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To Know is to Know the Value of a Variable

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Abstract

We develop an epistemic logic that can express *knowledge of a dependency between variables* (or complex terms). An epistemic dependency formula $K_a^{t_1,\ldots,t_n} t$ says that agent a knows the value of term t conditional on being given the values of terms $t_1, ..., t_n$. We add dynamic operators $[!t_1, \ldots, t_n]\phi$, capturing the effect of publicly (and simultaneously) announcing the values of terms $t_1, \ldots, t_n$. We prove completeness, decidability and finite model property.

Keywords: knowing what, knowledge de re, dynamic epistemic logic.

1 Introduction

In this paper we build on the work of Plaza [14,15], and Wang and Fan [24,25] on formalizing the notion of 'knowledge de re' (knowledge of an object, "knowledge what") over Kripke models [2]. We understand this as *knowing the value of a variable*. Here, a variable is what in first-order modal logic is called a "non-rigid designator" x, taking different values (in some fixed domain D) at different possible worlds. If we denote by $w(x)$ the value of variable x at world w, and we denote by $\sim_a$ the epistemic accessibility relation of some agent a, then Plaza's semantics for 'knowledge de re' is given by putting [3]: $w \models K_a x$ iff $\forall v \sim_a w\ (v(x) = w(x))$. This is a natural analogue of the usual semantics of "knowledge that" in epistemic logic: an agent knows the value of x if that value is the same in all her epistemic alternatives. When the range D of possible values of x is finite, then this operator is obviously reduceable to the usual one, via a finite disjunction $\bigvee_{d \in D} K_a(x = d)$. But in general this is not possible. Plaza [15] had a very simple axiomatization of this operator (in combination with the usual epistemic operator $K_a\varphi$ for "knowing that"), and claimed its completeness, based on a reduction to standard epistemic logic. He also extended

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[2] In its turn, the work of Wang and Fan builds on previous research in Security on knowledge of keys and passwords, e.g. [9,11,23].

[3] We use the the same symbol K_a for "knowledge what" as the usual epistemic operator for "knowledge that", and we use variables $x, y, \ldots$ to denote the non-rigid designators. Plaza, and Wang and Fan, use a different notation Kv_a for "knowledge what", and denote the non-rigid designators by constants c. In our framework, doing this would be very confusing, since we also have *rigid* designators, which are naturally denoted by constants.

this logic with public announcement operators [4] $[!\varphi]\psi$ (of which he is the main originator [14]), and used the resulting logic to treat the classical "Sum and Product" puzzle. But he could not prove completeness of this extended logic. Wang and Fan solved this problem by introducing a *conditional version* of the above operator $K_a^\varphi x$ ("conditionally knowing what"), with the intuitive meaning that agent a could find the value of x *if given* the additional information that φ was the case. The introduction of such a conditional operator allows one to "pre-encode" the dynamics of $[!\varphi]\psi$, following a strategy pioneered by van Benthem [20], obtaining Reduction Axioms that allow us to reduce any dynamic formula to a static one. Their completeness proof was very complex (in the multi-agent case), going via a detour through first-order intentional logic. A more natural type of announcement in this context is the action $!x$ of *publicly announcing the value of* x. In a recent talk (Univ. of Amsterdam 2015), Wang stated as an open question the problem of finding a complete axiomatization for a logic that combines the operators for "knowledge that" $K\phi$, "knowledge of a value" Kx, propositional public announcements $[!\phi]\psi$ and public announcements of values $[!x]\psi$. This problem remained open until now, despite efforts in this direction by van Eijck, Gattinger and Wang. [5]

In this paper we solve this problem, by introducing another kind of conditional version of the above operator. An *epistemic dependency formula* $K_a^{x_1,\dots,x_n}y$ says that an agent knows the value of some variable y *conditional* on being given the values of variables $x_1, ..., x_n$. The semantics is the obvious generalization of the above clause: if we use the abbreviation $w(\overrightarrow{x}) = v(\overrightarrow{x})$ for the conjunction $w(x_1) = v(x_1) \wedge \dots w(x_n) = v(x_n)$, then we put

$$w \models K_a^{x_1,\dots,x_n}y \quad \text{iff} \quad \forall v \sim_a w \ (w(\overrightarrow{x}) = v(\overrightarrow{x}) \Rightarrow v(y) = w(y))\,.$$

In words: an agent knows y given $x_1, \dots, x_n$ if the value of y is the same in all the epistemic alternatives that agree with the actual world on the values of $x_1, \dots, x_n$. This operator has connections with Dependence Logic [6] and allows us to "pre-encode" the dynamics of the value-announcement operator $[!x]\varphi$.

Besides the epistemic dependency formulas and the dynamic public value-announcement operator, we introduce a number of other formal innovations, that are useful for both technical and conceptual purposes. One is that, in addition to variables, we also allow *constants* (i.e. rigid designators) c, whose value is the same in all possible worlds, as well as more complex terms t (built

[4] We use the dynamic-logic style notation for this operator that is by now standard in Dynamic Epistemic Logic, which we regard as natural: this is a dynamic modality, capturing weakest precondition of an action exactly as in PDL, except that the action is the one of publicly announcing φ. Plaza uses the more opaque notation $\varphi + \psi$.

[5] After submitting the AiML abstract, we became aware of an unpublished draft by van Eijck, Gattinger and Wang, containing work in progress on a *partial* solution to this problem. The logic axiomatized there has knowledge de re operators Kx and value announcements $[!x]\psi$, but it *cannot* express the usual "knowledge that" $K\phi$, *nor* the usual (propositional) public announcements $[!\phi]\psi$, so it is *not* a complete solution to the above problem.

[6] See Section 4.

from variables and constants using functions). Moreover, we have *relational atoms* $R(t_1, \ldots, t_n)$ expressing relationships between terms, and in particular an *equality predicate* $t = t'$, which captures *identity of values* and plays an essential role in our system. Although statements $K_a t$ *cannot* in general be reduced to "knowledge that" statements $K_a\varphi$, formulas of the form $x = c$ can be used to provide "local reductions": semantically, at each possible world w, $K_a x$ is equivalent to $K_a(x = c)$, where c is denotes the value $w(x)$ of variable x in world w. In our axiomatic system, this "local reduction" takes the shape of our "Knowledge De Re" axiom, whose relevant instance in this case is the validity

$$(x = c) \Rightarrow (K_a x \Leftrightarrow K_a(x = c)).$$

In words: *when the value of x is c, then knowing the value of x is the same as knowing that this value is c.* Our Knowledge De Re axiom generalizes this to epistemic dependency formulas. Combined with our "Existence of Value" Rule (saying that *variables always have a value*), this allows us to prove complex properties of epistemic dependence (e.g. the well-known Armstrong axioms [2]) in a simple way, from basic epistemic axioms. It also allows us to provide a rather simple completeness proof, based on a variation of the canonical model construction, in which constants act as 'witnesses' for the values of variables.

Another technical innovation is that we include a special type of variables $?_\varphi$, storing the *truth value of formula* φ. On the one hand, this introduces another layer of (interesting) technical complexity, since terms of the form $?_\varphi$ are even "more non-rigid" than the generic variables x, in that they can change their value while this value is being learnt. Indeed, while x keeps its value when that value is publicly announced, terms $?_\varphi$ corresponding to Moore sentences φ (such as "$x = 0$ but you don't know it") may change their values after being learnt. On the other hand, the use of such "fluctuating variables" allow us to simplify the syntax, by reducing the usual 'knowledge that' operator to 'knowledge what', via the equivalence $K_a\varphi \Leftrightarrow (\varphi \wedge K_a?\varphi)$. This is in the spirit of the Schaffer quote above: to "know that" φ is to know the answer to the question "what is the truth value of φ?" So, unlike Plaza, and Wang and Fan, we do not need two epistemic operators: there is only one kind of knowledge, namely knowing the value of a variable. Similarly, propositional announcements $[!\varphi]\psi)$ can be reduced to learning the value of variable $?_\varphi$.

So one could say, without exaggerating too much, that all knowledge "is", or can at least be *represented as*, knowledge of the value of a variable. Hence, this paper's title: itself a paraphrase of Quine's famous *dictum.*[7] This epistemic-modeling variant seems less problematic than the original, ontological version! And, in fact, our formalism suggests that the unary knowledge operator is just a special case. The more general version of our motto is: *to know is to know the dependence between (values of) variables.* This fits well with the popular view of knowledge-acquisition as a process of *learning correlations* (with the goal of eventually *tracking causal relationships* in the actual world).

[7] "To be is, purely and simply, to be the value of a variable" [16].

2 The Logic of Epistemic Dependency

In the following, we assume given a finite set $\mathcal{A}$ of "agents", and four countable sets of symbols: a set P of *propositional atoms*; a set Var of *variables*; a set C of *constants*, among which there are two distinguished constants 0 and 1 (with $0 \neq 1$); a set $\mathcal{F}$ of *functional symbols* and a set $\mathcal{R}$ of *relational symbols*, together with an *arity map* $ar : \mathcal{F} \cup \mathcal{R} \to N^*$, associating to all symbols $f \in F, R \in Rel$ natural numbers $ar(f), ar(R) \in N^*$. $\mathcal{R}$ includes an *equality* symbol $=$, with $ar(=) = 2$. Intuitively, the difference between variables and constants is that constants are rigid designators, while variables are non-rigid: so a variable can take different values in different possible worlds, while a constant denotes the same objects in all the worlds of a given model. We use letters $p, q, \ldots$ to denote atoms in P, letters $x, y, \ldots$ to denote variables in Var, and letters $c, d, \ldots$ to denote constants in C. We denote by $\overrightarrow{x}$ finite strings $\overrightarrow{x} = (x_1, \ldots, x_k) \in Var^*$ of variables (of any length $k \geq 0$), and similarly use $\overrightarrow{c}$ to denote finite strings $\overrightarrow{c} = (c_1, \ldots, c_k) \in C^*$ of objects. We denote by λ the empty string.

Syntax. The *Logic of Epistemic Dependency* (LED) has a twofold syntax, consisting of a set $\mathcal{L} = \mathcal{L}(P, Var, C, F, ar)$ of propositional *formulas* φ and a set $\mathcal{T} = \mathcal{T}(P, Var, C, \mathcal{F}, \mathcal{R}, ar)$ of *terms* t, defined by double recursion:

$$\varphi ::= p \mid R(\overrightarrow{t}) \mid \varphi \to \varphi \mid K_a^{\overrightarrow{t}} t$$

$$t ::= x \mid \quad c \quad \mid \quad ?_\varphi \quad \mid f(\overrightarrow{t})$$

where $a \in \mathcal{A}$ are agents, $x \in Var$ are variables, $c \in C$ are constants, $t \in \mathcal{T}$ are terms, $\overrightarrow{t}$ are finite tuples of terms and $f \in \mathcal{F}, R \in \mathcal{R}$ are symbols of arity equal to the length of $\overrightarrow{t}$. We abbreviate $=(t, t')$ as $t = t'$.

