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# Quantum planes and quantum cylinders from Poisson homogeneous spaces

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**Abstract.** Quantum planes and a new quantum cylinder are obtained as quantization of Poisson homogeneous spaces of two different Poisson structures on classical Euclidean group  $E(2)$ .

## 1. Introduction

The concept of the homogeneous space of a group is, maybe, one of the most widespread mathematical concepts, lying at the very foundation of, for example, harmonic analysis and the symmetries of physical systems. Right from the beginning of the theory of quantum groups, clarifying the concept of a quantum homogeneous space for a quantum group has been considered to be of the utmost importance (see [11] for example), although up until now a complete theory has been absent. The purpose of this work is to study the homogeneous quantum spaces of Euclidean quantum groups through an analysis of the ‘semiclassical’ limit of the Poisson structure on the classical group. Our aim is to verify to what extent the results in [12] and [13] relating covariant Poisson structures on the sphere and the one-parameter family of quantum spheres [11] are still valid in this case (although we will not deal with the analytical aspects as these are extensively covered in those references). There are two different versions of the Euclidean quantum group, introduced in [15] and [3] respectively; in [3] it is also shown how they can both be obtained through a contraction procedure from  $SU_h(2)$ . The way in which they are related is explained in [1]. Quite a lot of work has been done on these groups, whose interest lies both in that they are an easy example of the non-semisimple and non-compact case and in physical applications. For example, in [16] the standard Euclidean quantum group is treated at an analytical level, in [15] its relations with  $q$ -special functions were first studied and later in [2] and [9] (the last of a series of papers and useful for further references)  $q$ -harmonic analysis is more thoroughly investigated. The roots of unity theory [4] for the standard case were examined in [5], and these were hypothesized [1] and proved [4] to be trivial in the non-standard case. Quantum homogeneous spaces in the standard case were studied in [2], through duality arguments involving the quantized universal enveloping algebra. The semiclassical limit of these results is here shown to fit into the framework of the classification of covariant Poisson structures. This is the content of section 2. Furthermore, in the non-standard case the semiclassical limit suggests quite naturally that homogeneous spaces should be of cylinder-type. The covariant Poisson structures on this cylinder are classified in section 3 and in section 4 an explicit quantum cylinder whose classical limit is the Poisson homogeneous cylinder is

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given through generators and relations, verifying conditions explained in [6] for quantum homogeneous spaces.

**2. The standard Poisson algebraic  $\mathcal{E}(2)$  and quantum planes**

The classical two-dimensional Euclidean group is usually written as the space of complex matrices of the form

$$\begin{pmatrix} v & n \\ 0 & 1 \end{pmatrix}$$

where  $|v| = 1$ .

Its function (polynomial) algebra can thus be seen as a commutative algebra on generators  $v, \bar{v}, n, \bar{n}$ , with the additional relation  $v\bar{v} = 1$ . The matrix form also immediately gives us the Hopf algebra structure which we explicitly write down as follows

$$\begin{aligned} \Delta v &= v \otimes v & \Delta \bar{v} &= \bar{v} \otimes \bar{v} \\ \Delta n &= \bar{v} \otimes n + n \otimes 1 & \Delta \bar{n} &= v \otimes \bar{n} + \bar{n} \otimes 1 \\ S(v) &= \bar{v} & S(n) &= -vn & S(\bar{n}) &= -\bar{v}\bar{n} \\ \varepsilon(v) &= 1 & \varepsilon(n) &= \varepsilon(\bar{n}) = 0 \end{aligned}$$

and the usual  $*$ -structure is given by

$$v^* = \bar{v} \quad n^* = \bar{n} \quad *^2 = \text{Id}.$$

Let us define on this function algebra the following quadratic Poisson bracket (a standard bracket used in what follows):

$$\{v, n\} = vn \quad \{v, \bar{n}\} = v\bar{n} \quad \{n, \bar{n}\} = n\bar{n} \quad \{v, \bar{v}\} = 0.$$

*Remark 2.1.* These formulae define a Poisson bracket on the algebra of polynomial functions on  $\mathcal{E}(2)$ . That is why we talk about an *algebraic* Poisson structure and not of a Lie–Poisson structure that should be given on the whole algebra of smooth functions. In the following we will sometimes drop the adjective algebraic.

