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Abstract

We construct new topological theories related to sigma models whose
target space is a seven-dimensional manifold of G_2 holonomy. We define
a new type of topological twist and identify the BRST operator and the
physical states. Unlike the more familiar six-dimensional case, our topo-
logical model is defined in terms of conformal blocks and not in terms of
local operators of the original theory. We also present evidence that one
can extend this definition to all genera and construct a seven-dimensional
topological string theory. We compute genus zero correlation functions
and relate these to Hitchin's functional for three-forms in seven dimen-
sions. Along the way we develop the analogue of special geometry for
G_2 manifolds. When the seven-dimensional topological twist is applied
to the product of a Calabi–Yau manifold and a circle, the result is an
interesting combination of the six-dimensional A and B models.

1   Introduction

Topological strings on Calabi–Yau manifolds describe certain solvable sectors
of superstrings. In particular, various BPS quantities in string theory can be

exactly computed using their topological twisted version. Also, topological strings provide simplified toy examples of string theories which are still rich enough to exhibit interesting stringy phenomena in a more controlled setting. There are two inequivalent ways to twist the Calabi–Yau σ-model which leads to the celebrated A and B models [1]. The metric is not a fundamental degree of freedom in these models. Instead, the A-model apparently only involves the Kähler moduli and the B-model only the complex structure moduli. However, the roles interchange once branes are included, and it has even been conjectured that there is a version of S-duality which maps the A-model to the B-model on the same Calabi–Yau manifold [2]. This is quite distinct from mirror symmetry which relates the A-model on X to the B-model on the mirror of X. Subsequently, several authors found evidence for the existence of seven- and/or eight-dimensional theories that unify and extend the A and B models [3–7]. This was one of our motivations to take a closer look at string theory on seven-dimensional manifolds of G_2 holonomy and see whether it allows for a topological twist. We were also motivated by other issues, such as applications to M-theory compactifications on G_2 manifolds, and the possibility of improving our understanding of the relation between supersymmetric gauge theories in three and four dimensions.

In this paper, we study the construction of a topological string theory on a seven-dimensional manifold of G_2 holonomy. Our approach is to define a topological twist of the σ model on G_2 manifolds. On such manifolds, the $(1,1)$ world-sheet supersymmetry algebra gets extended to a non-linear algebra, which has a $c = \frac{7}{10}$ minimal model sub-algebra [8]. We use this fact to define the topological twist of the σ model. This is a particular realization of a more generic result: on an orientable d-dimensional manifold which has holonomy group H which is a subgroup of $\mathrm{SO}(d)$, the coset CFT $\mathrm{SO}(d)_1/H_1$ with its chiral algebra appears as a building block of the corresponding sigma model, at least at large volume. It is natural to conjecture that this building block persists at finite volume (i.e., to all orders in α'). It therefore gives rise to extra structure in the world sheet theory which corresponds to geometrical constructions in the target space. For example, for Calabi–Yau threefolds, this extra structure is given by the $U(1)$ R-symmetry current, which can be used to Hodge decompose forms of total degree $p + q$ into (p, q) forms. The exterior derivative has a corresponding decomposition as $d = \partial + \bar{\partial}$, and physical states in the world sheet theory correspond to suitable Dolbeault co-homology groups $H^*_{\bar{\partial}}(X, V)$. A G_2 manifold has an analogous refinement of the de Rham co-homology [9]. Differential forms can be decomposed into irreducible representations of G_2. The exterior derivative can be written as the sum of two nilpotent operators $d = \check{d} + \hat{d}$, where $\check{d}$ and $\hat{d}$ are obtained from d by restricting its action on differential forms to two disjoint subsets of G_2 representations. This leads to a natural question: Is there a topologically

twisted theory such that the BRST operator in the left (or right) sector maps to $\check{d}$? We will see that the answer to this question is yes, and in this paper, we give the explicit construction of such a theory.

The outline of the paper is as follows. In Section 2, we start by reviewing σ models on target spaces of G_2 holonomy. We discuss the relation between covariantly constant p-forms on target spaces and holomorphic currents in the world sheet theory: every covariantly constant p-form leads to the existence of a chiral current supermultiplet [25] (at least classically). A G_2 manifold has a covariantly constant three- and four-form leading to extra currents in the chiral algebra extending it from a $(1,1)$ super-conformal algebra to a non-linear algebra generated by six currents. As expected, this algebra contains the chiral algebra of the coset $SO(7)_1/(G_2)_1$, which by itself is another $\mathcal{N} = 1$ superconformal algebra with central charge $c = \frac{7}{10}$. This is a minimal model, called the tri-critical Ising model, which plays a crucial role in defining the twisted theory. In fact, the tri-critical Ising model is what replaces the $U(1)$ R-symmetry of the $\mathcal{N} = 2$ superconformal algebra. The full $c = \frac{21}{2}$ Virasoro algebra with generators L_n splits into two commuting Virasoro algebras, $L_n = L_n^I + L_n^r$, with L_n^I the generators of the $c = \frac{7}{10}$ tri-critical Ising model. This means that we can label highest weight states by their L_0^I and $L_0^r = L_0 - L_0^I$ eigenvalues. We also review some facts about the tri-critical Ising model. In the NS sector, there are primary fields of weights $0, \frac{1}{10}, \frac{6}{10}$ and $\frac{3}{2}$ and in the Ramond sector, there are two primary fields of weights $\frac{7}{16}$ and $\frac{3}{80}$. We discuss the fusion rules in this model, which helps us identify the conformal block structure of various fields. This structure plays an important role in definition of the twisted theory.

In Section 3, we derive a unitarity bound for the algebra which provides a non-linear inequality (a BPS bound) between the total weight of the state and its tri-critical Ising model weight. We define a notion of chiral primary states for G_2 sigma model by requiring that they saturate this bound. We also discuss the special chiral primary states in the CFT which correspond to the metric moduli that preserve the G_2 holonomy.

In Section 4, we define the topological twisting of the G_2 σ-model. We define correlation functions in the twisted theory by relating them to certain correlation functions in the untwisted theory with extra insertion of a certain Ramond sector spin field. The twisting acts on different conformal blocks of the same local operators in a different way. We also define the BRST operator Q as a particular conformal block of the original $\mathcal{N} = 1$ supercharge. The BRST cohomology consists precisely of the chiral primary states. We discuss the chiral ring, descent relations and a suggestive localization argument which shows that the path integral localizes on constant maps. Finally, we

analyze some of the putative properties of the twisted stress tensor of the theory.

In Section 5, we go on to discuss the geometric interpretation of the BRST cohomology. To make this connection, we use the fact that p-forms on the G_2 manifold transforming in different G_2 representations correspond to operators in the CFT which carry different tri-critical Ising model weight (L_0^I eigenvalue). Using this we can identify how the BRST operator acts on p-forms. We find that the BRST cohomology in the left or the right moving sector is a Dolbeault type cohomology of the differential complex $0 \to \Lambda_1^0 \to \Lambda_7^1 \to \Lambda_7^2 \to \Lambda_1^3 \to 0$ where the differential operator is the usual exterior derivative composed of various projection operators to particular representations of G_2 as indicated by the subscript. When we combine the left and the right movers, the BRST cohomology is just as a vector space equal to the total de Rham cohomology $H^*(M)$. The BRST cohomology includes the metric moduli that preserve the G_2 holonomy. These are in one-to-one correspondence with elements of $H^3(M)$. We also compute three point functions at genus 0 and show that these can be written as appropriate triple derivatives of a suitable generalization of Hitchin's functional. To show this, we develop an analogue of special geometry for G_2 manifolds by defining co-ordinates on the moduli space of G_2 metrics as periods of the G_2 invariant three form and the dual four form. As in the case of Calabi–Yau manifolds, the dual periods are derivatives of a certain pre-potential, which is proportional to the Hitchin's functional. We also argue that the partition function should be viewed as a wave function in a quantum mechanics corresponding to the phase space $H^2 \oplus H^3 \oplus H^4 \oplus H^5$, where the symplectic form is given by integrating the wedge product of two forms over the seven manifold. We also consider the special case of the G_2 manifold being a product of Calabi–Yau and a circle and show that the twisted G_2 theory is an interesting and non-trivial combination of the A and the B models.

There is extensive literature about string theory and M-theory compactified on G_2 manifolds. The first detailed study of the world-sheet formulation of strings on G_2 manifolds appeared in [8]. The world-sheet chiral algebra was studied in some detail in [8, 10–12]. For more about type II strings on G_2 manifolds and their mirror symmetry, see e.g. [13–23]. A review of M-theory on G_2 manifolds with many references can be found in [24].

2 G_2 sigma models

A supersymmetric σ-model on a generic Riemannian manifold has $(1, 1)$ world-sheet supersymmetry. However, existence of covariantly constant

p-forms implies the existence of an extended symmetry algebra [25]. This symmetry algebra is a priori only present in the classical theory. Upon quantization, it could either be lost or it could be preserved up to quantum modifications. However, since the extended symmetry is typically crucial for many properties of the theory such as spacetime supersymmetry, it is natural to postulate the extended symmetry survives quantization. To determine the quantum version of the algebra, one can for example study the most general quantum algebra with the right set of generators. For the generators expected in the G_2 case this was done in [10] (though not with this motivation). It turns out that there is a two-parameter family of algebras with the right generators. By requiring the right value of the total central charge, and by requiring that it contains the tri-critical Ising model (which is crucial for space-time supersymmetry), both parameters are fixed uniquely leading to what we call the G_2 algebra.

Alternatively, one could have started with the special case of $\mathbb{R}^7$ as a model of a G_2 manifold in the infinite volume limit. This is simply a theory of free fermions and bosons, and one can easily find a quantum algebra with the right number of generators using the explicit form of the covariantly closed three- and four-form for G_2 manifolds written in terms of a local orthonormal frame. From this large volume point of view it is natural to expect the coset $\mathrm{SO}(7)_1/(G_2)_1$ to appear, since $\mathrm{SO}(7)_1$ is just a theory of free fermions and bosons. In any case, this leads to the same result for the G_2 algebra as the approach described in the previous paragraph. In the remainder of this section we will briefly describe the large volume approach.

2.1 Covariantly constant p-forms and extended chiral algebras

We start from a sigma model with $(1, 1)$ supersymmetry, writing its action in superspace:

$$S = \int d^2z\, d^2\theta (G_{\mu\nu} + B_{\mu\nu}) D_\theta \mathbf{X}^\mu D_{\bar\theta} \mathbf{X}^\nu \tag{2.1}$$

where

$$D_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}, \quad D_{\bar\theta} = \frac{\partial}{\partial\bar\theta} + \bar\theta\frac{\partial}{\partial\bar z}$$

and $\mathbf{X}$ is a superfield, which, on shell can be taken to be chiral:

$$\mathbf{X}^\mu = \phi^\mu(z) + \theta\psi^\mu(z).$$

For now, we set $B_{\mu\nu} = 0$. This model generically has $(1, 1)$ superconformal symmetry classically. The super stress-energy tensor is given by

$$\mathbf{T}(z, \theta) = G(z) + \theta T(z) = -\tfrac{1}{2} G_{\mu\nu} D_\theta \mathbf{X}^\mu \partial_z \mathbf{X}^\nu.$$

This $\mathcal{N} = (1,1)$ sigma model can be formulated on an arbitrary target space. However, generically the target space theory will not be supersymmetric. For the target space theory to be supersymmetric the target space manifold must be of special holonomy. This ensures that covariantly constant spinors, used to construct supercharges, can be defined. The existence of covariantly constant spinors on the manifold also implies the existence of covariantly constant p-forms given by

$$\phi_{(p)} = \epsilon^T \Gamma_{i_1 \ldots i_p} \epsilon \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \qquad (2.2)$$

This expression may be identically zero. The details of the holonomy group of the target space manifold dictate which p-forms are actually present.

The existence of such covariantly constant p-forms on the target space manifold implies the existence of extra elements in the chiral algebra [25]. For example, given a covariantly constant p-form, $\phi_{(p)} = \phi_{i_1 \ldots i_p} dx^{i_p} \wedge \cdots \wedge dx^{i_p}$ satisfying $\nabla \phi_{i_1 \ldots i_p} = 0$, we can construct a holomorphic superfield current given by

$$\mathbf{J}_{(p)}(z, \theta) = \phi_{i_1 \ldots i_p} D_\theta \mathbf{X}^{i_1} \cdots D_\theta \mathbf{X}^{i_p}$$

which satisfies $D_{\bar{\theta}} \mathbf{J}_{(p)} = 0$ on shell. In components, this implies the existence of a dimension $\frac{p}{2}$ and a dimension $\frac{p+1}{2}$ current. For example, on a Kähler manifold, the existence of a covariantly constant Kähler two-form $\omega = g_{i\bar{j}}(d\phi^i \wedge d\phi^{\bar{j}} - d\phi^{\bar{j}} \wedge d\phi^i)$ implies the existence of a dimension 1 current $J = g_{i\bar{j}}\psi^i\psi^{\bar{j}}$ and a dimension $\frac{3}{2}$ current $G'(z) = g_{i\bar{j}}(\psi^i \partial_z \phi^{\bar{j}} - \psi^{\bar{j}} \partial_z \phi^i)$, which add to the $(1,1)$ superconformal currents $G(z)$ and $T(z)$ to give a $(2,2)$ superconformal algebra. In fact, there is a non-linear extension of the $(2,2)$ algebra even in the case of Calabi–Yau by including generators corresponding to the (anti)holomorphic three-form. This algebra was studied in [26].

2.2   Extended algebra for $\mathbf{G_2}$ sigma models

A generic seven-dimensional Riemannian manifold has SO(7) holonomy. A G_2 manifold has holonomy which sits in a G_2 subgroup of SO(7). Under this embedding, the eight-dimensional spinor representation $\mathbf{8}$ of SO(7) decomposes into a $\mathbf{7}$ and a singlet of G_2:

$$\mathbf{8} \to \mathbf{7} \oplus \mathbf{1}$$

The singlet corresponds to a covariantly constant spinor ϵ on the manifold satisfying

$$\nabla \epsilon = 0.$$

For G_2 manifolds (2.2) is non-zero only when $p = 0, 3, 4$ and 7 since an anti-symmetrized product of p fundamentals (**7**) of SO(7) has a G_2 singlet for these p. The zero and the seven forms just correspond to constant functions and the volume form. In addition to these, there is a covariantly constant three-form $\phi^{(3)} = \phi_{ijk}^{(3)} dx^i \wedge dx^j \wedge dx^k$ and its Hodge dual four-form, $\phi^{(4)} = *\phi_{(3)} = \phi_{ijkl}^{(4)} dx^i \wedge dx^j \wedge dx^k \wedge dx^l$. By this discussion, the three-form implies the existence of a superfield current $\mathbf{J}_{(3)}(z, \theta) = \phi_{ijk}^{(3)} D_\theta \mathbf{X}^i D_\theta \mathbf{X}^j D_\theta \mathbf{X}^k \equiv \Phi + \theta K$. Explicitly, Φ is a dimension $\frac{3}{2}$ current

$$\Phi = \phi_{ijk}^{(3)} \psi^i \psi^j \psi^k \tag{2.3}$$

and K is its dimension 2 superpartner

$$K = \phi_{ijk}^{(3)} \psi^i \psi^j \partial \phi^k. \tag{2.4}$$

Similarly, the four-form implies the existence of a dimension 2 current

$$Y = \phi_{ijkl}^{(4)} \psi^i \psi^j \psi^k \psi^l \tag{2.5}$$

and its dimension $\frac{5}{2}$ superpartner

$$N = \phi_{ijkl}^{(4)} \psi^i \psi^j \psi^k \partial \phi^l. \tag{2.6}$$

However, as it will become clear later, instead of Y and N, it is more useful to use the following basis of chiral currents

$$X = -Y - \tfrac{1}{2} G_{ij} \psi^i \partial \psi^j \tag{2.7}$$

and its superpartner

$$M = -N - \tfrac{1}{2} G_{ij} \partial \phi^i \partial \psi^j + \tfrac{1}{2} G_{ij} \psi^i \partial^2 \phi^j. \tag{2.8}$$

So in summary, the G_2 sigma model has a chiral algebra generated by the following six currents

$$
\begin{array}{llll}
h = \frac{3}{2} & G(z) & \Phi(z) & \\
h = 2 & T(z) & K(z) & X(z) \\
h = \frac{5}{2} & & & M(z)
\end{array}
$$

These six generators form a closed algebra which appears explicitly, e.g., in [8, 11] (see also [12]). We have reproduced the algebra in Appendix B. As explained in the beginning of Section 2, the existence of this algebra can be taken as the definition of string theory on G_2 manifolds.

2.3   The tri-critical Ising model

An important fact, which will be crucial in almost all the remaining analysis, is that the generators Φ and X form a closed sub-algebra:

$$\Phi(z)\Phi(0) = -\frac{7}{z^3} + \frac{6}{z}X(0)$$

$$\Phi(z)X(0) = -\frac{15}{2z^2}\Phi(0) - \frac{5}{2z}\partial\Phi(0)$$

$$X(z)X(0) = \frac{35}{4z^4} - \frac{10}{z^2}X(0) - \frac{5}{z}\partial X(0).$$

Defining the supercurrent $G_I = \frac{i}{\sqrt{15}}\Phi$ and stress-energy tensor $T_I = -\frac{1}{5}X$ this is recognized to be the unique $\mathcal{N} = 1$ super-conformal algebra of the minimal model with central charge $c = \frac{7}{10}$ known as the tri-critical Ising model. This sub-algebra plays a similar role to the one played by the $U(1)$ R-symmetry in the case of Calabi–Yau target spaces. The extended chiral algebra contains two $\mathcal{N} = 1$ superconformal sub-algebras: the original one generated by (G, T) and the $\mathcal{N} = 1$ superconformal sub-algebra generated by (Φ, X).

In fact, with respect to the conformal symmetry, the full Virasoro algebra decomposes in two commuting Virasoro algebras: $T = T_I + T_r$ with

$$T_I(z)T_r(w) = \text{regular}. \tag{2.9}$$

This means we can classify conformal primaries by two quantum numbers, namely its tri-critical Ising model highest weight and its highest weight with respect to T_r: $|\text{primary}\rangle = |h_I, h_r\rangle$. The Virasoro modules decompose accordingly as

$$\mathcal{M}_{c=21/2} = \mathcal{M}^I_{c=7/10} \otimes \mathcal{M}^{\text{rest}}_{c=98/10}. \tag{2.10}$$

Notice that this decomposition is with respect to the Virasoro algebras and not with respect to the $\mathcal{N} = 1$ structures, which in fact do not commute. For example, the superpartner of Φ with respect to the full $\mathcal{N} = 1$ algebra is K whereas its superpartner with respect to the $\mathcal{N} = 1$ of the tri-critical Ising model is X.

2.4   Tri-critical Ising and unitary minimal models

We now review a few facts about the tri-critical Ising that we will use later in the paper.

Unitary minimal models are labelled by a positive integer $p = 2, 3, \ldots$ and occur only on the "discrete series" at central charges $c = 1 - \frac{6}{p(p+1)}$. The tri-critical Ising model is the second member ($p = 4$) which has central charge $c = \frac{7}{10}$. In fact, it is also a minimal model for the $\mathcal{N} = 1$ superconformal algebra.

The conformal primaries of unitary minimal models are labeled by two integers $1 \leq n' \leq p$ and $1 \leq n < p$. Primaries with label (n', n) and $(p + 1 - n', p - n)$ are identical and should be identified with each other. Therefore, there are in total $p(p-1)/2$ primaries in the theory. The weights of the primaries are conveniently arranged into a Kac table. The conformal weight of the primary $\Phi_{n'n}$ is $h_{n'n} = \frac{[pn' - (p+1)n]^2 - 1}{4p(p+1)}$. In the tri-critical Ising model ($p = 4$) there are six primaries of weights $0, \frac{1}{10}, \frac{6}{10}, \frac{3}{2}, \frac{7}{16}, \frac{3}{80}$. Next, we write the Kac table for the tri-critical Ising model. Beside the identity operator ($h = 0$) and the $\mathcal{N} = 1$ supercurrent ($h = \frac{3}{2}$) the NS sector (first and third columns) contains a primary of weight $h = \frac{1}{10}$ and its $\mathcal{N} = 1$ superpartner ($h = \frac{6}{10}$). The primaries of weights $\frac{7}{16}, \frac{3}{80}$ are in the Ramond sector (middle column).

The Hilbert space of the theory decomposes in a similar way, $\mathcal{H} = \oplus_{n,n'} \mathcal{H}_{n',n} \times \tilde{\mathcal{H}}_{n'n}$. A central theme in this work is that since the primaries $\Phi_{n'n}$ form a closed algebra under the OPE they can be decomposed into conformal blocks which connect two Hilbert spaces. Conformal blocks are denoted by $\Phi_{n',n,m'm}^{l',l}$ which describes the restriction of $\Phi_{n',n}$ to a map that only acts from $\mathcal{H}_{m',m}$ to $\mathcal{H}_{l',l}$. More details can be found in [27].

An illustrative example, which will prove crucial in what follows, is the conformal block structure of the primary $\Phi_{2,1}$ of weight $1/10$. General arguments show that the fusion rule of this field with any other primary $\Phi_{n'n}$ is $\phi_{(2,1)} \times \phi_{(n',n)} = \phi_{(n'-1,n)} + \phi_{(n'+1,n)}$. The only non-vanishing conformal blocks in the decomposition of $\Phi_{2,1}$ are those that connect a primary with the primary right above it and the primary right below in the Kac table, namely, $\phi_{2,1,n',n}^{n'-1,n}$ and $\phi_{2,1,n',n}^{n'+1,n}$. This can be summarized formally by defining the following decomposition[1]

$$\Phi_{2,1} = \Phi_{2,1}^{\downarrow} \oplus \Phi_{2,1}^{\uparrow}. \tag{2.12}$$

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[1] Perhaps the notation with $\downarrow$ and $\uparrow$ is a bit misleading. By $\Phi_{2,1}^{\downarrow}$, we mean that conformal block of $\Phi_{2,1}$ which maps

$$\mathcal{H}_0 \overset{\Phi_{2,1}^{\downarrow}}{\to} \mathcal{H}_{1/10} \overset{\Phi_{2,1}^{\downarrow}}{\to} \mathcal{H}_{6/10} \overset{\Phi_{2,1}^{\downarrow}}{\to} \mathcal{H}_{3/2}. \tag{2.11}$$

This is going down only in the first column of the Kac table, but is actually going up in the third column.

Table 1: Kac table for the tri-critical Ising model

$n' \setminus n$	1	2	3
1	**0**	$\mathbf{\frac{7}{16}}$	$\frac{3}{2}$
2	$\mathbf{\frac{1}{10}}$	$\frac{3}{80}$	$\frac{6}{10}$
3	$\frac{6}{10}$	$\frac{3}{80}$	$\frac{1}{10}$
4	$\frac{3}{2}$	$\frac{7}{16}$	0

Similarly, the fusion rule of the Ramond field $\Phi_{1,2}$ with any primary is $\phi_{(1,2)} \times \phi_{(n',n)} = \phi_{(n',n-1)} + \phi_{(n',n+1)}$ showing that it is composed of two blocks, which we denote as follows:

$$\Phi_{1,2} = \Phi_{1,2}^- \oplus \Phi_{1,2}^+. \qquad (2.13)$$

It is important here to specify on which half of the Kac table we are acting. We take $\phi_{(n',n)}$ to be either in the first column or in the top half of the second column, i.e., in the boldface region of table 1. With this restriction we denote by $\Phi_{1,2}^-$ the conformal block that takes us to the left in the Kac table and $\Phi_{1,2}^+$ the one that takes us to the right. Conformal blocks transform under conformal transformations exactly like the primary field they reside in but are usually not single-valued functions of $z(\bar{z})$. This splitting into conformal blocks plays a crucial role in the twisting procedure. The $+$ and $-$ labels will be clarified further when we consider the Ramond sector of the full G_2 algebra in Section 7.1 where we see that these labels correspond to Ramond sector ground states with different fermion numbers.