Semantics. A *model (for $\mathcal{L}$ and $\mathcal{T}$)* is a structure

$$M = (W, D, [0], [1], \sim_a, \| \bullet \|, \bullet(\bullet), \mathbf{f}, \mathbf{R})_{a \in \mathcal{A}, f \in \mathcal{F}, R \in \mathcal{R}}$$

where: W is a set of *possible worlds*; D is a set of *objects*, containing at least two designated objects $[0] \neq [1]$; $\sim_a \subseteq W \times W$ are equivalence relations, called *epistemic indistinguishability* relations; $\| \bullet \|$ is a *valuation function* mapping each atomic sentence $p \in P$ to a set $\|p\| \subseteq W$ of possible worlds; $\bullet(\bullet)$: $W \times (Var \cup C) \to D$ is a map associating to each world $w \in W$ and each variable or constant $\alpha \in Var \cup C$ some object $w(\alpha) \in D$, called *the value of α at world w*, and satisfying the requirement that *the value of each constant is the same in all the worlds*: i.e., $w(c) = w'(c)$ for all $c \in C$ and all $w, w' \in W$; and for all symbols $f \in \mathcal{F}, R \in \mathcal{R}$ of arity $ar(f) = n$, we are given n-ary maps $\mathbf{f} : D^n \to D$ and n-ary relations $\mathbf{R} \subseteq D^n$, with the *standard interpretation of equality* $=$ as the *diagonal* $\{(d, d) : d \in D\}$ of D.

For the semantics, we simultaneously define an *extended valuation* (the "truth map") $\|\varphi\|_M$ for all formulas φ, and an *extended value map* $w(t)_M$ for all terms t and all worlds $w \in W$. We will use the notation $w(\overrightarrow{t}) := (w(t_1), \ldots, w(t_k))$ for the string of values corresponding to any given string of terms $\overrightarrow{t} = (t_1, \ldots, t_k) \in \mathcal{T}^*$. The truth map is given for propositional atoms $p \in P$ by the valuation $\|p\|$, and extended to other formulas by recursively putting: $\|R(\overrightarrow{t})\| = \{w \in W | w(\overrightarrow{t}) \in \mathbf{R}\}$; $\|\varphi \to \psi\| = (W \setminus \|\varphi\|) \cup \|\psi\|$; $\|K_a^{\overrightarrow{t}} t'\| = \{w \in W | \forall v \in W (w \sim_a v \wedge w(\overrightarrow{t}) = v(\overrightarrow{t}) \Rightarrow w(t') = v(t'))\}$. The extended value map $w(t')$ is given by the value map $w(\alpha)$ for variables and constants $\alpha \in Var \cup C$, and extended to other terms by recursion: $w(?_\varphi) = [1]$ iff $w \in \|\varphi\|$; $w(?_\varphi) = 0$ iff $w \notin \|\varphi\|$; $w(f(\overrightarrow{t})) = \mathbf{f}(w(\overrightarrow{t}))$.

Abbreviations. We put $\top := (1 = 1)$; $\bot := (1 = 0)$: $\neg\varphi := \varphi \to \bot$; $\varphi \vee \psi := \neg\varphi \to \psi$; $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$; $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$; $K_a^{\overrightarrow{t}} \varphi := \varphi \wedge K_a^{\overrightarrow{t}} ?_\varphi$; $\langle K_a^{\overrightarrow{t}} \rangle \varphi := \neg K_a^{\overrightarrow{t}} \neg\varphi$; $K_a\varphi := K_a^\lambda \varphi$ (where λ is the empty string); $\langle K_a \rangle \varphi := \neg K_a \neg\varphi$; $K_a^\varphi \psi := K_a(\varphi \to \psi)$. We also put $K_a^{\overrightarrow{t}} \overrightarrow{t'} := \bigwedge_{1 \leq i \leq k} K_a^{\overrightarrow{t}} t'_i$, and $(\overrightarrow{t} = \overrightarrow{t'}) := \bigwedge_{1 \leq i \leq k} t_i = t'_i$, where k is the length of $\overrightarrow{t'}$.

Ground Terms A *ground term* is a term that contains no variables and no propositional formulas (hence, no ?); in other words, ground terms are built only from constants $c, d, \ldots \in C$ by recursively applying function symbols $f, g, \ldots$. Let us denote by $\mathcal{T}^0$ the set of all ground terms.

Propositional Substitution: For atoms $p \in P$ and formulas θ, the *substitution of p with θ* is an operation mapping every formula $\varphi \in \mathcal{L}$ into a new formula $\varphi[p/\theta] \in \mathcal{L}$, and similarly mapping every tuple of term $\overrightarrow{t} \in \mathcal{T}$ into a new tuple $\overrightarrow{t}[p/\theta]$, obtained by uniformly substituting p with θ as usual. [8]

Variable Substitution: For variables $x \in Var$ and terms $t \in \mathcal{T}$, the *substitution of x with t* is an operation mapping every formula $\varphi \in \mathcal{L}$ into a new formula $\varphi[x/t] \in \mathcal{L}$, and mapping every term $t' \in \mathcal{T}$ into a new term $t'[x/t]$, obtained by uniformly substituting x with t in the usual way. [9]

Example 1. Alice and Bob have each a natural number written on their foreheads. It is common knowledge that Alice's number x_a is the immediate successor of Bob's number x_b. Both are blindfolded, so nobody can see the numbers. The *model* has: $Var = \{x_a, x_b\}$, $D = C = N$ is the set of natural numbers; $\mathcal{F} = \{+, \times\}$ and $\mathcal{R} = \{=, >\}$ contain the usual operations and relations on N; the set W of worlds consists of all functions $w : Var \to N$, satisfying the given constraint $w(x_a) = w(x_b) + 1$; the epistemic relations are given by the universal relations: $\sim_a = \sim_b = W \times W$. Note that the sentence $\neg K_a x_a \wedge \neg K_b x_b \wedge K_a(x_a > x_b) \wedge K_b(x_a > x_b) \wedge K_a^{x_b} x_a \wedge K_b^{x_a} x_b$ is true in all worlds. So *nobody knows his/her number, but both know that Alice's number is larger, and both could come to know the numbers if given only the other's number.*

[8] More precisely: $p[p/\theta] := \theta$; $q[p/\theta] := q$; $(R(t_1, \ldots, t_n))[p/\theta] := R(t_1[p/\theta], \ldots, t_n[p/\theta])$; $(\varphi \to \psi)[p/\theta] := \varphi[p/\theta] \to \psi[p/\theta]$; $K_a^{t_1, \ldots, t_n} t[p/\theta] := K_a^{t_1[p/\theta], \ldots, t_n[p/\theta]} t[p/\theta]$; $\lambda[p/\theta] := \lambda$; $c[p/\theta] := c$; $x[p/\theta] := x$; $?_\varphi[p/\theta] := ?_{\varphi[p/\theta]}$; $f(t_1, \ldots, t_n)[p/\theta] := f(t_1[p/\theta], \ldots, t_n[p/\theta])$.

[9] I.e., $\lambda[x/t] := \lambda$; $c[x/t] := c$; $x[x/t] := t$; $y[x/t] := y$; $?_\varphi[x/t] := ?_{\varphi[x/t]}$; $f(t_1, \ldots, t_n)[x/t] := f(t_1[x/t], \ldots, t_n[x/t])$; $p[x/t] := p$; $(R(t_1, \ldots, t_n)[x/t] := R(t_1[x/t], \ldots, t_n[x/t])$; $(\varphi \to \psi)[x/t] := \varphi[x/t] \to \psi[x/t]$; $(K_a^{t_1, \ldots, t_n} t')[x/t] := K_a^{t_1[x/t], \ldots, t_n[x/t]} t'[x/t]$.

Proof system. The proof system LED consists of the following:

RULES:

- *Propositional Substitution*: From φ, infer $\varphi[p/\theta]$.
- *Variable Substitution*: From φ, infer $\varphi[x/t]$.
- *Modus Ponens Rule*: From φ and $\varphi \to \psi$, infer ψ.
- *Necessitation*: From φ, infer $K_a\varphi$.
- *Existence-of-Value Rule* (EVR):

$$\text{From } x = c \to \varphi \text{ , infer } \varphi, \text{ provided that } c \text{ does not occur in } \varphi.$$

AXIOMS:

- All the classical propositional tautologies.
- All the $S5$ axioms for K_a.
- *Knowledge De Re*:

$$(\overrightarrow{x} = \overrightarrow{c} \wedge y = d) \to \left(K_a^{\overrightarrow{x}} y \leftrightarrow K_a^{\overrightarrow{x}=\overrightarrow{c}} y = d\right)$$

- *Equality Axioms*:

$$x = x$$

$$x = y \to y = x$$

$$(x = y \wedge y = z) \to x = z$$

$$\overrightarrow{x} = \overrightarrow{y} \to f(\overrightarrow{x}) = f(\overrightarrow{y})$$

$$(x = y \wedge R(\overrightarrow{z}, x, \overrightarrow{w})) \to R(\overrightarrow{z}, y, \overrightarrow{w})$$

- *Characteristic Functions*:

$$?_\varphi = 1 \leftrightarrow \varphi,$$

$$?_\varphi = 0 \leftrightarrow \neg\varphi,$$

- *Knowledge of Functions*:

$$K_a^{\overrightarrow{x}} f(\overrightarrow{x})$$

In fact, two of the axioms are redundant: symmetry and transitivity of $=$ follow from the other axioms, but we chose to include them for convenience. We write $\vdash \psi$ if ψ is provable in the proof system LED. For any set of formulas Φ and any formula ψ, we write $\Phi \vdash \psi$ if there exist finitely many formulas $\phi_1, \ldots, \phi_n \in \Phi$ (for some $n \in N$) such that $\vdash (\phi_1 \wedge \ldots \phi_n) \to \psi$. We say that Φ is *logically closed* if, for every formula $\psi \in \mathcal{L}$, $\Phi \vdash \psi$ implies $\psi \in \Phi$. We say that Φ is *consistent* if $\Phi \nvdash \perp$, and that *a formula φ is consistent with* Φ if $\Phi \cup \{\varphi\}$ is consistent (equivalently: if $\Phi \nvdash \neg\varphi$).

Lemma 2.1 *For a set Φ of formulas, put $K_a^{\overrightarrow{t}} \Phi := \{K_a^{\overrightarrow{t}} \phi : \phi \in \Phi\}$. Then we have that:*

- *if $\Phi \vdash \psi$ then $K_a^{\overrightarrow{t}} \Phi \vdash K_a^{\overrightarrow{t}} \psi$.*
- *$\Phi \cup \{\psi\} \vdash \theta$ iff $\Phi \vdash (\psi \to \theta)$.*

Proposition 2.2 *Let φ be a formula and z be a variable that does* not *occur in the scope of any K_a-operator in φ. Then the following is provable in LED:*

$$\vdash (x = y \wedge \varphi[z/x]) \to \varphi[z/y].$$

NOTE The unrestricted version of the above schema is not valid! A counterexample is obtained by taking for instance φ to be the formula $K_a(z = c)$. This is related to the Phosphorus/Hesperus paradox.