*Proposition 2.2.* The formulae above define a Poisson algebraic structure on  $\mathcal{E}(2)$ .

*Proof.* Let us write the Poisson bivector corresponding to the bracket as follows:

$$w(v, n, \bar{n}) = vn\partial_v \wedge \partial_n + v\bar{n}\partial_v \wedge \partial_{\bar{n}} + n\bar{n}\partial_n \wedge \partial_{\bar{n}}.$$

By direct computation this Poisson bivector verifies the multiplicativity property:

$$w(gg') = ((L_g)'_{g'} \otimes (L_g)'_{g'})w(g') + ((R_{g'})'_g \otimes (R_{g'})'_g)(w(g))$$

where  $L_g$  and  $R_g$  stand respectively for left and right translations in  $\mathcal{E}(2)$ . □

Differentiating the Poisson bivector at the origin gives a coalgebra structure on the Lie algebra  $e(2)$ . Choosing  $J = \partial_v|_e, X = \partial_n|_e, Y = \partial_{\bar{n}}|_e$  as generators of the Lie algebra we have the coalgebra structure:

$$\delta(J) = 0 \quad \delta(X) = J \wedge X \quad \delta(Y) = J \wedge Y.$$

It is easy to show that such a structure is not a coboundary one.

The Poisson bivector mentioned above defines the associated homomorphism  $B_w$  from the cotangent to the tangent bundle of  $\mathcal{E}(2)$  that can be easily calculated to be

$$\begin{aligned} dv &\mapsto -v(n\partial_n + \bar{n}\partial_{\bar{n}}) = X_v \\ dn &\mapsto n(-v\partial_v + \bar{n}\partial_{\bar{n}}) = X_n \\ d\bar{n} &\mapsto \bar{n}(n\partial_n + v\partial_v) = X_{\bar{n}}. \end{aligned}$$

Thus we have the differentiable distribution of tangent subspaces:

$$(v, n, \bar{n}) \rightarrow (X_v, X_n, X_{\bar{n}})$$

whose integral manifolds are the symplectic leaves of the Poisson structure.

*Remark 2.3. The geometric description of symplectic leaves.* Let us observe that for a generic point of the space  $(v, n, \bar{n})$  we have

$$vn\bar{n}X_v + \bar{n}X_n + nX_{\bar{n}} = 0$$

and thus the tangent space is generically two dimensional.

All the points of the form  $(v, 0, 0)$  are zero-dimensional symplectic leaves of this Poisson distribution. The set of all these points is nothing other than  $S^1$  viewed as a subgroup of  $\mathcal{E}(2)$ , and thus, being a union of leaves, it is a Poisson algebraic subgroup. It is not difficult to see that this subgroup together with all its finite subgroups forms the only Poisson algebraic subgroup of  $\mathcal{E}(2)$ .

At this point it could be interesting to calculate the primitive spectrum of standard  $Fun_q(\mathcal{E}(2))$  to see whether it has a good relation with the symplectic foliation. The primitive spectrum of this algebra is not very difficult to discern after we have noticed that it can be given the form of an iterated skew (twisted) polynomial ring (see [7, 8]).

*Corollary 2.4.* The  $\mathcal{E}(2)$ -homogeneous space  $\mathbf{R}^2$  has a natural Poisson structure induced by the algebraic Poisson structure on  $\mathcal{E}(2)$ .

The projection map  $\pi : \mathcal{E}(2) \rightarrow \mathbf{R}^2$  is nothing other than the map that sends  $(v, n, \bar{n}) \rightarrow (\frac{1}{2}(n + \bar{n}), -i\frac{1}{2}(n - \bar{n}))$ . Denoting  $z$  and  $\bar{z}$  the coordinates on the plane it can then immediately be seen that the Poisson structure on the plane is

$$\{z, \bar{z}\} = z\bar{z}.$$

By definition such a structure is  $(\mathcal{E}(2), w)$ -covariant. We recall that by covariance we mean that the action map  $\phi : \mathcal{E}(2) \times \mathbf{R}^2 \rightarrow \mathbf{R}^2$  is a Poisson map when the first space is given the product Poisson structure.

