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NON-LINEAR FINITE *W*-SYMMETRIES AND APPLICATIONS IN ELEMENTARY SYSTEMS

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Non-linear finite W-symmetries and applications in elementary systems

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Contents

Abstract

In this paper it is stressed that there is no *physical* reason for symmetries to be linear and that Lie group theory is therefore too restrictive. We illustrate this with some simple examples. Then the theory of finite W-algebras, which is an important class of non-linear symmetries, is reviewed. In particular, we discuss both the classical and quantum theory and elaborate on several aspects of their representation theory. Some new results are presented. These include finite W coadjoint orbits, real forms and unitary representation of finite W-algebras and Poincaré–Birkhoff–Witt theorems for finite W-algebras. Also we present some new finite W-algebras that are not related to $sl(2)$ embeddings. At the end of the paper we investigate how one could construct physical theories, for example gauge field theories, that are based on non-linear algebras.

1. Introduction

The notion of symmetry is one of the most fundamental concepts in physics. Relativity theory for example is based on symmetry, namely Lorentz invariance. This symmetry is the mathematical expression of the postulate that physical laws are the same for all inertial observers. In particle physics the principles of symmetry provide a powerful overall framework. This allows successful classification and interpretation of an overwhelming amount of experimental data concerning the spectrum of elementary particles. Moreover, the interactions between elementary particles are completely determined through the principle of *local gauge invariance*, a central paradigm of modern particle theory. The highly successful standard model, unifying the electro-weak and strong interactions between elementary particles, is based on the local gauge group $U(1) \times SU(2) \times SU(3)$.

The mathematical theory concerning symmetry transformations in a physical system is group theory. It has been developed independently in different fields of mathematics. In the context of algebraic equations the idea of groups was used already by Lagrange in 1771, though the name 'group' was only introduced in 1830 by Galois. The second area in which it appeared was number theory, with Euler (1761) and Gauss (1801) as the most important contributors. The concept found its place in geometry in the middle of the 19th century, when Klein proposed it as a tool to classify certain new geometrical structures that had been discovered at the time. At the end of the 19th century, it was realized that these three group concepts were the same and this insight led to the formation of modern abstract group theory by Lie (1870). The name of Lie has been associated to continuous linear groups, now called Lie groups. It was Cartan who subsequently almost fully developed this subject, though he was rather isolated for a period of about thirty years.

The simplest symmetries are those in which the physical system is symmetric under a finite number of transformations or when it is invariant under displacement in a finite number of 'directions'. The natural mathematical structures describing such symmetries are discrete groups and finite-dimensional Lie groups respectively.

Group theory has been introduced in physics in the 1920s mostly through the work of Weyl and Wigner. Apart from the obvious description of symmetries in crystals, they realized that group theory is of utmost importance in quantum physics. Weyl writes in [1] on group theory in quantum physics: "It reveals essential features which are not contingent on a special form of the dynamics laws nor on special assumptions concerning the forces involved. We may well expect that it is just this part of quantum physics which is most certain of a lasting place."

Lie groups and their Lie algebras have a wide range of applications in physics. In fact, most symmetry considerations of physical systems have up to recently been based on the application of the theory of Lie groups and algebras. This is true for the Lorentz group in the theory of special relativity, as well as for the groups underlying gauge theories of the fundamental interactions between the basic constituents of matter. It is worth noting that these theories, based on linear symmetries, in fact have highly non-linear dynamics.

In mathematics, the development of the theory of Lie groups was continued with the work of Chevalley (1950), Serre (1966) and Dynkin. In 1967, Kac [2] and Moody independently generalized the theory of finite-dimensional Lie algebras to the infinite-dimensional case. These so-called affine Lie algebras have found remarkable applications in two-dimensional field theory and string theory [3].

For a long time, all efforts to develop the theory of symmetry in physics were restricted to the *linear* case, i.e. Lie groups and Lie algebras. However, it was realized recently that the 'Lie algebra' might be too narrow a concept from the physical point of view. Unfortunately, the extensive machinery developed for the analysis of linear symmetries largely breaks down in the non-linear domain. Nevertheless, in the eighties, led by developments in string theory and solvable models, non-linear symmetries gained importance in physics. In fact, this may be seen as part of the boom in the field of non-linear science in mathematics and physics which started roughly in the middle of the nineteenth century. Non-linear symmetries seem a logical next step in this field which has led to the discovery of theories concerning for example solitary waves (solitons), chaos and turbulence, but also non-linear gauge theories.

One type of non-linear algebra that has received much attention in recent years are the so called quantum groups. They are obtained from ordinary Lie algebras by deforming (quantizing) the co-Poisson structure present on any Lie algebra, or, equivalently, by deforming the space of functions on the group manifold. These algebras have some very interesting applications in conformal field theory but it is not this type that we will be concerned with in this paper.

Surprisingly enough, when in 1985 Zamolodchikov [4] took up the subject of non-linear algebras, he considered *infinite-dimensional W*-algebras. In his work on conformal field theory, he generalized the well-known Virasoro algebra, which is the infinite-dimensional Lie algebra associated with the conformal symmetries in two-dimensional space–time. What he found were *non-linear* infinite-dimensional algebras, that were called W-algebras. Subsequently, W-algebras were studied in the context of string theory, the theory of integrable systems and the theory of two-dimensional critical phenomena. For more details we refer the reader to the reviews [5–7] on W-algebras in conformal field theory.

One important question was and still is the classification of W-algebras, in other words, to make a complete list of all W-algebras. The most profitable approach to date is to apply the Drin-feld–Sokolov construction [8] to derive W-algebras starting from affine Lie algebras. It has been shown in [9] that this construction gives rise to a large class of W-algebras. This relation between W-algebras and affine Lie algebras in principle enables one to construct the W-algebra theory from the theory of affine Lie algebras.

At this point, the question arose whether finite-dimensional analogues of the infinite-dimensional W-algebras exist as well. The relevance of this question is apparent if one considers that the theory of the infinite-dimensional so-called loop groups and algebras can be derived from the finite-dimensional Lie groups and algebras underlying them. Loop groups are special cases of infinite-dimensional groups of smooth maps from some space–time manifold X to a finite-dimensional Lie group G, namely for $X = S^1$. The study of infinite-dimensional groups of smooth maps is a natural consequence of the combination of symmetry principles with locality or causality.

The group multiplication in such a group is just pointwise multiplication, i.e. if $f, g \in Map(X, G)$ and $a \in G$ then $(f \cdot g)(a) = f(a)g(a)$. In quantum field theory, groups of the form $Map(X, G)$ and their Lie algebras $Map(X, g)$ (where $\mathscr{G}$ is the Lie algebra of G) arise essentially in two different ways: through the principle of local gauge invariance, which is at the heart of modern high energy physics, and through the theory of current groups and algebras.

Unfortunately, for generic manifolds X, surprisingly little is known about the group $Map(X, G)$. Especially, the representation theory of these groups is still almost unexplored. This exception to this is the case $X = S^1$ we mentioned above, where $Map(S^1, G)$ and $Map(S^1, g)$ are called 'loop

groups' and 'loop algebras'. Loop groups and algebras arise in simplified models of quantum field theory in which space is taken to be one-dimensional and therefore also in string models of elementary particles. The study of loop groups and algebras is much simpler than when X is some, more complicated manifold. This is caused by the fact that they behave much like the ordinary finite-dimensional Lie groups and algebras that underlie them. This remarkable fact makes knowledge of the finite-dimensional theory essential for the study of the infinite-dimensional theory.

For the infinite-dimensional W-algebras it was not entirely clear what the finite algebras underlying W-algebras were and whether there was a finite version of W-theory at all. Considering the way in which W-algebras were first introduced into physics, they really do not seem to have any relation to the theory of loop groups and algebras. Nevertheless, the theory of W-algebras does have a finite counterpart as has been shown in [10, 11]. In fact, the finite theory is remarkably rich, and as with loop algebras, contains already several of the essential features of infinite-dimensional W-algebra theory.

The main objective of this paper is twofold. On the one hand, we wish to convey to the reader our view that non-linear symmetries are not only interesting in relation to theories that have a mathematical sophistication comparable to that of string theory, but that they play an important role throughout physics. In fact, finitely generated non-linear symmetries already show up in very elementary and famous physical systems.

Our second objective is to give a readable account of the classical and quantum theory of finite W-algebras and to present some new results.

The outline of the paper is as follows. After some general remarks on dynamical systems and symmetries, we illustrate by means of known examples, that *many basic physical systems have non-linear symmetry algebras*. In particular, the symmetry algebras of the two-dimensional aniso-tropic harmonic oscillator with frequency ratio $2:1$ and the Kepler problem are shown to be finite W-algebras.

Finite W-algebras can be constructed from finite Lie algebras by a procedure resembling the constraint formalism. Precisely how this works will be the subject of chapter 3. In this chapter, we construct and develop the classical theory of finite W-algebras. Also we show that many finite W-algebras contain known Lie algebras as subalgebras. The question, therefore, naturally arises whether there exist non-linear extensions of $su(3) \times su(2) \times u(1)$. The answer to this question turns out to be affirmative.

Having developed the classical theory, we turn to the quantum case and show how to quantize finite W-algebras using the BRST formalism. In some specific cases we explicitly construct the BRST cohomologies. Next, we consider the representation theory of quantum finite W-algebras. In order to define unitary highest weight representations for finite W-algebras it is necessary to consider real forms and Poincaré–Birkhoff–Witt theorems for these algebras. Having developed this part of the theory we describe the results on Kac-determinants and character formulas that were recently conjectured in [12]. Finally, we present some ideas about possible constructions of theories with finite W-symmetries. In particular, we give some topological theories having finite non-linear W-symmetries including the one recently constructed in [13], and we show how finite W-algebras are realized in one-dimensional generalized Toda theories. We conclude with yet another simple quantum mechanical example, which has a finite W-algebra as a spectrum generating algebra.

Most of the material in this review is based on existing results. New are the results described in Sections 4.3, 5.1, the Poincaré–Birkhoff–Witt theorem in Section 5.2, part of Section 6.1 and Sections 6.2.2 and 6.2.3.

2. Symmetries in simple physical systems

In this chapter we first briefly review some basic facts concerning algebras of conserved quantities in elementary mechanical systems. It is explicitly stressed that there is no reason, neither physical nor mathematical, why these algebras have to be linear. In fact, they are non-linear in general. We then illustrate this explicitly in some examples, namely, the harmonic oscillator and the Kepler problem.

2.1. Symmetry algebras

During the motion of a mechanical system, the generalized coordinates q^i and the velocities $\dot{q}^i$ vary in time. Nevertheless, there exist functions of these quantities whose values do not change but depend only on the initial conditions. Such functions are called 'integrals of motion'. Integrals of motion which do not depend explicitly on time are called conserved quantities. From now on we will restrict ourselves to conserved quantities.

Noether's theorem states that conserved quantities are related to symmetries. Symmetry transformations leave the action invariant, i.e. $\delta S = 0$. Therefore, the variation of the Lagrangian can only be equal to a total derivative $\delta L = (d/dt)\Lambda(q,t)$. Let us consider the symmetry transformation $q' = q + \delta q$. Then we find that

$$\delta L = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{d}{dt} \Lambda(q,t) \,, \tag{2.1}$$

if the Lagrangian only depends implicitly on time. Partial integration yields

$$\frac{d}{dt} \left(\Lambda(q,t) - \frac{\partial L}{\partial \dot{q}} \delta q \right) \delta L = \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \,. \tag{2.2}$$

The right-hand side is equal to zero because of the Euler–Lagrange equations. Therefore, if q satisfies the equations of motion, then

$$\frac{d}{dt} \left(\Lambda(q,t) - \frac{\partial L}{\partial \dot{q}} \delta q \right) = 0 \,. \tag{2.3}$$

The expression between the brackets is a conserved quantity. This is Noether's theorem.

In the above we considered a general transformation $q'(q)$. More particularly, this transformation can consist of a set of independent transformations in different 'directions', labeled by a parameter ε_a:

$$\delta q^i = \varepsilon_a \delta_a q^i \,. \tag{2.4}$$

The conserved quantities Q_a, also called 'Noether charges', that can be associated to all these symmetry transformations are defined as

$$Q_a = \Lambda_a(q,t) - \frac{\partial L}{\partial \dot{q}} \delta_a q \,, \tag{2.5}$$

such that

$$\frac{dQ_a}{dt} = 0 \,. \tag{2.6}$$

Now, classically, the time derivative of a quantity Q is given by

$$\frac{dQ}{dt} = \{H, Q\} \,, \tag{2.7}$$

where the Poisson bracket of the functions f and g is defined by

$$\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \,. \tag{2.8}$$

Conserved quantities are therefore characterized by the fact that they Poisson-commute with the Hamiltonian:

$$\{Q, H\} = 0 \,. \tag{2.9}$$

Let $\{Q_a\}$ be a set of independent[1] conserved quantities, i.e. $\{Q_a, H\} = 0$. It is clear that any polynomial $P(\{Q_a\})$ in the conserved quantities is conserved as well. Also it follows from the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \,, \tag{2.10}$$

that

$$\{H, \{Q_a, Q_b\}\} = 0 \,. \tag{2.11}$$

This means that the Poisson bracket of two conserved quantities is conserved. We call the set $\{Q_a\}$ of conserved quantities *closed* if

$$\{Q_a, Q_b\} = P_{ab} \,, \tag{2.12}$$

where P_{ab} is some function of the quantities $\{Q_a\}$. If the Poisson bracket contains some conserved quantities which are *not* present in the set $\{Q_a\}$, i.e. if this set is not closed, we can add these quantities to the set and thus *make* it closed.

The Poisson algebra (2.12) of conserved quantities will from now on be called the *classical symmetry algebra*. The quantities $\{Q_a\}$ are called the *generators* of the algebra. If an algebra has a finite number of generators, it is said to be 'finitely generated'. The transformations associated

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[1] Two conserved quantities $Q_1(q,p)$ and $Q_2(q,p)$ are called independent if the vectors $(\partial Q_1/\partial q, \partial Q_1/\partial p)$ and $(\partial Q_2/\partial q, \partial Q_2/\partial q)$ are linearly independent.

with the Noether charge Q_a of the coordinates q^i and momenta p_i are of course given by

$$\delta_a q^i = \{q^i, Q_a\}, \quad \delta_a p_i = \{p_i, Q_a\}.$$
(2.13)

In quantum mechanics, the equivalent of Eq. (2.7) is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H, Q],$$
(2.14)

where Q and H are now operators. As in the classical case, we define the symmetry algebra as the set of independent operators $\{Q_a\}$ which commute with the Hamiltonian and have the property that the commutation relations of Q_a and Q_b can again be expressed in terms of the operators $\{Q_a\}$.

As $[Q_a, H] = 0$ for a conserved quantity, the operator Q_a will transform eigenstates of H into (possibly different) eigenstates with equal energy. The Hilbert space of the system therefore decomposes into a direct sum of irreducible representations of the symmetry algebra generated by $\{Q_a\}$.

In the next section, we shall introduce *linear* and *non-linear* symmetries using some very simple and well-known physical systems.

2.2. Harmonic oscillators

Harmonic oscillators constitute a category of very basic systems in physics. They show up in virtually all problems with a finite or infinite number of degrees of freedom. The reason is that, upon linearizing a generic dynamical problem harmonic oscillators approximate any arbitrary potential in the neighbourhood of a stable equilibrium position, describing for example small vibrations of an atom in a crystalline lattice or a nucleon in a nucleus. On the other hand, the behavior of most continuous physical systems, such as the vibrations of an elastic medium or the electromagnetic field in a cavity, can be described as a superposition of an infinite number of harmonic oscillators. In this section, the idea of both linear and non-linear symmetries will be illustrated using these elementary systems.

2.2.1. The isotropic case

Let us consider a particle of mass M moving in a quadratic potential in the (x_1, x_2)-plane. The Hamiltonian of this system is given by

$$H = \frac{1}{2M}(p_1^2 + p_2^2) + \frac{1}{2}M\omega^2(x_1^2 + x_2^2),$$
(2.15)

where ω is the angular frequency of the oscillator. In the usual quantum description of this system one introduces the so called 'raising' and 'lowering' operators

$$a^\dagger = \sqrt{\frac{M\omega}{2\hbar}}\, x_1 - \mathrm{i}\sqrt{\frac{1}{2M\omega\hbar}}\, p_1, \quad a = \sqrt{\frac{M\omega}{2\hbar}}\, x_1 + \mathrm{i}\sqrt{\frac{1}{2M\omega\hbar}}\, p_1,$$

$$b^\dagger = \sqrt{\frac{M\omega}{2\hbar}}\, x_2 - \mathrm{i}\sqrt{\frac{1}{2M\omega\hbar}}\, p_2, \quad b = \sqrt{\frac{M\omega}{2\hbar}}\, x_2 + \mathrm{i}\sqrt{\frac{1}{2M\omega\hbar}}\, p_2,$$
(2.16)

where $\hbar$ is Planck's constant. From the canonical commutation relations between coordinates and momenta $[x_i, p_j] = i\hbar\delta_{ij}$, one can easily derive

$$[a, a^\dagger] = [b, b^\dagger] = 1 , \tag{2.17}$$

and all other commutators are zero.

In terms of raising and lowering operators, the Hamiltonian reads $H = (a^\dagger a + b^\dagger b + 1)\hbar\omega$. The Hilbert space of the system is spanned by the states

$$|p, q\rangle = (a^\dagger)^p (b^\dagger)^q |\Omega\rangle , \tag{2.18}$$

where p, q are non-negative integers and $|\Omega\rangle$ is the 'ground state' (which has the property that $a|\Omega\rangle = b|\Omega\rangle = 0$). The energy of the eigenstate $|p, q\rangle$ is $E_{p,q} = (p + q + 1)\hbar\omega$. From this we see that the states $|p - r, r\rangle$, where $r = 0, 1, \ldots, p$ all have the same energy, i.e. the energy eigenvalue $E_{p,q}$ has a $(p + q + 1)$-fold degeneracy. This leads one to conjecture the existence of a symmetry supplying extra quantum numbers and transforming eigenstates with the same energy into each other. We shall now discuss this symmetry. For notational convenience we take from now on $M = \omega = \hbar = 1$.

Consider the operators

$$S_+ = ab^\dagger ; \quad S_- = a^\dagger b ; \quad S_0 = b^\dagger b - a^\dagger a . \tag{2.19}$$

These quantities are conserved as is expressed by the equation $i\partial_t S_i = [S_i, H] = 0$ for $i = \pm, 0$. The commutation relations between the operators S_i themselves can be easily calculated using (2.17) and read

$$[S_0, S_\pm] = \pm 2S_\pm ; \quad [S_+, S_-] = S_0 . \tag{2.20}$$

Notice that the commutation relations of the operators S_i are again expressions in terms of S_i, so the algebra is closed. Furthermore, these expressions are *linear*, such that the algebra is linear. In fact, the commutator algebra (2.20) is nothing but $su(2)$, the simplest example of a non-abelian simple Lie algebra.

The theory of Lie algebras is well known and extensively described in the literature. Lie algebras can be related to symmetry groups, called *Lie groups*, by an exponential map. All elements g in the component connected to the unit element of a Lie group can be written as $g = \exp(\alpha^a t_a)$, where t_a are the generators of the Lie algebras, α^a are numbers and summation over the index a is understood.

The action of the generators S_+ and S_- of the $su(2)$ symmetry can be interpreted as follows. The state of the (quantum mechanical) particle in the plane is composed of oscillations in two directions. The operator S_+ decreases the oscillation in the x_2-direction and increases oscillation in the other one. It can continue this action, until the state of the particle consists only of oscillation in the x_1-direction. The operator S_- in its turn, squeezes the orbit of the particle towards oscillation in the x_2-direction.

What we can conclude from this section, is that the isotropic oscillator in two dimensions has $su(2)$ symmetry, which is larger than the obvious $so(2)$ rotation symmetry in the $x-y$ plane. Similarly, it can be shown that the symmetry of an n-dimensional isotropic harmonic oscillator is the Lie algebra $su(n)$ (or $u(n)$, if one also views the Hamiltonian itself as part of the symmetry algebra).

2.2.2. The anisotropic case

We will now consider a slightly more complicated case: the two-dimensional *anisotropic* harmonic oscillator. The Hamiltonian of the anisotropic oscillator is given by

$$H = \frac{1}{2M}(p_1^2 + p_2^2) + \tfrac{1}{2}M\omega_1^2 x_1^2 + \tfrac{1}{2}M\omega_2^2 x_2^2 . \tag{2.21}$$

Again we will take $M = \hbar = 1$. In order to have degeneracy in the energy levels of E, we take $\omega_1 = 1/m$ and $\omega_2 = 1/n$, where m and n are positive integers. In terms of the raising and lowering operators, which satisfy the commutation relations (2.17), the Hamiltonian reads

$$H = \frac{1}{m}\left(a^\dagger a + \frac{1}{2}\right) + \frac{1}{n}\left(b^\dagger b + \frac{1}{2}\right) . \tag{2.22}$$

If we consider the analogues of the operators (2.19) in this system, we find that the generators $S_\pm$ no longer commute with the Hamiltonian H. In other words, $S_\pm$ are not conserved. Therefore the algebra $su(2)$ is *not* a symmetry algebra of the anisotropic oscillator.

In order to arrive at the true symmetry algebra of the anisotropic oscillator, we consider, like [14,15], the following operators that *do* commute with H:

$$\tilde{j}_+ = a^m(b^\dagger)^n , \quad \tilde{j}_- = (a^\dagger)^m b^n , \quad \tilde{j}_0 = \frac{1}{n}\left(b^\dagger b + \frac{1}{2}\right) - \frac{1}{m}\left(a^\dagger a + \frac{1}{2}\right) . \tag{2.23}$$

These are the generators of the symmetry algebra of the anisotropic quantum harmonic oscillator with frequencies m and n positive integers, as was shown in [15].

For simplicity, let us consider the case that the frequencies have the fixed value $m = 2$ and $n = 1$. Calculation of the commutation relations of the generators (2.23) for these values of m and n produces:

$$[\tilde{j}_i, H] = 0 , \quad [\tilde{j}_0, \tilde{j}_\pm] = \pm 2\tilde{j}_\pm , \quad [\tilde{j}_+, \tilde{j}_-] = -3\tilde{j}_0^2 + 2H\tilde{j}_0 + H^2 - \tfrac{3}{4} , \tag{2.24}$$

with $i = 0, \pm$. After an invertible basis transformation, given by

$$j_+ = \frac{1}{\sqrt{3}}\tilde{j}_+ ; \quad j_- = \frac{-1}{\sqrt{3}}\tilde{j}_- ; \quad j_0 = \tilde{j}_0 - \tfrac{1}{3}H , \tag{2.25}$$

we obtain

$$[H, j_i] = 0 , \quad [j_0, j_\pm] = \pm 2j_\pm , \quad [j_+, j_-] = j_0^2 + C , \tag{2.26}$$

where $C = \tfrac{1}{4} - \tfrac{4}{9}H^2$. This commutator algebra is known in the literature as $W_3^{(2)}$ [10]. We conclude that the symmetry algebra of a two-dimensional anisotropic harmonic oscillator with a frequency ratio $m:n = 2:1$ is the non-linear finite W-algebra $W_3^{(2)}$. We will return to this algebra later in the paper. More generally, if we take m and n arbitrary positive integers we find that the symmetry algebra is non-linear whenever $m \neq n$ [15].

$W_3^{(2)}$ is an example of a *non-linear algebra*. The commutators cannot be written as linear combinations of the generators, but as linear combinations of products of generators. As it has a finite set of generators, it is finitely-generated.

In contrast to Lie algebras, one cannot associate a finite group to a non-linear algebra. The only group G which can be obtained with an exponential map from a non-linear algebra is generated not only by the generators of this algebra, but also by their polynomials and therefore is infinitely generated. Only then is the requirement met, that $\forall g_i \in G g_1 g_2 = g_3$, because of the Campbell–Baker–Hausdorff formula for the multiplication of exponential maps. While the elements of an algebra give rise to infinitesimal transformations, group elements correspond to finite symmetry transformations. Consequently, we cannot easily tell in the non-linear case how the transformation works globally, though we know the infinitesimal transformations on the local level. From the latter, one can, in principle, derive a differential equation whose solution for finite time corresponds to transformations of the 'finite W-group', but these differential equations are typically non-linear and very hard to solve explicitly.

Though the problem we considered is an elementary and a linear one, meaning that the equations of motion are linear, its symmetry algebra turns out to be non-linear. Surprisingly enough the study of these non-linear symmetry algebras has been taken up only recently. Most attention has focused on non-linear extensions of $su(2)$, see for example [16–24].

2.3. Coulomb potential

A large class of well-known dynamical systems have a Coulomb potential. Planetary motion and the motion of a charged particle in a Coulomb field are of this type. In this section, we will consider the symmetry algebra of this class of systems.

The Hamiltonian is given by

$$H = \frac{p^2}{2m} - \frac{\mu}{r}, \tag{2.27}$$

where μ is a fixed constant. Due to the spherical symmetry of the potential, this system is invariant under rotations in three dimensions. This invariance leads to the conservation of angular momentum L. We start on the classical level by considering the Poisson brackets of the dynamical variables, which are functions on the phase space with coordinates (r, p). The algebra is given by $\{L_i, L_j\} = \varepsilon_{ijk} L_k$ and is called $so(3)$ (or $su(2)$). It is a simple Lie algebra.

However, the angular momentum L is not the only conserved quantity in this system. Let us consider the vector R defined by

$$R = L \times p + \mu \frac{r}{r}, \tag{2.28}$$

with μ again the fixed constant, proportional to the central potential. Straightforward calculation shows that this, so-called 'Runge–Lenz vector,' also (Poisson) commutes with the Hamiltonian: $\{L_i, H\} = \{R_i, H\} = 0$. This additional symmetry differs from the $so(3)$ symmetry, which is a geometric symmetry, i.e. which can be expressed as mappings of configuration space alone. The symmetry transformations associated to the Runge–Lenz vector do not transform the coordinates and momenta separately. They act on the entire phase space. The term 'dynamical symmetries' is sometimes used for this type of symmetries.

In the example of planetary motion, the Runge–Lenz vector points along the major axis of the ellipsoid orbit of the planet and its magnitude is proportional to the eccentricity of the orbit.

In an analogous quantum case, i.e. the hydrogen atom, which is discussed in detail in [25], the fact that the angular momentum is not the only conserved quantity is reflected by the degeneracy in the spectrum. While the states of the Hamiltonian depend on three quantum numbers, n, l and m, the energy depends only on n: $E_n \sim 1/n^2$. The energy is independent of m because of the $so(3)$ symmetry, which corresponds to the conservation of angular momentum. The degeneracy with respect to this magnetic quantum number is present for any central potential. The absence of dependence on l suggests that there is another conserved quantity, which turns out to be the Runge–Lenz vector. This degeneracy occurs only if the potential is of the form $1/r$ and is thereby particular to the Coulomb potential.

Let us return to the classical symmetry algebra of L_i and R_i. In the first place, the algebra is only closed if we include H in the set of generators. Secondly, we see that it is actually non-linear:

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k \,, \quad \{R_i, R_j\} = -2\varepsilon_{ijk} H L_k \,,$$
$$\{R_i, L_j\} = \varepsilon_{ijk} R_k \,, \quad \{L_i, H\} = \{R_i, H\} = 0 \,. \tag{2.29}$$

Usually one linearizes (by a non-linear basis transformation) this algebra giving rise to the well-known hidden $so(4)$ symmetry in the hydrogen atom [25]. The algebra above describes the Kepler orbit in a three-dimensional Euclidean space. Now consider the same Kepler problem on a three-sphere S^3. In [26, 27] it is shown that the Poisson brackets between the components of the R vector become

$$\{R_i, R_j\} = \varepsilon_{ijk}(-2H + \lambda L^2) L_k \,, \tag{2.30}$$

where λ is the curvature of the sphere, which is equal to the inverse radius R of the sphere $\lambda = 1/R$. We will call this algebra the *Runge–Lenz algebra*. Later it will be shown that it is a finite W-algebra (which was first remarked in [28]).

In the previous sections we have shown that even simple and well-known physical systems may have non-linear W-symmetries. In Section 6.2 we will see yet another example, namely Toda systems. It seems therefore to be justified to embark on a more systematic study of these algebras and their representation theory. This will lead us to the theory of finite W-algebras.

3. Classical finite W-algebras

As we have seen in the first chapter, algebras of conserved quantities are always closed, but not necessarily linear. This is not merely an abstract mathematical possibility; we have seen that some of the simplest and most fundamental systems in physics exhibit non-linear symmetries. When trying to analyze these symmetry algebras however, or perhaps to construct their irreducible representations, one inevitably runs into trouble due to the fact that they are non-linear. New methods are therefore needed.

In this chapter, it will be shown that many non-linear algebras, including the ones we encountered in the previous chapter, can in fact be seen as 'reductions' of Lie algebras. This result clearly opens up a new possibility of analyzing them since the theory of Lie algebras is well developed. In

the present chapter, we restrict ourselves to the classical case and leave quantization to the next chapter.

The method we use in this chapter is to construct non-linear Poisson algebras from a canonical linear Poisson algebra associated to any Lie algebra, the Kirillov Poisson algebra. First we give a description in simple terms of the Kirillov Poisson algebra. Then we use a procedure very similar to the constraint formalism called 'Poisson reduction', to construct linear and non-linear algebras from this Poisson algebra. After that we discuss in detail a very interesting class of algebras which are derived using $sl(2)$ embeddings. These algebras are in general non-linear but they may contain linear subalgebras which one can predict rather easily. Using this we discuss how to obtain non-linear extensions of $SU(3) \times SU(2) \times U(1)$.

3.1. Kirillov Poisson structures

Consider a system with $SO(3)$ rotation invariance. The conserved quantities associated to this symmetry are the three components of the angular momentum L. If one calculates the Poisson brackets between two components of L, one finds that they satisfy

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k \ . \tag{3.1}$$

Now, let $\{I_a\}_{a=1}^3$ be the generators of $SO(3)$, i.e. I_a is the infinitesimal generator of rotations around the x^a-axis. The commutator between I_a and I_b is given by

$$[I_a, I_b] = \varepsilon_{abc} I_c \ . \tag{3.2}$$

Note that even though (3.1) is a Poisson relation and (3.2) is a commutator algebra, the *structure constants* in (3.1) and (3.2) are the same. That is, the components of the angular momentum together generate a Poisson algebra which is isomorphic to the Lie algebra of $SO(3)$.

This example illustrates a general principle: if G is a symmetry group of some physical system, then the Poisson brackets between the (Noether) conserved quantities $\{J_a\}$ have the same structure constants as the commutator brackets between the generators $\{t_a\}$ of G, i.e. if $[t_a, t_b] = f_{ab}^c t_c$, then

$$\{J_a, J_b\} = f_{ab}^c J_c \ , \tag{3.3}$$

which is called the Kirillov–Poisson algebra. The generators $\{t_a\}$ span the so-called 'Lie algebra' $\mathcal{G}$ of G. We conclude that *any* system with G symmetry contains the Kirillov Poisson structure as a subalgebra.

For later use, we will now give a somewhat more formal definition of the Kirillov Poisson structure associated to $\mathcal{G}$. The reader may wish to skip this part at first reading.

Let $\mathcal{G}$ be a Lie algebra, $\mathcal{G}^*$ its dual and $C^\infty(\mathcal{G}^*)$ the set of smooth functions on $\mathcal{G}^*$. The Kirillov–Poisson bracket between $F, G \in C^\infty(\mathcal{G}^*)$ is defined for all $\xi \in \mathcal{G}^*$ by

$$\{F, G\}(\xi) = \langle \xi, [\mathrm{grad}_\xi F, \mathrm{grad}_\xi G] \rangle \ , \tag{3.4}$$

where $\langle ., . \rangle$ denotes the usual contraction between $\mathcal{G}^*$ and $\mathcal{G}$ and $\mathrm{grad}_\xi F$ is uniquely defined by

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\xi + \varepsilon\xi')|_{\varepsilon=0} = \langle \xi', \mathrm{grad}_\xi F \rangle \ , \tag{3.5}$$

for all $\xi' \in \mathcal{G}^*$. Note that $\mathrm{grad}_\xi F$ is therefore an element of $\mathcal{G}$, which means that $[\mathrm{grad}_\xi F, \mathrm{grad}_\xi G]$ is well defined.

We can recover formula (3.1) as follows. Let $\{t_a\}$ be the basis of $\mathcal{G}$ and J_a the element of $C^\infty(\mathcal{G}^*)$ given by $J_a(\xi) \equiv \langle \xi, t_a \rangle$, then these functions satisfy (3.1).

3.2. Poisson reduction of the Kirillov Poisson structure

Finite W-algebras are constructed by applying a procedure very similar to the Dirac constraint formalism, called 'Poisson reduction', to the Kirillov Poisson structure. Essentially, what one does is to impose a set of first class constraints on the system. As usual, these first class constraints will generate gauge invariances. One, therefore, looks for gauge invariant quantities. In general, the set of gauge invariant quantities will be generated by a certain finite subset. These are the generators of the finite W-algebra. Calculating the Poisson brackets between these generators we find that the algebra of gauge invariant quantities is in general non-linear, i.e. the Poisson brackets close on polynomials of the generators, not on linear combinations. This is then the finite W-algebra. Let us now come to a more precise definition of finite W-algebras.

Let again $\mathcal{G}$ be some Lie algebra, $\mathcal{L} \subset \mathcal{G}$ some subalgebra and $\chi: \mathcal{L} \to \mathbb{C}$ a one-dimensional representation[2] of $\mathcal{L}$. Let $\{t_a\}$ and $\{t_\alpha\}$ be bases of $\mathcal{G}$ and $\mathcal{L}$ respectively, such that $\{t_\alpha\} \subset \{t_a\}$, and let $J_a \in C^\infty(\mathcal{G}^*)$ be defined by $J_a(\xi) = \langle \xi, t_a \rangle, \forall \xi \in \mathcal{G}^*$. Again let $K(\mathcal{G}) = (C^\infty(\mathcal{G}^*), \{.,.\})$ denote the Kirillov Poisson algebra associated to $\mathcal{G}$.

The first step is to constrain the functions J_α, corresponding to the subalgebra $\mathcal{L}$, to constant values:

$$\phi_\alpha \equiv J_\alpha - \chi(t_\alpha) = 0 \ . \tag{3.6}$$

Denote the hyper-surface in $\mathcal{G}^*$ determined by $\phi_\alpha = 0$ for all α by C. The set of functions on C is equal to the set of functions on $\mathcal{G}^*$ up to functions which are zero on C. Any function that is zero on all of C has the form $f^\alpha \phi_\alpha$. Let us denote the set of all these 'zero functions' by I:

$$I = \{f^\alpha \phi_\alpha \mid f^\alpha \in C^\infty(\mathcal{G})\} \ . \tag{3.7}$$

Since all elements of I are zero on all of C, there is no way somebody living on C can distinguish between the functions g and $g + f$ if f is an element of I. Mathematically, this is expressed by the equality

$$C^\infty(C) = C^\infty(\mathcal{G})/I \ . \tag{3.8}$$

This formula means that we are to identify all functions in $C^\infty(\mathcal{G}^*)$ that differ by an element of I.