Proposition 2.3 *(*Knowledge of Ground Terms and Ground Identities*). For all **ground terms** $t, t' \in \mathcal{T}^0$, all the instances of the following schema are provable in LED:*

$$\vdash K_a t;$$

$$\vdash t = t' \to K_a t = t'.$$

Proof. We prove the *first claim* by induction on t: *For the base step*, let $t := c$ be a constant. From the Knowledge De Re axiom, we get $\vdash x = c \to (K_a x \leftrightarrow K_a(x = c))$. By substituting c for x and using the first equality axiom, we get $\vdash K_a c \leftrightarrow K_a(c = c)$. But on the other hand, by applying Necessitation to the first equality axiom, we have $\vdash K_a(c = c)$, and hence we obtain $\vdash K_a c$. *For the inductive step*: consider a term of the form $f(\overrightarrow{t})$, where $\overrightarrow{t} = (t_1, \ldots, t_n)$ is a tuple of ground terms. By the induction hypothesis, we can assume that $\vdash K_a t_i$ for all $i = 1, n$. Using this and the Knowledge De Re axiom, we derive $\vdash \overrightarrow{t} = \overrightarrow{c} \to K_a \overrightarrow{t} = \overrightarrow{c}$. Combining this with $\vdash K_a \overrightarrow{t} = \overrightarrow{c} \to K_a f(\overrightarrow{t}) = f(\overrightarrow{c})$ (obtained by applying Necessitation, Kripke's axiom and Modus Ponens to the fourth equality axiom), we obtain $\vdash \overrightarrow{t} = \overrightarrow{c} \to K_a f(\overrightarrow{t}) = f(\overrightarrow{c}$. Combining this with the theorem $\vdash \left(\overrightarrow{t} = c \wedge f(\overrightarrow{c}) = d\right) \to K_a^{\overrightarrow{t}=\overrightarrow{c}} f(\overrightarrow{t}) = d$ (obtained from the axiom $\vdash K_a^{\overrightarrow{t}} f(\overrightarrow{t})$ and the Knowledge De Re axiom), we get that $\vdash \left(\overrightarrow{t} = c \wedge f(\overrightarrow{c}) = d\right) \to K_a f(\overrightarrow{t}) = d$. This, together with the obvious theorem $\vdash \overrightarrow{t} = c \to \left(f(\overrightarrow{t}) = d \to (\overrightarrow{t} = c \wedge f(\overrightarrow{c}) = d)\right)$ (an obvious consequence of the equality axioms), gives us $\vdash \overrightarrow{t} = c \to \left(f(\overrightarrow{t}) = d \to K_a f(\overrightarrow{t}) = d\right)$. Applying the the (EVR) rule, we get $\vdash f(\overrightarrow{t}) = d \to K_a f(\overrightarrow{t}) = d$, which by the Knowledge De Re axiom, yields $\vdash f(\overrightarrow{t}) = d \to K_a f(\overrightarrow{t})$. Applying again the (EVR) rule, we obtain $\vdash K_a f(\overrightarrow{t})$, as desired.

As for the second claim: given the first claim, we have $\vdash K_a t$ and $\vdash K_a t'$. This, together with (a suitable substitution instance of) the Knowledge De Re axiom and the conjunctivity of knowledge, gives us $\vdash (t = c \wedge t' = c) \to K_a(t = c \wedge t' = c)$, and hence (using equality axioms and the axioms of normal modal logic) $\vdash (t = c \wedge t' = c) \to K_a t = t'$. Together with $\vdash t = c \to (t = t' \to t' = c)$ (a consequence of the equality axioms), this yields $\vdash t = c \to (t = t' \to K_a t = t')$. By the (EVR) rule, we obtain $\vdash t = t' \to K_a t = t'$, as desired. $\square$

Proposition 2.4 *All the following theorems are provable in LED:*

$$\vdash K_a^{x_1,\dots,x_k} y \to K_a^{x_{\pi(1)},\dots,x_{\pi(k)}} y, \text{ for every permutation } \pi : \{1,\dots,k\} \to \{1,\dots,k\}$$

$$\vdash (K_a^{\overrightarrow{x}} \overrightarrow{y} \wedge K_a^{\overrightarrow{x},\overrightarrow{y}} \overrightarrow{z}) \to K_a^{\overrightarrow{x}} \overrightarrow{z}$$

$$\vdash K_a^{\overrightarrow{x}} \overrightarrow{y} \to K_a^{\overrightarrow{x},\overrightarrow{z}} \overrightarrow{y}$$

$$\vdash K_a^{\overrightarrow{x}} \overrightarrow{y} \to K_a^{\overrightarrow{x}} f(\overrightarrow{y})$$

$$\vdash \overrightarrow{x} = \overrightarrow{c} \to (K_a^{\overrightarrow{x}} \varphi \to K_a^{\overrightarrow{x}=\overrightarrow{c}} \varphi)$$

$$\vdash K_a^{\overrightarrow{x}} (\varphi \to \psi) \to (K_a^{\overrightarrow{x}} \varphi \to K_a^{\overrightarrow{x}} \psi)$$

$$\vdash K_a^{\overrightarrow{x}} \varphi \to \varphi$$

$$\vdash K_a^{\overrightarrow{x}} \varphi \to K_a^{\overrightarrow{x}} K_a^{\overrightarrow{x}} \varphi$$

$$\vdash \neg K_a^{\overrightarrow{x}} \varphi \to K_a^{\overrightarrow{x}} \neg K_a^{\overrightarrow{x}} \varphi$$

Proof. We only prove the first two formulas, the other proofs are similar. *For the first*, we use the obvious propositional validity $\vdash (x_1 = c_1 \wedge \dots x_k = c_k) \to (x_{\pi(1)} = c_{\pi(1)} \wedge \dots x_{\pi(k)} = c_{\pi(k)})$, together with two instances of Knowledge De Re axiom: $\vdash (x_1 = c_1 \wedge \dots x_k = c_k) \to (K_a^{x_1,\dots,x_k} y \leftrightarrow K_a^{x_1=c_1 \wedge \dots x_k = c_k} y)$, and $\vdash (x_1 = c_1 \wedge \dots x_k = c_k) \to (K_a^{x_{\pi(1)},\dots,x_{\pi(k)}} y \leftrightarrow K_a^{x_{\pi(1)}=c_{\pi(1)} \wedge \dots x_{\pi(k)}=c_{\pi(k)}} y)$. From these we derive $\vdash (x_1 = c_1 \wedge \dots x_k = c_k) \to (K_a^{x_1,\dots,x_k} y \to K_a^{x_{\pi(1)},\dots,x_{\pi(k)}} y)$, then apply repeatedly the (EVR) rule to obtain the desired conclusion.

For the second, we use three instances of Knowledge De Re axiom: $\vdash (\overrightarrow{x} = \overrightarrow{c} \wedge \overrightarrow{y} = \overrightarrow{d}) \to (K_a^{\overrightarrow{x}} \overrightarrow{y} \leftrightarrow K_a^{\overrightarrow{x}=\overrightarrow{c}} \overrightarrow{y} = \overrightarrow{d})$, $\vdash (\overrightarrow{x} = \overrightarrow{c} \wedge \overrightarrow{y} = \overrightarrow{d} \wedge \overrightarrow{x} = \overrightarrow{e}) \to (K_a^{\overrightarrow{x},\overrightarrow{y}} \overrightarrow{z} \leftrightarrow K_a^{\overrightarrow{x}=\overrightarrow{c} \wedge \overrightarrow{y}=\overrightarrow{d}} \overrightarrow{z} = \overrightarrow{e})$, and $\vdash (\overrightarrow{x} = \overrightarrow{c} \wedge \overrightarrow{z} = \overrightarrow{e}) \to (K_a^{\overrightarrow{x}} \overrightarrow{z} \leftrightarrow K_a^{\overrightarrow{x}=\overrightarrow{c}} \overrightarrow{z} = \overrightarrow{e})$. From these, together with the usual properties of the normal propositional operator $K_a\phi$ (and the fact that $K^\phi \psi$ is just an abbreviation for $K_a(\phi \to \psi)$), we obtain $\vdash (\overrightarrow{x} = \overrightarrow{c} \wedge \overrightarrow{y} = \overrightarrow{d} \wedge = \overrightarrow{e}) \to ((K_a^{\overrightarrow{x}} \overrightarrow{y} \wedge K_a^{\overrightarrow{x},\overrightarrow{y}} \overrightarrow{z}) \to K_a^{\overrightarrow{x}} \overrightarrow{z})$, then we apply the (EVR) rule. □

One can also easily verify that:

Proposition 2.5 *The following Necessitation-type rule for $K_a^{\overrightarrow{t}}$ is derivable in LED: if $\vdash \varphi$ then $\vdash K_a^{\overrightarrow{t}} \varphi$.*

"Pseudo-modalities": necessitation/possibility forms. For any finite string $s \in (\mathcal{L} \cup (\mathcal{A} \times \mathcal{T}^*))^*$, consisting of formulas $\phi \in \mathcal{L}$ and/or pairs $(a, \overrightarrow{t})$ of agents $a \in \mathcal{A}$ and strings $\overrightarrow{t} \in \mathcal{T}^*$ of terms, we define "pseudo-modalities" $[s]$ and $\langle s \rangle$, mapping any formula $\phi \in \mathcal{L}$ to formulas $[s]\phi \in \mathcal{L}$ (called a "necessity form") and $\langle s \rangle \phi \in \mathcal{L}$ (called a "possibility form"). The definition is by recursion, putting for necessity forms: $[\lambda]\phi := \phi$ for the empty string λ; $[\psi, s]\phi := \psi \to [s]\phi$; and $[(a, \overrightarrow{t}), s]\phi := K_a^{\overrightarrow{t}}[s]\phi$. As for possibility forms, we put $\langle s \rangle \phi := \neg [s] \neg \phi$.

Lemma 2.6 *For every necessity form $[s]$ there exists some formula $\psi \in \mathcal{L}$, such that for all $\theta \in \mathcal{L}$, we have:*

$$\vdash [s]\theta \quad \textit{iff} \quad \vdash \psi \to \theta.$$

Moreover, the same constants and variables occur in ψ as in s.

