Generic pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility

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Generic pricing of foreign exchange, inflation and stock options under stochastic interest rates and stochastic volatility.

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Abstract

In this paper we deal with the pricing of stock, foreign exchange and inflation options under stochastic interest rates and stochastic volatility. We consider a foreign exchange framework for the pricing inflation-indexed options in which the valuation of stock and foreign exchange options can be treated as a nested case. We assume multi-factor Gaussian rates for both the nominal (domestic) as the real (foreign) economy, which economies (currencies) can be exchanged against each other by means of the inflation index (exchange rate) which is driven by log-normal dynamics with a stochastic volatility component. Furthermore we allow for a general correlation structure between the drivers of the volatility, the inflation index, the nominal and the real rates. We derive explicit option pricing formulas for various securities, like vanilla call/put options, forward starting options, inflation-indexed swaps and inflation caps/floors. All these options can be valued in closed-form under Schöbel-Zhu (1999) stochastic volatility, whereas we device an (Monte Carlo) approximation in the form of a very effective control variate for the general Heston (1993) model.

Keywords: Inflation, Foreign Exchange, Stochastic volatility, Stochastic interest rates, Hybrids.

1 Introduction

The long maturity and hybrid derivative markets are developing more and more. Not only are increasingly exotic structures created, also the markets for plain vanilla derivatives are growing. One of the recent advances is the development of long maturity option markets across various asset classes; during the last years long maturity securities, such as Target Auto Redemption Notes (TARN) equity-interest rate options (e.g. see Caps (2007)), Power-Reverse Dual-Currency (PRDC) Foreign Exchange (FX) swaps (e.g. see Piterbarg (2005)) and inflation-indexed Limited Price Indices (LPI) structures (e.g see Mercurio (2005) or Mercurio and Moreni (2006)) have become increasingly popular. Whereas for FX, inflation and hybrid structures, which explicitly depend on future interest rates evolutions, it is immediately clear that the use of stochastic interest rates is crucial in a derivative pricing model, the addition of stochastic rates is also important for the pricing and in particularly the hedging of long maturity equity derivatives (e.g. see Bakshi et al. (2000)); first, the option's rho, which measures/hedges

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the interest rate risk of the derivative, is increasing with time to maturity. Secondly, the stochastic interest rates are important for exotic option pricing since the numeraire is the discount bond associated with the maturity of the option. Because the long term interest rates are to a reasonable degree correlated with FX/inflation/equity indices, the rates directly influence the pricing kernel used in exotic option pricing.

Most investment banks have now standardized a three-factor modeling framework to price cross-currency (i.e. FX and inflation) options (see Piterbarg (2005), Sippel and Ohkoshi (2002) or Jarrow and Yildirim (2003)), hereby the index follows a log-normal process, and the interest rates of the two currencies are driven by one-factor Gaussian (e.g. see Hull and White (1993)) models. The choice of Gaussian assumptions for the interest rates and the log-normality for the index has allowed for a very efficient, essentially closed-form, calibration to at-the-money options on the index, i.e. on the FX-rate or stock price. The assumption of log-normality for an index, though technically very convenient, does not find in justification in the financial equity markets (e.g. see Bakshi et al. (1997)), the FX markets (e.g see Piterbarg (2005), Caps (2007)) nor in the inflation markets (e.g. see Mercurio and Moreni (2006), Kenyon (2008)). In fact, the markets for these products exhibit a strong volatility skew or smile, implying log index returns deviating from normality and suggesting the use of skewed and heavier tailed distributions. Moreover many multi-currency structures (like LPIs or PRDCs) are particularly sensitive to volatility skews/smiles as they often incorporate multiple strikes as well as callable/knockout components. Hence appropriate exotic option pricing models, which need to quantify the volatility exposure in such structures, should at least be able to incorporate the smiles/skews in the vanilla markets. While various methods exist to incorporate volatility smiles (i.e. local volatility, stochastic volatility and/or jumps), the calibration of such models is by no means trivial. A skew-mechanism is normally applied to the forward index price (i.e. the FX-rate, CPI/Equity index), however to price multi-currency options also a term-structure involving various time points of the forward index is required. The incorporation of stochastic interest rates makes the connection between the two particularly non-trivial (e.g. see Piterbarg (2005) or Antonov et al. (2008)). Though the issue is important, Piterbarg (2005) even dubs it as 'perhaps even the most important current outstanding problems for quantitative research departments worldwide', there is remarkably little literature available on the subject even though the problem attracted both the attention of practitioners as well as from academia (e.g. see van der Ploeg (2007)).

Only very recently a few approaches were suggested. A local volatility approach is used in Piterbarg (2005) who derives approximating formulas for calibration. Andreasen (2006) combines Heston (1993) stochastic volatility with independent stochastic interest rates drivers and derives closed-form Fourier expressions for vanilla options. To correlate the independent rate drivers with the FX-rate Andreasen (2006) uses an indirect approach in the form of a volatility displacement parameter, which has some disadvantages as that it can lead to extreme model parameters (e.g. see Antonov et al. (2008)). The calibration of FX options stochastic interest rates with Heston (1993) stochastic volatility under a full correlation structure is undertaken in Antonov et al. (2008) who use Markovian projection to derive approximation formulas. Though their projection technique is elegant, the quality of their approximation deteriorates for larger maturities or more extreme model parameters. The exact pricing of FX options under Schöbel and Zhu (1999) stochastic volatility, single-factor Gaussian rates and a full correlation structure was only recently considered in van Haastrecht et al. (2008).

In this paper, building forth on the results of van Haastrecht et al. (2008), Antonov et al. (2008), Andreasen (2006) and Piterbarg (2005), we consider the pricing of foreign exchange, inflation and stock options under Schöbel and Zhu (1999) and Heston (1993) stochastic volatility and under multi-factor Gaussian interest rates with a full correlation structure. Since stock and FX options are a special

(nested) cases of inflation-indexed caps/floors[4] we will mainly focus on the pricing of inflation index derivatives. The stock and FX model option pricing formulas hence follow directly from our generalization of the foreign exchange inflation framework of Jarrow and Yildirim (2003). The setup of the paper is as follows: in section 3 the basic vanilla derivatives are being considered, in section 2 we introduce our new model and section 3.1 considers the pricing methodology. In section 4 we derive the characteristic functions (cf.) required for the Fourier-based pricing methods: under Schöbel and Zhu (1999) stochastic volatility we can derive the cf. of our model in closed-form, under Heston (1993) stochastic volatility it is extremely challenging derive the cf. of the general model in closed-form, nonetheless we demonstrate how the cf. of the special (uncorrelated) case can be used as a simple and efficient control variate for the general model. Finally, section 6 concludes.

2 The model

Before introducing the general model, we first consider the Jarrow and Yildirim (2003) model which can be seen as a special (degenerate) case of our model. The Jarrow and Yildirim (2003) framework for modeling inflation and real rates is based on a foreign-exchange analogy between the real of and the nominal economy. That is, the real rates are seen as interest rates in the real (foreign) economy, whereas the nominal rates represent the interest rates in the nominal (domestic) economy. The inflation index then represents the exchange rate between the nominal (domestic) and real (foreign) currency. There are several assumptions that can be made with respect to the evolution of these dynamics: we first discuss the classical Jarrow and Yildirim (2003) model, before turning to generalized model setups.

2.1 Special case: Jarrow-Yildirim (2003) model

Jarrow and Yildirim (2003) assume that the real-world evolution of the nominal and real instantaneous forward rates is given by HJM-dynamics, whereas the inflation index is log-normal distributed. Though several choices can be made with respect to the volatility structure within an HJM-model, Jarrow and Yildirim (2003) assume that the forward rate volatilities are given by $\sigma e^{a(T-t)}$. Using the equivalent formulation of the HJM-model in terms of instantaneous short rates then results in the following dynamics under the risk-neutral measure Q^n, see Jarrow and Yildirim (2003).

Proposition 2.1 *The Q_n dynamics of the instantaneous nominal rate $n(t)$, real rate $r(t)$ and the inflation index $I(t)$ are given by*

$$dn(t) = [\vartheta_n(t) - a_n n(t)]dt + \sigma_n dW_n(t), \tag{1}$$

$$dr(t) = [\vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]dt + \sigma_r dW_r(t), \tag{2}$$

$$dI(t) = I(t)[n(t) - r(t)]dt + \sigma_I I(t)dW_I(t), \tag{3}$$

with $a_n, a_r, \sigma_n, \sigma_r, \sigma_I$ positive parameters and where (W_n, W_r, W_I) is a Brownian motion under Q^n (i.e. with the nominal bank-account as numeraire) with correlations $\rho_{n,r}, \rho_{n,I}$ and $\rho_{r,I}$, and with $\vartheta_n(T)$ and $\vartheta_r(t)$ deterministic functions which are used to exactly fit the term structure the nominal and real interest rates.

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[4]In our framework an inflation option can be seen as forward-starting FX-option, hence the pricing of FX-option follows from the pricing of inflation option by setting the forward starting date equal to the current date. A stock option can be seen as an FX-option in which (possibly deterministic) foreign interest rates represent the continuous dividend yield.

Note that the covariance in (2) between the inflation and real rate term $\rho_{r,I}\sigma_I\sigma_r$, arises due to a change of the real to the nominal risk-neutral measure, e.g. see Geman et al. (1996). With this particular volatility structure, Jarrow and Yildirim (2003) thus assumed that both the nominal as real (instantaneous) rates followed Hull and White (1993) processes under their own risk-neutral measure. Moreover they showed that the real rate still follows an Ornstein-Uhlenbeck process under the nominal risk-neutral measure Q_n and that the inflation index $I(T)$ for each $t < T$ is log-normal distributed under Q_n, specifically one can write:

$$I(T) = I(t)\exp\left(\int_t^T [n(u) - r(u)]du - \frac{1}{2}\sigma_I(T - t) + \sigma_I[W_I(T) - W_I(t)]\right). \tag{4}$$

The main advantage of the Jarrow and Yildirim (2003) model is its tractability; one for example has analytical formulas for the prices of YYIIS (see Brigo and Mercurio (2006) pp.653, formula 16.15) and closed-form Black-like formulas for the prices of inflation-indexed caplets (see Brigo and Mercurio (2006) pp.663, formula 17.4). Though one can challenge the one-factor rate models, the biggest disadvantage of the Jarrow and Yildirim (2003) model for the pricing of inflation derivatives is most often the log-normal assumption of the inflation index, which does not find its justification in the markets, e.g. see Mercurio and Moreni (2006), Kenyon (2008) or Kruse (2007).

2.2 General model

In this section we will present a general model, which can be seen as an extension of the models of Jarrow and Yildirim (2003), van Haastrecht et al. (2008). That is, instead of one-factor Hull and White (1993) models for the instantaneous nominal and real rates, we let the short rate be driven by multiple (correlated) factors. To ease the notation, we use an equivalent additive formulation for Hull-White interest rates in terms of a sum of correlated Gaussian factors plus a deterministic function, i.e. we write the model into an affine factors formulation, e.g. Duffie et al. (2000) and Duffie et al. (2003). The deterministic factor can be chosen as to exactly fit the term structure of the nominal or real interest rates, e.g. see Brigo and Mercurio (2006) or Pelsser (2000). The nominal short interest rate be driven by K correlated Gaussian factors and the real short rate by M factors, the multi-factor Gaussian interest can hence be represented as:

$$n(t) = \varphi_n(t) + \sum_{i=1}^K x_n^i(t), \qquad r(t) = \varphi_r(t) + \sum_{j=1}^M x_r^j(t), \tag{5}$$

where $\varphi_n(t), \varphi_r(t)$ are the deterministic functions to fit the nominal and real term structure (in particular $\varphi_n(0) = n(0)$ and $\varphi_r(0) = r(0)$) and with $x_n^i(t)$, $x_r^j(t)$ the Gaussian factors which drive respectively the nominal and real rates.

The second extension in our model is that we make the volatility σ_I stochastic. Moreover we let this stochastic volatility factor, which we from now on denote by $v(t)$, be correlated with the instantaneous interest rates and the inflation index. Two popular choices within the stochastic volatility literature are the models of Heston (1993) and Schöbel and Zhu (1999). In the latter the volatility is modeled as an Ornstein-Ühlenbeck process

$$dv(t) = \kappa[\psi - v(t)]dt + \tau dW_v(t), \quad v(0) = v_0 \tag{6}$$

with κ, ψ, σ_v positive parameters and where $W_v(t)$ is a Brownian motion that is correlated with the other driving factors, especially the asset price. Note that we have a positive probability that $v(t)$ in (6)

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can become negative, which will only cause the correlation between $v(t)$ and the other driving factors to (temporarily) change sign.

The most popular stochastic volatility model, however, is the Heston (1993) model which mainly owns its popularity due to its analytical tractability. In the Heston model, the variance is modeled by the following Feller/CIR/square-root process

$$dv^2(t) = \kappa[\theta - v^2(t)]dt + \xi v(t)dW_\nu(t), \quad v^2(0) = v_0^2 \tag{7}$$

with κ, θ, ξ positive parameters and where W_ν represents again a Brownian that is correlated with the other model factors.

With the multi-factor Gaussian rates and with stochastic volatility a la Schöbel-Zhu or Heston, we come to the following proposition for the dynamics of our model.

Proposition 2.2 *The Q_n dynamics of the K-factor instantaneous nominal rate $n(t)$, M-factor real rate $r(t)$ and the inflation index $I(t)$ are given by*

$$dx_n^i(t) = -a_n^i x_n^i(t)dt + \sigma_n^i dW_{n_i}(t) \quad i = 1, \dots, K, \tag{8}$$

$$dx_r^j(t) = [-a_r^j x_r^j(t) - \rho_{I,x_r^j} v(t)\sigma_r^j]dt + \sigma_r^j dW_{r_j}(t) \quad j = 1, \dots, M, \tag{9}$$

$$dI(t) = I(t)[n(t) - r(t)]dt + v(t)I(t)dW_I(t) \tag{10}$$

with $a_n^i, a_r^j, \sigma_n^i, \sigma_r^j$ positive parameters, $v(t)$ the stochastic volatility factor with dynamics given by (6) or (7), and where $(W_{n_1}, \dots, W_{n_K}, W_{r_1}, \dots, W_{r_M}, W_\nu)$ is a Brownian motion under Q^n with (possibly) a full correlation structure.

The multi-factor Gaussian model is still very tractable; one, for example, has the following analytical formulas for the prices of zero-coupon bond options nominal risk-neutral measure Q_n:

$$P_n(t,T) = \mathbb{E}_n\left\{e^{-\int_t^T n(u)du}\right\} = A_n(t,T)e^{-\sum\limits_{i=0}^{K} B_n^i(t,T)x_n^i(t)}, \tag{11}$$

$$P_r(t,T) = \mathbb{E}_r\left\{e^{-\int_t^T r(u)du}\right\} = A_r(t,T)e^{-\sum\limits_{j=0}^{M} B_r^i(t,T)x_r^j(t)}. \tag{12}$$

where $A_n(t,T), A_r(t,T), B_n^i(t,T), B_r^i(t,T)$ are affine functions, see appendix B.1. A useful quantity for the pricing of inflation-indexed options will turn out the be the forward inflation index $I_F(t)$ under under the nominal T-forward measure for a general maturity T, i.e.

$$I_F(t) = I(t)\frac{P_r(t,T)}{P_n(t,T)}. \tag{13}$$

Hence since $I_F(T) = I(T)$, we can directly substitute the forward inflation index dynamics for the inflation index, to price European time-T options. In the following subsection we will derive the dynamics of $I_F(t)$ under the nominal T-forward measure.

