim \overline{\overline{\mathcal{N}}}(g)$. Since $\tilde{E}_r$ is dense in E_r(already proved), we may assume that $g \in \tilde{E}_r$. So, $\overline{\overline{\mathcal{N}}}(g)$ has precisely r hyperbolic attractors/repellors and does not admit 'saddle connections'. This must also be true for $\overline{\overline{\mathcal{N}}}(f)$, in contradiction with $f \notin \tilde{E}_r$.

The **'openess part' of Assertion 2**: This a direct consequence of the Assertion 1 (which has already been verified above).   □

Notes

1. An equilibrium for a C^1- vector field on $\mathbb{R}^2$ is called hyperbolic if the Jacobi matrix at this equilibrium has only eigenvalues with non vanishing real parts (cf. [10]).
2. i.e. $\tilde{D}F(x_1, x_2) \cdot DF(x_1, x_2) = \det(DF(x_1, x_2))I_2$, where I_2 stands for the 2×2-unit matrix.
3. I.e. each period is of the form $n\omega_1 + m\omega_2, n, m \in \mathbb{Z}$. In particular, $\mathrm{Im}\frac{\omega_2}{\omega_1} \neq 0$ (cf. [13,14]).
4. The pair of basic periods (ω_1, ω_2) for f is called *reduced* or *primitive* if $|\omega_1|$ is minimal among all periods for f, whereas $|\omega_2|$ is minimal among all periods ω for f with the property $\mathrm{Im}\frac{\omega}{\omega_1} > 0$ (cf. [14]).
5. If this is not the case, an (arbitrary small) shift of P along its diagonal, is always possible such that the resulting parallelogram satisfies our assumption (cf. Figure 11 and [13]).
6. Choose these neighbourhoods so that they are contained in the period parallelogram P, cf. Figure 11.
7. Note that, when perturbing $(\mathbf{a}, \mathbf{b})$ in the indicated way, the winding numbers $\eta(\mathbf{f}_{(\cdot, a, b)}(\gamma_1))$, $\eta(\mathbf{f}_{(\cdot, a, b)}(\gamma_2))$, and thus also λ^0, remain unchanged.
8. Note that at simple zeros an analytic function is conformal. In case of a pole, use also (6).

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