Proof. If $s = \lambda$, then take $\psi := \top$. Otherwise, $[s]\theta$ is just a sequence of symbols of the form $\psi \to \ldots$ and $K_a^{\vec{t}} \ldots$, followed at the end by θ. Starting from the left, we can "eliminate" one by one each knowledge symbol $K_a^{\vec{t}} \ldots$ by "pushing" it into the premise, using the fact [10] that: $\vdash \psi \to K_a^{\vec{t}} \phi$ holds iff $\vdash \langle K_a^{\vec{t}} \rangle \psi \to \phi$ holds. At the end of this process, we obtain a formula of the form $\psi \to \theta$. It is easy to see that ψ depends only on s, not on θ, and that moreover ψ contains the same constants and variables as s. □

Lemma 2.7 *Given $s \in (\mathcal{L} \cup (\mathcal{A} \times \mathcal{T}^*))^*$, $t \in \mathcal{T}$, $\varphi \in \mathcal{L}$, let c be a constant that does not occur in s, t or φ. Then the following rule is admissible in LED:*

$$\textit{if} \ \vdash [s](t = c \to \varphi) \quad \textit{then} \ \vdash [s]\varphi$$

Proof. Let ψ be the formula associated to s by the previous Lemma: so for all θ, $\vdash [s]\theta$ iff $\vdash \psi \to \theta$. Suppose now that we have $\vdash [s](t = c \to \varphi)$. Then we also get $\vdash \psi \to (t = c \to \varphi)$, and hence $\vdash t = c \to (\psi \to \varphi)$. Let x be a variable not occurring in t, φ or s (and hence, by the previous Lemma, not occurring in ψ either). Using (some substitution instance of one of) the Equality Axioms, we obtain $\vdash x = c \to (t = x \to (\psi \to \varphi))$. Since c does not to occur in s, t or φ, by the previous Lemma it doesn't occur in ψ either. By the (EVR) rule, we obtain $\vdash t = x \to (\psi \to \varphi)$. Using the Variable Substitution Rule (where we substitute t for x), we get $\vdash t = t \to (\psi \to \varphi)$. But we also have $\vdash t = t$ (by another of the Equality axioms), and hence $\vdash \psi \to \varphi$. Using again the previous Lemma, we obtain $\vdash [s]\varphi$. □

Theorem 2.8 *The proof system LED is* sound *and* strongly complete *(and hence the logic LED is* compact*). Moreover, this logic has the* strong finite model property*, and hence it is* decidable*.*

The rest of this section is dedicating to the proof of this theorem. For any countable set of constants C, let $\mathcal{L}_C$ be the language of LED based only on constants in C. A *C-theory* Φ is a consistent set of formulas in $\mathcal{L}_C$; here, "consistent" means consistent with respect to the proof system LED formulated for the language $\mathcal{L}_C$. A *maximal C-theory* is a C-theory Φ that is maximal (w.r.t. inclusion) among all C-theories. A *C-witnessed theory* is a C-theory Φ such that, for every term $t \in \mathcal{T}_C$, string $s \in (\mathcal{L}_C \cup (\mathcal{A} \times \mathcal{T}_C^*))^*$ and formula $\varphi \in \mathcal{L}_C$, if $\Phi \vdash [s](t = c \to \varphi)$ for all $c \in C$, then $\Phi \vdash [s]\varphi$. Equivalently: if whenever $\langle s \rangle \varphi$ is consistent with Φ, then there exists some $c \in C$ s.t. $\langle s \rangle (t = c \wedge \varphi)$

[10] This is an instance of the well-known fact that in the axiomatic system $S5$, a formula $\psi \to \Box\phi$ is a theorem iff the formula $\Diamond\psi \to \phi$ is a theorem.

is consistent with Φ. A *maximal C-witnessed theory* is a C-witnessed theory which is not a proper subset of any other C-witnessed theory.

For the completeness proof, we make use of the following three easily verifiable results:

Lemma 2.9 *If Φ is a C-theory and $\Phi \nvdash \neg\phi$, then $\Phi \cup \{\phi\}$ is also a C-theory. Moreover, if Φ is C-witnessed, then $\Phi \cup \{\phi\}$ is C-witnessed.*

Lemma 2.10 *If $\Phi_0 \subseteq \Phi_1 \subseteq \ldots \subseteq \Phi_n \subseteq \ldots$ is an increasing chain of C-theories, then $\bigcup_{n\in N} \Phi_n$ is a C-theory. Moreover, if all Φ_n are C-witnessed then $\bigcup_{n\in N} \Phi_n$ is C-witnessed.*

Lemma 2.11 *A C-theory Φ is a C-witnessed maximal C-theory iff it is a maximal C-witnessed theory.*

The completeness proof goes now via the following steps:

Lemma 2.12 *(*Lindenbaum Lemma*) Every C-witnessed theory Φ can be extended to a maximal C-witnessed theory $T_\Phi \supseteq \Phi$.*

Proof. Let $\phi_0, \phi_1, \ldots, \phi_n, \ldots$ be an enumeration of formulas in $\mathcal{L}_C$. We define an increasing chain $\Phi_0 \subseteq \Phi_1 \ldots \subseteq \Phi_n \subseteq \ldots$ of C-witnessed theories: first, put $\Phi_0 := \Phi$; then, given the witnessed C-theory Φ_n, put $\Phi_{n+1} := \Phi_n$ if $\Phi \vdash \neg\phi_n$, and put $\Phi_{n+1} := \Phi_n \cup \{\phi_n\}$ otherwise (if $\Phi_n \nvdash \neg\phi_n$). Finally, we put $T_\Phi := \bigcup_{n\in N} \Phi_n$. By Lemma 2.10, this is a C-witnessed theory. Moreover, it is also a maximal C-theory (since every formula consistent with T_Φ is in T_Φ), so it is a maximal C-witnessed theory. $\Box$

Lemma 2.13 *(*Extension Lemma*) Let C be a set of constants, and let $C' = \{c_0, c_1, \ldots, c_n, \ldots\}$ be a countable set of "fresh" constants, i.e. s.t. $C \cap C' = \emptyset$. Put $\tilde{C} = C \cup C'$. Then every C-theory Φ can be extended to a $\tilde{C}$-witnessed theory $\tilde{\Phi} \supseteq \Phi$, and hence (by Lindenbaum Lemma) to a maximal $\tilde{C}$-witnessed theory $T_\Phi \supseteq \Phi$.*

Proof. Let $\gamma_1, \ldots, \gamma_n \ldots$ be an enumeration of all the triplets of the form $\gamma_n = (s_n, t_n, \phi_n)$ consisting of any necessity form $s_n \in (\mathcal{L}_{\tilde{C}} \cup (\mathcal{A} \times \mathcal{T}^*_{\tilde{C}}))^*$, any term $t_n \in \mathcal{T}_{\tilde{C}}$ and formula $\phi_n \in \mathcal{L}_{\tilde{C}}$. For every such triplet $\gamma_n = (s_n, t_n, \phi_n)$, put $C'(n) =: \{c' \in C' : c'$ occurs in either s_n or t_n or $\phi_n\}$. Note that $C'(n)$ is always finite.

We now construct an increasing chain $\Phi_0 \subseteq \Phi_1 \ldots \subseteq \Phi_n \subseteq \ldots$ of $\tilde{C}$-theories, satisfying the following three properties: (1) $\Phi_0 = \Phi$; (2) for every $n \in N$, the set $C'_n := \{c' \in C' : c'$ occurs in $\Phi_n\}$ is finite; (3) for every triplet $\gamma_n = (s_n, t_n, \phi_n)$ in the above enumeration, if $\Phi_n \nvdash \neg\langle s_n\rangle\phi_n$, then $\exists m \in N$ s.t. $\langle s_n\rangle(t_n = c'_m \wedge \phi_n) \in \Phi_{n+1}$. The construction is by recursion. For $n = 0$, we put $\Phi_0 := \Phi$, which takes care of condition (1) above. At step $n+1$, let Φ_n be a $\tilde{C}$-theory satisfying clause (1) above, and let $\gamma_n = (s_n, t_n, \phi_n)$ be the n-th triplet in the above enumeration. We have two cases: (a) if we have $\Phi_n \vdash [s_n]\neg\phi_n$, then we put $\Phi_{n+1} := \Phi_n$; (b) in case that we have $\Phi_n \nvdash [s_n]\neg\phi_n$, then we choose m

to be the least natural number bigger [11] than the indices of all the constants in $C'(n) \cup C'_n$, and put $\Phi_n + 1 := \Phi_n \cup \{\langle s_n \rangle (t_n = c'_m \wedge \phi_n)\}$. To show that this gives us a C-theory, notice that c'_m doesn't occur in s_n, t_n, ϕ_n or Φ_n. If Φ_{n+1} were inconsistent, then we'd have $\Phi_n \vdash [s_n](t_n = c'_m \to \neg\phi_n)$, so $\exists \theta_1, \ldots, \theta_k \in \Phi_n$ s.t. $\vdash \theta_1 \to (\theta_n \to \cdots [s_n](t_n = c'_m \to \neg\phi_n))$ is a theorem in LED. But $c'_m \notin C'(n) \cup C'_n$, so c'_m doesn't occur in $s_n, t_n, \phi_n, \theta_1, \ldots, \theta_n$ (or in any other formula of Φ_n). By Lemma 2.7, we have that $\vdash \theta_1 \to (\theta_n \to \cdots [s_n]\neg\phi_n)$ is also a theorem in LED, and hence that $\Phi_n \vdash [s]\neg\phi_n$, contrary to our assumption (in case b). So in both cases Φ_{n+1} is a $\tilde{C}$-theory. It is also easy to see that it satisfies condition (2): in case (a) we have $C'_{n+1} = C'_n$ (finite by the inductive assumption); in case (b) we have $C'_{n+1} = C'_n \cup C'(n) \cup \{c'_m\}$ (still finite). Finally, it is obvious that condition (3) is satisfied.

Given now this increasing sequence $\Phi = \Phi_0 \subseteq \cdots \subseteq \Phi_n \subseteq \cdots$ of $\tilde{C}$-theories satisfying (1)-(3) above, take $\tilde{\Phi} := \bigcup_{n \in N} \Phi_n$. By Lemma 2.10, $\tilde{\Phi}$ is a $\tilde{C}$-theory, and it obviously includes $\Phi = \Phi_0$. Condition (3) above implies that $\tilde{\Phi}$ is $\tilde{C}$-witnessed. □

Together, the last three results imply that, in order to show completeness, it is enough to show that, for any countable set C of constants, *every maximal C-witnessed theory has a model*. We now proceed to prove this.

From now on, we fix the set of constants C, and we assume given a maximal C-witnessed theory T_0. For each term $t \in \mathcal{T} = \mathcal{T}_C$, we can define an equivalence relation $\sim^t$ on maximal C-witnessed theories T, T', by putting: $T \sim^t T'$ iff $\forall c \in C((t = c) \in T \Leftrightarrow (t = c) \in T')$. Put $\sim \ := \ \bigcap_{t \in \mathcal{T}^0} \sim^t$ (where recall that $\mathcal{T}^0$ is the set of ground terms). It is obvious that $\sim$ is also an equivalence relation on maximal C-witnessed theories.

In addition, we can define another equivalence relation $\equiv$ on the set of constants C by putting: $c \equiv c'$ iff $(c = c') \in T_0$. For any constant $c \in C$, let us denote by $[c] := \{c' \in C : c \equiv c'\}$ the equivalence class of c modulo $\equiv$.