Dynamics under the T-forward measure

Using the change of numeraire technique of Geman et al. (1996), we will now derive the dynamics of our model under the T-forward measure for a general maturity T. Note that under their risk-neutral measures the nominal and real discount bond prices follows the processes

$$\frac{dP_n(t,T)}{P_n(t,T)} = n(t)dt + \sum_{i=1}^{K} \sigma_n^i dW_{n_i}(t), \quad \frac{dP_r(t,T)}{P_r(t,T)} = r(t)dt + \sum_{j=1}^{M} \sigma_r^j dW_{r_j}(t), \tag{14}$$

hence, by an application of Ito's lemma, we find the following dynamics for the T-forward asset price process

$$\frac{dI_F(t)}{I_F(t)} = \sum_{i=1}^{K} \sigma_n^i B_n^i \left[\sum_{k=1}^{K} \rho_{x_n^i,x_n^k} \sigma_n^k B_n^k(t,T) - \sum_{j=1}^{M} \rho_{x_n^i,x_r^j} \sigma_r^j B_r^j(t,T) - \rho_{x_n^i,\nu} \nu(t) \right] dt \tag{15}$$

$$+ \nu(t) dW_I(t) + \sum_{i=1}^{K} \sigma_n^i dW_{n_i}(t) - \sum_{j=1}^{M} \sigma_r^j dW_{r_j}(t) \tag{16}$$

By definition the forward inflation rate is a martingale process under the nominal T-forward measure. This is achieved by defining the following transformations of the Brownian motion(s):

$$dW_{n_k} \mapsto dW_{n_i}^T - \sum_{i=1}^{K} \rho_{x_n^i,x_n^k} \sigma_n^i B_n^i(t,T) dt, \qquad\qquad k = 1, \ldots, K, \tag{17}$$

$$dW_{r_j} \mapsto dW_{r_j}^T - \sum_{i=1}^{K} \rho_{x_n^i,x_r^j} \sigma_n^i B_n^i(t,T) dt \qquad\qquad j = 1, \ldots, M, \tag{18}$$

$$dW_I \mapsto dW_I^T - \sum_{i=1}^{K} \rho_{x_n^i,I} \sigma_n^i B_n^i(t,T) dt \tag{19}$$

$$dW_\nu \mapsto dW_\nu^T + \sum_{i=1}^{K} \rho_{x_n^i,\nu} \sigma_n^i B_n^i(t,T) dt \tag{20}$$

Hence for the stochastic volatility dynamics under the T-forward dynamics in the Schöbel-Zhu case we obtain:

$$d\nu(t) = \kappa \left[\xi(t) - \nu(t) \right] dt + \tau dW_\nu^T(t), \tag{21}$$

$$\xi(t) = \psi + \sum_{i=1}^{K} \frac{\rho_{x_n^i,\nu} \sigma_n^i \tau}{\kappa} B_n^i(t,T) \tag{22}$$

while the Heston dynamics become

$$d\nu^2(t) = \kappa \left[\zeta(t) - \nu^2(t) \right] dt + \xi \nu(t) dW_\nu^T(t) \tag{23}$$

$$\zeta(t) = \theta + \sum_{i=1}^{K} \frac{\rho_{x_n^i,\nu} \sigma_n^i \xi \nu(t)}{\kappa} B_n^i(t,T) \tag{24}$$

Hence we come to the following proposition for the inflation dynamics under the nominal T-forward measure Q_n^T.

Proposition 2.3 *The Q_n^T dynamics of the K-factor instantaneous nominal rate n(t), M-factor real rate r(t) and the inflation index I(t) are given by*

$$dx_n^k(t) = \left[-a_n^k x_n^k - \sigma_n^k \sum_{i=1}^{K} \rho_{x_n^i,x_n^k} \sigma_n^i B_n^i(t,T) \right] dt + \sigma_n^k dW_{n_k}^T \qquad\qquad k = 1, \ldots, K, \tag{25}$$

$$dx_r^j(t) = \left[-a_r^j x_r^j - \sigma_r^j \rho_{I,x_r^j} \nu(t) - \sigma_r^j \sum_{i=1}^{K} \rho_{x_n^i,x_r^j} \sigma_n^i B_n^i(t,T) \right] dt + \sigma_r^j dW_{r_j}^T \qquad j = 1, \ldots, M, \tag{26}$$

$$\frac{dI_F(t)}{I_F(t)} = \nu(t) dW_I^T(t) + \sum_{i=1}^{K} \sigma_n^i dW_{n_i}^T(t) - \sum_{j=1}^{M} \sigma_r^j dW_{r_j}^T(t) \tag{27}$$

where $(W_{n_1}^T, \ldots, W_{n_K}^T, W_{r_1}^T, \ldots, W_{r_M}^T, W_\nu^T)$ is a Brownian motion under Q_n^T and with stochastic volatility dynamics as in (21) and (23).

We can simplify (27) by switching to logarithmic coordinates and rotating the Brownian motions $W_{n_1}^T, \ldots, W_{n_K}^T, W_{r_1}^T, \ldots, W_{r_M}^T$ and W_ν^T to $W_F^T(t)$. Defining

$$z(t) := \log(I_F(t)) \tag{28}$$

and an application of Ito's lemma yields

$$dz(t) = -\frac{1}{2} \nu_F^2(t) dt + \nu_F(t) dW_F^T(t), \tag{29}$$

with $\nu_F(t)$ the instantaneous variance of the forward inflation index (explicitly defined in (58)). For example notice that for Schöbel-Zhu volatility dynamics, we now have transformed the system of (2.1) of the variables $x_n^1(t), \ldots, x_n^K(t), x_r^1(t), \ldots, x_r^M(t), I(t), \nu(t)$ under the nominal risk-neutral measure to the system (29)-(21) of variables $z(t), \nu(t)$ under the T-forward measure. We can now for example use this latter system to determine characteristic function of log inflation rate in our model, see section 4.

3 Pricing and Applications

In this section we will briefly discuss the main inflation derivatives. We adopt the notation that is being used in Brigo and Mercurio (2006) and Mercurio (2005), to which authors we also refer an excellent overview of interest rate and inflation-indexed derivatives and models.

3.1 Pricing

We will now discuss the general option pricing framework for inflation, FX and stock options. That is, we briefly review the framework of Carr and Madan (1999) for the pricing of European option prices using Fourier inversion. Directly afterwards we show how this framework can be applied to value inflation, FX and stock derivatives. Under the risk-neutral measure Q (i.e. with the bank account as numeraire), we can write the following for the price $C_T(k)$ of an European option ($\omega = 1$ for a call, $\omega = -1$ for a put) maturing at time T, with strike $K = \exp(k)$, on an asset I:

$$C_T(k, \omega) = \mathbb{E}_n \left\{ e^{-\int_t^{T_i} n(u) du} \left[\omega \big(I(T) - K \big) \right]^+ \Big| \mathcal{F}_t \right\}, \tag{30}$$

and hence note that to price European options we only need the probability distribution of the T-forward stock price at time T. Therefore, instead of evaluating expected discounted payoff under the risk-neutral bank account measure, we can also change the underlying probability measure to evaluate this expectation under the T-forward probability measure Q^T (e.g. see Geman et al. (1996)). This is equivalent to choosing the T-discount bond as numeraire. Hence conditional on time t, we can evaluate the price of a European option ($\omega = 1$ for a call, $\omega = -1$ for a put) with strike $K = \exp(k)$ as

$$C_T(k, \omega) = P_n(t, T) \mathbb{E}_n^T \left\{ \left[\omega \big(I_F^T(T) - K \big) \right]^+ \Big| \mathcal{F}_t \right\} \tag{31}$$

where $P_n(t, T)$ denotes the price of a (pure) discount bond and $I_F^T(t) := \frac{I(t)}{P_n(t,T)}$ denotes the T-forward index price. The above expression can be numerically evaluated by means of a Fourier inversion of the

log-asset price characteristic function; following Carr and Madan (1999), Lewis (2001) and Lord and Kahl (2007), we can then write the call option price (30) with log strike k, in terms of the (T-forward) characteristic function ϕ_T of the T-forward log index price $z(T) := \log I_F^T(t)$, i.e.

$$C_T(k, \omega, \alpha) = \frac{P(t, T)}{\pi} \int_0^\infty \text{Re}\Big(e^{-(\alpha+iv)k}\psi_T(v, \omega, \alpha)\Big)dv + R\Big(I_F^T(t), K, \alpha(k)\Big), \tag{32}$$

where the residue term R equals

$$R\big(F, K, \alpha\big) := F \cdot 1_{\{\omega\alpha \leq 0\}} - K \cdot 1_{\{\omega\alpha \leq -1\}} - \frac{1}{2}\Big(F \cdot 1_{\{\alpha=0\}} - K \cdot 1_{\{\omega\alpha=-1\}}\Big), \tag{33}$$

with

$$\psi_T(v, \omega, \alpha) := \frac{\phi_T\big(v - (\omega\alpha + 1)i\big)}{(\omega\alpha + iv)(\omega\alpha + 1 + iv)}, \tag{34}$$

and where $\phi_T(u) := \mathbb{E}^{Q^T}\Big[\exp\big(iuz(T)\big)\big|\mathcal{F}_t\Big]$ denotes the T-forward characteristic function of the log index price. Note that (32) can be efficiently evaluated, i.e. either by direct integration or Fast Fourier Transformation, see for example Carr and Madan (1999), Lee (2004) or Lord and Kahl (2007). Thus for the pricing of call and put options, it suffices to know the characteristic function of the log price process.

3.2 Inflation derivatives

Before dealing with the pricing of inflation-index derivatives within the general model (2.2), we first discuss the main (vanilla) inflation-indexed securities. Hereby we adopt the notation that is being used in Brigo and Mercurio (2006) and Mercurio (2005), to which authors we also refer for an excellent overview of interest rate and inflation-indexed derivatives and models.

Inflation-indexed swaps

Given a set of payment dates $T_1, \ldots, T_M$, an inflation-indexed swap (IIS) is a swap where, on each date, party A pays party B the inflation rate over a predefined period, while party B pays party A a fixed rate. This inflation rate is calculated as the percentage return of the inflation index (e.g. HICP) over the time interval it applies to. The two main ISS contracts that are traded in the markets are the zero-coupon inflation-indexed swap (ZCIIS) and the year-on-year inflation-indexed swap (YYIIS). In the ZCIIS, the payoff at time T_M, assuming $T_M = M$ years, party B pays party A the fixed amount

$$N[(1 + K)^M - 1], \tag{35}$$

where K is the strike (e.g. the break-even inflation rate) and N the nominal value of the contract. In exchange, party A pays party B, at the time final time T_M, the floating amount of

$$N\Big[\frac{I(T_M)}{I_0} - 1\Big], \tag{36}$$

with $I(T_M), I_0$ the inflation/CPI index respectively at time T_M and T_0. In the YYIIS, at each time T_i, party B pays party A the fixed amount

$$N\phi_i K, \tag{37}$$

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where ϕ_i denotes the fixed-leg year fraction for the interval $[T_{i-1}, T_i]$, and N the nominal value of the YYIIS. In exchange, at each time T_i, party A pays party B the floating amount

$$N\psi_i\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right],\tag{38}$$

where ψ_i denotes the fixed leg year fraction for the interval $[T_{i-1}, T_i]$ ($T_0 := 0$).

Let P_n and P_r respectively denote the (zero-coupon) discount bond prices of the real and nominal economy, then standard no-arbitrage theory and some straightforward rewriting show that the price of an ZCIIS (zero-coupon inflation-indexed swap) can be expressed as

$$\mathbf{ZCIIS}(t, T_M, I_0, N, K) = N\left[\frac{I(t)}{I_0}P_r(t, T_M) - (1 + K)^M\right],\tag{39}$$

which quantity is model-independent. That is, the above price is not based on any specific assumptions on the evolution of the (real and nominal) interest rates, but simply follows from the absence of arbitrage. This is an important fact, since it allows us to strip, without ambiguity, real zero-coupon bond prices, from the quotes prices of ZCIIS. More specifically, given a set of market quotes of $K = K(T_M)$ at time $t = 0$, we can use equation (39) together with net present value (35) to determine discount bonds of the real economy, i.e.

$$P_r(0, T_M) = P_n(0, T_M)(1 + K(T_M))^M.\tag{40}$$

A completely different story applies to the valuation of a YYIIS (year-on-year inflation-indexed swap), which in fact depends on the evolution of the underlying quantities and hence its price is model dependent; note that the value at time $t < T_{i-1}$ of the payoff (38) at time T_i is

$$
\begin{aligned}
\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i\mathbb{E}_n\left\{e^{-\int_t^{T_i} n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right]\Big|\mathcal{F}_t\right\}\\
&= N\psi_i\mathbb{E}_n\left\{e^{-\int_t^{T_i} n(u)du}\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} - N\psi_i P_n(t, T_i),
\end{aligned}\tag{41}
$$

where $\mathbb{E}_n$ denotes the expectation under the nominal risk-neutral measure. We briefly comment on why the latter expectation is model dependent, first notice that

$$\mathbb{E}_n\left\{e^{-\int_t^{T_i} n(u)du}\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} = \mathbb{E}_n\left\{e^{-\int_t^{T_{i-1}} n(u)du}P_r(T_{i-1}, T_i)\Big|\mathcal{F}_t\right\},\tag{42}$$

hence we can interpret the expectation from (41) as the nominal price of a derivative that payoffs off (in nominal units), the real zero-coupon bond price $P_r(T_{i-1}, T_i)$ at time T_i. Alternatively we can also evaluate the latter expectation under a different measure, e.g. see Geman et al. (1996). Denote with Q_n^T as the nominal T-forward measure for some maturity T and let $\mathbb{E}_n^T$ represent the expectation under the corresponding measure, then we can write (42) as:

$$\mathbb{E}_n\left\{e^{-\int_t^{T_{i-1}} n(u)du}P_r(T_{i-1}, T_i)\Big|\mathcal{F}_t\right\} = P_n(t, T_{i-1})\mathbb{E}_n^{T_{i-1}}\left\{P_r(T_{i-1}, T_i)\Big|\mathcal{F}_t\right\},\tag{43}$$

where $\mathbb{E}_n^{T_{i-1}}$ denotes the expectation under the nominal T_{i-1} forward measure. If the nominal or real rates are deterministic, then this expectation would reduce to the present value (in nominal units) of the forward price of the real zero-coupon bond, i.e. we would then have

$$P_n(t, T_{i-1})\mathbb{E}^{T_{i-1}}\left\{P_r(T_{i-1}, T_i)\Big|\mathcal{F}_t\right\} = P_r(T_{i-1}, T_i)P_n(t, T_{i-1}).\tag{44}$$

However for inflation-linked derivative pricing purposes it is usually desirable (if not necessary) that real rates are stochastic, and the expectation of (41) is model dependent. In fact, if the nominal and real rates are correlated (and hence stochastic), the change of measure will change the drift of the real rate $r(t)$ and hence also the expectation of (43). In interest rate terms, this effect is known under the term convexity adjustment, e.g. see Pelsser (2000) or Brigo and Mercurio (2006). For example if one assumes one-factor Gaussian rates (as in the JY model), one will see this convexity effect for any non-zero correlation coefficient between the nominal and real rates.

Finally note that we can also evaluate the expectation of (41) under the T_i-forward measure, i.e.

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i P(t, T_i)\mathbb{E}_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} - N\psi_i P_n(t, T_i). \tag{45}$$

This latter interpretation, which expresses the YYIIS (year-on-year inflation-indexed swap) as the T_i-forward expectation of the return on the inflation index, is very useful for our pricing methodology (see section 3.1), because it expresses the price of a YYIIS in terms of the distribution of $\frac{I(T_i)}{I(T_{i-1})}$ under the T_{i-1}-forward measure.