It is easy to show that the constraints (3.6) are all 'first class'. It means that $\{\phi_\alpha, \phi_\beta\} \in I$, for all α, β. For this, remember that χ is a one-dimensional representation of $\mathcal{L}$, which means that $\chi([t_\alpha, t_\beta]) = \chi(t_\alpha)\chi(t_\beta) - \chi(t_\beta)\chi(t_\alpha) = 0$. On the other hand, $\chi([t_\alpha, t_\beta]) = f_{\alpha\beta}^\gamma \chi(t_\gamma)$. From this follows that $\{\phi_\alpha, \phi_\beta\} = f_{\alpha\beta}^\gamma \phi_\gamma$.

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[2] One can in principle also consider higher dimensional representations of $\mathcal{L}$ in some auxiliary algebra, see Section 4.3.4. Here, we will, for simplicity, restrict ourselves to the one-dimensional case.

Obviously, I is an ideal with respect to the (abelian) multiplication map in $K(\mathcal{G})$, since $hf^{\alpha}\phi_{\alpha} = \tilde{f}_{\alpha}\phi_{\alpha} \in I$ for all $h \in C^{\infty}(\mathcal{G})$, where $\tilde{f}_{\alpha} = hf^{\alpha}$.

I is also a Poisson subalgebra of $K(\mathcal{G})$. In order to see this, let $f = f^{\alpha}\phi_{\alpha}$ and $h = h_{\alpha}\phi_{\alpha}$ be elements of I. Then

$$\{f, g\} = \{f^{\alpha}, h^{\beta}\}\phi_{\alpha}\phi_{\beta} + \{f^{\alpha}, \phi_{\beta}\}h^{\beta}\phi_{\alpha} + \{\phi_{\alpha}, h^{\beta}\}f^{\alpha}\phi_{\beta} + f^{\alpha}h^{\beta}\{\phi_{\alpha}, \phi_{\beta}\} \ . \tag{3.9}$$

Obviously, the first three terms are again elements of I. That the last term is an element of I follows from the fact that the constraints $\{\phi_{\alpha}\}$ are all first class.

Nevertheless, I is not an ideal with respect to the Poisson bracket. Consequently, the Poisson bracket is *not* preserved if we divide out I. That is, the Poisson bracket on $\mathcal{G}$ does *not* induce one on C. Physically, this is equivalent to the statement that first class constraints induce non-physical gauge invariances, that have to be eliminated from the theory. Mathematically, one proceeds as follows. Define the maps

$$X_{\alpha}: C^{\infty}(\mathcal{G}^{*}) \to C^{\infty}(\mathcal{G}^{*}) \tag{3.10}$$

by

$$X_{\alpha}(f) = \{\phi_{\alpha}, f\} \ . \tag{3.11}$$

In geometric terms the X_{α} are the 'Hamiltonian vector fields' associated to the constraints $\{\phi_{\alpha}\}$, which can be interpreted as the derivative of some function $f \in C^{\infty}(\mathcal{G}^{*})$ along the direction of the gauge invariance.

Now let $[f]$ denote the equivalence class $f + I$, i.e. $[f] \in C^{\infty}(C)$. Then

$$X_{\alpha}(f + I) = \{\phi_{\alpha}, f\} + \{\phi_{\alpha}, I\} \subset X_{\alpha}(f) + I \ , \tag{3.12}$$

where we used the fact that I is a Poisson subalgebra of $K(\mathcal{G})$. Put differently, we therefore have

$$X_{\alpha}[f] = [X_{\alpha}(f)] \ , \tag{3.13}$$

which means that the maps X_{α} descend to well defined maps

$$X_{\alpha}: C^{\infty}(C) \to C^{\infty}(C) \ . \tag{3.14}$$

In geometric terms this is nothing but the statement that the Hamiltonian vector fields of the constraints are tangent to C.

Now define the set of functions which are constant under the flow of X_{α}, i.e. gauge invariant, as

$$\mathcal{W} = \{[f] \in C^{\infty}(C) \mid X_{\alpha}[f] = 0, \text{ for all } \alpha\} \ . \tag{3.15}$$

The point now is that the Kirillov–Poisson structure on $C^{\infty}(\mathcal{G}^{*})$ induces naturally a Poisson structure $\{.,.\}^{*}$ on $\mathcal{W}$. Let $[f], [h] \in \mathcal{W}$, then this Poisson structure is simply given by

$$\{[f], [h]\}^{*} = [\{f, h\}] \ . \tag{3.16}$$

Of course, one has to show that this Poisson structure is well defined. In order to do so we have to check two things: firstly, we have to check whether $[\{f, h\}] \in \mathcal{W}$ whenever $[f], [h] \in \mathcal{W}$, and

secondly, we have to show that the definition does not depend on the choice of the representatives f and h. Both these checks are easily carried out and one finds that indeed the Poisson bracket $\{.,.\}^*$ turns $\mathcal{W}$ into a Poisson algebra. The Poisson algebra found by reducing a Kirillov–Poisson algebra is called a *finite W-algebra*

$$W(\mathcal{G}, \mathcal{L}, \chi) \equiv (\mathcal{W}, \{.,.\}^*) . \tag{3.17}$$

In more physical terms, $\mathcal{W}$ is the set of gauge invariant quantities and the Poisson bracket between them is obtained by first calculating the Poisson bracket and then putting the constraints to zero.

It is clear that the number of finite W-algebras is huge. However, it is by no means clear that all non-linear algebras are of this type. In fact, this is almost certainly not the case. However, as we shall see, several interesting non-linear symmetry algebras encountered in physics, including the ones described in the previous chapter, *are* finite W-algebras. Anyway, in this paper we restrict our attention to finite W-algebras.

3.2.1. Examples

Let us now consider some examples in order to clarify the construction. First take $\mathcal{G} = sl(2)$, the set of traceless 2×2 matrices. This Lie algebra can be described as follows. It is the (complex or real) span of three generators t_+, t_- and t_0 with commutation relations $[t_0, t_\pm] = \pm 2t_\pm$ and $[t_+, t_-] = t_0$. The Kirillov–Poisson structure, therefore, reads

$$\{J_0, J_\pm\} = \pm 2J_\pm ; \quad \{J_+, J_-\} = J_0 . \tag{3.18}$$

The Lie algebra $sl(2)$ has several subalgebras. The most obvious one is the so-called Cartan subalgebra spanned by t_0. If we take $\mathcal{L}$ to be the Cartan subalgebra and $\chi = 0$, then from Eq. (3.6) we find $\phi = J_0$. The ideal I (see (3.7)) consists of elements of the form $f(J_0, J_+, J_-)\phi$, where f is an arbitrary smooth function in three variables. Since dividing out I corresponds to putting the constraints to zero, we find that $C^\infty(C)$ is isomorphic to the set of arbitrary smooth functions in J_+ and J_-

$$C^\infty(C) = \{f(J_+, J_-) \mid f \text{ is smooth}\} . \tag{3.19}$$

The next step in the construction is to find the set $\mathcal{W}$ of all elements in $C^\infty(C)$ that Poisson commute with ϕ (after imposing the constraint). As $\{\phi, J_+ J_-\} = J_+\{J_0, J_-\} + J_-\{J_0, J_+\} = -J_- J_+ + J_- J_+ = 0$, we find that $\mathcal{W}$ consists of functions that depend only on the combination $J_+ J_-$. However, as $\mathcal{W}$ has only one generator, $J_+ J_-$, it turns out to be an abelian algebra.

In fact this is what generically happens when we choose $\mathcal{G} = sl(2)$. If $\mathcal{L}$ is one of the so-called 'Borel subalgebras' $b_\pm$ spanned by t_0 and $t_\pm$, then there are effectively no degrees of freedom left, that is $\mathcal{W} = 0$. If one chooses $\mathcal{L}$ to be the span of t_+ or t_- (and $\chi = 0$), then $\mathcal{W}$ will again be an abelian algebra with one generator, J_0. The situation does not change for $\chi \neq 0$, because the number of generators stay the same. We conclude that there are no interesting finite W-algebras that can be derived from $sl(2)$.

Let us therefore turn to $\mathcal{G} = sl(3)$. This Lie algebra is spanned by eight elements $\{t_1, t_2, t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_3}, t_{-\alpha_1}, t_{-\alpha_2}, t_{-\alpha_3}\}$, where α_1, α_2 and α_3 denote the three positive root vectors of $sl(3)$.

The Kirillov–Poisson algebra reads

$$\{J_1, J_{\pm\alpha_1}\} = \pm 2J_{\pm\alpha_1}, \qquad \{J_2, J_{\pm\alpha_1}\} = \mp J_{\pm\alpha_1},$$
$$\{J_1, J_{\pm\alpha_2}\} = \mp J_{\pm\alpha_2}, \qquad \{J_2, J_{\pm\alpha_2}\} = \pm 2J_{\pm\alpha_2},$$
$$\{J_1, J_{\pm\alpha_3}\} = \pm J_{\pm\alpha_3}, \qquad \{J_2, J_{\pm\alpha_3}\} = \pm J_{\pm\alpha_3},$$
$$\{J_{\pm\alpha_1}, J_{\pm\alpha_2}\} = \pm J_{\pm\alpha_3}, \qquad \{J_{\pm\alpha_1}, J_{\mp\alpha_3}\} = \mp J_{\mp\alpha_2}, \qquad (3.20)$$
$$\{J_{\pm\alpha_2}, J_{\mp\alpha_3}\} = \pm J_{\mp\alpha_1}, \qquad \{J_{+\alpha_3}, J_{-\alpha_3}\} = J_1 + J_2,$$
$$\{J_{+\alpha_1}, J_{-\alpha_1}\} = J_1, \qquad \{J_{+\alpha_2}, J_{-\alpha_2}\} = J_2.$$

The most obvious choice for the subalgebra $\mathscr{L}$ is the Cartan subalgebra spanned by t_1 and t_2. Again take $\chi = 0$. According to (3.6), the constraints are then $\phi_1 = J_1$ and $\phi_2 = J_2$. Reasoning as before $C^\infty(C)$ is shown to consist of smooth functions in the variables $J_{\alpha_1}, J_{-\alpha_1}, J_{\alpha_2}, J_{-\alpha_2}, J_{\alpha_3}$ and $J_{-\alpha_3}$. In order to construct $\mathscr{W}$, we need to find functions of these variables that Poisson commute (after imposing the constraints) with ϕ_1 and ϕ_2. In principle, sl_3 has dimension eight. We have imposed two first class constraints, which brings the dimension of C down to six. As ϕ_1 and ϕ_2 are first class, they generate gauge invariances, or in other words, two of the six dimensions will correspond to gauge degrees of freedom. On eliminating these, which is essentially what constructing $\mathscr{W}$ amounts to, we are left with four dimensions. We now look for four independent elements of $C^\infty(C)$ that commute with ϕ_1 and ϕ_2. It is easy to construct them. They read

$$A_1 = J_{\alpha_1} J_{-\alpha_1}, \quad A_2 = J_{\alpha_2} J_{-\alpha_2}, \quad A_3 = J_{\alpha_3} J_{-\alpha_3}, \quad B = \tfrac{1}{2}(J_{\alpha_3} J_{-\alpha_1} J_{-\alpha_2} - J_{-\alpha_3} J_{\alpha_1} J_{\alpha_2}),$$
$$C = \tfrac{1}{2}(J_{\alpha_3} J_{-\alpha_1} J_{-\alpha_2} + J_{-\alpha_3} J_{\alpha_1} J_{\alpha_2}). \qquad (3.21)$$

Note that there are five invariant quantities. However, the relation

$$C^2 = A_1 A_2 A_3 + B^2 \qquad (3.22)$$

between the generators in (3.21) brings the number of independent dimensions back to four. In fact this relation defines the four-dimensional surface in five-dimensional Euclidean space (with coordinates C, B, A_1, A_2, A_3) on which the finite W-algebra lives.

The non-zero Poisson brackets between the generators (3.21) read

$$\{A_i, A_{i+1}\}^* = 2B, \qquad \{A_i, B\}^* = A_i(A_{i+1} - A_{i-1}), \qquad (3.23)$$

where i is a cyclic index, i.e. $i \in \{1, 2, 3\}$ and $i + 3 = i$.

Another reduction of $sl(3)$, and one that leads to a linear finite W-algebra, is the following. Take $\mathscr{L}$ to be the span of t_{α_2} and t_{α_3} and $\chi = 0$. The constraints read $J_{\alpha_2} = J_{\alpha_3} = 0$ and $C^\infty(C)$ consists of functions of $J_{\alpha_1}, J_{-\alpha_1}, J_1, J_2, J_{-\alpha_2}$ and $J_{-\alpha_3}$. Again, a simple counting argument, similar to the one given above, leads us to look for four independent generators of $\mathscr{W}$. It is easy to check that $J_{\alpha_1}, J_{-\alpha_1}, J_1$ and J_2 satisfy the requirement that they commute with the constraints. Now $J_{\alpha_1}, J_{-\alpha_1}$ and J_1 form an sl_2 algebra

$$\{J_1, J_{\pm\alpha_1}\}^* = \pm 2J_{\pm\alpha_1}, \qquad \{J_{\alpha_1}, J_{-\alpha_1}\}^* = J_1, \qquad (3.24)$$

while $s = J_1 + 2J_2$ commutes with all the other generators and thus forms a $u(1)$. Thus we have found that $sl(2) \oplus u(1)$ *is a reduction of* $sl(3)$. As we shall see later, when we discuss real forms of

finite W-algebras this also means that $su(2) \oplus u(1)$ is a reduction of $su(3)$:

$$su(3) \rightarrow su(2) \oplus u(1) . \tag{3.25}$$

This example is actually a special case of a more general class of reductions. Let δ be an arbitrary element of the Cartan subalgebra of a (semi) simple Lie algebra $\mathscr{G}$. We can use δ to define a so-called 'grading' on $\mathscr{G}$. Define $\mathscr{G}_n = \{x \in \mathscr{G} \mid [\delta, x] = nx\}$. It then follows that

$$\mathscr{G} = \bigoplus_n \mathscr{G}_n , \quad [\mathscr{G}_n, \mathscr{G}_m] \subset \mathscr{G}_{n+m} . \tag{3.26}$$

If we now take $\mathscr{L} = \bigoplus_{n>0} \mathscr{G}_n$ and $\chi = 0$, then all elements J_a such that $t_a \in \mathscr{G}_0$ will commute (after imposing the constraints) with all $\phi_\alpha \equiv J_\alpha$. This can be seen as follows. Let $t_a \in \mathscr{G}_0$, then $[t_\alpha, t_a] \in \mathscr{L}$, as follows from (3.26). As all J_α are constrained to zero, we find that $\{\phi_\alpha, J_a\} = \{J_\alpha, J_a\}$ is a linear combination of J_β which after imposing the constraints become zero. We conclude, therefore, that this type of reduction always leads to a finite W-algebra which is isomorphic to $\mathscr{G}_0$. The ones, that are most closely related to the infinite-dimensional W-algebras of conformal field theory.

The finite W-algebras considered in this section are associated to $sl(2)$ embeddings into $\mathscr{G}$. A given $sl(2)$ embedding fixes both $\mathscr{L}$ and χ which means that there is one finite W-algebra for every $sl(2)$ embedding. For $sl(n)$ the number of inequivalent $sl(2)$ embeddings is equal to the number of partitions of the number n and this is therefore the number of finite W-algebras of this type that one is able to extract from $sl(n)$.

3.2.2. sl_2-embeddings

In this section, we always take $\mathscr{G}$ to be a simple Lie algebra. This implies that the inner product $(x, y) = \text{Tr}(\text{ad}(x)\text{ad}(y))$ $(x, y \in \mathscr{G})$, the so called Cartan–Killing form, is non-degenerate, i.e. there does not exist an element $y \in \mathscr{G}$ such that $(x, y) = 0$ for all $x \in \mathscr{G}$. Therefore, any element f of $\mathscr{G}^*$ can be written as $f(.) = (x, .)$ for some $x \in \mathscr{G}$, or, in other words, we can identify $\mathscr{G}$ and $\mathscr{G}^*$. Due to this fact we can define the Kirillov–Poisson structure on $\mathscr{G}$ instead of on $\mathscr{G}^*$ which makes life easier in some respects. Consider for this $C^\infty(\mathscr{G})$ instead of $C^\infty(\mathscr{G}^*)$ and define the functions J^a on $\mathscr{G}$ by $J^a(t_b) = \delta^a_b$. The Poisson algebra satisfied by these quantities is

$$\{J^a, J^b\} = f^{ab}_c J^c , \tag{3.27}$$

where we have raised and lowered indices with the metric $g_{ab} = \text{Tr}(t_a t_b)$.

Now let there be given an $sl(2)$ subalgebra $\{t_0, t_+, t_-\}$ of $\mathscr{G}$,

$$[t_0, t_\pm] = 2t_\pm , \quad [t_+, t_-] = t_0 . \tag{3.28}$$

If we define the spaces

$$\mathscr{G}^{(n)} = \{x \in \mathscr{G} \mid [t_0, x] = nx\} , \tag{3.29}$$

then

$$\mathscr{G} = \bigoplus_n \mathscr{G}^{(n)} . \tag{3.30}$$

The decomposition (3.30) is again a 'grading' of $\mathscr{G}$, because

$$[\mathscr{G}^{(n)}, \mathscr{G}^{(m)}] \subset \mathscr{G}^{(n+m)} . \tag{3.31}$$

In general, the numbers n can be integers or half integers, that is $n \in \frac{1}{2}\mathbb{Z}$. As in the infinite-dimensional case, we would like to take

$$\mathscr{L} = \mathscr{G}^{(+)} \equiv \bigoplus_{n > 0} \mathscr{G}^{(n)} \tag{3.32}$$

together with $\chi(t_+) = 1$ and all others zero. However, there is a slight problem with this choice, because the commutator of two elements of $\mathscr{G}^{(\frac{1}{2})}$ may contain t_+, which is an element of $\mathscr{G}^{(1)}$. This means that χ would no longer be a one-dimensional representation of $\mathscr{G}^{(+)}$ and not all constraints would be first class. Classically, this need not be a problem, because one can eliminate second class constraints with the so-called Dirac bracket [29]. However, quantization becomes much more involved if not all constraints are first class. Fortunately, it is possible for many Lie algebras, among which all $sl(n)$, to replace $\mathscr{G}^{(+)}$ and χ by a new algebra $\mathscr{G}_+$ and a new one-dimensional representation $\chi : \mathscr{G}_+ \to \mathbb{C}$ which leads to the same finite W-algebra without having to resort to Dirac brackets. What one does is replace the 'grading element' t_0 by a new one, δ, which is also an element of the Cartan subalgebra [30, 29]. The element δ defines a grading of $\mathscr{G}$ which is different from the one in (3.30)

$$\mathscr{G} = \bigoplus_n \mathscr{G}_n \, , \tag{3.33}$$

where $\mathscr{G}_n = \{x \in \mathscr{G} \mid [\delta, x] = nx\}$. Essentially, δ has the property that it splits $\mathscr{G}^{(\frac{1}{2})}$ and $\mathscr{G}^{(-\frac{1}{2})}$ into two pieces:

$$\mathscr{G}^{(\frac{1}{2})} = \mathscr{G}_L^{(\frac{1}{2})} \oplus \mathscr{G}_R^{(\frac{1}{2})} \, , \quad \mathscr{G}^{(-\frac{1}{2})} = \mathscr{G}_L^{(-\frac{1}{2})} \oplus \mathscr{G}_R^{(-\frac{1}{2})} \tag{3.34}$$

and that

$$\mathscr{G}_0 = \mathscr{G}_L^{(-\frac{1}{2})} \oplus \mathscr{G}^{(0)} \oplus \mathscr{G}_L^{(+\frac{1}{2})} \, , \quad \mathscr{G}_{\pm 1} = \mathscr{G}^{(\pm 1)} \oplus \mathscr{G}_R^{(\pm \frac{1}{2})} \, , \tag{3.35}$$

i.e. 'half' of $(\pm \frac{1}{2})$ gets added to $\mathscr{G}^{(0)}$, which becomes $\mathscr{G}_0$ and the other half gets added to $\mathscr{G}^{(\pm 1)}$, which becomes $\mathscr{G} \pm 1$. Obviously,

$$\mathscr{G}_+ \oplus \mathscr{G}_L^{(\frac{1}{2})} = \mathscr{G}^{(+)} \, . \tag{3.36}$$

Furthermore, δ is chosen such that the grading of $\mathscr{G}$ is integral, i.e. all n in (3.33) are integers. The one-dimensional representation $\chi : \mathscr{G}_+ \to \mathbb{C}$ is taken to be the same as before, i.e. $\chi(t_+) = 1$ and zero everywhere else. What has been done here, effectively, is that the set of second class constraints (corresponding to $\mathscr{G}^{(\frac{1}{2})}$) has been split into two halves. The constraints in one half we still put to zero. One can choose δ such that they will be first class. The constraints in the other half are kept free. However, the degrees of freedom they represent can be gauged away by the gauge invariance generated by the first class constraints in the first half. It is not possible to find an element δ satisfying the requirements for all Lie algebras [30]. For $sl(n)$ however, there is no problem.

We now proceed as before. Let again $\{t_a\} = \{t_\alpha\} \cup \{t_{\bar{\alpha}}\}$ be a basis of $\mathscr{G}$, where $\{t_\alpha\}$ is a basis of $\mathscr{L} = \mathscr{G}_+$ and $\{t_{\bar{\alpha}}\}$ is a basis of $\mathscr{G}_0 \oplus \mathscr{G}_-$. The constraints are given as usual by $\phi^\alpha = J^\alpha - \chi(t_\alpha)$.

It is clear that elements of $C^\infty(C)$ are smooth functions in the variables $\{J^{\bar{\alpha}}\}$, as all the J^α have been constrained to constants. The next step is to find $\mathscr{W}$. For this we need to look for elements W of $C^\infty(C)$ such that

$$\{\phi^\alpha, W\} = 0 \, , \tag{3.37}$$

after imposing the constraints. This will then be the W-algebra associated to the $sl(2)$ embedding. In general, it is no easy task to find the complete set of elements W such that (3.37) holds. However, in the case of finite W-algebras derived from $sl(2)$ embeddings there turns out to be an algorithmic procedure for doing so. For this we have to consider the gauge transformations generated by the first class constraints ϕ^α. Let $x = x^a t_a$ be an arbitrary element of $\mathscr{G}$. The gauge transformations generated by ϕ^α then act as follows on x:

$$
\begin{aligned}
\delta_\alpha x &\equiv (\delta_\alpha J^a)(x)t_a \equiv \varepsilon\{\phi^\alpha, J^a\}(x)t_a = \varepsilon\{J^\alpha, J^a\}(x)t_a \\
&= \varepsilon f_b^{\alpha a} J^b(x)t_a = \varepsilon f_b^{\alpha a} x^c \delta_c^b t_a = \varepsilon f_b^{\alpha a} x^b t_a \\
&= \varepsilon g^{\alpha c} f_{cb}^a x^b t_a = [\varepsilon g^{\alpha c} t_c, x^b t_b] \\
&= [\varepsilon t^\alpha, x] .
\end{aligned}
\tag{3.38}
$$

Now, if $t_\alpha \in \mathscr{G}_+$, then $t^\alpha \in \mathscr{G}_-$, because $\mathscr{G}_+$ and $\mathscr{G}_-$ are non-degenerately paired by the Cartan–Killing form [29]. It therefore follows from (3.38) that the group of gauge transformations generated by the first class constraints is nothing but $G_- = \exp(\mathscr{G}_-)$ and that this group acts on $\mathscr{G}$ by group conjugation $x \to gxg^{-1}$, $g \in G_-$. Now let x be an element of C. In general it has the form

$$
x = t_+ + \sum_{\tilde{\alpha}} x^{\tilde{\alpha}} t_{\tilde{\alpha}} ,
\tag{3.39}
$$

where $x^{\tilde{\alpha}}$ are complex numbers. It can be shown [29] that the gauge freedom G_- can be completely fixed by bringing x to the form

$$
t_+ + \sum_{t_{\tilde{\alpha}} \in \mathscr{G}_{lw}} W^{\tilde{\alpha}} t_{\tilde{\alpha}} ,
\tag{3.40}
$$

where $\mathscr{G}_{lw} = \{x \in \mathscr{G} \mid [t_-, x] = 0\}$. Put differently, for every x of the form (3.39), there exists a unique $g \in G_-$ such that gxg^{-1} is of the form (3.40). Since the element (3.40) is the same for any element within a certain gauge orbit, the quantities W^α must be gauge invariant. This means that they commute with the constraints $\{\phi^\alpha, W^{\tilde{\alpha}}\} = 0$. By construction, the quantities $W^{\tilde{\alpha}}$ form a complete set of generators of the finite W-algebra in question. The only thing left is, therefore, to calculate the Poisson relations

$$
\{W^{\tilde{\alpha}}, W^{\tilde{\beta}}\}^* = \{W^{\tilde{\alpha}}, W^{\tilde{\beta}}\} .
\tag{3.41}
$$

The equality in Eq. (3.41) can be understood as follows. In principle, the bracket $\{.,.\}^*$ is simply $\{.,.\}$ where it is understood that we take ϕ^α *after* we have calculated the Poisson bracket. However, as $\mathscr{G}_0 \oplus \mathscr{G}_-$ is a subalgebra of $\mathscr{G}$ and since the quantities $W^{\tilde{\alpha}}$ will be polynomials only in $J^{\tilde{\alpha}}$, as follows from (3.39), we find that the Poisson bracket of $W^{\tilde{\alpha}}$ and $W^{\tilde{\beta}}$ will not involve any J^α. This justifies (3.41). It is now time for some examples.

3.2.3. Examples

As a simple example take $\mathscr{G} = sl_2$. Also take as a basis of $sl(2)$ the matrices t_1, t_2 and t_3 such that

$$
J^a t_a = \begin{pmatrix} J^2 & J^1 \\ J^3 & -J^2 \end{pmatrix} .
\tag{3.42}
$$

The Kirillov–Poisson structure then reads

$$\{J^2, J^1\} = -J^1 \ , \quad \{J^2, J^3\} = J^3 \ , \quad \{J^1, J^3\} = -2J^2 \ . \tag{3.43}$$

Obviously, there is only one non-trivial $sl(2)$ embedding into $sl(2)$, namely the algebra itself. Now, $sl(2)$ splits up into three subalgebras of grade $+1$, 0 and -1,

$$sl(2) = \mathscr{G}^{(+1)} \oplus \mathscr{G}^{(0)} \oplus \mathscr{G}^{(-1)} \tag{3.44}$$

given by the span of t_3, t_2 and t_1, respectively. Note that all grades are integral, which means that $\mathscr{G}_n \equiv \mathscr{G}^{(n)}$. The constraint becomes $\phi = J^1 - \chi(t_1) = J^1 - 1$, since $t_1 = t_+$. $C^\infty(C)$ is given by the smooth functions of J^2 and J^3. Next we look for gauge invariant functions, i.e. functions which Poisson commute with the constraint (after imposing $\phi = 0$). We do this by the method outlined above. Let g be an arbitrary element of G_-, i.e.

$$g = \exp \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} . \tag{3.45}$$

We can find the gauge invariant function W by solving

$$g(t_+ + J^2 t_2 + J^3 t_3) g^{-1} \equiv t_+ + W t_3 \tag{3.46}$$

for g and W. In matrix form, this equation reads

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} J^2 & 1 \\ J^3 & -J^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ W & 0 \end{pmatrix} . \tag{3.47}$$

This equation is satisfied if $a = J^2$. We then find that

$$W \equiv J^3 + J^2 J^2 \ . \tag{3.48}$$

Note that indeed $\{\phi, W\} = \{J^1, J^3 + J^2 J^2\} = -2J^2 + 2J^1 J^2$, such that after imposing $\phi = 0$ we have $\{\phi, W\} = 0$. As $\mathscr{W}$ has only one generator, the obtained W-algebra is trivial.

In order to illustrate the construction in a slightly less trivial case consider the so-called non-principal sl_2 embedding into sl_3. For sl_3 we choose the basis

$$J^a t_a = \begin{pmatrix} J^4 + \frac{1}{2}J^5 & J^2 & J^1 \\ J^6 & -2J^4 & J^3 \\ J^8 & J^7 & J^4 - \frac{1}{2}J^5 \end{pmatrix} . \tag{3.49}$$

The $sl(2)$ embedding is given by $t_0 = t_5$, $t_+ = t_1$ and $t_- = t_8$. $\mathscr{G}_{lw}$ is the span of t_8, t_7, t_6 and t_4. The grading of $sl(3)$ with respect to this $sl(2)$ subalgebra is

$$sl(3) = \mathscr{G}^{(-1)} \oplus \mathscr{G}^{(-\frac{1}{2})} \oplus \mathscr{G}^{(0)} \oplus \mathscr{G}^{(\frac{1}{2})} \oplus \mathscr{G}^{(1)} \ , \tag{3.50}$$

where

$$\mathscr{G}^{(-1)} = \mathrm{span}\{t_8\} \ , \quad \mathscr{G}^{(0)} = \mathrm{span}\{t_4, t_5\} \ , \quad \mathscr{G}^{(-\frac{1}{2})} = \mathrm{span}\{t_6, t_7\} \ ,$$
$$\mathscr{G}^{(1)} = \mathrm{span}\{t_1\} \ , \quad \mathscr{G}^{(\frac{1}{2})} = \mathrm{span}\{t_2, t_3\} \ . \tag{3.51}$$

Obviously, this grading is *not* integral. The element δ can be found in the appendix and reads in this case $\delta = \frac{1}{3} \text{diag}(1, 1, -2)$. The grading with respect to δ is

$$sl(3) = \mathscr{G}_{-1} \oplus \mathscr{G}_0 \oplus \mathscr{G}_1 , \tag{3.52}$$

where

$$\mathscr{G}_{-1} = \text{span}\{t_7, t_8\} , \quad \mathscr{G}_0 = \text{span}\{t_4, t_5, t_2, t_6\} , \quad \mathscr{G}_1 = \text{span}\{t_1, t_3\} . \tag{3.53}$$

Obviously,

$$\mathscr{G}_L^{(\frac{1}{2})} = \text{span}\{t_2\} , \quad \mathscr{G}_R^{(\frac{1}{2})} = \text{span}\{t_3\} . \tag{3.54}$$

The one dimensional representation is given by

$$\chi(t_1) = 1 , \quad \chi(t_3) = 0 . \tag{3.55}$$

According to the standard procedure, the constraints are

$$\phi^1 = J^1 - 1 = 0 , \quad \phi^3 = J^3 = 0 . \tag{3.56}$$

Again we want to find the gauge invariant functions in the variables $J^2, J^4, J^5, \ldots, J^8$. The group of gauge transformations G_- is given by

$$G_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} . \tag{3.57}$$

As before, we solve the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \begin{pmatrix} J^4 + \frac{1}{2}J^5 & J^2 & 1 \\ J^6 & -2J^4 & 0 \\ J^8 & J^7 & J^4 - \frac{1}{2}J^5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{pmatrix} = \begin{pmatrix} W^4 & 0 & 1 \\ W^6 & -2W^4 & 0 \\ W^8 & W^7 & W^4 \end{pmatrix}$$

for a, b and W^i. The result reads $a = \frac{1}{2}J^5$, $b = J^2$ and

$$\begin{aligned} W^4 = J^4 , \quad & W^7 = J^7 + \frac{1}{2}J^2 J^5 - 3J^2 J^4 , \\ W^6 = J^6 , \quad & W^8 = J^8 + \frac{1}{4}J^5 J^5 + J^2 J^6 , \end{aligned} \tag{3.58}$$

all of which can easily be shown to be gauge invariant. These are now the generators of the finite W-algebra.

Defining

$$\begin{aligned} j_+ = W^7 , \quad & j_- = \frac{4}{3}W^6 , \quad j_0 = -4W^4 \\ C = -\frac{2}{3}C_2 = & -\frac{4}{3}(W^8 + 3W^4 W^4) \end{aligned} \tag{3.59}$$

and calculating the Poisson brackets between these quantities we find

$$\{j_0, j_\pm\}^* = \pm 2j_\pm , \quad \{j_+, j_-\}^* = j_0^2 + C . \tag{3.60}$$

Note that this algebra is identical to the symmetry algebra of the two-dimensional anisotropic oscillator with frequency ratio $2:1$. It is a quadratic extension of $su(2)$. In fact, the Jacobi identities

are still satisfied if one replaces $j_0^2 + C$ by an arbitrary function of j_0 (see also the remark at the end of Section 2.2.2). It is not clear whether all these can be obtained from sl_2-embeddings. If one considers the sl_2-embedding in sl_{n+1} under which the fundamental representation $\underline{n+1}$ branches into $\underline{1} + \underline{n}$, one finds $\{j_+, j_-\}^* = j_0^n +$ other generators. This suggests that one can obtain all polynomial non-linear deformations of $su(2)$ as a suitable quotient of finite W-algebras related to sl_2-embeddings, but we have no proof of this.