Canonical Model The *canonical model for* T_0 is a model $M = (\Omega, D, [0], [1], \sim_a, \| \bullet \|, \bullet(\bullet), \mathbf{f}, \mathbf{R})_{a \in \mathcal{A}, f \in \mathcal{F}, R \in \mathcal{R}}$ for the language $\mathcal{L}_C$, defined as follows: the *state space* is $\Omega := \{T \subseteq \mathcal{L}_C : T \text{ maximal witnessed } \mathcal{L}_C\text{-theory with } T \sim T_0\}$; the set of *objects* is $D := \{[c] : c \in C\}$, where $[c]$ is the equivalence class of c modulo $\equiv$, and the equivalence classes $[0]$ and $[1]$ are the two designated objects; the *epistemic relations* are: $T \sim_a T'$ iff $\forall \varphi \in \mathcal{L}_C(K_a\varphi \in T \Rightarrow \varphi \in T')$. For $f \in \mathcal{F}$, we put $\mathbf{f}([c_1], \ldots, [c_n]) := [c]$ for $c_1, \ldots, c_n, c \in C$ with $(f(c_1, \ldots, c_n) = c) \in_0$; and for $R \in \mathcal{R}$, we put $\mathbf{R} := \{([c_1], \ldots, [c_n]) : R(c_1, \ldots, c_n) \in T_0$. The *valuation* is $\|p\| := \{T \in W : p \in T\}$. The *value* $T(\alpha)$ of $\alpha \in Var \cup C$ at world $T \in \Omega$ is given by $T(c) := [c]$ for $c \in C$, and $T(x) := [c]$, for $x \in Var$ and $c \in C$ with $(x = c) \in T$. It is easy to check that these definitions are independent of the choice of representatives, so M is indeed a *well-defined model* for $\mathcal{L}_C$.

[11] Such a number exists, due to the inductive assumption (2) above.

Lemma 2.14 *(*Intersection Lemma*) For all agents $a \in \mathcal{A}$ and finite strings of terms $\overrightarrow{t} = (t_1, \ldots, t_n)$, we have: $\sim_a^{\overrightarrow{t}} = \sim_a \cap \sim^{t_1} \cap \cdots \sim^{t_n}$.*

Proof. *The left-to-right inclusion*: the *LED*-theorem $\vdash K_a^{\overrightarrow{x}} \overrightarrow{y} \to K_a^{\overrightarrow{x}, \overrightarrow{z}} \overrightarrow{y}$ (proven in Proposition 2.4) yields by substitution $\vdash K_a \varphi \to K_a^{\overrightarrow{t}} \varphi$ and $\vdash K_a^{t_i} \varphi \to K_a^{\overrightarrow{t}} \varphi$, from which we obtain $\sim_a^{\overrightarrow{t}} \subseteq \sim_a$ and $\sim_a^{\overrightarrow{t}} \subseteq \sim^{t_i}$ (for all $i = 1, n$). *For the converse inclusion*: suppose $T, S \in \Omega$ satisfy $T \sim_a S$ and $T \sim^{t_i} S$ for all $i = 1, n$. To show that $T \sim_a^{\overrightarrow{t}} S$, let $K_a^{\overrightarrow{t}} \varphi \in T$. We need to show that $\varphi \in S$: for this, notice that, since T is C-witnessed (and , for each $i = 1, n$ there must exist constants $c_i \in C$ such that T is consistent with $t_i = c_i$. Since T is maximal, it follows that $(t_i = c_i) \in T$, and moreover that $(\overrightarrow{t} = \overrightarrow{c}) \in T$. By applying the theorem $\vdash \overrightarrow{t} = \overrightarrow{c} \to (K_a^{\overrightarrow{t}} \varphi \to K_a^{\overrightarrow{t} = \overrightarrow{c}} \varphi)$, we obtain that $K_a^{\overrightarrow{t} = \overrightarrow{c}} \varphi \in T$, i.e. $K_a(\overrightarrow{t} = \overrightarrow{c} \to \varphi) \in T$. This together with $T \sim_a S$, gives us that $(\overrightarrow{t} = \overrightarrow{c} \to \varphi) \in S$. But from $(\overrightarrow{t} = \overrightarrow{c}) \in T$ and $T \sim^{t_i} S$ for all $i = 1, n$, we derive that $(\overrightarrow{t} = \overrightarrow{c}) \in T$, and hence by closure of (the maximal theory T) under modus ponens, we obtain that $\varphi \in T$. $\square$

As a consequence of Lemma 2.14, *all $\sim_a^{\overrightarrow{t}}$ are equivalence relations.*

Lemma 2.15 *(*Diamond Lemma*) Let $T \in \Omega$, and let $a, \overrightarrow{t}, \varphi$ be such that $K_a^{\overrightarrow{t}} \varphi \notin T$. Then there exists some theory $S \in \Omega$ such that $T \sim_a^{\overrightarrow{t}} S$ but $\varphi \notin S$.*

Proof. Let $\Psi := \{\psi : K_a^{\overrightarrow{t}} \psi \in T\}$. We will show the following
Claim: The set $\Psi \cup \{\neg\varphi\}$ *is a C-witnessed theory.*

To prove this claim, we first need to show that this set is consistent. Suppose not; then there exist $\psi_1, \ldots, \psi_n \in \Psi$ (hence, $K_a^{\overrightarrow{t}} \psi_i \in T$ for all $i = 1, n$) such that $\vdash (\psi_1 \wedge \ldots \psi_n) \to \varphi$ is a theorem. But then we also have that $\vdash (K_a^{\overrightarrow{t}} \psi_1 \wedge \ldots K_a^{\overrightarrow{t}} \psi_n) \to K_a^{\overrightarrow{t}} \varphi$ and $(K_a^{\overrightarrow{t}} \psi_1 \wedge \ldots K_a^{\overrightarrow{t}} \psi_n) \in T$, hence $K_a^{\overrightarrow{t}} \varphi \in T$, in contradiction with our assumption (that $K_a^{\overrightarrow{t}} \varphi \notin T$).

Next, to show that $\Psi \cup \{\neg\varphi\}$ is C-witnessed, suppose that, for some triple (s', t', φ'), we have $\Psi \cup \{\neg\varphi\} \vdash [s'](t' = c \to \phi')$ for all $c \in C$. By a previous lemma, this gives that $\Psi \vdash (\neg\varphi \to [s'](t' = c \to \phi'))$ for all c, and by another lemma we obtain that $K_a^{\overrightarrow{t}} \Psi \vdash K_a^{\overrightarrow{t}} (\neg\varphi \to [s'](t' = c \to \phi'))$ (where recall that $K_a^{\overrightarrow{t}} \Psi = \{K_a^{\overrightarrow{t}} \psi : \psi \in \Psi\}$). But note that $K_a \Psi \subseteq T$, and hence we get $T \vdash K_a^{\overrightarrow{t}} (\neg\varphi \to [s'](t' = c \to \phi'))$ for all $c \in C$. Since $T \in \Omega$ is C-witnessed, it follows (by applying the C-witnessing condition to the necessitation form $((a, \overrightarrow{t}), \neg\varphi, s'))$ that $T \vdash K_a^{\overrightarrow{t}} (\neg\varphi \to [s']\phi')$, and hence by maximality that $K_a^{\overrightarrow{t}} (\neg\varphi \to [s']\phi') \in T$, hence $(\neg\varphi \to [s']\phi') \in \Psi$. From this we obtain that $\Psi \cup \{\neg\varphi\} \vdash [s']\phi'$, thus proving our Claim above.

Given the Claim, we can now use the Extension Lemma (in combination with Lindenbaum Lemma) to extend the set $\Psi \cup \{\neg\varphi\}$ to a maximal C-witnessed theory S. It is easy to see that we have $S \sim T \sim T_0$, hence $S \in \Omega$. We obviously have $(\neg\varphi) \in S$, so by consistency $\varphi \notin S$. Finally, $\Psi \subseteq S$ gives us that $T \sim_a^{\overrightarrow{t}} S$. $\square$

Lemma 2.16 *(*"Knowledge de Re" Lemma*) Let $T \in \Omega$, and let $a, \overrightarrow{t}, t'$ be such that $K_a^{\overrightarrow{t}} t' \notin T$. Then there exists some theory $S \in \Omega$ such that $T \sim_a^{\overrightarrow{t}} S$ but $T \not\sim^{t'} S$.*

Proof. Since T is a maximal C-witnessed theory, there exist $\overrightarrow{c} \in C^*$, $c \in C$ such that $\overrightarrow{t} = \overrightarrow{c}, t' = c' \in T$. By using the theorem $\vdash (\overrightarrow{t} = \overrightarrow{c} \wedge t' = c') \rightarrow (K_a^{\overrightarrow{t}} t' \leftrightarrow K_a(\overrightarrow{t} = \overrightarrow{c} \rightarrow t' = c'))$ (which is a substitution instance of the *Knowledge de Re* axiom) and the assumption that $K_a^{\overrightarrow{t}} t' \notin T$, we obtain that $K_a(\overrightarrow{t} = \overrightarrow{c} \rightarrow t' = c') \notin T$. By the Diamond Lemma, there exists some $S \in \Omega$ such that $T \sim_a^{\overrightarrow{t}} S$ but $(\overrightarrow{t} = \overrightarrow{c} \rightarrow t' = c') \notin S$. By the maximality of S, we get that $(\overrightarrow{t} = \overrightarrow{c} \wedge t' \neq c') \in S$, and hence that $T \sim_{t_i} S$ for all $i = 1, n$ but $T \not\sim_{t'} S$. Using also $T \sim_a^{\overrightarrow{t}} S$ and the Intersection Lemma above, we conclude that $T \sim_a^{\overrightarrow{t}} S$ (and $T \not\sim_{t'} S$), as desired. □

Lemma 2.17 *(*Truth Lemma*) Let $M = (\Omega, D, \sim_a, \| \bullet \|, \bullet(\bullet), \mathbf{f}\,)_{a \in \mathcal{A}, f \in F}$ be the canonical model for (some theory) T_0. Then for every formula φ and every term t, we have:*

(1) *$T \in \|\varphi\|_M$ iff $\varphi \in T$, and*

(2) *$T(t) = [c]$ iff $(t = c) \in T$.*

Proof. We prove both claims by simultaneous induction on the complexity [12] of formulas φ and terms t:

To prove (1): for *atomic formulas* $p \in P$, (1) is trivial. *For relational atoms* $R(\overrightarrow{t})$, let $\overrightarrow{[c]} \in D^*$ be s.t. $T(\overrightarrow{t}) = \overrightarrow{[c]}$, so by the induction hypothesis for (2) we have $(\overrightarrow{t} = \overrightarrow{c}) \in T$. Then we have the following sequence of equivalencies: $T \in \|R(\overrightarrow{t})\|_M$ iff $T(\overrightarrow{t}) \in \mathbf{R}$ iff $\overrightarrow{[c]} \in \mathbf{R}$ iff $R(\overrightarrow{[c]}) \in T_0$ iff (using $T \sim T_0$) $R(\overrightarrow{[c]}) \in T$ iff $R(\overrightarrow{[t]}) \in T$ (where at the last step we used the fact that $(\overrightarrow{t} = \overrightarrow{c}) \in T$ and the equality axioms). For *implicational formulas* $\phi \rightarrow \psi$, this goes as usual, using the properties of maximally consistent theories. For *epistemic formulas* $K_a^{\overrightarrow{t}} t'$, with $\overrightarrow{t} = (t_1, \ldots, t_n)$: to prove the *left-to-right implication*, suppose that $T \in \|K_a^{\overrightarrow{t}} t'\|_M$ but $(K_a^{\overrightarrow{t}} t') \notin T$. By the Knowledge de Re Lemma, there exists some $S \in \Omega$ such that $T \sim_a^{\overrightarrow{t}} S$ but $T \not\sim^{t'} S$. By the Intersection Lemma, we obtain that $T \sim_a S$ and $T \sim^{t_i} S$ for all $i \in \{1, \ldots, n\}$; i.e. for every $c \in C$ and $i \in \{1, \ldots, n\}$, we have that: $(t_i = c) \in T \Leftrightarrow (t_i = c) \in S$. By the induction hypothesis for claim (2), this implies that $T(\overrightarrow{t}) = S(\overrightarrow{t})$. From this, together with $T \sim_a S$ and the fact that $T \in \|K_a^{\overrightarrow{t}} t'\|_M$ (as well as the semantic clause for knowledge de re), we obtain that $T(t') = S(t')$. Applying again the induction hypothesis for (2), we get that

[12] Our notion of complexity is a function $comp : \mathcal{L}_C \cup \mathcal{T}_C \rightarrow N$, defined recursively by putting: $comp(p) = comp(c) = comp(x) = 0$, $comp(R(t_1, \ldots, t_n)) = 1 + max(comp(t_1), \ldots, comp(t_n))$, $comp(\phi \rightarrow \psi) = 1 + max(comp(\phi), comp(\psi))$, $comp(K_a^{t_1, \ldots, t_n} t') = 1 + max(comp(t_1), \ldots, comp(t_n), comp(t'))$, $comp(?_\phi) = 1 + comp(\phi)$, $comp(f(t_1, \ldots, t_n)) = 1 + max(comp(t_1), \ldots, comp(t_n))$.