Inflation-indexed caplets/floorlets

An inflation-indexed caplet can be seen as a call option on the inflation rate implied by the inflation (e.g. CPI) index. Analogously an inflation-indexed floorlet can be seen as put option on the same inflation rate. In formulas we can write the following for the payoff of an IICF (inflation-indexed cap/floorlet) at time T_i

$$N\phi_i\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+, \tag{46}$$

where N denotes the nominal value of the contract, κ the IICF strike, ϕ_i the year fraction for the interval $[T_{i-1}, T_i]$ and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet. Setting $K := 1 + \kappa$, standard no-arbitrage theory implies that the value of the payoff 46 at time $t \le T_{i-1}$ is

$$
\begin{aligned}
\mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) &= N\psi_i\mathbb{E}_n\left\{e^{-\int_t^{T_i} n(u)du}\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)\right]^+\Big|\mathcal{F}_t\right\} \tag{47}\\
&= N\psi_i P_n(t, T_i)\mathbb{E}_n^{T_i}\left\{\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)\right]^+\Big|\mathcal{F}_t\right\}.
\end{aligned}
$$

The pricing of an IICF (inflation-indexed cap/floorlet) is thus very similar to that of a forward starting (cliquet) option. In fact (47) is equivalent to a call option on the forward return of the inflation index, i.e. on the inflation rate.

Pricing

The crucial quantity for the pricing of the inflation-indexed derivatives in our model (2.2) is the log-return $z(T_{i-1}, T_i)$ of the inflation index over the interval $[T_{i-1}, T_i]$ under the T_i-forward measure $Q_n^{T_i}$, i.e.

$$z(T_{i-1}, T_i) := \log\left(\frac{I(T_i)}{I(T_{i-1})}\right), \tag{48}$$

and henceforth we assume that we explicitly know the characteristic function ϕ_{T_i} of $z(T_{i-1}, T_i)$

$$\phi_{T_{i-1},T_i}(u) = \mathbb{E}_n^{T_i}\left[\exp\left(iuz(T_{i-1}, T_i)\right)\big|\mathcal{F}_t\right]. \tag{49}$$

The derivation and explicit formulas of the characteristic function(s) are discussed in section 4.

Pricing of inflation-indexed swaps

The main two inflation-indexed swaps are the ZCIIS and the YYIIS. Recall that the zero-coupon swap is model independent and is simply given by no-arbitrage arguments, i.e. by (39). Given the characteristic function $\phi_{T_i}(u)$ from (49) of the log-inflation return under the T_i-forward measure, the pricing of a YYIIS is extremely simple. In fact recall from (45) that we have the following expression for the price of a YYIIS:

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i P(t, T_i)\mathbb{E}_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} - N\psi_i P_n(t, T_i), \tag{50}$$

and then note that the expectation in the above expression is nothing more than the characteristic function of the log-return evaluated in the complex-valued point $-i$,

$$\mathbb{E}_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} = \mathbb{E}_n^{T_i}\left\{\exp\left[i(-i)\log\left(\frac{I(T_i)}{I(T_{i-1})}\right)\right]\Big|\mathcal{F}_t\right\} = \phi_{T_i}(-i). \tag{51}$$

Hence the price of a YYIIS is just given by following simple expression:

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i P(t, T_i)\phi_{T_i}(-i) - N\psi_i P_n(t, T_i). \tag{52}$$

Pricing of inflation-indexed caplets/floorlets

The pricing of forward starting options like cliquets, attracted the recent attention of both practitioners as well as from academia (e.g. see Lucić (2003), Hong (2004) and Brigo and Mercurio (2006)). In this section we will show how one can price inflation call options in the framework of Carr and Madan (1999); working under the T_i-forward measure, we are in particular interested in the T_i-forward log return on the inflation index $z(T_{i-1}, T_i)$ between the times T_{i-1} and T_i:

$$z(T_{i-1}, T_i) := \log\frac{I(T_i)}{I(T_{i-1})} = z(T_i) - z(T_{i-1}), \tag{53}$$

where $I(T_i)$ and $z(T_i)$ respectively denote the inflation and log inflation index at time T_i. From (47) we know that we can express an inflation caplet as a call option on the forward return of the index. We can then place this directly in the Carr and Madan (1999) methodology of section 3.1. That is, we consider the log inflation return $z(T_{i-1}, T_i)$ and formulas (32)-(32), and then write the following for the price of an IICF (inflation-indexed caplet, $\omega = 1$ for a caplet, $\omega = -1$ for a floorlet):

$$
\begin{aligned}
\mathbf{IICplt}(t, T_{i-1}, T_i, \phi_i, K, N, \omega) &= N\phi_i P_n(t, T_i)\mathbb{E}_n^{T_i}\left\{\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)\right]^+\Big|\mathcal{F}_t\right\} \\
&= N\phi_i P_n(t, T_i)\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[e^{-(\alpha+iv)\log K}\psi_{T_{i-1}, T_i}(v, \omega, \alpha)\right]dv \\
&\quad + R\big(\exp(z(T_{i-1}, T_i)), K, \alpha(k)\big)
\end{aligned}
\tag{54}
$$

with $\psi_{T_{i-1}, T_i}(v, \omega, \alpha)$ a function of the characteristic function (under the T_i-forward measure) of the forward log-return between T_{i-1} and T_i as in (34) and with the residual term R as defined in (33). Alternatively the price of a floorlet can be expressed in terms of the corresponding caplet price (and vice versa) by means of a put-call parity, e.g. see Mercurio (2005). Given that we know the characteristic function, formula (54) provides an efficient and accurate way for determining the prices of inflation-indexed caps/floors. What remains is the derivation of this forward characteristic function, which we will discuss in section 4.

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3.3 FX and stock derivatives

The pricing of FX and stock derivatives within the general model (2.2) can be done using similar techniques as in the previous section with inflation-indexed derivatives. The main difference is that inflation-indexed derivatives are usually forward-starting options, whereas the vanilla FX and stock options do share this feature. In a way, one can therefore treat FX and stock options within the FX setup of our (2.2) as nested (degenerate) cases of inflation derivatives by choosing the forward-starting date equal to the current date and normalizing the stock/index price by $I(0)$, i.e. in accordance with (53). In a similar spirit, one can see a stock option as a FX option in which the foreign instantaneous interest rate represents the stochastic (or deterministic) continuous dividend rate of the stock.

For clarity we provide the pricing formulas for FX and stock options: working under the T-forward measure, the pricing formulas require the characteristic function

$$\phi_T(u) := \mathbb{E}^{Q^T}\left[\exp\left(iuz(T)\right)\middle|\mathcal{F}_t\right] \tag{55}$$

of the log index/FX-rate/stock price $z(T) := \log I(T)$. Equipped with this characteristic function, the time-T forward FX-rate $\mathrm{FFX}(T)$ (i.e. with convexity adjustment when the foreign interest rates are stochastic) is given by

$$\mathrm{FFX}(T) = \mathbb{E}^{Q^T}\left[I(T)\right] = \phi_T(-i). \tag{56}$$

Provided with the log-asset price characteristic function, one can immediately price a call/put option on the stock or FX-rate within 'Fourier-inversion' framework of section 3.1. More specifically, one can directly substitute the characteristic function for ϕ_T into the pricing formulas (32)-(34). Completely analogously to inflation-indexed options, one can price forward-starting (cliquet) options on the forward return of the FX-rate/stock index by substituting the characteristic function $\phi_{T_{i-1},T_i}(u)$ of the forward log return (53) into the pricing equations (32)-(34). We will discuss the derivation of both these characteristic functions in the next section.

4 Characteristic function of the model

In this section we will turn to the derivation of the characteristic function of the log inflation return under the nominal T-forward measure Q_n^T. For an inflation index which is driven by a Schöbel-Zhu stochastic volatility process, we are be able to derive a closed-form expression, whereas for the Heston stochastic volatility case we are able to approximate this characteristic function. Before turning to these derivations, we first turn to a volatility aspect of the inflation index and to the Gaussian interest rates, which treatment is common for both volatility choices.

Volatility driver and multi-factor Gaussian rates

To ease notation we introduce some matrix notation: let $\Sigma(t, T)$ denote the $1 \times (1+K+M)$ column vector of 'volatilities' driving the Brownian motion of the T-forward inflation index, with corresponding

$(1 + K + M) \times (1 + K + M)$ correlation matrix R, i.e.

$$\Sigma(t,T) = \begin{bmatrix} v(t) \\ \sigma_n^1 B_n^1(t,T) \\ \vdots \\ \sigma_n^K B_n^K(t,T) \\ -\sigma_r^1 B_r^1(t,T) \\ \vdots \\ -\sigma_r^M B_r^M(t,T) \end{bmatrix}, R = \begin{pmatrix} 1 & \rho_{x_n^1,v} & \cdots & \rho_{x_r^K,v} & \rho_{x_r^1,v} & \cdots & \rho_{x_r^M,v} \\ \rho_{x_n^1,v} & 1 & \cdots & \rho_{x_n^1,x_n^K} & \rho_{x_n^1,x_r^1} & \cdots & \rho_{x_n^1,x_r^M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \rho_{x_K^1,v} & \rho_{x_n^1,x_n^K} & \cdots & 1 & \rho_{x_n^K,x_r^1} & \cdots & \rho_{x_n^K,x_r^M} \\ \rho_{x_r^1,v} & \rho_{x_n^1,x_r^1} & \cdots & \rho_{x_n^K,x_r^1} & 1 & \cdots & \rho_{x_r^1,x_r^M} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{v,x_r^M} & \rho_{x_n^1,x_r^M} & \ddots & \rho_{x_K^K,x_r^M} & \rho_{x_r^1,x_r^M} & \cdots & 1 \end{pmatrix}, \tag{57}$$

Hence we can write the following for the instantaneous variance $v_F(t)$ of the inflation index under the T-forward measure:

$$v_F^2(t) = \Sigma'(t,T) R \Sigma(t,T). \tag{58}$$

Another useful expression is the integrated variance of the multi-factor Gaussian rate process; we can write the following for the instantaneous variance $v_{K,M}(t)$ of the sum of the rate processes:

$$v_{K,M}^2(t,T) = \sum_{i=2}^{K+M+1} \left(\Sigma^{(i)}(t,T)\right)^2 + 2\sum_{i=2}^{K+M+1}\sum_{j=i+1}^{K+M+1} R^{(i,j)} \Sigma^{(i)}(t,T)\Sigma^{(j)}(t,T) \tag{59}$$

with $\Sigma^{(i)}$ is the i-th element of the vector $\Sigma(t,T)$ and where $R^{(i,j)}$ denotes the element at row i and column j of the matrix R.

For the integrated rate variance $V_{K+M}(t,T)$ one has the following expression

$$V_{K,M}(t,T) := \int_t^T v_{K,M}^2(u,T)du = \sum_{i=2}^{K+M+1} C^{(i,i)} + 2\sum_{i=2}^{K+M+1}\sum_{j=i+1}^{K+M+1} R^{(i,j)} C^{(i,j)}, \tag{60}$$

where $C^{(i,j)}$ denotes the integrated covariance between the i-th and the the j-th element of the vector of rate volatilities $\Sigma(t)$. For the covariance between the first and the $K + M$-th element, one for example has

$$C^{(2,K+M+1)} := \int_t^T \left(\sigma_n^1 B_n^1(u,T)\right)\left(-\sigma_M^r B_r^M(u,T)\right)du \tag{61}$$

$$= -\frac{\sigma_n^1 \sigma_M^r}{a_n^1 a_r^M}\left((T-t) + \frac{e^{-a_n^1(T-t)} - 1}{a_n^1} + \frac{e^{-a_r^M(T-t)} - 1}{a_r^M} - \frac{e^{-(a_n^1+a_r^M)(T-t)} - 1}{a_n^1 + a_r^M}\right),$$

and for the special $i = j$ the above formula reduces to the integrated variance, for example

$$C^{(2,2)} := \int_t^T \left(\sigma_n^1 B_n^1(u,T)\right)^2 du = \left(\frac{\sigma_n^1}{a_n^1}\right)^2\left((T-t) + \frac{2}{a_n^1}e^{-a_n^1(T-t)} - \frac{1}{2a_n^1}e^{-2a_n^1(T-t)} - \frac{3}{2a_n^1}\right). \tag{62}$$

4.1 Schöbel-Zhu stochastic volatility

In this section we will determine the characteristic function (under the T-forward measure) of the forward log-inflation return $z(T_{i-1}, T_i)$ between times T_{i-1} and T_i. For this we first need to determine the characteristic function of the T-forward log-inflation rate z_T for a general maturity T. Building forth on the results of van Haastrecht et al. (2008), which authors derive the characteristic function for the one-factor Schöbel-Zhu-Hull-White model, we will derive its multi-factor generalization in the following subsection.

4.1.1 Characteristic function of the log-inflation rate

We will now determine the characteristic function of the reduced system (29), for which we shall use a partial differential approach; subject to the terminal boundary condition

$$f(T, z, v) = \exp\big(iuz(T)\big), \tag{63}$$

the Feynman-Kac theorem implies that the expected value of $\exp(iuz(T))$, equals the solution of the Kolmogorov backward partial differential equation for the joint probability distribution function $f(t, y, v)$, i.e.

$$f := f(t, z, v) = \mathbb{E}^{Q^T}\Big[\exp(iuz(T))\big|\mathcal{F}_t\Big]. \tag{64}$$

Thus the solution for f equals the characteristic function of the forward asset price dynamics (starting from z at time t). To obtain the Kolmogorov backward partial differential equation for the joint probability distribution function $f = f(t, y, v)$, we need to take into account the covariance term $dy(t)dv(t)$ which equals

$$
\begin{aligned}
dy(t)dv(t) &= \Big(v(t)dW_I^T(t) + \sum_{i=1}^{K}\sigma_n^i B_n^i(t, T)dW_{x_n^i}^T(t) - \sum_{j=1}^{M}\sigma_r^j B_r^j(t, T)dW_{x_r^j}^T(t)\Big)\Big(\tau dW_v^T(t)\Big) \\
&= \Big(\rho_{Iv}\tau v(t) + \sum_{i=1}^{K}\rho_{x_n^i v}\tau\sigma_n^i B_n^i(t, T) - \sum_{j=1}^{K}\rho_{x_r^j v}\tau\sigma_r^j B_r^j(t, T)\Big)dt.
\end{aligned}
\tag{65}
$$

The model we are considering is not an affine model in $y(t)$ and $v(t)$, but it is if we enlarge the state space to include $v^2(t)$:

$$dz(t) = -\frac{1}{2}v_F^2(t)dt + v_F(t)dW_F^T(t) \tag{66}$$

$$dv(t) = \kappa\big[\xi(u) - v(t)\big]dt + \tau dW_v^T(t) \tag{67}$$

$$dv^2(t) = 2v(t)dv(t) + \tau^2 dt = 2\kappa\Big(\frac{\tau^2}{2\kappa} + \xi(t)v(t) - v^2(t)\Big)dt + 2\tau v(t)dW_v(t) \tag{68}$$

Using (66) and (65), we obtain the following partial differential equation for $f(t, z, v)$:

$$
\begin{aligned}
0 = {} & f_t - \frac{1}{2}v_F^2(t)f_z + \kappa(\xi(t) - v(t))f_v + \frac{1}{2}v_F^2(t)f_{zz} \\
& + \Big(\rho_{Iv}\tau v(t) + \sum_{i=1}^{K}\rho_{x_n^i v}\tau\sigma^i B_n^i(t, T) - \sum_{j=1}^{M}\rho_{x_r^j v}\tau\sigma^j B_r^j(t, T)\Big)f_{zv} + \frac{1}{2}\tau^2 f_{vv}.
\end{aligned}
\tag{69}
$$

Solving this partial differential equation, subject to the terminal boundary condition (63), provides us with the characteristic function of the forward asset price dynamics (starting from z at time t). Due to the affine structure of our model, we come to the following proposition.

