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Taming Logic

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Abstract. In this paper, we introduce a general technology, called *taming*, for finding well-behaved versions of well-investigated logics. Further, we state completeness, decidability, definability and interpolation results for a multimodal logic, called *arrow logic*, with additional operators such as the *difference operator*, and *graded modalities*. Finally, we give a completeness proof for a strong version of arrow logic.

Key words: arrow logic, modal logic, completeness, decidability, difference operator, graded modalities

1. Taming

In this section, we argue that it is important to find nice (complete, decidable, etc.) versions of logics, and introduce a technology to achieve this goal.

1.1. Why tame?

There are interesting and well-investigated logics that do not behave in a nice way in some respects. Examples are the undecidability of classical first-order logic, *FOL*, and the incompleteness and undecidability of several versions of arrow logic, *AL*, cf. Definition 2.1.

One may argue that some of these features are necessary, e.g., in *FOL* we can build up whole of mathematics, so *FOL* must have a high complexity. However, *FOL* has several other applications when decidability would be a desirable property. For instance, (Andréka *et al.*, 1995c) proposes relativized versions of *FOL* as "modal fragments" of classical logic. These relativized versions have nicer properties than *FOL* itself, cf. (Andréka and Thompson, 1988) (in algebraic disguise),

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and (Marx and Venema, 1995), (Mikulás, 1995a) (in modal-logical disguise), and (Németi, 1986; 1992).

Our other example is arrow logic, (*AL*). *AL* as defined in (van Benthem, 1994a) is intended to be the core of logical systems for reasoning about dynamic aspects of the subject matter of our thinking, e.g., properties of processes, actions and programs. Thus, one of the basic intended areas for applications of *AL* is computer science. There decidability of *AL* is clearly a desirable property. The most interesting connective of *AL* – and certainly one that contributes to the dynamic character of the logic – is composition. If composition is an associative operator, *AL* is undecidable. Moreover, any non-trivial extension or strengthening of associative *AL* is undecidable, cf. (Andréka *et al.*, 1994a). Thus, it is natural to consider non-associative versions of *AL*. Indeed, most of them are decidable and complete, cf. below.

On the other hand, we would like the expressive power of the nice logic to be rather large. To achieve this, one can strengthen the logic by introducing new connectives while taking care not to lose the nice properties. Below we will give examples how to do this.

1.2. How to Tame Modal Logics?

To answer this question let us consider our two examples, *FOL* and *AL*, and try to understand what causes the undesirable properties of these logics.

(1) Consider classical first-order logic as a modal logic, cf. (Venema, 1992a; 1995). Then the frame condition corresponding to the commutativity of the quantifiers $\exists v_i \exists v_j \varphi \rightarrow \exists v_j \exists v_i \varphi$ is

$$\forall xyz((\mathsf{T}_i xz \ \&\ \mathsf{T}_j zy) \Rightarrow \exists z'(\mathsf{T}_j xz' \ \&\ \mathsf{T}_i z'y)),$$

where x, y, z, z' are "worlds", and T_k is the accessibility relation of the "modality" $\exists v_k$. In (Németi, 1992), it is argued that the above condition is the reason for the undecidability of *FOL*. In (Németi, 1992), (Marx, 1995), (Marx and Venema, 1995) and (Mikulás, 1995a) there are several decidable quantifier logics lacking this condition.

(2) Our other example is arrow logic. If we consider the associative version of *AL*, then the frames satisfy

$$\forall xyzuv((\mathsf{C}xyz \ \&\ \mathsf{C}yuv) \Rightarrow \exists w(\mathsf{C}xuw \ \&\ \mathsf{C}wvz)),$$

where C is the accessibility relation interpreting the binary modality $\bullet$, cf. Definition 2.1. The above frame condition corresponds to $(\varphi \bullet \psi) \bullet \chi \rightarrow \varphi \bullet (\psi \bullet \chi)$. Associativity of composition makes arrow logic undecidable, cf. (Andréka, 1991a), (Andréka *et al.*, 1994a) and Theorem 2.2 below.

Both of the above conditions are existential frame conditions. Thus, these examples suggest that some existential frame conditions may be dangerous, i.e., they may cause undecidability and non-finite axiomatizability. The first step in taming a logic is to get rid of those existential frame conditions that cause undesirable properties. *First step.* First we will get rid of all of the existential frame conditions, and then add back the innocent ones.

(I) Let $L(\mathsf{K})$ be a modal logic defined by a class K of Kripke frames. We can define a first-order language using the accessibility relations R_c of K as $n+1$-ary predicates for every n-ary modality c of L. We take all the substructures in the first-order model-theoretic sense of elements of K, cf. (Chang and Keisler, 1990). Then we get a class $\mathbf{Sub}\mathsf{K}$ of frames. We call the logic $L(\mathbf{Sub}\mathsf{K})$ the *core* of $L(\mathsf{K})$. If we consider the universal first-order theory of K, then it coincides with that of $\mathbf{Sub}\mathsf{K}$. Thus, in the logic $L(\mathbf{Sub}\mathsf{K})$, we got rid of the existential frame conditions of K.

There are several reasons to start the taming process with $L(\mathbf{Sub}\mathsf{K})$. First, in a sense, $L(\mathbf{Sub}\mathsf{K})$ is relatively close to $L(\mathsf{K})$, since all universal frame conditions are preserved. Second, as we saw above, in many cases some of the existential frame conditions are responsible for the ugly behaviour. So there is a chance that $L(\mathbf{Sub}\mathsf{K})$ has nicer properties than $L(\mathsf{K})$. Moreover, if we consider the class $\mathsf{Alg}(L(\mathsf{K}))$ of algebras* corresponding to the logic $L(\mathsf{K})$, cf. (Andréka *et al.*, 1995a; 1995b), then we can get the class $\mathsf{Alg}(L(\mathbf{Sub}\mathsf{K}))$ by a well-known and well-investigated operation called *relativization*, cf. (Henkin *et al.*, 1985). In many cases, this yields a class of algebras with nicer properties than the original class, reflecting the fact that $L(\mathbf{Sub}\mathsf{K})$ has nicer properties than $L(\mathsf{K})$ does.

(II) Although this procedure may yield nicer logics, it is a very drastic step. The situation is like taming a lion by pulling out all of its teeth. Since we got rid of *all* of the existential conditions and not just the dangerous ones, usually $L(\mathbf{Sub}\mathsf{K})$ is remarkably weaker than $L(\mathsf{K})$.

Thus, it is natural to try to find a class K' of frames such that $\mathsf{K} \subset \mathsf{K}' \subset \mathbf{Sub}\mathsf{K}$ and $L(\mathsf{K}')$ still has nice properties. In this way, we may get back some of the power of $L(\mathsf{K})$.

An example is pair arrow logic, cf. below. In pair arrow logic, $\mathbf{Sub}\mathsf{K}$ consists of all frames whose universes are arbitrary binary relations. If we make the requirement that the universes must be symmetric and reflexive relations (these properties are expressible by existential frame conditions, cf. below), then the logic of these frames still has the same nice properties, cf. Theorem 2.2.

Second step. Although we may strengthen the logics in the above way, the expressive power of these logics is usually strictly smaller than that of the original logic. For instance, there may be connectives that are not definable any more. The larger expressive power has obvious advantages. Beside that, the stronger logic may have nicer properties as well. For instance, the existence of the universal modality ensures that the logic has a deduction term, i.e., the deduction theorem holds. See (Simon, 1992) for more detail and motivation for strengthening.

* In case of a modal logic $L(\mathsf{K})$, this class can be defined as the class of subalgebras of products of complex algebras of members of K.