We shall now consider another example which turns out to be of unexpected physical significance. Take $\mathcal{G} = sl(4)$ and choose the basis $\{t_a\}$ such that

$$
J^a t_a = \begin{pmatrix}
\frac{1}{2}J^7 + J^8 + J^9 & J^5 + J^6 & J^2 + J^3 & J^1 \\
J^{10} + J^{11} & \frac{1}{2}J^7 - J^8 - J^9 & J^4 & J^2 - J^3 \\
J^{12} + J^{13} & J^{14} & -\frac{1}{2}J^7 + J^8 - J^9 & J^5 - J^6 \\
J^{15} & J^{12} - J^{13} & J^{10} - J^{11} & -\frac{1}{2}J^7 - J^8 + J^9
\end{pmatrix} . \quad (3.61)
$$

The $sl(2)$ embedding we consider is $t_0 = t_7$, $t_+ = t_2$, $t_- = t_{12}$. The grading is given by

$$
\mathcal{G} = \mathcal{G}^{(-1)} \oplus \mathcal{G}^{(0)} \oplus \mathcal{G}^{(1)} , \quad (3.62)
$$

where

$$
\mathcal{G}^{(-1)} = \operatorname{span}\{t_{12}, t_{13}, t_{14}, t_{15}\} , \quad \mathcal{G}^{(0)} = \operatorname{span}\{t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\} ,
$$
$$
\mathcal{G}^{(1)} = \operatorname{span}\{t_1, t_2, t_3, t_4\} , \quad \mathcal{G}_{lw} = \mathcal{G}^{(-1)} \oplus \operatorname{span}\{t_5, t_8, t_{10}\} . \quad (3.63)
$$

From (3.62) we see that the grading is integral, so there is no problem with second class constraints. The one-dimensional representation $\chi: \mathcal{G}^{(1)} \to \mathbb{R}$ is given by $\chi(t_2) = 1$ and $\chi(t_1) = \chi(t_3) = \chi(t_4) = 0$. The constraints therefore read $J^1 = J^3 = J^4 = 0$ and $J^2 = 1$. Performing the calculation of $\mathcal{W}$ as before, we find that the finite W-algebra associated to this $sl(2)$ embedding can be written as

$$
\{L^a, L^b\} = f_c^{ab} L^c , \quad \{L^a, R^b\} = f_c^{ab} R^c ,
$$
$$
\{R^a, R^b\} = (-2H - C_2) f_c^{ab} L^c , \quad (3.64)
$$

where $a = 1, 2, 3$ and f_c^{ab} are the structure constants of $sl(2)$. Note that this algebra is essentially the Runge–Lenz algebra which is the symmetry algebra of a particle moving on S^3 in a Coulomb potential, as we have seen in the previous chapter.

3.2.4. Classical Miura transformation

A careful examination of Eqs. (3.39) and (3.40) reveals that the gauge invariant generators $W^{\tilde{\alpha}}$ in general have a very specific form. Let $t_{\tilde{\alpha}}$ be a lowest weight vector, i.e. $[t_-, t_{\tilde{\alpha}}] = 0$, then

$$
W^{\tilde{\alpha}} = W_-^{\tilde{\alpha}} + W_0^{\tilde{\alpha}} , \quad (3.65)
$$

where $W_0^{\tilde{\alpha}}$ contains all terms that only contain $J^{\tilde{\beta}}$ and $t_{\tilde{\beta}} \in \mathcal{G}_0$, and $W_-^{\tilde{\alpha}}$ is the sum of the remaining terms. Furthermore, it turns out that $W_0^{\tilde{\alpha}} \neq 0$. From this last fact we can derive a very important result. As the Poisson bracket of two elements of degree zero is again of degree zero, we find that we must have

$$
\{W^{\tilde{\alpha}}, W^{\tilde{\beta}}\}_0 = \{W_0^{\tilde{\alpha}}, W_0^{\tilde{\alpha}}\} , \quad (3.66)
$$

where the subscript 0 in $\{.,.\}$ means that we throw away everything except the grade zero piece of (whatever comes out of) the Poisson bracket. What this equation says is that the $\{W_0^{\tilde{\alpha}}\}$ form an algebra which is isomorphic to the algebra satisfied by the $\{W^{\tilde{\alpha}}\}$. However, note that $W_0^{\tilde{\alpha}}$ only contains J^β of degree zero which means that what we have done is to embed the finite W-algebra into the Kirillov–Poisson algebra of the semi-simple Lie algebra $\mathcal{G}_0$ (denoted by $K(\mathcal{G}_0)$). The map

$$\mathcal{W} \to K(\mathcal{G}_0) \tag{3.67}$$

is called the 'classical (finite) Miura transformation'.

Let us consider an example. In the case of $sl(2)$ we have seen that $W = J^3 + J^0 J^0 \equiv W_- + W_0$, where $W_0 = J^0 J^0$. This example was trivial due to the fact that it had only one generator. More generally, however, in the case of the so called 'principal $sl(2)$ embeddings' into $sl(n)$ we get abelian finite W-algebras with $n - 1$ generators. The reason for this is that for principal embeddings, $\mathcal{G}_0$ is equal to the Cartan subalgebra, which is an abelian algebra. The W-algebra is therefore also abelian. The non-trivial cases arise for the non-principal embeddings.

Take for example again the case of $\mathcal{G} = sl(3)$ with the non-principal embedding. The generators $W^{\tilde{\alpha}}$ of this algebra were given in (3.58). As $\mathcal{G}_0$ was the span of t_4, t_5, t_2 and t_6 we find that the grade 0 pieces of the generators are given by:

$$W_0^4 = J^4 , \quad W_0^6 = J^6 , \quad W_0^7 = \tfrac{1}{2}J^2 J^5 - 3J^2 J^4 , \quad W_0^8 = \tfrac{1}{4}J^5 J^5 + J^2 J^6 . \tag{3.68}$$

Now, defining

$$h = -\tfrac{3}{2}J^4 - \tfrac{1}{4}J^5 , \quad e = J^2 , \quad f = J^6 , \quad s = \tfrac{1}{2}J^4 - \tfrac{1}{4}J^5 , \tag{3.69}$$

which satisfy the Poisson relations of $sl(2) \oplus u(1)$

$$\{h, e\} = e , \quad \{h, f\} = -f , \quad \{e, f\} = 2h , \tag{3.70}$$

we find, using Eq. (3.59), the following expressions of the generators j_0, j_+, j_- and C in terms of h, e, f and s:

$$j_0 = 2h - 2s , \quad j_+ = e(h - 3s) , \quad j_- = \tfrac{4}{3} f , \quad C = -\tfrac{4}{3}(h^2 + ef + 3s^2) . \tag{3.71}$$

Using the relations (3.70), one can easily verify that these expressions indeed satisfy the algebra (3.60). Note that due to the fact that the expressions in the right-hand side of (3.71) are quadratic in the cases of j_+ and C it is not possible to express h, e, f and s similarly in terms of $j_0, j_\pm, C$, i.e. to invert the Miura transformation. The reason for this is that the Miura transformation is in general a homomorphism rather than an isomorphism.

In the case of the Runge–Lenz algebra, which we constructed above as a reduction of $sl(4)$, $\mathcal{G}_0$ is given by $sl(2) \oplus sl(2) \oplus u(1)$. The explicit expressions of L^a and R^a in terms of the generators of $\mathcal{G}_0 = sl(2) \oplus sl(2) \oplus u(1)$ can be found in [6].

3.2.5. General form of sl(2) related finite W-algebras

Consider the plane in $\mathcal{G}$ made up of elements of the form

$$x = t_+ + \sum_{t_{\tilde{\alpha}} \in \mathcal{G}_{lw}} x^{\tilde{\alpha}} t_{\tilde{\alpha}} , \tag{3.72}$$

where $x^{\tilde{a}}$ are complex numbers. That is, we put all J^a to zero except of course J^+, corresponding to t_+, which we put to 1 and $J^{\tilde{a}}$ with $t_{\tilde{a}} \in \mathscr{G}_{lw}$, which we leave unconstrained. It can be shown [29] that this set of constraints (which is larger than the set of constraints that lead to the constraint surface C) is completely second class, that is, none of the above mentioned constraints are first class. This means that the original Poisson bracket on $\mathscr{G}$, the Kirillov–Poisson bracket, induces a 'Dirac bracket' on the set of elements of the form (3.72). Recall that for a set of second class constraints ϕ^i the Dirac bracket between two functions f and g is defined by

$$\{f, g\}_D = \{f, g\} - \{f, \phi^i\} \Delta_{ij} \{\phi^j, g\}, \tag{3.73}$$

where Δ_{ij} is the inverse of the matrix $\Delta^{ij} = \{\phi^i, \phi^j\}$.

Now, since the set of elements of the form (3.72) is nothing but $C/\mathscr{G}_-$, i.e. C with all the gauge freedom removed we find that the Dirac bracket algebra of the functions $J^{\tilde{a}}$ (for $t_{\tilde{a}} \in \mathscr{G}_{lw}$) must be isomorphic to the finite W-algebra associated to the $sl(2)$ embedding in question.

It is possible [29] to derive an elegant general formula for the Dirac bracket algebra on $\mathscr{G}_{lw}$, and thus of the finite W-algebra. Let $w \in \mathscr{G}_{lw}$ and let Q_1 and Q_2 be smooth functions on $\mathscr{G}_{lw}$. Define $\mathrm{grad}_w Q \in \mathscr{G}_{hw} \equiv \{x \in \mathscr{G} \mid [t_+, x] = 0\}$ by

$$(w', \mathrm{grad}_w Q) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} Q(w + \varepsilon w')|_{\varepsilon = 0}, \tag{3.74}$$

for all $w \in \mathscr{G}_{lw}$. This uniquely defines $\mathrm{grad}_w Q$, since $\mathscr{G}_{lw}$ and $\mathscr{G}_{hw}$ are non-degenerately paired by the Cartan–Killing form $(.,.)$. The finite W-algebra associated to the $sl(2)$ embedding in question is now (isomorphic to)

$$\{Q_1, Q_2\}(w) = \left(w, \left[\mathrm{grad}_w Q_1, \frac{1}{1 + L \circ \mathrm{ad}_w}, \mathrm{grad}_w Q_2\right]\right), \tag{3.75}$$

where $\mathrm{ad}_w(.) = [w, .]$. The operator L is essentially the inverse of ad_{t_+}. There is, nevertheless, a problem, since $\mathrm{Ker}(\mathrm{ad}_{t_+}) \neq 0$. It is, however, a 1–1 map from $\mathrm{Im}(\mathrm{ad}_{t_-})$ to the image of ad_{t_+}. One now takes L to be the inverse of ad_{t_+} in this domain of definition and then extends it to all of $\mathscr{G}$ by 0.

Note that formula (3.75) reduces to the ordinary Kirillov–Poisson bracket for the trivial $sl(2)$ embedding $t_0 = t_- = t_+ = 0$. For non-trivial $sl(2)$ embeddings however (3.75) is a highly non-trivial and in general non-linear Poisson structure as we have seen.

3.2.6. W-coadjoint orbits

In this section, we are going to construct the coadjoint orbits of finite W-algebras associated to sl_2 embeddings. First, we briefly discuss some general aspects of symplectic and coadjoint orbits and then we show how one can construct finite W symplectic orbits from the coadjoint orbits of the Lie algebra to which they are associated.

Suppose we are given a Poisson manifold, that is a manifold M together with a Poisson bracket $\{.,.\}$ on the space of smooth functions on M. In general, such a Poisson structure is not associated to a symplectic form on M since the Poisson bracket can be degenerate, i.e. there may exist functions that Poisson commute with all other functions. The Poisson bracket $\{.,.\}$ does however induce a symplectic form on certain submanifolds of M. In order to see this consider the set of

Hamiltonian vector fields

$$X_f(\cdot) = \{f, .\} \quad \text{for } f \in C^\infty(M) . \tag{3.76}$$

Note that the set of functions on M that Poisson commute with all other functions are in the kernel of the map $f \mapsto X_f$. From this it follows that in every tangent space the span of all Hamiltonian vector fields is only a subspace of the whole tangent space. We say that a Poisson structure is regular, if the span of the set of Hamiltonian vector fields has the same dimension in every tangent space. Obviously, a regular Poisson structure defines a tangent system on the manifold M and from the well known relation

$$[X_f, X_g] = X_{\{f,g\}} , \tag{3.77}$$

it then immediately follows that this system is integrable (in the sense of Frobenius). Therefore, M foliates into a disjoint union of integral manifolds of the Hamiltonian vector fields. Obviously, the restriction of the Poisson bracket to one of these integral manifolds is non-degenerate and therefore associated to a symplectic fom. They are therefore called symplectic leaves.

The symplectic leaves play an important role in the representation theory of Poisson algebras. A representation of a Poisson algebra is a symplectic manifold S together with a map π from the Poisson algebra to the space of smooth functions on S that is linear and preserves both the multiplicative and Poisson structure of the Poisson algebra. Also the Hamiltonian vector field associated to $\pi(f) \in C^\infty(S)$ must be complete (i.e. defined everywhere). A representation is called *irreducible*, if span of the set $\{X_{\pi(f)}(s) | f \in C^\infty(M)\}$ is equal to the tangent space of S in s for all $s \in S$. The role of the symplectic leaves is clarified by the following theorem [31]. If a representation of the Poisson algebra $C^\infty(M)$ is irreducible, then S is symplectomorphic to a covering space of a symplectic leaf of M.

From the above it follows that it is rather important to construct the symplectic leaves associated to finite W-algebras. Looking at Eq. (3.75) it is clear that constructing these symplectic orbits from scratch could be rather difficult. Luckily, we can use the fact that the symplectic orbits of the Kirillov–Poisson structure are known (by the famous Kostant–Souriau theorem they are nothing but the coadjoint orbits and their covering spaces) to construct these. The answer turns out to be extremely simple and is given in the following theorem:

Theorem. *Let $\mathcal{O}$ be a coadjoint orbit of the Lie algebra g, then the intersection of $\mathcal{O}$ and g_{lw} is a symplectic orbit of the finite W-algebra $W(g; t_0)$.*

Proof. Let again $C = \{x \in g \mid \phi^\alpha(x) = 0\}$. The Hamiltonian vector fields $X^\alpha = \{\phi^\alpha, .\}$ form an involutive system tangent to C and therefore C foliates. g_{lw} has one point in common with every leaf and denote the canonical projection from C to g_{lw}, which projects an element $x \in C$ to the unique point $x' \in g_{lw}$, by π. This map induces a map π_* from the tangent bundle TC of C to the tangent bundle Tg_{lw} of g_{lw}. Let f be a 'gauge invariant' function on C, i.e. $\{f, \phi_\alpha\}|_C = 0$ then obviously $X_f = \{f, .\}$ is a section of TC (i.e. as a vector field it is tangent to C in every point of C). It need not be an element of Tg_{lw} however, but using the gauge invariance we can project it back onto the gauge slice. This projection of X_f on to the gauge slice is given by $\pi_*(X_f) \in Tg_{lw}$. By construction, the Dirac bracket now has the property that $\{f, .\}^* = \pi_*(X_f)$. What we now need to show is that $\pi_*(X_f)$ is tangent to the coadjoint orbit $\mathcal{O}$. Well, obviously we have $\pi_*(X_f) = X_f + Y$, where Y is

a tangent to the gauge orbit. Since the spaces tangent to the gauge orbits are spanned by the Hamiltonian vector fields X^α, we find that Y is a tangent vector of $\mathcal{O}$. Because X_f is by definition a tangent vector of $\mathcal{O}$ we see that $\pi_*(X_f)$ is also tangent to the coadjoint orbit $\mathcal{O}$. What we can conclude is that the Hamiltonian vector fields on g_{lw} w.r.t. the reduced Poisson bracket are all tangent to the coadjoint orbit and therefore the symplectic orbits are submanifolds of the coadjoint orbits. The theorem now follows immediately.

Let us now recall some basic facts on coadjoint orbits of simple Lie algebras. Let f be an adinvariant function on g, i.e. $f(axa^{-1}) = f(x)$ for all $a \in G$ and $x \in g$. Then we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f(e^{tx} y e^{-tx})|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(y + t[x, y])|_{t=0}$$

$$= (\mathrm{grad}_y\, f, [x, y]) = ([\mathrm{grad}_y\, f, x], y)\,, \tag{3.78}$$

which means that

$$([\mathrm{grad}_y\, f, \mathrm{grad}_y h], y) = \{f, h\}(y) = 0\,. \tag{3.79}$$

We conclude that any adinvariant function Poisson commutes with all other functions. Conversely, any function that Poisson commutes with all others is adinvariant, because $\{f, h\} = 0$ for all h implies that $X_h(f) = 0$ for all h, where X_h is the Hamiltonian vector field associated to h, which means that the derivative of f in all directions tangent to a symplectic orbit are zero.

Consider now the Casimir functions $\{C_i\}_{i=1}^{\mathrm{rank}(g)}$ of the Lie algebra g. Certainly, these Poisson commute with all other functions and it can in fact be shown that the (co)adjoint (since we have identified the Lie algebra with its dual the adjoint and coadjoint orbits coincide) orbits of maximal dimension of the Lie algebra are given by their constant sets

$$C_\mu = \{x \in g \mid C_i(x) = \mu_i;\ \mu_i \in \mathbb{C};\ i = 1, \ldots, \mathrm{rank}(g)\}\,. \tag{3.80}$$

Their dimension is therefore $\dim_\mathbb{C}(g) - \mathrm{rank}(g)$.

Coadjoint orbits are very important to the representation theory of (semi)simple Lie groups. In order to see why let us recall the Borel–Weil–Bott (BWB) theorem [32]. Let G be a compact semisimple Lie group with maximal torus T and let R be a finite-dimensional irreducible representation of G. The space of highest weight vectors of R is one-dimensional (call it V) and furnishes a representation of T. The product space $G \times V$ is a trivial line bundle over G and its quotient by the action of T (where $(a, v) \sim (at, t^{-1}v)$ for $a \in G$ and $t \in T$) is a holomorphic line bundle over G/T (which is a complex manifold). Now, L admits a G action $(a, v) \to (a'a, v)$, where $a, a' \in G$ and $v \in V$, so the space of holomorphic sections of L is naturally a G representation. The BWB theorem now states that this representation is isomorphic to R. The BWB construction has, however, the restriction that it applies only to compact groups. There does exist however a generalization of the BWB construction called the coadjoint method of Kirillov. In this method one generalizes G/T to certain homogeneous spaces G/H which can be realized as coadjont orbits. As we have mentioned above, a coadjoint orbit carries a natural (G invariant) symplectic form ω inherited from the Kirillov–Poisson structure and can therefore be seen as a phase space of some classical mechanical system. One then attempts to quantize this symplectic manifold using the methods of geometric quantization [33]. For this one is supposed to construct a holomorphic line

bundle L over the coadjoint orbit such that the first Chern class of L is equal to the (cohomology class of) ω. If such a line bundle exists the coadjoint orbit is called quantizable. For a quantizable orbit the bundle L can then be shown to admit a hermitian metric whose curvature is equal to ω. The space of sections of L thus obtains a Hilbert space structure and is interpreted as the physical Hilbert space of the quantum system associated to the coadjoint orbit. It carries a unitary representation of the group G and in fact one attempts to construct all unitary representations in this way. Obviously this construction is a generalization of the BWB method. Note: the fact that generic groups have unitary irreducible representations only for a discrete set of highest weights is translated into the fact that only a discrete set of orbits is quantizable.

From the above it is clear that the symplectic (or W-coadjoint) orbits of finite W-algebras are extremely important especially in the study of global aspects of W-algebras. In order to see this, note that coadjoint orbits are homogeneous spaces of the Lie group G in question, and it is therefore tempting to interpret the symplectic orbits of W-algebras as W-*homogeneous spaces*. These naturally carry much information on the global aspects of W-transformations. In the notation used above the symplectic orbits of the finite W-algebra $W(g;t_0)$ are given by

$$\mathcal{O}_\mu = C_\mu \cap g_{lw} \,, \tag{3.81}$$

which are therefore (generically) of dimension $\dim_\mathbb{C}(g_{lw}) - \mathrm{rank}(g)$.

Let us give some examples. As we have seen, the finite W-algebras associated to principal sl_2 embeddings are trivial in the sense that they are Poisson abelian. This means that all Hamiltonian vector fields are zero and that all symplectic orbits are points. Again the simplest non-trivial case is the algebra $W_3^{(2)}$. The two independent Casimir functions of sl_3 are

$$C_1 = \tfrac{1}{2}\mathrm{Tr}(J^2) \quad \text{and} \quad C_2 = \tfrac{1}{3}\mathrm{Tr}(J^3) \,, \tag{3.82}$$

where $J = J^a t_a$. The spaces $\mathcal{O}_\mu$ can be constructed by taking for J the constrained and gauge fixed matrix

$$J = \begin{pmatrix} j & 0 & 1 \\ G_+ & -2j & 0 \\ T & G_- & j \end{pmatrix} \,. \tag{3.83}$$

One then easily finds

$$C_1 = 3j^2 + T \,, \quad C_2 = -2j^3 + G_+ G_- + 2jT \,. \tag{3.84}$$

Obviously $\dim(g_{lw}) = 4$ and introducing the variables $z_1 = j; z_2 = G_+; z_3 = G_-; z_4 = T$ we find that

$$\mathcal{O}_{\mu_1,\mu_2} = \{z \in \mathbb{C}^4 \,|\, 3z_1^2 + z_4 = \mu_1 \text{ and } 2z_1 z_4 + z_2 z_3 - 2z_1^3 = \mu_2\} \,. \tag{3.85}$$

The topological nature of these symplectic orbits becomes clearer if we insert $z_4 = \mu_1 - 3z_1^2$ into the second equation. We find that topologically $\mathcal{O}_{\mu_1,\mu_2}$ is equivalent to the two-dimensional surface in $\mathbb{C}^3$ determined by the equation

$$z_2 z_3 = \mu_2 - 2\mu_1 z_1 + 8z_1^3 \,. \tag{3.86}$$

Compare this to the coadjoint orbits of $sl(2, \mathbb{C})$ which are given by [34]

$$z_2 z_3 = z_1^2 + \mu \; . \tag{3.87}$$

It is well known that the coadjoint orbits of $sl(2, \mathbb{R})$ and $su(2)$, which are real forms of $sl(2, \mathbb{C})$, can be found by considering the intersection of these complex coadjoint orbits with the appropriate real subspaces of $sl(2, \mathbb{C})$.

In this section we have seen that it is easy to find the coadjoint orbits of finite W-algebras by using the fact that they are reductions of Lie algebras. Constructing these orbits would have been much more complicated, if not practically impossible, if this information had not been used.

3.2.7. Semi-simple subalgebras of finite W-algebras

As we have seen, most finite W-algebras are non-linear, i.e. the commutation relations (Poisson brackets) close on polynomials, not just linear combinations, of the generators. It is, however, not uncommon for a finite W-algebra to have a linear subalgebra. We have actually already seen an example of this in the case of the Runge–Lenz algebra which was a reduction of $sl(4)$. The generators L^a in this algebra formed an $sl(2)$ subalgebra of the Runge–Lenz algebra (which as a whole was non-linear). In this section, we will discuss how linear subalgebras arise and how one can predict them.

For this we need some basic facts on the theory of $sl(2)$ embeddings. As is well known [35] $sl(2)$ embeddings into $sl(n)$ are completely characterized by the way the fundamental (or defining) representation of $sl(n)$ branches into irreducible $sl(2)$ representations. Furthermore, all conceivable branchings are possible. So the fundamental (n-dimensional) representation of $sl(n)$ can branch into a direct sum of $n_1, n_2, n_3, \ldots$ dimensional $sl(2)$ representations as long as we have $n = n_1 + n_2 + n_3 + \cdots$. The number of inequivalent $sl(2)$ embeddings is therefore equal to the number of partitions of the number n. Conversely, any partition determines an $sl(2)$ embedding up to inner automorphisms.

We are now ready to explain when semi-simple subalgebras of finite W-algebras arise [29]. Suppose a certain $sl(2)$ representation occurs m times in the branching of the fundamental representation of $sl(n)$ then there will be an $sl(m)$ subalgebra in the resulting finite W-algebra. For example, in the case of the Runge–Lenz algebra one can easily check, using the explicit form of the generators t_0, t_+ and t_-, that the four-dimensional fundamental representation of $al(4)$ decomposes into a direct sum of two two-dimensional representations of $sl(2)$. The partition of 4 corresponding to this branching is $4 = 2 + 2$. The fact that the same representation occurs twice immediately leads us to expect that there will be an $sl(2)$ subalgebra in the resulting finite W-algebra. Indeed, we know this to be the case. Another, rather trivial example is the case of the trivial $sl(2)$ embedding $t_0 = t_+ = 0$. This case corresponds to the partition $n = 1 + 1 + \cdots + 1$, i.e. the singlet of $sl(2)$ is n-fold degenerate. We, therefore, expect an $sl(n)$ subalgebra. This is of course the original Lie algebra itself.

A much more interesting question is whether there exist non-linear extensions of the gauge group $SU(3) \times SU(2) \times U(1)$ of the standard model, i.e. finite W-algebras that contain its Lie algebra as a subalgebra. The answer turns out to be affirmative. The smallest W-algebra that contains such a subalgebra must be a reduction of at least $sl(7)$ because we need in the branching of the fundamental representation at least a double and a triple degeneracy. The smallest n for which it is possible to do this is 7. The branching that does the trick is $7 = 1 + 1 + 1 + 2 + 2$. Due to the fact

that the singlet '1' has a threefold degeneracy and the doublet '2' a double degeneracy we conclude that the finite W-algebra will contain an $sl(3)$ and an $sl(2)$ subalgebra. The $u(1)$ subalgebra will be there automatically due to the fact that the second order Casimir descends to the finite W-algebra and becomes a generator of it. The step from $sl(n)$ to $su(n)$ is then made by taking the appropriate real form of the W-algebra, but we will come back to this later.

We conclude that it is not difficult to construct non-linear extensions of the gauge group of the standard model. The next step would of course be to construct an actual gauge theory with this non-linear gauge algebra. In doing so one might be helped by the fact that it is a reduction of, and can be embedded into (by the Miura transformation), a Lie algebra. An obvious procedure would therefore be to 'break' the symmetry of the linear theory to a non-linear subsymmetry by adding terms to the action which are only invariant under the transformations generated by the non-linear gauge algebra. This is the principle on which for example Toda theories base their non-linear W-symmetry (we will come back to this later).

The W-algebra associated to the branching $7 = 2 + 2 + 1 + 1 + 1$ can be embedded into (the universal enveloping algebra of) $su(5) \oplus su(2)$. One method of writing down a gauge theory based on this W-algebra would therefore be to first consider an $su(5) \oplus su(2)$ gauge theory and then to add to its action terms invariant under the transformations generated by the W-algebra only. A further discussion of finite W gauge theories will be postponed until Chapter 6.

4. Quantum finite W-algebras

In principle, a complete quantum description of finite W-algebras involves two things: First we need to quantize the classical algebras constructed in the previous chapter by associating to them appropriate non-commutative associative algebras. We come to this in a moment. The second step is to construct the irreducible unitary representations of these quantum algebras on Hilbert spaces.

Unfortunately, the general problem of quantizing finite W-algebras is not solved. However, in certain special cases, like the W-algebras associated to $sl(2)$ embeddings, the quantization *is* known [29]. In this chapter we discuss these results and add some new ones on finite W-algebras that are obtained by reduction w.r.t. Cartan subalgebras. In the next chapter we discuss unitary highest weight representations of finite W-algebras and describe some conjectures on Kac determinants and character formulas.

In quantum mechanics, quantization amounts to replacing Poisson brackets by commutators, sometimes denoted by

$$\{\cdot,\cdot\} \to \frac{i}{\hbar}[\cdot,\cdot] \,. \tag{4.1}$$

In a mathematically more sophisticated language, this amounts to replacing a Poisson algebra, which is commutative and associative, $(\mathscr{A}_0, \{.,.\})$ by an associative but non-commutative algebra $\mathscr{A}$ depending on a parameter $\hbar$, such that

- $\mathscr{A}/\hbar\mathscr{A} \simeq \mathscr{A}_0$;
- if π denotes the natural map $\pi: \mathscr{A} \to \mathscr{A}/\hbar\mathscr{A} \simeq \mathscr{A}_0$, then

$$\{\pi(X), \pi(Y)\} = \pi((XY - YX)/\hbar) \,. \tag{4.2}$$

The first condition simply expresses the fact that in the limit $\hbar \to 0$ the original algebra is recovered from the new one, while the second one tells you how to extract the Poisson brackets of the old algebra from the commutators of the new algebra.

In most cases one has a set of generators for $\mathcal{A}_0$, and $\mathcal{A}$ is completely fixed by giving the commutation relations of these generators.

For example, let $\mathcal{A}_0$ be the Kirillov–Poisson algebra $K(\mathcal{G})$ associated to $\mathcal{G}$, then a quantization of this Poisson algebra is the algebra $\mathcal{A}$ generated by J^a and $\hbar$, subject to the commutation relations $[J^a, J^b] = \hbar f_c^{ab} J^c$ (note that here the J^a are no longer functions on $\mathcal{G}$ but the quantum objects associated to them). Obviously, the Jacobi identities are satisfied. Specializing to $\hbar = 1$, this algebra is precisely the universal enveloping algebra $\mathcal{U}\mathcal{G}$ of $\mathcal{G}$.

To find quantizations of finite W-algebras, one can first reduce the sl_n Kirillov–Poisson algebra, and then try to quantize the resulting algebras that we studied in the previous sections. On the other hand, one can also first quantize and then constrain. We will follow the latter approach, and thus study the reductions of the quantum Kirillov algebra

$$[J^a, J^b] = \hbar f_c^{ab} J^c \ . \tag{4.3}$$

We want to impose the same constraints on this algebra as we imposed previously on the Kirillov–Poisson algebra, to obtain the quantum versions of the finite W-algebras $(\mathcal{G}, \mathcal{L}, \chi)$. Imposing constraints on quantum algebras can be done using the BRST formalism [36], which is what we will use in the next sections. It is important to realize that we do by no means prove uniqueness of the quantization of the finite W-algebra. There may exist other, non-equivalent quantizations. The one that one finds by applying the BRST formalism starts with a specific quantization of the Kirillov algebra, namely (4.3), and leads to one specific quantization of the finite W-algebra. Clearly, this is the most natural one from the Lie-algebraic point of view, but it is an interesting open problem to find out whether or not other quantizations of finite W-algebras exist. One might, for example, examine what happens when one starts with a quantum group rather than the quantum-Kirillov algebra. This might lead to q-deformations of finite W-algebras, which can – if they exist – be seen as two-parameter quantizations of classical finite W-algebras.

4.1. BRST quantization of finite W-algebras

Recall that the constraints we want to impose read

$$J^\alpha - \chi(J^\alpha) = 0 \ . \tag{4.4}$$

To make this into good quantum constraints, χ must be promoted to a representation of the Lie algebra $\mathcal{L}$ into some associative algebra, which is a quantization of the Poisson algebra in which $\mathcal{L}$ was represented classically by means of χ. This associative algebra will be denoted by $\mathcal{U}_\chi$. In the cases where $\chi = 0$ or χ is a one-dimensional character, this is trivial. For all other cases we will simply assume this has been accomplished in some way or the other. Furthermore, we will take $\hbar = 1$ for simplicity; the explicit $\hbar$ dependence can be determined afterwards.

To set up the BRST framework we need to introduce anticommuting ghosts and antighosts c_α and b^α, associated to the constraints that we want to impose [36]. They satisfy $b^\alpha c_\beta + c_\beta b^\alpha = \delta^\alpha_\beta$ and generate the Clifford algebra $Cl(\mathcal{L} \oplus \mathcal{L}^*)$. The quantum-Kirillov algebra is just the universal

enveloping algebra $\mathcal{U}\mathcal{G}$, and the total space on which the BRST operator acts is $\Omega = \mathcal{U}\mathcal{G} \otimes \mathcal{U}_\chi \otimes Cl(\mathcal{L} \oplus \mathcal{L}^*)$. A $\mathbb{Z}$ grading on Ω is defined by $\deg(J^a) = \deg(\mathcal{U}_\chi) = 0$, $\deg(c_\alpha) = +1$ and $\deg(b^\alpha) = -1$, and we can decompose $\Omega = \bigoplus_k \Omega^k$ accordingly. The BRST differential on Ω is given by $d(X) = [Q, X]$, where Q is the BRST charge

$$Q = (J^\alpha - \chi(J^\alpha))c_\alpha - \tfrac{1}{2} f_\gamma^{\alpha\beta} b^\gamma c_\alpha c_\beta , \tag{4.5}$$

and $[.,.]$ denotes the graded commutator (as it always will from now on)

$$[A, B] = AB - (-1)^{\deg(A) \cdot \deg(B)} BA . \tag{4.6}$$

Note that $\deg(Q) = 1$. First-class constraints generate gauge transformations, and Q acts on generators precisely as gauge transformations, with the parameters replaced by the anticommuting ghosts c_α. Roughly, the antighosts b^α are needed to impose the first class constraints, whereas the ghosts c_α are needed to perform the gauge fixing on the constrained phase space.

The standard BRST complex associated to the first-class constraints is

$$\cdots \overset{d}{\to} \Omega^{k-1} \overset{d}{\to} \Omega^k \overset{d}{\to} \Omega^{k+1} \overset{d}{\to} \cdots \tag{4.7}$$

Of interest are the cohomology groups of this complex, $H^k(\Omega; d)$, defined by

$$H^k(\Omega; d) = \frac{\ker d : \Omega^{k-1} \to \Omega^k}{\operatorname{im} d : \Omega^k \to \Omega^{k+1}} . \tag{4.8}$$

By definition, the zeroth cohomology group is the quantization of the finite W-algebra. It is straightforward to verify that the zeroth cohomology is indeed a closed, associative algebra. To show that it is a quantization of the classical finite W-algebra is less straightforward, but can also be done. If the gauge group $\hat{H}$ generated by the first class constraints acts properly on the constrained phase space C, one expects that all higher cohomologies vanish, as they are generically related to singularities in the quotient $C/\hat{H}$. In particular, if χ is a one-dimensional character related to an sl_2 embedding, the higher cohomologies will vanish, but if it is zero, they will not. Later on we will see examples of this phenomenon. In the mathematics literature, the cohomology of the BRST complex is called the Hecke algebra $\mathcal{H}(\mathcal{G}, \mathcal{L}, \chi)$ associated to $(\mathcal{G}, \mathcal{L}, \chi)$. General Hecke algebras have not been computed, apart from those where χ is the one-dimensional character related to the principal sl_2 embedding in $\mathcal{G}$. In that case it was shown by Kostant [37] that the only non-vanishing cohomology is $H^0(\Omega; d)$ and that it is isomorphic to the center of the universal enveloping algebra. Recall that the center of the $\mathcal{U}\mathcal{G}$ is generated by the set of independent Casimirs of $\mathcal{G}$. This set is closely related to the generators of standard infinite W_n-algebras; in that case there is one W-field for each Casimir which form a highly non-trivial algebra [38]. We see that for finite W-algebras the same generators survive, but that they form a trivial abelian algebra.

Since in the cases where χ is a character related to an sl_2 embedding the BRST cohomology can be completely calculated, we will first restrict our attention to this situation. In particular, we will verify Kostants result, but also see that for non-principal sl_2 embeddings quantum finite W-algebras are non-trivial. Later we will come back to the more general situation, in particular to $\chi = 0$. Here, the general answer is not known, and related to some interesting and difficult open problems in mathematics. Therefore, we will restrict our attention to a few examples to sketch the general idea. But before that we first consider the cases related to sl_2 embeddings.