*Remark 2.5.* The Poisson structure just defined on the plane has a symplectic foliation consisting of two families of zero-dimensional leaves parametrized by points of two orthogonal lines and four two-dimensional leaves separated by those points. Observing that the usual quantum plane structure  $F_q$ , the algebra on two  $q$ -commuting generators, is a quantization of this Poisson bracket, it is no surprise that the primitive spectrum of this algebra is in bijective correspondence with the foliation (see [14] for the explicit expression of the primitive spectrum).

The next problem is that of classifying all  $(\mathcal{E}(2), w)$ -covariant Poisson structures on the plane. We will follow [17] where the covariance condition is rewritten at the infinitesimal level. We will denote by

$$\phi : \mathcal{E}(2) \times \mathbf{R}^2 \rightarrow \mathbf{R}^2$$

the action map, with  $\phi_x = \phi(\cdot, x) \forall x \in \mathbf{R}^2$  and  $\phi_g = \phi(g, \cdot) \forall g \in \mathcal{E}(2)$ . With this notation condition 2.2 of [17] states that covariant structures on the plane are in one-to-one correspondence with elements  $\rho \in \wedge^2 T_0(\mathbf{R}^2)$  such that

$$(\phi_0)_* \delta(X) + X \cdot \rho = 0 \quad \forall X \in \mathcal{G}_0$$

where  $\mathcal{G}_0$  is the tangent algebra of the rotation subgroup—the stabilizer of 0 in  $\mathbf{R}^2$ , and  $X \cdot \rho$  is given by

$$\left. \frac{d}{dt} ((\phi_{\exp(tX)})_* \rho) \right|_{t=0}.$$

In fact such a  $\rho$  can be extended to a Poisson bivector field on  $\mathbf{R}^2$  simply as

$$\rho(x) = (\phi_0)_{*,g}(w(g)) + (\phi_g)_{*,0}(\rho)$$

where  $g \in \mathcal{E}(2)$  is such that  $g \cdot 0 = x$ . Now, as  $\mathcal{G}_0$  is nothing but the algebra generated by  $J$  and  $\delta(J) = 0$ , the above condition can be rewritten as  $J \cdot \rho = 0$  (simply an invariance condition) which is always verified. We can then take  $\rho = k \partial_1 \wedge \partial_2$ , where the derivatives are calculated in 0. The corresponding bivector field is

$$(a, b) \mapsto k \partial_1 \wedge \partial_2$$

and the Poisson bracket on the plane is

$$\{z, \bar{z}\} = z\bar{z} + k.$$

Note that when  $\rho = 0$  we obtain exactly the Poisson structure on the plane given by corollary 2.4. We have then just proved the following.

*Proposition 2.6.* There is a one-parameter family of covariant Poisson structures on the plane, with respect to the standard Poisson structure on  $\mathcal{E}(2)$ , given by

$$\{z, \bar{z}\} = z\bar{z} + k.$$

*Remark 2.7.* In the case in which  $k \neq 0$  the symplectic foliation changes drastically. The zero-dimensional symplectic leaves are the points of the hyperbola  $z\bar{z} = -k$ , and they divide the plane into three two-dimensional symplectic leaves. Following this geometric picture we will refer to this case as the case of hyperbolic covariant structures. The behaviour of the symplectic foliation with respect to  $k$  can be described through the intersection between the plane  $z = 0$  and the hyperbolic cone  $xy = k$  (even in the quantum sphere case such a description is possible).

### 3. The non-standard Poisson structure and the cylinder

We still deal with the same function algebra but we want to consider another family of Poisson structures (which we shall refer to as non-standard Poisson structures) given in [10] and whose quantization gives the so-called non-standard Euclidean quantum group

$$\{v, n\} = \omega(1 - v) \quad \{v, \bar{n}\} = -\omega(v^2 - v) \quad \{n, \bar{n}\} = \omega(n - \bar{n})$$

where  $\omega$  is a non-zero complex number.

That this bracket gives a Poisson algebraic structure on  $\mathcal{E}(2)$  (or, in other words, a family of isomorphic structures) is proven in [10].