Thus, we will try to introduce new connectives to the logic without losing the nice properties. Examples are the universal modality $\diamond$, the difference operator D, and the graded modalities $\langle n \rangle$ that can be added to the nice versions of arrow logic without losing decidability, cf. Theorem 2.5 below.

The coordinate-wise versions $\langle n \rangle_i$ for $i < n$ and transitive closure can be added, too, with similarly positive results (to mention a few more examples).

2. Arrow logic

We illustrate our two-step strategy of taming logics using arrow logic. First we give the definitions of several versions of arrow logic. We will concentrate on its pair version and tame it. For more on arrow logic see (van Benthem, 1994a), (Marx, 1992; 1995), (Marx *et al.*, 1992), (Venema, 1992b), and (Andréka *et al.*, 1994a), (Mikulás *et al.*, 1995), (Mikulás, 1992; 1995b), (Simon, 1992). For a logic with the same connectives as arrow logic see the algebraic logic of binary relations $\mathcal{L}^x$ in (Tarski and Givant, 1987) p. 47.

DEFINITION 2.1. *Arrow logic, AL*, is defined as follows. Its connectives are the Booleans, the *identity* constant id, a unary connective $\otimes$ called *converse*, and a binary connective $\bullet$ called *composition*. The set of formulas is built up in the usual way using a denumerable set of propositional variables.

A structure $\langle W, \mathsf{C}, \mathsf{F}, \mathsf{I} \rangle$ is called an *arrow frame* if W is a non-empty set, I is a unary, F is a binary, and C is a ternary relation on W. An *arrow model* is an arrow frame together with a valuation v of the propositional variables. *Truth* of a formula φ at a world w in a model $\langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}, v \rangle$, in symbols $w \Vdash_v \varphi$, is defined as follows:

- $w \Vdash_v p \stackrel{\text{def}}{\iff} w \in v(p)$ for every propositional variable p,
- $w \Vdash_v \neg\varphi \stackrel{\text{def}}{\iff}$ not $w \Vdash_v \varphi$,
- $w \Vdash_v \varphi \wedge \psi \stackrel{\text{def}}{\iff} w \Vdash_v \varphi \;\&\; w \Vdash_v \psi$,
- $w \Vdash_v \varphi \bullet \psi \stackrel{\text{def}}{\iff} (\exists w', w'' \in W)\mathsf{C}ww'w'' \;\&\; w' \Vdash_v \varphi \;\&\; w'' \Vdash_v \psi$,
- $w \Vdash_v \otimes\varphi \stackrel{\text{def}}{\iff} (\exists w' \in W)\mathsf{F}ww' \;\&\; w' \Vdash_v \varphi$,
- $w \Vdash_v \mathsf{id} \stackrel{\text{def}}{\iff} \mathsf{I}w$.

Pair arrow logic is defined as follows. Its syntax is the same as that of *AL*. An arrow frame $\langle W, \mathsf{C}_W, \mathsf{F}_W, \mathsf{I}_W \rangle$ is a *pair frame* if the following holds. The universe W is a binary relation $W \subseteq U \times U$ for some set U, called the *base* of the frame, and the accessibility relations C_W, F_W, and I_W are relation composition, relation converse, and identity restricted to W. That is, for all $\langle x, x' \rangle, \langle y, y' \rangle, \langle z, z' \rangle \in W$,

- $\mathsf{C}_W \langle x, x' \rangle \langle y, y' \rangle \langle z, z' \rangle \stackrel{\text{def}}{\iff} x = y \;\&\; x' = z' \;\&\; y' = z$,
- $\mathsf{F}_W \langle x, x' \rangle \langle y, y' \rangle \stackrel{\text{def}}{\iff} x = y' \;\&\; x' = y$,
- $\mathsf{I}_W \langle x, x' \rangle \stackrel{\text{def}}{\iff} x = x'$.

The class of all pair frames will be denoted by PF. We define subclasses of PF as follows.

Let s, r, t abbreviate 'symmetry', 'reflexivity' and 'transitivity', respectively, and let $H \subseteq \{r, s, t\}$. PF_H denotes that subclass of PF where each element of PF_H satisfies the properties in H. Thus, e.g., $\mathsf{PF}_{r,s}$ consists of all pair frames with reflexive and symmetric universes. Let PF_{SQ} be the class of *square pair frames*, i.e., those with universe of the form $U \times U$.

Now we define several versions of pair arrow logic corresponding to the above classes of frames. The logic PAL_H is defined as the arrow logic of the class PF_H of frames: $PAL_H = L(\mathsf{PF}_H)$. That is, the universe of a frame for PAL_H is any binary relation satisfying the conditions in H. We will call these logics the *relativized versions* of (pair) arrow logic. $PAL_\emptyset$, that is, when $H = \emptyset$, is called the *completely relativized version* of pair arrow logic. PAL_{SQ} denotes the *square version* of pair arrow logic: $PAL_{SQ} = L(\mathsf{PF}_{SQ})$.

Validity of a formula in a model, $\langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}, v\rangle \models \varphi$ is defined in the usual way:

$$\langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}, v\rangle \models \varphi \overset{\text{def}}{\iff} (\forall w \in W) w \Vdash_v \varphi.$$

Given a class K of frames, the *(global) semantical consequence relation*, $\Gamma \models_{\mathsf{K}} \varphi$, is defined as follows. $\Gamma \models_{\mathsf{K}} \varphi$ iff for every model $\langle \mathcal{F}, v\rangle$ such that $\mathcal{F} \in \mathsf{K}$, if Γ is valid in $\langle \mathcal{F}, v\rangle$, then so is φ.* Thus, e.g., $\models_{\mathsf{PF}_{SQ}}$ denotes the semantical consequence relation in PAL_{SQ}.

Let us consider the strongest pair arrow logic PAL_{SQ}, and try to apply the taming strategy described in the previous section. In the above definition of PAL_{SQ}, we required that the universes of the frames are Cartesian spaces. Transitivity of the universe ensures that composition is associative, i.e., the following is a valid formula:

$$(\varphi \bullet \psi) \bullet \chi \leftrightarrow \varphi \bullet (\psi \bullet \chi).$$

Associativity causes both Hilbert-style incompleteness and undecidability of pair arrow logic, cf. (Andréka, 1991a) and (Andréka *et al.*, 1994a). Thus, to find nicer versions of pair arrow logic we should apply the "non-square approach", i.e., allow frames with non-square universes.

First step. (I) The core of PAL_{SQ} is the completely relativized pair arrow logic $PAL_\emptyset$. We got rid of the existential conditions by relativization, i.e., by defining the meanings of the formulas *relative* to an arbitrary binary relation. That is why associativity does not hold in $PAL_\emptyset$. To see more clearly why associativity does not hold in (many of) the relativized versions of arrow logic, we introduce the following notation: let $[\![\varphi]\!] \overset{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and for $X, Y \subseteq W$,

* Note that the semantical consequence relation can be defined *locally* as well, i.e., using truth at a world instead of validity in a model.

$X \circ_W Y \stackrel{\text{def}}{=} \{\langle w, w' \rangle \in W : \exists w''(\langle w, w'' \rangle \in X \;\&\; \langle w'', w' \rangle \in Y)\}$. Then, in a model for PAL_H with universe $W \subseteq U \times U$,

$$[\![\varphi \bullet \psi]\!] = [\![\varphi]\!] \circ_W [\![\psi]\!] = ([\![\varphi]\!] \circ_{U \times U} [\![\psi]\!]) \cap W.$$

That is, to get the meaning of composition in relativized models, we have to intersect the standard (or unrelativized) meaning with the universe of the model. There are non-transitive relations W such that $\langle w, w' \rangle \in W$, and for some x, $\langle w, x \rangle \in [\![\varphi]\!] \circ_W [\![\psi]\!]$ and $\langle x, w' \rangle \in [\![\chi]\!]$, while $[\![\psi]\!] \circ_W [\![\chi]\!] = \emptyset$. That is, $[\![(\varphi \bullet \psi) \bullet \chi]\!] \neq \emptyset = [\![\varphi \bullet (\psi \bullet \chi)]\!]$.