$(t' = c) \in T \Leftrightarrow (t' = c) \in S$ holds for all $c \in C$, i.e. $T \sim^{t'} S$ contrary to our assumption above. To show the *converse*: suppose that $(K_a^{\overrightarrow{t}} t') \in T$. Since T is C-witnessed there must exist $\overrightarrow{c} \in C^*, c' \in C$ such that $(\overrightarrow{t} = \overrightarrow{c}), (t' = c') \in T$. By the induction hypothesis for (2), we have $T(t_i) = [c_i]$ for all $i \in \{1, \ldots, n\}$, and also $T(t') = [c']$. To prove now that $T \in \|\varphi\|_M$, let $S \in \Omega$ be such that $T \sim_a S$ and $T(\overrightarrow{t}) = S(\overrightarrow{t})$. It is enough to prove that $T(t') = S(t')$. Using the Knowledge de Re axiom, and the fact that $(K_a^{\overrightarrow{t}} t') \in T$, we obtain that $(K_a^{\overrightarrow{t} = \overrightarrow{c}} t' = c') \in T$, i.e. $K_a(\overrightarrow{t} = \overrightarrow{c} \rightarrow t' = c') \in T$. Since $T \sim_a S$, we must have $(\overrightarrow{t} = \overrightarrow{c} \rightarrow t' = c') \in S$. From $T(\overrightarrow{t}) = S(\overrightarrow{t})$ and $T(t_i) = [c_i]$ for all i, we get that $S(t_i) = [c_i]$ for all i, and so by the induction hypothesis for (2) we have $(\overrightarrow{t} = \overrightarrow{c}) \in S$. This, together with $(\overrightarrow{t} = \overrightarrow{c} \rightarrow t' = c') \in S$, gives us that $(t' = c') \in S$. Applying again the induction hypothesis for the second claim, we obtain that $S(t') = [c'] = T(t')$, as desired.

To show (2): it is trivially true for variables $x \in Var$ and constants $c' \in C$. For terms of the form $?_\varphi$: we know that by definition $T(?_\varphi) = [1]$ holds iff $T \in \|\varphi\|_M$ holds, i.e. (by the induction hypothesis for (1)) iff $\varphi \in T$ iff $(?_\varphi \leftrightarrow 1) \in T$ (by the Characteristic Functions Axiom). A similar argument shows that $T(?_\varphi) = [0]$ iff $(?_\varphi \leftrightarrow 0) \in T$. Since $[0]$ and $[1]$ are the only possible values of $T(?_\varphi)$, we obtain the desired conclusion. For terms of the form $f(t_1, \ldots, t_n)$, let $c_1, \ldots, c_n \in C$ be s. t. $T(t_1) = [c_1], \ldots, T(t_n) = [c_n]$, i.e. $(t_1 = c_1), \ldots, (t_n = c_n) \in T$. (By the definition of the canonical value function, such constants must exist.) We have $T(f(t_1, \ldots, t_n) = \mathbf{f}(T(t_1), \ldots, T(t_n)) = \mathbf{f}([c_1], \ldots, [c_n]) = [f(c_1, \ldots, c_n)]$. Hence we have that: $T(f(t_1, \ldots, t_n) = [c]$ holds iff $f(c_1, \ldots, c_n)] = [c]$ holds, i.e. iff $(f(c_1, \ldots, c_n) = c) \in T$. □

In particular, T_0 is satisfied at world T_0 in M: *this finishes our proof of strong completeness.*

The *decidability proof* goes via the following two steps:

STEP 1: Reduction of LED validities to validities in a less expressive language. Let LED_0 be the language with the following syntax

$$\varphi ::= p \mid R(\overrightarrow{t}) \mid \varphi \rightarrow \varphi \mid K_a\varphi$$

$$t ::= x \mid \quad c \quad \mid f(\overrightarrow{t})$$

In other words: we only allow terms that do not contain characteristic functions $?_\phi$ and we only allow the usual (propositional) modalities. The semantics is the obvious one, with all constructs interpreted as in GK and with the epistemic modalities interpreted in the usual way (using the relations $\sim_a$).

In fact, for technical reasons it is convenient to also look at the *extended* language LED_1 obtained by adding the usual epistemic modalities to LED:

$$\varphi ::= p \mid R(\overrightarrow{t}) \mid \varphi \rightarrow \varphi \mid K_a^{\overrightarrow{t}} t \mid K_a\varphi$$

$$t ::= x \mid \quad c \quad \mid \quad ?_\varphi \quad \mid f(\overrightarrow{t})$$

(Once again, the semantics is the obvious one).

It is clear that LED_1 *and* LED_0 *are co-expressive*, since $K_a\varphi$ is equivalent to $\varphi \wedge K_a^\lambda ?_\varphi$. In contrast, LED_0 *is a less expressive language than LED (and hence than* LED_1*)*:

Counterexample. We show that the formula $K_a x$ is not equivalent to any formula in LED_0. Suppose, towards a contradiction that $K_a x$ is equivalent to some formula ϕ_0 in LED_0. Let C_0 be the finite set of constants occurring in ϕ_0, and $C_1 := C_0 \cup \{0,1\}$. Take a model M_1 with two distinct worlds $W_1 = \{w, w'\}$, four distinct objects $D = \{[0],[1],d,d'\}$, $f(\bullet) = [0]$ for all functions and arguments, $\sim_a = W_1 \times W_1$, $\|p\| = \emptyset$ for all p, and $w(x) = d$, $w'(x) = d'$ for all variables x. Take another model M_2 with only one world $W_2 = \{w''\}$, same D, f and $\|p\|$ as for M_1, but with $\sim_a = W_2 \times W_2$ and $w''(x) = d$ for all variables x. It is easy to see that the worlds w, w' and w'' satisfy exactly the same formulas in the language of LED_0 based only on constants in C_0. Hence, these three worlds are equivalent wrt the truth value of ϕ_0. However, $K_a x$ is true at w'', while being false at w and w'. This contradicts the equivalence between $K_a x$ and ϕ_0.

So *the modalities for knowledge of a value really increase the expressivity of our language*. Nevertheless, we can prove that *every validity of* LED_1 *"translates" to a validity of* LED_0:

Proposition 2.18 *("Validity Reduction") There exists a computable map* τ *from the language* LED_1 *to the language* LED_0*, such that, for every formula* φ *of* LED_1*, we have:*

$$\varphi \text{ is valid} \quad \text{iff} \quad \tau(\varphi) \text{ is valid.}$$

Proof. The proof is by induction, using another notion of complexity γ that *counts only the number of nested de re modalities and nested* ? *symbols.*[13] Note that every term t of LED_1 can be rewritten as $t = t_0[x_1/?_{\phi_1}, \ldots, x_n/?_{\phi_n}]$, for some term t_0 of LED_0 as well as some variables $x_1, \ldots, x_n$ and formulas $\phi_1, \ldots, \phi_n$ (in LED_1), with $\gamma(\phi_i) < \gamma(t)$. For $\overrightarrow{c} \in \{0,1\}^n$, we introduce the notations $t_0[\overrightarrow{c}] := t_0[x_1/c_1, \ldots, x_n/c_n]$, and $\overrightarrow{c}(\overrightarrow{\phi}) := \bigwedge_{i=1,n} c_i(\phi_i)$, where $c(\phi) := \phi$ for $c = 1$, and $c(\phi) = \neg\phi$ for $c = 0$. Now for any tuple of terms $\overrightarrow{t} = (t^1, \ldots, t^m)$ of LKG_1, let t_0^i (with $i \in \{1, \ldots, m\}$) be the corresponding terms in LED_0, with variables $x_1^i, \ldots, x_{n_i}^i$ and formulas $\phi_1^i, \ldots, \phi_{n_i}^i$ (for $i \in \{1, \ldots, m\}$), s.t. $t^i = t_0^i[x_1^i/?_{\phi_1^i}, \ldots, x_{n_i}^i/?_{\phi_{n_i}^i}]$ holds for all $i \in \{1, \ldots, m\}$. Then we put $(R(t^1, \ldots, t^m)_0 := \bigvee_{\overrightarrow{c}^1 \in \{0,1\}^{n_1}} \cdots \bigvee_{\overrightarrow{c}^m \in \{0,1\}^{n_m}} \left(\bigwedge_{i \in \{1,\ldots,m\}} \overrightarrow{c}^i(\overrightarrow{\phi}^i) \wedge R(t_0^1[\overrightarrow{c}^1], \ldots, t_0^m[\overrightarrow{c}^m]) \right)$.

Claim A: $R(\overrightarrow{t})$ *is logically equivalent to* $(R(\overrightarrow{t}))_0$.

(The proof is an easy verification.)

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[13] More precisely, we recursively put: $\gamma(p) = \gamma(x) = \gamma(c) = 0$, $\gamma(\varphi \to \psi) = max(\gamma(\varphi), \gamma(\psi))$, $\gamma(K_a\varphi) = \gamma(\neg\varphi) = \gamma(\varphi)$, $\gamma(K_a^{t_1,\ldots,t_n} t) = 1 + max(\gamma(t_1), \ldots, \gamma(t_n), \gamma(t))$, $\gamma(?_\varphi) = 1 + \gamma(\varphi)$, $\gamma(R(t_1, \ldots, t_n)) = \gamma(f(t_1, \ldots, f_n)) = max(\gamma(t_1), \ldots, \gamma(t_n))$.