The infinitesimal counterpart of the bracket is the coproduct on the Lie algebra  $e(2)$  given by

$$\delta(P_1) = 0 \quad \delta(P_2) = \omega P_2 \wedge P_1 \quad \delta(J) = \omega J \wedge P_2.$$

This is a coboundary coproduct by taking

$$\delta(X) = a d_X r \quad r = \omega J \wedge P_2.$$

We will call  $w'$  the Poisson bivector and  $B_{w'}$ , the associated homomorphism between tangent and cotangent spaces, is

$$\begin{aligned} dv &\mapsto \omega(v-1)\partial_n + \omega(v^2-v)\partial_{\bar{n}} = \omega(v-1)(\partial_n + v\partial_{\bar{n}}) = X_v \\ dn &\mapsto \omega(1-v)\partial_v + \omega(\bar{n}-n)\partial_{\bar{n}} = X_n \\ d\bar{n} &\mapsto \omega(v-v^2)\partial_v + \omega(n-\bar{n})\partial_n = X_{\bar{n}} \end{aligned}$$

from which we have the distribution of tangent subspaces that integrates to symplectic leaves.

*Remark 3.1.* For every  $(v, n, \bar{n})$  the relation

$$(v-v^2)X_n + (v-1)X_{\bar{n}} + (\bar{n}-n)X_v = 0$$

holds, showing that the distribution is, at most, two dimensional at every point. If we restrict ourselves to the point of  $S^1$  given by  $n = \bar{n} = 0$  as in the previous case, we see that the distribution is exactly two dimensional in these points and thus every point of the form  $(v, 0, 0)$  is contained in a two-dimensional symplectic leaf. Thus  $S^1$  is not a Poisson subgroup of non-standard Poisson  $\mathcal{E}(2)$ . If we restrict ourselves to points  $v = 1$  and  $n = \bar{n}$  we see that the distribution vanishes and thus all these points are zero-dimensional symplectic leaves. Again we are dealing with a subgroup (in matrix form this corresponds to upper triangular unimodular  $2 \times 2$  real matrices) and it is isomorphic to  $\mathbf{R}$ . Thus  $\mathbf{R}$  is a Poisson subgroup for the non-standard Poisson structure on  $\mathcal{E}(2)$ . This implies that the corresponding homogeneous space, a cylinder, has a canonical covariant Poisson structure. One further remark could be that in this case the discrete infinite groups are also a Poisson subgroup. However we have to note that, as the Poisson structure is defined only at the algebraic level, such subgroups are not closed, and thus cannot be given by annihilating an element of the function algebra.

Again it may be interesting to confront the whole symplectic foliation with the primitive ideal structure of its quantization, which could be computed using the fact that  $\mathcal{E}_\omega(2)$  is an iterated differential polynomial ring (as noted in [4]).

*Proposition 3.2.* The cylinder  $\mathcal{C}$ , viewed as a homogeneous space of the non-standard Poisson Euclidean group, can be parametrized as having its function algebra generated by  $v, \bar{v}$  and  $\bar{v}\bar{n} - vn = m$  with the covariant Poisson structure given by

$$\{v, m\} = -\omega(v^2 - 1).$$

*Proof.* It is sufficient to show that the two functions  $v$  and  $m$  parametrize the cylinder, the rest will follow immediately. That the function  $v$  is invariant with respect to the action of  $\mathcal{A}$  is obvious, and an easy calculation also shows the invariance of  $m$ .  $\square$

*Remark 3.3.* From the explicit Poisson bracket in 3.3 one can obtain the corresponding symplectic foliation on the cylinder. The distribution of tangent subspaces is

$$\begin{aligned} dv &\mapsto \omega(v^2 - 1)\partial_m = X_v \\ dm &\mapsto -\omega(v^2 - 1)\partial_v = X_m \end{aligned}$$

and thus all the points on the line  $v = 1$  are zero-dimensional symplectic leaves and all the other points belong to a unique two-dimensional leaf.