The completely relativized version $PAL_\emptyset$ behaves much nicer than the square version PAL_{SQ}, cf. Theorem 2.2.

(II) The logic $PAL_\emptyset$ is much weaker than PAL_{SQ}. For instance, $\varphi \bullet \mathsf{id} \leftrightarrow \varphi$ is not valid in $PAL_\emptyset$, while it is valid in PAL_{SQ}. The reason for this is that we allowed frames with irreflexive universes, i.e., the existential frame condition $\forall x \exists y(\mathsf{I}z \wedge \mathsf{C}xxz)$ does not hold in PF. Another example is the formula $\otimes\otimes\varphi \leftrightarrow \varphi$, and the corresponding frame condition is $\forall x \exists! y(\mathsf{F}xy \wedge \mathsf{F}yx)$ implying symmetry of the universe of the pair frame. We may consider pair frames with reflexive and/or symmetric universes getting back some of the existential frame conditions of PF_{SQ}, and strengthening the logic this way. Then we get the logics PAL_H ($H \subseteq \{r, s\}$). These logics still have the nice properties, cf. Theorem 2.2 below.

We recall that by a *Hilbert-style calculus* we mean an inference system given by a finite set of axiom schemata and inference rules such that the rules do not contain any side conditions.* For the definitions of the other metalogical notions we refer to (Andréka *et al.*, 1995a; 1995b), (Chang and Keisler, 1990) and (Marx, 1995).

The various parts of the following theorem have been proved by H. Andréka, R. Kramer, R. Maddux, M. Marx and I. Németi. For precise reference we refer to (Marx, 1995).

THEOREM 2.2. Let $H \subseteq \{r, s, t\}$ be arbitrary. Then

1. PAL_H has a strongly sound and strongly complete Hilbert-style calculus iff $t \notin H$;
2. PAL_H is decidable iff $t \notin H$;
3. PAL_H has the Craig interpolation property iff $t \notin H$;
4. PAL_H has the Beth definability property iff $t \notin H$.

We mention that the above negative results hold for PAL_{SQ}, since the theories of PF_{SQ} and $\mathsf{PF}_{r,s,t}$ are the same. (Because every $\mathsf{PF}_{r,s,t}$-frame consists of a disjoint union of square frames.)

* For instance, the calculi containing the irreflexivity rule for the difference operator, cf. (Gabbay, 1981) and (Venema, 1992a), are not Hilbert-style calculi.

Second step. In PAL_{SQ} there are connectives that are not definable in PAL_H ($H \subseteq \{r, s\}$). Consider the universal modality $\Diamond$ interpreted as:

$$[\![\Diamond\varphi]\!] \stackrel{\text{def}}{=} \{w \in W : (\exists w' \in W) w' \in [\![\varphi]\!]\}.$$

In PAL_{SQ}, $\Diamond\varphi$ can be defined as $\top \bullet \varphi \bullet \top$ while in PAL_H it is not definable, cf. (Andréka, 1991b). The $\Diamond$ is really useful from a theoretical point of view, since a deduction term is definable using $\Diamond$, cf. (Simon, 1992).

To strengthen our relativized logics, we may re-introduce connectives which were definable in the square logic, or we may even add new connectives. We will add the *difference operator* D to PAL_H, obtaining PAL_H^{D}. The interpretation of D is:

$$[\![\mathsf{D}\varphi]\!] \stackrel{\text{def}}{=} \{w \in W : (\exists w' \in W) w \neq w' \;\&\; w' \in [\![\varphi]\!]\}.$$

Note that $\mathsf{D}\varphi$ is definable in PAL_{SQ} by $(\top \bullet \varphi \bullet \neg\mathsf{id}) \vee (\neg\mathsf{id} \bullet \varphi \bullet \top)$, cf. (Venema, 1992b), and that $\Diamond\varphi$ is definable by D: $\mathsf{D}\varphi \vee \varphi$. For more on the difference operator see (Gargov *et al.*, 1987), (Koymans, 1989), (de Rijke, 1993) and (Sain, 1988).

We will prove that adding D to $PAL_{r,s}$ does not ruin completeness. We prove the following theorem in the next section. An algebraic proof can be found in (Marx *et al.*, 1994) and (Mikulás, 1995b).

THEOREM 2.3. For $H = \{r, s\}$, PAL_H^{D} has a strongly sound and strongly complete Hilbert-style calculus.

We conjecture that the above theorem remains true for any $H \subseteq \{r, s\}$. For the case $t \in H$, (Andréka, 1991a) gives a negative answer.

For decidability, a similar result holds (for the decidability proof, cf. (Andréka *et al.*, 1994b) or (Mikulás, 1995b)).

THEOREM 2.4. The difference logic PAL_H^{D} is decidable iff $t \notin H$.

We can also add the so-called *graded modalities* $\langle n \rangle$ for $n \in \omega \setminus 1$:

$$[\![\langle n \rangle\varphi]\!] \stackrel{\text{def}}{=} \begin{cases} W & \text{if } |[\![\varphi]\!]| \geq n \\ \emptyset & \text{otherwise.} \end{cases}$$

We denote this new logic by PAL_H^{grad}. Note that $\langle n \rangle$ is definable in PAL_{SQ} if $n \in \{1, 2\}$, cf. (Marx, 1995), and that D and $\langle 1 \rangle$, $\langle 2 \rangle$ are definable in terms of each other. For motivation concerning graded modalities, their application in computer science, epistemic and probability logic we refer to (van der Hoek, 1992). We can add all the $\langle n \rangle$ to PAL_H ($t \notin H$) without losing decidability. The next theorem is due to H. Andréka, Sz. Mikulás and I. Németi, its proof can be found in (Andréka *et al.*, 1994b) and (Mikulás, 1995b).

THEOREM 2.5. The graded logic PAL_H^{grad} is decidable iff $t \notin H$.

Unfortunately, the Craig interpolation and Beth definability properties are not preserved after strengthening. M. Marx, I. Németi and I. Sain showed that PAL_H^{D} and PAL_H^{grad} do not have these two properties for any $H \subseteq \{r, s, t\}$, see (Marx, 1995).

Let us conclude this section with some remarks. As we mentioned above, there are relativized versions of *FOL* that behave nicely. We will not deal with this problem here. However, FOL_3^2, the 3 variable fragment of *FOL* with binary predicates is equivalent to PAL_{SQ}, cf. (Henkin *et al.*, 1985) and (Tarski and Givant, 1987). Thus, whenever we obtain results about PAL_{SQ}, these results apply to FOL_3^2 as well. For more information on taming *FOL*, we refer to (Andréka *et al.*, 1995c) and (Mikulás, 1995a).

Note that, for $n > 1$, $\langle n\rangle$ is not a modality in the following sense: it does not distribute over disjunction, i.e., the following is not a valid formula:

$$\langle n\rangle(\varphi \vee \psi) \leftrightarrow (\langle n\rangle\varphi \vee \langle n\rangle\psi).$$

However, we can add modalities $\Diamond_n$ to PAL_H such that the two logics become equivalent. That is, the connectives (with their intended meanings) of one logic can be defined in terms of the connectives of the other logic, cf. (Andréka and Németi, 1994). The interpretation of $\Diamond_n$ is:

$$[\![\Diamond_n(\varphi_0, \dots, \varphi_{n-1})]\!] \stackrel{\mathrm{def}}{=} \begin{cases} W & \text{if } (\exists w_0, \dots, w_{n-1} \in W)(\forall i \in n) \\ & \quad w_i \in [\![\varphi_i]\!] \;\&\; (\forall j \neq i) w_i \neq w_j \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to see that $\Diamond_n$ distributes over disjunction in each of its arguments.