Given this claim, let φ be any formula in LED_1. We can obviously bring it to conjunctive normal form, i.e. establish a validity $\models \varphi \Leftrightarrow \bigwedge_i \bigvee_j \phi^{ij}$, where the formulas ϕ^{ij} are of one the following "basic" forms p, $\neg p$, $R(\overrightarrow{t})$, $\neg R(\overrightarrow{t})$, $K_a^{\overrightarrow{t}} t'$, $\neg K_a^{\overrightarrow{t}} t'$, $K_a\psi$, or $\neg K_a\psi$. Let $\mathcal{T}_\varphi$ be the set of all terms occurring in this normal form, and let F be an *injective* map that associates to each term $t \in \mathcal{T}_\varphi$ some "fresh" constant $F(t) \in C \setminus \mathcal{T}_\varphi$ (such that $F(t) \neq F(t')$ for $t \neq t'$). For any string $\overrightarrow{t} = (t_1, \ldots, t_n)$ of terms in $\mathcal{T}_\varphi$, put $F(\overrightarrow{t}) := (F(t_1), \ldots, F(t_n))$. We now associate to each of the above "basic" formulas ϕ^{ij} some corresponding formulas ϕ_0^{ij}, as follows: $\phi_0^{ij} := \phi^{ij}$ if ϕ^{ij} is the form p or $K_a\psi$; $\phi_0^{ij} := (R(\overrightarrow{t}))_0$ (as defined above) if ϕ^{ij} is of the form $R(\overrightarrow{t})$; $\phi_0^{ij} := K_a\left(\overrightarrow{t} = F(\overrightarrow{t}) \Rightarrow t' = F(t')\right)$ if ϕ^{ij} is of the form $K_a^{\overrightarrow{t}} t'$; and finally $(\phi^{ij})_0 := \neg\psi_0$, if $\phi^{ij} = \neg\psi$ with ψ of one of the forms p, $R(\overrightarrow{t})$, $K_a\varphi$ or $K_a^{\overrightarrow{t}} t'$.

We associate now to our formula φ above a new formula φ_0 of lower γ-complexity, by putting

$$\varphi^0 \ := \ \left(\bigwedge_{t \in \mathcal{T}_\phi} t = F(t) \right) \Rightarrow \bigwedge_i \bigvee_j \phi_0^{ij}.$$

It is now easy to verify the following:

Claim B: If φ is not in LED_0, then $\gamma(\varphi^0) < \gamma(\varphi)$.

Finally, we can prove the key step of our "Validity Reduction":

Claim C: φ is valid iff φ^0 is valid.

Proof of Claim C: Note the validity $\models \left(\bigwedge_{t \in \mathcal{T}_\phi} t = F(t)\right) \Rightarrow (\phi^{ij} \Leftrightarrow \phi_0^{ij})$. (This is obvious when ϕ^{ij} is of the form p, $\neg p$, $K_a\psi$ or $\neg K_a\psi$; it follows from Claim A when ϕ^{ij} is of the form $R(\overrightarrow{t})$ or $\neg R(\overrightarrow{t})$; and it follows from the Knowledge de Re Axiom when ϕ^{ij} is of the form $K_a^{\overrightarrow{t}} t'$ or $\neg K_a^{\overrightarrow{t}} t'$.) Using the normal form of φ, we obtain the following validity:

$$(*) \ \models \ \left(\bigwedge_{t \in \mathcal{T}_\varphi} t = F(t) \right) \Rightarrow (\varphi \Leftrightarrow \bigwedge_i \bigvee_j \phi_0^{ij}).$$

To prove now *one direction of Claim C*, assume that φ is valid. Using (*), it follows that $\left(\bigwedge_{t \in \mathcal{T}_\phi} t = F(t)\right) \Rightarrow \bigwedge_i \bigvee_j \phi_0^{ij}$ is valid, i.e. φ_0 is valid. For the *other direction*: assume that φ_0 is valid, and let $M = (W, D, [0], [1], \sim_a, \|\bullet\|, \bullet(\bullet), \mathbf{f}, \mathbf{R})_{a \in \mathcal{A}, f \in \mathcal{F}, R \in \mathcal{R}}$ be a model and $w_0 \in W$ be any world. We can change this to a different model $M' = (W, D, [0], [1], \sim_a, \|\bullet\|, \bullet(\bullet)', \mathbf{f}, \mathbf{R})_{a \in \mathcal{A}, f \in \mathcal{F}, R \in \mathcal{R}}$, where we changed only the value map (and only at w_0) by putting $w_0(c)' = w_0(t)$ whenever $F(t) = c$ with $t \in \mathcal{T}_\varphi$, and $w(c)' = w(c)$ in rest. Since F is injective, this gives us a well-defined value map. The change doesn't affect the values of the terms $t \in \mathcal{T}_\varphi$ (since they don't contain any of the fresh constants whose value was changed), so we have $w_0(t)' = w_0(t) = w_0(F(t))'$ for all these terms, hence $w_0 \in |\bigwedge_{t\in\mathcal{T}_\phi} t = F(t)|_{M'}$. Using (*) and the fact that φ_0 is valid, it follows that $w_0 \in |\varphi\|_{M'}$. But φ contains none of the constants whose values were changed, so its truth value was not affected by the change, i.e. we also have $w_0 \in |\varphi\|_M$. Since M and w_0 are arbitrary, we conclude that φ is valid.

Applying repeatedly the last two Claims, we get an immediate proof of Proposition 2.18, by induction on $\gamma(\varphi)$. □

Thus, we have reduced the problem of proving FMP for *LED* to the corresponding problem for the simpler language LED_0.

STEP 2: Finite Model Property for LED_0

Proposition 2.19 *The logic LED_0 has* (strong) finite model property*: every satisfiable formula φ_0 is satisfiable in a finite model.*

Proof. Let φ_0 be a satisfiable formula in a language $\mathcal{L} = \mathcal{L}(P, Var, C, \mathcal{F}, \mathcal{R}, ar)$ for LED_0, and let $M = (W, D, [0], [1], \sim_a, \| \bullet \|, \bullet(\bullet), \mathbf{f}, \mathbf{R})_{a\in\mathcal{A}, f\in\mathcal{F}, R\in\mathcal{R}}$ be a model and $w_0 \in W$ be a world such that $w_0 \in \|\varphi_0\|_M$. Take $\Sigma \subseteq \mathcal{L} \cup \mathcal{T}$ be the *smallest* set of formulas and terms in LED_0 that contains φ_0, 0 and 1, and is closed under subterms and subformulas.[14] It is easy to see that Σ *is finite.*

Let us put $\mathcal{T}_\Sigma := \mathcal{T} \cap \Sigma$, $\mathcal{L}_\Sigma := \mathcal{L} \cap \Sigma$, $P_\Sigma := P \cap \Sigma$, $Var_\Sigma := Var \cap \Sigma$, $C_\Sigma := C \cap \Sigma$. We now define an equivalence relation $\cong$ on W by putting: $w \cong v$ iff $\forall\varphi \in \Sigma(w \in \|\varphi\|_M \Leftrightarrow v \in \|\varphi\|_M)$. For any $w \in W$ we denote by $|w| := \{v \in W : w \cong v\}$ the $\cong$-equivalence class of w, and we put $W_\Sigma := \{|w| : w \in W\}$ for the set of all $\cong$-equivalence classes. Note that W_Σ *is finite.* Fix now some arbitrary well-ordering $<$ of W. For every class $w \in W_\Sigma$, we denote by w_0 the first element of the class $|w|$ (wrt $<$). Let $D_0 := \{w(t) : |w| \in W_\Sigma, t \in \mathcal{T}_\Sigma\} \cup \{[0], [1]\}$. Note that D_0 is *finite.* If $D \setminus D_0 \neq \emptyset$, then choose some $d^0 \in D \setminus D_0$, and put $D_\Sigma := D_0 \cup \{d^0\}$. If however $D \setminus D_0 = \emptyset$, then put $D_\Sigma := D_0 = D$. Note that in both cases D_Σ is a *finite subset of* D.

We now define a "filtrated" model

$$M_\Sigma = (W_\Sigma, D_\Sigma, [0]_\Sigma, [1]_\Sigma, \sim_a^\Sigma, \| \bullet \|_\Sigma, \bullet(\bullet), \mathbf{f}_\Sigma, \mathbf{R}_\Sigma)_{a\in\mathcal{A}, f\in\mathcal{F}, R\in\mathcal{R}}.$$

by taking W_Σ and D_Σ as above, and putting $[0]_\Sigma := [0]$; $[1]_\Sigma := [1]$; $w \sim_a^\Sigma v$ iff $\forall K_a\phi \in \Sigma\,(w \in \|K_a\phi\|_M \Leftrightarrow v \in \|K_a\phi\|_M)$; $\|p\|_\Sigma := \{|w| : w \in \|p\|_M\}$ for $p \in P_\Sigma$, and $\|p\|_\Sigma := \emptyset$ for $p \in P \setminus P_\Sigma$; $|w|(\alpha) := w(\alpha)$ for $\alpha \in Var_\Sigma \cup C_\Sigma$, and $|w|(\alpha) := d^0$ for $\alpha \in (Var \cup \Sigma) \setminus (Var_\Sigma \cup C_\Sigma)$; $\mathbf{f}_\Sigma(d_1, \ldots, d_n) := \mathbf{f}(d_1, \ldots, d_n)$ if there exists a term $f(t_1, \ldots, t_n) \in \Sigma$ and a world $|w| \in W_\Sigma$ s.t. $|w|(t_i) = d_i$ for all $i \in \{1, \ldots, n\}$; and $\mathbf{f}_\Sigma(d_1, \ldots, d_n) := d^0$ otherwise; finally, $\mathbf{R}_\Sigma := \mathbf{R} \cap (D_\Sigma \times D_\Sigma)$. Note that $\mathbf{f}_\Sigma : D_\Sigma \to D_\Sigma$ is a *well-defined* function, $=_\Sigma$ is the diagonal $\{(d, d) : d \in D_\Sigma\}$, and M_Σ *is indeed a finite model.*

Claim D ("*Term Lemma*"): For every term $t \in \mathcal{T}_\Sigma$ and $w \in W$, we have $|w|(t)_\Sigma = w(t)$.

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[14] More precisely, Σ has to satisfy the following closure conditions: (1) if $(\varphi \to \psi) \in \Sigma$ then $\varphi, \psi \in \Sigma$; (2) if $K_a\varphi \in \Sigma$ then $\varphi \in \Sigma$; (3) if $f(t_1, \ldots, t_n) \in \Sigma$ then $t_1, \ldots, t_n \in \Sigma$; (4) if $R(t_1, \ldots, t_n) \in \Sigma$ then $t_1, \ldots, t_n \in \Sigma$.

Proof of Claim D: Proof by induction: the base case is by definition; for the inductive step: $|w|(f(t_1,\dots,t_n) = f_\Sigma(|w|(t_1),\dots,|w|(t_n)) = f_\Sigma(w(t_1),\dots,w(t_n)) = f(w(t_1),\dots,w(t_n)) = w(f(t_1,\dots,t_n))$, where we used the induction hypothesis, the definition of f_Σ and $f(t_1,\dots,t_n) \in \mathcal{T}_\Sigma$.