4.2. Quantum finite W-algebras from sl_2 embeddings

As said previously, we want to compute the cohomology of $(\Omega; d)$. Unfortunately, this is a difficult problem to approach directly. We therefore use a well known trick of cohomology theory, namely we split the complex $(\Omega; d)$ into a double complex and calculate the cohomology via a spectral sequence argument. We will just sketch the idea, a more detailed treatment can be found in [11]. Crucial is that the operator d can be decomposed into two anti-commuting pieces. Write $Q = Q_0 + Q_1$, with

$$Q_1 = J^\alpha c_\alpha - \tfrac{1}{2} f_\gamma^{\alpha\beta} b^\gamma c_\alpha c_\beta , \quad Q_0 = -\chi(J^\alpha) c_\alpha , \tag{4.9}$$

and define $d_0(X) = [Q_0, X]$, $d_1(X) = [Q_1, X]$. Then one can verify by explicit computation that $d_0^2 = d_0 d_1 = d_1 d_0 = d_1^2 = 0$. Associated to this decomposition is a bigrading of $\Omega = \bigoplus_{k,l} \Omega^{k,l}$ defined by

$$\deg(J^\alpha) = (-k, k) , \quad \text{if } t_a \in \mathcal{G}_k$$

$$\deg(c_\alpha) = (k, 1 - k) , \quad \text{if } t_\alpha \in \mathcal{G}_k \tag{4.10}$$

$$\deg(b^\alpha) = (-k, k - 1) , \quad \text{if } t_\alpha \in \mathcal{G}_k ,$$

with respect to which d_0 has degree $(1,0)$ and d_1 has degree $(0,1)$. Thus $(\Omega^{k,l}; d_0; d_1)$ has the structure of a double complex. Explicitly, the action of d_0 and d_1 is given by

$$d_1(J^a) = f_b^{\alpha a} J^b c_\alpha , \quad d_1(c_\alpha) = -\tfrac{1}{2} f_\alpha^{\beta\gamma} c_\beta c_\gamma , \quad d_1(b^\alpha) = J^\alpha + f_\gamma^{\alpha\beta} b^\gamma c_\beta ,$$

$$d_0(J^a) = d_0(c_\alpha) = 0 , \quad d_0(b^\alpha) = -\chi(J^\alpha) . \tag{4.11}$$

To simplify the algebra, it is advantageous to introduce

$$\hat{J}^a = J^a + f_\gamma^{a\beta} b^\gamma c_\beta . \tag{4.12}$$

Our motivation is to introduce these new elements $\hat{J}^a$ is twofold: first, similar expressions were encountered in a study of the effective action for W_3 gravity [39], where it turned out that the BRST cohomology for the infinite W_3 algebra case could conveniently be expressed in terms of $\hat{J}$'s; second, such expressions were introduced for the J^a's that live on the Cartan subalgebra of g in [40], and simplified their analysis considerably. In terms of $\hat{J}$ we have

$$d_1(\hat{J}^a) = f_\gamma^{\alpha a} \hat{J}^\gamma c_\alpha , \quad d_1(c_\alpha) = -\tfrac{1}{2} f_\alpha^{\beta\gamma} c_\beta c_\gamma , \quad d_1(b^\alpha) = \hat{J}^\alpha ,$$

$$d_0(\hat{J}^a) = -f_\gamma^{a\beta} \chi(J^\gamma) c_\beta , \quad d_0(c_\alpha) = 0 , \tag{4.13}$$

$$d_0(b^\alpha) = -\chi(J^\alpha) . \tag{4.14}$$

Now that we have a double complex, we can apply the techniques of spectral sequences [41] to it, in order to compute the cohomology of $(\Omega; d)$. The idea behind spectral sequences is to find a series of complexes $(\Omega_j; D_j)$, $j \geq 0$, such that $\Omega_0 = \Omega$, such that $\Omega_{j+1} = H^*(\Omega_j, D_j)$ and such that $\Omega_\infty = H^*(\Omega; d)$. In practice this is only a convenient technique if the spectral sequence degenerates at some point, which means that Ω_j does not change anymore for j larger than some j_0. If a complex is actually a double complex, like in our case, two natural spectral sequences exist. One of these has $D_0 = d_0$ and $D_1 = d_1$, the other one has $D_0 = d_1$ and $D_1 = d_0$.

The first people to propose using the theory of spectral sequences in the setting of W-algebras were Feigin and Frenkel [40]. In fact, in this way they computed the BRST cohomology in the infinite-dimensional case but only for the special example of the principal sl_2 embedding (which are known to lead to the W_N algebras). However, their calculation has the drawback that it is very difficult to generalize it to arbitrary embeddings and that it constructs the cohomology in an indirect way (via commutants of screening operators). Their spectral sequence was based on a double complex, within the notation introduced above, $D_0 = d_1$ and $D_1 = d_0$. In [29] it was first proposed to calculate the BRST cohomology using the other spectral sequence with $D_0 = d_0$ and $D_1 = d_1$. As it turns out this has drastic simplifying consequences for the calculation of the cohomology.

4.2.1. The BRST cohomology

The following theorem [29] gives the BRST cohomology on the level of vector spaces.

Theorem. *As before let $\mathscr{G}_{lw} \subset \mathscr{G}$ be the kernel of the map* $\mathrm{ad}_{t_-} : \mathscr{G} \to \mathscr{G}$. *Then the BRST cohomology is given by the following isomorphisms of vector spaces*

$$H^k(\Omega;d) \simeq (\mathscr{U}\mathscr{G}_{lw})\delta_{k,0} \ . \tag{4.15}$$

The computation of the BRST cohomology is simplified considerably due to the introduction of the new set of generators $\hat{J}^a$. The simplification arises due to the fact that $H^*(\Omega;d) \simeq H^*(\Omega_{\mathrm{red}};d)$, where Ω_{red} is the subalgebra of Ω generated by $\hat{J}^{\bar{\alpha}}$ and c_α. We will not prove this here, but note that it is nothing but the statement that the antighosts b^α impose the constraints. The subalgebra Ω_{red} is a quantum version of the constraint phase space we have seen previously. The reduced complex $(\Omega_{\mathrm{red}};d)$ is described by the following set of relations:

$$d_1(\hat{J}^{\bar{\alpha}}) = f_{\bar{\gamma}}^{\alpha\bar{\alpha}}\hat{J}^{\bar{\gamma}}c_\alpha \ , \quad d_1(c_\alpha) = -\tfrac{1}{2}f_\alpha^{\beta\gamma}c_\beta c_\gamma \ , \quad d_0(\hat{J}^{\bar{\alpha}}) = -f_\gamma^{\bar{\alpha}\beta}\chi(J^\gamma)c_\beta \ ,$$
$$d_0(c_\alpha) = 0 \ , \quad [\hat{J}^{\bar{\alpha}},\hat{J}^{\bar{\beta}}] = f_{\bar{\gamma}}^{\bar{\alpha}\bar{\beta}}\hat{J}^{\bar{\gamma}} \ , \quad [\hat{J}^{\bar{\alpha}},c_\beta] = -f_\beta^{\bar{\alpha}\gamma}c_\gamma \ , \quad [c_\alpha,c_\beta] = 0 \ . \tag{4.16}$$

At this stage we apply the spectral sequence technique. The inconvenient choice $D_0 = d_1$ and $D_1 = d_0$ would lead us to compute the cohomology of d_1 on Ω_{red}. This turns out to be very hard, and in addition the spectral sequence will not degenerate ever. In other words, we would have to compute cohomologies infinitely many times in order to find the final answer. The choice $D_0 = d_0$ and $D_1 = d_1$ is much more convenient, it turns out that the spectral sequence already degenerates after the first step, i.e. after taking the D_0 cohomology, so that $\Omega_\infty = \Omega_1$. Therefore, all that remains is to compute the d_0 cohomology of Ω_{red}. To get an idea of the d_0 cohomology, we rewrite $d_0(\hat{J}^{\bar{\alpha}})$ as

$$d_0(\hat{J}^{\bar{\alpha}}) = -\mathrm{Tr}([\chi(J^\gamma)t_\gamma,t^{\bar{\alpha}}]t^\beta c_\beta) = -\mathrm{Tr}([t_+,t^{\bar{\alpha}}]t^\beta c_\beta) \ . \tag{4.17}$$

From this it is clear that $d_0(\hat{J}^{\bar{\alpha}}) = 0$ for $t^{\bar{\alpha}} \in \mathscr{G}_{hw}$. Furthermore, since $t_{\bar{\alpha}} \in \mathscr{G}_0 \oplus \mathscr{G}_-$ and $\dim(\mathscr{G}_{lw}) = \dim(\mathscr{G}_0)$, it follows that for each β there is a linear combination $a(\beta)_{\bar{\alpha}}\hat{J}^{\bar{\alpha}}$ with $d_0(a(\beta)_{\bar{\alpha}}\hat{J}^{\bar{\alpha}}) = c_\beta$. This can be used to prove that

$$H^k(\Omega_{\mathrm{red}};d_1) = \bigotimes_{t_{\bar{\alpha}} \in \mathscr{G}_{lw}} \mathbb{C}[\hat{J}^{\bar{\alpha}}]\delta_{k,0} = (\mathscr{U}\mathscr{G}_{lw})\delta_{k,0} \ . \tag{4.18}$$

Because the spectral sequence stops here, the total cohomology of $(\Omega; d)$ is the same as (4.18), and this proves theorem. We see that the ghosts c_α have been 'used' to keep only the generators in $\mathcal{G}_{lw}$. This is the quantum counterpart of the gauge fixing procedure we saw previously.

As expected, there is only cohomology of degree zero, and furthermore, the elements of $\mathcal{G}_{lw}$ are in one-to-one correspondence with the components of $\mathcal{G}$ that made up the lowest weight gauge in Chapter 1. Therefore, $H^*(\Omega; d)$ really is a quantization of the finite W-algebra. What remains to be done is to compute the algebraic structure of $H^*(\Omega; d)$. What we have computed in the previous theorem is an isomorphism of vector spaces rather than an isomorphism of algebras. The only thing that (4.15) tells us is that the product of two elements a and b of bidegree $(-p, p)$ and $(-q, q)$ is given by the product structure of $\mathcal{U}\mathcal{G}_{lw}$, modulo terms of bidegree $(-r, r)$ with $r < p + q$. To find these lower terms we need explicit representatives of the generators of $H^0(\Omega; d)$ in Ω. Such representatives can be constructed using the so-called tic-tac-toe construction [41], another important ingredient of the theory of spectral sequences: take some $\phi_0 \in \mathcal{G}_{lw}$, of bidegree $(-p, p)$. Then $d_0(\phi_0)$ is of bidegree $(1 - p, p)$. Since $d_1 d_0(\phi_0) = -d_0 d_1(\phi_0) = 0$, and there is no d_1 co-homology of bidegree $(1 - p, p)$, $d_0(\phi_0) = d_1(\phi_1)$ for some ϕ_1 of bidegree $(1 - p, p - 1)$. Now repeat the same steps for ϕ_1, giving a ϕ_2 of bidegree $(2 - p, p - 2)$, such that $d_0(\phi_1) = d_1(\phi_2)$. Note that $d_1 d_0(\phi_1) = -d_0 d_1(\phi_1) = -d_0^2(\phi) = 0$. In this way, we find a sequence of elements ϕ_l of bidegree $(l - p, p - l)$. The process stops at $l = p$. Let

$$W(\phi) = \sum_{l=0}^{p} (-1)^l \phi_l. \tag{4.19}$$

Then $dW(\phi) = 0$, and $W(\phi)$ is a representative of ϕ_0 in $H^0(\Omega; d)$. The algebraic structure of $H^0(\Omega; d)$ is then determined by calculating the commutation relations of $W(\phi)$ in Ω, where ϕ_0 runs over a basis of $\mathcal{G}_{lw}$. This is the quantum finite W-algebra. The non-linearity comes from the fact that the ϕ_l are polynomials of order $l + 1$ in the hatted generators $\hat{J}$.

Let us now give an example of the construction described above.

4.2.2. Example

Consider again the non-principal sl_2 embedding into sl_3 associated to the following partition of the number 3: $3 = 2 + 1$. We constructed the classical W-algebra associated to this embedding earlier. We shall now quantize this Poisson algebra by the methods developed above. Take the following basis of sl_3:

$$r_a t_a = \begin{pmatrix} \dfrac{r_4}{6} - \dfrac{r_5}{2} & r_2 & r_1 \\[2mm] r_6 & -\dfrac{r_4}{3} & r_3 \\[2mm] r_8 & r_7 & \dfrac{r_4}{6} + \dfrac{r_5}{2} \end{pmatrix}. \tag{4.20}$$

Remember that (in the present notation) the sl_2 embedding is given by $t_+ = t_1$, $t_0 = -t_5$ and $t_- = t_8$. The nilpotent subalgebra $\mathcal{G}_+$ is spanned by $\{t_1, t_3\}$, $\mathcal{G}_0$ by $\{t_2, t_4, t_5, t_6\}$ and $\mathcal{G}_-$ by $\{t_7, t_8\}$. The d_1 cohomology of Ω_{red} is generated by $\{\hat{J}^4, \hat{J}^7, \hat{J}^6, \hat{J}^8\}$, and using the tic-tac-toe construction

one finds representatives for these generators in $H^0(\Omega_{\text{red}}; d)$:

$$W(\hat{J}^4) = \hat{J}^4 , \quad W(\hat{J}^6) = \hat{J}^6 , \quad W(\hat{J}^7) = \hat{J}^7 - \tfrac{1}{2}\hat{J}^2\hat{J}^5 - \tfrac{1}{2}\hat{J}^4\hat{J}^2 + \tfrac{1}{2}\hat{J}^2 ,$$

$$W(\hat{J}^8) = \hat{J}^8 + \tfrac{1}{4}\hat{J}^5\hat{J}^5 + \hat{J}^2\hat{J}^6 - \hat{J}^5 . \tag{4.21}$$

Let us illustrate the tic-tac-toe construction by working out the form of $W(\hat{J}^7)$ in somewhat more detail. Starting with $\hat{J}^7$, we find

$$d_1(\hat{J}^7) = \hat{J}^2 c_1 - \tfrac{1}{2}\hat{J}^4 c_3 - \tfrac{1}{2}\hat{J}^5 c_3 . \tag{4.22}$$

To find a ϕ_1 such that $d_0(\phi_1)$ equals this, one can either write down the most general form of ϕ_1 with arbitrary coefficients,

$$\phi_1 = a_1\hat{J}^2\hat{J}^2 + a_2\hat{J}^2\hat{J}^5 + a_3\hat{J}^5\hat{J}^5 + a_4\hat{J}^4\hat{J}^2 + a_5\hat{J}^2\hat{J}^6$$

$$+ a_6\hat{J}^5\hat{J}^4 + a_7\hat{J}^5\hat{J}^6 + a_8\hat{J}^2 + a_9\hat{J}^5 , \tag{4.23}$$

or make a more clever guess using the form of $d_0(\hat{J}^5) = 2c_1$ and $d_0(\hat{J}^2) = -c_3$. In our case, this immediately tells us that $a_1 = a_3 = a_5 = a_6 = a_7 = 0$, and that $a_2 = \tfrac{1}{2}$ and $a_4 = \tfrac{1}{2}$. The value of a_8 and a_9 have to be fixed by explicit computation, and one finds

$$\phi_1 = \tfrac{1}{2}\hat{J}^2\hat{J}^5 + \tfrac{1}{2}\hat{J}^4\hat{J}^2 - \tfrac{1}{2}\hat{J}^2 , \tag{4.24}$$

leading to the form of $W(\hat{J}^7)$ in (4.21).

Let us introduce another set of generators

$$C = -\tfrac{4}{3}W(\hat{J}^8) - \tfrac{1}{9}W(\hat{J}^4)W(\hat{J}^4) - 1 ,$$

$$j_0 = -\tfrac{2}{3}W(\hat{J}^4) + 1 , \quad j_+ = W(\hat{J}^7) , \quad j_- = \tfrac{4}{3}W(\hat{J}^6) . \tag{4.25}$$

The commutation relations between these generators are given by

$$[j_0, j_+] = 2j_+ , \quad [j_0, j_-] = -2j_- , \quad [j_+, j_-] = j_0^2 + C ,$$

$$[C, j_+] = [C, j_-] = [C, j_0] = 0 . \tag{4.26}$$

These are precisely the same as the relations for the finite $W_3^{(2)}$ algebra given in [10] and (2.26). Notice that in this case the quantum relations are identical to the classical ones. The explicit $\hbar$ dependence can be recovered simply by multiplying the right-hand sides of (4.26) by $\hbar$.

In the appendix we also discuss the explicit quantization of all the finite W-algebras that can be obtained from sl_4. There one does encounter certain quantum effects, i.e. the quantum relations will contain terms of order $\hbar^2$ or higher.

4.2.3. Quantum Miura transformation

In Section 3.2.4 we saw that there exists a realization of classical finite W-algebras in terms of the Kirillov–Poisson algebra $\mathcal{K}(\mathcal{G}_0)$. A similar Miura transformation can be constructed in the quantum case, using the explicit form of the generators as described at the end of Section 4.2.1. Denote by $W(\phi)^{0,0}$ the part of $W(\phi)$ of bidegree $(0, 0)$, so it is $(-1)^p\phi_p$ in the notation of (4.19). The quantum Miura transformation is simply given by the map $W(\phi) \to W(\phi)^{0,0}$. As was shown in [11], this map is an isomorphism of algebras. The generators $\hat{J}^{\bar{\alpha}}$, with $\bar{\alpha}$ restricted to $\mathcal{G}_0$, form an algebra that is isomorphic to $\mathcal{U}\mathcal{G}_0$, the universal enveloping algebra of $\mathcal{G}_0$. As we will see later, these

facts play an important role in the representation theory of finite W-algebras. Let us illustrate the quantum Miura transformation for the example we considered in the previous section. If we introduce $s = (\hat{J}^4 + 3\hat{J}^5)/4$, $h = (\hat{J}^5 - \hat{J}^4)/4$, $f = 2\hat{J}^6$ and $e = \hat{J}^2/2$, then h, e, f form an sl_2 Lie algebra with $[h, e] = e$, $[h, f] = -f$ and $[e, f] = 2h$ while s commutes with everything. The quantum Miura transformation one finds from (4.25) by restricting these expressions to their bidegree $(0, 0)$ part reads, in terms of s, e, f, h,

$$C = -\tfrac{4}{3}(h^2 + \tfrac{1}{2}ef + \tfrac{1}{2}fe) - \tfrac{4}{9}s^2 + \tfrac{4}{3}s - 1 \; ,$$
$$H = 2h - \tfrac{2}{3}s + 1 \; , \quad E = -2(s - h - 1)e \; , \quad F = \tfrac{2}{3}f \; .$$

(4.27)

4.3. Quantum finite W-algebras not from sl_2 embeddings

As promised, we now turn to the more difficult case of finite W-algebras that cannot be obtained from sl_2 embeddings. Important is the case where we take $\chi = 0$, which we will study first. In general these BRST cohomologies are very difficult to calculate. One exception to the rule is the case where one chooses $\mathscr{L}$ to be the Cartan subalgebra (we considered the classical counterpart of this case in the previous chapter). We explicitly quantize this algebra in the case of $sl(3)$.

4.3.1. $\chi = 0$

The BRST charge in this case reads

$$Q = J^x c_x - \tfrac{1}{2}f_{\bar{\gamma}}^{x\beta}b^\gamma c_x c_\beta \; .$$

(4.28)

The action of the differential d is given by

$$d(J^a) = f_b^{xa}J^b c_x \; , \quad d(c_x) = -\tfrac{1}{2}f_x^{\beta\gamma}c_\beta c_\gamma \; , \quad d(b^x) = J^x + f_{\bar{\gamma}}^{x\beta}b^\gamma c_\beta \; .$$

(4.29)

To simplify the algebra, it is again advantageous to introduce

$$\hat{J}^a = J^a + f_{\bar{\gamma}}^{a\beta}b^\gamma c_\beta \; .$$

(4.30)

In terms of $\hat{J}$ we have

$$d(\hat{J}^a) = f_{\bar{\gamma}}^{xa}\hat{J}^{\bar{\gamma}}c_x \; , \quad d(c_\alpha) = -\tfrac{1}{2}f_x^{\beta\gamma}c_\beta c_\gamma \; , \quad d(b^x) = \hat{J}^x \; .$$

(4.31)

Here $\bar{\gamma}$ runs over a basis of the orthocomplement $\mathscr{L}^\perp$ of $\mathscr{L}$, $\mathscr{G} = \mathscr{L}^\perp \oplus \mathscr{L}$. Notice that in contrast to the cases obtained from sl_2 embeddings, $\mathscr{L}^\perp$ need not be a closed Lie algebra itself. This makes life extra complicated.

Although we have made a simplification by going to the hatted generators $\hat{J}^a$, they satisfy more complicated commutation relations than the original generators J^a. They read

$$[\hat{J}^a, \hat{J}^b] = f_c^{ab}\hat{J}^c - b^x c_\beta(f_{\bar{\gamma}}^{\beta b}f_x^{\bar{\gamma}a} + f_{\bar{\gamma}}^{a\beta}f_x^{\bar{\gamma}b}) \; ,$$
$$[c_\alpha, \hat{J}^a] = f_\alpha^{a\beta}c_\beta \; , \quad [b^\beta, \hat{J}^a] = f_\alpha^{\beta a}b^x \; .$$

(4.32)

Let us define $\mathscr{F}^p$ as the vector space spanned by all products of at most p generators $\hat{J}^a$ and arbitrarily many ghosts and antighosts. Clearly, the differential d preserves this space. Furthermore, the relations (4.32) show that $\mathscr{F}^p\mathscr{F}^q \subset \mathscr{F}^{p+q}$ and that the induced algebraic structure on the quotient $\mathscr{F}^p/\mathscr{F}^{p-1}$ is precisely that of a set of commuting variables $\hat{J}^a$ and anti-commuting

variables b^α, c_α. These are just the classical relations between the variables. The subspaces $\mathscr{F}^p$ define a so-called filtration of Ω, and associated to such a filtration is another spectral sequence [42]. What we have shown here is that the computation of the first term in this particular sequence is equivalent to the computation of the cohomology of d assuming classical commutation relations between the $\hat{J}^a$ and the ghosts. It would be great if the spectral sequence would degenerate after this first step, but although we expect that this happens in many cases, we do not have a general proof. The problem is that certain higher cohomologies will survive, and to show that the spectral sequence degenerates one has to do an explicit computation of all the higher operators D_i. To avoid this problem we will from now on assume that the spectral sequence indeed degenerates after the first step, but keep in mind that in all explicit examples this has to be verified explicitly.

The problem has now basically been reduced to a 'classical cohomology' computation. The following observation is that we can (like in the sl_2 case) go to a reduced vector space consisting of polynomials in c_α and $\hat{J}^{\bar{\gamma}}$ (using the Künneth theorem, see also [29]). Let $V_{\mathscr{L}^\perp}$ denote the commutative algebra freely generated by the $\hat{J}^{\bar{\gamma}}$. $V_{\mathscr{L}^\perp}$ is nothing but the constrained phase space we have seen previously. We want to compute the cohomology of d on $V_{\mathscr{L}^\perp} \otimes \wedge \mathscr{L}^*$, the last term $\wedge \mathscr{L}^*$ representing the ghosts c_α. This particular cohomology problem is a well-known one in mathematics. Since $\mathscr{L}^\perp$ is an $\mathscr{L}^*$ module in the obvious way, so is $V_{\mathscr{L}^\perp}$, and the cohomologies we are computing are nothing but the Lie algebra cohomologies $H^n(\mathscr{L}^*, V_{\mathscr{L}^\perp})$. Alternatively, they can be described in terms of the Ext groups $\mathrm{Ext}^n_{\mathscr{U}\mathscr{L}^*}(\mathbb{C}, V_{\mathscr{L}^\perp})$, which might be computable if we had a resolution of $V_{\mathscr{L}^\perp}$ in terms of injective $\mathscr{U}\mathscr{L}^*$ modules, but we do not know how to write down such a resolution. In any case, if the spectral sequence degenerates after one step and we would know these Lie algebra cohomologies, to obtain the final full result for the original problem still requires an additional amount of work. First, we have to use a tic-tac-toe type of construction to find the explicit generators, and secondly, we have to compute their commutation relations. In the latter we have to be careful to identify terms which differ by a BRST exact expression. Altogether the result of this procedure will be that the BRST cohomology is a quantum deformation of the classical Lie algebra cohomology, granted that the spectral sequence degenerates after the first step.

Extra subtleties further arise if one wants to write down a generating basis for the quantum finite W-algebra. The number of generators may be much smaller than the number of generators of the corresponding classical Poisson algebra. This is due to the fact that the Poisson algebra is commutative, and the quantum algebra is not. Take for example $\mathscr{G} = sl_2$, and $\mathscr{L} = 0$. In this trivial case the classical algebra is the Kirillov–Poisson algebra based on sl_2, with generators J^-, J^0, J^+. The quantum algebra is the quantum-Kirillov algebra (4.3), but this one is generated by just J^+ and J^-, by virtue of the relation $J^0 = 1/\hbar [J^+, J^-]$. This relation becomes singular in the limit $\hbar \to 0$, explaining the apparent discrepancy between the two. Similar phenomena take place in conformal field theory, see [43].

Since the zeroth cohomology $H^0(\mathscr{L}^*, V_{\mathscr{L}^\perp})$ is simply the space $(V_{\mathscr{L}^\perp})^{\mathscr{L}^*}$ of $\mathscr{L}^*$ invariants of $V_{\mathscr{L}^\perp}$, and the quantum W-algebra is by definition the zeroth cohomology, we find that the quantum finite W algebra, assuming the spectral sequence degenerates, is as a vector space isomorphic to $(V_{\mathscr{L}^\perp})^{\mathscr{L}^*}$. To give a basis of this space with its classical multiplication rule is a problem of the so-called invariant theory. The answer to this question is known only for special $V_{\mathscr{L}^\perp}$ and $\mathscr{L}^*$. Again, this resembles closely the situation in conformal field theory as explained in [43]. Thus, even classically we cannot in general give a minimal set of generators of the algebra $(\mathscr{G}, \mathscr{L}, 0)$.

To illustrate these rather abstract statements we will now give a few examples.

4.3.2. The algebra $(\mathcal{G}, \mathcal{G}^+, 0)$

Suppose that we have chosen $\mathcal{L}$ as in the case of an sl_2 embedding, but with $\chi = 0$. This problem is interesting for a variety of reasons. Firstly, in the case of the principal sl_2 embedding it is an interesting open problem in mathematics. And secondly, it is the first term in the 'difficult' spectral sequence that we encountered before in the case where χ was a one-dimensional character, and might shed some light on the W-algebras obtained from sl_2 embeddings. More generally, we can look at any $\mathcal{G}^+$ which is the positive eigenspace of some grading element δ of the Cartan subalgebra. The nice feature of this case is that $\mathcal{L}^\perp = \mathcal{G}^- \oplus \mathcal{G}^0$ is a closed subalgebra. That implies that we can rigorously reduce the cohomology problem to one for the subalgebra generated by $\hat{J}^\gamma$ and the ghosts c_χ. The classical zeroth cohomology can easily be worked out, it is just the Kirillov–Poisson algebra based on $\mathcal{G}^0$. According to the spectral sequence argument above, the quantum algebra can as a vector space be at most isomorphic to this. On the other hand, by explicit computation one verifies that the quantum-Kirillov algebra based on $\mathcal{G}^0$ is contained in the quantum algebra. Combining these two facts proves

$$H^0(\mathcal{G}, \mathcal{G}^+, 0) = \mathcal{U}\mathcal{G}^0 \ . \tag{4.33}$$

The really difficult problem is to compute the higher cohomologies here. For this, we only know the answer for sl_2, if $\mathcal{G}^+$ is generated by J^+. Then

$$H^1(sl_2, \mathcal{G}^+, 0) = c\mathbb{C}[J^-] \ . \tag{4.34}$$

4.3.3. The algebra $(\mathcal{G}, \mathcal{T}, 0)$

In this example we choose $\mathcal{L} = \mathcal{T}$ to be the maximal torus of $\mathcal{G}$. Now, $\mathcal{L}^\perp$ is no longer a closed subalgebra. However, we are in the fortunate circumstances that $\hat{J}^a = J^a$. This means in particular that the quantum algebra is generated by expressions

$$J_{\alpha_1} J_{\alpha_2} \ldots J_{\alpha_n} \ , \tag{4.35}$$

where J_{α_i} is the generator J^γ associated to the (positive or negative) root α_i, and $\sum \alpha_i = 0$. To find a minimal basis is a separate problem. For example, for sl_3 a set of generators is

$$
\begin{aligned}
A_1 &= J_{\alpha_1} J_{-\alpha_1} \ , \quad A_2 = J_{\alpha_2} J_{-\alpha_2} \ , \quad A_3 = J_{\alpha_1 + \alpha_2} J_{-\alpha_1 - \alpha_2} \ , \\
B &= \tfrac{1}{2}(J_{-\alpha_1} J_{-\alpha_2} J_{\alpha_1 + \alpha_2} - J_{\alpha_1} J_{\alpha_2} J_{-\alpha_1 - \alpha_2}) \ , \\
C &= \tfrac{1}{2}(J_{-\alpha_1} J_{-\alpha_2} J_{\alpha_1 + \alpha_2} + J_{\alpha_1} J_{\alpha_2} J_{-\alpha_1 - \alpha_2}) \ .
\end{aligned} \tag{4.36}
$$

The BRST charge in this case is

$$Q = c_0 H_0 + c_1 H_1 \ , \tag{4.37}$$

where $H_0 = [J_{\alpha_1}, J_{-\alpha_1}]$ and $H_1 = [J_{\alpha_2}, J_{-\alpha_2}]$. This is an example where the commutators do not close in themselves, but only modulo BRST exact expressions. For example, when working out $[A_2, C]$ one encounters the expression $[J_{\alpha_2}, J_{-\alpha_2}]$, which lives in the Cartan subalgebra. Elements of the Cartan subalgebra can be moved to either the left or the right of an expression. Only if they stand to the left or right will the resulting expression be BRST exact (it is the d of an anti-ghost times something) and can be put equal to zero. In this way we find $[A_2, C]$ is purely BRST exact, so

the commutator vanishes. The final result for the quantum algebra can be conveniently expressed as follows. Introduce

$$\tilde{B} = B + \tfrac{1}{2}\hbar(A_1 - A_2 + A_3) \, .$$ (4.38)

Then the brackets are

$$[A_i, A_{i+1}] = 2\hbar\tilde{B} \, , \quad [A_i, \tilde{B}] = \hbar(A_{i+1}A_i - A_i A_{i-1}) \, ,$$ (4.39)

which is indeed a quantum deformation of the classical result. The generators satisfy the following relation

$$C^2 - \tilde{B}^2 = \tfrac{1}{2}(A_1 A_2 A_3 + A_3 A_2 A_1) + \tfrac{1}{4}\hbar(A_1 - A_2 + A_3)^2 \, .$$ (4.40)

It is amusing to note that in this particular case with $\mathscr{L}$ equal to a maximal torus there is an exact result for the zeroth cohomology,

$$H^0(\mathscr{G}, \mathscr{T}, 0) = (\mathscr{U}\mathscr{G})^{\mathscr{T}}/\mathscr{T} \, .$$ (4.41)

In the right hand side, we first perform a gauge fixing by looking at the centralizer of $\mathscr{T}$. In this resulting algebra, $\mathscr{T}$ generates a left ideal and we can divide out by this ideal, corresponding to imposing the constraints. Thus we have interchanged the usual order of first imposing constraints and then gauge fixing.

4.3.4. Further finite W-algebras

The last case we have not considered so far is when χ is some representation of $\mathscr{L}$ in a non-trivial algebra. An example is to take $\mathscr{G} = sl_3$, $\mathscr{L}$ equal to the upper triangular matrices, and a representation of χ in the one-dimensional Heisenberg algebra generated by p and q with $[p,q] = 1$. Explicitly, one can take $\chi(J_{\alpha_1}) = p$, $\chi(J_{\alpha_2}) = q$, and $\chi(J_{\alpha_1 + \alpha_2}) = 1$. In this case one presumably recovers $W_3^{(2)}$, the p and q corresponding to a set of auxiliary variables that can also be introduced in the infinite-dimensional case [44]. In fact, from the infinite-dimensional case [45] we know that the analysis for higher dimensional representations χ may become more complicated in that the spectral sequence does not degenerate after the first step any more. For certain higher dimensional χ that are inspired by sl_2 embeddings, one can nevertheless still compute the cohomology exactly. For all other type of representations χ, little is known, and this remains a region consisting of a vast number of new unexplored finite W-algebras. If $\mathscr{U}_\chi$ is isomorphic to a copy of $\mathscr{U}\mathscr{L}$, and χ is the isomorphism between the two, then the cohomology reduces essentially to $\mathscr{U}\mathscr{G}^{\mathscr{L}}$, the space of $\mathscr{L}$ invariants in $\mathscr{G}$. For some cases, this space of invariants has been studied in [46]. In addition, we want to mention that there exists an alternative method to quantize W-algebras [31]. Using this method one finds a faithful $*$-representation of the maximally non-compact real form (see next chapter) of a given finite W-algebra.

For applications to physics we are interested in the representation theory of quantum finite W-algebras. This is the subject of the next chapter.

5. The representation theory

We now turn to the representation theory for finite W-algebras. Physically, the representations that are most interesting are the unitary irreducible representations on Hilbert spaces. However, as

finite W-algebras are non-linear it is not completely obvious what unitary representations are. Usully, a representation of a group is called unitary if all group elements are represented by unitary operators on some Hilbert space. For finite W-algebras there is no such thing as an abstract group which one can get by exponentiating the algebra. Nevertheless, it is possible to define the concept of unitarity for finite W-algebras. The reason for this is that if the generators of a finite W-algebra are represented by (finite) matrices one can exponentiate them without runing into trouble because $\exp(A)$ converges for any matrix A. Therefore, even though there is no formal grouplike object associated to a finite W-algebra we *can* exponentiate its elements in any given representation. Now, not every element of a complex finite W-algebra can exponentiate to a unitary matrix, as we will explain in a moment (this is also true for the Lie algebras). It is therefore necessary to consider real forms of finite W-algebras before one addresses the problem of constructing unitary repesentations.

Deriving finite W-algebras from compact (real) Lie algebras such as $su(n)$ is complicated by the fact that these Lie algebras do not admit Gaussian decompositions. The question therefore arises whether the constructions developed up to now are physically interesting at all, as non-linear algebras involving compact Lie algebras seem not to be included. In fact they *are* included. The point is that one should consider the Lie algebras discussed above as *complex* Lie algebras, that is $sl(n)$ means $sl(n; \mathbb{C})$. Applying the reduction procedures one obtains complex finite W-algebras. Only then should questions about compactness and unitarity be addressed by studying the real forms of the complex W-algebra. As we shall see, a given finite W-algebra admits many real forms not all of which admit unitary representations.