3. Proof of Theorem 2.3

This section is devoted to the proof of the completeness Theorem 2.3: we provide an explicit derivation system for the logic $PAL_{r,s}^{\mathsf{D}}$. We start with an outline of the proof. Define the following class of pair frames:

$$\mathsf{PFD}_{r,s} \stackrel{\mathrm{def}}{=} \{\mathcal{F} = \langle V, \mathsf{C}_V, \mathsf{F}_V, \mathsf{I}_V, \neq\rangle : \langle V, \mathsf{C}_V, \mathsf{F}_V, \mathsf{I}_V\rangle \in \mathsf{PF}_{r,s}\}.$$

Here $\neq$ denotes inequality, and is the accessibility relation of the D-operator. By definition, $PAL_{r,s}^{\mathsf{D}}$ is the logic of the class $\mathsf{PFD}_{r,s}$. Below we define a class of arrow frames KD by means of Sahlqvist formulas such that for every set of $PAL_{r,s}^{\mathsf{D}}$-formulas $\Gamma \cup \{\varphi\}$, we have $\Gamma \models_{\mathsf{KD}} \varphi$ if and only if $\Gamma \models_{\mathsf{PFD}_{r,s}} \varphi$. Sahlqvist's theorem gives us a completeness result with respect to the "abstract" class KD. But then we also have a completeness result with respect to the class of pair frames $\mathsf{PFD}_{r,s}$.

In our proof strategy we follow the two-step taming process described above. First we define by means of Sahlqvist formulas a class K of arrow frames such that K equals the closure under zigzagmorphic images (to be defined below) of the class of pair frames $\mathsf{PF}_{r,s}$. This will give us an explicit Hilbert-style derivation system for $PAL_{r,s}$. We use some correspondence theory to get a clear first-order description of K. In this part we just state results which are known and well documented in the literature.

In the second part, we use the results about the classes K and $\mathsf{PF}_{r,s}$. We define the class KD of "abstract" frames and show that KD equals the closure under disjoint unions and zigzagmorphic images of the class of pair frames $\mathsf{PFD}_{r,s}$. In this part, where we are mainly concerned with the difference operator, we use "copying techniques" which are applied in various places in the literature concerning the D-operator (cf., e.g., (Koymans, 1989), (Gargov *et al.*, 1987), (de Rijke, 1993), (Sain, 1988)). This finishes the outline of the proof.

We will frequently use the generalization of Sahlqvist's theorem (cf., (Sahlqvist, 1975)) to arbitrary modal logics as described in (de Rijke and Venema, 1991). Sahlqvist's theorem specifies a large set of modal formulas, so-called Sahlqvist formulas, with the following two properties:

- every Sahlqvist formula φ corresponds to an effectively obtainable first-order formula φ^* in the first-order language of the frames of the logic such that, for every frame $\mathcal{F}$:

  $$\mathcal{F} \models \varphi \iff \mathcal{F} \models \varphi^*$$

- for every set Γ of Sahlqvist formulas, the derivation system consisting of the K-axioms,* Γ, and the rules modus ponens, universal generalization and substitution, is strongly sound and complete for $\models_{\mathsf{K}}$, where $\mathsf{K} = \{\mathcal{F} : \mathcal{F} \models \Gamma\}$.

3.1. First Taming Step: Axiomatizing $PAL_{r,s}$

Define the following set of arrow-logical formulas:

(A_1) $\top \bullet (\top \bullet (\mathsf{id} \wedge \varphi)) \to \top \bullet (\mathsf{id} \wedge \varphi)$
(A_2) $\varphi \bullet \mathsf{id} \leftrightarrow \varphi$
(A_3) $\otimes\otimes\varphi \leftrightarrow \varphi$
(A_4) $\otimes(\varphi \bullet \psi) \leftrightarrow \otimes\psi \bullet \otimes\varphi$
(A_5) $\otimes\varphi \bullet \neg(\varphi \bullet \psi) \wedge \psi \to \bot$.

Let K denote the class of all arrow frames satisfying the axioms $A_1, \ldots, A_5$. Note that all of $A_1, \ldots, A_5$ are in Sahlqvist form. The conditions which correspond to these axioms are the following:

* This set of axioms includes enough propositional tautologies, and the formulas ensuring that the modalities distribute over disjunction.

(C_1) $\forall xyzv((\mathsf{C}xyz \wedge \mathsf{C}zzv \wedge \mathsf{I}v) \rightarrow \mathsf{C}xxv)$
(C_2) $\forall x\exists y(\mathsf{I}z \wedge \mathsf{C}xxz)$
$(C_{2'})$ $\forall xyz(\mathsf{C}xyz \wedge \mathsf{I}z \rightarrow x = y)$
(C_3) $\forall x\exists!y(\mathsf{F}xy \wedge \mathsf{F}yx)$
(C_4) $\forall xyz(\exists w(\mathsf{F}xw \wedge \mathsf{C}wyz) \leftrightarrow \exists y'z'(\mathsf{F}y'y \wedge \mathsf{F}z'z \wedge \mathsf{C}xz'y'))$
(C_5) $\forall xyzv((\mathsf{C}xyz \& \mathsf{F}yv) \rightarrow \mathsf{C}zvx)$.

PROPOSITION 3.1. Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle$ be an arrow frame, then

$$\mathcal{F} \in \mathsf{K} \iff \mathcal{F} \models C_1, \ldots, C_5.$$

Proof. By the correspondence part of Sahlvist's theorem using, e.g., the Sahlqvist algorithm described in (de Rijke and Venema, 1991). □

Now we will take a closer look at arrow frames satisfying these conditions, because that will be useful later on.

If an arrow frame satisfies conditions (C_1) – (C_5) then there are three total functions living in this frame (cf. Proposition 3.2 below). They are defined as follows.

$$\begin{array}{lcl} \mathsf{f}x = y & \overset{\text{def}}{\iff} & \mathsf{F}xy \\ x_l = y & \overset{\text{def}}{\iff} & \mathsf{C}xyx \wedge \mathsf{I}y \\ x_r = y & \overset{\text{def}}{\iff} & \mathsf{C}xxy \wedge \mathsf{I}y \end{array}$$

So, $\mathsf{f}x$ gives us the converse arrow of x, and the functions $(\cdot)_l$ and $(\cdot)_r$ (l for left and r for right) give us the left and the right "endpoints" of an arrow.

It is convenient to have explicit symbols in our language corresponding to the two defined functions. We use the following convention: $\mathsf{s}_0^1\,\varphi$ abbreviates $(\mathsf{id} \wedge \varphi) \bullet \top$, and $\mathsf{s}_1^0\,\varphi$ stands for $\top \bullet (\mathsf{id} \wedge \varphi)$. We also define abbreviations for their conjugates* dom and ran as follows: $\mathsf{dom}\varphi$ stands for $(\mathsf{id} \wedge (\varphi \bullet \top))$, and $\mathsf{ran}\varphi$ for $(\mathsf{id} \wedge (\top \bullet \varphi))$. Their meanings are given by the following equations. This is easy to see by writing out the definitions.

$$\begin{array}{rcl} [\![\mathsf{s}_0^1\,\varphi]\!] & = & \{x : x_l \in [\![\varphi]\!]\} \\ [\![\mathsf{dom}\varphi]\!] & = & \{x_l : x \in [\![\varphi]\!]\} \\ [\![\mathsf{s}_1^0\,\varphi]\!] & = & \{x : x_r \in [\![\varphi]\!]\} \\ [\![\mathsf{ran}\varphi]\!] & = & \{x_r : x \in [\![\varphi]\!]\} \\ [\![\otimes\varphi]\!] & = & \{x : \mathsf{f}x \in [\![\varphi]\!]\} \end{array}$$

PROPOSITION 3.2. Every arrow frame which satisfies the conditions (C_1)–(C_5) also satisfies conditions (T_0)–(T_5) below.