Proposition 4.1 *The characteristic function of domestic T-forward log inflation-rate of the our model with Schöbel and Zhu (1999) stochastic volatility is given by the following closed-form solution:*

$$f(t, z, v) = \exp\Big[A(u, t, T) + B(u, t, T)z(t) + C(u, t, T)v(t) + \frac{1}{2}D(u, t, T)v^2(t)\Big], \tag{70}$$

where:

$$A(u,t,T) = \int_t^T \left\{ \left[\kappa\psi + (1+iu)\sum_{i=1}^K \rho_{x_n^i \nu}\tau\sigma_n^i B_n^i(t,T) - iu\sum_{j=1}^M \rho_{x_r^j \nu}\tau\sigma_r^j B_r^j(t,T) \right] C(u,s,T) \right. \tag{71}$$

$$\left. + \frac{1}{2}\tau^2\big(C^2(u,s,T) + D(u,s,T)\big)\right\} ds,$$

$$B(u,t,T) = B := iu, \tag{72}$$

$$C(u,t,T) = -\frac{u(i+u)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}\left\{ \gamma_0\big(1 + e^{-2\gamma(T-t)}\big) \right. \tag{73}$$

$$+ \sum_{i=1}^K \left[\big(\gamma_3^i - \gamma_4^i e^{-2\gamma(T-t)}\big) - \big(\gamma_5^i e^{-a_n^i(T-t)} - \gamma_6^i e^{-(2\gamma+a_n^i)(T-t)}\big) - \gamma_7^i e^{-\gamma(T-t)} \right]$$

$$\left. - \sum_{j=1}^M \left[\big(\gamma_8^j - \gamma_9^j e^{-2\gamma(T-t)}\big) - \big(\gamma_{10}^j e^{-a_r^j(T-t)} - \gamma_{11}^j e^{-(2\gamma+a_r^j)(T-t)}\big) - \gamma_{12}^j e^{-\gamma(T-t)} \right]\right\}$$

$$D(u,t,T) = -u(i+u)\frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \tag{74}$$

and with:

$$\gamma = \sqrt{(\kappa - \rho_{I\nu}\tau B)^2 - \tau^2(B^2 - B)}, \qquad \gamma_0 = \frac{\kappa\psi}{\gamma} \tag{75}$$

$$\gamma_1 = \gamma + (\kappa - \rho_{I,\nu}\tau B), \qquad \gamma_2 = \gamma - (\kappa - \rho_{I,\nu}\tau B),$$

$$\gamma_3^i = \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_1 + \rho_{x_n^i\nu}\sigma_n^i\tau(1+B)}{a_n^i\gamma}, \qquad \gamma_4^i = \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_2 - \rho_{x_n^i,\nu}\sigma_n^i\tau(1+B)}{a_n^i\gamma},$$

$$\gamma_5^i = \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_1 + \rho_{x_n^i,\nu}\sigma_n^i\tau(1+B)}{a_n^i(\gamma - a_n^i)}, \qquad \gamma_6^i = \frac{\rho_{I,x_n^i}\sigma_n^i\gamma_2 - \rho_{x_n^i,\nu}\sigma_n^i\tau(1+B)}{a_n^i(\gamma + a_n^i)},$$

$$\gamma_8^j = \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_1 + \rho_{x_r^j,\nu}\sigma_r^j\tau B}{a_r^j\gamma}, \qquad \gamma_9^j = \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_2 - \rho_{x_r^j,\nu}\sigma_r^j\tau B}{a_r^j\gamma},$$

$$\gamma_{10}^j = \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_1 + \rho_{x_r^j,\nu}\sigma_r^j\tau B}{a_r^j(\gamma - a_r^j)}, \qquad \gamma_{11} = \frac{\rho_{I,x_r^j}\sigma_r^j\gamma_2 - \rho_{x_r^j,\nu}\sigma_r^j\tau B}{a_r^j(\gamma + a_r^j)},$$

$$\gamma_7^i = (\gamma_3^i - \gamma_4^i) - (\gamma_5^i - \gamma_6^i), \qquad \gamma_{12}^j = (\gamma_8^j - \gamma_9^j) - (\gamma_{10}^j - \gamma_{11}^j)$$

Proof See appendix A.

Using the above characteristic function of log-inflation index under the T-forward measure, we are in the following section able to derive the forward starting characteristic of the log-inflation index return.

4.1.2 Characteristic function of log-inflation index return

Recently the pricing of forward starting options attracted the attention of both practitioners as well as from academia e.g. see Lucić (2003), Hong (2004), van Haastrecht et al. (2008) and in an inflation context Mercurio and Moreni (2006) and Kruse (2007). In this section we will consider the pricing of forward starting options like inflation caplets within the general model setup combined with Schöbel-Zhu volatility. In particular, using the framework of Carr and Madan (1999), as described in section

3.1, it suffices to know the characteristic function of the following log-inflation index return under the T_i-forward measure:

$$z(T_{i-1}, T_i) := \log\left(\frac{I(T_i)}{I(T_{i-1})}\right) = \log I(T_i) - \log I(T_{i-1}) \tag{76}$$

since $I(t) = I_F(t)\frac{P_n(t,T_i)}{P_r(t,T_i)}$, we can also express this return in terms of the forward inflation rate $I_F(t)$ or equivalently in terms of the forward log inflation rate $z(t)$, i.e.

$$z(T_{i-1}, T_i) = z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i). \tag{77}$$

We are then interested in the characteristic function $\phi_{T_{i-1}, T_i}(u)$ of the log-inflation index return $z(T_{i-1}, T_i)$ under the T_i forward measure, i.e.

$$\phi_{T_{i-1}, T_i}(u) := \mathbb{E}^{Q^T}\Big[\exp\big(iu(z(T_i) - z(T_{i-1}))\big)\big|\mathcal{F}_t\Big]. \tag{78}$$

First define,

$$\Lambda := \exp\Big(iu\big[z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)\big]\Big) \tag{79}$$

hence by using the tower law for conditional expectations and the (conditional) characteristic function of our model (70), we obtain the following expression for the characteristic function of the (forward) log-return:

$$
\begin{aligned}
\phi_{T_{i-1}, T_i}(u) &= \mathbb{E}_n^{T_i}\Big\{\Lambda\big|\mathcal{F}_t\Big\} = \mathbb{E}_n^{T_i}\Big\{\mathbb{E}_n^{T_i}\Big[\Lambda\big|\mathcal{F}_{T_{i-1}}\Big]\Big|\mathcal{F}_t\Big\} \\
&= \mathbb{E}_n^{T_i}\Big\{\exp\Big(iu\big[-z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)\big]\Big) \\
&\quad \cdot \mathbb{E}_n^{T_i}\big[\exp[iuz(T_i)]\big|\mathcal{F}_{T_{i-1}}\big]\Big|\mathcal{F}_t\Big\} \\
&= \exp\Big(iu\big[A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i)\big] + A(u, T_{i-1}, T_i)\Big) \tag{80} \\
&\quad \cdot \mathbb{E}_n^{T_i}\Big\{\exp\Big(iu\big[\sum_{k=1}^{K} B_n^k(T_{i-1}, T_i)x_n^k(T_{i-1}) - \sum_{j=1}^{M} B_r^j(T_{i-1}, T_i)x_r^j(T_{i-1})\big] \\
&\quad + C(u, T_{i-1}, T_i)v(T_{i-1}) + \frac{1}{2}D(u, T_{i-1}, T_i)v^2(T_{i-1})\Big)\Big|\mathcal{F}_t\Big\}
\end{aligned}
$$

Though latter expectation depends only on the (correlated) Gaussian variates $x_n^k(T_{i-1}), x_r^j(T_{i-1}), v(T_{i-1})$, we also have that the integrated volatility process $\int_t^{T_{i-1}} v(u)du$ arises in the real rate processes $x_r^j(T_{i-1})$, e.g. see proposition 2.3. To this end, we decompose $x_r^j(T_{i-1})$ into

$$
\begin{aligned}
x_r^j(T_{i-1}) &= V^j(T_{i-1}) + \widetilde{x_r^j}(T_{i-1}) \tag{81} \\
V^j(T_{i-1}) &:= \rho_{I,x_r^j}\sigma_r^j \int_t^{T_{i-1}} e^{-a_r^j(T_{i-1}-u)}v(u)du \\
&\sim N\Big(\mu_V^j(t, T_{i-1}), \sigma_V^j(t, T_{i-1})\Big) \tag{82} \\
\widetilde{x_r^j}(T_{i-1}) &= \mu_r^j(t, T_{i-1}) + \sigma_r^j \int_t^{T_{i-1}} e^{-a_r^j(T_{i-1}-u)}dW_{r_j}^{T_i}(u), \\
&\sim N\Big(\mu_r^j(t, T_{i-1}), \sigma_r^j(t, T_{i-1})\Big), \tag{83}
\end{aligned}
$$

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where $\mu_r^j(t, T_{i-1}), \sigma_r^j(t, T_{i-1}), \mu_V^j(t, T_{i-1})$ and $\sigma_V^j(t, T_{i-1})$ as defined in (146), (147), (155) and (156), see appendix B.2.

Hence we find that the characteristic function (80) is of the following Gaussian-quadratic form:

$$\exp\left(iu\left[A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i)\right] + A(u, T_{i-1}, T_i)\right)$$

$$\cdot \mathbb{E}_n^{T_i}\left\{\exp\left(iu\left[\sum_{k=1}^{K} B_n^k(T_{i-1}, T_i)x_n^k(T_{i-1}) - \sum_{j=1}^{M} B_r^j(T_{i-1}, T_i)\left(V^j(T_{i-1}) + \widetilde{x_r^j}(T_{i-1})\right)\right]\right.\right.$$

$$\left.\left. + C(u, T_{i-1}, T_i)v(T_{i-1}) + \frac{1}{2}D(u, T_{i-1}, T_i)v^2(T_{i-1})\right)\bigg|\mathcal{F}_t\right\}$$

$$=: \quad \mathbb{E}_n^{T_i}\left\{\exp\left[a_0 + a'X + X'BX\right]\right)\right\}, \tag{84}$$

with a_0 a constant, a' a row-vector, B a matrix and where X follows a multivariate standard normal distribution with correlation matrix S. Thus the random vector X consists of the $1 + K + 2M$ driving elements $v, x_n^1, \ldots, x_n^K, x_r^1, \ldots, x_r^M, V^1, \ldots, V^M$. Note that since we are only dealing with one quadratic term (i.e. $v^2(T_{i-1})$), we can reduce the quadratic form (84) of the random vector X to

$$\mathbb{E}_n^{T_i}\left\{\exp\left[a_0 + a'X + b_0 X^{(1)^2}\right]\right\}, \tag{85}$$

where the constants a_0, b_0, the column-vector a and the correlation matrix S of the Gaussian vector X, can be easily be recognized from (84) and are explicitly defined in appendix B.4.

Using standard theory on Gaussian-quadratic forms (e.g. see Glasserman (2003) or Feuerverger and Wong (2000)) we can now easily find an explicit solution for (84). Recalling that (84) is equivalent to the characteristic function (80) of the forward return on the log inflation index, we come to the following proposition. Hence we come to the following proposition with for the forward characteristic function.

Proposition 4.2 *Let C be a matrix (with typical element $c_{i,j}$) satisfying $C'C = S$ (e.g. by a Cholesky decomposition), define*

$$p_j := \sum_{i=1}^{1+K+2M} c_{i,j}a^{(i)}, \tag{86}$$

$$q_1 := \sum_{i=1}^{1+K+2M} c_{i,1}^2 b_0, \tag{87}$$

with correlation matrix S, column-vector a and constant b_0 as defined in appendix B.4. Provided that $q_1 < \frac{1}{2}$, the characteristic function of the forward log return $z(T_{i-1}, T_i)$ (76) under the T_i-forward measure is given by the following closed-form solution:

$$\phi_{T_{i-1},T_i}(u) = \frac{\exp\left(a_0 + \frac{p_1^2}{2q_1(1-2q_1)} - \frac{p_1^2}{4q_1} + \sum_{j=2}^{1+K+2M} \frac{p_j^2}{2}\right)}{\sqrt{1-2q_1}}. \tag{88}$$

Proof Since (84) is equivalent to (80), the characteristic function of the forward return on the log inflation index is given by an explicit solution of the Gaussian-quadratic form (84), which is given by standard theory on quadratic forms, e.g. see Glasserman (2003) or Feuerverger and Wong (2000). $\square$

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Equipped with the characteristic function of the log-inflation index return, the prices of year-on-year inflation-indexed swaps and inflation-indexed caps/floors are directly obtained by the formulas (52) and (54).

4.2 Heston stochastic volatility

The characteristic function-based pricing method in our model with Heston (1993) stochastic volatility will turn out to be somewhat more involved than under Schöbel and Zhu (1999) stochastic volatility. In fact for the general model Heston (1993) stochastic volatility we need to resort to approximate methods for the pricing of inflation-indexed options.

Recall from (27) and (23) that the general model dynamics with Heston (1993) volatility under the T-forward measure Q_n^T are given by

$$\frac{dI_F(t)}{I_F(t)} = v(t)dW_I^T(t) + \sum_{i=1}^{K} \sigma_n^i dW_{n_i}^T(t) - \sum_{j=1}^{M} \sigma_r^j dW_{r_j}^T(t) \tag{89}$$

$$dv^2(t) = \kappa\Big[\zeta(t) - v^2(t)\Big]dt + \xi v(t)dW_v^T(t) \tag{90}$$

with $\zeta(t) = \theta + \sum_{i=1}^{K} \frac{\rho_{x_n^i,v}\sigma_n^i\xi v(t)}{\kappa}B_n^i(t,T)$ and where $(W_{n_1}^T, \ldots, W_{n_K}^T, W_{r_1}^T, \ldots, W_{r_M}^T, W_v^T)$ is a Brownian motion with possible full correlation structure under Q_n^T. In principle one can then pursue the same steps as in the model with Schöbel and Zhu (1999) volatility to derive the characteristic function of the log-inflation rate, that is solving the Kolmogorov backward equation for the log-inflation rate for a certain boundary condition. However, due to the square-root volatility process, the Heston partial differential equation in combination with correlated Gaussian rates is unfortunately not affine any more. Hence, contrary to the previous model, there is (as far as we know) no exact closed-form expression for the characteristic function for this model. Nevertheless, in case we make the simplifying assumption that the rate processes are perpendicular to the stochastic volatility and the asset price processes, one can easily find an closed-form solution for its characteristic function. For the general case, we consider two alternative pricing methods

1. A projection of characteristic function the general model onto the uncorrelated case.

2. A control variate based pricing technique that uses an uncorrelated case, for example the projection of method (1) usually serves as a very powerful control.

The setup of the following section is therefore as follows: we first discuss the pricing for the log-inflation rate and the log-inflation index return in the model with uncorrelated Heston (1993) stochastic volatility. Then we show a projection technique of the general case onto the uncorrelated model. Finally, though the projection already works quite well, we also discuss the use of the approximate model as control variate in a Monte Carlo pricing procedure of the exact model.

4.2.1 Characteristic function of the log-inflation rate: uncorrelated case

For the derivation of the characteristic function of the uncorrelated model (i.e. with rate processes perpendicular to the variance and asset price process), we will use two propositions.

First of all, let $z_{HE}(t) = \log \frac{I_{HE}(t)P_r(t,T)}{P_n(t,T)}$ denote the T-forward log-asset price, with dynamics

$$dz_{HE}(t) = -\frac{1}{2}v^2(t) + v(t)dW_I^T(t), \tag{91}$$

$$dv^2(t) = \kappa\big[\theta - v^2(t)\big]dt + \xi v(t)dW_v^T(t), \tag{92}$$

one then has following proposition regarding the characteristic function of $z_{HE}(t)$.