*Remark 3.4.* Now we want to classify the covariant structures on the cylinder as in proposition 2.6. Again multiplicative  $(\mathcal{E}(2), w')$ -Poisson structures are characterized by their value at one point, say  $v = 1, m = 0$ . At this point a bivector has the form  $\varrho = k\partial_v \wedge \partial_m$  and, with the notation used in 2.5, has to fulfil the invariance condition

$$(\phi_{(1,0)})_*\delta(P_1) + P_1 \cdot \varrho = 0$$

that reduces to the condition  $P_1 \cdot \varrho = 0$  due to the triviality of  $\delta(P_1)$ . This condition is always verified. This implies that the space of covariant Poisson structures has the form of an affine one-dimensional space given that adjoining the structure of proposition 3.2 there is an invariant bivector field extending  $\varrho$ . Explicitly the possible Poisson structures are listed in the following.

*Proposition 3.5.* There is a one-parameter family of covariant Poisson structures on the cylinder, with respect to the non-standard Poisson  $\mathcal{E}(2)$ , given by

$$\{v, m\} = -\omega(v^2 - 1) + k.$$

*Remark 3.6.* As for the plane the case  $k \neq 0$  shows a completely different symplectic foliation. Just note that the bracket is trivial when

$$\omega v^2 + \omega + k = 0$$

from which we obtain the solutions

$$v = \pm\sqrt{(1 + \omega^{-1}k)}.$$

The condition that  $v$  is on the unit circle can be expressed as  $v\bar{v} = 1$ . Apart from the singular situation in which  $\omega^{-1}k \in i\mathbf{R}$ , there is always a special value of  $k \neq 0$  for which  $1 + \omega^{-1}k$  (and thus its square roots) belong to the unit circle. Thus for every complex  $\omega$  which is not purely imaginary there are always two values of  $v$  for which the Poisson bracket is degenerate. The corresponding symplectic foliation is then given by two lines of points and two two-dimensional symplectic leaves between them. For all the other values of  $k$  the distribution is two dimensional in every point and so the Poisson bracket induces a symplectic form on the whole cylinder. Geometrically the leaves can be described as an intersection between the cylinder itself and lines belonging to a one-parameter family of planes all intersecting in a fixed line of the cylinder.

#### 4. Quantum homogeneous cylinder

We want now to pass from the semiclassical situation of Poisson homogeneous spaces to quantum homogeneous spaces, as defined for example in [6]. The standard  $\mathcal{E}_q(2)$  has been treated in [2], where connections with the theory of spherical functions are also explained, so that in what follows we will restrict ourselves to the non-standard case where the quantization of the given Poisson structure can be obtained, as in [10], simply substituting the Poisson

bracket with the commutator, leaving the coalgebra structure unchanged. Let us just note the connections between [2] and proposition 2.6. The natural covariant Poisson structure is exactly the semiclassical limit of the usual quantum plane and is, in fact, the only quantum homogeneous spaces obtained by ‘quotient’ with respect to a proper quantum subgroup, incidentally the quantization of the unique Poisson subgroup of  $\mathcal{E}(2)$ . The quantum hyperboloid of [2] has, as its semiclassical limit, the covariant Poisson structure of 2.6 with  $k = -2$  (and an analysis of symplectic leaves gives a clear meaning to the appearance of the word ‘hyperbolic’ in its name). The fact that covariant Poisson structures constitute a complete one-parameter family either suggests the possibility of other quantum planes of which the two given in [13] should be, in a sense, paradigmatic, or asks for an explanation of the failure in quantizing Poisson homogeneous spaces for values of  $k$  different from 0 and  $-2$ .

Let us move to the non-standard case. Let us recall [1] that the non-standard quantum group is the Hopf algebra with the same coproduct, counit and antipode as the standard quantum group and commutation relations obtained substituting the Poisson bracket with the Lie bracket in the formulae at the beginning of section 3.

The classical projection  $\pi : G \rightarrow M$  from one Poisson–Lie group to its homogeneous space corresponds to an injective map on the function algebra level  $\hat{\pi} : \mathcal{F}(M) \rightarrow \mathcal{F}(G)$ . In remark 3.6 it was stated that we can see this map as a map between the quantizations and, requiring it to be an algebra map, we find the quantized commutation relations for the Poisson homogeneous space. It is obvious how this works in the quantum plane case. This implies that we can give the following definition.