Claim E ("*Filtration Lemma*") For all formulas $\phi \in \Sigma$, we have:

$$|w| \in \|\phi\|_{M_\Sigma} \text{ iff } w \in \|\phi\|_M, \quad \text{for every } w \in W.$$

Proof of Claim E: Proof by induction on ϕ. All steps go as in the classical proof of the Filtration Lemma (for the logic $S5$), except for the relational atoms $R(t_1,\dots,t_n) \in \Sigma$, for which we have $t_1,\dots,t_n \in \mathcal{T}_\Sigma$, and thus Claim D can be applied. So we have the sequence of equivalencies: $|w| \in \|R(t_1,\dots,t_n\|_{M_\Sigma}$ iff $(|w|(t_1),\dots,|w|(t_n)) \in \mathbf{R}_\Sigma$ iff (by definition of R_Σ) $(|w|(t_1),\dots,|w|(t_n)) \in \mathbf{R}$ iff (by Claim D) $(w(t_1),\dots,w(t_n)) \in \mathbf{R}$ iff $w \in \|R(t_1,\dots,t_n)\|_M$.

This finishes our proof that LED_0 has FMP. □

Putting together Step 1 and Step 2, we conclude that LED *also has FMP*, and thus (being also axiomatizable) it is *decidable*.

3 Learning the Value of a Variable

We now extend LED with *public value-announcement operators* $\langle ! \overrightarrow{t} \rangle$ for every tuple $\overrightarrow{t} \in \mathcal{T}^*$. These operators act on both formulas and terms. The syntax of *Public Announcement Logic of Epistemic Dependency* (PALED) is given by:

$$\varphi ::= p \mid R(\overrightarrow{t}) \mid \varphi \to \varphi \mid K_a^{\overrightarrow{t}} t \mid \langle ! \overrightarrow{t} \rangle \varphi$$

$$t ::= x \mid \quad c \quad \mid \quad ?_\varphi \quad \mid f(\overrightarrow{t}) \mid \langle ! \overrightarrow{t} \rangle t$$

where x are variables, c are constants, t are term in $\mathcal{T}$, $\overrightarrow{t}$ are finite tuples of terms, and f and R are symbols of arity equal to the length of $\overrightarrow{t}$.

The operations of *propositional substitution* and *variable substitution* can be extended in the obvious way to the new formulas and terms.[15]

Semantics. Our notion of model is the same as for LED. For every model M, we define an *extended valuation* (truth map) $\|\varphi\|$, an *extended value map* $w(t)_M$, and the *update* $M^{\overrightarrow{t}}$ *of model* M *with any finite string of terms* $\overrightarrow{t} \in \mathcal{T}^*$. The truth map and the extended value map are defined as for LED, except that we add the clauses:

$$\| \langle ! \overrightarrow{t} \rangle \varphi \|_M \;=\; \|\varphi\|_{M^{\overrightarrow{t}}}, \quad \text{and} \quad w(\langle ! \overrightarrow{t} \rangle t')_M \;=\; w(t')_{M^{\overrightarrow{t}}}.$$

For the update $M^{\overrightarrow{t}} := (W, \sim^a_{M^{\overrightarrow{t}}}, \| \bullet \|, \bullet(\bullet), \mathbf{f}, \mathbf{R})$, we *leave all the components the same*, except for *changing the epistemic relations* as follows:

$$\sim^a_{M^{\overrightarrow{t}}} \;=\; \{(w,s) \in W \times W | w \sim^a s, w(\overrightarrow{t})_M = s(\overrightarrow{t})_M\}.$$

[15] Formally, we put: $(\langle !t_1,\dots,t_n\rangle\varphi)[p/\theta] := \langle !t_1[p/\theta],\dots,t_n[p/\theta]\rangle(\varphi[p/\theta])$; $(\langle !t_1,\dots,t_n\rangle t')[p/\theta] := \langle !t_1[p/\theta],\dots,t_n[p/\theta]\rangle(t'[p/\theta])$; $(\langle !t_1,\dots,t_n\rangle\varphi)[x/t] := \langle !t_1[x/t],\dots,t_n[x/t]\rangle(\varphi[x/t])$; $(\langle !t_1,\dots,t_n\rangle t')[x/t] := \langle !t_1[x/t],\dots,t_n[x/t]\rangle(t'[x/t])$.

So all *agents jointly learn the values of* $\overrightarrow{t}$, and nothing else changes.

Example 2 The formula $\langle !x_a\rangle(K_a x_b \wedge K_b x_b)$ is true in (all worlds of) the model from Example 1 above. This can be verified by performing an update $!x_a$, which removes epistemic arrows between worlds having different values for x_a and checking that $K_a x_b \wedge K_b x_b$ holds in the updated model. So, after the value of Alice's number is announced, everybody will know Bob's number.

Propositional Public Announcements. The standard (propositional) public announcement formulas from PAL can be defined as abbreviations in our syntax, by putting: $\langle !\phi\rangle\psi := \phi \wedge \langle !(?\phi)\rangle\psi$, and $[!\phi]\psi := \phi \rightarrow \langle !(?\phi)\rangle\psi$.

Proof system. We obtain a complete system for $PALED$ by restricting the Substitution Rules to static contexts, and adding a Necessitation Rule for announcements, as well as Reduction Axioms. More precisely:

(i) *Restricted Propositional Substitution*: From φ, infer $\varphi[p/\theta]$, *provided that* p *doesn't occur in the scope of any dynamic operator in* φ.

(ii) *Restricted Variable Substitution*: From φ, infer $\varphi[x/t]$, *provided that* x *doesn't occur in the scope of any dynamic operator in* φ.

(iii) *All the other axioms and rules of the system LED*.

(iv) *Necessitation Rule for Announcements*: From $\vdash \varphi$ infer $\vdash \langle !\overrightarrow{t}\rangle\varphi$.

(v) *Propositional Reduction Axioms* [16]:

$$\langle !\overrightarrow{t}\rangle p \;\leftrightarrow\; p$$

$$\langle !\overrightarrow{t}\rangle R(t_1,\ldots,t_n) \;\leftrightarrow\; R(\langle !\overrightarrow{t}\rangle t_1,\ldots,\langle !\overrightarrow{t}\rangle t_n)$$

$$\langle !\overrightarrow{t}\rangle(\varphi \rightarrow \psi) \;\leftrightarrow\; \Big(\langle !\overrightarrow{t}\rangle\varphi \rightarrow \langle !\overrightarrow{t}\rangle\psi\Big)$$

$$\langle !\overrightarrow{t}\rangle K_a^{t_1,\ldots,t_n} t' \;\leftrightarrow\; K_a^{\overrightarrow{t},\langle !\overrightarrow{t}\rangle t_1,\ldots,\langle !\overrightarrow{t}\rangle t_n}\langle !\overrightarrow{t}\rangle t'$$

(vi) *Term Reduction Axioms*:

$$\langle !\overrightarrow{t}\rangle c \;=\; c$$

$$\langle !\overrightarrow{t}\rangle x \;=\; x$$

$$\langle !\overrightarrow{t}\rangle ?_{\varphi} \;=\; ?_{\langle !\overrightarrow{t}\rangle\varphi}$$

$$\langle !\overrightarrow{t}\rangle f(t_1,\ldots,t_n) \;=\; f(\langle !\overrightarrow{t}\rangle t_1,\ldots \langle !\overrightarrow{t}\rangle t_n)$$

Applying the Reduction Axiom iteratively, we can eliminate all dynamic operators in the usual way, and thus prove:

[16] Note that the third reduction axiom is the axiom K for announcements. This is usually stated for *universal* modalities $[\alpha]\psi$, but these coincide with the existential ones in our case, since value announcements are *deterministic* actions (whose transition relations are *functions*). Combining this axiom with the Necessitation Rule for announcements, one can show that formulas obtained by prefixing provably equivalent formulas with dynamic operators are provably equivalent. This is needed to apply Reduction Axioms repeatedly in order to gradually reduce (and eventually eliminate) *nested* announcement operators, e.g. $\langle \overrightarrow{t}^1\rangle\langle \overrightarrow{t}^2\rangle\psi$.

Theorem 3.1 The proof system $PALED$ is sound and weakly complete for the logic $PALED$. Moreover, $PALED$ has the same expressivity as LED.

4 Comparison with other work

Our epistemic dependency formulas are closely connected to Dependency Logic [18,19]. Note that $K_a^{x_1,\ldots,x_n} y$ expresses only a "local" dependency (at the actual world): this is reflected in the fact that this attitude is *not introspective* (i.e. $K_a^{x_1,\ldots,x_n} y$ does *not* imply $K_a K_a^{x_1,\ldots,x_n} y$). However, its "introspective version" $K_a K_a^{x_1,\ldots,x_n} y$ gives a more "global" dependency (across all the epistemically-possible worlds), thus capturing *knowledge of the dependency.* It is easy to see that $w \models K_a K_a^{x_1,\ldots,x_n} y$ is equivalent to the assertion that the *dependence atom* $=(x_1,\ldots,x_n,y)$ holds at the "team" $\{v : w \sim_a v\}$ comprising the set of (variable assignments associated to all) epistemic alternatives of w. But note that LED is *decidable*, in contrast to most variants of Dependence Logic!

Our logic has also interesting relations with the so-called 'erotetic logics' [8,12,26], including inquisitive logics [10]. First, as Hintikka [13], Schaffer [17], Aloni and others [1] argued, all types of knowledge (knowing that, knowing what, knowing who, knowing how, knowing whether, knowing which) are special cases of knowing the answer to a question: "*All knowledge involves a question. To know is to know the answer*" ([17]: 401). Second, every "variable" (mapping worlds to a set D of values) induces a partition of the state space; so variables could be used to represent any partitional question. Knowing the answer to a question is the same as knowing the value of the corresponding map.[17] Our epistemic dependency formulas capture an 'epistemic' version of *interrogative implication*, as studied in inquisitive logics. But the variable representation gives us *more information*: if we identify the values in D with abstract "answers", then we can compare answers for different questions, and thus formalize phenomena such as "knowing the answer without knowing the question", that cannot be dealt with in standard inquisitive semantics. We believe that an account of *questions as functions* from worlds to (sets of) "answers" gives a better model for interrogatives then the usual inquisitive representation.

In contrast to both Inquisitive Logic and Dependency Logic, our approach preserves the "classicality" of propositional calculus, and re-interprets the non-classical features in terms of *modal-epistemic* operators. As in Dynamic Epistemic Logic [14,7,5,22,20], and its most natural interrogative versions [3,21,6,4], our semantics is "modal" in the usual sense, with formulas evaluated at worlds, rather than at sets of worlds. We think of this as an advantage of our approach, suggesting that questions (and variable dependency) can be understood without denying the classical logical principles. To know is to know the answer. But the logic of Aristotle, Boole and Frege is not dead just yet.

[17] As for the "non-partitional" questions (having non-unique complete answers), as considered in Inquisitive Semantics, they could be represented in our framework as functions from worlds to *sets* of answers in $\mathcal{P}(D)$. It would be interesting to study the epistemic logic of "knowing *an* answer" (rather than 'the' answer) in this generalized framework.

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