3   Chiral primaries, moduli and a unitarity bound

Having discussed this $c = \frac{7}{10}$ sub-algebra we now turn to the full G_2 chiral algebra. We first identify a set of special states which will turn out to saturate a unitarity bound for the full G_2 algebra. We call these the chiral primary states. This name seems appropriate since the representations built on chiral primary states are "short" whereas the generic representation is "long." The chiral primary states include the moduli of the compactification, i.e., the metric and B-field moduli that preserve the G_2 holonomy.

3.1 Chiral primary states

The chiral algebra associated with manifolds of G_2 holonomy[2] allows us to draw several conclusions about the possible spectrum of such theories. It is useful to decompose the generators of the chiral algebra in terms of primaries of the tri-critical Ising model and primaries of the remainder (2.10). The commutation relations of the G_2 algebra imply that some of the generators of the chiral algebra decompose as [8]: $G(z) = \Phi_{2,1} \otimes \psi_{14/10}$, $K(z) = \Phi_{3,1} \otimes \psi_{14/10}$ and $M(z) = a\Phi_{2,1} \otimes \chi_{24/10} + b[X_{-1}, \Phi_{2,1}] \otimes \psi_{14/10}$, with ψ, χ primaries of the indicated weights in the T_r CFT and a, b constants.

The Ramond sector ground states on a seven-dimensional manifold (so that the corresponding CFT has $c = 21/2$) have weight $\frac{7}{16}$. This implies that these states, which are labeled by two quantum numbers (the weights under the tri-critical part and the remaining CFT), are $\left|\frac{7}{16}, 0\right\rangle$ and $\left|\frac{3}{80}, \frac{2}{5}\right\rangle$. The existence of the $\left|\frac{7}{16}, 0\right\rangle$ state living just inside the tri-critical Ising model is crucial for defining the topological theory. Coupling left and right movers, the only possible RR ground states compatible with the G_2 chiral algebra[3] are a single $\left|\frac{7}{16}, 0\right\rangle_{\mathrm{L}} \otimes \left|\frac{7}{16}, 0\right\rangle_{\mathrm{R}}$ ground state and a certain number of states of the form $\left|\frac{3}{80}, \frac{2}{5}\right\rangle_{\mathrm{L}} \otimes \left|\frac{3}{80}, \frac{2}{5}\right\rangle_{\mathrm{R}}$. For a further discussion of the RR ground states see also Section 7.1 and Appendix C.

By studying operator product expansions of the RR ground states using the fusion rules

$$\frac{7}{16} \times \frac{7}{16} = 0 + \frac{3}{2}$$
$$\frac{7}{16} \times \frac{3}{80} = \frac{1}{10} + \frac{6}{10}$$

we get the following "special" NSNS states

$$|0,0\rangle_{\mathrm{L}} \otimes |0,0\rangle_{\mathrm{R}}, \quad \left|\frac{1}{10}, \frac{2}{5}\right\rangle_{\mathrm{L}} \otimes \left|\frac{1}{10}, \frac{2}{5}\right\rangle_{\mathrm{R}}, \quad \left|\frac{6}{10}, \frac{2}{5}\right\rangle_{\mathrm{L}}$$

$$\otimes \left|\frac{6}{10}, \frac{2}{5}\right\rangle_{\mathrm{R}} \quad \text{and} \quad \left|\frac{3}{2}, 0\right\rangle_{\mathrm{L}} \otimes \left|\frac{3}{2}, 0\right\rangle_{\mathrm{R}} \tag{3.1}$$

corresponding to the four NS primaries $\Phi_{n',1}$ with $n' = 1, 2, 3, 4$ in the tri-critical Ising model. Note that for these four states there is a linear relation between the Kac label n' of the tri-critical Ising model part and the total

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[2]We loosely refer to it as "the G_2 algebra" but it should not be confused with the Lie algebra of the group G_2.

[3]Otherwise the spectrum will contain a one-form which will enhance the chiral algebra [8]. Geometrically this is equivalent to demanding that $b_1 = 0$.

conformal weight $h_{\text{total}} = \frac{n'-1}{2}$. In fact, in Section 3.3, we show that similar to the BPS bound in the $\mathcal{N} = 2$ case, primaries of the G_2 chiral algebra satisfy a (non-linear) bound of the form

$$h_I + h_r \geq \frac{1 + \sqrt{1 + 80h_I}}{8} \tag{3.2}$$

which is precisely saturated for the four NS states listed before. We will therefore refer to those states as "chiral primary" states. Just like in the case of Calabi–Yau, the $\frac{7}{16}$ field maps Ramond ground states to NS chiral primaries and is thus an analog of the "spectral flow" operators in Calabi–Yau.

3.2 Moduli

It was shown in [8] that the upper components

$$\tilde{G}_{-1/2}\left|\frac{1}{10}, \frac{2}{5}\right\rangle_{\text{L}} \otimes G_{-1/2}\left|\frac{1}{10}, \frac{2}{5}\right\rangle_{\text{R}}$$

correspond to exactly marginal deformations of the CFT preserving the G_2 chiral algebra

$$\left\{G_{-1/2}, \mathcal{O}_{1/10,2/5}\right\} = \mathcal{O}_{0,1}. \tag{3.3}$$

and as such, correspond to the moduli of the G_2 compactification. As we will see in more detail later, there are $b_2 + b_3$ such moduli.

Geometrically, the metric moduli are deformations of the metric (δg_{ij}) that preserve Ricci flatness (these deformations also preserve the G_2 structure). Such deformations satisfy the Lichnerowicz equation:

$$\Delta_L \delta g_{ij} \equiv -\nabla^2 \delta g_{ij} + 2R_{mijn}\delta g^{mn} + 2R^k_{(i}\delta g_{j)k} = 0. \tag{3.4}$$

That there are b_3 solutions to this equation (up to diffeomorphisms) can be seen by relating (3.4) to an equation for a three-form ω which is constructed out of δg via δg_{ij}: $\omega_{ijk} = \phi_{l[ij}\delta g^l_{k]}$. Indeed, it can be shown [28] that for every solution of (3.4) modulo diffeomorphisms there is a corresponding harmonic three-form:

$$\Delta_{\text{L}}\delta g = 0 \leftrightarrow \Delta\omega = 0. \tag{3.5}$$

A natural question is if Δ_{L} can be written as the square of some first-order operator. Such a construction exists if the manifold supports a covariantly constant spinor ϵ_0. We can construct a spinor valued one-form out of δg_{ij} as $\delta g_{ij}(\Gamma^i \epsilon_0)dx^j$. This is a section of $S(M) \otimes T^*M$ where $S(M)$ is the spin

bundle. There is a natural $\not{D}$ operator acting on this vector bundle. It can be shown that $\not{D}^\dagger \not{D} = \Delta_\mathrm{L}$, which then reduces (3.4) to

$$\not{D}\left(\delta g_{ij} \Gamma^i \epsilon_0 dx^j \right) = 0 \tag{3.6}$$

which was shown to imply

$$\nabla_i \delta g_{jk} \ \phi^{ij}{}_l = 0 \tag{3.7}$$

in [29]. This first-order condition for the metric moduli will be beautifully reproduced from our analysis later of the BRST cohomology of our topologically twisted sigma model.

There is another quick way to see how the condition of being chiral primary implies the first-order condition (3.7). This is done using the zero mode of the generator $K(z)$ of the G_2 algebra. In the next section we will find that $K_0 = 0$ for chiral primaries using some explicit calculations. One can also show this more generally, since the K_0 eigenvalue of highest weight states of the G_2 algebra can be determined in terms of their L_0 and X_0 eigenvalues by using the fact that the null ideal in (B.19) has to vanish when acting on such states (see Appendix B). Again this leads to the conclusion that $K_0 = 0$ for chiral primaries. Now in the large volume limit the operator $\mathcal{O}_{1/10,2/5\mathrm{L}} \times \mathcal{O}_{1/10,2/5\mathrm{R}}$, corresponds to the operator $\delta g_{ij} \psi_\mathrm{L}^i \psi_\mathrm{R}^j$.[4]   The K_0 eigenvalue is then easily extracted from the double pole in the OPE

$$K(z)\mathcal{O}_{1/10,2/5\mathrm{L}}(0) \sim \cdots + \frac{\nabla_i \delta g_{jk} \, \phi^{ij}{}_l \mathcal{O}_{1/10,2/5\mathrm{L}}(0)}{z^2} + \cdots . \tag{3.8}$$

We see that $K_0 = 0$ implies precisely the first-order condition (3.7) which is a nice consistency check of the framework.

3.3   A unitarity bound

The G_2 algebra has highest weight representations, made from a highest weight vector that is annihilated by all positive modes of all the generators. First, notice that when acting on highest weight states, the generators L_0, X_0 and K_0 commute[5] so a highest weight state can be labeled by the three eigenvalues l_0, x_0, k_0.[6] In addition, $l_0 \geq 0$, $x_0 \leq 0$, and k_0 is purely imaginary. The first two conditions follow from unitarity (recall that $-5X$

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[4]The tri-critical Ising model weight of this operator can be computed to be $\frac{1}{10}$ by taking the OPE of it with X and then extracting the second-order pole.

[5]The only subtlety is the $[X_0, K_0]$ commutator. It does not vanish in general, but it does vanish when acting on highest weight states.

[6]As we mentioned in the previous subsection, k_0 is determined in terms of l_0 and x_0 by requiring the vanishing of the null ideal (B.19) when acting on these states. We ignore this in this subsection, though it does alter the analysis.

is the stress tensor of the tri-critical Ising model), the last condition follows from the hermiticity conditions on K_0: $K_m^\dagger = -K_{-m}$.

Next, we want to derive some bounds on l_0, x_0, k_0 that come from unitarity. In particular, we consider the three states $\{G_{-1/2}|l_0, x_0, k_0\rangle$, $\Phi_{-1/2}|l_0, x_0, k_0\rangle$, $M_{-1/2}|l_0, x_0, k_0\rangle\}$ and we consider the matrix $\mathcal{M}$ of inner products of these states with their hermitian conjugates.[7] This matrix can be worked out using the commutation relations and we find

$$\mathcal{M} = \begin{pmatrix} 2l_0 & k_0 & l_0 + 2x_0 \\ -k_0 & -6x_0 & -5k_0/2 \\ 2x_0 + l_0 & 5k_0/2 & l_0/2 + 4x_0 - 8x_0 l_0 \end{pmatrix}. \tag{3.9}$$

This matrix is indeed hermitian, and unitarity implies that the eigenvalues of this matrix should be non-negative. In particular, the determinant should be non-negative

$$\det \mathcal{M} = (8l_0 - 6x_0 - 8l_0 x_0)k_0^2 + 24x_0^2(4l_0^2 - l_0 + x_0). \tag{3.10}$$

The piece between parentheses before k_0^2 is always positive, and k_0^2 is always negative. Therefore we should in particular require that (for $x_0 \neq 0$)

$$4l_0^2 - l_0 + x_0 \geq 0 \tag{3.11}$$

which implies

$$l_0 \geq \frac{1 + \sqrt{1 - 16x_0}}{8}. \tag{3.12}$$

Changing basis to eigenvalues of T_r, T_I (see 2.9) the bound (3.12) becomes

$$h_I + h_r \geq \frac{1 + \sqrt{1 + 80h_I}}{8}. \tag{3.13}$$

This bound will turn out to play an important role. When the bound is saturated, we will call the corresponding state "chiral primary" in analogy to states saturating the BPS bound in $\mathcal{N} = 2$. Since in the NS sector of the tri-critical Ising model, $h_I = 0, \frac{1}{10}, \frac{6}{10}, \frac{3}{2}$ chiral states have total $h_I + h_r$ scaling dimension $0, \frac{1}{2}, 1, \frac{3}{2}$ which exactly match the special NSNS states 3.1. We will see that just like for $\mathcal{N} = 2$ theories it is exactly those chiral states that survive the topological twist. Indeed, in the Coulomb gas approach they became weight zero after the twist. It is interesting to see that the definition of chiral primaries involves a non-linear identity. This reflects the fact that the G_2 chiral algebra is non-linear. Since $\det \mathcal{M} = 0$ for chiral primaries, a suitable linear combination of the three states used in building $\det \mathcal{M}$ vanishes. In other words, chiral primaries are annihilated by a combination

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[7]This analysis assumes that x_0 is strictly negative otherwise $\Phi_{-1/2}|l_0, 0, k_0\rangle$ vanishes. For x_0 we remove this state and consider the matrix of inner products of the remaining two states, which leads to exactly the same conclusion.

of fermionic generators and the representations built from chiral primaries will be smaller than the general representation, as expected for BPS states.

When the bound (3.13) is saturated, $\det \mathcal{M}$ can only be non-negative as long as $k_0 = 0$. Thus, chiral primaries necessarily have $k_0 = 0$, and we will mostly suppress the quantum number k_0 in the remainder.

4   Topological twist

To construct a topologically twisted CFT, we usually proceed in two steps. First we define a new stress-energy tensor, which changes the quantum numbers of the fields and operators of the theory under Lorentz transformations. Secondly, we identify a nilpotent scalar operator, usually constructed out of the supersymmetry generators of the original theory, which we declare to be the BRST operator. Often this BRST operator can be obtained in the usual way by gauge fixing a suitable symmetry. If the new stress tensor is exact with respect to the BRST operator, observables (which are elements of the BRST cohomology) are metric independent and the theory is called topological. In particular, the twisted stress tensor should have a vanishing central charge.

4.1   Review of twisting the Calabi–Yau σ-model

In practice [30, 31], for the $\mathcal{N} = 2$ theories, an n-point correlator on the sphere in the twisted theory can conveniently be defined[8] as a correlator in the *untwisted* theory of the same n operators plus two insertions of a spin-field, related to the space-time supersymmetry charge, that serves to trivialize the spin bundle. For a Calabi–Yau threefold target space there are two SU(3) invariant spin-fields which are the two spectral flow operators $\mathcal{U}_{\pm 1/2}$. This discrete choice in the left and the right moving sectors is the choice between the $+(-)$ twists [1] which results in the difference between the topological A/B models.

The action for the σ-model on a Calabi–Yau is given by

$$S = \int d^2 z \frac{1}{2} g_{i\bar{j}} \left(\partial x^i \bar{\partial} x^{\bar{i}} + \partial x^{\bar{i}} \bar{\partial} x^i \right) + g_{i\bar{j}} \left(i \psi_-^{\bar{j}} D \psi_-^i + i \psi_+^{\bar{j}} \bar{D} \psi_+^i \right)$$
$$+ R_{i\bar{j}k\bar{l}} \psi_+^i \psi_+^{\bar{j}} \psi_-^k \psi_-^{\bar{l}}. \tag{4.1}$$

---

[8]Up to proper normalization.

Twisting this σ-model corresponds to adding a background gauge field for the $U(1)$ which acts on the complex fermions. Effectively, we change the covariant derivative from $D = \partial + \frac{\omega}{2}$ to $D' = \partial + \frac{\omega}{2} + A$, where we set the background value of $A = \frac{\omega}{2}$. Similarly, $\bar{D}$ changes to $\bar{D}' = \bar{\partial} + \frac{\omega}{2} \pm \bar{A}$, where the $+$ sign refers to the B twist and the $-$ sign refers to the A twist. This has the effect of changing the action in the following way:

$$\delta S = \int g_{i\bar{j}} \psi^i_+ \psi^{\bar{j}}_+ \frac{\bar{\omega}}{2} \pm g_{i\bar{j}} \psi^i_- \psi^{\bar{j}}_- \frac{\omega}{2}. \tag{4.2}$$

Just considering the left moving sector, and bosonizing the ψ_+'s by defining $g_{i\bar{j}} \psi^i_+ \psi^{\bar{j}}_+ = i\sqrt{d}\partial\phi$, where d is the complex dimension of the Calabi–Yau, we find

$$\delta S = \int g_{i\bar{j}} \psi^i_+ \psi^{\bar{j}}_+ \frac{\omega}{2} = -i\frac{\sqrt{d}}{2} \int \phi\partial\omega = +i\frac{\sqrt{d}}{2} \int \phi R.$$

On a genus g Riemann surface, we can choose R such that it has δ-function support at $2 - 2g$ points. So, for example, on a sphere, we get

$$e^{-\delta S} = e^{i(\sqrt{d}/2)\phi(0)} e^{i(\sqrt{d}/2)\phi(\infty)}$$

which implies that correlation functions in the twisted theory are related to those in the untwisted theory by $2 - 2g$ insertions of the operator (also known as the spectral flow operator) $e^{i(\sqrt{d}/2)\phi}$:

$$\langle \cdots \rangle_{\text{twisted}} = \left\langle e^{i(\sqrt{d}/2)\phi(\infty)} \cdots e^{i(\sqrt{d}/2)\phi(0)} \right\rangle_{\text{untwisted}}.$$

This effectively adds a background charge for the field ϕ of magnitude $Q = \sqrt{d}$, changing the central charge of the CFT

$$c = \tfrac{3}{2} \times 2d \rightarrow 1 - 3Q^2 + 3d - 1 = 0$$

which is what we expect in a topological theory.

4.2   The G_2 twist on the sphere

We can apply a similar procedure to the G_2 σ-model. The role of the operator $e^{i(\sqrt{d}/2)\phi}$ will be played by the conformal block $\Phi^+_{1,2}$ of the primary with conformal weight $\frac{7}{16}$ which creates the state $\left|\frac{7}{16}, 0\right\rangle$. Notice that this state sits entirely inside the tri-critical Ising model. Indeed, also in the case of Calabi–Yau manifolds, the spectral flow operator $e^{i(\sqrt{d}/2)\phi}$, sits purely within the $U(1) = \frac{U(d)}{SU(d)}$ part. In G_2 manifolds, the coset $\frac{SO(7)_1}{(G_2)_1}$ (with central charge $\frac{7}{10}$) plays the same role as the $U(1)$ sub-algebra in $\mathcal{N} = 2$. We therefore suggest (refining a similar suggestion of [8]) that correlation functions of the

twisted theory are defined in terms of the untwisted theory as

$$\langle V_1(z_1)\dots V_n(z_n)\rangle_{\text{twisted}}^{\text{plane}}$$
$$\equiv \prod_{i=1}^{n} z_i^{(h_i-\tilde{h}_i)} \langle \Sigma(\infty)V_1(z_1)\dots V_n(z_n)\Sigma(0)\rangle_{\text{untwisted}}^{\text{plane}} \qquad (4.3)$$

where, $(h)\tilde{h}$ are the weights with respect to the (un)twisted stress tensor respectively[9] and Σ is the conformal block

$$\Sigma = \Phi_{1,2}^{+} \qquad (4.4)$$

defined in (2.13).

In [8] further arguments were given, using the Coulomb gas representation of the minimal model, that there exists a twisted stress tensor with vanishing central charge. Those arguments, which are briefly reviewed in Appendix A, are problematic because the Coulomb gas representation really adds additional degrees of freedom to the minimal model. To properly restrict to the minimal model, one needs to consider cohomologies of BRST operators defined by Felder [27]. The proposed twisted stress tensor of [8] does not commute with Felder's BRST operators and therefore it does not define a bona fide operator in the minimal model. In addition, a precise definition of a BRST operator for the topological theory was lacking in [8].

We will proceed differently. We formulate our discussion purely in terms of the tri-critical Ising model itself without ever referring to the Coulomb gas representation, except by way of motivation and intuition. We will propose a BRST operator, study its cohomology, and then use 4.3 to compute correlation functions of BRST invariant observables. The connection to target space geometry will be made. We will then comment on the extension to higher genus and on the existence of a topologically twisted G_2 string.

4.3   The BRST operator

The basic idea is that the topological theory for G_2 sigma models should be formulated in terms of its (non-local)[10] conformal blocks and not in

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[9]The product $\prod_{i=1}^{n} z_i^{(h_i-\tilde{h}_i)}$ comes about from the mapping between the flat cylinder and the sphere. Note that this is not the same as computing the expectation value of $V_1(z_1)\dots V_n(z_n)$ in the Ramond ground state $\Sigma(0)|0\rangle$ because we insert the same operator at $0, \infty$ and not an operator and its BPZ conjugate.

[10]It should be stressed that this splitting into conformal blocks is non-local in the sense that conformal blocks may be multi-valued functions of $z(\bar{z})$.

terms of local operators. By using the split (2.12) into conformal blocks, we can split any field whose tri-critical Ising model part contains just the conformal family $\Phi_{2,1}$ into its up and down parts. For example, the $\mathcal{N} = 1$ supercurrent $G(z)$ can be split as

$$G(z) = G^\downarrow(z) + G^\uparrow(z). \tag{4.5}$$

We claim that $G^\downarrow$ is the BRST current and $G^\uparrow$ is a candidate for the anti-ghost.[11] The basic $\mathcal{N} = 1$ relation

$$G(z)G(0) = (G^\downarrow(z) + G^\uparrow(z))(G^\downarrow(0) + G^\uparrow(0)) \sim \frac{2c/3}{z^3} + \frac{2T(0)}{z} \tag{4.6}$$

proves the nilpotency of this BRST current (and of the candidate anti-ghost) because the right-hand side contains descendants of the identity operator only and has trivial fusion rules with the primary fields of the tri-critical Ising model and so $(G^\downarrow)^2 = (G^\uparrow)^2 = 0$.

An algebraic formulation of the decomposition (4.5) starts from defining projection operators. Any state in the theory can be labeled by its eigen-values under the two commuting (2.9) Virasoro modes of T_I, T_r and perhaps some additional quantum numbers needed to completely specify the state. We denote by $P_{n'}$ the projection operator on the sub-space of states whose tri-critical Ising model part lies within the conformal family of one of the four NS primaries $\Phi_{n',1}$. The image of $P_{n'}$ is $\mathcal{H}_{n',1}$ which we abbreviate here to $\mathcal{H}_{n'}$. The corresponding weights of the primary fields in the tri-critical Ising model by $\Delta(n')$. Thus, $\Delta(1) = 0$, $\Delta(2) = \frac{1}{10}$, $\Delta(3) = \frac{6}{10}$ and $\Delta(4) = \frac{3}{2}$. This is summarized by the equation

$$\Delta(n') = \frac{(2n' - 3)(n' - 1)}{10}. \tag{4.7}$$

The four projectors add to the identity

$$P_1 + P_2 + P_3 + P_4 = 1 \tag{4.8}$$

because this exhausts the list of possible highest weights in the NS sector of the tri-critical Ising model.[12]

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[11]Incidently, the Coulomb gas representation indeed assigns the expected conformal weights after the twist (see Appendix A).

[12]For simplicity, we will set $P_{n'} = 0$ for $n' \leq 0$ and $n' \geq 5$, so that we can simply write $\sum_{n'} P_{n'} = 1$ instead of (4.8).

We can now define our candidate BRST operator in the NS sector more rigorously

$$Q = G^{\downarrow}_{-1/2} \equiv \sum_{n'} P_{n'+1} G_{-1/2} P_{n'}. \tag{4.9}$$

The nilpotency $Q^2 = 0$ is easily proved:

$$Q^2 = \sum_{n'} P_{n'+2} G^2_{-1/2} P_{n'} = \sum_{n'} P_{n'+2} L_{-1} P_{n'} = 0 \tag{4.10}$$

where we could replace the intermediate $P_{n'+1}$ by the identity because of the property (4.5) and the last equality follows since L_{-1} maps each $\mathcal{H}_{n'}$ to itself.

4.4 BRST cohomology and chiral operators

Having defined the BRST operator, we can now compute its cohomology. We first derive the condition on the tri-critical Ising model weight h_I and its total weight for it to be annihilated by Q. Then we go on to defining the operator cohomology, which correspond to operators (or conformal blocks of operators) $\mathcal{O}$ satisfying $\{Q, \mathcal{O}\} = 0$. We mostly work in the NS sector. Perhaps it is more appropriate to work in the Ramond sector since the topological theory computations are done in the Ramond sector of the untwisted theory (see also Section 4.9). We assume here that a version of spectral flow exists which will map the NS sector to the Ramond sector. We discuss such a spectral flow in Appendix F.