The next problem we address is defining the concept of a highest weight representation. For this one needs a Poincaré–Birkhoff–Witt (PBW) like theorem for finite W-algebras. Using this we discuss some conjectures for the Kac determinants and character formulas for finite W highest weight modules. These conjectures were proposed in [12], although a discussion of the PBW theorem, of real forms of finite W-algebras and some other details are lacking in [12].

5.1. Real forms and unitary representations

Let $\mathscr{W}^{\mathbb{C}}$ denote a complex finite W-algebra (where we explicitly specified the fact that the algebra is a module over the complex numbers) obtained from a complex Lie algebra by the construction discussed in the previous sections. A representation $\rho : \mathscr{W}^{\mathbb{C}} \to \operatorname{lin}(\mathscr{H})$, where $(\mathscr{H}, \langle ., . \rangle)$ is a (complex Hilbert space), can never be such that $\exp(\rho(W))$ is a unitary operator on $\mathscr{H}$ for all $W \in \mathscr{W}$. One can see this as follows. For $\exp(\rho(W))$ to be unitary, we must have

$$\exp(\rho(W))^{\dagger} = \exp(\rho(W))^{-1} , \qquad (5.1)$$

from which follows

$$\rho(W)^{\dagger} = -\rho(W) . \qquad (5.2)$$

Suppose (5.2) holds for $W \in \mathscr{W}$. Now consider $W' = iW$. We have $\rho(W')^{\dagger} = (i\rho(W))^{\dagger} = -i\rho(W)^{\dagger} = \rho(W')$. From this it follows that $\exp(\rho(W'))$ is *not* unitary. We see that W and iW can never simultaneously give rise to a unitary operator.

What one needs to do is go to 'real forms' of $\mathscr{W}^{\mathbb{C}}$. It is clear from the argument above, that if $\exp(\rho(W))$ is unitary, then $\exp(\rho(\alpha W))$ is also unitary, *provided α is real*. Real subspaces of

$\mathscr{W}$ therefore have a chance of being represented unitarily. These real vector spaces should, however, be invariant with respect to taking commutators. This requirement leads one to consider the so-called anti-involutions of order 2 of $\mathscr{W}$.

A two-anti-involution of a complex finite W-algebra $\mathscr{W}^{\mathbb{C}}$ is a map $\omega: \mathscr{W}^{\mathbb{C}} \to \mathscr{W}^{\mathbb{C}}$ such that

(1) $\omega^2 = 1$,

(2) $\omega(\alpha w_1 + \beta w_2) = \bar{\alpha}\omega(w_1) + \bar{\beta}\omega(w_2)$,   (5.3)

(3) $\omega(w_1 w_2) = \omega(w_2)\omega(w_1)$,

where α and β are complex numbers and $w_1, w_2 \in \mathscr{W}^{\mathbb{C}}$. From (1) it follows that ω has two eigenvalues: ± 1. Consider the negative eigenspace of ω

$$\mathscr{W}^{\mathbb{C}}_- = \{W \in \mathscr{W} \mid \omega(W) = -W\} .$$ (5.4)

This space is actually a real subspace of $\mathscr{W}^{\mathbb{C}}$, as one can see as follows: let $W \in \mathscr{W}$ and $\alpha \in \mathbb{C}$, then $\omega(\alpha W) = -\bar{\alpha}W$. If αW is to be an element of $\mathscr{W}^{\mathbb{C}}_-$, we must have $\bar{\alpha} = \alpha$. We conclude that αW is only an element of $\mathscr{W}^{\mathbb{C}}_-$ if α is real.

$\mathscr{W}^{\mathbb{C}}_-$ is also closed under commutation, because

$$\omega([w_1, w_2]) \equiv [\omega(w_2), \omega(w_1)]$$ (5.5)

by property (3). Therefore, if $w_1, w_2 \in \mathscr{W}^{\mathbb{C}}_-$, we find

$$\omega([w_1, w_2]) = -[w_1, w_2] ,$$ (5.6)

which means that $[w_1, w_2]$ is an element of $\mathscr{W}^{\mathbb{C}}_-$ if w_1 and w_2 are. $\mathscr{W}^{\mathbb{C}}_-$ is therefore actually a subalgebra of $\mathscr{W}^{\mathbb{C}}$. We have thus found a closed real subalgebra of $\mathscr{W}^{\mathbb{C}}$ that stands a chance of admitting unitary representations. From now on we denote the space $\mathscr{W}^{\mathbb{C}}_-$ by $\mathscr{W}^{\mathbb{R}}_\omega$, where we have made the ω dependence explicit and in analogy with the Lie algebra case we call it a 'real form' of $\mathscr{W}^{\mathbb{C}}$.

A unitary representation of the real form $\mathscr{W}^{\mathbb{R}}_\omega$ is now defined as a representation of $\mathscr{W}^{\mathbb{R}}_\omega$ in some (complex) Hilbert space $\mathscr{H}$ (with inner product $\langle .,. \rangle$), such that for all elements $W \in \mathscr{W}^{\mathbb{R}}_\omega$

$$\exp(\rho(W))^\dagger = \exp(\rho(W))^{-1} ,$$ (5.7)

or, equivalently,

$$\rho(W)^\dagger = \rho(\omega(W)) \equiv -\rho(W) .$$ (5.8)

In general, not all real forms of a given $\mathscr{W}^{\mathbb{C}}$ algebra will admit unitary representations and in this sense the situation is similar to that in group theory.

Let us look at some examples. Consider again the (complex) finite W-algebra

$$[j_0, j_\pm] = \pm 2j_\pm , \quad [j_+, j_-] = j_0^2 + C .$$ (5.9)

This algebra has two independent anti-involutions ω_1 and ω_2, given by

$$\omega_1(j_0) = j_0 , \quad \omega_2(j_0) = -j_0 ,$$

$$\omega_1(j_\pm) = j_\mp , \quad \omega_2(j_\pm) = \pm j_\pm ,$$ (5.10)

$$\omega_1(C) = C , \quad \omega_2(C) = C .$$

The negative eigenvalue eigenspace of ω_1 is spanned by

$$\tfrac{1}{2}(-i)^{n-1}(\sigma_{k_0}\dots\sigma_{k_n}-\sigma_{k_n}\dots\sigma_{k_0}) \tag{5.11}$$

and

$$\tfrac{1}{2}(-i)^n(\sigma_{k_0}\dots\sigma_{k_n}+\sigma_{k_n}\dots\sigma_{k_0})\,, \tag{5.12}$$

for $0 \le p \le n k_p \in \{1,\dots,4\}$, and $n = 0, 1, 2\dots$, where

$$\sigma_1 = ij_0\,, \quad \sigma_2 = j_+ - j_-\,, \quad \sigma_3 = i(j_+ + j_-)\,, \quad \sigma_4 = iC\,. \tag{5.13}$$

The non-zero commutation relations between the generators σ_k read

$$[\sigma_1,\sigma_2] = 2\sigma_3\,, \quad [\sigma_2,\sigma_3] = 2\sigma_4 - 2i\sigma_1^2\,, \quad [\sigma_3,\sigma_1] = 2\sigma_2\,. \tag{5.14}$$

This is the real form of (5.9) with respect to ω_1. The unitary irreducible representations of this algebra were constructed in [10]. The two-dimensional representations read

$$\sigma_1 = \begin{pmatrix} i(x+1) & 0 \\ 0 & i(x-1) \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & \sqrt{2x} \\ -\sqrt{2x} & 0 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 0 & i\sqrt{2x} \\ i\sqrt{2x} & 0 \end{pmatrix}, \qquad \sigma_4 = \begin{pmatrix} -i(1+x^2) & 0 \\ 0 & -i(1-x^2) \end{pmatrix}, \tag{5.15}$$

where $x > 0$ is real. Note that $\exp(\alpha_a \sigma_a)$ is unitary for all real numbers α_a.

Now consider ω_2. In this case, the negative eigenvalue eigenspace is spanned by

$$\tfrac{1}{2}(-i)^{n-1}(S_{k_0}\dots S_{k_n}-S_{k_n}\dots S_{k_0}) \tag{5.16}$$

and

$$\tfrac{1}{2}(-i)^n(S_{k_0}\dots S_{k_n}+S_{k_n}\dots S_{k_0})\,, \tag{5.17}$$

with for $0 \le p \le n k_p \in \{3, +, -, 0\}$, and $n = 0, 1, 2\dots$, where

$$S_3 = j_0\,, \quad S_+ = ij_+\,, \quad S_- = j_-\,, \quad S_0 = iC\,. \tag{5.18}$$

The non-zero commutation relations between the generators read

$$[S_3,S_\pm] = \pm 2S_\pm\,, \quad [S_+,S_-] = iS_3^2 + S_0\,. \tag{5.19}$$

This real form cannot have a non-trivial finite-dimensional unitary representation. In order to see this, assume that v_0 is an eigenvector of S_3 (where we omit explicit reference to the representation ρ). As $S_3^\dagger = -S_3$ in a unitary representation, all eigenvalues of S_3 are purely imaginary, i.e. $S_3 v_0 = (i\lambda)v_0$, where $\lambda \in \mathbb{R}$. Now, consider the vector $S_\pm v_0$. It is also an eigenvector of S_3, as follows from

$$S_3 S_\pm v_0 = (S_\pm S_3 \pm 2S_\pm)v_0 = (i\lambda \pm 2)S_\pm v_0\,, \tag{5.20}$$

but with eigenvalue $i\lambda + 2$. This contradicts the fact that S_3 only has purely imaginary eigenvalues, therefore $S_\pm v_0$ must be zero. This means that the representation is at most one-dimensional.

5.1.1. Real forms of general finite W-algebras

In the previous section we have demonstrated that real forms of a complex finite W-algebra can be obtained from anti-involutions ω. Since our construction of finite W-algebras is based on the computation of a certain BRST cohomology, the question arises whether real forms of the Lie algebra lying above can be used to construct ones of the finite W-algebra. In the example we just considered, one can, however, easily convince oneself that none of the two anti-involutions comes from any of the three real forms of sl_3 (corresponding to $sl(3, \mathbb{R})$, $su(3)$ and $su(1,2)$). To understand what causes this, notice that if the BRST operator is given by $d(X) = [Q, X]$, then an anti-involution ω of $\mathscr{G}$ descends to an anti-involution of $H^*(d)$ iff $H^*(d) = H^*(\omega(d))$, where $\omega(d)(X) = [\omega(Q), X]$. In particular, a sufficient condition for this to happen is that $\omega(Q)$ is proportional to Q. Put differently, an anti-involution on $\mathscr{G}$ gives rise to one on the finite $\mathscr{W}$ algebra $(\mathscr{G}, \mathscr{L}, \chi)$, if $\omega(\mathscr{L}) = \mathscr{L}$ and $[\omega, \chi] = 0$. For example, this shows that the anti-involution $\omega(X) = -X$ of $\mathscr{G}$ always gives rise to an anti-involution of $(\mathscr{G}, \mathscr{H}, 0)$.[3] It is unknown whether or not any other anti-involutions of these algebras exist, and since we will not discuss their representation theory in detail, we will, for the remainder of this section, focus on those W-algebras that can be obtained from sl_2 embeddings. Let us denote the generators of sl_2 by $\{t_-, t_0, t_+\}$, and the corresponding images in $\mathscr{G}$ by J^-, J^0, J^+. Furthermore, we will assume that the one-dimensional representation χ is given by $\chi(J^+) = \xi$, where now ξ is an arbitrary complex number. The Lie algebra $\mathscr{G}$ can be decomposed in terms of the half-integral eigenvalues of $\mathrm{ad}(J^0)$ as $\mathscr{G} = \bigoplus_n \mathscr{G}^{(n)}$ (see also (3.30)). Naively, one would think that the choice of ξ is irrelevant, since there is an automorphism of $\mathscr{G}$ where every element of $\mathscr{G}^{(n)}$ is rescaled by λ^n for some non-zero complex number λ, and one can use this to put $\xi = 1$. However, this rescaling does not scale the generators in terms of the anti-commuting variables b^α, c_α and the J^α contains terms with different $\mathrm{ad}(J^0)$ eigenvalues. Therefore, varying ξ gives a one-parameter family of finite W-algebras that are not necessarily isomorphic. An interesting situation arises when ξ is purely imaginary. In that case, the anti-involution $\omega(X) = -X$ of $\mathscr{G}$ satisfies $[\omega, \chi] = 0$ and gives rise to an anti-involution on the corresponding finite W-algebra. In particular, the second anti-involution ω_2 in (5.10) can be obtained in this way, since for the embedding that describes the algebra $W_3^{(2)}$ it turns out that the resulting algebras are isomorphic for all ξ. More explicitly, for arbitrary ξ the algebra (5.9) becomes

$$[j_0, j_\pm] = \pm 2j_\pm , \quad [j_+, j_-] = \xi^{-1}j_0^2 + C . \tag{5.21}$$

One sees that indeed for purely imaginary ξ an anti-involution is obtained by sending each generator X to $-X$. A subsequent basis transformation $C \to \xi^{-1}C$ and $j_+ \to \xi^{-1}j_+$ yields exactly the anti-involution ω_2 in (5.10). Notice that taking ξ imaginary brings one automatically in the basis described in Eqs. (5.18) and (5.19).

In general, we will call the finite W-algebras that one gets for imaginary ξ, with their corresponding anti-involution, the maximal non-compact real form of $(\mathscr{G}, \mathscr{L}, \chi)$. In analogy with the corresponding situation for Lie algebras, these algebras do not have any interesting finite-dimensional unitary representations. The argument is the same as the one we presented for the case of $W_3^{(2)}$ at the end of the previous section.

---

[3] We assume here and in the sequel that a basis of $\mathscr{G}$ with real structure constants has been chosen, typically the one corresponding to the maximal non-compact real form of $\mathscr{G}$, which for $\mathscr{G} = sl(n, \mathbb{C})$ is $sl(n, \mathbb{R})$.

To have a more interesting representation theory, we would like to have the analogue of the compact real form for finite W-algebras, like the anti-involution ω_1 in (5.10). The construction of this real form is somewhat more involved, since it cannot be induced from one on $\mathcal{G}$. Let us denote the parabolic subalgebra $\bigoplus_{n \leq 0} \mathcal{G}^{(n)}$ of $\mathcal{G}$ by $\mathcal{P}$, the centralizer of sl_2 in $\mathcal{G}$ by $\mathcal{K}$, the restriction of the Cartan subalgebra to $\mathcal{K}$ by $\mathcal{H}_\mathcal{K}$ and the orthocomplement of $\mathcal{H}_K$ in $\mathcal{H}$ by $\mathcal{H}_S$ (so $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_K$). The centralizer of $\mathcal{K}$ and $\mathcal{G}$ is a direct sum of some subalgebra $\mathcal{G}_S$ (with Cartan subalgebra $\mathcal{H}_S$) and the $u(1)$ factors of $\mathcal{K}$. This algebra $\mathcal{G}_S$ plays an important role in the discussion of the representation theory [12]. In most cases, $\mathcal{G}_S$ is the subalgebra of $\mathcal{G}$ in which the sl_2 is principally embedded. $\mathcal{K}$ is precisely the semi-simple subalgebra of the finite W-algebra described in Section 3.2.7.

In Section 4.2.1 we saw that the computation of the BRST cohomology could be reduced to one for a reduced complex, where only the J^a which are not in $\mathcal{L}$ appear. If there are no degree $\frac{1}{2}$ subspaces in $\mathcal{G}$, so that $\mathcal{G}^{\pm(1/2)} = 0$ and the grading provided by t_0 is integral, then exactly the J^a that correspond to the parabolic algebra $\mathcal{P}$ appear. If the degree $\frac{1}{2}$ subspaces are non-zero, then there exists an other, equivalent formulation of the BRST cohomology involving auxiliary fields, and one can go to a modified reduced complex, where still only generators of $\mathcal{P}$ appear. Rather than trying to reduce an anti-involution of the whole Lie algebra $\mathcal{G}$, we can also try to start with one of the reduced complex and reduce that to the finite W-algebra. It is straightforward to analyze what conditions such an anti-involution has to satisfy and one finds that any anti-involution ω of the parabolic algebra $\mathcal{P}$ gives rise to one of the finite W-algebra $(\mathcal{G}, \mathcal{L}, \chi)$ if $\omega(J^-) = (\xi/\bar{\xi})J^-$.

To prove the existence of anti-involutions is not so easy in general. For example, the existence of the Cartan involution for ordinary Lie algebras becomes only transparent in certain distinguished bases of the Lie algebra, like the Cartan–Weyl basis. We do not have similar distinguished bases of $\mathcal{P}$ at our disposal, so that we can only conjecture the following.

There exists an anti-involution on $\mathcal{P}$ which restricts to the Cartan involution on $\mathcal{K}$ and to minus the identity on $\mathcal{H}_S$. We will call the corresponding real form of the finite W-algebra the compact real form.

In all known examples such an anti-involution exists. For example, in the example in the previous section the anti-involution of $\mathcal{P}$ corresponding to the anti-involution ω_1 in (5.10) is

$$\omega_1 \begin{pmatrix} a-b & 0 & 0 \\ c & -2a & 0 \\ e & d & a+b \end{pmatrix} = \begin{pmatrix} a+b & 0 & 0 \\ d & -2a & 0 \\ e & c & a-b \end{pmatrix}. \tag{5.22}$$

Another example is the finite W-algebra corresponding to the sl_2 embedding in sl_4 under which the fundamental representation of sl_4 decomposes as $2 \oplus 2$. In that case

$$\omega \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \begin{pmatrix} B^t & 0 \\ C^t & A^t \end{pmatrix}, \tag{5.23}$$

where A, B, C are two by two matrices.

It is an open problem to classify all possible real forms of finite W-algebras, and to prove the existence of a compact real form in general.

5.2. Highest weight representations

In this section we work out the construction of highest weight representations of finite W-algebras, and the conjectured form of the Kac determinant and the Kazhdan–Lusztig conjecture for such representations [12]. The latter relates the characters of highest weight representations to those of irreducible representations.

The definition of highest weight representations of Lie algebras uses the decomposition of the Lie algebra into generators corresponding to the positive roots, the negative roots and the Cartan subalgebra. A highest weight module is built by acting on a particular vector with arbitrary negative root generators. This vector is by definition annihilated by all the positive root generators, and has specific eigenvalues with respect to the generators of the Cartan subalgebra. If the Hamiltonian of a physical system would be part of the Cartan subalgebra, a physical motivation to look at such representations would be that their energy is bounded from below.

To imitate this construction, we need a similar type of decomposition of the finite W-algebra. The part of the Cartan subalgebra which survives in the finite W-algebra is precisely $\mathcal{H}_{\mathcal{N}}$, and every generator of the finite W-algebra has well-defined eigenvalues with respect to the adjoint action of the generators of $\mathcal{H}_{\mathcal{N}}$. Thus we can associate to every generator of the W-algebra a root in the root space $\mathcal{H}_{\mathcal{N}}^{*}$, which is the dual of $\mathcal{H}_{\mathcal{N}}$. To find out what are the positive and negative roots, we assume there is an automorphism of the Lie algebra $\mathcal{G}$ which maps $\mathcal{G}_{S}$ into a subalgebra generated by $\{E_{\pm \alpha_{i}}, H_{\alpha_{i}}\}_{i \in S}$, where S is a subset of the set of simple roots of $\mathcal{G}$. From now on we will assume we are in a basis of $\mathcal{G}$ where this is the case. Any root α of $\mathcal{G}$ can be orthogonally projected onto an element (called $\pi(\alpha)$) of the root space $\mathcal{H}_{\mathcal{N}}^{*}$. The image of the positive roots of $\mathcal{G}$ under this projection (except those which project to zero) is what we will call the positive roots of $\mathcal{H}_{\mathcal{N}}$. This then defines the following decomposition of the generators of the finite W-algebra

$$\mathcal{W} = \mathcal{W}_{-} \otimes \mathcal{W}_{0} \otimes \mathcal{W}_{+} . \tag{5.24}$$

The same decomposition can be written down not just for the generators, but also for the universal enveloping algebra of the W-algebra. Typically, $\mathcal{W}_{0}$ contains $\mathcal{H}_{\mathcal{N}}$, a certain number of central elements that commute with everything and do not give new roots (one of such generators is the remnant of the quadratic Casimir of the original Lie algebra), and other elements whose adjoint action cannot be diagonalized.

The properties of such a root-space type decomposition of the generators of the finite W-algebra are different from those of Lie algebras. Here, there can be root spaces of dimension larger than one, and one does not have subalgebras isomorphic to sl_{2} corresponding to any root. In addition, in the case of Lie algebras $[E_{\alpha}, E_{\beta}]$ vanishes if $\alpha + \beta$ is not a root or zero. Finite W-algebras, on the other hand, satisfy non-linear relations, so even if $\alpha + \beta$ is not a root, one can still have relations of the type $[E_{\alpha+\gamma}, E_{\beta-\gamma}] \sim E_{\alpha} E_{\beta}$.

An important property that finite W-algebras share with Lie algebras is that the Poincaré–Birkhoff–Witt theorem still holds. Choose a basis of the finite W-algebra so that the generators are in one-to-one correspondence with elements of $\mathrm{ker}(\mathrm{ad}_{t_{-}})$ and have a well defined non-negative half-integer eigenvalue with respect to $-\mathrm{ad}_{t_{0}}$. For example, choose the generators in one-to-one correspondence with the lowest weights of the irreducible sl_{2} representations in which $\mathcal{G}$ decomposes under the action of the embedded sl_{2}. If this basis is $\{W_{i}\}_{i \in I}$, we can order the index set I in some arbitrary fashion, and the Poincaré–Birkhoff–Witt theorem states that a basis for the finite

W-algebra (considered as a vector space) is given by all expressions of the type

$$W_{i_1} W_{i_2} \dots W_{i_n} , \tag{5.25}$$

where $i_1 \leq i_2 \dots \leq i_n$. To prove this, we first note that from the explicit form of the generators, as obtained from the tic-tac-toe construction, one can see that the commutator of two generators of $-\mathrm{ad}_{t_0}$ eigenvalue λ_1 and λ_2 does not contain generators or products of generators of $-\mathrm{ad}_{t_0}$ eigenvalue larger than $\lambda_1 + \lambda_2$. More precisely, the linear terms have eigenvalues $\leq \lambda_1 + \lambda_2$, the quadratic terms have eigenvalues $\leq \lambda_1 + \lambda_2 - 1$, the third order terms have eigenvalues $\leq \lambda_1 + \lambda_2 - 2$, etc. Armed with this observation the theorem can now be proven using double induction on the number of generators and their $-\mathrm{ad}_{t_0}$ eigenvalue.

A particular useful ordering of the generators of the finite W-algebra is to put all generators in $\mathscr{W}_-$ to the left, those corresponding to $\mathscr{W}_0$ in the middle and those corresponding to $\mathscr{W}_+$ to the right. Then one can define a highest-weight representation in the usual fashion, by acting with the W-algebra on a particular vector $|\lambda\rangle$, which (i) is annihilated by the generators in $\mathscr{W}_+$ and (ii) has certain eigenvalues with respect to the generators in $\mathscr{W}_0$. The Poincaré–Birkhoff–Witt theorem then tells us that the resulting module is spanned by the vectors

$$W_{i_1} W_{i_2} \dots W_{i_n} |\lambda\rangle , \tag{5.26}$$

where $i_1 \leq i_2 \dots \leq i_n$, and the W_i are elements of $\mathscr{W}_-$. This agrees precisely with the standard definition.

At this stage we should discuss how one constructs highest weight representations of finite W-algebras in terms of representations of $\mathscr{G}$. One possibility is to use the BRST procedure once more. The finite W-algebra was the BRST cohomology of a certain complex constructed out of $\mathscr{G}$, and it is possible to extend this complex to one where the Lie algebra is replaced by a representation of the Lie algebra. The cohomology of this complex then automatically gives representations of the finite W-algebra. This procedure has some disadvantages, however. Firstly, it is rather cumbersome if one has to compute a new cohomology for every representation, and secondly, finite-dimensional representations of $\mathscr{G}$ typically yield just the trivial representation of the finite W-algebra. For a discussion, see [6].

A better procedure is to take explicit representatives for the generators of the finite W-algebra, and to use those to obtain finite W-representations as subspaces of representations of the universal enveloping algebra of $\mathscr{G}$.[4] This only works if the generators form an exact closed algebra, not one up to BRST exact terms.[5] The only realization which has this property is the one that one gets from the tic-tac-toe construction in Section 4.2.1. This is not yet the complete story, however. One would also like to get highest weight representations of the finite W-algebra from highest weight representations of $\mathscr{G}$. However, from (4.25) one sees that C contains a lowering operator of $\mathscr{G}$,

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[4] Strictly speaking, one also needs a representation of a b^α, c_α Clifford algebra, for which one can take the module obtained by acting on a vector $|0\rangle$ that is annihilated by all the c_α. In that case one never has to deal with the Clifford algebra explicitly, and we will ignore it from now on.

[5] It would, for instance, be nice if one could express the generator of the finite W-algebra corresponding to the quadratic Casimir simply as $\eta_{ab} J^a J^b$, but then the finite W-algebra no longer closes and it is in general not clear whether the other generators can be modified in order to close the algebra.

namely J^8. Hence C is not automatically diagonal on a highest weight of $\mathscr{G}$. To achieve this we also have to use the quantum Miura transformation, which tells us that we can consistently restrict the generators to their bidegree $(0,0)$ part. If we do this, then C will be diagonal on a highest weight of $\mathscr{G}$, and we end up with a highest weight representation of the finite W-algebra. Thus, to summarize, we can construct highest weight representations of finite W-algebras via

highest weight representations of $\mathscr{G}$

$\downarrow$ restriction

highest weight representations of $\mathscr{G}_0$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5.27)

$\downarrow$ Miura

highest weight representations of $\mathscr{W}$

Assume from now on that we are in the basis of $\mathscr{G}_S$ where $\mathscr{G}_S$ is generated by $\{E_{\pm\alpha_i}, H_{\alpha_i}\}_{i\in S}$. The above construction of highest weight representations of the finite W-algebra leads to a non-linear parametrization of the eigenvalues of the generators of $\mathscr{W}_0$ in terms of the weights of $\mathscr{G}$. This is a correct counting of degrees of freedom: $\mathscr{W}_0$ consists of generators obtained from $\mathscr{G}_S$ in addition to those of $\mathscr{H}_{\mathscr{X}}$. Since the sl_2 is principally embedded in $\mathscr{G}_S$, we get $\mathrm{rank}(\mathscr{G}_S)$ generators in $\mathscr{W}_0$ from this, so that $\dim(\mathscr{W}_0) = \mathrm{rank}(\mathscr{G}_S) + \dim(\mathscr{H}_{\mathscr{X}}) = \dim(\mathscr{H}_S) + \dim(\mathscr{H}_{\mathscr{X}}) = \mathrm{rank}(\mathscr{G})$. The $\mathscr{W}_0$ eigenvalues corresponding to $\mathscr{H}_{\mathscr{X}}$ are linear in terms of the $\mathscr{G}$ weight Λ. The $\mathscr{W}_0$ eigenvalues corresponding to the generators obtained from $\mathscr{G}_S$ will typically be non-linear, and be reminiscent of the Casimirs of $\mathscr{G}_S$. Indeed, the conjectured form of the Kac determinant, to be discussed in a moment, shows that the $\mathscr{W}_0$ eigenvalues correspond to the W_S invariants that can be constructed out of Λ; W_S is the Weyl group of $\mathscr{G}_S$, and its action on the weights Λ will be discussed later.

As we said previously, every generator of the finite W-algebra has well-defined eigenvalues with respect to the adjoint action of $\mathscr{H}_{\mathscr{X}}$, and this defines for every W-generator a root in $\mathscr{H}_{\mathscr{X}}^*$. The set of roots corresponding to the generators of $\mathscr{W}_+$ will be denoted by $\Delta_+^{\mathscr{W}}$. We will write $\Delta_+^{\mathscr{W}} = \{(\bar{\alpha}_i, \mu_i)\}_{i\in I}$, where $\bar{\alpha}_i \in \mathscr{H}_{\mathscr{X}}^*$, and μ_i labels the potential degeneracy of such a root. The corresponding generators of $\mathscr{W}_+$ will be denoted by $W_{\bar{\alpha}}^\mu$, and those of $\mathscr{W}_-$ by $W_{-\bar{\alpha}}^\mu$. Furthermore, we will by $\bar{\Delta}_+^S$ denote the set of positive roots of $\mathscr{G}$ minus the set of positive roots of $\mathscr{G}_S$, and denote by π the canonical (orthogonal) projection from the root space of $\mathscr{G}$ to that of $\mathscr{X}$. On $\mathscr{H}_{\mathscr{X}}^*$ we can define the Kostant partition function $P(\bar{\alpha})$, which is the number of inequivalent series of non-negative integers $\{t_i\}_{i\in I}$ such that $\bar{\alpha} = \sum_i t_i \bar{\alpha}_i$. Now consider the highest weight module generated by $|\Lambda\rangle$. It consists of the vectors

$$W_{-\bar{\alpha}_1}^{\mu_1} W_{-\bar{\alpha}_2}^{\mu_2} \cdots W_{-\bar{\alpha}_n}^{\mu_n} |\Lambda\rangle = W_{\{-\bar{\alpha}\}}^{\{\mu\}} |\Lambda\rangle \,, \qquad\qquad\qquad (5.28)$$

with $(\bar{\alpha}_1, \mu_1) \geq (\bar{\alpha}_2, \mu_2) \geq \ldots \geq (\bar{\alpha}_n, \mu_n)$ with respect to some ordering of the roots of $\Delta_+^{\mathscr{W}}$. The states with $\sum_i \bar{\alpha}_i = \bar{\beta}$ span some finite-dimensional vector space $V_{\bar{\beta}}$. If ω denotes the anti-involution corresponding to the compact real form of $\mathscr{W}$, and we define $|\Lambda\rangle^\dagger = \langle\Lambda|$ and $\langle\Lambda|\Lambda\rangle = 1$, then unitarity of the highest weight representation implies that the $\dim(V_{\bar{\beta}}) \times \dim(V_{\bar{\beta}})$ dimensional matrix

$$M_{\bar{\beta}}(\Lambda)_{\{\mu',\bar{\alpha}'\},\{\mu,\bar{\alpha}\}} = \langle\Lambda|\omega(W_{\{-\bar{\alpha}'\}}^{\{\mu'\}})W_{\{-\bar{\alpha}\}}^{\{\mu\}}|\Lambda\rangle \qquad\qquad\qquad (5.29)$$

is hermitian and has only positive eigenvalues. Reducibility of the highest weight representation is indicated by zero eigenvalues of this matrix, and both unitarity and reducibility can be studied by

looking at the so-called Kac determinant of this matrix, which we denote by $M_{\bar{\beta}}(\Lambda)$. In [12] the following form of the Kac determinant was conjectured[6]

$$
M_{\bar{\beta}}(\Lambda) = K(\bar{\beta}, \Lambda) \prod_{k>0} \prod_{\alpha \in \bar{\Delta}_+^s} \left(\langle \Lambda + \rho_{\mathcal{W}}, \alpha \rangle - \frac{k}{2} \langle \alpha, \alpha \rangle \right)^{P(\bar{\beta} - k\pi(\alpha))}, \tag{5.30}
$$

where $K(\bar{\beta}, \Lambda)$ is a positive constant and $\langle \, , \rangle$ is the usual positive definite invariant inner product on $\mathcal{H}^*$. The vector $\rho_{\mathcal{W}}$ that appears in (5.30) depends on the details of the Miura transformation and the sl_2 embedding. To construct it, we look at the generator of the finite W-algebra which is written in the tic-tac-toe form of Section 4.2.1, and which differs from the quadratic Casimir $\eta_{ab} J^a J^b$ of $\mathcal{G}$ by a BRST-exact quantity. The bidegree $(0,0)$ of this generator, which appears in the Miura transformation, has a certain eigenvalue on the highest weight state $|\Lambda\rangle$. The eigenvalue in terms of $|\Lambda\rangle$ is of the form

$$
a \langle \Lambda + \rho_{\mathcal{W}}, \Lambda + \rho_{\mathcal{W}} \rangle + b, \tag{5.31}
$$

for some constants a, b, and this defines $\rho_{\mathcal{W}}$. One of the properties of $\rho_{\mathcal{W}}$ is that the $\mathcal{W}_0$ eigenvalues are invariant under the following action of W_S

$$
w \cdot \Lambda = w(\Lambda + \rho_{\mathcal{W}}) - \rho_{\mathcal{W}}. \tag{5.32}
$$

This is consistent with (5.30), since one easily verifies that $M_{\bar{\beta}}(w \cdot \Lambda) = M_{\bar{\beta}}(\Lambda)$ for $w \in W_S$. Thus the Kac determinant can be rewritten as a polynomial function of the $\mathcal{W}_0$ eigenvalues.

To illustrate the use of the Kac determinant we will now work it out in more detail for the finite W-algebra described in Section 4.2.2.

5.2.1. Example

In this example the algebra $\mathcal{G}_S$ is generated by $\{t_1, t_5, t_8\}$, which is not generated by $\{H_{\alpha_i}, E_{\pm\alpha_i}\}_{i \in S}$. Therefore we first perform an automorphism of the sl_3 to make sure that $\mathcal{G}_S$ is generated by $\{H_{\alpha_i}, E_{\pm\alpha_i}\}$. This automorphism is

$$
X \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} X \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.33}
$$

and the new basis of sl_3 is

$$
r_a t_a = \begin{pmatrix} \dfrac{r_4}{6} - \dfrac{r_5}{2} & r_1 & r_2 \\[2ex] r_8 & \dfrac{r_4}{6} + \dfrac{r_5}{2} & r_7 \\[2ex] r_6 & r_3 & -\dfrac{r_4}{3} \end{pmatrix}. \tag{5.34}
$$

---

[6] In [12], the constant K is omitted and it is implicitly assumed that one has chosen an embedding and a Miura transformation such that $\rho_{\mathcal{W}} = \rho$.