Proposition 4.3 *Conditional on time t, the characteristic function $\phi_{HE}(u)$ of the T-forward log-asset price $z_{HE}(T)$ of the classical* Heston *(1993) model is given by*

$$\phi_{HE}(u) := \exp\Big[iuz_{HE}(t) + A_{HE}(u,t,T) + B_{HE}(u,t,T)v^2(t)\Big] \tag{93}$$

where:

$$A_{HE}(u,t,T) := \theta\kappa\xi^{-2}\Big((\kappa - \rho\xi iu - d)T - 2\log(\frac{1 - g_2 e^{-dT}}{1 - g_2})\Big), \tag{94}$$

$$B_{HE}(u,t,T) := \xi^{-2}(\kappa - \rho\xi iu - d)\frac{1 - e^{-dT}}{1 - g_2 e^{-dT}} \tag{95}$$

and with:

$$d := \sqrt{(\rho\xi iu - \kappa)^2 + \xi^2(iu + u^2)}, \tag{96}$$

$$g_2 := \frac{\kappa - \rho\xi iu - d}{\kappa - \rho\xi iu + d}. \tag{97}$$

Proof For the proof we refer to Heston (1993) or Gatheral (2005). □

Note that in the literature one can find two (mathematically) equivalent formulations for the Heston characteristic function: the one presented above can for example be found in Albrecher et al. (2005) or Gatheral (2005) and is free of a numerical difficulty called branch cutting, while another representation can be found in the original Heston paper Heston (1993) or Kahl and Jäckel (2005), which may cause some numerical difficulties for certain model parameters, see Albrecher et al. (2005).

The second proposition concern the interest rates part of the inflation dynamics. To this end, define

$$R_{K,M}(t,T) := -\frac{1}{2}V_{K,M}(t,T) + \int_t^T \Big[\sum_{i=1}^K \sigma_n^i B_n^i(u,T)dW_{n_i}^T(u) - \sum_{j=1}^M \sigma_r^j B_r^j(u,T)dW_{r_j}^T(u)\Big]du, \tag{98}$$

we then come to the following proposition of the characteristic function of $R_{K,M}(t,T)$.

Proposition 4.4 *The characteristic function of $\phi_R(u)$ of $R_{K,M}(t,T)$ (98) is given by*

$$\phi_{K,M}(u) := \exp\Big[-\frac{1}{2}u(i + u)V_{K,M}(t,T)\Big]. \tag{99}$$

Proof First note that each of the factors $\int_t^T \sigma_n^i B_n^i(u,T)dW_{n_i}^T(u)du$ follows a Gaussian distribution with mean 0, hence $R_{K,M}(t,T)$ as sum of Gaussian variates is also Gaussian with mean $-\frac{1}{2}V_{K,M}(t,T)$. Using Fubini and Ito's isometry it then follows that $R_{K,M}(t,T)$ is normally distributed with mean $-\frac{1}{2}V_{K,M}(t,T)$ and variance $V_{K,M}(t,T)$ as explicitly given by (60). Moreover, the characteristic function $\phi_{K,M}(u)$ of $R_{K,M}(t,T)$ follows directly as consequence of this normality. □

With the results from propositions 4.3 and 4.4, we can now easily determine the characteristic function of the log-inflation index in the uncorrelated model, which results in the following proposition.

Proposition 4.5 *The characteristic function $\phi_F(u)$ for the log-inflation index $\log I_F(t)$ of the uncorrelated JY-HE dynamics (89) is given by the following closed-form expression:*

$$
\begin{aligned}
\phi_F(u) &= \phi_{HE}(u) \cdot \phi_{K,M}(u) \\
&= \exp\left[iuz_{HE}(t) + A_{HE}(u,t,T) + B_{HE}(u,t,T)v^2(t) - \frac{1}{2}V_{K,M}(t,T)\left(iu + u^2\right)\right]
\end{aligned}
\tag{100}
$$

Proof Since the Brownian motions driving the Heston dynamics $z_{HE}(t)$, i.e. $W_I^T(t)$ and $W_v^T(t)$, are uncorrelated with the Brownian motions that drive the rate process $R_{K,M}(t,T)$, i.e. $W_{n_i}^T(u)$ and $W_{r_j}^T(u)$, we have that we can write for the log-inflation index dynamics $\log I_F(t)$ of the dynamics of (29) (or equivalently of (89)) as the sum of the above two processes, i.e.

$$
\log I_F(t) = z_{HE}(t) + R_{K,M}(t,T).
$$

Since the driving Brownian motions are uncorrelated, we then have that $z_{HE}(t)$ is independent of $R_{K,M}(t,T)$ and furthermore that the characteristic function $\phi_F(u)$ of $\log I_F(t)$ is given by the product of the characteristic functions of $z_{HE}(t)$ and $R_{K,M}(t,T)$. □

4.2.2 Characteristic function of the log-inflation index return: uncorrelated case

We will now derive the (forward-starting) characteristic function of the log-inflation index return. Just as in our model from section 4.1.2, we follow Hong (2004) and van Haastrecht et al. (2008); that is, we consider the characteristic function $\phi_{T_{i-1},T_i}(u)$ of the log-inflation index return

$$
z(T_{i-1}, T_i) := \log\left(\frac{I(T_i)}{I(T_{i-1})}\right) = z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i).
\tag{101}
$$

In particular we want to resolve the characteristic function $\phi_{T_{i-1},T_i}(u)$ of $z(T_{i-1}, T_i)$ under the T_i-forward measure; using similar arguments (e.g. the tower law for conditional expectations) as in (80), we can obtain the following expression of the forward-starting characteristic function in our (uncorrelated) model:

$$
\begin{aligned}
\phi_{T_{i-1},T_i}(u) &= \mathbb{E}_n^{T_i}\Bigg\{\exp\Big(iu\Big[-z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)\Big]\Big) \\
&\qquad \cdot \mathbb{E}_n^{T_i}\Big[\exp[iuz(T_i)]\big|\mathcal{F}_{T_{i-1}}\Big]\Big|\mathcal{F}_t\Bigg\} \\
&= \exp\Big(A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2}V_{K,M}(t,T)\left(iu + u^2\right)\Big) \\
&\qquad \cdot \mathbb{E}_n^{T_i}\Bigg\{\exp\Big(iu\Big[-\log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)\Big] \\
&\qquad\qquad +B_{HE}(u, T_{i-1}, T_i)v^2(T_{i-1})\Big)\Big|\mathcal{F}_t\Bigg\}.
\end{aligned}
\tag{102}
$$

Hence since the rate processes $x_n^k(T_{i-1})$ and $x_r^j(T_{i-1})$ are independent of the variance process $v^2(T_{i-1})$, we have

$$
\begin{aligned}
\phi_{T_{i-1},T_i}(u) &= \exp\Big(A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2}V_{K,M}(t,T)\left(iu + u^2\right)\Big) \\
&\qquad \cdot \mathbb{E}_n^{T_i}\Big\{\exp\Big(iu\Big[-\log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i)\Big]\big|\mathcal{F}_t\Big\} \\
&\qquad \cdot \mathbb{E}_n^{T_i}\Big\{\exp\Big[B(u, t, T_{i-1})v^2(T_{i-1})\Big]\big|\mathcal{F}_t\Big\}.
\end{aligned}
\tag{103}
$$

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Hence its remains to evaluate the expectations in the latter expression; since the first expectation can be seen as the characteristic function of the log-bond prices, we have the following proposition.

Proposition 4.6 *The characteristic function $\phi_{K,M}(u)$ of the log bond prices in (103) under the T_i-forward measure is given by*

$$\phi_{K,M}(u) = \exp\left[iuh_0 - \frac{u^2}{2}h^{'}S_R h\right],\tag{104}$$

with the constant h_0, column vector h and correlation matrix S_R respectively as defined in (169), (170) and (171).

Proof Note that one can write

$$-\log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i) =: h_0 + h^{'}Z_R,\tag{105}$$

with Z_R the random Gaussian vector consisting of the normalized stochastic parts of the Gaussian factors $x_n^1, \ldots, x_n^K, x_r^1, \ldots, x_r^M$. Therefore (105) is nothing more than the characteristic function of the normal distribution $h_0 + h^{'}Z_R$, which is given by expression (104).

Alternatively, one can see this expectation as a special case of the Gaussian-quadratic form (84) of the model in proposition 4.2, i.e. without the volatility components $v(t)$ and $V^j(t)$). $\square$

For the calculation of the second expectation of (103) we will use the following property of the square root process $v^2(T_{i-1})$.

Proposition 4.7 *Provided that $2cy < 1$, the moment-generating function $\phi_{v^2}(y)$ of $v^2(T_{i-1})$ is given by*

$$\phi_{v^2}(y) = I\!E\left[\exp(yv^2(T_{i-1}))\right] = \frac{\exp(\frac{cy\lambda}{1-2cy})}{(1-2cy)^{\frac{2\kappa\theta}{\xi^2}}},\tag{106}$$

where

$$c \quad := \quad \frac{\xi^2(1 - e^{-\kappa(T_{i-1}-t)})}{4\kappa},\tag{107}$$

$$\lambda \quad := \quad \frac{4\kappa e^{-\kappa(T_{i-1}-t)}v(s)}{\xi^2(1 - e^{-\kappa(T_{i-1}-t)})}.\tag{108}$$

Proof The proposition follows directly from the fact that variance process $v^2(T_{i-1})$ is distributed as a constant c times a non-central chi-squared distribution with $\frac{4\kappa\theta}{\xi^2}$ degrees of freedom and non-centrality parameter λ, e.g. see Cox et al. (1985). $\square$

Hence we come to the following proposition for the characteristic function $\phi_{T_{i-1},T_i}(u)$ as in expression (103).

Proposition 4.8 *The forward-starting characteristic function $\phi_{T_{i-1},T_i}(u)$ of the model (2.2) with uncorrelated Heston (1993) stochastic volatility is given by the following closed-form expression:*

$$\phi_{T_{i-1},T_i}(u) = \exp\left(A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2}V_{K,M}(t,T)\left(iu + u^2\right)\right)\phi_{K,M}(u)\phi_{v^2}\left(B(u,t,T_{i-1})\right)\tag{109}$$

with $A_{HE}(u,t,T_{i-1})$ and $B_{HE}(u,t,T_{i-1})$ as defined in equations (94) and (95), and with $\phi_{K,M}(u)$ and $\phi_{v^2}(u)$ as in proposition (4.6) and 4.7.

Proof The characteristic function (109) of the forward log-inflation index return follows directly by evaluating the two expectations of (103). The first expectation of (103) equals the characteristic-generating function $\phi_{K,M}(u)$ of the log bond prices (105). The second expectation equals the moment-generating function ϕ_{v^2} of the shifted non-central chi-squared distributed random variable $v^2(T_{i-1})$, evaluated in the point $B(u,t,T_{i-1})$. $\square$

4.2.3   Projecting of the general case onto the uncorrelated model

Since in the general Heston model setup (i.e. with a full correlation structure) the affine structure is destroyed, it is challenging to find the characteristic function of the log-inflation index; we are not aware of a closed-form expression for characteristic function in the Heston model with correlated Gaussian rates. Nevertheless one can try to approximate the general dynamics by a simpler process for which a closed-form pricing expression does exists. Where a heuristic approach based on moment-matching techniques was suggested by van Haastrecht (2007), a more rigorous projection method was recently suggested by Antonov et al. (2008), which uses a Markovian projection technique of the general model onto the (affine) uncorrelated model. After the projected parameters are determined, one can just use the uncorrelated model and corresponding pricing formulas to price stock, foreign exchange and inflation derivatives. Though the Markovian projection technique is fast and works well for mild parameter settings and short maturities (i.e. when the 'distance' between the models is relatively small), the projection is rather involved and deteriorates for longer maturities and more extreme model parameters (i.e. when the 'distance' is relatively large), in particular for a large index-rate correlation in combination with a high volatility of the rates. For details on the Markovian projection and numerical results of the approximation, we refer the reader to Antonov et al. (2008).

4.2.4   Monte Carlo pricing method for the general model

Instead of approximating the prices of vanilla options in the general Heston setup, e.g. by a projection technique as touched upon in subsection 4.2.3, one can also entail a Monte Carlo procedure to price these options. Where the approximation formulas can be rather biased for certain model settings (e.g. see the discussion in subsection 4.2.3), a Monte Carlo estimate has the advantage that it converges to the true option price in the limit for the number of sample paths. Moreover a Monte Carlo procedure is generic and is straightforward to implement (if not already implemented for exotic option pricing). The main practical disadvantage of a Monte Carlo calibration procedure, is the speed with which vanilla option can be calculated within some error measure; since one repeatedly needs to update an error function of the 'distance' between model and market vanilla prices, the speed to calculate these model option prices is rather important. Even though one can price multiple options (e.g. on different times) with one Monte Carlo run, the use of closed-form option pricing formulas is often much faster. Nevertheless, with the use modern-day variance reduction techniques and the ever-growing computational power (in particular the fact that Monte Carlo procedure can be easily parallelized over multiple processers), we expect Monte Carlo techniques to become even more popular in the near future.

In this section we present an very effective control variate estimator for the pricing of vanilla options the general Heston dynamics. To demonstrate its efficiency, we take the pricing of a vanilla call option as example. To benchmark the numerical results against the Markovian projection, we consider the same hybrid equity-interest rate (stock) example as in Antonov et al. (2008). The setup of this section is as follows: we first discuss the control variate technique for the general model, after which we demonstrate which variance reductions can be expected and discuss its numerical efficiency.

Uncorrelated price as control variate estimator

As discussed in section 4.2.4, Monte Carlo pricing procedures might be easy to implement and quite generic, but often lack of speed and are hence sometimes being considered as 'brute-force'. Nowadays, however, a whole variety of variance reductions techniques are available to boost the computational efficiency of the Monte Carlo run, e.g. see Glasserman (2003) or Jäckel (2002) for an overview of such methods. One of these variance reductions techniques is the control variate estimator. The key

idea behind this technique is that we can use the error in estimating a similar quantity (from which we know the theoretical value) as a control to correct for the Monte Carlo error for the unknown quantity, see Glasserman (2003). The effectiveness of such a control variate depends explicitly on the correlation between the control and the to be estimated price. Thus if the control contains many information of the estimated price, it can correct quite a lot of Monte Carlo noise in the resulting estimator (and vice versa). Mathematically, it can be shown that, if the correlation between control and the standard Monte Carlo estimator are correlated with correlation coefficient ρ in combination with an optimal control parameter, one obtains (on average) a variance reduction of

$$\mathbf{VR}(\rho) = \frac{1}{1 - \rho^2}, \tag{110}$$

which can be enormous as $\rho \to 1$, e.g. see Glasserman (2003).

Before turning to the control variate estimator, we first introduce some notation. Let $\overline{C^0}, \overline{C^\rho}$ and C_i^0, C_i^ρ respectively denote the expected (European) call option price and the simulated call option prices for the general (superscript ρ) and the uncorrelated (superscript 0) dynamics. Since we know the call option price C^0 of the uncorrelated price in closed-form by inverting (109), and usually this price is largely correlated with the call option price C^ρ of the general model, we propose to use C^0 as a control for C^ρ; since the prices are highly correlated, we expect to see large variance reductions of the control variate estimator $\widetilde{C^\rho}(b)$ over the ordinary estimator C^ρ, i.e. from formula (110). The resulting control variate estimator $\widetilde{C^\rho}(b)$ is given by

$$\widetilde{C^\rho}(b) = \frac{1}{n} \sum_{i=1}^{n} (C_i^\rho - b(C_i^0 - \mathbb{E}[C^0])), \tag{111}$$

where b is a real coefficient. The optimal coefficient b^* that minimizes the variance of (111) can easily by calculated and is explicitly given by

$$b^* = \frac{\sigma_{C^\rho}}{\sigma_{C^0}} \rho_{C^0, C^\rho} = \frac{\mathrm{Cov}[C^0, C^\rho]}{\mathrm{Var}[C^0]}. \tag{112}$$

Note that one often also needs to estimate b^* from the simulations and this might induce some bias in the effectiveness (110) of the control variate. However, as discussed in Glasserman (2003), this bias is often very small; in case ρ_{C^0, C^ρ} is close to one and $\sigma_{C^\rho} \approx \sigma_C$ (which more than often is the case), it might even be a more efficient to just set b^* equal to one (since one does not have to estimate b^*, see Glasserman (2003). In section 5.1 the quality of the control variate estimator is investigated.