4.4.1 State cohomology

As a first step in the analysis of the BRST cohomology, we consider the action of Q on highest weight states $|h_I, h_r\rangle = |\Delta(k), h_r\rangle$ of the full algebra. Because Q is a particular conformal block of the supercharge $G_{-1/2}$, to extract the action of Q on a state, we first act with $G_{-1/2}$ on the state and then project on to the term. As discussed previously, the $\mathcal{N} = 1$ supercurrent G can be decomposed as $\Phi_{2,1} \otimes \psi_{14/10}$. The fusion rules of the tri-critical Ising model then imply that

$$\begin{aligned} G_{-1/2}\big|\Delta(k), h_r\big\rangle = c_1\big|\Delta(k-1), h_r - \Delta(k-1) + \Delta(k) - \tfrac{1}{2}\big\rangle \\ + c_2\big|\Delta(k+1), h_r - \Delta(k+1) + \Delta(k) - \tfrac{1}{2}\big\rangle \end{aligned} \tag{4.11}$$

where the two states on the right are highest weight states of the L_m, X_m sub-algebra (but not necessarily of the full G_2 algebra) and which are normalized to have unit norm. Then by definition

$$Q\big|\Delta(k), h_r\big\rangle = c_2\big|\Delta(k+1), h_r - \Delta(k+1) + \Delta(k) - \tfrac{1}{2}\big\rangle. \tag{4.12}$$

Using the G_2 algebra (Appendix B), we find that

$$\langle \Delta(k), h_r | G_{1/2} G_{-1/2} | \Delta(k), h_r \rangle = 2(\Delta(k) + h_r) = |c_1|^2 + |c_2|^2. \qquad (4.13)$$

The first answer is obtained using $\{G_{1/2}, G_{-1/2}\} = 2L_0$, the second follows from (4.11). In a similar way we compute

$$\langle \Delta(k), h_r | G_{1/2} X_0 G_{-1/2} | \Delta(k), h_r \rangle = 9\Delta(k) - h_r - 10\Delta(k)(\Delta(k) + h_r)$$
$$= -5\Delta(k-1)|c_1|^2 - 5\Delta(k+1)|c_2|^2. \qquad (4.14)$$

We can use (4.13) and (4.14) to solve for c_1 and c_2 up to an irrelevant phase. In particular, we find that the highest weight state is annihilated by Q, which is equivalent to $c_2 = 0$, if

$$9\Delta(k) - h_r - 10\Delta(k)(\Delta(k) + h_r) = -10\Delta(k-1)(\Delta(k) + h_r). \qquad (4.15)$$

We can rewrite this as

$$\Delta(k) + h_r = \frac{10\Delta(k)}{10\Delta(k) + 1 - 10\Delta(k-1)} = \frac{k-1}{2} = \frac{1 + \sqrt{1 + 80\Delta(k)}}{8} \qquad (4.16)$$

where we used (4.7). This is precisely the unitarity bound (3.13). Therefore, the only highest weight states that are annihilated by Q are the chiral primaries that saturate the unitarity bound. It is gratifying to see a close parallel with the other examples of topological strings in four and six dimensions.[13] We have shown so far that all states that are primary under the L_m, X_m sub-algebra and are annihilated by $G_{\frac{1}{2}}$ are annihilated by Q if they saturate the unitarity bound. These states, need not be primary with respect to the full G_2 algebra. This is implied by the condition $|c_1|^2 \geq 0$ in (4.13) and (4.14).

Of course, to study the full BRST cohomology, much more work is required, and in particular we would want to prove that BRST closed descendants are always BRST exact. We do not have such a proof, but some partial evidence is given in Section 4.6. In the RR sector it is much easier to analyze the BRST cohomology and there one immediately sees that the cohomology consists of just the RR ground states (see Section 4.9).

---

[13]Strictly speaking the previous derivation is not quite correct for $k = 1, 4$, since $\Delta(0)$ and $\Delta(5)$ do not exist. If they would appear, then the corresponding representations would not be unitary, since they lie outside the Kac table. This implies that the only representations with either $k = 0$ or $k = 3$ that can appear in the theory necessarily have $h_r = 0$, and these are indeed annihilated by the BRST operator.

The geometric meaning of the BRST cohomology will become clear in the next section. In the remainder of this section, we collect various other technical aspects of the twisted CFT. Readers more interested in the more geometrical aspects can jump to Section 5.

4.4.2 Operator cohomology

Let $\mathcal{O}_{n',h,\alpha}$ be the local operator corresponding to the state $|\Delta(n'),h,\alpha\rangle$.[14] Generically, Q does not commute with the local operators $\mathcal{O}_{\Delta(1),0}$, $\mathcal{O}_{\Delta(2),2/5}$, $\mathcal{O}_{\Delta(3),2/5}$ and $\mathcal{O}_{\Delta(4),0}$ corresponding to the chiral states $|0,0\rangle$, $\left|\frac{1}{10},\frac{2}{5}\right\rangle$, $\left|\frac{6}{10},\frac{2}{5}\right\rangle$, $\left|\frac{3}{2},0\right\rangle$ (for brevity we will denote those four local operators just by their tri-critical Ising model Kac index $\mathcal{O}_i$, $i = 1,2,3,4$). This is because the topological G_2 CFT is formulated not in terms of local operators of the untwisted theory but in terms of non-local conformal blocks. It is straightforward to check that the following blocks,

$$\mathcal{A}_{n'} = \sum_m P_{n'+m-1} \mathcal{O}_{n'} P_m \tag{4.17}$$

which pick out the maximal "down component" of the corresponding local operator, do commute with Q and are thus in its operator cohomology. For example writing explicitly $Q = P_4 G_{-1/2} P_3 + P_3 G_{-1/2} P_2 + P_2 G_{-1/2} P_1$ it follows trivially from the definition of the projectors $P_I P_J = P_I \delta_{I,J}$ that Q commutes with $\mathcal{A}_4 = P_4 \mathcal{O}_4 \, P_1$. To get some familiarity with the notation we work out another example,

$$\{Q, \mathcal{A}_2\} = \sum_{n'} P_{n'+1} \left(G_{-\frac{1}{2}} P_{n'} \mathcal{O}_2 + \mathcal{O}_2 P_{n'} G_{-\frac{1}{2}} \right) P_{n'-1}$$

$$= \sum_{n'} P_{n'+1} \left(\{G_{-\frac{1}{2}}, \mathcal{O}_2\} \right) P_{n'-1} = \sum_{n'} P_{n'+1} \mathcal{O}_{\Delta(1),1} P_{n'-1} = 0 \tag{4.18}$$

where we repeatedly use the property (4.5) and the existence of the marginal operators (3.3). Note that we have not shown that the blocks (4.17) exhaust the Q cohomology but presumably this is indeed the case.

This algebraic characterization of the conformal blocks corresponding to chiral primaries fits nicely with the Coulomb gas approach where the tri-critical Ising model vertex operator (i.e., block) of the chiral primaries was identified in (A) to be exactly the unscreened vertex that created the maximal "down" shift in the Kac table.

---

[14] Here α is a formal label that might be needed to completely specify a state.

4.5   The chiral ring

In a close parallel to what happens in theories with $\mathcal{N} = 2$ SUSY, the conformal blocks which commute with Q form a ring under the OPE. Due to the simplicity of the tri-critical Ising model there are in fact just two non-trivial checks which are $\mathcal{A}_2(z)\mathcal{A}_2(0)$ and $\mathcal{A}_2(z)\mathcal{A}_3(0)$. For example,

$$
\begin{aligned}
\mathcal{A}_2(z)\mathcal{A}_3(0) &= P_4\mathcal{O}_2(z)P_3\mathcal{O}_3(0)P_1 = P_4\mathcal{O}_2(z)\mathcal{O}_3(0)P_1 \\
&= P_4\mathcal{O}_4(0)P_1 = \mathcal{A}_4(0).
\end{aligned}
\tag{4.19}
$$

The second equality follows because P_1 projects on the identity and the third due to the unitarity bound (3.13), which for chiral primaries is just the linear relation (4.16), implying that in the OPE of two chiral primaries there can be no poles and the leading regular term is automatically also a chiral primary.

4.6   An sl(2|1) sub-algebra

We can construct an interesting sl(2|1) sub-algebra of the full algebra, whose commutation relations are identical to the lowest modes of the $N = 2$ algebra. To construct this sub-algebra, we define

$$
G_r^{\uparrow} = \sum_k P_{k-1}G_rP_k, \quad G_r^{\downarrow} = \sum_k P_{k+1}G_rP_k, \quad J_0 = L_0 - \{G_{-1/2}^{\downarrow}, G_{1/2}^{\uparrow}\}.
\tag{4.20}
$$

Using properties of the G_2 algebra, and Jacobi identities, we can show that the algebra generated by $G_{\pm 1/2}^{\downarrow}$, $G_{\pm 1/2}^{\uparrow}$, L_0, $L_{\pm 1}$ and J_0 closes and forms the algebra sl(2|1). Notice that $Q \equiv G_{-1/2}^{\downarrow}$ is one of the generators of this algebra. We know that sl(2|1) has short and long representations, and any state in the BRST cohomology must necessarily be a highest weight state of a short representation. This shows that sl(2|1) descendants are never part of the BRST cohomology. This is a hint that the only elements of the BRST cohomology are the chiral primaries, but to prove this we would need to extend this reasoning to include also elements which are descendants with respect to the other generators of the G_2 algebra, or require us to determine the precise form of the anti-ghost and twisted stress tensor.

Position independence of correlators. Notice that the generators of translations on the plane, namely, L_{-1} and $\tilde{L}_{-1}$ are BRST exact:

$$
L_{-1} = \left\{Q, G_{-1/2}^{\uparrow}\right\}.
\tag{4.21}
$$

It follows that, in the topological G_2 theory, genus zero correlation functions of chiral primaries between BRST closed states are position independent. This is a crucial ingredient of topological theories.

4.7   A twisted Virasoro algebra?

Before, we constructed an sl(2|1) algebra, and it is natural to ask if it can be extended to a full $N = 2$ algebra. This seems unlikely, but one definitely expects to find at least all the modes of a twisted stress tensor, which is essential for the construction of a topological string theory on higher genus Riemann surfaces. Since genus zero amplitudes are independent of the locations of the operators, this suggests that such a twisted stress tensor should indeed exist.

The construction of the sl(2|1) algebra immediately yields a candidate for the twisted stress tensor, namely

$$\tilde{L}_m \equiv \{Q, G^{\uparrow}_{m+\frac{1}{2}}\} \equiv \{Q, G_{m+\frac{1}{2}}\}. \tag{4.22}$$

This definition seems to work at first sight. For example,

$$\tilde{L}_{-1} = L_{-1} \tag{4.23}$$

as expected for a twisted energy-momentum tensor. In addition,

$$[\tilde{L}_{-1}, \tilde{L}_m] = (-1 - m)\tilde{L}_{m-1}, \tag{4.24}$$

which is the correct commutation relation for a Virasoro algebra. In addition, $[\tilde{L}_m, \tilde{L}_{-m}]$ annihilates chiral primaries, as expected for a twisted energy-momentum tensor with zero central charge. However, there is no obvious reason why the other commutation relations should be valid. Some extremely tedious calculations reveal that (assuming that we did not make any mistakes in the lengthy algebra) when acting on primaries of the full G_2 algebra

$$\tilde{L}_0 \big| \Delta(k+1), h_r \big\rangle = \frac{4k-2}{4k-1} \left((\Delta(k+1) + h_r) - \frac{k}{2} \right) \big| \Delta(k+1), h_r \big\rangle \tag{4.25}$$

and

$$[\tilde{L}_2, \tilde{L}_{-2}] \big| \Delta(k+1), h_r \big\rangle = c_k \left((\Delta(k+1) + h_r) - \frac{k}{2} \right)$$
$$\times \left(-1485 + 2868k + 2644k^2 - 3392k^3 - 640k^4 + 512k^5 \right.$$
$$\left. - 72k(\Delta(k+1) + h_r) \right) \big| \Delta(k+1), h_r \big\rangle \tag{4.26}$$

with

$$c_k = \frac{4k-2}{(k+1)(2k+3)(4k-11)(4k-1)^2(4k+9)}. \tag{4.27}$$

This clearly shows that $[\tilde{L}_2, \tilde{L}_{-2}] \neq 4\tilde{L}_0$. In addition, we see the shift in $\tilde{L}_0$ would live entirely in the tri-critical part were it not for the prefactor $\frac{(4k-2)}{(4k-1)}$ that appears. Having the twist purely in the tri-critical piece is appealing, as this can easily be implemented in the Coulomb gas formulation, but further work is required to prove that such a twisted energy-momentum tensor indeed exists and is BRST exact. This proposal is apparently not quite the correct one.

4.8   Moduli and descent relations

As mentioned in Section 3.1 the upper components $\tilde{G}_{-1/2} \left|\frac{1}{10}, \frac{2}{5}\right\rangle_\mathrm{L} \otimes G_{-1/2} \left|\frac{1}{10}, \frac{2}{5}\right\rangle_\mathrm{R}$ were shown in [8] to be exactly marginal deformations of the CFT preserving the G_2 chiral algebra. We also saw that they are in one-to-one correspondence with the b_3 metric moduli of the G_2 manifold. Once we include the B-field the number of such moduli will turn out to be $b_2 + b_3$ as we will see in Section 5.2. Since both the ordinary and the topologically-twisted theories should exist on an arbitrary manifold of G_2 holonomy it is important to check that the moduli space of deformations of the two theories agrees. So far we have seen that the interesting objects in the twisted theory are given in terms of non-local objects of the original one. We will now demonstrate that nevertheless the two theories have the same moduli space of deformations. In a fashion identical to (2.12) we can split the local field $\mathcal{O}_2$ that creates the chiral primary state $\left|\frac{1}{10}, \frac{2}{5}\right\rangle$ as

$$\mathcal{O}_2 = \mathcal{O}_2^\downarrow + \mathcal{O}_2^\uparrow = \sum_m P_{m+1}\mathcal{O}_2 P_m + \sum_m P_{m-1}\mathcal{O}_2 P_m. \qquad (4.28)$$

The first term coincides with $\mathcal{A}_2$ which corresponds to a chiral operator in the twisted theory so in particular $\{Q, \mathcal{A}_2\} = 0$. Also, a computation similar to (4.18) shows that $\{G_{-1/2}^\uparrow, \mathcal{O}_2^\uparrow\} = 0$. Using this we compute

$$\begin{aligned}
[Q, \{G_{-1/2}, \mathcal{O}_2\}] &= \left[Q, \left\{G_{-1/2}^\downarrow + G_{-1/2}^\uparrow, \ \mathcal{O}_2^\downarrow + \mathcal{O}_2^\uparrow\right\}\right] \\
&= \left[Q, \left\{Q, \mathcal{O}_2^\uparrow\right\}\right] + \left[Q, \left\{G_{-1/2}^\uparrow, \mathcal{A}_2\right\}\right] \\
&= \left[Q, \left\{G_{-1/2}^\uparrow, \mathcal{A}_2\right\}\right] \\
&= \left[\left\{Q, G_{-1/2}^\uparrow\right\}, \mathcal{A}_2\right] \\
&= \left[L_{-1}, \mathcal{A}_2\right] = \partial\mathcal{A}_2.
\end{aligned} \qquad (4.29)$$

In other words, we showed that $\partial\mathcal{A}_2 = \{Q, \text{something}\}$, and the something is the $(1,0)$-form $\{G_{-1/2}, \mathcal{O}_2\}$. This is a conventional operator that does not

involve any projectors. If we combine this also with the right-movers, we find that the deformations in the action of the topological string are exactly the same as the deformations of the non-topological string.

4.9 The Ramond sector

We have previously given evidence, though no rigorous proof, that the cohomology in the NS sector of $G^{\downarrow}_{-1/2}$ is given by the chiral primaries. In the R sector the situation is somewhat different. There is an obvious candidate for a BRST operator in the R sector, namely $Q = G^{\downarrow}_0$. Perhaps this is an even better candidate, as it is the zero-mode of a field (as it should be in a twisted theory), and because our twisting essentially boils down to doing computations in the R sector. It is not immediately clear that there is an easy map between the action of $G^{\downarrow}_0$ in the R sector and the action of $G^{\downarrow}_{-1/2}$ in the NS sector. This would require us to have a suitable isomorphism between the NS and R sector. Such an isomorphism does exist and is sometimes referred to as spectral flow (discussed more in Appendix F), however it is not at all clear that this maps $G^{\downarrow}_{-1/2}$ to $G^{\downarrow}_0$. It does however map R ground states to chiral primaries, so this is further evidence that the BRST cohomology in the NS sector consists of chiral primaries and nothing else.

As an aside, notice that in the NS sector we found an sl(2|1) sub-algebra using some of the modes of $G^{\uparrow}$ and $G^{\downarrow}$. In the R sector this is no longer the case. In the R sector the only easy calculation we can readily do is that

$$\{G^{\downarrow}_0, G^{\uparrow}_0\} = L_0 - \frac{7}{16}. \tag{4.30}$$

This in particular implies that the $G^{\downarrow}_0$ cohomology is given by the R ground states. This is an exact statement. Therefore, $G^{\downarrow}_0$ looks like an excellent candidate BRST operator. It also has the nice property that the right-hand side of (4.30) is the most natural definition of L^{twisted}_0 in the R sector in contrast to the situation in the NS sector.

4.10 Localization

It can be shown quite generally [1] that the path integral localizes to fixed points of the BRST symmetry. For the usual case of the A and B models, this implies that only holomorphic and constant maps contribute, respectively. To derive a similar statement for the topological G_2 sigma model, we start

by writing the action as

$$S = \int d^2z \frac{1}{2} g_{IJ} \partial x^I \bar{\partial} x^J + g_{IJ} \Big(i\psi_{\rm L}^{\uparrow J} D\psi_{\rm L}^{\downarrow I} + i\psi_{\rm L}^{\downarrow J} D\psi_{\rm L}^{\uparrow I}$$
$$+ i\psi_{\rm R}^{\uparrow J} \bar{D}\psi_{\rm R}^{\downarrow I} + i\psi_{\rm R}^{\downarrow J} \bar{D}\psi_{\rm R}^{\uparrow I} \Big) + R_{IJKL} \psi_{\rm R}^{\uparrow I} \psi_{\rm R}^{\downarrow J} \psi_{\rm L}^{\uparrow K} \psi_{\rm L}^{\downarrow L}.$$

This action has the fermionic symmetry

$$\delta x^I = i\epsilon_{\rm L} \psi_{\rm L}^{\downarrow I} + i\epsilon_{\rm R} \psi_{\rm R}^{\downarrow I}$$
$$\delta \psi_{\rm L}^{\uparrow I} = -\epsilon_{\rm L} \partial x^J - \epsilon_{\rm R} \psi_{\rm R}^{\downarrow K} \Gamma_{KM}^I \psi_{\rm L}^{\uparrow M}$$
$$\delta \psi_{\rm L}^{\downarrow I} = -\epsilon_{\rm R} \psi_{\rm R}^{\downarrow K} \Gamma_{KM}^I \psi_{\rm L}^{\downarrow M}$$
$$\delta \psi_{\rm R}^{\downarrow I} = -\epsilon_{\rm L} \psi_{\rm L}^{\downarrow K} \Gamma_{KM}^I \psi_{\rm R}^{\downarrow M}$$
$$\delta \psi_{\rm R}^{\uparrow I} = -\epsilon_{\rm R} \bar{\partial} x^J - \epsilon_{\rm L} \psi_{\rm L}^{\downarrow K} \Gamma_{KM}^I \psi_{\rm R}^{\uparrow M}.$$

The fixed points of this symmetry satisfy $\partial x^I = \bar{\partial} x^I = 0$, which implies that the path integral localizes on constant maps. Of course, we should take this analysis with a grain of salt: the decomposition of the world sheet fermions ψ^I into conformal blocks $\psi^{\uparrow I} + \psi^{\downarrow I}$ is inherently quantum mechanical and hence it is problematic to use this decomposition in path integral arguments. Nevertheless, we take this argument as at least suggestive that we are localizing on constant maps.

5   Relation to geometry

For a G_2 manifold, differential forms of any degree can be decomposed into irreducible representations of G_2

$$\Lambda^0 = \Lambda_1^0 \quad \Lambda^1 = \Lambda_7^1$$
$$\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2 \quad \Lambda^3 = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3$$

This is described in more detail in Appendix D. In a similar spirit as Hodge theory, this decomposes the cohomology groups as $H^p = \oplus_{\mathcal{R}} H_{\mathcal{R}}^p(M)$ where the sum is over G_2 representations $\mathcal{R}$. The cohomology turns out to depend solely on the representation $\mathcal{R}$ and not on the degree p [28]. For a proper compact G_2 manifold, $H^1(M) = 0$ and so there is no cohomology in the seven-dimensional representation of G_2. Also, $b_1^3 = 1$, corresponding to a unique closed three-form ϕ which defines the G_2 structure. There are only two independent Betti numbers left unknown, namely b_{14}^2 which is equal to the usual second Betti number b_2 and $b_{27}^3 = b_3 - 1$ with no known restrictions on these numbers.

5.1 Dolbeault complex for G_2 manifolds

It is possible to define a refinement of the de Rham complex, in a spirit somewhat similar to Dolbeault cohomology, as follows:

$$0 \to \Lambda_1^0 \xrightarrow{\check{D}} \Lambda_7^1 \xrightarrow{\check{D}} \Lambda_7^2 \xrightarrow{\check{D}} \Lambda_1^3 \to 0 \tag{5.1}$$

where $\check{D}$ is the usual exterior derivative when acting on zero-forms, but is the composition of the exterior derivative and projection to the **7** and **1** representations of G_2 when acting on one and two forms, respectively:

$$\check{D}(\alpha) = \pi_7^2(d\alpha) \quad \text{for } \alpha \in \Lambda^1$$
$$\check{D}(\beta) = \pi_1^3(d\beta) \quad \text{for } \beta \in \Lambda^2$$

where the projection operators π_r^p are defined in Appendix D. In local coordinates, these expressions become

$$\left(\check{D}(\alpha)\right)_{\mu\nu} dx^\mu \wedge dx^\nu = 3\partial_{[\mu}A_{\nu]}\phi_\rho^{\mu\nu}\phi_{\eta\chi}^\rho dx^\eta \wedge dx^\chi \quad \alpha = A_\mu dx^\mu$$

$$\left(\check{D}(\beta)\right)_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho = \partial_{[\xi}B_{\eta\chi]}\phi^{\xi\eta\chi}\phi_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho \quad \beta = B_{\mu\nu}dx^\mu \wedge dx^\nu$$

We will next see that the cohomology of this differential complex maps to the BRST cohomology in the left (or right) moving sector. The differential operator $\check{D}$ maps to the BRST operator $G_{-1/2}^\downarrow$. This gives a nice and natural geometric meaning to the BRST operator, and clearly shows we are on the right track.

5.2 The BRST cohomology geometrically

In the previous section, we argued that the BRST cohomology consists of the chiral primary operators of our conformal field theory. We now proceed to study the sigma model description of these operators and the geometric meaning of the chiral ring.

To determine whether an operator corresponds to a chiral primary, we need to find its L_0 and X_0 quantum numbers. Often in topological theories, this calculation can be reduced to operators built out of non-derivative fields only. In our case we also expect this to be the case, since all elements in the cohomology are in one-to-one correspondence to R ground states. Also, the argument that the path integral localizes on constant maps indicates that only zero modes appear.

So we proceed by analyzing the action of the BRST operator at the level of operators that do not contain any derivatives of fields. In the left-moving sector, such operators are in one-to-one correspondence with p-forms on the target space:

$$\omega_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \leftrightarrow \omega(x^\mu)_{i_1,\ldots,i_p} \psi^{i_1} \ldots \psi^{i_p}. \tag{5.2}$$

The same is obviously also true in the right-moving sector, but for simplicity we analyze the left-moving sector first.