* Two unary modalities are conjugates if the truth definition of the one uses the inverse of the accessibility relation of the other. An example is the pair $\{F, P\}$ from tense logic. Note that $\otimes$ is its own conjugate.

(T_0) f, $(.)_l$ and $(.)_r$ are total functions, and f is idempotent
(T_1) $\forall x(\mathsf{I}x \rightarrow x = \mathsf{f}(x) = x_l = x_r)$
(T_2) $\forall x(x_l = (\mathsf{f}x)_r \wedge x_r = (\mathsf{f}x)_l)$
(T_3) $\forall xyz(\mathsf{C}xyz \rightarrow x_l = y_l \wedge y_r = z_l \wedge z_r = x_r)$
(T_4) $\forall xyzv(\mathsf{C}xyz \wedge \mathsf{F}zv \rightarrow \mathsf{C}yxv)$
(T_5) $\forall xyz(\mathsf{C}xyz \wedge \mathsf{I}x \rightarrow \mathsf{F}zy)$

Proof. Cf. (Marx, 1995). □

Since in the frames from K, the relation F will be a total function, we will denote it from now on by the function symbol f.

Before we go to the D-operator, we need to state one more result. For that we need the notion of a zigzagmorphism (this notion is also known under the names p-morphism or bounded morphism, cf. (Bull and Segerberg, 1984)).

DEFINITION 3.3. Let $\mathcal{F} = \langle V, \mathsf{C}^{\mathcal{F}}, \mathsf{f}^{\mathcal{F}}, \mathsf{I}^{\mathcal{F}}\rangle$ and $\mathcal{G} = \langle W, \mathsf{C}^{\mathcal{G}}, \mathsf{f}^{\mathcal{G}}, \mathsf{I}^{\mathcal{G}}\rangle$ be two arrow frames. A function $h : V \longrightarrow W$ is called a *zigzagmorphism* if it satisfies the following conditions:

- $\mathsf{Id}^{\mathcal{F}} x \iff \mathsf{Id}^{\mathcal{G}} h(x)$
- $h(\mathsf{f}^{\mathcal{F}} x) = \mathsf{f}^{\mathcal{G}} h(x)$
- $\mathsf{C}^{\mathcal{F}} xyz \Rightarrow \mathsf{C}^{\mathcal{G}} h(x)h(y)h(z)$
- $\mathsf{C}^{\mathcal{G}} h(x)yz \Rightarrow (\exists y', z' \in V)(h(y') = y \;\&\; h(z') = z \;\&\; \mathsf{C}^{\mathcal{F}} xy'z')$.

If h is a surjective function, then $\mathcal{G}$ is called a *zigzagmorphic image* of $\mathcal{F}$.

For K a class of frames, we use **Zig**K to denote the class of all zigzagmorphic images of members of K.

Let M $= \langle \mathcal{F}, v\rangle$ and N $= \langle \mathcal{G}, v'\rangle$ be two arrow-logical models. Then h is a zigzagmorphism from M to N, if in addition to the above conditions, it satisfies:

- $x \in v(p) \iff h(x) \in v'(p)$, for every propositional variable p.

We recall from (Bull and Segerberg, 1984) the standard modal-logical result that (local) truth is preserved under zigzagmorphisms between models.

The next proposition relates the two classes K and $\mathsf{PF}_{r,s}$ of frames.

PROPOSITION 3.4. K $= \mathbf{Zig}\mathsf{PF}_{r,s}$.

Proof. An explicit proof can be found in (Marx and Venema, 1995). The proposition originates, in algebraic disguise, with (Maddux, 1982). □

Now we can state a Hilbert-style completeness result for $PAL_{r,s}$. Let $\vdash_{K_1}$ denote the derivation system for arrow logic consisting of the K-axioms, the axioms $A_1, \ldots, A_5$, and the rules modus ponens, universal generalization and substitution.

THEOREM 3.5 *(Soundness and Completeness for $PAL_{r,s}$).* Let $\Gamma \cup \{\varphi\}$ be any set of AL-formulas. Then,

$$\Gamma \vdash_{K_1} \varphi \iff \Gamma \models_{\mathsf{PF}_{r,s}} \varphi.$$

Proof. By the completeness part of Sahlqvist's theorem, $\vdash_{K_1}$ is strongly sound and complete with respect to the class K. It is easy to check soundness of $\vdash_{K_1}$ with respect to the class $\mathsf{PF}_{r,s}$. For completeness, assume $\Gamma \not\vdash_{K_1} \varphi$. We have to prove $\Gamma \not\models_{\mathsf{PF}_{r,s}} \varphi$. By the Sahlqvist completeness result, there is a frame $\mathcal{F} \in \mathsf{K}$, a valuation v and a world w such that $\langle \mathcal{F}, v\rangle \models \Gamma$ and $w \Vdash_v \neg\varphi$. By Proposition 3.4, $\mathcal{F}$ is a zigzagmorphic image of a pair frame $\mathcal{G} \in \mathsf{PF}_{r,s}$. Let this zigzagmorphism be denoted by h. Let the valuation v' on $\mathcal{G}$ be defined as:

$$v'(p) = \{w' : h(w') \in v(p)\}, \text{ for every propositional variable } p.$$

Then the model $\langle \mathcal{F}, v\rangle$ is a zigzagmorphic image of the model $\langle \mathcal{G}, v'\rangle$. Thus, for every world w' of $\mathcal{G}$ and formula ψ, $w' \Vdash_{v'} \psi$ iff $h(w') \Vdash_v \psi$. This means that $\langle \mathcal{G}, v'\rangle \models \Gamma$. On the other hand, let w' be the pre-image of w, i.e., $h(w') = w$. Then $w' \Vdash_{v'} \neg\varphi$ by $w \Vdash_v \neg\varphi$. That is, we found a pair model witnessing $\Gamma \not\models_{\mathsf{PF}_{r,s}} \varphi$. □

With Theorem 3.5 we finished our first step in the taming process. Now we apply the second step, and add the difference operator to the just tamed logic.

3.2. SECOND TAMING STEP: AXIOMATIZING $PAL^{\mathsf{D}}_{r,s}$

We now give a characterization of the class $\mathsf{PFD}_{r,s}$ in the same spirit as above for $\mathsf{PF}_{r,s}$. Define the following $PAL^{\mathsf{D}}_{r,s}$ formulas:

(A_6) $\varphi \wedge \mathsf{D}\psi \rightarrow \mathsf{D}(\psi \wedge \mathsf{D}\varphi)$
(A_7) $\mathsf{DD}\varphi \rightarrow (\varphi \vee \mathsf{D}\varphi)$
(A_8) $\otimes\varphi \rightarrow \Diamond\varphi$
(A_9) $\varphi \bullet \psi \rightarrow \Diamond\varphi \wedge \Diamond\psi$
(A_{10}) $\mathsf{s}^1_0(\mathsf{dom}(\varphi \bullet \mathsf{dom}\psi)) \rightarrow \varphi \bullet \psi \vee \mathsf{s}^0_1\,\mathsf{Dran}\psi$
(A_{11}) $\mathsf{D}\varphi \leftrightarrow \mathsf{s}^1_0\,\mathsf{Ddom}\varphi \vee \mathsf{s}^0_1\,\mathsf{Dran}\varphi.$

Again all these formulas are in Sahlqvist form, as is easy to verify. Let $\vdash_{K_2}$ be the extension of the derivation system $\vdash_{K_1}$ with the axioms (A_6),..., (A_{11}) and distribution and universal generalization for the D-operator. We are ready to formulate an explicit version of Theorem 2.3.