5 Applications and Numerical Results

In this section, we look at two applications of the model; first, for an equity example and with Heston (1993) stochastic volatility, we test the quality of the control variate estimator $\widetilde{C^\rho}$ of (111), compare it to the Markovian projection technique of Antonov et al. (2008) and discuss its practical applicability in a Monte Carlo calibration and/or pricing procedure. Secondly, we consider two applications (one with Schöbel and Zhu (1999) and one with Heston (1993) stochastic volatility) in which we calibrate our model to FX (option) market data. The example explicitly takes into account the pronounced long-term FX implied volatility skew/smile that is present in the markets. Finally the results are compared and analyzed.

5.1 Quality of the control variate estimator

To test the numerical quality of the control variate estimator $\widetilde{C^\rho}$ of (111), we turn to the pricing of (European) call options under the general hybrid Heston dynamics. To this end we consider two different parameter settings, listed in table 1 below.

Example	κ	ξ	ρ_{I,v^2}	$v(0)$	θ	y_r	y_q	a_n	σ_n	ρ_{I,x_n^1}	ρ_{v^2,x_n^1}
case I	2.0	1.0	-0.3	0.09	0.09	0.04	0.0	0.03	0.007	(∗)	0.0
case II	0.25	0.625	-0.4	0.0625	0.0625	0.05	0.02	0.05	0.01	0.30	0.15

Table 1: Numerical test cases for the Control Variate estimator (111). y_r denotes the continuous (constant) interest rate yield, y_q the continuous (constant) dividend yield, the (∗) indicates that we vary this parameter during the experiments and in all cases $I(0) = 100$.

Both test cases roughly correspond to parameter settings that are likely to be encountered in medium to long maturity equity markets. The first test case is prevalent in the existing literature: similar Heston parameter settings, in a pure equity context, are considered in Broadie and Kaya (2006), Lord et al. (2008) and Andersen (2007). The second test case is taken from Antonov et al. (2008) wherein it serves to test their Markovian projection approximation, i.e. as touched upon in section 4.2.3. Using these test cases, we first look at the quality of the control as function of the equity rates correlation coefficient and secondly we investigate the efficiency the control variate estimator (111) as function of the option maturity and compare it with the Markovian Projection technique of Antonov et al. (2008).

Results for case I

Though the uncorrelated price is often highly correlated with the price of the general model, the efficiency is dependent on the specific model parameters. For example notice that for $\rho_{I,x_1^n} = \rho_{v^2,x_1^n} = 0\%$, the control variate estimator is exact, because in that case the uncorrelated price equals the required estimate. Though the effectiveness depends on both correlation parameters, the impact of the correlation rate-vol is usually much smaller than the impact of the rate-stock correlation, e.g. see Antonov et al. (2008) or van Haastrecht et al. (2008). Moreover, from a practical point of view, the rate-stock parameter is most important for the pricing and hedging of hybrid equity-interest rate securities. We therefore restrict ourselves to investigate the impact of the rate-stock parameter on the quality of the control variate estimator: we look at the (empirical) variance reductions for a three year call option with an ATMF (at-the-money-forward) strike level of 100% for different ρ_{I,x_1^n}. The results can be found in table 2 below,

ρ_{I,x_1^n}	$\widehat{\rho_{C^0,C^\rho}}$	$\widehat{b}$	Var. Red.	ρ_{I,x_1^n}	$\widehat{\rho_{C^0,C^\rho}}$	$\widehat{b}$	Var. Red.
-0.9	99.859%	0.960	**356**	0.9	99.864%	1.034	**367**
-0.8	99.911%	0.965	**562**	0.8	99.913%	1.031	**574**
-0.7	99.940%	0.970	**839**	0.7	99.941%	1.027	**852**
-0.6	99.960%	0.974	**1254**	0.6	99.961%	1.024	**1268**
-0.5	99.974%	0.979	**1937**	0.5	99.974%	1.020	**1950**
-0.4	99.984%	0.983	**3188**	0.4	99.984%	1.016	**3202**
-0.3	99.992%	0.987	**5888**	0.3	99.992%	1.012	**5902**
-0.2	99.996%	0.992	**13597**	0.2	99.996%	1.008	**13614**
-0.1	99.999%	0.996	**55209**	0.1	99.999%	1.004	**55252**
0	100%	1	∞	-	-	-	-

Table **2**: Expected variance reductions when using the Control variate estimator of (111) instead of the standard Monte Carlo estimator. For various values of ρ_{I,x_1^n} the expected reduction for a three-year call option with an at-the-money strike is calculated using the estimates $\widehat{b}$ and $\widehat{\rho_{C^0,C^\rho}}$ respectively for the optimal control coefficient and correlation between the control and the estimated quantity. Parameter settings from **case I** of table **1**. Results based on 50.000 pseudo-random paths.

From the above table (the case $\rho_{I,x_1^n} = 1$ does not constitute in a valid correlation matrix and is hence omitted), we can see that the control is in all cases very effective, i.e. resulting in large to huge variance reductions. As expected, the variance reductions become larger for smaller absolute values of ρ_{I,x_1^n}; for the case $\rho_{I,x_1^n} = 0$, the control is perfect and results in a zero variance control variate estimator, whereas for larger values of $|\rho_{I,x_1^n}|$ the correlation between the 'uncorrelated' and 'correlated' option prices is smaller and therefore reduces the effectiveness of the control, as is being theoretically underpinned by formula (110). Thus from table **2** we can see that the effectiveness of the control, i.e. the resulting variance reduction, depends to a large extent on the absolute value of the correlation 'between' interest rates and equity underlying. Finally, it is worthwhile mentioning that because the ρ_{C^0,C^ρ} and $\frac{\sigma_{C^\rho}}{\sigma_{C^0}}$ the (estimated) optimal coefficients $\widehat{b}$ are also close to one. In such a situation it might be more efficient to just set $b^* = 1$ and consequently save the computational effort in estimating $\widehat{\rho_{C^0,C^\rho}}$, see Glasserman (2003).

Results for case II

The second test case of table **1**, consists of an experiment where investigate the variance reductions of (111) over the standard Monte Carlo estimator for European call options of different maturities and strikes. Furthermore, since the same parameters are being used in Antonov et al. (2008), we can use these results to draw a comparison between the Monte Carlo control variate estimator and the Markovian projection technique. The numerical results can be found in table **3** below,

Maturity in years	Strike	Sim. Vol. (std. dev.)* (Ordinary MC)	MP error*	Var. Red. (CV variate MC)
1	86.07	24.45 (0.06)	0.04	**6 381**
1	92.77	22.25 (0.05)	0.02	**5 884**
1	100.00	20.36 (0.05)	-0.04	**5 717**
1	107.79	19.42 (0.05)	-0.08	**6 549**
1	116.18	19.67 (0.06)	-0.03	**7 938**
3	77.12	22.61 (0.08)	0.03	**661**
3	87.82	20.05 (0.08)	0.01	**622**
3	100.00	17.95 (0.09)	-0.04	**629**
3	113.87	17.23 (0.13)	-0.09	**763**
3	129.67	18.02 (0.18)	-0.09	**985**
5	71.50	21.89 (0.06)	0.06	**250**
5	84.56	19.43 (0.05)	0.02	**240**
5	100.00	17.49 (0.06)	-0.05	**246**
5	118.26	16.83 (0.08)	-0.11	**295**
5	139.85	17.55 (0.12)	-0.13	**371**
10	62.23	21.55 (0.07)	0.06	**98**
10	78.89	19.52 (0.07)	0.00	**100**
10	100.00	18.01 (0.08)	-0.10	**106**
10	126.77	17.41 (0.11)	-0.19	**124**
10	160.70	17.75 (0.16)	-0.24	**152**
20	51.13	22.28 (0.06)	0.03	**54**
20	71.50	20.91 (0.06)	-0.05	**55**
20	100.00	19.94 (0.06)	-0.17	**57**
20	139.85	19.44 (0.09)	-0.27	**63**
20	195.58	19.40 (0.13)	0.35	**72**

Table **3**: Variance reductions for **case I** of table **1** using 50000 pseudo-random paths. Reported is the variance reduction factor ('Var. Red.'), i.e. the fraction between the variance of the control variate (111) and the standard Monte Carlo estimator. The starred results, i.e. from the simulated volatility and standard deviations ('Sim. Vol. (std. dev.)*') and the errors of the Heston DV Markovian projection ('MP error*'), were taken from Antonov et al. (2008).

From the above table, we can see that the control variate estimator by far outperforms the ordinary Monte Carlo estimator; for short to moderate maturity options the control variate shows large to huge variance reduction factors varying from 629 to 7938. For middle to long term option options, the variance reductions are smaller, but still quite reasonable with reductions from 54 to 371. If we look at the variance reductions over different strike levels, the differences are somewhat smaller. It is worthwhile to notice that, for a fixed maturity, the control variate is most effective for out-of-money options, which are usually the hardest options to value by (plain) Monte Carlo.

We conclude the analysis, by comparing the Monte Carlo control variate estimator (111) with the Markovian Projection technique. The results of the best projection technique of Antonov et al. (2008) is denoted by Heston DV (displaced volatility) and can be found in the fourth column of table **3**. The most crucial difference between the methods is that the Markovian Projection technique is in principle a biased approximation, whereas the control variate is unbiased and converges to the true price. However, in practice one often only has a limited available computational budget and one

will also notice bias in the Monte estimates as a consequence of the limited number of simulations, which bias could be larger than the error in the approximation. Essentially the choice between both methods therefore constitutes of a weigh-off between speed and accuracy, which might differ across applications and products. Nevertheless let us consider one concrete example; for example consider the pricing of a ten-year option ATMF call option and for argument-sake assume that the Monte Carlo volatility of 18.01 is in fact the true volatility and hence the Markovian Projection error is 0.10. We can then ask ourselves how many simulations are needed to improve the error of this approximation in at least 90% of the cases. By definition, 90% of all the spanned confidence intervals should contain the 'true' price of 18.01, hence to improve the MP error, we should aim to get the standard deviation of the Monte Carlo estimated volatility smaller than $\frac{0.10}{\Psi^{-1}(95\%)} = 0.061$ (or equivalently a variance smaller than 0.061^2). Using the fact that the Black and Scholes (1973) ATMF price is close to linear as a function of the volatility, taking the standard deviation 0.08 of the simulated volatility and the variance reduction factor 108 of the above table and assuming a convergence rate of the Monte Carlo of one over square root of the number of simulations N, one can find that one needs

$$M = \frac{\text{Var}_N}{\text{Var}_{\text{REQ}}} \frac{N}{\text{VR}} = \frac{0.08^2}{0.061^2} \frac{50000}{108} = 802$$

simulations to improve upon the MP error in 90% of the cases, with VR the variance reduction factor and where Var_{REQ} represents the required variance corresponding to a confidence level $1 - \alpha = 90\%$. Should we for example take $\alpha = 50\%$, one can find that on average one only has to use 134 simulations to perform 'equally well' as the MP projection. Hence due to the large variance reductions, only a very moderate amount of simulations is needed to come up with a good estimate. Though the above analysis is too small the draw very strong conclusions about the comparison between the MP projection technique and the control variate, the main conclusion we do like to draw is that only a moderate amount of simulations is required to obtain reliable price/volatility estimates for the above call options: in most situations a couple of thousand paths will suffice to obtain prices that lie within typical bid-ask spreads.

Finally we would also like to point out that the MP projection might also be used in conjunction with the control variate estimator (111) in a model calibration procedure; a first point (in future research) could be to investigate the quality of the MP projection as control for the exact dynamics. Secondly, in a practical implementation one might first use the MP approximation to calibrate the model (which consists of most of the iterations) and consecutively use the control variate to refine the (near) optimal parameters found in the previous minimization. Please note hereby that (for each new parameter guess) one only needs a single Monte Carlo run to price all options simultaneously. In this way (assuming one uses a sufficiently large number of paths in the last few optimization steps using the Monte Carlo) one can get the best of both worlds, i.e. the speed of an approximating formula combined with the accuracy of the control variate estimator.

5.2 Calibration to FX market

We will test our model by calibrating it to FX option market data. To this end, we consider the same vanilla FX data (see appendix C) as is being considered in Piterbarg (2005) which uses this set for the calibration of his local volatility model. In an elegant paper, Piterbarg (2005) concludes that for the pricing and managing of exotic FX derivatives (i.e. PRDCs), it is essential to take the FX implied volatility skew/smile into account; hence though FX model setups may differ, i.e. local volatility in Piterbarg (2005), Heston (1993) stochastic volatility with independent stochastic interest rate drivers in Andreasen (2006) and our stochastic volatility model with multi-factor Gaussian rates and Heston

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(1993) or Schöbel and Zhu (1999) volatility under a full correlation structure, all these models share the essential feature of explicitly accounting for the FX skew/smile.

For the calibration results of our model we consider the same interest rate and correlation parameters as in Piterbarg (2005); that is, the interest curves in the domestic (Japanese yen) and foreign (US dollar) economies are given by

$$
\begin{aligned}
P_n(0, T) &= \exp\left(-0.02 \cdot T\right), \\
P_r(0, T) &= \exp\left(-0.05 \cdot T\right),
\end{aligned}
$$

and the one-factor Hull and White (1993) interest rate parameters for the interest rate evolutions in both currencies are given by

$$
\begin{aligned}
a_n(t) &:= 0.0\%, \quad \sigma_n(t) := 0.0\%, \\
a_n(t) &:= 5.0\%, \quad \sigma_n(t) := 1.2\%.
\end{aligned}
$$

The correlation parameters are given by

$$
\rho_{n,r} = 25.00\%, \quad \rho_{I,n} = \rho_{I,r} = -15.00\%, \quad \rho_{n,v} = \rho_{r,v} = 0.00\%.
$$

Note that our stochastic volatility model has the additional flexibility of correlating the domestic of foreign exchanges with the volatility drivers (i.e. through $\rho_{n,v}$ or $\rho_{r,v}$), however for simplicity we fix them to zero here. The initial spot FX rate (yen per dollar) is set at 105.00. The ten expiry dates that are being considered in the calibration, and the seven strikes that are being considered per date, are given in table 6 of appendix C. For each maturity T_n, the strikes $K_i(T_n)$ are being computed using the formula

$$
K_i(T_n) = F(0, T_n) \exp\left(0.1 \cdot \delta_i \sqrt{T_n}\right), \qquad \delta_i \in \{-1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5\}. \tag{113}
$$

In particular, note that the fourth strike level corresponds to the forward FX rate for that date. Using the above parameter setup, we calibrate our model(s) (2.2), i.e. first with Schöbel and Zhu (1999) stochastic volatility and then with Heston (1993) stochastic volatility, to the FX data set as described in appendix C. The calibration results are reported in the following section.