The group G_2 acts on the tangent space of the manifold, and the space of p-forms at a point can be decomposed in G_2 representations as explained before. Since X_0 and L_0 are G_2 singlets, they take the same value in each of these representations. Some further explicit calculations[15] involving the precise form of X_0 then reveal that the quantum numbers associated to each representation are

	1	**7**	**14**	**27**
$p=0$	$\lvert 0,0 \rangle$			
$p=1$		$\left\lvert \frac{1}{10}, \frac{2}{5} \right\rangle$		
$p=2$		$\left\lvert \frac{6}{10}, \frac{2}{5} \right\rangle$	$\lvert 0,1 \rangle$	
$p=3$	$\left\lvert \frac{3}{2},0 \right\rangle$	$\left\lvert \frac{11}{10}, \frac{2}{5} \right\rangle$		$\left\lvert \frac{1}{10}, \frac{7}{5} \right\rangle$
$p=4$	$\lvert 2,0 \rangle$	$\left\lvert \frac{16}{10}, \frac{2}{5} \right\rangle$		$\left\lvert \frac{6}{10}, \frac{7}{5} \right\rangle$
$p=5$		$\left\lvert \frac{21}{10}, \frac{2}{5} \right\rangle$	$\left\lvert \frac{3}{2},1 \right\rangle$	
$p=6$		$\left\lvert \frac{26}{10}, \frac{2}{5} \right\rangle$		
$p=7$	$\left\lvert \frac{7}{2},0 \right\rangle$			

$$\tag{5.4}$$

This table also nicely reflects the two maps which take a p-form ω into a $p+3$ form given by $\omega \wedge \phi$ and into a $p+4$ form $\omega \wedge *\phi$ (see Appendix D). When restricted to G_2 representations, these operators are either identically zero or act as isomorphisms. They translate to the action of $\Phi_{-3/2}$ and X_{-2} at the level of states. Notice that chiral primaries appear only in four places in (5.4), and precisely those differential forms enter into (5.1). Of course, this is not a coincidence, as we will see next.

---

[15]As an example, we determine the X_0 eigenvalue of the operator $A(X)_\mu \psi^\mu$ which corresponds to the one-form $A(X)_\mu dx^\mu$. Using the expression for $X(z)$ in (2.7), the X_0 eigenvalue is given by the coefficient of the second-order pole in the OPE

$$X(z).\Big(A(X)_\mu \psi^\mu(0)\Big) \sim \cdots - \frac{1}{2}\frac{A(X)_\mu \psi^\mu}{z^2} + \cdots \tag{5.3}$$

which gives the X_0 eigenvalue of this operator to be $-\frac{1}{2}$ and the tri-critical Ising model weight $\frac{1}{10}$.

In order to construct the precise form of these states, we need to project the relevant forms on to appropriate G_2 representation. All such projectors can be constructed in terms of the three-form ϕ and its Hodge dual, as explained in Appendix D. To find their precise form, various identities satisfied by ϕ are useful, such as

$$\phi_c{}^{de}\phi_{de}{}^f = \tfrac{1}{6}\delta_c^f$$

$$\phi_{ab}{}^{cd}\phi_{cd}{}^e = \tfrac{1}{6}\phi_{ab}{}^e$$

$$\phi_{ab}{}^c\phi_c{}^{de} = \frac{2}{3}\phi_{ab}{}^{de} + \frac{1}{36}(\delta_a^d\delta_b^e - \delta_a^e\delta_b^d)$$

$$\phi_{ab}{}^{cd}\phi_{cd}{}^{ef} = \frac{1}{12}\phi_{ab}{}^{ef} + \frac{1}{144}(\delta_a^e\delta_b^f - \delta_a^f\delta_b^e)$$

$$\phi_{abc}\phi^{abc} = \tfrac{7}{6}$$

$$\phi_{[ab}{}^{cd}\phi_{e]cd} = \phi_{abe}$$

$$\tfrac{1}{2}\phi_{[ab}{}^{cd}\phi_{e]cd}{}^f = -\tfrac{1}{4}\phi_{abe}{}^f. \tag{5.5}$$

In these equations, antisymmetrization over n indices does not include a factor of $1/n!$. They are also useful in order to compute the X_0 eigenvalue in each representation. Notice, however, that the exact quantum X_0 eigenstates cannot in general be written in terms of fields without derivatives, typically one needs to add some quantum corrections involving fewer fermions and a few derivatives as well.

This table allows us to extract the precise action of the BRST operator on the operators that do not involve derivatives. For example,

$$G_{-1/2}A_\mu(X)\psi^\mu = \frac{1}{2}\partial_{[\nu}A_{\mu]}\psi^\nu\psi^\mu + A_\mu(X)\partial X^\mu. \tag{5.6}$$

In the calculation we get a covariant derivative, however this is equal to the ordinary derivative when acting on forms as an exterior derivative. To extract the action of $G^\downarrow_{-\frac{1}{2}}$, we first observe the second term has $X_0 = 0$ and therefore only contributes to $G^\uparrow_{-\frac{1}{2}}$. The first term has a part transforming in the **7** of G_2 and a part transforming in the **14** of G_2, and according to (5.4) we need to project on the **7** to obtain the action of $G^\downarrow_{-1/2}$. The relevant projection operator is $P_{ab}{}^{de} = 6\phi_{ab}{}^c\phi_c{}^{de}$, and we finally get

$$G^\downarrow_{-1/2}A_\mu(X)\psi^\mu = 3\partial_{[\nu}A_{\mu]}\phi^{\nu\mu}{}_\rho\phi^\rho{}_{\alpha\beta}\psi^\alpha\psi^\beta. \tag{5.7}$$

It is clear by inspection of table (5.4) that chiral primaries, i.e., non-trivial elements of the BRST cohomology, can either be singlet zero- or three-forms, or one- or two-forms transforming in the **7** of G_2.

By repeating (5.7) for the two-form $B_{\mu\nu}\psi^\mu\psi^\nu$ and the three-form $\phi_{\mu\nu\alpha}$ $\psi^\mu\psi^\nu\psi^\alpha$, the kernel of Q_{BRST} in the left-moving sector is then seen to consist of

$$1$$
$$A_\mu\psi^\mu \quad \text{with } \phi_\rho{}^{\mu\nu}\partial_{[\mu}A_{\nu]} = 0$$
$$B_{\mu\nu}\psi^\mu\psi^\nu \quad \text{with } \phi^{\rho\mu\nu}\partial_{[\rho}B_{\mu\nu]} = 0$$
$$\phi_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho. \tag{5.8}$$

We should still remove the image of $G^\downarrow_{-1/2}$, which means identifying, e.g.,

$$A_\mu \sim A_\mu + \partial_\mu C \tag{5.9}$$

and

$$B_{\alpha\beta} \sim B_{\alpha\beta} + 3\partial_{\left[\nu}D_{\mu\right]}\phi^{\nu\mu}{}_\rho\phi^\rho{}_{\alpha\beta} \tag{5.10}$$

for arbitrary C, D_μ.

It is interesting to note that the BRST cohomology in the left-moving sector is just the Dolbeault-type cohomology of the $\check{D}$ operator that we defined in the previous subsection. The BRST operator $G^\downarrow_{-1/2}$ naturally maps to the operator $\check{D}$. In fact, the table (5.4) reveals the existence of two other differential complexes. One of these is related to the complex in (5.1) by the Hodge duality. The other one is a new complex

$$0 \to \Lambda^2_{14} \xrightarrow{\tilde{D}} \Lambda^3_7 \oplus \Lambda^3_{27} \xrightarrow{\tilde{D}} \Lambda^4_7 \oplus \Lambda^4_{27} \xrightarrow{\tilde{D}} \Lambda^5_{14} \to 0 \tag{5.11}$$

where the differential operator $\tilde{D}$ is the composition of the ordinary exterior derivative with appropriate projection operators (defined in Appendix D). This new complex does not consist of chiral primaries and does not seem to

	1	**7**	**14**	**27**
$p = 0$	$\lvert 0,0\rangle$			
$p = 1$		$\lvert\frac{1}{10},\frac{2}{5}\rangle$		
$p = 2$		$\lvert\frac{6}{10},\frac{2}{5}\rangle$	$\lvert 0,1\rangle$	
$p = 3$	$\lvert\frac{3}{2},0\rangle$	$\lvert\frac{11}{10},\frac{2}{5}\rangle$		$\lvert\frac{1}{10},\frac{7}{5}\rangle$
$p = 4$	$\lvert 2,0\rangle$	$\lvert\frac{16}{10},\frac{2}{5}\rangle$		$\lvert\frac{6}{10},\frac{7}{5}\rangle$
$p = 5$		$\lvert\frac{21}{10},\frac{2}{5}\rangle$	$\lvert\frac{3}{2},1\rangle$	
$p = 6$		$\lvert\frac{26}{10},\frac{2}{5}\rangle$		
$p = 7$	$\lvert\frac{7}{2},0\rangle$			

Figure 1: Differential complexes and the BRST cohomology.

play any role in the twisted theory we are considering, but it would still be interesting to know whether it has a distinguished geometric interpretation (see figure 1).

If we do not combine left and right movers, the cohomology is almost trivial. As we noted earlier, compact G_2 manifolds have $b_1 = 0$ and therefore there is no cohomology in the seven-dimensional representation of G_2. As a consequence, only the identity and the three-form survive if we do not include right-movers.

However, once we combine left- and right-movers, we obtain a more interesting cohomology. The two-form B and one-form A are in one-to-one correspondence via $B_{\mu\nu} = \phi_{\mu\nu}{}^\alpha A_\alpha$ so it is sufficient to consider only the combination of the left- and right moving one-forms. Each of them transforms in the **7** of G_2, and $\mathbf{7} \otimes \mathbf{7} = \mathbf{1} + \mathbf{7} + \mathbf{14} + \mathbf{27}$. We get one non-trivial class from **1**, none from **7**, b_2 from **14** and $b_3 - 1$ from **27**. In total, we get $b_2 + b_3$, corresponding to the non-trivial B-field and metric deformations of the G_2 manifold. This is indeed the set of moduli that we expect to find in a topological theory. If we replace the left or right movers by a two-form, these results do not change. We also get a contribution to the cohomology from the left-moving zero-/three-form times the right-moving zero-/three-form. The total cohomology is

$$
\begin{aligned}
\text{0-form} \ &\times\ \text{0-form} \rightarrow b_0 \\
\text{1-form} \ &\times\ \text{1-form} \rightarrow b_2 + b_3 \\
\text{2-form} \ &\times\ \text{2-form} \rightarrow b_4 + b_5 \\
\text{3-form} \ &\times\ \text{3-form} \rightarrow b_7
\end{aligned}
\tag{5.12}
$$

plus another copy of this if we allow the left and right levels not to match each other. Either way, we get one or two copies of the full cohomology $H^*(M)$ of M.

We can verify whether we recover known results about the metric moduli of G_2 manifolds. According to this, metric and B-field moduli should be given by operators of the form

$$
(\delta g_{\mu\nu} + \delta B_{\mu\nu})\psi_{\mathrm{R}}^\mu \psi_{\mathrm{L}}^\nu
\tag{5.13}
$$

with

$$
\phi_\alpha{}^{\lambda\mu}(\nabla_{[\lambda}\delta g_{\mu]\nu} + \nabla_{[\lambda}\delta B_{\mu]\nu}) = 0.
\tag{5.14}
$$

Metric moduli are indeed known to satisfy this equation (3.7) as pointed out in [29]. To verify that B-moduli also satisfy (5.14), we first use the fact

that ϕ is covariantly constant to rewrite

$$\phi_\alpha{}^{\lambda\mu}(\nabla_{[\lambda}\delta B_{\mu]\nu}) = \nabla_{[\lambda}(\delta B_{\mu]\nu}\phi_\alpha{}^{\lambda\mu}). \tag{5.15}$$

Since B-moduli transform in the **14** of G_2, they also obey (see Appendix D)

$$\delta B_{\lambda\mu}\phi_\alpha{}^{\lambda\mu} = 0. \tag{5.16}$$

We can therefore replace the right-hand side of (5.15) by

$$\nabla_{[\lambda}(\delta B_{\mu\nu]}\phi_\alpha{}^{\lambda\mu}) = \partial_{[\lambda}\delta B_{\mu\nu]}\phi_\alpha{}^{\lambda\mu} = 0 \tag{5.17}$$

since B-moduli are closed two-forms. This shows that the B-moduli also satisfy (5.14) and the BRST cohomology consists exactly of the metric and the B-field moduli.

5.3   Correlation functions

In this section we explicitly compute some simple correlation functions in the G_2 sigma model by working in the classical, large volume approximation.

As we discussed already, the operator cohomology contains only operators that map $\mathcal{H}_i$ to $\mathcal{H}_j$ with $i \leq j$. Therefore, only a finite set of correlation functions will be non-zero. Let us first consider the left-movers only, and consider a three-point function of three operators $\mathcal{O}_k = A_\mu^k \psi^\mu$, with $k = 1, 2, 3$, and we assume each to be in the BRST cohomology. This boils down to the calculation of

$$\langle V_{7/16,+}\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3 V_{7/16,+}\rangle \tag{5.18}$$

in the untwisted theory. This object turns out to be a four-point function in the R-sector

$$\langle \Phi_0 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_R \tag{5.19}$$

because $V_{7/16,+}^\dagger = V_{7/16,-} = \Phi_0 V_{7/16,+}$. The operator Φ is $\phi_{\alpha\beta\gamma}\psi^\alpha\psi^\beta\psi^\gamma$, and from the contractions we obtain for the correlator something proportional to

$$\phi_{\alpha\beta\gamma}g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}A_\mu^1 A_\nu^2 A_\rho^3. \tag{5.20}$$

The inverse metrics arise due to the fact that in this approximation the fermion two-point function is proportional to the inverse metric.

Combining left- and right- movers, relabeling everything in terms of metric and B-field moduli, and including an integral over the seven manifold from the zero mode of X^μ, we finally obtain for the three-point function for

metric and B-field moduli

$$\mathcal{F}_{3\text{-point}} = \int_M d^7x \sqrt{g}\phi_{\alpha\beta\gamma}(\delta_1 g^{\alpha\alpha'} + \delta_1 b^{\alpha\alpha'})(\delta_2 g^{\beta\beta'} + \delta_2 b^{\beta\beta'})$$
$$\times (\delta_3 g^{\gamma\gamma'} + \delta_3 b^{\gamma\gamma'})\phi_{\alpha'\beta'\gamma'}. \tag{5.21}$$

To analyze this expression a bit further, we drop the B-field moduli. In addition, we will take a suitable set of coordinates t_i on the moduli space of G_2 metrics, and denote by Y_i the operator corresponding to sending $t_i \to t_i + \delta t_i$. In other words, the three-point function reads

$$\langle Y_i Y_j Y_k \rangle = \int_M d^7x \sqrt{g}\phi_{\alpha\beta\gamma} \frac{\partial g^{\alpha\alpha'}}{\partial t_i} \frac{\partial g^{\beta\beta'}}{\partial t_j} \frac{\partial g^{\gamma\gamma'}}{\partial t_k} \phi_{\alpha'\beta'\gamma'}. \tag{5.22}$$

One might expect, based on general arguments (see, e.g., [32]), that this is the third derivative of some prepotential if suitable "flat" co-ordinates are used. For example, consider the manifold $M = T^7$ and choose coordinates such that ϕ is linear in them. We find that

$$\langle Y_i Y_j Y_k \rangle = -\frac{1}{21} \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \int \phi \wedge *\phi. \tag{5.23}$$

This strongly suggests that the same results should also be valid on general G_2 manifolds. In fact, in the next subsection, we will develop a version of "special geometry" for G_2 manifolds and show that with an appropriate definition of flat co-ordinates for the moduli space of G_2 metrics, the three-point function can be written as in (5.23)

The action

$$S = \int \phi \wedge *\phi \tag{5.24}$$

also appears in [33], where it was shown that the critical points of this functional, viewed as a functional on the space of three-forms in a given cohomology class, are precisely the three-forms of G_2 manifolds. It was also the starting point of topological M-theory in [4]; see also [3]. It is tempting to speculate that our topological G_2 string provides the framework to quantize topological M-theory, which by itself is not yet a well-defined quantum theory.

5.4 G_2 special geometry

To prove in full generality a relation between our topological three-point function and the Hitchin functional we need to develop a version of "special geometry" for G_2 manifolds.

First of all we define

$$\mathcal{I} = \int \phi \wedge *\phi, \tag{5.25}$$

which will be a functional on the space of G_2 metrics (or on the space of the corresponding three-forms).

The most natural choice for flat co-ordinates, as our torus example also suggests, is to choose periods, as we do in the case of the six-dimensional topological string. We thus pick a symplectic basis of homology three-cycles C_A and dual four cycles D^A, and define co-ordinates on the moduli space of G_2 metrics as

$$t^A = \int_{C_A} \phi. \tag{5.26}$$

For the dual periods we introduce the notation

$$F_A = \int_{D^A} *\phi. \tag{5.27}$$

It is perhaps tempting to write

$$\phi = t^A \chi_A \tag{5.28}$$

with χ_A a basis of three-forms Poincare dual to the four-cycles D_A. This is not quite correct as the detailed form of ϕ will in general differ from (5.28) by an exact three-form. In most calculations, this exact three-form drops out, but it is important to keep in mind that ϕ cannot simply be expanded linearly in a given basis of cohomology.

Continuing, we can also write F_A as

$$F_A = \int *\phi \wedge \partial_A \phi. \tag{5.29}$$

Furthermore, by a generalization of the Riemann bilinear identities we find that

$$\mathcal{I} = t^A F_A. \tag{5.30}$$

Let us now take one derivative of $\mathcal{I}$. We readily obtain

$$\partial_B \mathcal{I} = F_B + t^A \partial_B F_A. \tag{5.31}$$

We can also perform straightforward explicit computations by using the canonical expressions for ϕ and $*\phi$ in local co-ordinates:

$$\phi = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}, \tag{5.32}$$

$$*\phi = dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247}. \tag{5.33}$$

Here, $dx^{ijk} = e^i \wedge e^j \wedge e^k$, with $e^i = e^i_\mu dx^\mu$ a local orthonormal frame, i.e., a set of vielbeins in which the metric becomes $g_{\mu\nu} = e^i_\mu e^i_\nu$. To find the variation of various quantities with respect to t^A, we will need to vary the

vielbeins. We notice that up to SO(7) rotations rotating the e^i into each other

$$\partial_A e_\mu^a = \tfrac{1}{2}\partial_A h_{\mu\nu} h^{\nu\lambda} e_\lambda^a. \tag{5.34}$$

Then, using the explicit expressions for ϕ and $*\phi$ in terms of the vielbeins, we find

$$F_A = \int *\phi \wedge \partial_A \phi = \frac{3}{2}\partial_A h_{\mu\nu} h^{\mu\nu} \mathcal{I} \tag{5.35}$$

and

$$t^B \partial_A F_B = \int \partial_A *\phi \wedge \phi = 2\partial_A h_{\mu\nu} h^{\mu\nu} \mathcal{I} \tag{5.36}$$

which implies that

$$t^A \partial_B F_A = \tfrac{4}{3} F_B. \tag{5.37}$$

We conclude from (5.31) and (5.37) that

$$F_B = \tfrac{3}{7}\partial_B \mathcal{I}. \tag{5.38}$$

Thus we see that $\mathcal{I}$ is homogeneous of degree 7/3 in the co-ordinates t^A, which can also easily be verified explicitly, but more importantly we have found that the dual periods are the derivatives of a single function, the prepotential F, which is given by

$$F = \tfrac{3}{7}\mathcal{I}. \tag{5.39}$$

Next we turn to the second derivative of $\mathcal{I}$. From the above we readily obtain

$$\int \partial_A \phi \wedge \partial_B *\phi = \int \partial_B \phi \wedge \partial_A *\phi = 3\int \phi \wedge \partial_A \partial_B *\phi = \frac{3}{7}\partial_A \partial_B \mathcal{I}. \tag{5.40}$$

We can evaluate the first expression most easily, by varying the vielbeins that appear in the standard expression for ϕ and $*\phi$, and by counting the resulting terms. We find

$$\int \partial_A \phi \wedge \partial_B *\phi = \frac{1}{2}\int \sqrt{g}(\partial_A h_{\mu\nu} h^{\mu\nu}\partial_B h_{\rho\sigma} h^{\rho\sigma} - \partial_A h_{\mu\nu} h^{\nu\rho}\partial_B h_{\rho\sigma} h^{\sigma\mu}). \tag{5.41}$$

On the other hand, by using the third identity in (5.5) we deduce

$$\int \sqrt{g}\phi_{abc}\partial_A h^{aa'}\partial_B h^{bb'} h^{cc'}\phi_{a'b'c'} = \frac{1}{36}\int \sqrt{g}(\partial_A h_{\mu\nu} h^{\mu\nu}\partial_B h_{\rho\sigma} h^{\rho\sigma}$$
$$- \partial_A h_{\mu\nu} h^{\nu\rho}\partial_B h_{\rho\sigma} h^{\sigma\mu}). \tag{5.42}$$

Combining (5.40), (5.41) and (5.42) we finally obtain

$$\int \sqrt{g}\phi_{abc}\partial_A h^{aa'}\partial_B h^{bb'} h^{cc'}\phi_{a'b'c'} = \frac{1}{42}\partial_A \partial_B \mathcal{I}. \tag{5.43}$$

Therefore, the second derivatives of $\mathcal{I}$ closely resemble the expression for the three-point function we obtained from the topological string.

Turning finally to the third derivative, this analysis is a bit more tedious. In analogy with (5.40) we have

$$\int \partial_A \phi \wedge \partial_B \partial_C * \phi = -\frac{3}{2} \int \phi \wedge \partial_A \partial_B \partial_C * \phi = \frac{3}{7} \partial_A \partial_B \partial_C \mathcal{I}. \qquad (5.44)$$

The first expression is again the most useful one to manipulate, and we do this as before in terms of a representation in a local orthonormal flat frame (i.e., vielbeins). We again use the variation of the vielbein as given in (5.34). We find a new feature, namely we now also will run into double derivatives of the metric, due to the double derivative acting on $*\phi$ in the first expression in (5.44). We can get rid of this double derivative as follows. We write ∂_{BC} for the double derivative acting on a single vielbeins only. Then it is easy to see that

$$\int \partial_A \phi \wedge \partial_{BC} * \phi = \int \partial_{BC} \phi \wedge \partial_A * \phi. \qquad (5.45)$$

Now notice that $\partial_B \partial_C = \partial_{BC} + \partial'_{BC}$, where ∂'_{BC} is defined such that the two derivatives never act on the same vielbein. Thus, for example,

$$\begin{aligned} \partial_{BC} e^1 \wedge e^2 &\equiv \partial_B \partial_C e^1 \wedge e^2 + e^1 \wedge \partial_B \partial_C e^2 \\ \partial'_{BC} e^1 \wedge e^2 &\equiv \partial_B e^1 \wedge \partial_C e^2 + \partial_C e^1 \wedge \partial_B e^2 \end{aligned} \qquad (5.46)$$

and clearly these two add up to $\partial_B \partial_C$. Because ϕ is linear in t^A (5.28), we can replace in (5.45) $\partial_{BC} \phi = \partial_B \partial_C \phi - \partial'_{BC} \phi = -\partial'_{BC} \phi$. So we obtain,

$$\partial_A \partial_B \partial_C \mathcal{I} = \frac{7}{3} \left(\int \partial_A \phi \wedge \partial'_{BC} * \phi - \int \partial'_{BC} \phi \wedge \partial_A * \phi \right). \qquad (5.47)$$

In this expression no double derivatives of the metric appear anymore. However, it contains a priori all kinds of contractions of the three single derivatives of the metric. To determine the detailed form of the result, we took (5.47), wrote $*\phi$ in terms of ϕ using the seven-dimensional completely antisymmetric ϵ tensor, and expanded (5.47) in terms of all possible contractions that can appear. After a significant amount of tedious algebra we found, quite surprisingly, that almost all terms cancel, and that we are left with the simple final result

$$\partial_A \partial_B \partial_C \mathcal{I} = -21 \int \sqrt{g} \phi_{abc} \partial_A h^{aa'} \partial_B h^{bb'} \partial_C h^{cc'} \phi_{a'b'c'}. \qquad (5.48)$$

This proves that our topological three-point function is indeed the third derivative of a single function, which is precisely the Hitchin functional, viewed as a function on the space of G_2 metrics! Notice that (5.48) is valid both for the rather trivial modulus which corresponds to rescaling ϕ, as well

as for the $b_3 - 1$ moduli which live in the **27** of G_2. For the latter moduli an expression similar to (5.48) was written down in [34], where it was used to describe fibrations of G_2 manifolds by co-associative submanifolds. These three-point functions were called Yukawa couplings in that paper, though the relation with the physical Yukawa couplings in M-theory was not given. Our results show that the cubic coupling (5.48), which is the topological three-point function, is indeed closely related to the physical Yukawa couplings that one obtains in compactifying M-theory on G_2 manifolds. This is because the Kähler potential of the resulting four-dimensional theory is essentially the logarithm of $\mathcal{I}$, and Yukawa couplings are given by the third derivative of the Kähler potential. A more detailed discussion can be found in Section 7.3.