THEOREM 3.6 *(Soundness and Completeness for $PAL^{\mathsf{D}}_{r,s}$).* Let $\Gamma \cup \{\varphi\}$ be any set of PAL^{D}-formulas. Then,

$$\Gamma \vdash_{K_2} \varphi \iff \Gamma \models_{\mathsf{PFD}_{r,s}} \varphi.$$

Proof. By Proposition 3.9 below, using the same argument as in the proof of the completeness Theorem 3.5. □

The rest of this section is devoted to a proof of Proposition 3.9 below. We proceed in the same way as in the case of $PAL_{r,s}$, so we start by stating the frame correspondents of the axioms.

Expand the first-order language of arrow frames with a binary relation R, which denotes the accessibility relation of the D-operator (R is not necessarily $\neq$). That is, in a model with valuation v,

$$x \Vdash_v \mathsf{D}\varphi \overset{\text{def}}{\iff} (\exists y)\mathsf{R}xy \;\&\; y \Vdash_v \varphi.$$

The conditions corresponding to the axioms $(A_6), \ldots, (A_{11})$ are the following:

(C_6) $\forall xy(\mathsf{R}xy \to \mathsf{R}yx)$
(C_7) $\forall xyz((\mathsf{R}xy \wedge \mathsf{R}yz) \to (x = z \vee \mathsf{R}xz))$
(C_8) $\forall x(x = \mathsf{f}x \vee \mathsf{R}x\mathsf{f}x)$
(C_9) $\forall xyz(\mathsf{C}xyz \to ((x = y \vee \mathsf{R}xy) \wedge (x = z \vee \mathsf{R}xz)))$
(C_{10}) $\forall xyz((x_l = y_l \wedge y_r = z_l) \to (\mathsf{C}xyz \vee \mathsf{R}x_r z_r))$
(C_{11}) $\forall xy(\mathsf{R}xy \leftrightarrow (\mathsf{R}x_l y_l \vee \mathsf{R}x_r y_r))$.

PROPOSITION 3.7. Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle$ be an expanded arrow frame satisfying the conditions $C_1, \ldots, C_5$. Then, for $6 \leq i \leq 11$, we have

$$\mathcal{F} \models (A_i) \iff \mathcal{F} \models (C_i).$$

Proof. By the correspondence part of Sahlqvist's theorem. We show the "hard side" of (C_{11}) as an example. Assume $\mathcal{F} \models (A_{11})$. We want to show that $\mathcal{F} \models (C_{11})$, so assume $\mathsf{R}xy$, for some x, y in the domain of $\mathcal{F}$. Let $\mathrm{M} = \langle \mathcal{F}, v\rangle$ be a model with $v(p) = \{y\}$. Then $x \Vdash_v \mathsf{D}p$, whence $x \Vdash_v \mathsf{s}^1_0\, \mathsf{D}\mathsf{dom}\varphi \vee \mathsf{s}^0_1\, \mathsf{D}\mathsf{ran}\varphi$. Suppose $x \Vdash_v \mathsf{s}^1_0\, \mathsf{D}\mathsf{dom}\varphi$. Unraveling the truth-definition, we obtain that there exists a z such that $\mathsf{R}x_l z$ and $z \Vdash_v \mathsf{dom}p$. But then, by our chosen valuation and the fact that $\mathcal{F}$ satisfies $C_1, \ldots, C_5$, z must be y_l, so we have $\mathsf{R}x_l y_l$, as desired. The other side is proved similarly. □

Define the following class of arrow frames expanded with a binary relation R:

$$\mathsf{KD} \overset{\text{def}}{=} \{\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle : \mathcal{F} \models (C_1) - (C_{11})\}.$$

Later we need the following facts about this class.

PROPOSITION 3.8. The following theorems follow from conditions (C_1)–(C_{10}):

(D_1) $\forall xyz((\neg\mathsf{R}x_l y_l \wedge y_r = z_l \wedge z_r = x_r) \to \mathsf{C}xyz)$
(D_2) $\forall xyz((x_l = y_l \wedge \neg\mathsf{R}y_r z_l \wedge z_r = x_r) \to \mathsf{C}xyz)$
(D_3) $\forall xy((x_l = y_l \wedge \neg\mathsf{R}x_r y_r) \to x = y)$
(D_4) $\forall xy((\neg\mathsf{R}x_l y_l \wedge x_r = y_r) \to x = y)$.

Conditions (D_1) and (D_2) are just variants of (C_{10}). (D_3) and (D_4) express the fact that if $x_l = y_l$, $x_r = y_r$ and one of the two pairs is R-irreflexive, then x equals y.

Proof. Use $(C_{10}), (C_5), (T_4)$ and (T_2) to show (D_1) and (D_2). Assume the antecedent of (D_3). Use (T_1) and (T_2) to derive that $(x_l)_l = x_l \wedge \neg \mathsf{R} x_r (\mathsf{f}y)_l \wedge (\mathsf{f}y)_r = (x_l)_r$. Then (D_2) implies that $\mathsf{C} x_l x \mathsf{f} y$ and (T_5) that $\mathsf{ff}y = x$. But then, idempotence of f implies $x = y$. (D_4) follows easily from (D_3). □

We are almost ready to formulate the core lemma of the completeness proof. A zigzagmorphism between arrow frames expanded with the relation R is a zigzagmorphism for the arrow frames with the extra condition that (in the terminology of Definition 3.3):

- $\mathsf{R}^{\mathcal{F}} xy \Rightarrow \mathsf{R}^{\mathcal{G}} h(x)h(y)$
- $\mathsf{R}^{\mathcal{G}} h(x)y \Rightarrow \exists y'(h(y') = y \;\&\; \mathsf{R}^{\mathcal{F}} xy')$.

Let (AU) denote the frame condition $\forall xy(x \neq y \rightarrow \mathsf{R}xy)$.

PROPOSITION 3.9. (i) Each $\mathcal{F} \in \mathsf{KD}$ consists of a disjoint union of frames satisfying (AU).

(ii) Each $\mathcal{F} \in \mathsf{KD}$ which satisfies (AU) is a zigzagmorphic image of some $\mathcal{G} \in \mathsf{PFD}_{r,s}$.

Proof. (i) Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle \in \mathsf{KD}$. Define a binary relation $\sim$ on W as follows

$$x \sim y \overset{\text{def}}{\Longleftrightarrow} x = y \vee \mathsf{R}xy.$$

Conditions (C_6) and (C_7) imply that $\sim$ is an equivalence relation. We denote the equivalence class of x by $\bar{x} \overset{\text{def}}{=} \{y \in W : x \sim y\}$. Define for each equivalence class, a frame $\mathcal{F}_{\bar{x}} \overset{\text{def}}{=} \langle \bar{x}, \mathsf{C}', \mathsf{f}', \mathsf{I}', \mathsf{R}'\rangle$ such that the relations are the restrictions to $\bar{x}$. We claim that each $\mathcal{F}_{\bar{x}} \models (AU)$ and $\mathcal{F}$ is a disjoint union of the system of frames $\{\mathcal{F}_{\bar{x}} : x \in F\}$, by which we prove part (i) of the lemma. The first part of the claim is immediate, for the second it suffices to show that each $\mathcal{F}_{\bar{x}}$ is a subframe of $\mathcal{F}$ generated by $\bar{x}$, which is precisely the point of conditions (C_8) and (C_9). This finishes part (i) of the lemma.