5.2.1 Calibration results using Schöbel-Zhu stochastic volatility

In this section we look how well the model (2.2), i.e. with Schöbel and Zhu (1999) and Heston (1993) stochastic volatility, fits the market. The FX option market volatilities are given in table 7 of appendix C. Using a local minimization method, we then calibrate the model to the various maturities by minimizing the differences between model and market implied volatilities. The differences are reported in table 4 and 5 below.

	strike 1	strike 2	strike 3	strike 4	strike5	strike 6	strike 7
0.5	-0.11%	0.00%	0.03%	-0.02%	-0.09%	0.00%	0.28%
1	-0.18%	0.00%	0.08%	0.00%	-0.18%	-0.14%	0.22%
3	-0.47%	0.00%	0.29%	0.30%	0.00%	-0.16%	0.02%
5	-0.42%	0.00%	0.25%	0.27%	0.04%	-0.12%	0.00%
7	-0.74%	0.00%	0.57%	0.80%	0.56%	-0.07%	-0.81%
10	-0.67%	0.00%	0.48%	0.69%	0.55%	0.12%	-0.49%
15	-0.45%	0.00%	0.22%	0.22%	0.02%	-0.35%	-0.82%
20	-0.83%	-0.27%	0.07%	0.22%	0.18%	0.00%	-0.33%
25	-1.07%	-0.44%	0.00%	0.26%	0.34%	0.26%	0.04%
30	-1.29%	-0.53%	0.00%	0.35%	0.51%	0.52%	0.39%

Table **4**: Differences, in implied Black volatilities, between market and model values using Schoöbel-Zhu stochastic volatility.

The model produces a very good fit to the market, as can be seen from table **4**, with differences smaller than 0.50% in most points and with an excellent fit around the at-the-money-forward volatilities and the slope of the volatility skews for each maturity. The model produces similar calibration results as the models of Piterbarg (2005) and Andreasen (2006). The low-strike (in-the-money call) options are underestimated by the model, which seems to have slight difficulties in fitting the tails of the implied volatility structure, suggesting the addition of an extra factor, e.g. a trivial extension including Poisson-type jumps. Nonetheless, the smiles produced by the model are much closer to the market than a log-normal model would indicate. In particular the fit is much better than a log-normal model for in- and out-the-money options.

5.2.2 Calibration results using Heston stochastic volatility

In the second test case, we look how well the model (2.2) with Heston (1993) stochastic volatility fits the market. For simplicity we consider uncorrelated stochastic volatility, as we can then directly price the required FX options in closed form. Nonetheless, the calibration results to call option prices should be very similar as it is shown in Antonov et al. (2008), that the parameters of the general model can often be projected onto parameters of the uncorrelated model, while to a large extent preserving these option prices. We then fit the model by minimizing the differences between model and market implied volatilities, which calibration results can be found in table **5** below.

	strike 1	strike 2	strike 3	strike 4	strike5	strike 6	strike 7
0.5	-0.15%	0.00%	0.05%	0.00%	-0.11%	-0.05%	0.19%
1	-0.20%	0.00%	0.09%	0.00%	-0.13%	0.00%	0.41%
3	-0.25%	0.00%	0.09%	0.00%	-0.18%	-0.06%	0.43%
5	-0.19%	0.00%	0.07%	0.00%	-0.19%	-0.23%	0.02%
7	-0.11%	0.00%	0.03%	0.00%	-0.09%	-0.14%	0.05%
10	-0.04%	0.01%	0.01%	0.01%	-0.02%	-0.05%	0.00%
15	-0.01%	0.01%	0.00%	-0.01%	0.00%	-0.01%	0.00%
20	-0.30%	0.00%	0.13%	0.13%	0.00%	-0.24%	-0.60%
25	-0.29%	0.00%	0.16%	0.21%	0.15%	-0.03%	-0.30%
30	-0.72%	-0.28%	0.00%	0.16%	0.20%	0.11%	-0.07%

Table **5**: Differences, in implied Black volatilities, between market and model values using Heston stochastic volatility.

From the above table, we can see that the model again produces a very good fit to the market, with differences now even smaller than 0.30% in most points and with excellent fits across moneyness and maturities. It seems that Heston (1993) model is slightly better in capturing the extreme FX skew we calibrating against and in a way is able to capture both the volatility part of the at-the-money prices, as well as the extreme event part of in- and out-the-money prices. Alternatively, one can argue that the addition of an extra factor is still needed for the pricing of certain exotic options (e.g. see van der Ploeg (2006) and Fouque et al. (2000)), which discussion is however beyond the scope of this article. It is shown in Piterbarg (2005) and Andreasen (2006), that it is of crucial importance to toke the FX skew into account for the pricing and managing of exotic FX structures like PRDCs (power reverse dual contracts) or cliquets. Therefore, since the skews/smiles generated by our stochastic volatility models are much closer to the market than produced by a log-normal model, we can conclude our stochastic volatility model(s) (2.2) is better suited to price and manage these exotic FX structures. Finally, though the models of Piterbarg (2005) and Andreasen (2006) account for the FX skew, our model(s) stands out as we model stochastic volatility (versus local volatility used in Piterbarg (2005)) and stochastic interest rates, where we allow all driving model factors to be instantaneously correlated with each other (versus independent Gaussian rates used in Andreasen (2006)). Having the flexibility to correlate the underlying FX-rate with both stochastic volatility and stochastic interest rates yields a realistic model, which is of practical importance for the pricing and hedging of options with a long-term FX exposure.

6   Conclusion

We have developed a new model incorporating stochastic volatility and multi-factor Gaussian interest rates under a full correlation structure of all driving model factors. The model is suitable for the pricing and hedging of multi-currency structures which are particularly sensitive to future interest rates evolutions and volatility skews/smiles. Such options include the popular equity-interest rate TARN options, inflation LPI options and PRDC FX swaps. Since an inflation option could be seen as an forward starting FX option and a stock option as an FX option wherein the foreign rates represent the dividend yield, we have focussed on the pricing of inflation-indexed derivatives: the pricing of vanilla FX and stock options follows directly as a nested case. By inverting the characteristic function of the forward log-inflation index price or the forward log-inflation index return, we have shown how call/put options, forward starting options, year-on-year inflation-indexed swaps and inflation-indexed

caps/floors can be valued in closed-form. Under Schöbel and Zhu (1999) stochastic volatility, using its affine properties, we were able to derive the corresponding characteristic functions in closed-form, hence the pricing of these options is exact. Under Heston (1993) stochastic volatility, these can only be derived under special (i.e. zero) correlation assumptions. Nonetheless the pricing formulas derived for this uncorrelated case are directly applicable by either using a projection of the general model onto the uncorrelated case, or by using it as a control variate for the general model which results in such large variance reductions that its incorporation in the calibration procedure becomes a more than viable option.

A Deriving the characteristic function of the log 'Schöbel-Zhu' inflation rate

In this appendix we will prove that the partial differential equation (69), i.e.

$$
\begin{aligned}
0 \;=\; & f_t - \frac{1}{2}v_F^2(t)f_z + \kappa(\xi(t) - v(t))f_v + \frac{1}{2}v_F^2(t)f_{zz} \\
& + \Big(\rho_{Iv}\tau v(t) + \sum_{i=1}^{K}\rho_{x_n^i v}\tau\sigma_n^i B_n^i(t,T) - \sum_{j=1}^{M}\rho_{x_r^j v}\tau\sigma_r^j B_r^j(t,T)\Big)f_{zv} + \frac{1}{2}\tau^2 f_{vv},
\end{aligned}
\tag{114}
$$

subject to the terminal boundary condition $f(T,y,\sigma) = \exp(iuy(T))$ has a solution given by (70)-(75); to solve this differential equation, we use the ansatz (70), find the corresponding partial derivatives and substitute these in (69). We then obtain a system of ordinary differential equations that is similar to the one-factor model as in van Haastrecht et al. (2008) and which can be solved using similar methods.

Expanding $v_F^2(t)$ according to (58) and collecting the terms for $y(t)$, $v(t)$ and $\frac{1}{2}v^2(t)$ yields the following system of ordinary differential equations for the functions $A(u,t,T),\ldots,D(u,t,T)$:

$$
0 \;=\; \frac{\partial B(u,t,T)}{\partial t} \quad \Rightarrow B(u,t,T) := B,
\tag{115}
$$

$$
0 \;=\; \frac{\partial D(u,t,T)}{\partial t} - 2(\kappa - \rho_{xv}\tau B)D(u,t,T) + \tau^2 D^2(t) + (B^2 - B),
\tag{116}
$$

$$
\begin{aligned}
0 \;=\; & \frac{\partial C(u,t,T)}{\partial t} + (\rho_{xv}\tau B - \kappa + \tau^2 D)C(u,t,T) \\
& + \Big\{\sum_{i=1}^{K}[\rho_{Ix_n^i}\sigma_n^i B_n^i(t,T)] - \sum_{j=1}^{M}[\rho_{Ix_r^j}\sigma_r^j B_r^j(t,T)]\Big\}(B^2 - B) \\
& + \Big\{\kappa\xi(t) + \Big(\sum_{i=1}^{K}[\rho_{x_n^i v}\tau\sigma_n^i B_n^i(t,T)] - \sum_{j=1}^{M}[\rho_{x_r^j v}\tau\sigma_r^j B_r^j(t,T)]\Big)B\Big\}D(u,t,T),
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
0 \;=\; & \frac{\partial A(u,t,T)}{\partial t} + \Big[\kappa\xi(t) + \sum_{i=1}^{K}\rho_{x_n^i v}\tau\sigma_n^i B_n^i(t,T)B - \sum_{j=1}^{M}\rho_{x_r^j v}\tau\sigma_r^j B_r^j(t,T)B\Big]C(u,t,T) \\
& + \frac{1}{2}\tau^2(C^2(u,t,T) + D(u,t,T)) + \frac{1}{2}(B^2 - B)v_{K,M}^2(t,T),
\end{aligned}
\tag{118}
$$

with $v_{K,M}^2(t,T)$ the instantaneous variance of the Gaussian rate processes, see (59). As already hinted at in equation (115), it immediately that follows $B(u,t,T) = B$ equals a constant since its derivative is zero. Subject to the boundary condition (63) we then find

$$
B = iu.
\tag{119}
$$

The second equation (116) yields a Riccati equation with constant coefficients and boundary condition $D(u,T,T) = 0$ which is equivalent to the PDE for the D-term in the SZHW model (see van Haastrecht

et al. (2008)) and has the following solution:

$$D(u, t, T) = -u(i + u)\frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \tag{120}$$

$$\text{with: } \gamma = \sqrt{(\kappa - \rho_{xv}\tau B)^2 - \tau^2(B^2 - B)}, \tag{121}$$

$$\gamma_1 = \gamma + \frac{1}{2}q_1 = \gamma + (\kappa - \rho_{xv}\tau B), \tag{122}$$

$$\gamma_2 = \gamma - \frac{1}{2}q_1 = \gamma - (\kappa - \rho_{xv}\tau B). \tag{123}$$

The third equation (117) for $C(u, t, T)$ looks pretty daunting, but is merely a first order linear differential equation of the form $\frac{\partial C(u,t,T)}{\partial t} + g(t)C(u, t, T) + h(t) = 0$, with associated boundary condition $C(u, T, T) = 0$. Hence we can represent a solution for $C(u, t, T)$ as:

$$C(u, t, T) = \int_t^T h(s) \exp[\int_t^s g(v)dv]ds, \tag{124}$$

$$\text{with: } g(v) = -(\kappa - \rho_{xv}\tau B) + \tau^2 D(u, v, T), \tag{125}$$

$$h(s) = \left(\kappa\xi(u) + \sum_{i=1}^K (\rho_{x_n^i v}\tau\sigma_n^i B_n^i(s, T) - \sum_{j=1}^M \rho_{x_r^j v}\tau\sigma_r^j B_r^j(s, T))B\right)D(u, s, T)$$

$$+ \left(\sum_{i=1}^K \rho_{Ix_n^i}\sigma_n^i B_n^i(s, T) - \sum_{j=1}^M \rho_{Ix_r^j}\sigma_r^j B_r^j(s, T)\right)(B^2 - B)$$

$$= \kappa\psi D(u, s, T)$$

$$+ \sum_{i=1}^K \left\{\rho_{Ix_n^i}\sigma_n^i B_n^i(s, T)(B^2 - B) + \left[\rho_{x_n^i v}(1 + B)\tau\sigma_n^i B_n^i(s, T)\right]D(u, s, T)\right\}$$

$$- \sum_{j=1}^M \left\{\rho_{Ix_r^j}\sigma_r^j B_r^j(s, T)(B^2 - B) + \left[\rho_{x_r^j v}B\tau\sigma_r^j B_r^j(s, T)\right]D(u, s, T)\right\}. \tag{126}$$

We first consider the integral over g: dividing equation (125) by $D(u, t, T)$, rearranging terms and integrating we find the surprisingly simple solution:

$$\int g(v)dv = \int -(\kappa - \rho_{xv}\tau B) + \tau^2 D(u, v, T)dv$$

$$= \int (\kappa - \rho_{xv}\tau B) - \frac{(B^2 - B)}{D(u, v, T)} - \frac{\partial D(u, v, T)}{\partial v}\frac{1}{D(u, v, T)}dv$$

$$= \log(\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}) + c, \tag{127}$$

where γ, γ_1 and γ_2 are defined in (75) and with c denotes the integration constant. Hence taking the exponent and filling in the required integration boundaries yields

$$\exp\left[\int_t^s g(v)dv\right] = \frac{\gamma_1 e^{\gamma(T-s)} + \gamma_2 e^{-\gamma(T-s)}}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}, \tag{128}$$

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Hence substituting this expression into (124) we find (after a long but straightforward calculation) for $C(u, t, T)$:

$$
\begin{aligned}
C(u, t, T) \;=\; & -\frac{u(i + u)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}} \Big\{ \gamma_0 \big(1 + e^{-2\gamma(T-t)}\big) \\
& + \sum_{i=1}^{K} \Big[\big(\gamma_3^i - \gamma_4^i e^{-2\gamma(T-t)}\big) - \big(\gamma_5^i e^{-a_n^i(T-t)} - \gamma_6^i e^{-(2\gamma + a_n^i)(T-t)}\big) - \gamma_7^i e^{-\gamma(T-t)} \Big] \\
& - \sum_{j=1}^{M} \Big[\big(\gamma_8^j - \gamma_9^j e^{-2\gamma(T-t)}\big) - \big(\gamma_{10}^j e^{-a_r^j(T-t)} - \gamma_{11}^j e^{-(2\gamma + a_r^j)(T-t)}\big) - \gamma_{12}^j e^{-\gamma(T-t)} \Big] \Big\}
\end{aligned}
\tag{129}
$$

with the constants $\gamma, \gamma_0, \ldots, \gamma_{12}^j$ as defined in (75).

Finally, by integration equation (119), we find the following expression for $A(u, t, T)$:

$$
\begin{aligned}
A(u, t, T) \;=\; & \int_t^T \Big[\frac{1}{2}(B^2 - B) v_{K,M}^2(t, T) + \kappa \xi(s) C(u, s, T) + \frac{1}{2}\tau^2 (C^2(u, s, T) + D(u, s, T)) \Big] ds \\
\;=\; & -\frac{1}{2} u(i + u) V_{K,M}(t, T) \\
& + \int_t^T \Big\{ \Big[\kappa \psi + (1 + iu) \sum_{i=1}^{K} \rho_{x_n^i v} \tau \sigma_n^i B_n^i(t, T) - iu \sum_{j=1}^{M} \rho_{x_r^j v} \tau \sigma_r^j B_r^j(t, T) \Big] C(u, s, T) \\
& + \frac{1}{2} \tau^2 \big(C^2(u, s, T) + D(u, s, T)\big) \Big\} ds,
\end{aligned}
\tag{130}
$$

where $V_{K,M}(t, T)$ is the integrated variance of the multi-factor Gaussian rates which can found by simple integration, see (58). It is possible to write a closed-form expression for the remaining integral in (130). As the ordinary differential equation for $D(u, s, T)$ is exactly the same as in the Heston (1993) or Schöbel and Zhu (1999) model, it will involve a complex logarithm and should therefore be evaluated as outlined in Lord and Kahl (2008) in order to avoid any discontinuities. The main problem however lies in the integrals over $C(u, s, T)$ and $C^2(u, s, T)$, which will involve the Gaussian hypergeometric $_2F_1(a, b, c; z)$. The most efficient way to evaluate this hypergeometric function (according to *Numerical Recipes*, Press and Flannery (1992)) is to integrate the defining differential equation. Since all of the terms involved in $D(u, s, T)$ are also required in $C(u, s, T)$, numerical integration of the second part of (130) seems to be the most efficient method for evaluating $A(u, t, T)$. Hereby we conveniently avoid any issues regarding complex discontinuities altogether.