5.5   Inclusion of the B-field

We next want to see what happens when we include the B-field. There is only one relevant correlator

$$\int \sqrt{g}\phi_{abc}\partial_p B^{aa'}\partial_q B^{bb'}\partial_C h^{cc'}\phi_{a'b'c'}, \tag{5.49}$$

since the correlators involving one or three B-field insertions vanish identically due to symmetry/anti-symmetry properties of the index contractions. We introduced co-ordinates s^p on the space $H^2(M)$ of B-fields, but still need to specify how they are defined. To simplify the above expression, we first observe that since $B^{bb'}$ lives in the **14** of G_2 (the B-field is a closed two-form and the only non-trivial second cohomology transforms as in the **14** dimensional representation of G_2), which means $\phi_{abb'}B^{bb'} = 0$. Therefore, we can antisymmetrize over a, a', b, b' in the previous expression so that it becomes

$$\frac{1}{24}\int \sqrt{g}\phi_{c[ab}\phi_{a'b']c'}\partial_p B^{aa'}\partial_q B^{bb'}\partial_C h^{cc'}. \tag{5.50}$$

Next, we can use the following identity

$$\phi_{a[bc}\phi_{b'c']a'} = -\tfrac{4}{9}g_{a[b}\phi_{cb'c']a'} - \tfrac{4}{9}g_{a'[b}\phi_{cb'c']a} - \tfrac{2}{9}\delta_{aa'}\phi_{[bcb'c']} \tag{5.51}$$

which we can prove in a local orthonormal frame. Inserting (5.51) into (5.50) leads to

$$\int \sqrt{g}\phi_{abc}\partial_p B^{aa'}\partial_q B^{bb'}\partial_C h^{cc'}\phi_{a'b'c'} = -\frac{1}{9}\frac{\partial^3}{\partial t^C \partial s^p \partial s^q}\int \sqrt{g}\phi^{abcd}B_{ab}B_{cd} \tag{5.52}$$

where it is crucial that we choose our coordinates s^a such that the periods of $B \wedge B$ along all four cycles are purely quadratic expressions in terms of the

s^p that do not depend on the t^A. We can rewrite (5.52) more compactly as

$$\int \sqrt{g}\phi_{abc}\partial_p B^{aa'}\partial_q B^{bb'}\partial_C h^{cc'}\phi_{a'b'c'} = -\frac{1}{216}\frac{\partial^3}{\partial t^C \partial s^p \partial s^q}\int B \wedge B \wedge \phi \tag{5.53}$$

which is manifestly invariant under $B \to B + dV$. The expression on the right-hand side of (5.53) also appeared in [34] as defining a nice quadratic form on the space of B-fields; here we see that it arises naturally from the topological G_2 string. Also notice that this term is purely cubic in the coordinates, so fourth and higher derivatives of this term will vanish identically.

The final generating functional of all correlation functions is an extension of the Hitchin's functional to include the B-fields:

$$\mathcal{I}_{\text{tot}} = \int \phi \wedge *\phi + \frac{7}{72}\int B \wedge B \wedge \phi. \tag{5.54}$$

5.6   What are we quantizing?

From the earlier discussion it seems clear that the prepotential $\mathcal{I}$ of the topological string theory that we are studying can be viewed as a wave function in the Hilbert space that one obtains by quantization of the symplectic space $H^2(M,\mathbb{R}) \oplus H^3(M,\mathbb{R}) \oplus H^4(M,\mathbb{R}) \oplus H^5(M,\mathbb{R})$, with symplectic structure $\omega(\delta\alpha,\delta\beta) = \int \delta\alpha \wedge \delta\beta$. For the six-dimensional topological string, this point of view was taken in [35], see also [3, 36], and it was shown that this is the natural way to understand the holomorphic anomaly. In our case we do not have a holomorphic anomaly, so it is not clear how compelling the interpretation of $\mathcal{I}$ as a wave function is, see also Section 8.1. Still, it is interesting to pursue this idea a little bit and therefore we will now briefly study the wave function interpretation restricting to the metric degrees of freedom only, i.e., we restrict ourselves to $H^3 \oplus H^4$.

In order to be able to define suitable covariant derivatives we first define a Kähler potential

$$K = -\frac{3}{7}\log \mathcal{I}. \tag{5.55}$$

This is, up to a numerical factor, precisely the Kähler potential of the four-dimensional (4D) theory obtained by compactifying M-theory on a G_2 manifold (see Section 7.3). In fact, the expression in (5.53) corresponds to the gauge couplings of the 4D theory[16]   so that at tree level our topological

---

[16]More precisely [37], the gauge couplings are proportional to $\left(t^A \frac{\partial^3}{\partial t^A \partial s^p \partial s^q}\int B \wedge B \wedge \phi\right)$ and the θ terms are given by $\left(p^A \frac{\partial^3}{\partial t^A \partial s^p \partial s^q}\int B \wedge B \wedge \phi\right)$ where p^A are moduli coming

string computes both the Kähler potential and the gauge couplings of the low energy effective field theory.

We can use the Kähler potential to define a covariant derivative

$$\nabla_A \phi = \partial_A \phi + \partial_A K \phi \qquad (5.56)$$

which has the property that $\nabla_A \phi$ lives purely in the **27** of G_2. In other words, the covariant derivative projects out the G_2 singlet contribution. Similarly, we can define a covariant derivative of $*\phi$ via

$$\nabla_A * \phi = \partial_A * \phi + \tfrac{4}{3}\partial_A K * \phi. \qquad (5.57)$$

A useful observation is that

$$\nabla_A * \phi = - * \nabla_A \phi \qquad (5.58)$$

which can be derived using the calculations done in the preceding sections, but which also follows from the identity [33]

$$\delta * \phi = * \left(\tfrac{4}{3}\pi_1(\delta\phi) + \pi_7(\delta\phi) - \pi_{27}(\delta\phi) \right) \qquad (5.59)$$

where π_1, π_7 and π_{27} are the appropriate projections on the corresponding G_2 representations, and $\delta\phi$ is an arbitrary variation.

Turning back to $H^3 \oplus H^4$, we wish to consider the quantization of this space with respect to the symplectic form

$$\omega = \int_M \delta\alpha_3 \wedge \delta\alpha_4 \qquad (5.60)$$

for $(\alpha_3, \alpha_4) \in H^3 \oplus H^4$.

The simplest quantization, the analog of the real polarization in the case of the B-model, is to define

$$p^A = \int_{C_A} \alpha_3, \quad q_A = \int_{D^A} \alpha_4 \qquad (5.61)$$

for which the symplectic form becomes simply

$$\omega = \sum_A dp^A \wedge dq_A. \qquad (5.62)$$

This structure is manifestly independent of the G_2 structure of the manifold, i.e., it is background independent.

_____

from the C field in M-theory:

$$C = \sum_{a=1}^{h_2} A^a \wedge \partial_a B + p^A \partial_A \phi.$$

Here A^a are the h_2 gauge fields in the four-dimensional theory. We will come back to this in Section 7.3.

Next, we introduce a different set of co-ordinates. We pick a fixed reference G_2 structure ϕ and choose

$$(\alpha_3, \alpha_4) = (x^A \partial_A \phi, y^A *_\phi \partial_A \phi). \tag{5.63}$$

We put the subscript ϕ on $*$ to indicate that this is defined with respect to the reference G_2 structure. Notice that $*_\phi \partial_A \phi$ is closed, this follows from the identity

$$*\partial_A \phi = -\partial_A * \phi - \tfrac{7}{3}\partial_A K * \phi \tag{5.64}$$

and since $d * \phi = 0$ it is clear that $d\partial_A * \phi = 0$ as well, so that the right-hand side of (5.64) is indeed closed.

Combining (5.40) and (5.64) we find that the symplectic form becomes

$$\omega = e^{-7K/3} \partial_A \partial_B K dx^A \wedge dy^B \tag{5.65}$$

so that after quantization

$$\left[x^A, y^B\right] = -i e^{7K/3} K^{AB} \tag{5.66}$$

with K^{AB} the inverse of $K_{AB} \equiv \partial_A \partial_B K$.

As we vary the background the quantization changes. The co-ordinate x^A is independent of the background [in fact, $x^A = p^A$ defined in (5.61)], since ϕ is linear in the background co-ordinates t^A (up to possible an exact form). However, y^A changes. Its variation follows by imposing [35]

$$\frac{\partial \alpha_4}{\partial t^B} = 0. \tag{5.67}$$

After some straightforward algebra we obtain

$$\partial_A y^D - \tfrac{7}{3}\partial_A K y^D = -K_{ABC} K^{CD} y^B, \tag{5.68}$$

where $K_{ABC} \equiv \partial_A \partial_B \partial_C K$. It is interesting to observe that the answers are naturally expressed in terms of the Kähler potential K.

Equation (5.68) implies that y eigenstates satisfy

$$\partial_A |y\rangle = \left(-K_{ABC} K^{BD} \frac{\partial}{\partial y^D} Y^C + \frac{7}{3} K_A \frac{\partial}{\partial Y^B} Y^B\right) |y\rangle. \tag{5.69}$$

The topological string wave function $\psi(y) = \langle \psi_{\text{top}} | y \rangle$ will then satisfy a similar differential equation, given that $|\psi_{\text{top}}\rangle$ does not depend on the choice of background G_2 structure. This is the analog of the holomorphic anomaly for the G_2 string.

From here on there are many different polarizations one can study. We can combine x^A and y^A in complex coordinates and work with the corresponding coherent states, to be closer to what we do in the case of a Calabi–Yau

manifold. We can also separate out the overall rescalings of the metric and parametrize

$$(\alpha_3, \alpha_4) = (\xi\phi + x^i \nabla_i \phi, \zeta * \phi + y^j *_\phi \nabla_j \phi). \tag{5.70}$$

The symplectic form, in these co-ordinates, becomes

$$\omega = e^{-\frac{7K}{3}} \left(d\xi \wedge d\zeta + (\partial_i \partial_j K - \partial_i K \partial_j K) dx^i \wedge dx^j \right). \tag{5.71}$$

The rest of the analysis will be similar to what we did before and we will not work out the details here. It will be an interesting question to see whether we can use these differential equations to make an educated guess about the higher genus contributions to the wave function.

To summarize, the topological G_2 string can be viewed as a wave function associated to a certain Lagrangian submanifold of the symplectic space $H^2 \oplus H^3 \oplus H^4 \oplus H^5$. The Lagrangian submanifold consists of the points

$$\left(B, \phi, \frac{7}{3} *_\phi \phi + \frac{7}{72} B \wedge B, \frac{7}{36} B \wedge \phi \right) \tag{5.72}$$

where ϕ runs over the space of G_2 metrics and B over $H^2(M)$.

5.7 Topological G_2 strings on $CY \times S^1$

An interesting example to study is the topological G_2 string on $CY \times S^1$. Because of the S^1, this seven-manifold is not a generic G_2 manifold. Whereas generic G_2 manifolds have no supersymmetric two-cycles, $CY \times S^1$ does have such two-cycles and therefore world-sheet instantons will contribute to the theory. In addition, the analysis of the BRST cohomology will be modified since $H^1(CY \times S^1, \mathbb{R}) = \mathbb{R}$. We will postpone a detailed discussion of these issues to another occasion, and here mainly focus on the metric and B-field moduli of $CY \times S^1$.

Any manifold of the form $CY \times S^1$ has a natural G_2 structure of the form

$$\phi = \mathrm{Re}(e^{i\alpha}\Omega) + R\omega \wedge d\theta$$
$$*\phi = R\,\mathrm{Im}(e^{i\alpha}\Omega) \wedge d\theta + \tfrac{1}{2}\omega \wedge \omega \tag{5.73}$$

where θ is a periodic variable with period 2π, $e^{i\alpha}$ is an arbitrary phase, R is the radius of the S^1 and Ω and ω are the holomorphic three-form and Kähler

form on the Calabi–Yau manifold. These are not completely independent, but have to obey

$$i \int \Omega \wedge \bar{\Omega} = \frac{4}{3} \int \omega \wedge \omega \wedge \omega. \tag{5.74}$$

The G_2 BRST complex in say the left-moving sector, acting at the level of zero modes, involves among others the following differentials:

$$\Omega^0(M, \mathbb{R}) \xrightarrow{d} \Omega^1(M, \mathbb{R}) \xrightarrow{*\phi \wedge d} \Omega^6(M, \mathbb{R}) \xrightarrow{d} \Omega^7(M, \mathbb{R}). \tag{5.75}$$

where we used the identification of $\mathbf{7}$ in $\Omega^2(M, \mathbb{R})$ with $\Omega^6(M, \mathbb{R})$ and of $\mathbf{1}$ in $\Omega^3(M, \mathbb{R})$ with $\Omega^7(M, \mathbb{R})$ [see table (5.4)]. The complex (5.75) is equivalent to (5.1) for any G_2 manifold. Thus, the full BRST cohomology is obtained by combining two complexes of the form (5.75), one for the left-movers and one for the right-movers. If we specialize to the case of a Calabi–Yau manifold times a circle using (5.73), (5.75) reduces to a certain complex involving the differential forms on the Calabi–Yau manifold. We are not aware of any literature on Calabi–Yau manifolds where such a complex appears, and this shows that the topological G_2 twist is not in a straightforward way related to the usual topological twist for Calabi–Yau manifolds.

More generally, complexes of the form (5.75) can be constructed for any special holonomy manifold by simply replacing ϕ by a suitable covariantly closed differential form. It is an interesting question whether such complexes in general give rise to a new geometric understanding of special holonomy manifold.

Turning back to the $CY \times S^1$ case, the metric moduli of $CY \times S^1$ include the $2h^{1,2}$ complex structure moduli and $h^{1,1}$ Kähler moduli of the Calabi–Yau, but also the radius of the circle R. The total number of metric moduli is therefore $\dim H^3(CY \times S^1, \mathbb{R}) - 1$. The number of three-form moduli is, however, equal to $\dim H^3(CY \times S^1, \mathbb{R})$. The difference is the parameter α in (5.73). Strictly speaking α does not correspond to an element of the BRST cohomology, and we should therefore remove the period of ϕ corresponding to α from our consideration, but since nothing turns out to depend on α we may as well work with the full set of $\dim H^3(CY \times S^1)$ periods. The modulus R on the other hand is physical, and this has some interesting consequences for the relation between the topological G_2 string and the A- and B-model topological string on the Calabi–Yau manifold.

To study the topological G_2 string and its relation to the A and B models, we choose a basis of three-cycles A^I, B_I with intersection number $(A^I, B_J) = \delta^I_J$ on the Calabi–Yau manifold. Similarly, we choose a basis of two-cycles

C^a and dual four-cycles D_a. The cycles on $CY \times S^1$ are then given by

$$
\begin{aligned}
&\text{twocycles} : C^a \\
&\text{threecycles} : C^a \times S^1, \quad A^I, \quad B_I \\
&\text{fourcycles} : D_a, \quad B_I \times S^1, \quad -A^I \times S^1 \\
&\text{fivecycles} : D_a \times S^1.
\end{aligned}
\tag{5.76}
$$

The prepotential of the topological G_2 string also depends on the B-field. To take this into account we need to improve the four-form to

$$
*\phi \to \phi^{(4)} \equiv -R\,\mathrm{Im}(e^{i\alpha}\Omega) \wedge d\theta - \frac{1}{2}\mathrm{Re}\left(\omega + \frac{i}{2}B\right) \wedge \left(\omega + \frac{i}{2}B\right). \tag{5.77}
$$

The various periods, which define co-ordinates on the moduli space of G_2 metrics, are given by

$$
\begin{aligned}
b^a &= \int_{C^a} B \\
k^a &= \int_{C^a \times S^1} \phi \\
q^I &= \int_{A^I} \phi \\
p_I &= \int_{B_I} \phi \\
\frac{3}{7}\frac{\partial \mathcal{I}}{\partial k^a} &= \int_{D_a} \phi^{(4)} \\
\frac{3}{7}\frac{\partial \mathcal{I}}{\partial q^I} &= \int_{B_I \times S^1} \phi^{(4)} \\
\frac{3}{7}\frac{\partial \mathcal{I}}{\partial p_I} &= \int_{-A^I \times S^1} \phi^{(4)} \\
\frac{1}{2}\frac{\partial \mathcal{I}}{\partial b^a} &= \int_{D_a \times S^1} B \wedge \phi.
\end{aligned}
\tag{5.78}
$$

Now, we want to relate these variable to the quantities that appear naturally in the A and the B models on the Calabi–Yau manifold. If we denote by $\mathcal{F}^A$ and $\mathcal{F}^B$ the suitably normalized prepotentials of the A and the B models, then these obey

$$
\begin{aligned}
X^I &= \int_{A^I} \Omega \\
\frac{\partial \mathcal{F}^B}{\partial X^I} &= \int_{B_I} \Omega
\end{aligned}
$$

$$t^a = \int_{C^a} \omega + \frac{i}{2} B$$

$$\frac{\partial \mathcal{F}^A}{\partial t^a} = \int_{D_a} \left(\omega + \frac{i}{2} B \right)^2 \tag{5.79}$$

with X^I and t^a the complex structure and complexified Kähler moduli. By comparing (5.78) and (5.7) we can now determine the relation between $\mathcal{I}$ and $\mathcal{F}^A$ and $\mathcal{F}^B$. This is somewhat subtle due to the appearance of the parameter R in ϕ and $\phi^{(4)}$. R itself is not an independent period but it appears in (5.78) in a non-trivial way. We should also keep in mind that in (5.7) Ω and ω are constrained by (5.74), so that the variables X^I and t^a obey a non-trivial constraint. To reformulate this constraint we denote

$$P(X^I, \bar{X}^I) = 3i \int \Omega \wedge \bar{\Omega}, \quad Q(t^a, \bar{t}^a) = 4 \int \omega^3 \tag{5.80}$$

so that the constraint is that $P(X^I, \bar{X}^I) = Q(t^a, \bar{t}^a)$. A comparison of the periods yields the following set of equations (we put $\alpha = 0$ here, but it can be trivially put back into the equations by replacing $\Omega \to e^{i\alpha}\Omega$)

$$b^a = 2\,\mathrm{Im}(t^a)$$

$$k^a = 2\pi R\,\mathrm{Re}(t^a)$$

$$q^I = \mathrm{Re}\left(X^I\right)$$

$$p_I = \mathrm{Re}\left(\partial_I \mathcal{F}^B\right)$$

$$\frac{3}{7}\frac{\partial \mathcal{I}}{\partial k^a} = -\frac{1}{2}\mathrm{Re}\left(\partial_a \mathcal{F}^A\right)$$

$$\frac{3}{7}\frac{\partial \mathcal{I}}{\partial q^I} = -2\pi R\,\mathrm{Im}\left(\partial_I \mathcal{F}^B\right)$$

$$\frac{3}{7}\frac{\partial \mathcal{I}}{\partial p_I} = 2\pi R\,\mathrm{Im}\left(X^I\right)$$

$$\frac{1}{2}\frac{\partial \mathcal{I}}{\partial b^a} = 2\pi R\,\mathrm{Im}\left(\partial_a \mathcal{F}^A\right). \tag{5.81}$$

To solve this system of equations, we first express $P(X^I, \bar{X}^I)$ in terms of q^I, p_I. As is well-known, in terms of q^I, p_I P is equal to the Legendre transform of the imaginary part of $\mathcal{F}^B$,

$$P\left(p_I, q^I\right) = 3i \int \Omega \wedge \bar{\Omega} = 12\left(\mathrm{Im}(\mathcal{F}^B) - p_I\,\mathrm{Im}(X^I)\right)_{q^I = \mathrm{Re}(X^I),\, p_I = \mathrm{Re}(\partial_I \mathcal{F}^B)}. \tag{5.82}$$

We cannot express $Q(t^a, \bar{t}^a)$ in terms of k^a directly, due to the factor of R that appears in the relation between k^a and t^a. However, the following is a

function of just the k^a:

$$S(k^a) = 4 \int (2\pi R\omega)^3. \tag{5.83}$$

The constraint $P = Q$ now implies that R is a non-trivial function of q^I, p_I, k^a, given by

$$2\pi R(p_I, q^I, k^a) = \left(\frac{S(k^a)}{P(p_I, q^I)} \right)^{\frac{1}{3}}. \tag{5.84}$$

We also define

$$T\left(p_I, q^I, k^a, b^a\right) = 12 \operatorname{Re} \left(\mathcal{F}^A \right)_{t^a = k^a/2\pi R(p_I, q^I, k^a) + ib^a/2} \tag{5.85}$$

so that

$$S(k^a) = \left(2\pi R \left(p_I, q^I, k^a\right) \right)^3 T\left(p_I, q^I, k^a, b^a\right)_{b^a = 0}. \tag{5.86}$$

We now claim that

$$\mathcal{I} = 2\pi R(p_I, q^I, k^a) \left(-\frac{7}{36} P\left(p_I, q^I\right) - \frac{7}{72} T\left(p_I, q^I, k^a, b^a\right) \right)$$

$$= -\frac{7}{3}(2\pi R) \left(\left(\operatorname{Im}(\mathcal{F}^B) - p_I \operatorname{Im}(X^I) \right) + \frac{1}{2} \operatorname{Re}\left(\mathcal{F}^A \right) \right). \tag{5.87}$$

This shows that the prepotential of the topological G_2 string is indeed a combination of the A- and B-model topological string, but the complex and Kähler moduli of the Calabi–Yau manifold get mixed in a rather intricate way due to the presence of the radius R. R is closely related to the volume of the Calabi–Yau manifold, and it would be interesting to see if this is related to and/or can resolve the gravitational anomaly found in the one-loop calculation in the six-dimensional Hitchin system in [38]. The non-trivial role that R plays in this also manifests itself in the analysis of four-dimensional supergravity, see e.g., [39].

To show that (5.87) solves (5.81) is somewhat complicated due to the dependence of R on p_I, q^I, k^a. However, one may check that

$$\frac{\partial \mathcal{I}}{\partial (2\pi R)} = -\frac{7}{36} \left(P\left(p_I, q^I\right) - T\left(p_I, q^I, k^a, b^a\right)_{b^a = 0} \right) \tag{5.88}$$

where it is important to differentiate not just the explicit R that appears in (5.87), but also the R that appears in the definition of T in (5.85). The right-hand side of (5.88) is precisely the original constraint (5.74) and therefore vanishes identically. In other words, the radius seems to play the role of a Lagrange multiplier that imposes the volume constraint (5.74). Because of this, we can treat R as a constant when verifying (5.81), and with this simplification it is straightforward to verify that (5.87) solves (5.81).

From (5.87) we also find, using (5.84) and (5.86), that

$$\mathcal{I}_{b^a=0} = -\frac{7}{12} S(k^a)^{1/3} P\left(p_I, q^I\right)^{2/3}. \qquad (5.89)$$

Thus, the topological G_2 string is not just the sum of the A and B models, but it can also be written as the product of fractional powers of the A and B models. It would be interesting to know whether either the combinations (5.87) and (5.89) have any distinguished meaning for six-dimensional topological strings.

6   The topological G_2 string

We have so far been considering a topologically twisted σ-model of maps from a sphere into a G_2 manifold. However, on higher genus Riemann surfaces, there is nothing interesting to compute in the σ-model. To get interesting amplitudes, we need to couple the σ-model to two-dimensional gravity, and integrate over the moduli space of Riemann surfaces. This will define the topological G_2 string. In the following, we first give a preliminary discussion of the topological σ-model at higher genus and then construct a measure on the moduli space of Riemann surfaces to define the topological string amplitudes.