Strictly speaking J's are elements of the dual Lie algebra $\mathscr{G}^*$. We identify the latter with the Lie algebra via the pairing $(X, Y) = \text{tr}(XY)$. Furthermore, highest weight representations of $\mathscr{G}$ correspond to lowest weight representations of $\mathscr{G}^*$, and it is the highest weight representations of $\mathscr{G}^*$ that give the highest weight representations of the finite W-algebra. To correct for this, we apply the Chevalley automorphism $H \rightarrow -H$, $E_\alpha \leftrightarrow E_{-\alpha}$ to $\mathscr{G}$. This leads to the following identification between the J's and sl_3 matrices

$$
r_a J^a = \begin{pmatrix} -r_4 + r_5 & r_1 & r_2 \\ r_8 & -r_4 - r_5 & r_7 \\ r_6 & r_3 & 2r_4 \end{pmatrix} . \tag{5.35}
$$

The semi-simple subalgebra $\mathscr{K}$ is a $u(1)$, and is spanned by t_4 which is proportional to $2H_{\alpha_2} + H_{\alpha_1}$. Therefore, the root space $\mathscr{K}_{\mathscr{X}}^*$ is spanned by $\alpha_2 + \frac{1}{2}\alpha_1$. Furthermore, we find that $\bar{\Delta}_+^S = \{\alpha_2, \alpha_1 + \alpha_2\}$, and that $\pi(\alpha_2) = \pi(\alpha_1 + \alpha_2) = \alpha_2 + \frac{1}{2}\alpha_1$. Hence the positive roots of the W-algebra correspond to positive multiples of $2\alpha_1 + \alpha_1$, and we see that $\mathscr{W}_0$ is generated by C and j_0, that $\mathscr{W}_+$ is generated by j_+ with root $\alpha_2 + \frac{1}{2}\alpha_1$ and that $\mathscr{W}_-$ is generated by j_- with root $-(\alpha_2 + \frac{1}{2}\alpha_1)$. The anti-involution corresponding to the compact real form is the same as ω_1 in (5.33).

The explicit form of the generators as given in (4.21) and (4.25) can now be used to deduce what the eigenvalues of j_0 and C on a sl_3 highest weight state $|\Lambda\rangle$ are. If we parametrize the weight Λ as $q_1\lambda_1 + q_2\lambda_2$, where $\lambda_{1,2}$ are the fundamental weights of sl_3, we find for the j_0 and C eigenvalues h, c respectively,

$$
h = 1 + \frac{1}{3}(2q_1 + 4q_2) , \quad c = -\frac{4}{9}(q_1^2 + q_2^2 + q_1 q_2) - \frac{4}{3}q_2 - 1 . \tag{5.36}
$$

Alternatively, c can be written as $-\frac{2}{3}\langle \Lambda + \alpha_2, \Lambda + \alpha_2\rangle + \frac{1}{3}$, and we deduce that $\rho_{\mathscr{W}} = \alpha_2$. Therefore, element w of the Weyl group acts on a weight Λ as

$$
w \cdot \Lambda = w(\Lambda + \rho_{\mathscr{W}}) - \rho_{\mathscr{W}} . \tag{5.37}
$$

The Weyl group is $\mathbb{Z}_2$ and is generated by the reflection s_{α_1} in the line perpendicular to α_1. It acts on q_1, q_2 as

$$
s_{\alpha_1}(q_1) = -q_1 + 2 , \quad s_{\alpha_1}(q_2) = q_1 + q_2 + 1 , \tag{5.38}
$$

and leaves h, c invariant.

For the Kac determinant at $\bar{\beta} = p(\alpha_2 + \frac{1}{2}\alpha_1)$ we find

$$
M_p(q_1, q_2) = K_p(q_1, q_2) \prod_{k=1}^{p} (q_2 + 2 - k)(q_1 + q_2 + 1 - k) , \tag{5.39}
$$

which can in agreement with our expectations be rewritten in terms of c, h as

$$
M_p(q_1, q_2) = K_p(q_1, q_2) \left(\frac{3}{4}\right)^p \prod_{k=1}^{p} (c + (h + 1 - k)^2 + \frac{1}{3}(k^2 - 1)) . \tag{5.40}
$$

This result can be compared with the results in [10] by working out explicit commutation relations. Then one finds out that $K_p(q_1, q_2) = p!(\frac{4}{3})^p$.

The Kac determinant can now be used to obtain information about unitary and reducible representations of the finite W-algebra.

Unitarity of a representation implies that the Kac determinant is positive for every $\bar{\beta}$. In this example the representation is spanned by $\{ j_-^l |\Lambda\rangle \}_{l\geq 0}$ and the Kac determinant gives us the norm

$$M_p(q_1, q_2) = \langle \Lambda | j_+^p j_-^p |\Lambda\rangle ,\tag{5.41}$$

so that positivity of the Kac determinant is here equivalent to unitarity. It follows that the highest weight representation is unitary if (i) $q_2 < \min(-q_1, -1)$ or (ii) there exists a non-negative integer r such that $\max(r - 2, r - 1 - q_1) < q_2 < \min(r - 1, r - q_1)$. If the Kac determinant has a zero at some level p, then the norm $j_-^p |\Lambda\rangle$ is zero and this vector can be consistently put equal to zero, leading to a finite-dimensional representation of the finite W-algebra. The condition for this is that either $q_2 = p - 2$ or $q_1 + q_2 = p - 1$. This leaves one parameter free which we parametrize by an arbitrary real number x as $q_1 + 2q_2 = \frac{3}{2}(p + x - 2)$, so that $h = p + x - 1$ and $c = \frac{1}{3}(1 - p^2) - x^2$. When are these finite-dimensional representations unitary? From the Kac determinant we see that if $q_2 = p - 2$ then $q_1 + q_2 + 1 - k$ must be positive for $0 \leq k \leq p - 1$, and if $q_1 + q_2 = p - 1$ then $q_2 + 2 - k$ must be positive for $0 \leq k \leq p - 1$. Therefore we find that (i) $q_2 = p - 2$ and $q_1 > 0$ or (ii) $q_1 + q_2 = p - 1$ and $q_2 > p - 3$. Both possibilities imply the same condition for x, namely $\frac{3}{2}(p + x - 2) > 2p - 4$, or $x > \frac{1}{3}(p - 2)$. Finally, let us look at the maximal reducible representations. These are representations where the Kac determinant has a maximal number of zeroes, corresponding to finite-dimensional reducible representations of the finite W-algebra. Such representations exist if for some integer l satisfying $0 < l < p$ we have (i) $q_2 + 2 - p = 0$ and $q_1 + q_2 + 1 - l = 0$ or (ii) $q_1 + q_2 + 1 - p = 0$ and $q_2 + 2 - l = 0$. In terms of x this implies $p + l - 3 = \frac{3}{2}(p + x - 2)$, or $x = \frac{1}{3}(2l - p)$. For this value of x the finite W-algebra has a p-dimensional representation with an l-dimensional subrepresentation.

5.3. Characters of finite W-algebras

In this final part of the description of the representation theory of finite W-algebras we will take a closer look at the characters of these representations. In the description of the Kac determinant we explained that every highest weight representation can be decomposed into finite-dimensional vector spaces $V_{\bar{\beta}}$, and the Kac determinant was defined as $\det\langle v_i | v_j\rangle$, where the $|v_i\rangle$ formed a certain distinguished basis of $V_{\bar{\beta}}$. The formal character of a highest weight representation R with highest weight Λ of a finite W-algebra is now defined as follows

$$\mathrm{ch}\, R = \sum_{\bar{\beta}} \dim(V_{\bar{\beta}}) \, \mathrm{e}^{\pi(\Lambda) - \bar{\beta}} ,\tag{5.42}$$

where π is the orthogonal projection of the weight space of $\mathcal{G}$ on that of $\mathcal{K}$. This is a formal expression in the sense that it contains the ill-defined object $\exp(\bar{\beta})$. The best way to think of these objects is that ultimately we want to view the character as a function on $\mathcal{H}_{\mathcal{K}}$, and for $\lambda \in \mathcal{H}_{\mathcal{K}}$ the function $\exp(\bar{\beta})$ is defined as $\exp(\bar{\beta})(\lambda) = \exp(\langle \bar{\beta}, \lambda\rangle)$. In particular, these exponentials satisfy the usual property $\exp(\bar{\alpha})\exp(\bar{\beta}) = \exp(\bar{\alpha} + \bar{\beta})$. The combination $\pi(\Lambda) - \bar{\beta}$ which appears in the exponent in (5.42) is exactly what gives the eigenvalues of the generators in $\mathcal{H}_{\mathcal{K}}$, when acting on the states in $V_{\bar{\beta}}$.

The Verma module $M(\Lambda)$ with highest weight Λ is by definition the highest weight module that is freely generated by $|\Lambda\rangle$, so that in particular we keep all the states with zero norm. Every highest weight module with highest weight Λ is a quotient of $M(\Lambda)$, and the quotient of $M(\Lambda)$ and its maximal proper submodule $M'(\Lambda)$ is an irreducible highest weight module $L(\Lambda) = M(\Lambda)/M'(\Lambda)$. All irreducible highest weight modules are of the form $L(\Lambda)$ for some Λ, and our goal in this section is to study the characters of the modules $L(\Lambda)$ in terms of the characters of the Verma modules $M(\Lambda)$.

The characters of the Verma modules $M(\Lambda)$ are easily obtained, since they are freely generated by $\mathcal{W}_-$, and we can use the Poincaré–Birkhoff–Witt theorem to write down a basis for $M(\Lambda)$ (see (5.28)). From this one deduces that

$$\operatorname{ch} M(\Lambda) = \frac{e^{\pi(\Lambda)}}{\prod_{\tilde{\alpha} \in \Delta_+^{\mathcal{W}}} (1 - e^{-\tilde{\alpha}})} . \tag{5.43}$$

Non-trivial submodules of $M(\Lambda)$ can arise if $M(\Lambda)$ contains a singular vector, which is a vector (not equal to $|\Lambda\rangle$) annihilated by all generators in $\mathcal{W}_+$. If we denote this vector by $|\Lambda'\rangle$, then the Verma module $M(\Lambda)$ contains the submodule $M(\Lambda')$. If the latter would happen to be the maximal proper submodule of $M(\Lambda)$, we would find that $L(\Lambda) = M(\Lambda)/M(\Lambda')$. Since the character of a quotient of two modules is simply the difference of the two characters, this would lead to the following identity for characters

$$\operatorname{ch} L(\Lambda) = \operatorname{ch} M(\Lambda) - \operatorname{ch} M(\Lambda') . \tag{5.44}$$

More general situations can occur, of course. For example, if $M(\Lambda)$ contains three singular vectors $|\Lambda_1\rangle$, $|\Lambda_2\rangle$ and $|\Lambda_3\rangle$, and $|\Lambda_3\rangle$ is contained in both $M(\Lambda_1)$ and $M(\Lambda_2)$, then the maximal proper submodule is generated by $M(\Lambda_1)$ and $M(\Lambda_2)$, but it would not be correct to subtract their characters from $M(\Lambda)$ to get the character of $L(\Lambda)$, since we would have subtracted the vectors in $M(\Lambda_3)$ twice. Rather, the correct formula is

$$\operatorname{ch} L(\Lambda) = \operatorname{ch} M(\Lambda) - \operatorname{ch} M(\Lambda_1) - \operatorname{ch} M(\Lambda_2) + \operatorname{ch} M(\Lambda_3) . \tag{5.45}$$

A further complication is the possible existence of subsingular vectors. These are vectors that become singular after modding out a module generated by singular vectors. In that case one also needs to subtract the submodules generated by these subsingular vectors. Altogether this leads to an expression for the character of $L(\Lambda)$ as a linear combination of those of $M(\Lambda)$ with integer coefficients. Since we know the latter explicitly (5.43), this immediately yields the explicit characters of $L(\Lambda)$.

The character formula conjectured in [12] deals with the maximally degenerate representations of the finite W-algebra. These are representations for which the Kac determinant has a maximal number of vanishing factors. Vanishing factors in the Kac determinant are closely related to the existence of singular vectors (the latter have zero norm), and an alternative definition of maximally degenerate representations is those representations with a maximum number of singular vectors. For these representations the character formulas will be the most complicated ones. A factor $\langle \Lambda + \rho_{\mathcal{W}}, \alpha \rangle - (k/2)\langle \alpha, \alpha \rangle$ in the Kac determinant vanishes (for $\mathcal{G} = sl_n$) if $\langle \Lambda + \rho_{\mathcal{W}}, \alpha \rangle$ is a positive integer, since every root has length two. If ρ denotes one half of the sum of the positive roots, then $\langle \rho, \alpha_i \rangle = 1$ for all simple roots α_i. Therefore, if $\Lambda + \rho_{\mathcal{W}} = \lambda + \rho$, and λ is a dominant integral weight (a linear combination of the fundamental weights with nonnegative integer coefficients),

then $\langle \Lambda + \rho_{\mathcal{W}}, \alpha \rangle$ will be a positive integer for any root α. From now on we will restrict our attention to such weights.

The Weyl group W of $\mathcal{G}$ acts on Λ via (5.32). The subgroup W_S does not change the representation of the finite W-algebra, so from that point of view we may identify weights and their images under the action of W_S. Thus, the orbit of the weight under the Weyl group gives a set of weights of the finite W-algebra in one-to-one correspondence with the coset $W_S \backslash W$, and the conjecture is that these are precisely all the singular vectors of the representation of the finite W-algebra.

The Weyl group is generated by the reflections in the hyperplanes perpendicular to the simple roots. Every element of the Weyl group can be written as the product of such reflections, and the minimal number of reflections in simple roots needed to obtain an element w of the Weyl group is called the length of that element, $l(w)$. A particular ordering of the elements of the Weyl group plays an important role in the character formula, the so-called Bruhat ordering (see e.g. [47]). Write $w' \to w$ if $w = tw'$ and $l(w) > l(w')$, where t is the reflection in the hyperplane perpendicular to some root (not necessarily simple). Then we say that $w' < w$, if there is a sequence $w' \to w_1 \to \cdots \to w_{m-1} \to w$. Now given a Weyl group W_S associated to some subset S of the set of simple roots, we define W^S as the set of $w \in W$ such that $l(sw) > l(w)$ for all reflections s in any hyperplane perpendicular to any root (not necessarily simple) of $\mathcal{G}_S$. Any element $w \in W$ can be uniquely written as uv with $u \in W_S$ and $v \in W^S$, and W^S is a set of representatives for the coset $W_S \backslash W$. We will identify $W_S \backslash W$ with the set W^S and give it a partial ordering by restricting the Bruhat ordering on W to W^S.

We can now write down the Kazhdan–Lusztig conjecture for finite W-algebras, as proposed in [12]. It reads

$$\operatorname{ch} L(\tau \cdot (w_{\max} \cdot \Lambda)) = \sum_{\sigma \le \tau \in W_S \backslash W} (-1)^{l(\sigma)} (-1)^{l(\tau)} \tilde{P}^S_{\sigma\tau}(1) \operatorname{ch} M(\sigma \cdot (w_{\max} \cdot \Lambda)) . \tag{5.46}$$

Here, $w_{\max}$ is the longest element of W (so that $w_{\max} \cdot \Lambda + \rho_{\mathcal{W}} - \rho$ is an anti-dominant weight), and $\tilde{P}^S_{\sigma,\tau}(x)$ are the so-called dual relative Kazhdan–Lusztig polynomials associated to W_S, see [48, 49]. In [48] this character identity is proven for ordinary simple Lie algebras, when S is the empty set.

One of the implications of this character formula is that the singular vectors in $M(\Lambda)$ are in one-to-one correspondence with the elements of $W_S \backslash W \sim W^S$, and that the Bruhat ordering tells us precisely which singular vectors can be obtained from which other singular vectors by acting on them with $\mathcal{W}_-$.

To conclude this section we illustrate this character formula with the example studied in Sections 4.2.2 and 5.2.1. The Weyl group of sl_3 is S_3 and the Bruhat ordering on S_3 coincides with the ordering with respect to the length of the elements, i.e. $w < w' \leftrightarrow l(w) < l(w')$. If we denote the reflections in the lines perpendicular to the simple roots α_1 and α_2 by s_1 and s_2, then $W_S = \{1, s_1\}$ and $W^S = \{1, s_2, s_2 s_1\}$. We represent the highest weight as in Section 5.2.1 as $\Lambda = q_1 \lambda_1 + q_2 \lambda_2$. The maximal Weyl group element in $s_1 s_2 s_1$ and by explicit computation one finds that

$$w_{\max} \cdot \Lambda = (-1 - q_2)\lambda_1 + (-1 - q_1)\lambda_2 ,$$

$$s_2 \cdot (w_{\max} \cdot \Lambda) = (-q_1 - q_2)\lambda_1 + (q_1 - 3)\lambda_2 , \tag{5.47}$$

$$s_2 s_1 \cdot (w_{\max} \cdot \Lambda) = (2 - q_1)\lambda_1 + (q_1 + q_2 - 1)\lambda_2 .$$

In view of (5.38) representations with weight $s_2 s_1 \cdot (w_{\max} \cdot \Lambda)$ can be identified with those with weight $s_1 s_2 s_1 \cdot (w_{\max} \cdot \Lambda)$ which is just $\Lambda = q_1 \lambda_1 + q_2 \lambda_2$, and we will do so in the equations that follow. The values of the dual relative Kazhdan–Lusztig polynomials are given by $\tilde{P}^S_{\sigma\tau}(1) = 1$ for all $\sigma \leq \tau$, with one exception. If $\sigma = 1$ and $\tau = s_2 s_1$ then it vanishes. Substituting everything in the character formula yields three equations

$$\operatorname{ch} L(-1 - q_2, -1 - q_1) = \operatorname{ch} M(-1 - q_2, -1 - q_1)$$

$$\operatorname{ch} L(-q_1 - q_2, q_1 - 3) = \operatorname{ch} M(-q_1 - q_2, q_1 - 3) - \operatorname{ch} M(-1 - q_2, -1 - q_1) \qquad (5.48)$$

$$\operatorname{ch} L(q_1, q_2) = \operatorname{ch} M(q_1, q_2) - \operatorname{ch} M(-q_1 - q_2, q_1 - 3) \,.$$

These identities agree with the picture of the representation of this finite W-algebra as sketched at the end of Section 5.2.1. An explicit expression for the character of $L(q_1, q_2)$ is now easily obtained. The positive root in $\Delta^{\mathscr{W}}_+$ is $\alpha_2 + \frac{1}{2}\alpha_1 = \frac{3}{2}\lambda_2$, and $\pi(\Lambda) = (q_2 + \frac{1}{2}q_1)\lambda_2$. Thus

$$\operatorname{ch} L(q_1, q_2) = \frac{\mathrm{e}^{(2q_2 + q_1)\lambda_2/2} - \mathrm{e}^{(q_1 - q_2 - 6)\lambda_2/2}}{1 - \mathrm{e}^{-3\lambda_2/2}} \,. \qquad (5.49)$$

The formal limit $\lambda_2 \to 0$ of the right-hand side yields $q_2 + 2$ and this is the dimension of $L(q_1, q_2)$. If we parametrize as in the end of Section 5.2.1 the maximal reducible representations by $q_1 = l - 2$ and $q_2 = p - l - 1$, we find that the dimension of $L(q_1, q_2)$ is l. This is indeed correct since for these values of q_1, q_2 the p-dimensional finite representation had an l-dimensional subrepresentation which is precisely $L(q_1, q_2)$.

6. Constructing theories with finite W-symmetries

In Chapter 2 we have seen some finite W-algebras that appear in simple physical systems. This happened more or less by coincidence, a priori we had no reason to expect a finite W-algebra to show up (although anisotropic harmonic oscillators have quite generally extended non-linear symmetry algebras for rational frequency ratios, see e.g. [15]). Here we want to pose the reverse question, namely, can one given some finite W-algebra construct theories that have this W-algebra as their symmetry algebra? An additional interesting question is whether or not one can build gauge theories based on finite W-algebras. If we would succeed in constructing a gauge theory for a finite W-algebra that contains $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, this might be a new candidate for a Grand Unified Theory. Such finite W-algebras certainly exist, as we briefly explained at the end of Section 3.2.7. Unfortunately, we have not succeeded in constructing a gauge theory for finite W-algebras except in one dimension, and some topological type theories in other dimensions. In this section, we want to sketch some of the ideas and problems involved in the construction of theories with finite W-symmetries. This is definitely not a finished chapter in the theory of finite W-algebras, and it will have more the character of a series of remarks than that of a finished theory.

6.1. Problems with finite W-symmetries

We start with some particular (classical) finite W-algebra given by the Poisson brackets

$$\{W_\alpha, W_\beta\} = P_{\alpha\beta}\{W_\gamma\} \,, \qquad (6.1)$$

where the $P_{\alpha\beta}$ are certain polynomials in W_γ. Suppose that we have some field theory with fields ϕ_i with a finite W-symmetry. This means that there are transformation rules $\delta\phi_i = \sum_\alpha \varepsilon_\alpha \delta_\alpha \phi_i$ that leave the action invariant, where ε_α is the constant parameter corresponding to the generator W_α. Associated to these transformation rules are a set of conserved currents j_α^μ that can be found in the usual way through the Noether procedure. The corresponding conserved charges, that generate the symmetries, are

$$Q_\alpha = \int d^{d-1}x\, j^0 \,. \tag{6.2}$$

The statement that the theory is invariant under the finite W-algebra (6.1) means that the conserved charges have to satisfy precisely (6.1), i.e.

$$\{Q_\alpha, Q_\beta\} = P_{\alpha\beta}(Q_\gamma) \,. \tag{6.3}$$

It is hard to see how one could realize (6.3). The left-hand side contains two integrations $\int d^{d-1}x$, and one of these disappears after integrating over the delta function that arises in the equal-time Poisson brackets. Hence, the left-hand side contains exactly one integration $\int d^{d-1}x$. The right-hand side contains as many integrations $\int d^{d-1}x$ as the degree of $P_{\alpha\beta}$. These two facts can only be made to agree with each other if (i) the finite W-algebra is linear or (ii) $d = 1$. Case (i) is precisely what we are not interested in, since that brings us back in the realm of ordinary Lie algebras, and we will come back to $d = 1$ later. Therefore, it seems that we have some kind of no-go theorem in dimensions $d > 1$. What else could we try to do? First, the above argument assumes the symmetries are local. If one drops this assumption, it might still be possible to do something. We have not analyzed this possibility, partly because of the problems in dealing with theories with non-local symmetries. A second possibility is to change the definition of what we mean by a theory with finite W-symmetries, and this will be the subject of the next section. The reader who has some knowledge of the corresponding situation with 'infinite' W-algebras in two dimensions may wonder how these escape the no-go theorem given above. The reason is that the 'infinite' W-algebras have an infinite number of generators, and this makes it possible to convert the integrations that remain in the right-hand side of (6.3) into infinite sums of generators. Since we restrict attention to finite W-algebras with a finite number of generators, this does not provide a way out either.

6.1.1. Another definition of finite W-symmetries

In [50] an attempt was made to write down a gauge theory for a non-linear algebra which is a deformation of $su(2)$. This construction involved a set of scalar fields, which are not needed in the gauge theory of pure $su(2)$. This can be rephrased in the language of the previous paragraph by saying that rather than looking for a theory which has (6.3) as its symmetry algebra, we look at one which has the following algebra of symmetries

$$\{Q_\alpha, Q_\beta\} = P_{\alpha\beta}^\gamma(T_\delta)Q_\gamma \,, \tag{6.4}$$

where the T_δ are extra scalars, one for each generator of the finite W-algebra. The T_δ are added by hand. Now the left- and right-hand sides of (6.4) contain the same number of integrations and the objection of the previous paragraph no longer applies. In some sense one might say that a minimal finite W-multiplet necessarily involves an extra set of scalar fields. We have not yet specified what

the polynomials $P_{\alpha\beta}^{\gamma}$ are. A first guess could be to require

$$P_{\alpha\beta}^{\gamma}(Q_\delta)Q_\gamma = P_{\alpha\beta}(Q_\delta) , \tag{6.5}$$

so that (6.4) becomes identical to (6.3) upon identifying Q_α with T_α. It is, however, not clear that with this choice the Jacobi identities are satisfied. To verify the Jacobi identities, we need also the bracket

$$\{Q_\alpha, T_\beta\} = S_{\alpha\beta}(T_\gamma) . \tag{6.6}$$

Furthermore, we assume that the equal time Poisson bracket of T_α with T_β vanishes. Then the Jacobi identities give two differential equations for S and P. If we choose $P_{\alpha\beta}^{\gamma}$ as in (6.5), we cannot give an explicit solution for S or even prove that a polynomial solution always exists. A much more natural choice is

$$S_{\alpha\beta}(T_\gamma) = P_{\alpha\beta}(T_\gamma) , \quad P_{\alpha\beta}^{\gamma}(T_\delta) = \frac{\partial}{\partial T_\gamma} P_{\alpha\beta}(T_\delta) , \tag{6.7}$$

since now the Jacobi identities $\{Q_\alpha, \{Q_\beta, Q_\gamma\}\} + \text{cycl} = 0$ and $\{Q_\alpha, \{Q_\beta, T_\gamma\}\} + \text{cycl} = 0$ follow directly from the Jacobi identities for the original W-algebra. These conventions differ slightly from those in [50], but agree after rescaling the deformation parameter in [50] by a factor of two. The bracket (6.6) with $S_{\alpha\beta}$ as in (6.7) also appeared in [13], where a study of theories with non-linear gauge symmetries was performed and the two-dimensional non-linear gauge theory (6.25) was first discovered. A particular nice feature of the algebra (6.4) and (6.6) is that if the original finite W-algebra contains a semi-simple Lie subalgebra as in Section 3.2.7, it is preserved in our new 'W-inspired' Poisson algebra, and the new algebra could still serve as a starting point for a non-linear Grand Unified Theory.

Is it possible to realize the algebra (6.4), (6.6) in some field theory? Although this would not be a finite W-invariant theory in the narrow sense, it would at least be a finite W-inspired theory. If there are no further fields in the theory apart from the scalars T_α, it is natural to look at sigma-models with Lagrangian density

$$\mathscr{L} = -\tfrac{1}{2} G^{\alpha\beta}(T_\gamma)\partial^\mu T_\alpha \partial_\mu T_\beta - V(T_\gamma) . \tag{6.8}$$

Invariance under the transformations generated by Q_γ yields the equations

$$\tfrac{1}{2}\partial^\rho G^{\alpha\beta} P_{\rho\gamma} + \tfrac{1}{2} G^{\rho\beta}\partial^\alpha P_{\rho\gamma} + \tfrac{1}{2} G^{\alpha\rho}\partial^\beta P_{\rho\gamma} = 0 , \quad \partial^\rho V P_{\rho\gamma} = 0 . \tag{6.9}$$

The equation for the potential simply states that V is in the center of the finite W-algebra. If the latter is generated by certain polynomials $C_i(W_\alpha)$, then V is a function of these C_i. The center of a finite W-algebra from an sl_2 embedding consists of the Casimirs of the original Lie algebra, since these clearly commute with the BRST operator. Of course, depending on the explicit representatives one chooses to represent the W-algebra generators, the generators of the center of the W-algebra will in general differ from the Casimirs by certain BRST-exact terms.

The solution to the first equation in (6.9) is less obvious. If the symmetry algebra is a Lie algebra then one can choose $G^{\alpha\beta}$ to be an invariant inner product on the Lie algebra. However, Lie algebras are special in the sense that Q_α and T_α transform in the same way under W-transformations. In general, Q_α transforms in the same way as $\partial_\mu T_\alpha$, which differs from the way T_α transforms. We have examined two examples of non-linear algebras in detail. One of them is the finite W-algebra

$W_3^{(2)}$ and the other one is the following non-linear deformation of sl_2[7]

$$[j_0, j_\pm] = \pm j_\pm , \quad [j_+, j_-] = \phi'(j_0) ,$$ (6.10)

where $\phi(j_0)$ is neither a linear nor quadratic function of j_0. The center of this algebra is (classically) generated by $C_1 = 2j_+ j_- + \phi(j_0)$. The finite $W_3^{(2)}$ algebra can be written in the same way as (6.10), where now $\phi = \frac{1}{3} j_0^3 + C j_0$ and C is an additional generator that commutes with everything. The center of this algebra is generated by $C_1 - j_+ j_- + \phi$ and $C_2 = C$. In both cases we have explicitly solved the differential equation for $G^{\alpha\beta}$ and we have found that the only allowed kinetic terms are in either case of the form

$$G^{ij}(C_k) \partial^\mu C_i \partial_\mu C_j .$$ (6.11)

These are not really interesting, since they only induce dynamics for the gauge invariant combinations of T_α. It would be interesting to know whether a similar result holds for all non-linear algebras.

We can rephrase what went wrong in a different language as follows. The derivative $\partial_\mu T_\alpha$ transforms under Q_γ as

$$\{Q_\gamma, \partial_\mu T_\alpha\} = \partial^\beta P_{\gamma\alpha}(T) \partial_\mu T_\beta .$$ (6.12)

Let us call a tensor which transforms in this way a covariant tensor. The transformation rule for a contravariant tensor Y^α can now be determined by requiring $\partial_\mu T_\alpha Y^\alpha$ to be invariant, yielding

$$\{Q_\gamma, Y^\alpha\} = \partial^\alpha P_{\beta\gamma}(T) Y^\beta .$$ (6.13)

The lack of interesting invariant actions of the type (6.8) can now be rephrased by saying that there is no interesting symmetric rank-two contravariant tensors. For Lie algebras the Killing metric provides such a tensor, but for non-linear W-algebras this property ceases to exist. Interestingly, there is a natural rank-two contravariant tensor in the game, namely the inverse $P^{\alpha\beta}(T)$ of the tensor $P_{\alpha\beta}(T)$. The latter is not invertible in general, due to the fact that the non-linear algebra may have a center. If it has a center we will treat the generators of the center as numbers rather than generators, so that $P^{\alpha\beta}$ can contain inverses of these, and the indices α, β range only over those generators that are not in the center of the finite W-algebra. It is now a straightforward exercise to verify that $P^{\alpha\beta}$ indeed transforms as a rank-two contravariant tensor. Since it is anti-symmetric rather than symmetric we cannot use it to construct an invariant action of the type (6.8), but it can be used to construct an invariant action in two dimensions involving an ε-tensor

$$S = \int d^2 x \, \varepsilon^{\mu\nu} P^{\alpha\beta}(T) \partial_\mu T_\alpha \partial_\nu T_\beta .$$ (6.14)

This is a topological action in two dimensions, as it does not depend on the two-dimensional metric. We will come back to this and other related actions after we have introduced gauge fields for finite W symmetries, since we are after all looking for gauge theories.

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[7] Non-linear algebras of this type are among the few examples of non-linear algebras that are similar to finite W-algebras and that have been studied previously, see remark at the end of Section 2.2.2.

6.1.2. Gauge fields for finite W-symmetries

Gauge fields appear if we want to make finite W-symmetries local, i.e. we allow the parameter of the gauge transformations to be space-time dependent. To find the transformation rules for the gauge fields one can for example propose a covariant derivative for T_α, $D_\mu T_\alpha = \partial_\mu T_\alpha - h_\mu^\beta R_{\beta\alpha}(T)$, and require that $D_\mu T_\alpha$ satisfies the same transformation rule under local W-transformations as $\partial_\mu T_\alpha$ under global ones, in other words, it should still transform as a covariant tensor. This leads to the following result

$$D_\mu T_\alpha = \partial_\mu T_\alpha - h_\mu^\beta P_{\beta\alpha}(T) , \quad \delta h_\mu^\beta = \partial_\mu \varepsilon^\beta - \varepsilon^\eta \partial^\beta P_{\eta\gamma}(T) h_\mu^\gamma , \tag{6.15}$$

where ε^γ is the local parameter for the finite W-transformation generated by Q_γ. With these definitions one finds indeed a structure such as (6.12),

$$\delta(D_\mu T_\alpha) = \varepsilon^\eta \partial^\gamma P_{\eta\alpha}(T) D_\mu T_\gamma . \tag{6.16}$$

It is not at all clear, however, that (6.16) (or (6.12)) is the appropriate way to define an object that transforms 'covariantly' under finite W-transformations. Firstly, it is not clear how to define another covariant derivative $D_v^{(2)}$ so that $D_v^{(2)} D_\mu T_\alpha$ transforms also covariantly, as $\partial_v \partial_\mu T_\alpha$ transforms differently from $\partial_\mu T_\alpha$, already under finite W-transformations. Secondly, one could as well consider modified transformation rules for h_μ^β such as

$$\delta h_\mu^\beta = \partial_\mu \varepsilon^\beta - \varepsilon^\eta \partial^\beta P_{\eta\gamma}(T) h_\mu^\gamma + \varepsilon^\eta (D_\mu T_\gamma) \Sigma_\eta^{\beta\gamma}(T) , \tag{6.17}$$

to obtain for $D_\mu T_\alpha$ the following transformation rule

$$\delta(D_\mu T_\alpha) = \varepsilon^\eta (\partial^\gamma P_{\eta\alpha}(T) + P_{\alpha\beta}(T) \Sigma_\eta^{\beta\gamma}) D_\mu T_\gamma . \tag{6.18}$$

If we can choose Σ in such a way that $D_\mu T_\alpha$ transforms in the same way as T_α, then we can immediately write down an invariant kinetic term for the T_α, based on one of the elements of the center of the finite W-algebra. We have, however, no clue whether this can be done or whether it is sensible. Alternatively, one could try to fix Σ by requiring the gauge algebra to close on the gauge fields. One can easily compute that

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] h_\mu^\beta = \delta_{\varepsilon_3} h_\mu^\beta - \varepsilon_1^\eta \varepsilon_2^\sigma (D_\mu T_\lambda) X_{\eta\sigma}^{\mu\lambda} , \tag{6.19}$$

where $\varepsilon_3^\beta = \partial^\beta P_{\gamma\eta} \varepsilon_1^\eta \varepsilon_2^\sigma$ and

$$X_{\eta\sigma}^{\mu\lambda} = -\partial^\beta P_{\eta\gamma} \Sigma_\sigma^{\gamma\lambda} + \partial^\beta P_{\sigma\gamma} \Sigma_\eta^{\gamma\lambda} + \partial^\lambda P_{\sigma\gamma} \Sigma_\eta^{\beta\gamma} - \partial^\lambda P_{\eta\gamma} \Sigma_\sigma^{\beta\gamma} + P_{\gamma\theta} \Sigma_\sigma^{\theta\lambda} \Sigma_\eta^{\beta\gamma}$$

$$- P_{\gamma\theta} \Sigma_\eta^{\theta\lambda} \Sigma_\sigma^{\beta\gamma} + P_{\sigma\gamma} \partial^\gamma \Sigma_\eta^{\beta\lambda} - P_{\eta\gamma} \partial^\gamma \Sigma_\sigma^{\beta\lambda} - \partial^\gamma P_{\sigma\eta} \Sigma_\gamma^{\beta\lambda} - \partial^\lambda \partial^\beta P_{\sigma\eta} . \tag{6.20}$$

Clearly, the gauge algebra closes if $X_{\eta\sigma}^{\mu\lambda} = 0$, but the significance of this equation for Σ remains to be seen. Finally, one would like to write down an invariant action for the gauge fields, preferably in terms of a generalized curvature, which would be a non-linear version of Yang–Mills theory. It would be interesting to analyze for each of the possible transformation rules for the gauge fields whether or not such invariant actions exist. An attempt in this direction was made in [50], where an invariant action was constructed for a non-linear deformation of $su(2)$, to first order in the deformation parameter. However, it is not clear whether their result can be extended systematically to an arbitrary order in the deformation parameter. Another more successful attempt [13], based on the rules (6.15), led to a genuine invariant action in two dimensions (see (6.25) below), but unfortunately, since it is in two dimensions and of topological type, not a very useful generalization

of Yang–Mills theory. Still, it might lead to new geometrical structures and a better understanding of W symmetries, in the next section we describe the action given in [13] and a number of generalizations of it.