(ii) Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle \in \mathsf{KD}$ satisfy (AU). The proof of part (ii) consists of two steps. First we show that $\mathcal{F}$ is a zigzagmorphic image of a pair frame expanded with a relation R which satisfies (AU). In the second step we make the R relation irreflexive, thereby turning it into the inequality relation. These two steps are given in the schema below.

$$\begin{array}{lccccc} & & \text{STEP A} & & \text{STEP B} & \\ \text{full language} & \mathcal{F} \in \mathsf{KD} & \xleftarrow{l} & \mathcal{G} \in \mathsf{KD} & \xleftarrow{p} & \mathcal{H} \in \mathsf{PFD}_{r,s} \\ & & & \mathcal{G} \models (AU) & & \\ & \vdots & & \vdots & & \\ \text{D-free reduct} & \mathcal{F}^* \in \mathsf{K} & \xleftarrow{l^*} & \mathcal{G}^* \in \mathsf{PF}_{r,s} & & \end{array}$$

By Proposition 3.4 we may assume that the R-free reduct $\mathcal{F}^*$ of $\mathcal{F}$ is a zigzagmorphic image, say by function l^*, of a pair frame $\mathcal{G}^* = \langle V^*, \mathsf{C}_{V^*}, \mathsf{f}_{V^*}, \mathsf{I}_{V^*}\rangle$ for some reflexive and symmetric relation V^* with base U^*.

STEP A. The problem with the representation $\mathcal{G}^*$ is that it may contain two different pairs which get mapped to the same point in $\mathcal{F}$ which is not R reflexive. This will prevent extending the zigzagmorphism l^* to one for R as well. We will create a new pair frame $\mathcal{G}$ in which this problem is eliminated.

Define an equivalence relation $\equiv$ on the base U^* as follows:

$$(\forall u, v \in U^*) : u \equiv v \overset{\text{def}}{\Longleftrightarrow} u = v \text{ or } \neg \mathsf{R} l^*\langle u, u\rangle l^*\langle v, v\rangle.$$

CLAIM 1. (i) $\equiv$ is an equivalence relation.

(ii) $u \equiv v \Rightarrow l^*\langle u, u\rangle = l^*\langle v, v\rangle$.

Proof of Claim. Immediate because $\mathcal{F} \models (AU)$. □

Define

$$\begin{aligned} U &\overset{\text{def}}{=} U^*/{\equiv} \\ V &\overset{\text{def}}{=} \{\langle u/{\equiv}, v/{\equiv}\rangle \in U \times U : \langle u, v\rangle \in V^*\}. \end{aligned}$$

Define a function $l : V \longrightarrow W$ as $l\langle u/{\equiv}, v/{\equiv}\rangle \overset{\text{def}}{=} l^*\langle u', v'\rangle$ for some $u' \in u/{\equiv}$ and $v' \in v/{\equiv}$. Note that, by the definition of V, for every pair $\langle u/{\equiv}, v/{\equiv}\rangle \in V$, there exists a pair $\langle u', v'\rangle \in V^*$ such that $u \equiv u'$ and $v \equiv v'$. Hence l is defined for every element in V. The next claim states that this is a real definition.

CLAIM 2. l is well defined, i.e., for every $\langle u, v\rangle, \langle u', v'\rangle \in V^*$, if $u \equiv u'$ and $v \equiv v'$ then $l^*\langle u, v\rangle = l^*\langle u', v'\rangle$.

Proof of Claim. Suppose $\langle u, v\rangle, \langle u', v'\rangle \in V^*$ and $u \equiv u'$ and $v \equiv v'$. We have four cases, according to whether $u = u'$ and $v = v'$. If $u = u'$ and $v = v'$ the statement is trivial. So assume otherwise.

Case 2: $[u \neq u' \;\&\; v \neq v']$. Then, by the definition of $\equiv$, $\neg \mathsf{R} l^*\langle u, u\rangle l^*\langle u', u'\rangle$ and $\neg \mathsf{R} l^*\langle v, v\rangle l^*\langle v', v'\rangle$. Since l^* is a zigzagmorphism for the relational operators, this means that we have $\neg \mathsf{R}(l^*\langle u, v\rangle)_l(l^*\langle u', v'\rangle)_l$ and $\neg \mathsf{R}(l^*\langle u, v\rangle)_r(l^*\langle u', v'\rangle)_r$.

Then condition (C_{11}) implies that $\neg \mathsf{R}l^*\langle u,v\rangle l^*\langle u',v'\rangle$, so by (AU), $l^*\langle u,v\rangle = l^*\langle u',v'\rangle$.

Cases 3 and 4: $[u = u' \ \&\ \neg \mathsf{R}l^*\langle v,v\rangle l^*\langle v',v'\rangle]$ and $[\neg \mathsf{R}l^*\langle u,u\rangle l^*\langle u',u'\rangle \ \&\ v = v']$. These cases are solved in a similar way, but now using (D_3) and (D_4) from Proposition 3.8. □

To finish the first step of the proof, define the pair frame $\mathcal{G} = \langle V, \mathsf{C}_V, \mathsf{f}_V, \mathsf{I}_V, \mathsf{R}_V\rangle$ where we set $\mathsf{R}_V xy \overset{\text{def}}{\iff} \mathsf{R}l(x)l(y)$, and $\mathsf{C}_V, \mathsf{f}_V$ and I_V are relational composition, converse and identity restricted to V, respectively. The next claim states that we have accomplished our first goal.

CLAIM 3. (i) V is a reflexive and symmetric relation.

(ii) $\mathcal{G} \models x \neq y \to \mathsf{R}_V xy$.

(iii) The function l is a zigzagmorphism from $\mathcal{G}$ onto the frame $\mathcal{F}$.

Proof of Claim. (i) is immediate by the definition of V.

(ii) We will denote $u/\equiv$ by $\bar{u}$. Suppose that $\neg \mathsf{R}_V\langle \bar{u},\bar{v}\rangle\langle \bar{u'},\bar{v'}\rangle$ for some $\langle \bar{u},\bar{v}\rangle$ and $\langle \bar{u'},\bar{v'}\rangle \in V$. We have to show that $\bar{u} = \bar{u'}$ and $\bar{v} = \bar{v'}$. We compute:

$$\begin{array}{lcl}
\neg \mathsf{R}_V\langle \bar{u},\bar{v}\rangle\langle \bar{u'},\bar{v'}\rangle & \iff & \text{(using well-definedness of } l)\\
\neg \mathsf{R}l^*\langle u,v\rangle l^*\langle u',v'\rangle & \overset{(C_{11})}{\iff} & \text{(since } l^* \text{ is a zigzagmorphism)}\\
\neg \mathsf{R}l^*\langle u,u\rangle l^*\langle u',u'\rangle & & \\
\quad \& \ \neg \mathsf{R}l^*\langle v,v\rangle l^*\langle v',v'\rangle & \Rightarrow & \text{(by the definition of } \equiv)\\
\bar{u} = \bar{u'} \text{ and } \bar{v} = \bar{v'}. & &
\end{array}$$

(iii) All steps in this proof, except homomorphism of C_V, are straightforward, since equivalent pairs are mapped to the same place, cf. Claim 2. We show that l is a homomorphism for C_V. Suppose $\langle \bar{u},\bar{v}\rangle, \langle \bar{u},\bar{w}\rangle, \langle \bar{w},\bar{v}\rangle \in V$. We have to show that $\mathsf{C}l\langle \bar{u},\bar{v}\rangle l\langle \bar{u},\bar{w}\rangle l\langle \bar{w},\bar{v}\rangle$ holds. By definition of V, we have $u,u',v,v',w,w' \in U^*$, $\{\langle u,v\rangle, \langle u',w\rangle, \langle w',v'\rangle\} \subseteq V^*$ and $u \equiv u', w \equiv w'$ and $v \equiv v'$. By the definition of l it is sufficient to show that $\mathsf{C}l^*\langle u,v\rangle l^*\langle u',w\rangle l^*\langle w',v'\rangle$ holds.