6.1   Twisting the σ-model at higher genus

Generalizing the sphere computation to higher genera [30, 31], n-point correlators on a genus-g Riemann surface in the twisted theory are defined as a correlator in the untwisted theory of the same n operators plus $(2 - 2g)$ insertions of the spin-field that is related to the space-time supersymmetry charge. For a Calabi–Yau three-fold target space on a Riemann surface with $g > 1$ the meaning of this prescription is to insert $2g - 2$ of the conjugate spectral flow operator $(e^{-i(\sqrt{d}/2)\phi}$ in the notation of Section 4.1). To generalize this to the G_2 situation, we will do something similar. However, there is only a single G_2 invariant spinfield. This is where the decomposition in conformal blocks in Section 2.3 is useful: the spin-field $\Phi_{1,2}$ (which corresponds to the particular Ramond sector ground state $\left|\frac{7}{16}, 0\right\rangle$) could be decomposed in a block $\Phi_{1,2}^+$ and in a block $\Phi_{1,2}^-$ [see equations (2.14) and (A.8), and also Section 7.1]. At genus zero we needed two insertions of $\Phi_{1,2}^+$, so the natural guess is that at genus g we need $2g - 2$ insertions of $\Phi_{1,2}^-$. We will demonstrate shortly that with this guess the topological G_2 strings are indeed "critical" in seven dimensions.

6.2 Topological strings

To go from a topological σ-model to topological strings, we need to integrate over the moduli space of Riemann surfaces, $\mathcal{M}_g$. To construct a measure on the moduli space of Riemann surfaces, we need an anti-ghost $G_*^\uparrow$, such that $\{Q, G_*^\uparrow\} = T$ where T is the twisted stress tensor and Q is the BRST operator. We use the notation $G_*^\uparrow$ for the anti-ghost because the conformal block $G^\uparrow$ defined previously almost does the job, as discussed in Section 4.7. In the following, we assume that a suitable modification $G_*^\uparrow$ of $G^\uparrow$ exists which we can use to define the topological string amplitudes. With this important assumption we can define the genus-g free energy F_g of the G_2 topological string by integrating over the $3g - 3$-dimensional moduli space of genus-g Riemann surfaces $\mathcal{M}_g$ along with $3g - 3$ insertions of the anti-ghost folded against Beltrami differentials giving the appropriate measure of integration

$$F_g = \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} |(\mu_i, G_*^\uparrow)|^2 \right\rangle_g \tag{6.1}$$

where the folded anti-ghosts are defined by integrating them over the genus-g world-sheet against the Beltrami differentials $(\mu_i, G_*^\uparrow) = \int d^2z \mu_i(z) G_*^\uparrow(z)$.

Critical dimension. The usual topological strings on Calabi–Yau manifolds have a "critical dimension" $d = 6$ (complex dimension 3). This is because essentially all the higher genus free energies F_g vanish when the target space is a complex manifold of (complex) dimension other than 3. The G_2 string is critical in seven dimensions. Indeed, we can use the fusion rules of the tri-critical Ising model to show that there is a non-vanishing contribution to correlation functions of $2g - 2$ $\Phi_{1,2}$'s and $3g - 3$ $G^\uparrow$. We can also show that their correlation functions are non-zero by considering the Coulomb gas representation of the tri-critical Ising model (which is useful to compute correlation functions). From that perspective the $2g - 2$ insertions of $\Phi_{1,2}^-$ and $3g - 3$ insertions of $G^\uparrow$ yield a total ϕ charge of

$$(2 - 2g)\frac{5}{2\sqrt{10}} + (3g - 3)\frac{2}{\sqrt{10}} = (g - 1)\frac{1}{\sqrt{10}} \tag{6.2}$$

which is exactly the correct amount needed to cancel the existing background charge $\left(\frac{1}{\sqrt{10}}\right)$ of the tri-critical Ising model on a genus-g Riemann surface. Here we used that the anti-ghost $G^\uparrow$ has weight two in the Coulomb gas representation (see Appendix A).

The G_2 topological string partition function is defined as an asymptotic series in a coupling constant λ

$$\mathcal{Z} = e^{\mathcal{F}}, \quad \text{where } \mathcal{F} = \sum_{g=0}^{\infty} \lambda^{2-2g} F_g. \qquad (6.3)$$

The descent relations introduced in Section 4.8 enable us to now define correlation functions of chiral primaries just like in the $\mathcal{N} = 2$ topological string.

7   Physics in three dimensions

Since we are discussing type II string theory compactified on a manifold of G_2 holonomy, we expect the topological G_2 string to be of relevance for the resulting three-dimensional effective field theory. In this section we will explore some properties of this effective field theory and how they are related to topological G_2 strings. Since G_2 compactifications preserve four supercharges, the resulting three-dimensional theory will have $\mathcal{N} = 2$ supersymmetry.

7.1   Massless fields and the GSO projection

We are dealing with an odd-dimensional compactification of string theory. Therefore, the GSO projection is particularly subtle. In order to define it, we need a notion of fermion number. We will first define this in the NS sector of the internal CFT corresponding to the sigma model on the G_2 manifold. As discussed in some detail in [8], we can assign a fermion number to a state by assigning a fermion number to the tri-critical Ising part of the state. In the NS sector, there is a tri-critical Ising model notion of fermion number in which we associate fermion number $(-1)^{n+1}$ for states in Hilbert space $\mathcal{H}_n$ with $n = 1, \ldots, 4$ ($n = 1$ corresponding to the identity, $n = 2$ to the primary $\frac{1}{10}$, etc). The fermion number in the three-dimensional spacetime part of the compactification in the NS sector is the usual one.

In the R sector, things are less straightforward. In three dimensions, the representations of the Clifford algebra are two-dimensional, and there are no chiral spinors. The same holds true in seven dimensions. Therefore, in order to have a well-defined fermion number, we need to take a reducible representation of the Clifford algebra in three dimensions which consists of two spinors which we will call $|3, +\rangle$ and $|3, -\rangle$ where the sign indicates

fermion number. Similarly, we need two spinors coming from the seven-dimensional part, which we will call $|7, +\rangle$ and $|7, -\rangle$. The zero modes of the three-dimensional fermions map $|3, +\rangle$ to $|3, -\rangle$ and vice versa. With this doubling we have a well-defined action of $(-1)^F$ given by $(-1)^F |3, \pm\rangle = \pm |3, \pm\rangle$. A similar remark applies to the seven-dimensional part. When we combine the three- and seven-dimensional part, we find that if we take all possible combinations, we obtain a reducible representation. The smallest irreducible representation, which still allows for a proper action of $(-1)^F$, is obtained by taking, e.g., the combinations

$$|\chi, +\rangle = |3, +\rangle \otimes |7, +\rangle + |3, -\rangle \otimes |7, -\rangle$$
$$|\chi, -\rangle = |3, +\rangle \otimes |7, -\rangle + |3, -\rangle \otimes |7, +\rangle \tag{7.1}$$

where fermion number acts as $(-1)^F |\chi, \pm\rangle = \pm |\chi, \pm\rangle$. The GSO projection projects on one of the two chiralities and results in a single two-component spinor in three dimensions. From the right-movers we get another two-component spinor and this is how we arise at $N = 2$ supersymmetry in three dimensions.[17]

If we just quantize the seven-dimensional sigma model, the previous concept suggests that we get two copies of each R representation, together with a label $\pm$. The natural interpretation from the point of view of the tri-critical Ising model, is that $\pm$ corresponds to the decomposition of R ground states into two conformal blocks. In this way, the fusion rules of the tri-critical Ising model can be made to agree with the fermion number assignment, up to an extra minus sign for the product of two fields in the RR sector. For example,

$$\left[\frac{7}{16}, \pm\right] \otimes \left[\frac{7}{16}, \mp\right] = [0, +]$$
$$\left[\frac{7}{16}, \pm\right] \otimes \left[\frac{7}{16}, \pm\right] = \left[\frac{3}{2}, -\right]$$
$$\left[\frac{7}{16}, \pm\right] \otimes \left[\frac{3}{80}, \mp\right] = \left[\frac{6}{10}, +\right]$$
$$\left[\frac{7}{16}, \pm\right] \otimes \left[\frac{3}{80}, \pm\right] = \left[\frac{1}{10}, -\right], \tag{7.2}$$

etc.

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[17]Notice that this also resolves the peculiar feature that representations in the R sector (discussed in Appendix C) of the G_2 algebra can be one-dimensional, but once we combine left- and right-movers they should be two-dimensional. As this shows, the R sector really involves two-dimensional representations, and the left-right sector four-dimensional ones. No strange enhancement is necessary once we combine left- and right-movers.

Using these fusion rules, it is easy to see that tree level correlation functions only vanish if the total $(-1)^F$ of the operators in the correlation function is equal to $(-1)^p$, where $p = n_R/2$ is half the number n_R of R fields. This applies to both the left- and right-movers separately. At higher genus correlation functions also involve a choice of spin structure.

We can now also properly define operators like $G^\downarrow$ and $G^\uparrow$ in the R sector. We decompose the R Hilbert space as

$$\mathcal{H}_R \equiv \mathcal{H}_{R,1} \oplus \mathcal{H}_{R,2} \oplus \mathcal{H}_{R,3} \oplus \mathcal{H}_{R,4} = \mathcal{H}_{7/16,+} \oplus \mathcal{H}_{3/80,-} \oplus \mathcal{H}_{3/80,+} \oplus \mathcal{H}_{7/16,-} \tag{7.3}$$

and define the up and down projections exactly as in the case of the NS sector in terms of the action on $\mathcal{H}_i$. For example, $G^\downarrow$ will only map $\mathcal{H}_i \to \mathcal{H}_{i+1}$.

7.2 Relation of the topological G_2 string to physical amplitudes

An important application of topological strings stems from the realization [1, 30, 31] that its amplitudes agree with certain amplitudes of the physical superstring. The usual topological strings on Calabi–Yau manifolds compute F-terms in four-dimensional compactification of the physical superstrings. A natural question is: What physical amplitudes does the topological G_2 string compute in three-dimensional $\mathcal{N} = 2$ compactifications of superstring theories? As we will see, at genus zero, the topological string indeed computes certain Yukawa couplings. However, at higher genus, unlike the usual topological string theories, the topological G_2 string does not compute F-terms in three dimensions. As we will see, this failure to compute such terms can be traced to the absence of chiral spinors in three dimensions.

Comactification of type II superstrings on G_2 holonomy manifolds leads to $\mathcal{N} = 2$ supergravity in three dimensions, where a single supercharge arises from each world sheet chirality. The (e.g., left moving) supersymmetry generator is constructed according to the standard FMS ansatz [40]

$$Q^\alpha = \oint e^{-\varphi/2} \Big(S^\alpha_{3+} \Sigma_+ + S^\alpha_{3-} \Sigma_- \Big) \tag{7.4}$$

where $S_{3\pm}$ is a spin-field in $R^{1,2}$ (corresponding to the states $|3, \pm\rangle$ in Section 7.1) and $\Sigma_\pm$ are operators corresponding to the states $|7, \pm\rangle$ in Section 7.1. Also, φ is the bosonized super-ghost arising in the standard BRST quantization of type II superstrings.

Which physical amplitudes can we possibly relate to the topological string? These should be amplitudes involving Ramond sector vertex operators which, in their G_2 factor have the field Σ_- inserted an appropriate number of times to give a topological amplitude.[18] In addition, in order to have some non-trivial dynamics in three dimensions, we need a field which sits in $(\mathbf{3}, \mathbf{1})$ of $SO(3) \times G_2 \subset SO(10)$. A singlet under the $SO(3)$ factor would imply a non-dynamical degree of freedom in three dimensions.

The RR sector. The RR vertex operators have spinor bilinears. We are looking for singlets under G_2. These will come from the spinor bilinears made out of the covariantly constant spinor on the G_2 manifold. As discussed before, this can only generate a three-form or a four-form. All other combinations vanish. Then, there remains a unique field which sits in the $(\mathbf{3}, \mathbf{1})$ of $SO(3) \times G_2$. For types IIA and IIB, this corresponds to a scalar field ρ such that

$$\text{type IIA} \quad \partial_\mu \rho = \int_{M_7} F_{RR}^{(4)} \wedge *\phi,$$

$$\text{type IIB} \quad \partial_\mu \rho = \int_{M_7} F_{RR}^{(5)} \wedge \phi. \tag{7.5}$$

where ϕ is the three-form that defines the G_2 structure. The vertex operator (in type IIB) corresponding to these spacetime fields in the $\frac{-1}{2}$ picture is

$$V^i = e^{-(\varphi + \tilde\varphi)/2}\left(S_{3+}^\alpha (\tau_{\alpha\beta}^i) \tilde S_{3+}^\beta \Sigma_+ \tilde\Sigma_+ + S_{3-}^\alpha (\tau_{\alpha\beta}^i) \tilde S_{3-}^\beta \Sigma_- \tilde\Sigma_- \right) \tag{7.6}$$

where (non)tilde denotes (left) right-movers and τ^i are the Pauli matrices.[19]

At first sight, it might seem that $2g - 2$ insertions of this operator would twist the G_2 part of the CFT by appropriate insertions of the spin field Σ_-. However, this is of course incorrect, because the vertex operator in (7.6) is a sum of two terms. Therefore, in addition to getting terms with $\Sigma_-^{2g-2} \tilde\Sigma_-^{2g-2}$ which can be simply related to the topological amplitudes, we get terms with $\Sigma_+^{2g-2} \tilde\Sigma_+^{2g-2}$ insertions and also all possible cross-terms which are non-topological in nature. At a generic genus, generally these non-topological

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[18] In the case of Calabi–Yau three-folds, analogous amplitudes which are related to the topological string consist of $2g - 2$ graviphotons, which suggests a F-term in the four-dimensional effective action of the form W^{2g}, where W is the Weyl super-multiplet of $\mathcal{N} = 2$ supergravity. Here, W is the chiral superfield of $\mathcal{N} = 2$ supergravity multiplet whose first component is the graviphoton field strength $T_{\mu\nu}$. In components, the W^{2g} term gives a coupling between two gravitons and $2g - 2$ graviphotons: $R^2 T^{2g-2}$, and it can be shown that the coefficient of this term is the topological string partition function $F_g(t, \bar t)$.

[19] For type IIA, we need to change $\tilde\Sigma_\pm$ to $\tilde\Sigma_\mp$.

terms are non-vanishing, with the result that the total amplitude is non-topological in nature. For type II strings on Calabi–Yau manifolds, there is a natural way to restrict to one of the two terms in such a vertex operator 7.6, and that is by looking at self-dual (or anti-self-dual) graviphoton field strengths. In three dimensions, there is no natural way to restrict to one of the two terms in the vertex operator. Therefore, we conclude that generically, the topological string does not seem to compute F-terms in the three-dimensional effective action. There is an exception, though, at genus 0.

7.3   Tree level effective action and the topological G_2 string

In order to describe the three-dimensional effective action it is convenient to first work with 11-dimensional supergravity compactification on G_2 manifolds down to four dimensions. The three-dimensional action can then be obtained by a dimensional reduction. The four-dimensional theory has b_3 chiral multiplets and b_2 vector multiplet. The scalars in the chiral multiplets are complex combinations of the metric moduli and the three form 11-dimensional C-field moduli: $S^A = t^A + ip^A$, where p^A is defined in footnote 16. The Kähler potential for the scalars is a function of the real part of S^A and is given by [37]

$$K(S + \bar{S}) = -3 \log \left(\frac{1}{7} \int \phi \wedge *\phi \right) \tag{7.7}$$

The kinetic terms for the b_2 gauge fields are given by

$$\text{Im} \int d^4x d^2\theta \tau_{ab} W^a_\alpha W^{\alpha b} \tag{7.8}$$

which can be dimensionally reduced to three dimensions

$$S = \text{Im} \int d^3x d^2\theta \tau_{ab} W^a_\alpha W^{b\alpha} \tag{7.9}$$

where W^a_α is the field strength superfield, the gauge coupling is $\tau_{ab} = S^A \partial_A \partial_a \partial_b \left(\frac{36}{7} \mathcal{I}_{\text{tot}} \right)$, where $\mathcal{I}_{\text{tot}}$ is defined in equation (5.54) and $\partial_a = \frac{\partial}{\partial s^a}$.

   This action is written in terms of dimensionally reduced four-dimensional vector multiplet as an integral over a chiral half of superspace. In three dimensions, vector multiplets are dual to the chiral multiplet and it is interesting to determine the Kähler potential for these chiral multiplets. To this end, we need to perform the duality transformation and it is convenient to do this directly in superspace. Four-dimensional vector multiplets are not the most convenient way to define gauge theories in three dimensions. Gauge theories in three dimensions are usually formulated in terms of linear

multiplets. We therefore first rewrite (7.9) in terms of linear multiplets G^a in terms of which the action becomes

$$S = \int d^3x d^4\theta (\tau_{ab}(S) + \bar{\tau}_{ab}(\bar{S})) G^a G^b. \tag{7.10}$$

We can write the B-field as $G^a \omega_a$ and $\phi = (S^A + \bar{S}^A)\chi_A$, where ω_a and χ_A are bases of H^2 and H^3 respectively, of the G_2 manifold. Then, the superspace action can be formally written as

$$S = \int d^3x d^4\theta \int B \wedge B \wedge \phi \tag{7.11}$$

which is exactly the second term which appears in $\mathcal{I}_{\text{total}}$.

To perform the duality transformation explicitly between the linear and the chiral multiplets (see e.g., [41]), we can even start from a more general action

$$S = \int d^3x d^4\theta f(G^a, S, \bar{S}). \tag{7.12}$$

This action can be rewritten as

$$S = \int d^3x d^4\theta f(\tilde{G}^a, S, \bar{S}) - \tilde{G}^a (Y_a + \bar{Y}_a) \tag{7.13}$$

where the superfields $\tilde{G}^a$ are unconstrained real superfields, and the Y_a are chiral superfields. Extremizing the action with respect to Y_a constrains $\tilde{G}^a$ to be linear superfields from which we obtain (7.12) again. We can also vary this action with respect to $\tilde{G}^a$ which yields the equation

$$Y_a + \bar{Y}_a = \frac{\partial f(\tilde{G}^a, S, \bar{S})}{\partial \tilde{G}^a}. \tag{7.14}$$

By solving for $\tilde{G}^a$ in terms of S and $\bar{S}$ and substituting in (7.13) gives the dual description in terms of a Kähler potential $K(Y_a + \bar{Y}_a, S, \bar{S})$ for the chiral multiplets Y_a:

$$S = \int d^3x d^4\theta K(Y_a + \bar{Y}_a, S, \bar{S}). \tag{7.15}$$

Here, K is the Legendre transform of f. For our case (7.10), $f = \left(\tau_{ab}(S) + \bar{\tau}_{ab}(\bar{S})\right)\tilde{G}^a \tilde{G}^b$, so

$$K(Y_a + \bar{Y}_a, S, \bar{S}) = (Y_a + \bar{Y}_a)\left(\Re\tau(S)^{-1}\right)^{ab}(Y_b + \bar{Y}_b). \tag{7.16}$$

This is simply the Legendre transform of (7.11) with respect to the B-field moduli.

8   Discussion, open questions, and future directions

In this concluding section, we list and discuss several interesting issues and future directions.

8.1   The coupling constant

The partition function for the ordinary topological string on Calabi–Yau manifolds is better thought of as a wave function. This picture emerges from the holomorphic anomaly, where the holomorphic anomaly equation is interpreted as describing the change in basis (an infinitesimal fourier transform) in the quantum mechanics whose phase space is given by $H^3(M)$ [35]. It remains an interesting question whether the partition function of our topological string should naturally have a wave function interpretation. In our case, there is no corresponding holomorphic anomaly equation. Also, when we consider our topological string on $\text{CY} \times S^1$, it naturally contains both the holomorphic and anti-holomorphic A and B models. These facts suggest an interpretation as a partition function as opposed to a wave function.

However, we also argued in Section 5.6 that we could view the topological G_2 string as a wavefunction corresponding to a Lagrangian submanifold of $H^2 + H^3 + H^4 + H^5$. From this perspective, it is interesting to note that we can naturally incorporate the string coupling in the framework. Consider again our function

$$\mathcal{I} = \frac{1}{g_s^2} \int \phi \wedge *\phi + \frac{7}{24g_s^2} \int B \wedge B \wedge \phi \qquad (8.1)$$

where we have now included the string coupling constant. We can associate to it a Lagrangian submanifold of $H^*(M)$ which now also includes H^0 and H^7, namely

$$\left(\frac{1}{g_s}, B, \phi, \frac{\partial \mathcal{I}}{\partial \phi}, \frac{\partial \mathcal{I}}{\partial B}, \frac{\partial \mathcal{I}}{\partial \frac{1}{g_s}} \right). \qquad (8.2)$$

In this way the string coupling gets naturally associated to $H^0(M)$. This is similar to what is done in the A model. In the B model, the string coupling is related to one particular component of H^3, namely the one proportional to the holomorphic three-form. At first sight, it does not seem to be the case here. However, as discussed in Appendix D, there is an isomorphism between H^0 and H_1^3, i.e., those elements of the third cohomology which transform as the singlet under the group G_2. The moduli space has a projective structure. We can view the t^A defined in (5.26) as providing real projective coordinates on the $b_{27}^3 = b_3 - 1$ dimensional moduli space of G_2 metrics which correspond

to deformations of the G_2 structure which are not rescalings of the metric. The partition function of the topological G_2 string is then a section of a real-line bundle of degree $\frac{7}{3}$. Though this is not the structure that we find in the topological string, it may naturally emerge when we try to lift it to M-theory.

8.2 Strong coupling limit

The construction of the topological string theory that we have given is a perturbative one. The strong coupling limit and a non-perturbative completion remains an interesting question. A strong coupling limit, if well defined, could naturally be topological M-theory [3–5]. An obvious strong coupling limit is one where we scale ϕ with $\lambda^{3/7}$ and g_s with λ, after which we send $\lambda \to \infty$. This does not change the form of $\mathcal{I}$. It is not clear whether the result should be viewed as a string theory. In fact, it is perhaps more appropriate to think of this topological theory as describing certain sector of M-theory compactification on G_2 manifolds down to four dimensions. The number of variables that remain will be one-less compared to the number of variables in three dimensions—we lose the degree of freedom corresponding to the rescaling of ϕ, the three-form which defines the G_2 structure; or equivalently, the string coupling.

Another limit we can study is the theory on $CY \times S^1$. In this case we can try to decompactify the S^1, which is related via a 9–11 flip to the strong coupling limit before. Since R depends non-trivially on all moduli, it is not immediately clear what is a natural set of variables that survives. Perhaps we should keep all H^3 except the class proportional to ϕ, as we do for the complex structure in the B-model?

8.3 Relation to black holes and Hitchin flows

Notice that our function $P(q^I, p_I)$ (5.82) is the Legendre transform of the free energy of the B-model, which is exactly the expression that appears in the recent discussions of the relation between topological strings and black hole entropy [42]. This is perhaps not that surprising given that $P(q^I, p_I)$ is the volume of the CY at the horizon of the black hole through the attractor mechanism. Yet, one may wonder whether the circle in the seven-dimensional theory on $CY \times S^1$ can be interpreted as a Euclidean time direction so that the theory can be directly viewed as a thermal system with non-zero entropy, giving a microscopic description of the black hole

entropy. Perhaps our topological twist can be interpreted as counting BPS states in a black hole background.

In [43], domain wall solutions of $\mathcal{N} = 2$ gauged four-dimensional super-gravity were constructed, where the supergravity theory was obtained by the dimensional reduction of type IIA on "half-flat" six manifolds. These are manifolds which have a particular type of SU(3) structure. The domain walls are determined by flow equations which govern the dependence of scalars (corresponding to the moduli of the internal manifold) in the direction transverse to the domain wall. These flow equations were shown to be equivalent to Hitchin's flow equations, which implies that the transverse direction to the domain wall combines with the internal manifold to give a G_2 manifold. A natural question is whether the black hole attractor flows have a similar interpretation in terms of Hitchin flows which may then admit a re-interpretation of these in terms of a manifold with G_2 structure. We leave this interesting point for a future investigation.