6.1.3. Topological actions with finite W-symmetry

At the end of section (6.1) we already gave an example of a topological theory that is invariant under global finite W-transformations. Since we determined the covariant derivative by requiring that $D_\mu T_\alpha$ transforms as a covariant tensor, we can immediately write down an action invariant under local W symmetries, simply by replacing in (6.14) the derivatives by covariant derivatives, yielding

$$S = \int d^2 x \, \varepsilon^{\mu\nu} P^{\alpha\beta}(T) D_\mu T_\alpha D_\nu T_\beta \ . \tag{6.21}$$

The equations of motion obtained by varying this action with respect to h_μ^α read $D_\mu T_\alpha = 0$. Under a finite W-transformation it transforms as (6.16), and hence finite W-symmetries map solutions of the equations of motion into other solutions. Since the equations of motion follow purely from the h_α^μ-dependent part of (6.21), this suggests that the h_α^μ-dependent as well as the h_α^μ-independent part of (6.21) are both separately invariant under finite W-transformations. This is indeed the case, as long as the two-surface we integrate over has no boundary. To write down the various possibilities and their transformation rules in the most transparent way, introduce the one forms $h^\alpha \equiv h_\mu^\alpha dx^\mu$, so that for example the covariant derivative of T_α becomes the one-form $D T_\alpha = d T_\alpha - h^\beta P_{\beta\alpha}$. We next introduce the following three two-forms

$$\mathcal{F}_I = P^{\alpha\beta}(T) d T_\alpha \wedge d T_\beta \ , \quad \mathcal{F}_{II} = P^{\alpha\beta}(T) D T_\alpha \wedge D T_\beta \ ,$$

$$\mathcal{F}_{III} = 2 T_\alpha d h^\alpha + P_{\alpha\beta}(T) h^\alpha \wedge h^\beta \ . \tag{6.22}$$

The integral $\int d^2 x \, \mathcal{F}_{II}$ is nothing but (6.21), and up to an exact form which we added for convenience, $\mathcal{F}_I$ and $\mathcal{F}_{III}$ are, respectively, the h^α-independent and h^α-dependent parts of $\mathcal{F}_{II}$,

$$\mathcal{F}_{II} = \mathcal{F}_I - \mathcal{F}_{III} + 2 d(T_\alpha h^\alpha) \ . \tag{6.23}$$

By virtue of the Jacobi identity $\mathcal{F}_I$ is closed, $d\mathcal{F}_I = 0$, but $\mathcal{F}_{II}$ and $\mathcal{F}_{III}$ are not. Furthermore, the transformation rules for the three two-forms under local finite W-transformations read

$$\delta\mathcal{F}_I = 2 d(\varepsilon^\gamma d T_\gamma) \ , \quad \delta\mathcal{F}_{II} = 0 \ , \quad \delta\mathcal{F}_{III} = 2 d(\varepsilon^\eta h^\gamma (P_{\eta\gamma} - T_\alpha \partial^\alpha P_{\eta\gamma})) \ . \tag{6.24}$$

Starting from these identities one can now easily write down a variety of topological actions in arbitrary dimensions by wedging together suitable combinations of these two-forms, like an arbitrary number of $\mathcal{F}_I$'s, or an arbitrary number of $\mathcal{F}_{II}$'s. One can also couple each of the three to a U(1) Yang–Mills field strength. Another possibility is to write down non-topological highly interacting theories like $\int d^4 x \, \mathcal{F}_{II} \wedge * \mathcal{F}_{II}$, where $*$ denotes the Hodge star. The last action we want to mention is the integral of $\mathcal{F}_{III}$,

$$S_{BF} = \int d^2 x \, \mathcal{F}_{III} \ . \tag{6.25}$$

This action was proposed in [13] where it was called a version of two-dimensional dilaton gravity. If the finite W-algebra is a quadratic extension of the Poincaré algebra, this action describes a Yang–Mills-like formulation of R^2 gravity with dynamical torsion [51]. Interestingly enough, if the finite W-algebra is a Lie algebra, the two-form $\mathscr{F}_{III}$ is equal to $\mathrm{Tr}(BF)$, where B is an adjoint scalar and F the usual Yang–Mills curvature. The gauge theory described by the corresponding action (6.25) is sometimes called topological BF-theory and is related to Reidemeister–Ray–Singer torsion [52] and is a tool to study moduli spaces of flat connections [53, 54]. It would be extremely interesting to see if (6.25) actually describes a non-linear generalization of all these geometrical structures. This would require a much better understanding of the notion of global W-transformations and the concept of a fiber bundle with a non-linear structure group, and these topics certainly deserve a further investigation.

6.1.4. Other possibilities in $d > 1$

In this final section on dimensions larger than one, we will briefly indicate some other possible approaches to the construction of invariant actions for finite W-algebras.

- In the construction sketched so far, we did not use the fact that our finite W-algebras were obtained from a Lie algebra by imposing constraints. It would be nice if we could somehow use this fact to our advantage. Suppose we have at our disposal a theory which is invariant under some global symmetry algebra $\mathscr{G}$. Associated to these $\mathscr{G}$ transformations are certain conserved charges whose Poisson brackets form precisely $\mathscr{G}$. We can impose the constraints that some of these charges have to be equal to either zero or one, but since these constraints are non-local, it is not obvious what the best way to impose them is. Adding the constraints with a Lagrange multiplier to the action makes the action non-local and difficult to handle.

- Alternatively, if we would start with an action with a local symmetry based on the algebra $\mathscr{G}$, we could try to impose constraints on the gauge fields. Putting some components of the gauge field equal to one breaks Lorentz invariance, since the gauge field transforms as a vector, and not as a scalar. Putting components equal to zero is only compatible with Lorentz invariance if we put the same components equal to zero for all A_μ, $\mu = 0, \ldots, d - 1$. This is such a strong requirement that typically no interesting non-linear symmetry will be left. A possible way around these obstructions is to perform some twisting, so that A_μ transforms in a non-standard way under Lorentz transformations. Whether or not this is possible remains to be explored, but this is for example what one does in two dimensions to get infinite W-algebras.

- Previously, we have examined the representation theory of finite W-algebras in some detail. Given some n-dimensional unitary representation of a finite W-algebra, one can always take fields ϕ_i, $i = 1, \ldots, n$ that transform in this representation, and write down a finite W-invariant term $\phi_i^* \phi_i$. These are clearly not very interesting, as they are invariant under $\mathrm{U}(n)$ and the finite W-algebra is realized as a subalgebra of $\mathrm{U}(n)$. Now one can write down terms $\phi_i^* M_{ij} \phi_j$ which are also invariant under $\mathrm{U}(n)$ if M transforms in the adjoint representation. An interesting question is whether one can come up with some constrained field M_{ij} so that this term is no longer $\mathrm{U}(n)$ invariant, but still invariant under finite W-transformations, so that it can be used to break $\mathrm{U}(n)$ to a finite W-algebra. Another way to phrase this question is whether the finite W-orbit on M_{ij} is the same as the $\mathrm{U}(n)$ orbit, or strictly smaller?

- Can one use the fact that many finite W-algebras look as if they are deformations of some Lie algebra? There is a one-to-one correspondence between generators of a finite W-algebra

associated to an sl_2 embedding, and the generators of the Lie algebra $\mathscr{G}_0$ (see Section 3.2.4), given by the requirement that they belong to the same sl_2 orbit. We do not know whether a finite W-algebra can always be seen as a deformation of $\mathscr{G}_0$, and whether this would be useful for the construction of actions.

- The universal enveloping algebra of a finite W-algebra can be viewed as an infinite-dimensional Lie algebra, subject to additional relations. Can one construct a theory invariant under this infinite-dimensional Lie algebra and systematically impose the extra relations?

Fortunately, the obstruction explained in Section 6.1 does not apply in one dimension. For the remainder, we will examine some of the possibilities that exist in $d = 1$.

6.2. Finite W-invariance in $d = 1$

In one dimension it is possible to realize finite W-algebras in terms of conserved charges. However, this fact in itself is not yet sufficient to construct a theory invariant under finite W-symmetries. For that one needs a realization of finite W-algebras in terms of known objects, such as creation and annihilation operators or generators of a Lie algebra. Or alternatively, one can use the fact that finite W-algebras were obtained by imposing constraints on the generators of a Lie algebra. We will now briefly examine these possibilities in turn.

6.2.1. Imposing constraints

Suppose we have some action which is invariant under the global symmetry algebra $\mathscr{G}$. Associated to these global symmetries is a set of local conserved charges that obey $\{Q_\alpha, Q_\beta\} = f_{\alpha\beta}^\gamma Q_\gamma$. Then we can immediately write down an action in which we impose the constraints necessary to get the finite W-algebra $(\mathscr{G}, \mathscr{L}, \chi)$ by adding to the action the term

$$\int dt\, \mathrm{Tr}(A(Q - \chi(Q))) + \cdots , \tag{6.26}$$

where A is an $\mathscr{L}$ valued gauge field, and $Q = Q_\alpha T^\alpha$. The dots indicate possible terms that are of higher order in A. This action has a local gauge invariance generated by the first class constraints $Q - \chi(Q)$, under which A transforms as a kind of gauge field. How A transforms exactly depends on the details of the theory. If we perform a BRST gauge fixing of this gauge symmetry then the action becomes the original action plus a free ghost action, and the BRST operator that generates the BRST symmetries of this action is precisely the one we used to analyze quantum finite W-algebras, with J replaced by Q. In particular, the Hilbert space of the theory is given by the BRST cohomology and carries a representation of the finite W-algebra. This is a genuine finite W-invariant theory, but the W-invariance is only apparent on the level of the Hilbert space.

In some cases, one starts with theories whose symmetry algebra contains two copies of $\mathscr{G}$. When this happens it is possible to impose constraints on both algebras, and it can happen that one can explicitly integrate out the Lagrange multipliers, thus yielding an action without BRST symmetry but with a finite W-invariance. One particular example is to start with the action of a point particle moving in a group manifold. This will lead to the celebrated Toda theories, and we will describe this example in some more detail.

One starts with the action for a free particle moving on the group manifold G (the Lie algebra of G is $\mathscr{G}$). The metric on G is given by extending the Cartan–Killing metric $(t_a, t_b) = \text{Tr}(t_a t_b)$ on the Lie algebra $\mathscr{G}$ all over G in a left-right invariant way. This leads to the familiar action

$$S[g] = \frac{1}{2} \int dt \, \text{Tr}\left(g^{-1} \frac{dg}{dt} g^{-1} \frac{dg}{dt} \right) . \tag{6.27}$$

It satisfies the following identity

$$S[gh] = S[g] + S[h] + \int dt \, \text{Tr}\left(g^{-1} \frac{dg}{dt} \frac{dh}{dt} h^{-1} \right) , \tag{6.28}$$

from which one immediately deduces the equations of motion,

$$\frac{d}{dt}\left(g^{-1} \frac{dg}{dt} \right) = \frac{d}{dt}\left(\frac{dg}{dt} g^{-1} \right) = 0 . \tag{6.29}$$

The action (6.27) is invariant under $g \to h_1 g h_2$ for constant elements $h_1, h_2 \in G$. This leads to the conserved currents $J, \bar{J}$ given by

$$J = \frac{dg}{dt} g^{-1} \equiv J^a t_a \quad \text{and} \quad \bar{J} = g^{-1} \frac{dg}{dt} \equiv \bar{J}^a t_a . \tag{6.30}$$

The equations that express the conservation of these currents in time coincide with the equations of motion of the system, so fixing the values of these conserved quantities completely fixes the orbit of the particle once its position on $t = 0$ is specified. In this sense, the free particle on a group is a completely integrable system. The conserved quantities form a Poisson algebra [55]

$$\{J^a, J^b\} = f_c^{ab} J^c , \tag{6.31}$$

with similar equations for $\bar{J}$. This is precisely the Kirillov–Poisson bracket we used as a starting point for the construction of finite W-algebras. These were obtained by imposing constraints on the Poisson algebra (6.31), and we want to do the same here to get systems with finite W symmetry. Actually, we already have the first explicit example at our disposal here. If we consider the trivial embedding of sl_2 in SL_n, then the finite W-algebra is the Kirillov–Poisson algebra (6.31). The action (6.27) is the generalized Toda theory for the trivial embedding. The conserved currents of this generalized Toda theory form a Poisson algebra that is precisely the finite W-algebra associated to the trivial embedding.

Finite W-algebras were obtained by imposing a set of first class constraints $\pi_{\mathscr{L}}(J) = \chi(J)$, where $\pi_{\mathscr{L}}$ is the projection on $\mathscr{L}$. Here we want to impose the same constraints, together with similar constraints on $\bar{J}$,

$$\pi_{\mathscr{L}^*}(\bar{J}) = \chi^*(\bar{J}) \quad \text{and} \quad \pi_{\mathscr{L}}(J) = \chi(J) . \tag{6.32}$$

There are two equivalent ways to deal with these constraints. One can either reduce the equations of motion, or reduce the action for the free particle. Let us first reduce the equations of motion, where we restrict our attention to the finite W-algebras obtained from an sl_2 embedding. If $G_{\pm}$ denote the subgroups of G with Lie algebra $\mathscr{L}, \mathscr{L}^*$, and G_0 the subgroup with Lie algebra $\mathscr{G}_0$, then almost every element g of G can be decomposed as $g = g_- g_0 g_+$, where $g_{\pm, 0}$ are elements of the

corresponding subgroups, because G admits a generalized Gauss decomposition $G = G_- G_0 G_+$ {strictly speaking $G_- G_0 G_+$ is only dense in G, but we will ignore this subtlety in the remainder}. Inserting $g = g_- g_0 g_+$ into (6.32) we find

$$g_0^{-1} t_+ g_0 = \frac{dg_+}{dt} g_+^{-1} , \quad g_0 t_- g_0^{-1} = g_-^{-1} \frac{dg_-}{dt} . \tag{6.33}$$

In the derivation of these equations one uses that $\pi_{\mathscr{P}}(g_- t_+ g_-^{-1}) = t_+$, and a similar equation with $\pi_{\mathscr{P}^*}$ and t_-, which follow from the fact that $t_\pm$ have degree ± 1. The constrained currents look like

$$J = g_- \left(t_+ + \frac{dg_0}{dt} g_0^{-1} + g_0 t_- g_0^{-1} \right) g_-^{-1} , \quad \bar{J} = g_+^{-1} \left(t_- + g_0^{-1} \frac{dg_0}{dt} + g_0^{-1} t_+ g_0 \right) g_+ . \tag{6.34}$$

The equations of motion now become

$$0 = g_-^{-1} \frac{dJ}{dt} g_- = \frac{d}{dt} \left(\frac{dg_0}{dt} g_0^{-1} \right) + [g_0 t_- g_0^{-1}, t_+] ,$$

$$0 = g_+ \frac{d\bar{J}}{dt} g_+^{-1} = \frac{d}{dt} \left(g_0^{-1} \frac{dg_0}{dt} \right) + [t_-, g_0^{-1} t_+ g_0] , \tag{6.35}$$

which are generalized finite Toda equations as will be shown in a moment.

Alternatively, one can reduce the action by writing down the following gauged version of the action

$$S[g, A_+, A_-] = \frac{1}{2} \int dt \, \mathrm{Tr} \left(g^{-1} \frac{dg}{dt} g^{-1} \frac{dg}{dt} + A_+^2 + A_-^2 \right)$$

$$+ \int dt \, \mathrm{Tr}(A_-(J - \chi(J)) + A_+(\bar{J} - \chi^*(\bar{J})) + A_- g A_+ g^{-1}) . \tag{6.36}$$

This action is invariant under the following transformations

$$g \to h_- g h_+ , \quad A_- \to h_- A_- h_-^{-1} - \frac{dh_-}{dt} h_-^{-1} , \quad A_+ \to h_+^{-1} A_+ h_+ - h_+^{-1} \frac{dh_+}{dt} , \tag{6.37}$$

where $h_\pm$ are arbitrary elements of $G_\pm$. We assume here that χ is either constant or zero. If χ is some higher dimensional representation, one needs in addition to (6.36) a Lagrangian describing these additional degrees of freedom. In the case where the finite W-algebra comes from an sl_2 embedding we can use the gauge invariance to put $g_+ = g_- = e$ (where e is the unit element of the group G) in the Gauss decomposition of g, thus we can take $g = g_0 \in G_0$. Then from the equations of motion for $A_\pm$ we find $A_+ = g_0^{-1} t_+ g_0$ and $A_- = g_0 t_- g_0^{-1}$ (The terms A_+^2 and A_-^2 are not present in this case). Substituting these back into the action it reduces to

$$S[g_0] = \frac{1}{2} \int dt \, \mathrm{Tr} \left(g_0^{-1} \frac{dg_0}{dt} g_0^{-1} \frac{dg_0}{dt} \right) - \int dt \, \mathrm{Tr}(g_0 t_- g_0^{-1} t_+) . \tag{6.38}$$

The equations of motion for this action are indeed given by (6.35), showing the equivalence of the two approaches.

This generalized Toda action describes a particle moving on G_0 in some background potential. Two commuting copies of the finite W-algebra leave the action (6.38) invariant and act on the space of solutions of the equations of motion (6.35).[8] An explicit proof of this will be given in the next section. This action is only given infinitesimally, because we do not know how to exponentiate finite W-algebras. One can, however, sometimes find subspaces of the space of solutions that constitute a minimal orbit of the W-algebra, see for example [56] where this was worked out for the infinite W_3 algebra.

For the principal embeddings of sl_2 in sl_n, the equations of motion reduce to ordinary finite Toda equations of the type

$$\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \exp\left(\sum_{j=1}^{n-1} K_{ij} q_j\right) = 0 \, , \tag{6.39}$$

where $i = 1, \ldots, n - 1$, K_{ij} is the Cartan matrix of sl_n, and $g_0 = \exp(q_i H_i)$.

The general solution of the equations of motion (6.35) can be constructed as follows. Let $h_0^{(1)}, h_0^{(2)}$ be elements of G_0. Let X_0 be an arbitrary element of g_0. If $g_0(t)$ is defined by the Gauss decomposition

$$g_-(t)g_0(t)g_+(t) = h_0^{(1)} \exp t(X_0 + (h_0^{(1)})^{-1}t_+ h_0^{(1)} + h_0^{(2)}t_-(h_0^{(2)})^{-1})h_0^{(2)} \, , \tag{6.40}$$

then $g_0(t)$ is the most general solution of (6.35). The easiest way to study the action of the finite W-algebra on these solutions, is to use the explicit transformation rules (6.58). This might provide a valuable tool in the study of the solutions (6.40).

In the case when $\chi = 0$ in (6.36) it is not so straightforward to pick a good gauge for g, unless $\mathscr{L} = \mathscr{G}^+$, which corresponds to the case described in Section 4.3.2. Then a good gauge choice is to pick g in G_0, and integrating out $A_\pm$ is trivial. The result is

$$S[g_0] = \frac{1}{2} \int \mathrm{d}t \, \mathrm{Tr}\left(g_0^{-1} \frac{\mathrm{d}g_0}{\mathrm{d}t} g_0^{-1} \frac{\mathrm{d}g_0}{\mathrm{d}t}\right) . \tag{6.41}$$

The symmetry of this theory is $\mathscr{G}_0 \times \mathscr{G}_0$, which is in perfect agreement with (4.33).

Finally, let us present an action which has the finite W-algebra obtained by setting the Lie algebra generators in a Cartan subalgebra equal to zero, as discussed in Section 4.3.3. We take the action (6.36), but since it is not easy to find a good gauge choice for g, we put $A_+ = 0$, i.e. we impose only constraints on J, not on $\bar{J}$. This is a special case of (6.26). A good gauge choice is then for example $g = g_- g_+$, where $g_\pm \in G_\pm$ and $G = G_- TG_+$, is a standard Gauss decomposition of G. Integrating out A yields the following action

$$S[g_-, g_+] = \int \mathrm{d}t \, \mathrm{Tr}\left(g^{-1} \frac{\mathrm{d}g_-}{\mathrm{d}t} \frac{\mathrm{d}g_+}{\mathrm{d}t} g_+^{-1}\right) - \int \mathrm{d}t \, \mathrm{Tr}\left(\pi_{\mathscr{T}}\left(g_- \frac{\mathrm{d}g_+}{\mathrm{d}t} g_+^{-1} g_-^{-1}\right)^2\right) , \tag{6.42}$$

where $\pi_{\mathscr{T}}$ is the projection on the Lie algebra $\mathscr{T}$.

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[8] More precisely, the symmetries of (6.38) form an algebra that is on-shell isomorphic to a finite W-algebra.

Any of the actions in this section can in principle be used as a starting point for a theory with local finite W-symmetries in one dimension. We are not going to discuss this here; basically one can use the same techniques as one uses in two-dimensions to gauge infinite W-algebras, see e.g. [7].

6.2.2. Realizations of finite W-algebras in terms of lie algebras

The Miura transformation provides a realization of a finite W-algebra that comes from an sl_2 embedding in terms of the generators of $\mathcal{G}_0$ (see Section 3.2.4). Can one, given such a realization, find an action invariant under finite W-transformations? One way is to first express the generators of $\mathcal{G}_0$ in terms of oscillators (see [6,57]), thereby providing a Fock realization of the finite W-algebra. Subsequently, one can try to use the method in the next section to find an invariant action. Alternatively, one can try to use the realization in terms of $\mathcal{G}_0$. An obvious guess is to look for actions of the type

$$S[g_0] = \frac{1}{2} \int dt\, \mathrm{Tr}\left(g_0^{-1} \frac{dg_0}{dt} g_0^{-1} \frac{dg_0}{dt}\right) - \int dt\, V(g_0)\,. \tag{6.43}$$

The kinetic part of this action has a $\mathcal{G}_0 \times \mathcal{G}_0$ invariance, and the problem is now to find a potential $V(g_0)$ that reduces this invariance to a finite W-invariance. To be able to say something more we first need to work out some properties of the polynomials in terms of $J_0 = (dg_0/dt)g_0^{-1}$ that the Miura transformation gives us and that form a realization of the finite W-algebra.

The (classical) Miura transformation can be described as follows. We start with the Kirillov–Poisson algebra (6.31), and decompose J as $J_- + J_0 + J_+$, and in addition we will decompose $J_- = J_{-1} + J_{-2} + \cdots$ according to (3.33). After imposing the constraints we get $J_{\mathrm{constr}} = J_- + J_0 + t_+$. The finite W-algebra is the Poisson algebra of the polynomial. $P(J_-, J_0)$ that are gauge invariant under the gauge transformations generated by the first class constraints on the constrained phase space, i.e. $P(J_-, J_0)$ satisfies

$$\mathrm{Tr}\left(\left(\frac{\delta P(J_-, J_0)}{\delta J_-} + \frac{\delta P(J_-, J_0)}{\delta J_0}\right)[J_- + J_0 + t_+, \varepsilon_-]\right) = 0\,, \tag{6.44}$$

where ε_- is an arbitrary parameter with values in $\mathcal{G}_-$. We can rewrite (6.44) as

$$\pi_+\left[\frac{\delta P(J_-, J_0)}{\delta J_-} + \frac{\delta P(J_-, J_0)}{\delta J_0}, J_- + J_0 + t_+\right] = 0\,, \tag{6.45}$$

with π_+ the projection on $\mathcal{G}_+$. The Miura transformation does not give $P(J_-, J_0)$, but just $P(J_0) = P(J_-, J_0)|_{J_-=0}$. If we insert $J_- = 0$ in (6.45) and then project it onto $\mathcal{G}_{+1}$, we find

$$[Q(J_0), J_0] + \left[\frac{\delta P(J_0)}{\delta J_0}, t_+\right] = 0\,, \tag{6.46}$$

with $Q(J_0) = (\delta P(J_-, J_0)/\delta J_{-1})|_{J_-=0}$.

Under a small W-transformation generated by $P(J_0)$, the potential $V(g_0)$ transforms as

$$\delta V(g_0) = \varepsilon\, \mathrm{Tr}\left(\frac{\delta P(J_0)}{\delta J_0} \frac{\delta V(g_0)}{\delta g_0 g_0^{-1}}\right)\,. \tag{6.47}$$

For the perturbed action to be W-invariant, we want this variation to be a total derivative. In order to be able to use the identity (6.46) in (6.47), we need to require that

$$\frac{\delta V(g_0)}{\delta g_0 g_0^{-1}} = [t_+, R(g_0)] \ . \tag{6.48}$$

This allows us to rewrite

$$\delta V(g_0) = \varepsilon \operatorname{Tr}(Q(J_0)[R(g_0), J_0]) \ . \tag{6.49}$$

If, in addition to (6.48), we require that

$$[R(g_0), J_0] = \frac{\mathrm{d}}{\mathrm{d}t} T(g_0) \tag{6.50}$$

for some functional $T(g_0)$, then $\delta V(g_0)$ is a total time derivative modulo the equation of motion $\mathrm{d}J_0/\mathrm{d}t = 0$ of the unperturbed part of (6.43). We can then modify the transformation rule of g_0 to cancel the equation of motion terms in $\delta V(g_0)$, but this gives rise to a new variation of the potential $V(g_0)$, and we have to check that this is again a total time derivative. Clearly, this is a somewhat cumbersome procedure and it is not clear that it terminates. One obvious solution to (6.48) and (6.50) is the potential $V(g_0) = \operatorname{Tr}(g_0 t_- g_0^{-1} t_+)$, with $R(g_0) = g_0 t_- g_0^{-1}$ and $T(g_0) = -R(g_0)$. We do not know whether other simple solutions to (6.48) and (6.50) exist. Rather than verifying step by step that (6.38) is invariant under finite W-transformations, we will give a direct proof of this. This proof, given below, will show that the procedure sketched in this section is in general not very efficient for finding invariant actions, but may provide some clues for finding other and better techniques.

The Toda action was obtained by imposing constraints on both J and $\bar{J}$. The corresponding constraints first brought J in the form $J = J_- + J_0 + t_+$, and subsequently in the form (6.34), where g_- is a function of g_0 determined by (6.33). If we substitute this special constrained form of J in the gauge invariant polynomials $P(J_0, J_-)$, then the g_- dependence disappears, since we are precisely interested in G_- invariant polynomials, and we find the following polynomials in terms of g_0 and its time derivatives:

$$P(g_0) \equiv P(J_0, J_-)|_{J_0 = (\mathrm{d}g_0/\mathrm{d}t)g_0^{-1}, J_{-1} = g_0 t_- g_0^{-1}, J_{-2} = 0, J_{-3} = 0, \ldots} \ . \tag{6.51}$$

We claim that these are precisely the conserved quantities of the Toda theory. To prove this, we take (6.45) and deduce from it that we must in particular have

$$\operatorname{Tr}\left(J_{-1}\left[\frac{\delta P(J_-, J_0)}{\delta J_-} + \frac{\delta P(J_-, J_0)}{\delta J_0}, J_- + J_0 + t_+\right]\right) = 0 \ . \tag{6.52}$$

If we put $J_{-2} = J_{-3} = \cdots = 0$ in this equation, and denote

$$P(J_{-1}, J_0) = P(J_-, J_0)|_{J_{-2} = 0, J_{-3} = 0, \ldots} \tag{6.53}$$

etc., we find

$$\mathrm{Tr}\left(J_{-1}\left(\left[\frac{\delta P(J_{-1},J_0)}{\delta J_{-1}},J_0\right]+\left[\frac{\delta P(J_{-2},J_{-1},J_0)}{\delta J_{-2}}\bigg|_{J_{-2}=0},J_{-1}\right]\right.\right.$$

$$\left.\left.+\left[\frac{\delta P(J_{-1},J_0)}{\delta J_0},t_+\right]\right)\right)=0\,.\tag{6.54}$$

The middle term in this equation drops out, and the remainder can be rewritten as

$$\mathrm{Tr}\left(\frac{\delta P(J_{-1},J_0)}{\delta J_{-1}}\,[J_0,J_{-1}]+\frac{\delta P(J_{-1},J_0)}{\delta J_0}\,[J_0,t_+]\right)=0\,.\tag{6.55}$$

Now if we put $J_0=(\mathrm{d}g_0/\mathrm{d}t)g_0^{-1}$ and $J_{-1}=g_0t_-g_0^{-1}$, then

$$\frac{\mathrm{d}J_0}{\mathrm{d}t}=[t_+,J_{-1}]\,,\qquad\frac{\mathrm{d}J_{-1}}{\mathrm{d}t}=[J_0,J_{-1}]\,.\tag{6.56}$$

The first equation is the Toda equation of motion, and the second one is a straightforward algebraic identity. Using these, (6.55) can be rewritten as

$$\mathrm{Tr}\left(\frac{\delta P(J_{-1},J_0)}{\delta J_{-1}}\frac{\mathrm{d}J_{-1}}{\mathrm{d}t}+\frac{\delta P(J_{-1},J_0)}{\delta J_0}\frac{\mathrm{d}J_0}{\mathrm{d}t}\right)=0\,,\tag{6.57}$$

and this is nothing but $(\mathrm{d}/\mathrm{d}t)P(g_0)$, proving that $p(g_0)$ is a conserved quantity in the Toda theory.

The transformation rules that leave (6.38) invariant can now immediately be deduced. They read

$$\delta g_0=\varepsilon\left(\frac{\delta P(J_{-1},J_0)}{\delta J_0}\bigg|_{J_0=(\mathrm{d}g_0/\mathrm{d}t)g_0^{-1},\,J_{-1}=g_0t_-g_0^{-1}}\right)g_0\,.\tag{6.58}$$

To verify that these transformations form indeed a finite W-algebra, we compute the variation of another conserved quantity $Q(g_0)=Q(J_{-1},J_0)|_{J_0=(\mathrm{d}g_0/\mathrm{d}t)g_0^{-1},\,J_{-1}=g_0t_-g_0^{-1}}$ under the finite W-transformations generated by $P(g_0)$

$$\delta Q=\mathrm{Tr}\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta P}{\delta J_0}\right)+\left[\frac{\delta P}{\delta J_0},J_0\right]\right)\frac{\delta Q}{\delta J_0}+\left[\frac{\delta P}{\delta J_0},J_{-1}\right]\frac{\delta Q}{\delta J_{-1}}\right)\,.\tag{6.59}$$

Using the exact identity (6.55) and the Toda equations of motion, we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta P}{\delta J_0}\right)=\left[\frac{\delta P}{\delta J_{-1}},J_{-1}\right]\tag{6.60}$$

modulo equations of motion. Inserted into (6.59) this yields

$$\delta Q=\mathrm{Tr}\left(\left[\frac{\delta P}{\delta J_{-1}},J_{-1}\right]\frac{\delta Q}{\delta J_0}+\left[\frac{\delta P}{\delta J_0},J_0\right]\frac{\delta Q}{\delta J_0}+\left[\frac{\delta P}{\delta J_0},J_{-1}\right]\frac{\delta Q}{\delta J_{-1}}\right)\,.\tag{6.61}$$

This corresponds exactly to the brackets of the finite W-algebra, with $J_{-2}=\cdots=0$. Therefore, the symmetry transformations (6.58) form a finite W-algebra modulo field equations, i.e. the algebra is on-shell isomorphic to a finite W-algebra.

6.2.3. Realizations of finite W-algebras in terms of oscillators

Given a realization of a finite W-algebra in terms of a set of oscillators, or more general in terms of the generators of an algebra $\mathscr{A}$, one can try to compute the centralizer of the W-algebra in $\mathscr{A}$, and take any of the generators of the centralizer as the generator of time translations in some physical system. For this to make sense one wants the generator to be Hermitian, so that it can be identified with the Hamiltonian. Any action whose Hamiltonian is identical to this one is an action invariant under W-transformations, provided the commutation relations found by canonical quantization agree with those given by the algebraic structure of $\mathscr{A}$. For any realization, the center of the finite W-algebra is always a subalgebra of the centralizer in $\mathscr{A}$, and these are the first candidates to look at.

As an example, we can take the oscillator realization of $W_3^{(2)}$ in Section 2.2.2. It is given by

$$j_+ = \frac{1}{\sqrt{3}} a^2 b^\dagger \,, \quad j_- = \frac{-1}{\sqrt{3}} a(b^\dagger)^2 \,, \quad j_0 = \tfrac{2}{3} b^\dagger b - \tfrac{2}{3} a^\dagger a \,,$$

$$C = -(\tfrac{1}{3} a^\dagger a + \tfrac{2}{3} b^\dagger b)^2 - (\tfrac{1}{3} a^\dagger a + \tfrac{2}{3} b^\dagger b) \,. \tag{6.62}$$

The center of $W_3^{(2)}$ contains C, and the Hamiltonian of the anisotropic harmonic oscillator is given by $\tfrac{3}{4}\sqrt{1 - 4C}$. Although this is non-polynomial in terms of C, it is polynomial in terms of the oscillators and belongs to the centralizer of the finite W-algebra. The other generator of the center of $W_3^{(2)}$ is $C_3 = j_0^3 + 2j_0 + 3j_0 C + 3j_+ j_- + 3j_- j_+$, which is equal to $h/3 - 16h^3/27$ with h the Hamiltonian. This illustrates the fact that the centralizer of $W_3^{(2)}$ in the oscillator algebra is generated by h.[9] It would be interesting to know whether the example of the anisotropic harmonic oscillator can somehow be obtained as a reduction of a system with $sl(3)$ symmetry. This would open the door for the construction of many more examples of quantum mechanical systems with finite W-symmetry.