*Definition 4.1.* The quantum cylinder  $\mathcal{C}_\omega$  is the algebra generated by the elements  $v, \bar{v}$  and  $m$  with the relations:

$$v\bar{v} = \bar{v}v = 1 \quad vm = mv - \omega(v^2 - 1) \quad \bar{v}m = m\bar{v} + \omega(\bar{v} - \bar{v}^2)$$

and with the  $*$ -structure:

$$v^* = \bar{v} \quad m^* = -m.$$

*Proposition 4.2.* The quantum cylinder is a quantum homogeneous space of non-standard  $\mathcal{E}_\omega(2)$  in the sense of [6], i.e. it is a  $*$ -invariant right coideal. Furthermore the elements  $\{v^r m^s : r \in \mathbb{Z}, s \in \mathbb{N}\}$  provide a basis as a vector space for this algebra.

*Proof.* Obviously the algebra  $\mathcal{C}_\omega$  is  $*$ -invariant. To prove that it is a right coideal in  $Fun_q(\mathcal{E}(2))$  is sufficient to perform straightforward calculations on the generators. We have

$$\begin{aligned} \Delta v &= v \otimes v \in \mathcal{E}_\omega(2) \otimes \mathcal{C}_\omega \\ \Delta m &= 1 \otimes m + \bar{v}\bar{n} \otimes \bar{v} - vn \otimes v \in \mathcal{E}_\omega(2) \otimes \mathcal{C}_\omega. \end{aligned}$$

To prove the linear independence of elements  $v^r m^s$  let us first observe that  $\mathcal{C}_\omega$  is a skew polynomial ring [7] and explicitly  $\mathcal{C}_\omega = R[m; \delta]$  where  $R = \mathbb{C}[v, v^{-1}]$  and  $\delta(v) = (v^2 - 1)$ . Thus it is possible to define the degree in  $m$ , that we will denote  $\deg_m$ , for every element in  $\mathcal{C}_\omega$  and prove that if  $n$  and  $u$  are in  $\mathcal{C}_\omega$  then

$$\deg_m(nu) = \deg_m(n) + \deg_m(u).$$

It is then clear that there cannot be linear relations involving monomials with  $\deg_m \neq 0$ . On the other hand linear relations involving only elements of  $m$ -degree 0 are linear relations in  $\mathbb{C}[v, v^{-1}]$  and thus they should be trivial.



Let us observe that  $C_\omega$ , being a skew polynomial ring, is easily shown to verify good algebraic properties through very general arguments. For example it is a Noetherian integral domain.  $\square$

In [6] it is shown how, in a fixed Hopf algebra  $\mathcal{H}_q$ , it is possible to construct a Galoisian reciprocity between  $*$ -invariant subalgebras and right coideals on one side and Hopf  $*$ -ideals on the other side through the assignments:

$$\Sigma : B \mapsto \mathcal{A}_B = \langle (S^n - \varepsilon \mathbf{1})(b), b \in B, n \in \mathbf{Z} \rangle$$

$$\Pi : \mathcal{A} \mapsto B_{\mathcal{A}} = \{b : (\pi \otimes \text{id}) \circ \Delta(b) = 1 \otimes b\}$$

where  $\pi$  is the projection from the whole Hopf algebra onto the quotient with respect to  $\mathcal{A}$ . It is then natural to give the following definition.

*Definition 4.3.* The closure of a quantum homogeneous space  $\mathcal{C}$  in the Hopf algebra  $\mathcal{H}_q$  is the homogeneous space  $\mathcal{B} = \Pi \circ \Sigma(\mathcal{C})$ .

*Proposition 4.4.* The closure of the quantum cylinder  $C_\omega$  is the quantum homogeneous space corresponding to the quantum subgroup of non-standard  $\mathcal{E}_\omega(2)$  given by the Hopf- $*$ -ideal  $\mathcal{I} = \langle v - 1, n - \bar{n} \rangle$ .