Notice that in M-theory on G_2 manifolds there are no supersymmetric black holes, so we do not expect the existing relation between topological strings and BPS black holes to generalize to this setup.

8.4   An analog of KS theory?

The topological A and B models are defined perturbatively in an on-shell formalism which studies maps from the world sheet to a target space. Perturbative computations can be done using world-sheet methods. However, for the B-model, there is a target space "string field theory" (though for the B-model, this reduces to a field theory), namely the Kodaira Spencer theory which presumably yields exactly the same results as the world-sheet calculations. This is a theory of complex structure deformations of the Calabi–Yau manifold. The fundamental variable of Kodaira Spencer theory corresponds to an infinitesimal change of the complex structure of the Calabi–Yau manifold. The equation of motion of this theory is equivalent to the complex structure being integrable. The action, which can be written down by following the standard rules of string field theory [44], consists of a quadratic kinetic term and a cubic interaction term. There are no higher point interaction terms since four and higher point correlation functions in the world-sheet theory vanish.

One may hope that the target space theory of the topological G_2 string is a seven-dimensional theory of deformations of G_2 structures, a version of the Kodaira Spencer theory that lives in seven dimensions. The fundamental

variable should be an infinitesimal metric deformation, i.e., a symmetric two-tensor $A_{\mu\nu}$. If we again follow the standard string field theory logic, the action would take the form

$$S = S_2(A) + S_3(A) \tag{8.3}$$

with $S_2(A) \sim \int A \frac{G_0^-}{b_0^-} A = \int A \frac{G_0^\downarrow}{G_{*0}^\uparrow} A$ and with

$$S_3(A) = \int d^7 x \sqrt{g} \phi^{\alpha\beta\gamma} A_{\alpha\alpha'} A_{\beta\beta'} A_{\gamma\gamma'} \phi^{\alpha'\beta'\gamma'}. \tag{8.4}$$

The equation of motion of this theory, if correct, should correspond to the equation for integrability of A to a G_2 metric. Such a quadratic equation is unknown to us so it would be interesting to study further. Notice that for the A-model such a simple cubic theory does not exist.

There is yet another theory in the case of the B-model which has been proposed as a possible equivalent space-time theory, which is a six-dimensional Hitchin functional. This is proposed in [4] and studied and refined in [38]. In the latter paper it is also pointed out that the six-dimensional Hitchin theory has a one-loop gravitational anomaly which again suggests that complex and Kähler moduli cannot be treated independently. This agrees well with the analysis of our model on $CY \times S^1$ and clearly it is worth trying to understand whether our theory on $CY \times S^1$ is free of any such one-loop anomalies. What is confusing and begs for clarification is the fact that the six-dimensional theory has a Kodaira Spencer formulation and a Hitchin formulation and both are supposed to reproduce the prepotential (see also [3]), whereas in seven dimensions, we only have the prepotential itself and that is the Hitchin functional. It would be quite interesting if the seven-dimensional Hitchin functional would also be the effective spacetime theory, since that would mean that prepotential obtained from Hitchin's functional would again be Hitchin's functional. We clearly need to sort all this out if we want to make progress in "topological M-theory" (see also [3–5]).

8.5   Branes

Though our theory does not have world-sheet instantons (since there are no supersymmetric two-cycles), it does have supersymmetric branes, namely $0, 3, 4$ and 7 branes, that will give rise to non-perturbative corrections. Presumably, the formulation of topological M-theory is in terms of topological membranes. However, strings and membranes are dual in seven dimensions. It is for these reasons that the 3 brane is specially interesting. Its world-volume theory is a candidate topological membrane theory that might

give rise to an alternative definition of a seven-dimensional theory (see also [45,46] for further discussions of membranes in G_2 manifolds). In some examples one can see that membranes should play an important role. For example, if one considers topological strings on orientifolds of CY compactifications, one finds a version of Gromow–Witten invariants coming from oriented and unoriented string world-sheets. As the theory is equivalent to M-theory on $(CY \times S^1)/\mathbb{Z}_2$, from the M-theory point of view we are counting membranes wrapping the S^1 [47]. We leave a detailed discussion of the branes in the theory to a future publication.

8.6   Open problems and future directions

There are several further open problems. Perhaps the most important one is to find a twisted stress tensor which is crucial for the definition of the topological string beyond genus zero. It is also interesting to understand the geometric meaning of the higher genus amplitudes. In the case of the A-model, the higher genus amplitudes roughly compute the number of holomorphic maps from a genus g Riemann surface into the Calabi–Yau. Such an interpretation is less clear for the B-model for $g > 1$ (the genus 0 result reproduce the special geometry relations and the genus 1 result is related to the holomorphic Ray–Singer torsion). For example, are there interesting indices (like the elliptic genus) that we can define and study in this context? Perhaps related to this, we would like to understand better the localization arguments.

Mirror symmetry for G_2 manifolds will be interesting to investigate in the context of our topological twist. A version of mirror symmetry for G_2 manifolds was studied in [14, 19, 20, 23]. In [20], an analog of Witten index was introduced that counts the total number of ground states and not just ground states weighted with $(-1)^F$, where F counts the fermion number. This was defined by using a Z_2 automorphism L of the G_2 algebra under which the currents K and Φ change signs, and the index was defined as $\mathrm{Tr}(L(-1)^F)$. This index will count the total number of chiral primary states in our topological theory. In fact, in [23], it was argued that acting with L in the left sector and the identity in the right sector corresponds to the mirror automorphism of the G_2 algebra, which can then be geometrically interpreted as mirror symmetry for G_2 manifolds.

We list several other related questions that still remain open. For example, are there other relations to the low energy effective action? Is there a Berkovits formulation in three dimensions? Is the Dolbeault-like complex for G_2 manifolds that corresponds to the BRST cohomology in the left or

the right sector useful in other contexts? It is also perhaps worthwhile to investigate more concrete world-sheet models of theories based on the G_2 algebra, e.g., using minimal models and discrete torsion, see e.g., [15–23]. It is also interesting to extend this construction to more general setting which involve turning on the NS–NS background fields. As discussed in [48], this setup involves a study of $G_2 \times G_2$ structures, and it would be interesting to understand how our topological twist is modified in this context.

A natural extension of this work is to study topological strings on spin(7) manifolds. This may reveal interesting extensions of Hitchin's functionals to such manifolds.

Acknowledgments

Appendix A   The Coulomb gas representation

A useful (though subtle) representation of minimal models is the "Coulomb gas" representation. Much of the evidence pointing at a possible topological twisting for G_2 manifolds was constructed in [8] using this approach. For reasons that will become apparent defining the topological theory in this representation is very difficult. Although we proceeded in the main text to define the topological construction in an independent way which avoids many of the complications of the Coulomb gas representation, we summarize it here for completeness as well as for a useful source of intuition for the results we obtained in the main text.

In the Coulomb gas representation minimal model primaries are represented as vertex operators in a theory of a scalar coupled to a background charge. The holomorphic energy momentum tensor in such theories is given by

$$T(z) = -\frac{1}{2} \left(\partial\phi(z)\partial\phi(z) + iQ\partial^2\phi(z) \right) \tag{A.1}$$

with central charge

$$c = 1 - 3Q^2. \tag{A.2}$$

Primaries are the "vertex operators"

$$V_{n'n}(z) \equiv e^{i\alpha_{n'n}\phi(z)} \tag{A.3}$$

where

$$\alpha_{n'n} = \frac{1}{\sqrt{2}}[(n'-1)\alpha_- + (n-1)\alpha_+]. \tag{A.4}$$

The conformal dimension of these operators

$$h(V_{n'n}) = \frac{1}{2}\alpha_{n'n}(\alpha_{n'n} + Q). \tag{A.5}$$

In the tri-critical Ising model we choose $Q = \frac{1}{\sqrt{10}}$ which sets $\alpha_+ = \frac{4}{\sqrt{10}}$ and $\alpha_- = -\frac{5}{\sqrt{10}}$ and one can easily verify that (A.5) correctly reproduces the conformal weights inside the tri-critical Ising model.

An important subtlety arises because one can construct two weight 1 vertex operators $V_\pm \equiv V_{\pm 1, \mp 1} = e^{-i\sqrt{2}\alpha_\pm}$ called screening operators. Integrating $V_\pm$ against the vertex operators (A.3) gives screened vertex operators which have the same conformal weight as (A.3) but a different "charge" under $\phi \to \phi + \text{const}$. More precisely, these operators are defined as

$$V_{n'n}^{r'r}(z) = \int \prod_{i=1}^{r'} du_i \prod_{j=1}^{r} dv_j V_{n'n}(z) V_+(u_1) \cdots V_+(u_{r'}) V_-(v_1) \cdots V_-(v_r) \tag{A.6}$$

where the contours of the u and v integrations have been defined carefully in [27]. Each screened vertex operator $V_{n'n}^{r'r}$ corresponds to a different conformal block of the operator $V_{n'n}$. So, e.g., in (2.12), the two conformal blocks, in the Coulomb gas picture are given by

$$\Phi_{2,1}^\uparrow = P_\vdash V_{21}^{10} P_\vdash + P_\dashv V_{21}^{00} P_\dashv, \quad \Phi_{2,1}^\downarrow = P_\vdash V_{21}^{00} P_\vdash + P_\dashv V_{21}^{10} P_\dashv \tag{A.7}$$

where we have been careful to put in projectors $P_\vdash$ and $P_\dashv$. $P_\vdash$ projects to the states corresponding to the first column of the Kac table and the first two entries of the second column, whereas $P_\dashv$ projects to the last two entries of the middle column and the third column of the Kac table. In this way we unambiguously embed the minimal model Hilbert space in the Hilbert space of the scalar field. Similarly, for the conformal blocks of $\Phi_{1,2}$ we have the following Coulomb gas representations:

$$\Phi_{1,2}^+ = P_\vdash V_{12}^{00} P_\vdash + P_\dashv V_{12}^{01} P_\dashv, \quad \Phi_{1,2}^- = P_\vdash V_{12}^{01} P_\vdash + P_\dashv V_{12}^{00} P_\dashv. \tag{A.8}$$

In the Coulomb gas representation of the tri-critical Ising model, the field ϕ has a background charge $Q = \frac{1}{\sqrt{10}}$. If we just consider the subspace of the Hilbert space corresponding to the projection $P_\vdash$, we can write

$P_\vdash V_{12}^{00} P_\vdash = e^{i\frac{5}{2\sqrt{10}}\phi}$ and then in this sector, insertions of two Σ fields on a sphere effectively changes the background charge from

$$Q = \frac{1}{\sqrt{10}} \to \frac{6}{\sqrt{10}}. \tag{A.9}$$

The central charge of the total CFT changes from $c = \frac{21}{2}$ to zero:

$$c = \frac{3}{2} \times 7 = \frac{7}{10} + \frac{98}{10} \to 1 - 3\left(\frac{6}{\sqrt{10}}\right)^2 + \frac{98}{10} = 0 \tag{A.10}$$

which hints strongly at the existence of a topological theory.

Changing the background charge changes the weights of various fields. The change in weight depends on the charge of the field. In fact, since different conformal blocks of the same field carry different charges, their weights shift by different amounts after the twist. *The twisting acts differently on the conformal blocks of the same operator.* For example, the new weights of some of the blocks after the twist are

$$\begin{aligned} G^\downarrow \to 1, \quad & G^\uparrow \to 2 \\ M^\downarrow \to 2, \quad & M^\uparrow \to 3. \end{aligned} \tag{A.11}$$

Using (A.5) one finds the conformal weights of Coulomb gas vertex operators in the twisted theory shifted

$$V_{21}^{00} = e^{-2i/\sqrt{10}}, V_{31}^{00} = e^{-4i/\sqrt{10}} \to -\frac{2}{5}$$

$$V_{31}^{00} = e^{-6i/\sqrt{10}}, \mathbf{1} \to 0$$

$$V_{21}^{10} \sim e^{2i/\sqrt{10}} \to \frac{3}{5}. \tag{A.12}$$

Notice that the blocks corresponding to the unscreened vertex operators in the Coulomb gas representation, dressed with the appropriate weight in the remainder CFT of the "chiral" states (3.1) become weight 0 after the twist. Similar arguments were used in [8].

A few words about the Coulomb gas approach are however in order. The Hilbert space of the free theory with a background charge is larger than that of the minimal model. To go from the free theory to the minimal model, we need to consider cohomologies of approach BRST operators defined by Felder [27]. So while the Coulomb gas representation is useful in doing computations, it cannot be used to construct new operators unless they commute with Felder's BRST operators. We thus emphasize that these arguments should be taken as inspirational rather than rigorous.

Appendix B   The G_2 algebra

The G_2 algebra is given by [8]

$$\{G_n, G_m\} = \frac{7}{2}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0} + 2L_{n+m} \tag{B.1}$$

$$[L_n, L_m] = \frac{21}{24}(n^3 - n)\delta_{n+m,0} + (n - m)L_{n+m} \tag{B.2}$$

$$[L_n, G_m] = \left(\frac{1}{2}n - m\right)G_{n+m} \tag{B.3}$$

$$\{\Phi_n, \Phi_m\} = -\frac{7}{2}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0} + 6X_{n+m} \tag{B.4}$$

$$[X_n, \Phi_m] = -5\left(\frac{1}{2}n - m\right)\Phi_{n+m} \tag{B.5}$$

$$[X_n, X_m] = \frac{35}{24}(n^3 - n)\delta_{n+m,0} - 5(n - m)X_{n+m} \tag{B.6}$$

$$[L_n, X_m] = -\frac{7}{24}(n^3 - n)\delta_{n+m,0} + (n - m)X_{n+m} \tag{B.7}$$

$$\{G_n, \Phi_m\} = K_{n+m} \tag{B.8}$$

$$[G_n, K_m] = (2n - m)\Phi_{n+m} \tag{B.9}$$

$$[G_n, X_m] = -\frac{1}{2}\left(n + \frac{1}{2}\right)G_{n+m} + M_{n+m} \tag{B.10}$$

$$\{G_n, M_m\} = -\frac{7}{12}\left(n^2 - \frac{1}{4}\right)\left(n - \frac{3}{2}\right)\delta_{n+m,0}$$
$$+ \left(n + \frac{1}{2}\right)L_{n+m} + (3n - m)X_{n+m} \tag{B.11}$$

$$[\Phi_n, K_m] = \frac{3}{2}\left(m - n + \frac{1}{2}\right)G_{n+m} - 3M_{n+m} \tag{B.12}$$

$$\{\Phi_n, M_m\} = \left(2n - \frac{5}{2}m - \frac{11}{4}\right)K_{n+m} - 3 : G\Phi :_{n+m} \tag{B.13}$$

$$[X_n, K_m] = 3(m + 1)K_{n+m} + 3 : G\Phi :_{n+m} \tag{B.14}$$

$$[X_n, M_m] = \left[\frac{9}{4}(n + 1)\left(m + \frac{3}{2}\right) - \frac{3}{4}\left(n + m + \frac{3}{2}\right)\left(n + m + \frac{5}{2}\right)\right]G_{n+m}$$
$$- \left[5(n + 1) - \frac{7}{2}\left(n + m + \frac{5}{2}\right)\right]M_{n+m} + 4 : GX :_{n+m} \tag{B.15}$$

$$[K_n, K_m] = -\frac{21}{6}(n^3 - n)\delta_{n+m,0} + 3(n - m)(X_{n+m} - L_{n+m}) \tag{B.16}$$

$$[K_n, M_m] = \left[\frac{11}{2}(n+1)\left(n+m+\frac{3}{2}\right) - \frac{15}{2}(n+1)n\right]\Phi_{n+m}$$
$$+ 3 : GK :_{n+m} -6 : L\Phi :_{n+m} \tag{B.17}$$

$$\{M_n, M_m\} = -\frac{35}{24}\left(n^2 - \frac{1}{4}\right)\left(n^2 - \frac{9}{4}\right)\delta_{n+m,0} + \left[\frac{3}{2}(n+m+2)(n+m+3)\right.$$
$$\left. - 10\left(n+\frac{3}{2}\right)\left(m+\frac{3}{2}\right)\right]X_{n+m} + \left[\frac{9}{2}\left(n+\frac{3}{2}\right)\left(m+\frac{3}{2}\right)\right.$$
$$\left. - \frac{3}{2}(n+m+2)(n+m+3)\right]L_{n+m}$$
$$- 4 : GM :_{n+m} +8 : LX :_{n+m}. \tag{B.18}$$

An important property of the algebra is the fact that it contains a null ideal, generated by $[10, 11]$

$$\mathcal{N} = 4(GX) - 2(\Phi K) - 4\partial M - \partial^2 G. \tag{B.19}$$

This null ideal has various consequences. For example, it allows us to determine the eigenvalue of K_0 on highest weight states in terms of their L_0 and X_0 eigenvalues. Thus, K_0 is not an independent quantum number in the theory.

In [10] a two-parameter family of chiral algebras was found, with the same generators as the G_2 algebra. However, the G_2 algebra is the only one among this family which has the right central charge $c = 21/2$ and contains the tri-critical Ising model as a sub-algebra. The latter is needed for space-time supersymmetry, and therefore the G_2 algebra appears to be uniquely fixed by these physical requirements.

The representation theory of the G_2 algebra was studied in some detail in [12]. Both in the NS and R sectors there are short and long representations. We will discuss the representations of the latter in the next Section (Appendix C). In the NS sector the short representations correspond to what we called chiral primaries, whereas in the R sector the short representations correspond to R ground states.

Character formulae for the G_2 algebra are unknown. In [15] the partition functions for string theory on particular non-compact G_2 manifolds were found, and from these one can extract candidate character formulas for some of the representations of the G_2 algebra. It would be nice to have general explicit expressions for the characters. One may try to obtain these by using the fact that the G_2 algebra can be obtained by quantum Hamiltonian reduction (see e.g., [49]) from the affine super Lie algebra based on $D(2, 1, \alpha)$, as suggested in [50]. Following the strategy in [51] one expects that the characters can be expressed in terms of highest weight characters

of the $D(2,1,\alpha)$ affine super Lie algebra, but we have not explored this in this paper.

Appendix C   R sector

In this section we will be completely pedantic. In the R sector we have the following commutation relations of the zero modes (L_0 commutes with everything)

$$\{G_0, G_0\} = 2\left(L_0 - \frac{7}{16}\right)$$
$$\{G_0, \phi_0\} = K_0$$
$$[G_0, X_0] = -\frac{1}{4}G_0 + M_0$$
$$\{G_0, M_0\} = \frac{1}{2}\left(L_0 - \frac{7}{16}\right) \tag{C.1}$$
$$[G_0, K_0] = K_0$$

$$[X_0, K_0] = \frac{3}{2}K_0 - 3\phi_0 G_0$$
$$[X_0, \phi_0] = 0$$
$$[X_0, M_0] = \frac{21}{16}G_0 - \frac{9}{4}M_0 + 4G_0 X_0 \tag{C.2}$$

$$[K_0, \phi_0] = -\frac{3}{4}G_0 + 3M_0$$
$$[K_0, M_0] = 3G_0 K_0 - 6\phi_0\left(L_0 - \frac{7}{16}\right)$$
$$\{\phi_0, \phi_0\} = \frac{7}{8} + 6X_0$$
$$\{\phi_0, M_0\} = \frac{7}{4}K_0 - 3G_0\phi_0$$
$$\{M_0, M_0\} = \frac{21}{8}\left(L_0 - \frac{7}{16}\right) + 8\left(L_0 - \frac{7}{16}\right)X_0 - 4G_0 M_0. \tag{C.3}$$

In addition, there is the operator

$$\mathcal{N} = \tfrac{3}{2}M_0 - 3K_0\phi_0 + 6G_0 X_0 \tag{C.4}$$

which should be null when acting on highest weight states. To extract this algebra from the operator product expansion one needs to use a suitable normal ordering prescription. One may check that this algebra is consistent with hermiticity, associativity and yields the right spectrum for X_0.

To build representations, we first consider a highest weight vector of the form $|7/16, h_r\rangle$. One may check that $(\frac{7}{4}G_0 + M_0)|7/16, h_r\rangle$ has X_0 eigenvalue equal to $-99/16$. This is outside the Kac table for the tri-critical Ising model. Therefore, this vector has to be null. Given this null vector, we find that the representation a priori has four states remaining. Notice that, as we will discuss momentarily, these representations may still be reducible.

We introduce the basis

$$
\begin{pmatrix}
\left|\dfrac{7}{16}, h_r\right\rangle \\[2ex]
\left(-\dfrac{17}{4}G_0 + M_0\right)\left|\dfrac{7}{16}, h_r\right\rangle \\[2ex]
\phi_0\left|\dfrac{7}{16}, h_r\right\rangle \\[2ex]
\left(-\dfrac{17}{4}G_0 + M_0\right)\phi_0\left|\dfrac{7}{16}, h_r\right\rangle
\end{pmatrix}.
\tag{C.5}
$$

In this basis the various generators look like (with $\hat{l} = L_0 - \frac{7}{16}$)

$$
G_0 = \begin{pmatrix}
0 & -6\hat{l} & 0 & 0 \\[1ex]
-\dfrac{1}{6} & 0 & 0 & 0 \\[1ex]
0 & 0 & 0 & -6\hat{l} \\[1ex]
0 & 0 & -\dfrac{1}{6} & 0
\end{pmatrix}
$$

$$
M_0 = \begin{pmatrix}
0 & -\dfrac{27}{2}\hat{l} & 0 & 0 \\[1ex]
\dfrac{7}{24} & 0 & 0 & 0 \\[1ex]
0 & 0 & 0 & -\dfrac{27}{2}\hat{l} \\[1ex]
0 & 0 & \dfrac{7}{24} & 0
\end{pmatrix}
$$

$$
\phi_0 = \begin{pmatrix}
0 & 0 & -\dfrac{49}{8} & 0 \\[1ex]
0 & 0 & 0 & \dfrac{7}{8} \\[1ex]
1 & 0 & 0 & 0 \\[1ex]
0 & -\dfrac{1}{7} & 0 & 0
\end{pmatrix}
$$

$$
X_0 = \begin{pmatrix} -\dfrac{35}{16} & 0 & 0 & 0 \\ 0 & -\dfrac{3}{16} & 0 & 0 \\ 0 & 0 & -\dfrac{35}{16} & 0 \\ 0 & 0 & 0 & -\dfrac{3}{16} \end{pmatrix}
$$

$$
K_0 = \begin{pmatrix} 0 & 0 & 0 & \dfrac{63}{2}\hat{l} \\ 0 & 0 & \dfrac{7}{8} & 0 \\ 0 & -\dfrac{36}{7}\hat{l} & 0 & 0 \\ -\dfrac{1}{7} & 0 & 0 & 0 \end{pmatrix}. \tag{C.6}
$$

There is a two-parameter family of possible metrics compatible with unitarity, namely

$$
g = \begin{pmatrix} \dfrac{8a}{49} & 0 & -ib & 0 \\ 0 & \dfrac{288a\hat{l}}{49} & 0 & -36i\hat{l}b \\ ib & 0 & a & 0 \\ 0 & 36i\hat{l}b & 0 & 36\hat{l}a \end{pmatrix}. \tag{C.7}
$$

These representations are not irreducible. Indeed, we can go to an eigenbasis of ϕ_0. To do this we define a new basis as

$$
\begin{pmatrix} \dfrac{7i}{\sqrt{8}} & 0 & 1 & 0 \\ -\dfrac{7i}{\sqrt{8}} & 0 & 1 & 0 \\ 0 & -\dfrac{7i}{\sqrt{8}} & 0 & 1 \\ 0 & \dfrac{7i}{\sqrt{8}} & 0 & 1 \end{pmatrix} \begin{pmatrix} \left| \dfrac{7}{16}, h_r \right\rangle \\ \left(-\dfrac{17}{4}G_0 + M_0 \right) \left| \dfrac{7}{16}, h_r \right\rangle \\ \phi_0 \left| \dfrac{7}{16}, h_r \right\rangle \\ \left(-\dfrac{17}{4}G_0 + M_0 \right) \phi_0 \left| \dfrac{7}{16}, h_r \right\rangle \end{pmatrix}. \tag{C.8}
$$

Then the generators become

$$
G_0 = \begin{pmatrix} 0 & 0 & 0 & -6\hat{l} \\ 0 & 0 & -\hat{l} & 0 \\ 0 & -\dfrac{1}{6} & 0 & 0 \\ -\dfrac{1}{6} & 0 & 0 & 0 \end{pmatrix}
$$

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & -\dfrac{27}{2}\hat{l} \\ 0 & 0 & -\dfrac{27}{2}\hat{l} & 0 \\ 0 & \dfrac{7}{24} & 0 & 0 \\ \dfrac{7}{24} & 0 & 0 & 0 \end{pmatrix}$$

$$\phi_0 = \begin{pmatrix} \dfrac{7i}{\sqrt{8}} & 0 & 0 & 0 \\ 0 & -\dfrac{7i}{\sqrt{8}} & 0 & 0 \\ 0 & 0 & \dfrac{i}{\sqrt{8}} & 0 \\ 0 & 0 & 0 & -\dfrac{i}{\sqrt{8}} \end{pmatrix}$$

$$X_0 = \begin{pmatrix} -\dfrac{35}{16} & 0 & 0 & 0 \\ 0 & -\dfrac{35}{16} & 0 & 0 \\ 0 & 0 & -\dfrac{3}{16} & 0 \\ 0 & 0 & 0 & -\dfrac{3}{16} \end{pmatrix}$$

$$K_0 = \begin{pmatrix} 0 & 0 & 0 & -\dfrac{18i}{\sqrt{2}}\hat{l} \\ 0 & 0 & \dfrac{18i}{\sqrt{2}}\hat{l} & 0 \\ 0 & \dfrac{i}{\sqrt{8}} & 0 & 0 \\ -\dfrac{i}{\sqrt{8}} & 0 & 0 & 0 \end{pmatrix}. \tag{C.9}$$

The metric becomes

$$g = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c2 & 0 & 0 \\ 0 & 0 & 36c_2\hat{l} & 0 \\ 0 & 0 & 0 & 36c_1\hat{l} \end{pmatrix} \tag{C.10}$$

where c_1, c_2 are arbitrary constants related to a, b in some way which is not terribly important. We therefore see that the representation splits into two complex conjugate ones which are each two-dimensional. For $\hat{l} \neq 0$ this is the complete story, i.e., the zero modes are represented as two complex

conjugate two-dimensional representations. One is spanned by the first and fourth vectors, the other one by the second and the third.