To conclude, let us mention one more possibility. So far we tried to find systems with a finite W-algebra as its symmetry algebra. We could also be less restrictive, and demand that it is just a spectrum generating algebra. For our example of $W_3^{(2)}$, this would mean that we can also take j_0 as our Hamiltonian, since j_+ and j_- map j_0 eigenstates to j_0 eigenstates. Incidentally, an explicit example where this is the case is known [58]. Consider a sequence of Schrödinger operators $L_j = A_j^+ A_j^- + \lambda_j$ where $A_j^\pm = \pm(\mathrm{d}/\mathrm{d}x) + f_j(x)$ and λ_j is a constant. If $L_j A_j^+ = A_j^+ L_{j+1}$ and $A_j^- L_j = L_{j+1} A_j^-$, then the $A_j^\pm$ can be used to map eigenstates of L_j into those of L_{j+1} and vice versa. Therefore, if we know the spectrum of L_j we also know that of L_{j+1}. This technique to construct new exactly solvable Schrödinger operators from old ones is known as the factorization method. An interesting situation arises when one imposes a kind of periodic boundary condition on the chain of operators L_j, namely if one requires $L_{j+N} = L_j + \mu$ and $\lambda_{j+N} = \lambda_j + \mu$ for some parameter μ. If we denote L_1 by j_0 and define

$$j_+ = A_1^+ \ldots A_N^+ \,, \quad j_- = A_N^- \ldots A_1^- \,, \tag{6.63}$$

---

[9] If one allows non-polynomial expressions in terms of the oscillators, this is no longer true, as can be seen from the example $e^{\pi i a^\dagger a}$, which commutes with all generators of $W_3^{(2)}$.

then the following relations hold

$$[j_0, j_+] = \mu j_+ , \quad [j_0, j_-] = -\mu j_- ,$$

$$j_+ j_- = \prod_{k=1}^{N} (j_0 - \lambda_k) , \quad j_- j_+ = \prod_{k=1}^{N} (j_0 - \lambda_k + \mu) .$$

(6.64)

In particular, for $N = 3$, the commutator of j_+ and j_- will be a quadratic polynomial in j_0, exactly as in the $W_3^{(2)}$ algebra. The Hamiltonian j_0 in (6.64) is $-(d^2/dx^2) + f_1^2 + f_1' + \lambda_1$, where f_1 satisfies the following set of coupled differential equations

$$-f_1' + f_1^2 + \lambda_1 = f_2^2 + f_2' + \lambda_2 ,$$

$$-f_2' + f_2^2 + \lambda_2 = f_3^2 + f_3' + \lambda_3 ,$$

(6.65)

$$-f_3' + f_3^2 + \lambda_3 = f_1^2 + f_1' + \lambda_1 + \mu .$$

To solve these, first notice that the sum of these three equations is $2(f_1' + f_2' + f_3') + \mu = 0$, from which one derives $f_2 + f_3 = -f_1 - \mu x/2 + k$, with k some integration constant. Next, the second equation in (6.65) can be rewritten as $(f_3 - f_2)(f_3 + f_2) + (f_3 + f_2)' + \lambda_3 - \lambda_2 = 0$, or equivalently $(f_3 - f_2)(-f_1 - \mu x/2 + k) + (-f_1' - \mu/2) + \lambda_3 - \lambda_2 = 0$. We thus have one equation for $f_2 + f_3$ and one for $f_2 - f_3$ in terms of f_1. Solving these for f_2 and f_3 gives upon substituting these back into the differential equations (6.65) a differential equation for f_1, which turns out to be the Painlevé-IV equation. The corresponding potential in j_0 is then a one-gap potential. It would be interesting to see what role the representations of $W_3^{(2)}$ play in the spectrum of j_0.

The periodicity conditions for the operators L_j allow for a natural q-deformation [59]. The corresponding Schrödinger operators have a spectrum generating algebra which is a q-deformation of (6.64), and in particular for $N = 3$ one finds a q-deformation of $W_3^{(2)}$. The issue whether one can q-deform arbitrary finite W-algebras is an entirely different story, which we will not discuss here, but it is amusing to see that a q-deformation of $W_3^{(2)}$ can still occur in simple quantum mechanical systems.

Acknowledgements

Appendix

In this appendix we discuss the finite W-algebras that can be obtained from sl_4. Using the standard constructions developed in this chapter we calculate their relations and the quantum Miura maps.

Case 1: $4 = 2 + 1 + 1$

The basis of sl_4 we use to study this quantum algebra is

$$r_a t_a = \begin{pmatrix} \frac{1}{2}r_{10} - \frac{1}{8}r_8 - \frac{1}{8}r_9 & r_{11} & r_{12} & r_{15} \\ r_5 & \frac{3}{8}r_8 - \frac{1}{8}r_9 & r_7 & r_{14} \\ r_4 & r_6 & -\frac{1}{8}r_8 + \frac{3}{8}r_9 & r_{13} \\ r_1 & r_2 & r_3 & -\frac{1}{2}r_{10} - \frac{1}{8}r_8 - \frac{1}{8}r_9 \end{pmatrix}. \tag{A.1}$$

The sl_2 embedding is given by $t_+ = t_{15}, t_0 = t_{10}$ and $t_- = \frac{1}{2}t_1$. The nilpotent algebra $\mathcal{G}_+$ is spanned by $\{t_{13}, t_{14}, t_{15}\}$, $\mathcal{G}_0$ by $\{t_4, \ldots, t_{12}\}$ and $\mathcal{G}_-$ by $\{t_1, t_2, t_3\}$. The d_1 cohomology of Ω_{red} is generated by $\hat{J}^a$ for $a = 1, \ldots, 9$. Representatives that are exactly d-closed, are given by $W(\hat{J}^a) = \hat{J}^a$ for $a = 4, \ldots, 9$, and by

$$W(\hat{J}^1) = \hat{J}^1 + \hat{J}^4 \hat{J}^{12} + \hat{J}^5 \hat{J}^{11} + \frac{1}{4} \hat{J}^{10} \hat{J}^{10} + \frac{\hbar}{2} \hat{J}^{10},$$

$$W(\hat{J}^2) = \hat{J}^2 + \hat{J}^6 \hat{J}^{12} + \frac{1}{2} \hat{J}^8 \hat{J}^{11} + \frac{1}{2} \hat{J}^{10} \hat{J}^{11} + \hbar \hat{J}^{11}, \tag{A.2}$$

$$W(\hat{J}^3) = \hat{J}^3 + \hat{J}^7 \hat{J}^{11} + \frac{1}{2} \hat{J}^9 \hat{J}^{12} + \frac{1}{2} \hat{J}^{12} \hat{J}^{10} + \frac{\hbar}{2} \hat{J}^{12}.$$

Introduce a new basis of fields as follows:

$$U = \frac{1}{4}(W(\hat{J}^8) + W(\hat{J}^9)), \quad H = \frac{1}{4}(W(\hat{J}^8) - W(\hat{J}^9)), \quad F = -W(\hat{J}^7), \quad E = -W(\hat{J}^6),$$

$$G_1^- = -W(\hat{J}^3), \quad G_1^+ = W(\hat{J}^2), \quad G_2^- = W(\hat{J}^5), \quad G_2^+ = W(\hat{J}^4), \tag{A.3}$$

$$C = W(\hat{J}^1) + \frac{1}{2}EF + \frac{1}{2}FE + H^2 + \frac{1}{2}U^2 + 2\hbar U.$$

If we compute the commutators of these expressions, we find that C commutes with everything, $\{E, F, H\}$ form a sl_2 subalgebra and $G_i^\pm$ are spin $\frac{1}{2}$ representations for this sl_2 subalgebra. U represents an extra $u(1)$ charge. The nonvanishing commutators, with $\hbar$ dependence, are

$$[E, F] = 2\hbar H, \quad [H, E] = \hbar E, \quad [H, F] = -\hbar F,$$

$$[U, G_1^\pm] = \hbar G_1^\pm, \quad [U, G_2^\pm] = -\hbar G_2^\pm,$$

$$[H, G_i^\pm] = \pm\frac{\hbar}{2} G_i^\pm, \quad [E, G_i^-] = \hbar G_i^+, \quad [F, G_i^+] = \hbar G_i^-, \tag{A.4}$$

$$[G_1^+, G_2^+] = -2\hbar E(U + \hbar), \quad [G_1^-, G_2^-] = 2\hbar F(U + \hbar),$$

$$[G_1^+, G_2^-] = \hbar(-C + EF + FE + 2H^2 + \frac{3}{2}U^2 + 2HU) + \hbar^2(2H + 3U),$$

$$[G_1^-, G_2^+] = \hbar(C - EF - FE - 2H^2 - \frac{3}{2}U^2 + 2HU) + \hbar^2(2H - 3U).$$

Let us also present the quantum Miura transformation for this algebra. In this case, $\mathcal{G}_0 = sl_3 \oplus u(1)$. Standard generators of $\mathcal{G}_0$ can be easily identified. A generator of $u(1)$ is $s = \frac{1}{2}\hat{J}^8 + \frac{1}{2}\hat{J}^9 + 2\hat{J}^{10}$, and the sl_3 generators are $e_1 = \hat{J}^5$, $e_2 = \hat{J}^6$, $e_3 = \hat{J}^4$, $f_1 = \hat{J}^{11}$, $f_2 = \hat{J}^7$, $f_3 = \hat{J}^{12}$, $h_1 = -\frac{1}{2}\hat{J}^8 + \frac{1}{2}\hat{J}^{10}$ and $h_2 = \frac{1}{2}\hat{J}^8 - \frac{1}{2}\hat{J}^9$. The convention is such that the commutation

relations between $\{e_i, f_i, h_i\}$ are the same as those of the corresponding matrices defined by

$$a_i e_i + b_i f_i + c_i h_i = \hbar \begin{pmatrix} c_1 & a_1 & a_3 \\ b_1 & c_2 - c_1 & a_2 \\ b_3 & b_2 & -c_2 \end{pmatrix}. \tag{A.5}$$

The quantum Miura transformation reads

$$U = \tfrac{1}{6}(s - 2h_2 - 4h_1), \quad H = \tfrac{1}{2}h_2, \quad F = -f_2, \quad E = -e_2,$$

$$G_1^- = -f_2 f_1 - \tfrac{1}{3}(s - 2h_2 - h_1 + 3\hbar) f_3, \quad G_1^+ = e_2 f_3 + \tfrac{1}{3}(s + h_2 - h_1 + 3\hbar) f_1,$$

$$G_2^- = e_1, \quad G_2^+ = e_3, \tag{A.6}$$

$$C = \left(\frac{1}{24} s^2 + \frac{\hbar}{2} s\right) + \tfrac{1}{2}(e_1 f_1 + f_1 e_1 + e_2 f_2 + f_2 e_2 + e_3 f_3 + f_3 e_3)$$

$$\quad + \tfrac{1}{3}(h_1^2 + h_1 h_2 + h_2^2).$$

In C we again recognize the second Casimir of sl_3. It is a general feature of finite W-algebras that they contain a central element C, whose Miura transform contains the second Casimir of $\mathcal{G}_0$. C is the finite counterpart of the energy momentum tensor that every infinite W-algebra possesses.

Case 2: $4 = 2 + 2$
A convenient basis to study this case is

$$r_a t_a = \begin{pmatrix} \dfrac{r_6}{4} + \dfrac{r_{10}}{2} & \dfrac{r_5}{2} + \dfrac{r_8}{2} & r_{12} & r_{14} \\[2mm] \dfrac{r_7}{2} + \dfrac{r_9}{2} & -\dfrac{r_6}{4} - \dfrac{r_{11}}{2} & r_{13} & r_{15} \\[2mm] r_1 & r_2 & \dfrac{r_6}{4} - \dfrac{r_{10}}{2} & \dfrac{r_5}{2} - \dfrac{r_8}{2} \\[2mm] r_3 & r_4 & \dfrac{r_7}{2} - \dfrac{r_9}{2} & -\dfrac{r_6}{4} + \dfrac{r_{11}}{2} \end{pmatrix}. \tag{A.7}$$

The sl_2 embedding is given by $t_+ = t_{12} + t_{15}$, $t_0 = t_{10} - t_{11}$ and $t_- = \tfrac{1}{2}(t_1 + t_4)$. The subalgebra $\mathcal{G}_+$ is spanned by $\{t_{12}, \ldots, t_{15}\}$, $\mathcal{G}_0$ by $\{t_5, \ldots, t_{11}\}$ and $\mathcal{G}_-$ by $\{t_1, \ldots, t_4\}$. The d_1 cohomology of Ω_{red} is generated by $\hat{J}^1, \ldots, \hat{J}^7$. The d-closed representatives are $W(\hat{J}^a) = \hat{J}^a$ for $a = 5, 6, 7$, and

$$W(\hat{J}^1) = \hat{J}^1 - \tfrac{1}{4}\hat{J}^5 \hat{J}^9 + \tfrac{1}{4}\hat{J}^7 \hat{J}^8 + \tfrac{1}{4}\hat{J}^8 \hat{J}^9 + \tfrac{1}{4}\hat{J}^{10} \hat{J}^{10} + \frac{3\hbar}{4}\hat{J}^{10} - \frac{\hbar}{4}\hat{J}^{11},$$

$$W(\hat{J}^2) = \hat{J}^2 + \tfrac{1}{4}\hat{J}^5 \hat{J}^{10} + \tfrac{1}{4}\hat{J}^5 \hat{J}^{11} - \tfrac{1}{4}\hat{J}^6 \hat{J}^8 + \tfrac{1}{4}\hat{J}^8 \hat{J}^{10} - \tfrac{1}{4}\hat{J}^{10} \hat{J}^{11} + \frac{\hbar}{2}\hat{J}^8,$$

$$W(\hat{J}^3) = \hat{J}^3 - \tfrac{1}{4}\hat{J}^6 \hat{J}^9 - \tfrac{1}{4}\hat{J}^7 \hat{J}^{10} - \tfrac{1}{4}\hat{J}^7 \hat{J}^{11} + \tfrac{1}{4}\hat{J}^9 \hat{J}^{10} - \tfrac{1}{4}\hat{J}^9 \hat{J}^{11} + \frac{\hbar}{2}\hat{J}^9, \tag{A.8}$$

$$W(\hat{J}^4) = \hat{J}^4 + \tfrac{1}{4}\hat{J}^5 \hat{J}^9 - \tfrac{1}{4}\hat{J}^7 \hat{J}^8 + \tfrac{1}{4}\hat{J}^8 \hat{J}^9 + \tfrac{1}{4}\hat{J}^{11} \hat{J}^{11} + \frac{\hbar}{4}\hat{J}^{10} - \frac{3\hbar}{4}\hat{J}^{11}.$$

To display the properties of the algebra as clearly as possible, we introduce a new basis of fields

$$H = \tfrac{1}{2}W(\hat{J}^6), \quad E = -W(\hat{J}^7), \quad F = -W(\hat{J}^5),$$

$$G^+ = W(\hat{J}^3), \quad G^0 = W(\hat{J}^1) - W(\hat{J}^4), \quad G^- = -W(\hat{J}^2),$$

$$C = W(\hat{J}^1) + W(\hat{J}^4) + \tfrac{1}{8}W(\hat{J}^6)W(\hat{J}^6) + \tfrac{1}{4}W(\hat{J}^5)W(\hat{J}^7) \tag{A.9}$$

$$+ \tfrac{1}{4}W(\hat{J}^7)W(\hat{J}^5) + \frac{\hbar}{4}W(\hat{J}^6).$$

Here, C is the by now a familiar central element, $\{E, H, F\}$ form an sl_2 algebra and $\{G^+, G^0, G^-\}$ form a spin 1 representation with respect to this sl_2 algebra. The non-vanishing commutators are

$$[E, F] = 2\hbar H, \quad [H, E] = \hbar E, \quad [H, F] = -\hbar F,$$

$$[E, G^0] = 2\hbar G^+, \quad [E, G^-] = \hbar G^0, \quad [F, G^+] = \hbar G^0, \quad [F, G^0] = 2\hbar G^-,$$

$$[H, G^+] = \hbar G^+, \quad [H, G^-] = -\hbar G^-,$$

$$[G^0, G^+] = \hbar(-CE + EH^2 + \tfrac{1}{2}EEF + \tfrac{1}{2}EFE) - 2\hbar^3 E, \tag{A.10}$$

$$[G^0, G^-] = \hbar(-CF + FH^2 + \tfrac{1}{2}FFE + \tfrac{1}{2}FEF) - 2\hbar^3 F,$$

$$[G^+, G^-] = \hbar(-CH + H^3 + \tfrac{1}{2}HEF + \tfrac{1}{2}HFE) - 2\hbar^3 H.$$

Since $\mathcal{G}_0 = sl_2 \oplus sl_2 \oplus u(1)$, the quantum Miura transformation expresses this algebra in terms of generators $\{e_1, h_1, f_1\}, \{e_2, h_2, f_2\}, s$ of $\mathcal{G}_0$. The relation between these generators and the $\hat{J}^a$ are: $s = \hat{J}^{10} - \hat{J}^{11}$, $h_1 = \tfrac{1}{2}(\hat{J}^6 + \hat{J}^{10} + \hat{J}^{11})$, $h_2 = \tfrac{1}{2}(\hat{J}^6 - \hat{J}^{10} - \hat{J}^{11})$, $e_1 = \tfrac{1}{2}(\hat{J}^7 + \hat{J}^9)$, $e_2 = \tfrac{1}{2}(\hat{J}^7 - \hat{J}^9)$, $f_1 = \tfrac{1}{2}(\hat{J}^5 + \hat{J}^8)$ and $f_2 = \tfrac{1}{2}(\hat{J}^5 - \hat{J}^8)$. The commutation relations for these are $[e_1, f_1] = \hbar h_1$, $[h_1, e_1] = 2\hbar e_1$, $[h_1, f_1] = -2\hbar f_1$, and similar for $\{e_2, h_2, f_2\}$. For the quantum Miura transformation one then finds

$$H = \tfrac{1}{2}(h_1 + h_2), \quad E = -e_1 - e_2, \quad F = -f_1 - f_2,$$

$$G^+ = \tfrac{1}{2}e_1 h_2 - \tfrac{1}{2}e_2 h_1 + \tfrac{1}{4}s(e_1 - e_2) + \hbar(e_1 - e_2),$$

$$G^0 = f_1 e_2 - f_2 e_1 + \tfrac{1}{4}s(h_1 - h_2) + \hbar(h_1 - h_2), \tag{A.11}$$

$$G^- = \tfrac{1}{2}f_1 h_2 - \tfrac{1}{2}f_2 h_1 - \tfrac{1}{4}s(f_1 - f_2) - \hbar(f_1 - f_2),$$

$$C = (\tfrac{1}{8}s^2 + \hbar s) + \tfrac{1}{2}(e_1 f_1 + f_1 e_1 + e_2 f_2 + f_2 e_2) + \tfrac{1}{4}(h_1^2 + h_2^2).$$

The infinite-dimensional version of this algebra is one of the 'covariantly coupled' algebras that have been studied in [60]. The finite algebra (A.10) is almost a Lie algebra. If we assign particular values to C and to the second Casimir $C_2 = (H^2 + \tfrac{1}{2}EF + \tfrac{1}{2}FE)$ of the sl_2 subalgebra spanned by $\{E, H, F\}$, then (A.10) reduces to a Lie algebra. For a generic choice of the values of C and C_2 this Lie algebra is isomorphic to $sl_2 \oplus sl_2$. An interesting question is, whether similar phenomena occur for different covariantly coupled algebras.

Case 3: $4 = 3 + 1$

The last non-trivial non-principal sl_2 embedding we consider is $\underline{4}_4 \simeq \underline{3}_2 \oplus \underline{1}_2$. We choose yet another basis

$$
r_a t_a = \begin{pmatrix} \dfrac{r_5}{12} - \dfrac{r_6}{3} & r_8 & r_{11} & r_{12} \\[2mm] r_4 & -\dfrac{r_5}{4} & r_{13} & r_{14} \\[2mm] \dfrac{r_3}{2} - \dfrac{r_9}{2} & r_{10} & \dfrac{r_5}{12} + \dfrac{r_6}{6} + \dfrac{r_7}{2} & r_{15} \\[2mm] r_1 & r_2 & \dfrac{r_3}{2} + \dfrac{r_9}{2} & \dfrac{r_5}{12} + \dfrac{r_6}{6} - \dfrac{r_7}{2} \end{pmatrix} , \tag{A.12}
$$

in terms of which the sl_2 embedding is $t_+ = t_{11} + t_{15}$, $t^0 = -3t_6 + t_7$ and $t_- = 2t_3$. The subalgebra $\mathcal{G}_+$ is generated by $\{t_{11}, \ldots, t_{15}\}$, $\mathcal{G}_-$ is generated by $\{t_1, t_2, t_3, t_9, t_{10}\}$, and $\mathcal{G}_0$ is generated by $\{t_4, \ldots, t_8\}$. The d_1 cohomology is generated by $\hat{J}^1, \ldots, \hat{J}^5$, and d-closed representatives are given by $W(\hat{J}^4) = \hat{J}^4$, $W(\hat{J}^5) = \hat{J}^5$, and

$$
W(\hat{J}^1) = \hat{J}^1 + \tfrac{1}{6}\hat{J}^3\hat{J}^6 - \tfrac{1}{12}\hat{J}^6\hat{J}^3 + \tfrac{1}{4}\hat{J}^7\hat{J}^3 + \hat{J}^4\hat{J}^{10} + \tfrac{1}{4}\hat{J}^6\hat{J}^9 - \tfrac{1}{4}\hat{J}^7\hat{J}^9 - \tfrac{1}{3}\hat{J}^4\hat{J}^5\hat{J}^8
$$

$$
\quad - \tfrac{1}{3}\hat{J}^4\hat{J}^6\hat{J}^8 - \tfrac{1}{108}\hat{J}^6\hat{J}^6\hat{J}^6 + \tfrac{1}{12}\hat{J}^6\hat{J}^7\hat{J}^7 + \frac{31\hbar}{12}\hat{J}^4\hat{J}^8 - \frac{3\hbar}{4}\hat{J}^9 + \frac{5\hbar}{48}\hat{J}^6\hat{J}^6
$$

$$
\quad + \frac{\hbar}{6}\hat{J}^7\hat{J}^6 - \frac{3\hbar}{16}\hat{J}^7\hat{J}^7 - \frac{3\hbar^2}{8}\hat{J}^7 - \frac{7\hbar^2}{24}\hat{J}^6 ,
$$

$$
W(\hat{J}^2) = \hat{J}^2 - \tfrac{1}{2}\hat{J}^3\hat{J}^8 - \tfrac{1}{3}\hat{J}^5\hat{J}^{10} - \tfrac{1}{6}\hat{J}^6\hat{J}^{10} + \tfrac{1}{2}\hat{J}^7\hat{J}^{10} - \tfrac{1}{2}\hat{J}^8\hat{J}^9 + \frac{3\hbar}{2}\hat{J}^{10} + \tfrac{1}{9}\hat{J}^5\hat{J}^5\hat{J}^8 \tag{A.13}
$$

$$
\quad + \tfrac{1}{9}\hat{J}^5\hat{J}^6\hat{J}^8 + \tfrac{1}{36}\hat{J}^6\hat{J}^6\hat{J}^8 - \tfrac{1}{4}\hat{J}^7\hat{J}^7\hat{J}^8 - \hbar\hat{J}^5\hat{J}^8 - \frac{\hbar}{2}\hat{J}^6\hat{J}^8 - \frac{\hbar}{2}\hat{J}^7\hat{J}^8 + 2\hbar^2\hat{J}^8 ,
$$

$$
W(\hat{J}^3) = \hat{J}^3 + \hat{J}^4\hat{J}^8 + \tfrac{1}{12}\hat{J}^6\hat{J}^6 + \tfrac{1}{4}\hat{J}^7\hat{J}^7 + \frac{\hbar}{2}\hat{J}^7 - \frac{\hbar}{2}\hat{J}^6 .
$$

We introduce a new basis

$$
U = \tfrac{1}{4}W(\hat{J}^5), \quad G^+ = W(\hat{J}^4), \quad G^- = W(\hat{J}^2), \quad S = W(\hat{J}^1),
$$

$$
C = W(\hat{J}^3) + \tfrac{1}{24}W(\hat{J}^5)W(\hat{J}^5) - \frac{\hbar}{2}W(\hat{J}^5) . \tag{A.14}
$$

In this case, the fields are not organized according to sl_2 representations, because the centralizer of this sl_2 embedding in sl_4 does not contain an sl_2. Again C is a central element, and the

nonvanishing commutators are

$$[U, G^+] = \hbar G^+ , \quad [U, G^-] = -\hbar G^- ,$$

$$[S, G^+] = \hbar G^+ \left(-\frac{2}{3} C + \frac{20}{9} U^2 - \frac{43\hbar}{9} U + \frac{29\hbar^2}{27} \right) ,$$

$$[S, G^-] = \hbar \left(\frac{2}{3} C - \frac{20}{9} U^2 + \frac{43\hbar}{9} U - \frac{29\hbar^2}{27} \right) G^- , \tag{A.15}$$

$$[G^+, G^-] = \hbar S - \frac{4\hbar}{3} CU + \frac{3\hbar^2}{4} C + \frac{88\hbar}{27} U^3 - \frac{17\hbar^2}{2} U^2 + \frac{25\hbar^2}{6} U .$$

This is the first example where the brackets are no longer quadratic, but contain third order terms. For the sake of completeness, let us also give the quantum Miura transformation for this algebra. We identify generators $\{e, f, h\}, s_1, s_2$ of $sl_2 \oplus u(1) \oplus u(1) = \mathcal{G}_0$ via $f = \hat{J}^8$, $e = \hat{J}^4$, $h = \frac{1}{3}(\hat{J}^5 - \hat{J}^6)$, $s_1 = \frac{1}{3}(2\hat{J}^6 + \hat{J}^5)$ and $s_2 = \hat{J}^7$. The only non-trivial commutators between these five generators are $[e, f] = \hbar h$, $[h, e] = 2\hbar e$ and $[h, f] = -2\hbar f$. The quantum Miura transformation now reads

$$U = \tfrac{1}{4} s_1 + \tfrac{1}{2} h , \quad G^+ = e , \quad G^- = \left(\tfrac{1}{4} s_1^2 + \tfrac{1}{2} s_1 h + \tfrac{1}{4} h^2 - \tfrac{1}{4} s_2^2 - \frac{\hbar}{2}(3s_1 + 3h + s_2) + 2\hbar^2 \right) f,$$

$$S = -\tfrac{1}{12} e(8s_1 + 4h - 31\hbar)f - \tfrac{1}{108}(s_1 - h)^3 + \tfrac{1}{12}(s_1 - h)s_2^2 + \frac{5\hbar}{48}(s_1 - h)^2$$

$$+ \frac{\hbar}{6} s_2(s_1 - h) - \frac{3\hbar}{16} s_2^2 - \frac{3\hbar^2}{8} s_2 - \frac{7\hbar^2}{24}(s_1 - h) , \tag{A.16}$$

$$C = (\tfrac{1}{2} ef + \tfrac{1}{2} fe + \tfrac{1}{4} h^2) + \left(\tfrac{1}{4} s_2^2 + \frac{\hbar}{2} s_2 \right) + (\tfrac{1}{8} s_1^2 - \hbar s_1) .$$

This completes our list of finite quantum W-algebras from sl_4.

References

[1] H. Weyl, The Theory of Groups and Quantum Mechanics (Dover, New York, 1928).
[2] V.G. Kac, Infinite-dimensional Lie algebras – An introduction (Cambridge University Press, Cambridge, 1983) and references therein.
[3] P. Goddard and D. Olive, Int. J. Mod. Phys. A 1 (1986) 303.
[4] A.B. Zamolodchikov, Theor. Math. Phys. 65 (1985) 1205.
[5] P. Bouwknegt and K. Schoutens, Phys. Rep. 223 (1993) 183.
[6] T. Tjin, Ph.D. thesis, Amsterdam, 1993.
[7] J. de Boer, Ph.D. thesis, Utrecht, 1993.
[8] V. Drinfeld and V. Sokolov, J. Sov. Math. 30 (1984) 1975.
[9] F.A. Bais, T. Tjin and P. van Driel, Nucl. Phys. B 357 (1991) 632.
[10] T. Tjin, Phys. Lett. B 292 (1992) 60.
[11] J. de Boer and T. Tjin, Comm. Math. Phys. 158 (1993) 485.
[12] K. de Vos and P. van Driel, preprint DAMTP-93-66, UCLA/93/TEP/46, hep-th/9312016.
[13] N. Ikeda and K.-I. Izawa, Prog. Theor. Phys. 90 (1993) 237.

[14] J.M. Jauch and E.L. Hill, Phys. Rev. 57 (1940) 641.
[15] D. Bonatsos, C. Daskaloyannis, P. Kolokotronis and D. Lenis, preprint DEM-NT 94-36, THES-TP 26/94, hep-th/9411218.
[16] M. Rocek, Phys. Lett. B 255 (1991) 554.
[17] S.P. Smith, Trans. Am. Math. Soc. 322 (1990) 285.
[18] A.P. Polychronakos, Mod. Phys. Lett. A 5 (1990) 2325.
[19] C. Daskaloyannis, J. Phys. A 24 (1991) L789.
[20] Ya.I. Granovskii, A.S. Zhedanov and O.B. Grakhovskaya, Addition rule for non-linear algebras, Phys.Lett. B 278 (1992) 85.
[21] A.S. Zhedanov, Mod. Phys. Lett. A 7 (1992) 507.
[22] C. Delbecq and C. Quesne, J. Phys. A 26 (1993) L127.
[23] D. Bonatsos, C. Daskaloyannis and C. Kolokotronis, J. Phys. A 26 (1993) L871.
[24] F. Pan, J. Math. Phys. 34 (1994) 5065.
[25] M. Bander and C. Itzykson, Rev. Mod. Phys. 38 (1966) 330; (Part II) 346.
[26] P.W. Higgs, J. Phys. A 12 (1979) 309.
[27] Ya.I. Granovskii, A.S. Zhedanov and I.M. Lutsenko, Theor. Math. Phys. 91 (1992) 604.
[28] P. Bowcock, preprint DTP-94-5, hep-th/9403157.
[29] J. de Boer and T. Tjin, Comm. Math. Phys. 158 (1993) 485.
[30] L. Fehér, L. O'Raifeartaigh, Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. 222 (1992) 1.
[31] N.P. Landsman, J. Geom. Phys., to appear; hep-th/9305088; preprint DAMTP to appear; private communication.
[32] R. Bott, Ann. Math. 66 (1957) 203.
[33] N. Woodhouse, Geometric Quantization (Oxford University Press, Oxford 1980); J. Sniatycki, Geometric Quantization and Quantum Mechanics (Springer, New York, 1980); V. Guillemin and S. Sternberg, Invent. Math. 67 (1982) 515.
[34] E. Witten, Comm. Math. Phys. 114 (1988) 1.
[35] E.B. Dynkin, Amer. Math. Soc. Transl. 6 (1957) 111.
[36] B. Kostant and S. Sternberg, Ann. Phys. 176 (1987) 49.
[37] B. Kostant, Inv. Math. 48 (1978) 101.
[38] F. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. B 304 (1988) 348.
[39] J. de Boer and J. Goeree, Phys. Lett. B 274 (1992) 289; Nucl. Phys. B 381 (1992) 329; Nucl. Phys. B 401 (1993) 348; Nucl. Phys. B 401 (1993) 369.
[40] B. Feigin and E. Frenkel, Phys. Lett. B 246 (1990) 75; E. Frenkel, Harvard preprint 1991; thesis, Harvard University.
[41] R. Bott and L.W. Tu, Graduate Texts in Math., Vol. 82 (Springer, New York, 1982).
[42] J. McCleary, User's guide to spectral sequences, Mathematics Lecture Series 12 (Publish or Perish, Washington, 1985).
[43] J. de Boer, L. Fehér and A. Honecker, Nucl. Phys. B 420 (1994) 409.
[44] M. Bershadsky, Comm. Math. Phys. 139 (1991) 71.
[45] A. Sevrin and W. Troost, Phys. Lett. B 315 (1993) 304.
[46] F. Barbarin, E. Ragoucy and P. Sorba, preprint ENSLAPP-AL-489/94, hep-th/9410114.
[47] J.E. Humphreys, Cambridge Studies in Adv. Math. 29 (1992).
[48] D. Kazhdan and G. Lusztig, Invent. Math. 53 (1979) 165.
[49] S.I. Gelfand and R. MacPherson, Lecture Notes in Math. 924 (1982) 1.
[50] K.J. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Phys. Lett. B 255 (1991) 549.
[51] N. Ikeda and K.-I. Izawa, Prog. Theor. Phys. 89 (1993) 1077.
[52] A. Schwarz, Lett. Math. Phys. 2 (1978) 247.
[53] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209 (1991) 129.
[54] E. Witten, J. Geom. Phys. 9 (1992) 303.
[55] G. Papadopoulos, Phys. Lett. B 295 (1992) 44.
[56] Z. Bajnok, L. Palla and G. Takács, Nucl. Phys. B 385 (1992) 329.
[57] B. Kostant, Lecture Notes in Math. 466 (1974) 101.
[58] A.P. Veselov and A.B. Shabat, Functional Anal. Appl. 27 (1993) 81.
[59] V. Spiridonov, Comm. Theor. Phys. 2 (1993) 149.
[60] T. Tjin and P. van Driel, preprint ITFA-91-04.

[61] P. Schaller and T. Strobl, Poisson structure induced (topological) field theories, Mod. Phys. Lett A 9 (1994) 3129, hep-th/9405110.

[62] P. Schaller and T. Strobl, Quantization of field theories generalizing gravity Yang-Mills systems on the cylinder, in proceedings 'Integrable models and strings', Helsinki 1993, gr-qc/9406027.

[63] P. Schaller and T. Strobl, Poisson sigma models: a generalization of 2-d gravity Yang-Mills systems, TUW-94-21, hep-th/9411163.

[64] A.Yu. Alekseev, Peter Schaller and T. Strobl, The topological G/G WZW model in the generalized momentum representation, ETH-TH-95-14, hep-th/9505012.

[65] P. Schaller and T. Strobl, A brief introduction to Poisson sigma models, TUW-95-15, hep-th/9507020.

Note added in proof

In a series of papers [61–65] a very interesting class of two-dimensional topological field theories has been studied, that contain as a special case the BF theory for finite W-algebras described by equation (6.25). To each manifold N with a Poisson structure (i.e. a smooth section of $\wedge^2 TN$ satisfying the Jacobi identities) one can associate such a topological field theory. A finite W-algebra with k generators defines a Poisson structure on $\mathbb{R}^k$, and the corresponding Poisson sigma model is precisely (6.25). Poisson sigma models contain many interesting topological field theories as subcases, such as two-dimensional Yang-Mills theory and the G/G gauged Wess–Zumino–Witten theory. The Hilbert space of Poisson sigma models turns out to be determined by the integral symplectic leaves of N. For finite W-algebras we analyzed the symplectic leaves in Section 3.2.6. It would be very interesting to determine which of these are integral, as this would immediately provide us with the Hilbert space of (6.25). The same leaves also play an important role in the representation theory of finite W-algebras, suggesting that the Hilbert space and partition function of (6.25) should have a representation theoretical interpretation, and this might ultimately lead to a geometrical proof of some of the conjectures in Chapter 5.