*Proof.* The notation in the last line of the proposition means that  $\mathcal{I}$  is the ideal generated by elements enclosed in brackets. We want to verify that  $\mathcal{I}$  is really a Hopf- $*$ -ideal. First we have to verify that this ideal is a bilateral coideal, i.e.

$$\Delta \mathcal{I} = \mathcal{I} \otimes \mathcal{E}_\omega(2) + \mathcal{E}_\omega(2) \otimes \mathcal{I} \quad \varepsilon(\mathcal{I}) = 0.$$

It is sufficient to prove it for the generators. The second condition is trivial. The first follows from

$$\Delta(v - 1) = v \otimes v - 1 \otimes 1 = (v - 1) \otimes 1 + v \otimes (v - 1)$$

$$\Delta(n - \bar{n}) = \bar{v} \otimes n + n \otimes 1 - v \otimes \bar{n} - \bar{n} \otimes 1 = \bar{v} \otimes (n - \bar{n}) + (n - \bar{n}) \otimes 1 + (\bar{v} - v) \otimes \bar{n}.$$

We next have to prove  $S$ -invariance. Again on generators we have

$$S(v - 1) = (\bar{v} - 1) = -\bar{v}(1 - v)$$

$$S(n - \bar{n}) = m = \bar{v}(\bar{n} - n) - (v - 1)(\bar{v} + 1)n$$

so that  $S(\mathcal{I}) \subset \mathcal{I}$ . As for the  $*$ -structure we have

$$(v - 1)^* = \bar{v} - 1 = -\bar{v}(v - 1) \in \mathcal{I} \quad (n - \bar{n})^* = n^* - \bar{n}^* = -(n - \bar{n}) \in \mathcal{I}.$$

Next we have to verify that the closure of  $C_\omega$ , as defined in (4.3), coincides with the set

$$B_{\mathcal{I}} = \{b \in \mathcal{E}_\omega(2) : (\pi \otimes \text{id}) \circ \Delta(b) = 1 \otimes b\}$$

where  $\pi : \mathcal{E}_\omega(2) \rightarrow \mathcal{E}_\omega(2)/\mathcal{I}$  is the natural projection. Direct calculations show that indeed  $v$  and  $m$  belong to this set and thus that  $C_\omega$  is contained in  $B_{\mathcal{I}}$ . For example:

$$(\pi \otimes \text{id})(\Delta m) = 1 \otimes m + \pi(\bar{v}(\bar{n} - n)) \otimes \bar{v} + \pi((\bar{v} - v)n) \otimes v.$$

The stabilizer ideal of  $C_\omega$  is

$$\mathcal{A}_{C_\omega} = \Sigma(C_\omega) = \langle S^n(c) - \varepsilon(c)\mathbf{1}, b \in C_\omega, n \in \mathbf{Z} \rangle.$$

Let us note that  $(S - \varepsilon \mathbf{1})(m) = n - \bar{n}$  and  $(S - \varepsilon \mathbf{1})(\bar{v}) = v - 1$  so that the generators of  $\mathcal{I}$  belong to  $\mathcal{A}_{C_\omega}$  proving  $\mathcal{I} \subset \mathcal{A}_{C_\omega}$ . On the other hand from general arguments  $C_\omega \subset B_{\mathcal{I}}$  implies the opposite inclusion so that

$$\mathcal{A}_{C_\omega} = \mathcal{I}.$$

This proves that the closure of  $C_\omega$  is exactly  $B_{\mathcal{I}}$ .  $\square$

## 5. Conclusions

Analysing the Poisson structures and their Poisson homogeneous spaces for the Euclidean group we have recovered the semiclassical limit of the results in [2] for quantum homogeneous planes, showing that, in this limit, there is a one-parameter family of planes, as is the case for spheres [12, 13]. Furthermore we have determined a quantization of a Poisson structure on the cylinder compatible with a quantum homogeneous structure with respect to a non-standard quantum Euclidean group, showing that it can be recovered from an explicit quantum subgroup. Again we have shown that there is a one-parameter family of covariant Poisson structures on this space. The method of looking at the semiclassical limit situation for hints about the nature of general quantum homogeneous spaces could perhaps be helpful in all those situations in which the function algebra is explicitly given through generators and relations.

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