In the case we have R ground states, i.e., $\hat{l} = 0$, we see that the system degenerates further. We can consistently decouple the third and fourth vectors and find two complex conjugate one-dimensional representations of the algebra. These correspond to the $h_I = \frac{7}{16}R$ ground state that is purely internal. In this representation, $G_0 = M_0 = K_0 = 0$.

The null module generated by the third and fourth vectors also provides two one-dimensional complex conjugate representations. Taking c_1 and c_2 to scale as $\frac{1}{\hat{l}}$, we see that this gives rise to one-dimensional representations of the form $|\frac{3}{80}, \frac{2}{5}\rangle$. In these representations also $G_0 = M_0 = K_0 = 0$.

In short, in the R sector we have massless and massive representations. If we combine the left- and right-movers, things change a little bit. We cannot use eigenvectors of ϕ_0 and $\bar{\phi}_0$ with non-zero eigenvalue simultaneously, since that is inconsistent with $\{\phi_0, \bar{\phi}_0\} = 0$. The smallest unitary representation of this algebra is two-dimensional. Therefore, combining left and right massless representations leads to a two-dimensional representation. Combining massless and massive to a four-dimensional representation, and combining two massive representations to a eight-dimensional representation.

Appendix D   Decomposition of differential forms into irreps of G_2

In this appendix, we review the decomposition of differential forms into irreducible representations of the group G_2. Our discussion follows the one in [52].

For a G_2 manifold, differential forms of any degree can be decomposed into irreducible representations of G_2

$$\Lambda^0 = \Lambda_1^0 \quad \Lambda^1 = \Lambda_7^1$$
$$\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2 \quad \Lambda^3 = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3.$$

This decomposition is compatible with the Hodge star operation, so $*\Lambda_m^n = \Lambda_m^{7-n}$. It is useful to define this decomposition into irreducible representations explicitly.

Two- and five-forms. The two-forms decompose into a **7** and **14** of G_2. These spaces can be characterized as follows:

$$\Lambda^2_7 = \{\omega \in \Lambda^2; *(\phi \wedge \omega) = 2\omega\}$$
$$\Lambda^2_{14} = \{\omega \in \Lambda^2; *(\phi \wedge \omega) = -\omega\}.$$

It is useful to write expressions for projector operators π_7 and π_{14}. These project on to the appropriate subspaces:

$$\pi^2_7(\omega) = \frac{\omega + *(\phi \wedge \omega)}{3}$$
$$\pi^2_{14}(\omega) = \frac{2\omega - *(\phi \wedge \omega)}{3}$$

where the superscript 2 on π^2_k indicates that this is the projector when acting on two-forms. In local co-ordinates, these can be written as

$$(\pi^2_7)^{de}_{ab} = 6\phi^c_{ab}\phi^{de}_c = 4\phi^{de}_{ab} + \tfrac{1}{6}(\delta^d_a\delta^e_b - \delta^e_a\delta^d_b)$$
$$(\pi^2_{14})^{ef}_{ab} = -4\phi^{ef}_{ab} + \tfrac{1}{3}(\delta^e_a\delta^f_b - \delta^d_a\delta^e_b).$$

Similarly, for five forms, we have the decomposition:

$$\Lambda^5_7 = \{\omega \in \Lambda^5; \phi \wedge *\omega = 2\omega\}$$
$$\Lambda^5_{14} = \{\omega \in \Lambda^5, \phi \wedge *\omega = -\omega\}$$

which implies the projectors

$$\pi^5_7(\omega) = \frac{\omega + \phi \wedge *\omega}{3}$$
$$\pi^5_{14}(\omega) = \frac{2\omega - \phi \wedge *\omega}{3}.$$

Three- and four-forms. The three-forms decompose into **1**-, **7**- and **27**-dimensional representations of G_2. Explicitly, these spaces are given by

$$\Lambda^3_1 = \{\omega \in \Lambda^3 : \phi \wedge (*(*\phi \wedge \omega)) = 7\omega\}$$
$$\Lambda^3_7 = \{\omega \in \Lambda^3; *(\phi \wedge *(\phi \wedge \omega)) = -4\omega\}$$
$$\Lambda^3_{27} = \{\omega \in \Lambda^3; \phi \wedge \omega = *\phi \wedge \omega = 0\}.$$

We also define projection operators:

$$\pi^3_1(\omega) = \tfrac{1}{7}\phi \wedge (*(*\phi \wedge \omega))$$
$$\pi^3_7(\omega) = -\tfrac{1}{4}(*(\phi \wedge *(\phi \wedge \omega)))$$
$$\pi^3_{27}(\omega) = \omega - \pi^3_1(\omega) - \pi^3_7(\omega).$$

For four forms, we have the decomposition

$$\Lambda_1^4 = \{\omega \in \Lambda^4 : *\phi \wedge (*(\phi \wedge \omega)) = 7\omega\}$$
$$\Lambda_7^4 = \{\omega \in \Lambda^4; \phi \wedge *(\phi \wedge *\omega) = -4\omega\}$$
$$\Lambda_{27}^4 = \{\omega \in \Lambda^4; \phi \wedge \omega = *\phi \wedge \omega = 0\}$$

and the projectors

$$\pi_1^4(\omega) = \tfrac{1}{7} * \phi \wedge (*(\phi \wedge \omega))$$
$$\pi_7^4(\omega) = -\tfrac{1}{4}(\phi \wedge *(\phi \wedge *\omega))$$
$$\pi_{27}^4(\omega) = \omega - \pi_1^4(\omega) - \pi_7^4(\omega).$$

There are natural G_2-equivariant isomorphisms between these spaces. For example, the map $\omega \to \phi \wedge \omega$ is an isomorphism between $\Lambda_r^p \cong \Lambda_r^{p+3}$ if $\phi \wedge \omega_p$ is non-zero when $\omega \in \Lambda_r^p$:

$$\Lambda_1^0 \cong \Lambda_1^3 \quad \Lambda_7^1 \cong \Lambda_7^4$$
$$\Lambda_7^2 \cong \Lambda_7^5 \quad \Lambda_{14}^2 \cong \Lambda_{14}^5$$
$$\Lambda_7^3 \cong \Lambda_7^6 \quad \Lambda_1^4 \cong \Lambda_1^7.$$

Also, the map $\omega \to *\phi \wedge \omega$ is an isomorphism between $\Lambda_r^p \cong \Lambda_r^{p+4}$ when $*\phi \wedge \omega_p$ is non-zero when $\omega \in \Lambda_r^p$:

$$\Lambda_1^0 \cong \Lambda_1^4 \quad \Lambda_7^1 \cong \Lambda_7^5$$
$$\Lambda_7^2 \cong \Lambda_7^6 \quad \Lambda_1^3 \cong \Lambda_1^7$$

Appendix E   Some correlation functions

We can use the expression (4.3) to compute some correlation functions in the twisted theory in terms of correlation functions of the untwisted theory. For example, the two-point function of operators

$$\mathcal{O}_2 = \Phi_{2,1} \otimes \psi_h, \quad \mathcal{O}_3 = \Phi_{3,1} \otimes \psi_h$$

can be written in terms of a four-point function of the tri-critical Ising model

$$\langle \mathcal{O}_2(z_1)\mathcal{O}_3(z_2)\rangle = z_1^{-1/2} z_2^{-1}(z_1 - z_2)^{-2h}$$
$$\times \langle \Phi_{1,2}(\infty)\Phi_{2,1}(z_1)\Phi_{3,1}(z_2)\Phi_{1,2}(0)\rangle_{\text{tri-critical}}$$
$$= \frac{c}{(z_1 - z_2)^{2h-4/5}}$$

where c is a constant. This is independent of the position if $h = \tfrac{2}{5}$, which is what we need for the operators $\mathcal{O}_2$ and $\mathcal{O}_3$ to be chiral in the topological

theory. This correlation functions gets contributions from only one conformal block, precisely the one that is kept in the topological theory. On the other hand, consider the two-point function of operators whose tri-critical Ising model weight is $\frac{1}{10}$:

$$\mathcal{O} = \Phi_{2,1} \otimes \psi_h.$$

The two-point function of this operator with itself can be written in terms of a four-point function of the tri-critical Ising model:

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2) \rangle = z_1^{-1/2} z_2^{-1/2} (z_1 - z_2)^{-2h}$$
$$\times \langle \Phi_{1,2}(\infty)\Phi_{2,1}(z_1)\Phi_{2,1}(z_2)\Phi_{1,2}(0) \rangle_{\text{tri-critical}}$$
$$= \frac{c}{(z_1 - z_2)^{2h+1/5}} \times \frac{z_1 + z_2}{z_1 z_2}.$$

This is not even translationally invariant! However, it is easy to see that the conformal block that contributes to this correlation function is

$$\langle \Phi_{1,2} \mathcal{O}^{\uparrow} \mathcal{O}^{\downarrow} \Phi_{1,2} \rangle$$

but $\mathcal{O}^{\uparrow}$ is not a chiral operator. Correlation functions of chiral operators obey all the properties of a usual CFT. However, correlation functions of non-chiral operators in the twisted theory are not that of a CFT. This is qualitatively different from what happens in the usual $\mathcal{N} = 2$ twisting. In that case, the twisted theory makes sense as a CFT, even before we restrict ourselves to chiral operators. This intermediate CFT does not seem to exist for us.

Appendix F   Spectral flow and the twist

Whether or not the twisted stress tensor exists, and if so what its precise form is remains for now an open problem. In the case of Calabi–Yau manifolds, the existence of spectral flow was useful in order to construct the twisted stress tensor, so it is worth considering what precisely the analog of spectral flow is in our case.

Spectral flow, a word used rather loosely, refers to a particular isomorphism between the R and NS sectors of an $N = 2$ conformal field theory. What it does is easily illustrated in case of a free scalar field φ. Denote by $\hat{p} = i \oint \partial\varphi$ the zero mode of the momentum operator, and by $\hat{x}$ the conjugate coordinate. Then spectral flow by the amount η is simply implemented by the operator

$$S_1 = e^{i\eta\hat{x}}. \tag{F.1}$$

Spectral flow maps representations with momentum eigenvalue p to representations with momentum eigenvalue $p + \eta$. If we bosonize the $U(1)$ current

in $N = 2$ theories then this S_1 precisely implements what is usually referred to as spectral flow.

This is not quite the same as the statement that some particular R operator generates spectral flow. In that case, we are talking about an operator in the theory, and not a simple object constructed out of zero modes only such as S_1. It is this full operator, and not S_1, that appears in the generator of space-time supersymmetry. It is again easy to illustrate this in the case of a free scalar field. Instead of S_1 we consider the operator

$$S_2 = \oint \frac{dz}{z^{\eta q + 1}} e^{i\eta\phi} : \mathcal{H}_p \to \mathcal{H}_{\eta + p} \qquad \text{(F.2)}$$

acting on representations with momentum eigenvalue p and mapping them to representations of eigenvalue $p + \eta$. On highest weight states, S_1 and S_2 are identical, but on descendants they are not. The new stress tensors obtained by spectral flow are obtained using S_1. One can also define new stress tensors using the action of S_2, simply as $L'_n = S_2^{-1} L_n S_2$, but this is not usually done. One can explicitly work out the difference between the two prescriptions, but that is not very insightful. The modes of the twisted stress tensors of the A and B models are linear combinations of the modes of the initial stress tensor and its spectrally flown version. This is spectral flow with respect to S_1. Whether the twisted stress tensor has any relation to the new stress tensor obtained through S_2 is not known.

In the case of G_2 manifolds, the situation is different. We no longer have a version of S_1, but we do have a version of S_2, where the exponential of the field is now replaced by the R vertex operator $V_{7/16,+}$. It maps chiral primaries to R ground states and vice versa. It should induce an isomorphism between the NS and R sectors of the theory, otherwise the theory would not be space-time supersymmetric. In particular, this implies that we can define a new stress tensor in say the NS sector via $L'_n = S_2^{-1} L_n S_2$. Clearly, L'_0 annihilates all chiral primaries and is a good candidate for the zero mode of a twisted stress tensor. Whether the highest modes of L'_n can also be used to construct the modes of a twisted stress tensor still remains to be worked out, even in the case of Calabi–Yau manifolds. We leave this as an interesting direction to explore.

References

[1] E. Witten, *Mirror manifolds and topological field theory*, Preprint, arXiv:hep-th/9112056.

[2] A. Neitzke and C. Vafa, $N = 2$ *strings and the twistorial Calabi–Yau*, Preprint, arXiv:hep-th/0402128; N. Nekrasov, H. Ooguri and C. Vafa,

S-duality and topological strings, JHEP **0410** (2004), 009, arXiv:hep-th/0403167.

[3] A.A. Gerasimov and S.L. Shatashvili, *Towards integrability of topological strings. I: three-forms on Calabi–Yau manifolds*, JHEP **0411** (2004), 074, arXiv:hep-th/0409238.

[4] R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa, *Topological M-theory as unification of form theories of gravity*, Preprint, arXiv:hep-th/0411073.

[5] N. Nekrasov, *A la recherche de la m-theorie perdue. Z theory: chasing m/f theory*, Preprint, arXiv:hep-th/0412021.

[6] P.A. Grassi and P. Vanhove, *Topological M theory from pure spinor formalism*, Preprint, arXiv:hep-th/0411167.

[7] L. Anguelova, P. de Medeiros and A. Sinkovics, *On topological F-theory*, Preprint, arXiv:hep-th/0412120.

[8] S.L. Shatashvili and C. Vafa, *Superstrings and manifold of exceptional holonomy*, Preprint, arXiv:hep-th/9407025.

[9] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Math. Series **201**, Longman, Harlow, 1989; R. Reyes, *Some special geometries defined by Lie Groups*, PhD Thesis, Oxford University, 1993; M. Fernandez and L. Ugarte, *Dolbeault cohomology for G_2-manifolds*, Geometriae Dedicata, **70**, issue (1) (1998), 57–86.

[10] R. Blumenhagen, *Covariant construction of $N = 1$ superW algebras*, Nucl. Phys. B **381** (1992), 641.

[11] J.M. Figueroa-O'Farrill, *A note on the extended superconformal algebras associated with manifolds of exceptional holonomy*, Phys. Lett. B **392** (1997), 77, arXiv:hep-th/9609113.

[12] D. Gepner and B. Noyvert, *Unitary representations of SW(3/2,2) superconformal algebra*, Nucl. Phys. B **610** (2001), 545, arXiv:hep-th/0101116; B. Noyvert, *Unitary minimal models of SW(3/2,3/2,2) superconformal algebra and manifolds of G(2) holonomy*, JHEP **0203** (2002), 030, arXiv:hep-th/0201198.

[13] B.S. Acharya, *$N = 1$ M-theory-heterotic duality in three dimensions and Joyce manifolds*, Preprint, arXiv:hep-th/9604133.

[14] B.S. Acharya, *On mirror symmetry for manifolds of exceptional holonomy*, Nucl. Phys. B **524** (1998), 269, arXiv:hep-th/9707186.

[15] T. Eguchi and Y. Sugawara, *CFT description of string theory compactified on non-compact manifolds with G(2) holonomy*, Phys. Lett. B **519** (2001), 149, arXiv:hep-th/0108091.

[16] R. Roiban and J. Walcher, *Rational conformal field theories with G(2) holonomy*, JHEP **0112** (2001), 008, arXiv:hep-th/0110302.

[17] T. Eguchi and Y. Sugawara, *String theory on G(2) manifolds based on Gepner construction*, Nucl. Phys. B **630** (2002), 132, arXiv:hep-th/0111012.

[18] R. Blumenhagen and V. Braun, *Superconformal field theories for compact G(2) manifolds*, JHEP **0112** (2001), 006, arXiv:hep-th/0110232; *Superconformal field theories for compact manifolds with spin(7) holonomy*, JHEP **0112** (2001), 013, arXiv:hep-th/0111048.

[19] M. Aganagic and C. Vafa, *G(2) manifolds, mirror symmetry and geometric engineering*, Preprint, arXiv:hep-th/0110171.

[20] R. Roiban, C. Romelsberger and J. Walcher, *Discrete torsion in singular G(2)-manifolds and real LG*, Adv. Theor. Math. Phys. **6** (2003), 207, arXiv:hep-th/0203272.

[21] K. Sugiyama and S. Yamaguchi, *Coset construction of noncompact Spin(7) and G(2) CFTs*, Phys. Lett. B **538** (2002), 173, arXiv:hep-th/0204213.

[22] T. Eguchi, Y. Sugawara and S. Yamaguchi, *Supercoset CFT's for string theories on non-compact special holonomy manifolds*, Nucl. Phys. B **657** (2003), 3, arXiv:hep-th/0301164.

[23] M.R. Gaberdiel and P. Kaste, *Generalised discrete torsion and mirror symmetry for G(2) manifolds*, JHEP **0408** (2004), 001, arXiv:hep-th/0401125.

[24] B.S. Acharya and S. Gukov, *M theory and singularities of exceptional holonomy manifolds*, Phys. Rept. **392** (2004), 121, arXiv:hep-th/0409191.

[25] P.S. Howe and G. Papadopoulos, *Holonomy groups and W symmetries*, Commun. Math. Phys. **151** (1993), 467, arXiv:hep-th/9202036.

[26] S. Odake, *Extension of $N = 2$ superconformal algebra and Calabi–Yau compactification*, Mod. Phys. Lett. A **4** (1989), 557.

[27] G. Felder, *Best approach to minimal methods*, Nucl. Phys. B **317** (1989), 215, erratum ibid. B **324** (1989), 548.

[28] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, 2000.

[29] X. Dai, X. Wang and G. Wei, *On the stability of Riemannian manifold with parallel spinors*, Preprint, arXiv:math.dg/0311253.

[30] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Commun. Math. Phys. **165** (1994), 311, arXiv:hep-th/9309140.

[31] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory*, Nucl. Phys. B **413** (1994), 162, arXiv:hep-th/9307158.

[32] R. Dijkgraaf, H. Verlinde and E. Verlinde, *Topological strings in $D < 1$*, Nucl. Phys. B **352** (1991), 59.

[33] N. Hitchin, *The geometry of three-forms in six and seven dimensions*, Preprint, arXiv:math.dg/0010054; N. Hitchin, *Stable forms and special metrics*, Preprint, arXiv:math.dg/0107101.

[34] J.H. Lee and N.C. Leung, *Geometric structures on G(2) and spin(7)-manifolds*, Preprint, arXiv:math.dg/0202045.

[35] E. Witten, *Quantum background independence in string theory*, Preprint, arXiv:hep-th/9306122.

[36] E. Verlinde, *Attractors and the holomorphic anomaly*, Preprint, arXiv:hep-th/0412139.

[37] J. Gutowski and G. Papadopoulos, *Moduli spaces and brane solitons for M theory compactifications on holonomy G(2) manifolds*, Nucl. Phys. B **615** (2001), 237, arXiv:hep-th/0104105.

[38] V. Pestun and E. Witten, *The Hitchin functionals and the topological B-model at one loop*, Preprint, arXiv:hep-th/0503083.

[39] T.W. Grimm and J. Louis, *The effective action of type IIA Calabi–Yau orientifolds*, Preprint, arXiv:hep-th/0412277.

[40] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, Nucl. Phys. B **271** (1986), 93.

[41] N.J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, *Hyperkahler metrics and supersymmetry*, Commun. Math. Phys. **108** (1987), 535.

[42] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, Phys. Rev. D **70** (2004), 106007, arXiv:hep-th/0405146.

[43] C. Mayer and T. Mohaupt, *Domain walls, Hitchin's flow equations and G(2)-manifolds*, Class. Quant. Grav. **22** (2005), 379, arXiv:hep-th/0407198.

[44] E. Witten, *Chern-Simons gauge theory as a string theory*, Prog. Math. **133** (1995), 637, arXiv:hep-th/9207094.

[45] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, Preprint, arXiv:hep-th/9907026.

[46] C. Beasley and E. Witten, *A note on fluxes and superpotentials in M-theory compactifications on manifolds of G(2) holonomy*, JHEP **0207** (2002), 046, arXiv:hep-th/0203061.

[47] V. Bouchard, B. Florea and M. Marino, *Counting higher genus curves with crosscaps in Calabi–Yau orientifolds*, JHEP **0412** (2004), 035, arXiv:hep-th/0405083; V. Bouchard, B. Florea and M. Marino, *Topological open string amplitudes on orientifolds*, JHEP **0502** (2005), 002, arXiv:hep-th/0411227.

[48] F. Witt, *Generalised G_2-manifolds*, Preprint, arXiv:math.dg/0411642; F. Witt, *Special metric structures and closed forms*, Preprint, arXiv:math.dg/0502443.

[49] J. de Boer and T. Tjin, *Quantization and representation theory of finite W algebras*, Commun. Math. Phys. **158** (1993), 485, arXiv:hep-th/9211109; J. de Boer and T. Tjin, *The relation between quantum W algebras and Lie algebras*, Commun. Math. Phys. **160** (1994), 317, arXiv:hep-th/9302006.

[50] S. Mallwitz, *On SW minimal models and N=1 supersymmetric quantum Toda field theories*, Int. J. Mod. Phys. A **10** (1995), 977, arXiv:hep-th/9405025.

[51] E. Frenkel, V. Kac and M. Wakimoto, *Characters and fusion rules for W algebras via quantized Drinfeld–Sokolov reductions*, Commun. Math. Phys. **147** (1992), 295.

[52] S. Karigiannis, *Deformations of G_2 and spin(7) structures on manifolds*, Preprint, arXiv: math.DG/0301218.