There are several cases, depending on why the points are equivalent. One easy case is this: if $u = u', w = w'$ and $v = v'$, then, since l^* is a homomorphism, we have $\mathsf{C}l^*\langle u,v\rangle l^*\langle u,w\rangle l^*\langle w,v\rangle$. In all other cases, for at least one of the three pairs of equivalent points, the reflexive pairs at those points are mapped to an R irreflexive arrow. The next claim will help us out.

CLAIM 4. If $\mathcal{F} \in \mathsf{KD}$ and $\mathcal{F} \models (AU)$ then

$$\mathcal{F} \models [x_l = y_l \wedge y_r = z_l \wedge z_r = x_r \wedge (\neg \mathsf{R}x_l y_l \vee \neg \mathsf{R}y_r z_l \vee \neg \mathsf{R}z_r x_r)] \to \mathsf{C}xyz.$$

In words: if $x_l = y_l$ & $y_r = z_l$ & $z_r = x_r$ and at least one of the pairs $\langle x_l,y_l\rangle, \langle y_r z_l\rangle, \langle z_r,x_r\rangle$ is R irreflexive, then x can be decomposed into y and z.

Proof of Claim. We have shown already two cases: (D_1) and (D_2). Use (AU) to prove all other possibilities from $(C_{10}), (D_1)$ and (D_2). □

We show with an example how this claim helps us. Suppose $\neg \mathsf{R} l^*\langle u,u\rangle l^*\langle u',u'\rangle$ and $w = w'$ and $v = v'$. Because l^* is a zigzagmorphism, we have $l^*\langle u,u\rangle = (l^*\langle u,v\rangle)_l$, and similarly for the others. This implies that

$$\begin{aligned}&\neg \mathsf{R}(l^*\langle u,v\rangle)_l(l^*\langle u',w\rangle)_l \wedge (l^*\langle u',w\rangle)_r\\ &\quad = (l^*\langle w',v'\rangle)_l \wedge (l^*\langle w',v'\rangle)_r = (l^*\langle u,v\rangle)_r.\end{aligned}$$

So by the above claim, $\mathsf{C} l^*\langle u,v\rangle l^*\langle u',w\rangle l^*\langle w',v'\rangle$. Hence also $\mathsf{C} l\langle \bar{u},\bar{v}\rangle l\langle \bar{u},\bar{w}\rangle$ $l\langle \bar{w},\bar{v}\rangle$, which is what we had to prove. □

STEP B. Since the frame $\mathcal{G}$ constructed in the previous step is a pair frame, we only have to make sure that R_V becomes the inequality. Since $\mathcal{G} \models (AU)$, it suffices to make the R_V relation *irreflexive*. Define the following two sets:

$$\begin{aligned}
BAD &\stackrel{\text{def}}{=} \{u \in U : \mathsf{R}_V\langle u,u\rangle\,\langle u,u\rangle\}\\
COPIES &\stackrel{\text{def}}{=} \{\langle u',u'\rangle : u \in BAD\} \cup\\
&\qquad \{\langle v,u'\rangle, \langle u',v\rangle : \langle u,v\rangle \in V, u \in BAD\ \&\ u \neq v\}
\end{aligned}$$

Without loss of generality we may assume that *COPIES* is disjoint from V. Let $\mathcal{H} = \langle H, \mathsf{C}_H, \mathsf{f}_H, \mathsf{I}_H, \neq\rangle$ be given by the set H, defined as $H \stackrel{\text{def}}{=} V \cup COPIES$. It is easily verified that H is a reflexive and symmetric relation, so $\mathcal{H} \in \mathsf{PFD}_{r,s}$.

Define a function $p : H \longrightarrow V$ as the unique function such that

- p restricted to V is the identity function
- $p(\langle u',u'\rangle) \stackrel{\text{def}}{=} \langle u,u\rangle$ if $u \in BAD$
- $p(\langle u',v\rangle) \stackrel{\text{def}}{=} \langle u,v\rangle$ and $p(\langle v,u'\rangle) \stackrel{\text{def}}{=} \langle v,u\rangle$ if $u \neq v$ and $u \in BAD$.

The next claim states that for R_V we did enough, that is, we *only* copied R_V reflexive arrows.

CLAIM 5. (i) $(\forall x \in V) : (\mathsf{R}_V xx \iff$ there exists a copy of x in *COPIES*).

(ii) $(\forall x,y \in H) : ((x \neq y\ \&\ p(x) = p(y)) \Rightarrow \mathsf{R}_V p(x)p(y))$.

Proof of Claim. (i) Suppose $\mathsf{R}_V\langle u,v\rangle\langle u,v\rangle$ for some $\langle u,v\rangle \in V$. If $u = v$ then the claim holds by definition. So, suppose $u \neq v$. Then:

$$\begin{aligned}
&\mathsf{R}_V\langle u,v\rangle\langle u,v\rangle \stackrel{(C_{11})}{\iff}\\
&\mathsf{R}_V\langle u,u\rangle\langle u,u\rangle \text{ or } \mathsf{R}_V\langle v,v\rangle\langle v,v\rangle \iff\\
&u \in BAD \text{ or } v \in BAD \iff\\
&\langle u',v\rangle \in COPIES \text{ or } \langle u,v'\rangle \in COPIES.
\end{aligned}$$

(ii) follows from (i), since two pairs of H can only be mapped to the same pair in V if they are copies of each other. □

CLAIM 6. p is a zigzagmorphism from $\mathcal{H}$ onto $\mathcal{G}$.

Proof of Claim. Clearly p is surjective. That p is a zigzagmorphism for R_V is immediate by Claim 5. For I and f this is straightforward to check. For C observe that, if $\{\langle u,v\rangle, \langle u,w\rangle, \langle w,v\rangle\} \subseteq H$, then either they all are in V, or one pair is in V and the other two are in *COPIES*. With these two steps we have finished the proof of Proposition 3.9, because our original frame $\mathcal{F}$ will be a zigzagmorphic image of the frame $\mathcal{H}$ by the function given by the composition of l and p. □

4. Concluding remarks

In this section, we summarize our taming strategy, and mention some related results.

The taming strategy consists of two steps. First we try to get rid of those properties (frame conditions) which cause the ugly behavior of the logic. Second we try to strengthen the logic, e.g., by (re-)introducing connectives.

In this paper we showed how this strategy works for arrow logic. The same strategy can be applied to first-order logic as well. For instance, it is possible to generalize the notions of reflexivity and symmetry to relations of higher rank than two. In (Németi, 1992) it is shown that the completely relativized and the "reflexive" and/or "symmetric" versions of first-order logic are decidable. The "symmetric" and "reflexive" version of first-order logic has a strongly sound and strongly complete Hilbert-style calculus as well, cf. (Andréka and Thompson, 1988) and (Marx and Venema, 1995). Moreover, the above decidability results hold after strengthening by adding more connectives (e.g., substitutions, graded modalities, etc.) to these weakened versions of first-order logic, cf. (Marx and Venema, 1995) and (Mikulás, 1995a).

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