B Analytical properties of the Gaussian factors driving the asset price process

In this section we will discuss some properties of the processes that drive the asset price dynamics. That is, we discuss the pricing of bonds under multi-factor Gaussian interest rates (section B.1) and the moments of the Gaussian interest rates processes and the Ornstein-Uhlenbeck distributed volatility process under the T-forward measure (section B.2).

B.1 Zero-coupon bond prices under multi-factor Gaussian rates

In this appendix we briefly review zero-coupon bond prices of the Gaussian multi-factor rate model, i.e. one has the following analytical formulas for the zero-coupon bond prices (e.g. see Brigo and Mercurio (2006) for the two-factor model, which easily extends to the multi-factor case):

$$P_n(t,T) \;=\; \mathbb{E}_n\left\{e^{-\int_t^T n(u)du}\right\} = A_n(t,T)e^{-\sum\limits_{i=0}^{K} B_n^i(t,T)x_n^i(t)}, \tag{131}$$

$$A_n(t,T) \;=\; \frac{P_n^M(0,T)}{P_n^M(0,t)}\exp\left\{\frac{1}{2}\Big[V_n(t,T)-V_n(0,T)+V_n(0,t)\Big]\right\}, \tag{132}$$

$$B_n^i(t,T) \;=\; \frac{1-e^{-a_n^i(T-t)}}{a_n^i}, \tag{133}$$

and with completely analogous expressions for the real bond prices $P_r(t,T)$ and affine terms $A_r(t,T)$, $B_r(t,T)$. For the integrated rate variances $V_i(t,T)$, one also has closed-form expressions. To this end we let (just as in section 4) $C^{(i,j)}$ and $R^{(i,j)}$ respectively denote the integrated covariance and correlation between the i-th and j-th element of the vector of rate volatilities $\Sigma(t)$ of (57). Hence one can then express the integrated rate variance as

$$V_n(t,T) \;=\; \sum_{i=2}^{K+1} C^{(i,i)} + 2\sum_{i=2}^{K+1}\sum_{j=i+1}^{K+1} R^{(i,j)}C^{(i,j)}, \tag{134}$$

$$V_r(t,T) \;=\; \sum_{i=K+2}^{K+M+1} C^{(i,i)} + 2\sum_{i=K+2}^{K+M+1}\sum_{j=i+1}^{K+M+1} R^{(i,j)}C^{(i,j)}. \tag{135}$$

Analytical expressions for the covariances can be found in section 4.

B.2 Moments of the interest rate and volatility processes

In this appendix, we will derive the moments of the stochastic factors that drive the nominal, real and volatility rate. Since all factors follow Ornstein-Uhlenbeck processes, the moments can be found relatively easy.

Moments of the volatility process

By integrating the T_i forward dynamics of (21) conditional on $v(t)$, we obtain

$$v(T_{i-1}) = v(t)e^{-\kappa(T_{i-1}-t)} + \int_t^{T_{i-1}} \xi(u)e^{-\kappa(T_{i-1}-u)}du + \tau\int_t^{T_{i-1}} e^{-\kappa(T_{i-1}-u)}dW_v^T(u), \tag{136}$$

<div align="center">35</div>

where $\xi(u) := \psi + \sum_{i=1}^{K} \frac{\rho_{x_n^i,\nu}\sigma_n^i\tau}{a_n^i\kappa}[1 - e^{-a_n^i(T_i-u)}]$. From Ito's isometry, we then have the that the mean and variance of ν under the T_i-forward measure are given by:

$$\mu_\nu \;=\; \nu(t)e^{-\kappa(T_{i-1}-t)} + \Big(\psi + \sum_{i=1}^{K} \frac{\rho_{x_n^i,\nu}\sigma_n^i\tau}{a_n^i\kappa}\Big)\Big(1 - e^{-\kappa(T_i-t)}\Big)$$

$$+ \sum_{i=1}^{K} \frac{\rho_{x_n^i,\nu}\sigma_n^i\tau}{a_n^i(\kappa+a_n^i)}\Big[e^{-a_n^i(T_i-t)-\kappa(T_{i-1}-t)} - e^{-a_n^i(T_i-T_{i-1})}\Big], \tag{137}$$

$$\sigma_\nu^2 \;=\; \frac{\tau^2}{2\kappa}\Big(1 - e^{-2\kappa(T_{i-1}-t)}\Big). \tag{138}$$

Moments of the rate processes

Conditional on time t, one can integrate the rate dynamics of $x_n^i(T_{i-1})$ and $x_r^j(T_{i-1})$, from time t to T_{i-1}, to obtain the following following explicit solutions (see also Pelsser (2000) or Brigo and Mercurio (2006))

$$x_n^k(T_{i-1}) \;=\; x_n^k(t)e^{-a_n^k(T_{i-1}-t)} - M_{n_k}^{T_i}(t,T_{i-1}) + \sigma_n^k\int_t^{T_{i-1}} e^{-a_n^k(T_{i-1}-u)}dW_{n_k}^{T_i}(u), \tag{139}$$

$$x_r^j(T_{i-1}) \;=\; x_r^j(t)e^{-a_r^j(T_{i-1}-t)} - M_{r_j}^{T_i}(t,T_{i-1}) + \sigma_r^j\int_t^{T_{i-1}} e^{-a_r^j(T_{i-1}-u)}dW_{r_j}^{T_i}(u), \tag{140}$$

where

$$M_{n_k}^{T_i}(t,T_{i-1}) \;=\; \int_t^{T_{i-1}}\Big[\sigma_n^k\sum_{i=1}^{K}\rho_{x_n^i,x_n^k}\sigma_n^i B_n^i(u,T_i)\Big]e^{-a_n^k(T_{i-1}-u)}du$$

$$=\; \sigma_n^k\frac{1-e^{-a_n^k(T_{i-1}-t)}}{a_n^k}\sum_{i=1}^{K}\rho_{x_n^i,x_n^k}\frac{\sigma_n^i}{a_n^i}$$

$$-\; \sigma_n^k\sum_{i=1}^{K}\rho_{x_n^i,x_n^k}\frac{\sigma_n^i}{a_n^i(a_n^k+a_n^i)}\Big[e^{-a_n^i(T_i-T_{i-1})} - e^{-a_n^i(T_i-t)-a_n^k(T_{i-1}-t)}\Big], \tag{141}$$

$$M_{r_j}^{T_i}(t,T_{i-1}) \;=\; \int_t^{T_{i-1}}\Big[\rho_{I,x_r^j}\nu(u)\sigma_r^j + \sigma_r^j\sum_{i=1}^{K}\rho_{x_n^i,x_r^k}\sigma_n^i B_n^i(u,T_i)\Big]e^{-a_r^j(T_{i-1}-u)}du$$

$$=:\; \widetilde{M_\nu^{T_i}}(t,T_{i-1}) + \widetilde{M_{r_j}^{T_i}}(t,T_{i-1}). \tag{142}$$

In the last step we decompose $M_{r_j}^{T_i}(t,T_{i-1})$ into a deterministic part, denoted by $\widetilde{M_{r_j}^{T_i}}(t,T_{i-1})$ and a stochastic part depending on $\nu(u)$, denoted by $\widetilde{M_\nu^{T_i}}(t,T_{i-1})$. The calculation of the $\widetilde{M_{r_j}^{T_i}}(t,T_{i-1})$-term is similar to the nominal interest rate case and results in the following expression:

$$\widetilde{M_{r_j}^{T_i}}(t,T_{i-1}) \;=\; \sigma_r^j\frac{1-e^{-a_r^j(T_{i-1}-t)}}{a_r^j}\sum_{i=1}^{K}\rho_{x_n^i x_r^j}\frac{\sigma_n^i}{a_n^i} \tag{143}$$

$$-\; \sigma_r^j\sum_{i=1}^{K}\rho_{x_n^i x_r^j}\frac{\sigma_n^i}{a_n^i(a_r^j+a_n^i)}\Big[e^{-a_n^i(T_i-T_{i-1})} - e^{-a_n^i(T_i-t)-a_r^j(T_{i-1}-t)}\Big].$$

Hence from Ito's isometry we then have that the mean and variance of $x_n^k(T_{i-1})$ and $\widetilde{x_r^j}(T_{i-1})$ (conditional on time t) are respectively given by

$$\mu_n^k(t, T_{i-1}) = x_n^k(t)e^{-a_n^k(T_{i-1}-t)} - M_{n_k}^{T_i}(t, T_{i-1}) \tag{144}$$

$$\left(\sigma_n^k(t, T_{i-1})\right)^2 = \frac{(\sigma_n^k)^2}{2a_n^k}\left(1 - e^{-2a_n^k(T_{i-1}-t)}\right) \tag{145}$$

$$\mu_r^j(t, T_{i-1}) = x_r^j(t)e^{-a_r^j(T_{i-1}-t)} - \widetilde{M_{r_j}^{T_i}}(t, T_{i-1}) \tag{146}$$

$$\left(\sigma_r^j(t, T_{i-1})\right)^2 = \frac{(\sigma_r^j)^2}{2a_r^j}\left(1 - e^{-2a_r^j(T_{i-1}-t)}\right). \tag{147}$$

Hence it remains to determine the moments of $\widetilde{M_V^{T_i}}(t, T_{i-1})$, i.e. of

$$\widetilde{M_V^{T_i}}(t, T_{i-1}) = \sigma_r^j \rho_{I,x_r^j} \int_t^{T_{i-1}} v(u)e^{-a_r^j(T_{i-1}-u)}du. \tag{148}$$

By substituting the explicit solution (136) for $v(u)$ one obtains the following three integrals:

$$\sigma_r^j \rho_{I,x_r^j} v(t) \int_t^{T_{i-1}} e^{-\kappa(u-t)}e^{-a_r^j(T_{i-1}-u)}du \tag{149}$$

$$\sigma_r^j \rho_{I,x_r^j} \int_t^{T_{i-1}} \left[\int_t^u \xi(s)e^{-\kappa(u-s)}ds\right]e^{-a_r^j(T_{i-1}-u)}du \tag{150}$$

$$\sigma_r^j \rho_{I,x_r^j} \tau \int_t^{T_{i-1}} \left[\int_t^u e^{\kappa(u-s)}dW_V^{T_i}(s)\right]e^{-a_r^j(T_{i-1}-u)}du \tag{151}$$

The integral of (149) resolves into

$$v(t)\frac{\sigma_r^j \rho_{I,x_r^j}}{(a_r^j - \kappa)}\left[e^{-\kappa(T_{i-1}-t)} - e^{-a_r^j(T_{i-1}-t)}\right]. \tag{152}$$

By using Fubini's theorem to interchange the order of integration, one can find that the integral of (150) resolves into

$$\sigma_r^j \rho_{I,x_r^j} \frac{\kappa e^{-a_r^j(T_{i-1}-t)} + (a_r^j - \kappa) - a_r^j e^{-\kappa(T_{i-1}-t)}}{(a_r^j - \kappa)\kappa a_r^j}\left[\psi + \sum_{i=1}^{K} \frac{\rho_{x_n^i,V}\sigma_n^i \tau}{a_n^i \kappa}\right] \tag{153}$$

$$+ \frac{\sigma_r^j \rho_{I,x_r^j}}{\kappa(a_r^j - \kappa)} \sum_{i=1}^{K} \frac{\rho_{x_n^i,V}\sigma_n^i \tau}{(\kappa + a_n^i)a_n^i(a_r^j + a_n^i)}\left\{(a_r^j + a_n^i)e^{-\kappa(T_{i-1}-1)-a_n^i(T_i-t)}\right.$$
$$\left. -(\kappa + a_n^i)e^{-a_r^j(T_{i-1}-t)-a_n^i(T_i-t)} - (a_r^j - \kappa)e^{-a_n^i(T_i-T_{i-1})}\right\}.$$

Again by changing the order integration, we find that the following expression holds for the stochastic integral of (151):

$$\frac{\sigma_r^j \rho_{I,x_r^j} \tau}{(a_r^j - \kappa)} \int_t^{T_{i-1}} \left[e^{-\kappa(T_{i-1}-s)} - e^{-a_r^j(T_{i-1}-s)}\right]dW_V^{T_i}(s). \tag{154}$$

Hence from Ito's isometry, we have that $\widetilde{M_v^{T_i}}(t, T_{i-1})$ of (148) is normally distributed with mean $\mu_V^j(t, T_{i-1})$ and variance $\left(\sigma_V^j(t, T_{i-1})\right)^2$ given by

$$
\begin{aligned}
\mu_V^j(t, T_{i-1}) &= \sigma_r^j \rho_{I, x_r^j} v(t) \left[e^{-\kappa(T_{i-1}-t)} - e^{-a_r^j(T_{i-1}-t)} \right] \\
&\quad + \sigma_r^j \rho_{I, x_r^j} \frac{\kappa e^{-a_r^j(T_{i-1}-t)} + (a_r^j - \kappa) - a_r^j e^{-\kappa(T_{i-1}-t)}}{(a_r^j - \kappa)\kappa a_r^j} \left[\psi + \sum_{i=1}^{K} \frac{\rho_{x_n^i, v} \sigma_n^i \tau}{a_n^i \kappa} \right] \\
&\quad + \frac{\sigma_r^j \rho_{I, x_r^j}}{\kappa(a_r^j - \kappa)} \sum_{i=1}^{K} \frac{\rho_{x_n^i, v} \sigma_n^i \tau}{(\kappa + a_n^i) a_n^i (a_r^j + a_n^i)} \Big\{ (a_r^j + a_n^i) e^{-\kappa(T_{i-1}-1)-a_n^i(T_i-t)} \\
&\qquad - (\kappa + a_n^i) e^{-a_r^j(T_{i-1}-t)-a_n^i(T_i-t)} - (a_r^j - \kappa) e^{-a_n^i(T_i-T_{i-1})} \Big\}.
\end{aligned}
\tag{155}
$$

$$
\begin{aligned}
\left(\sigma_V^j(t, T_{i-1})\right)^2 &= \left(\frac{\sigma_r^j \rho_{I, x_r^j} \tau}{(a_r^j - \kappa)} \right)^2 \Big\{ \frac{1}{2\kappa} + \frac{1}{2a_r^j} - \frac{2}{(\kappa + a_r^j)} \\
&\quad - \frac{e^{-2\kappa(T_{i-1}-t)}}{2\kappa} - \frac{e^{-2a_r^j(T_{i-1}-t)}}{2a_r^j} + \frac{2e^{-(\kappa+a_r^j)(T_{i-1}-t)}}{(\kappa + a_r^j)} \Big\}
\end{aligned}
\tag{156}
$$

B.3 Terminal correlations between the driving factors

In this section we provide simple analytical expressions for the (terminal) correlations between the driving model factors, $v, x_n^1, \ldots, x_n^K, x_r^1, \ldots, x_r^K, V^1, \ldots, V^M$, from the current time t to time T_{i-1}. To this end, we consider the following explicit solutions for these Gaussian processes:

$$
v(T_{i-1}) = O(dt) + \tau \int_t^{T_{i-1}} e^{-\kappa(T_{i-1}-u)} dW_v^{T_i}(u),
\tag{157}
$$

$$
x_n^k(T_{i-1}) = O(dt) + \sigma_n^k \int_t^{T_{i-1}} e^{-a_n^k(T_{i-1}-u)} dW_{n_k}^{T_i}(u),
\tag{158}
$$

$$
x_r^j(T_{i-1}) = O(dt) + \sigma_r^j \int_t^{T_{i-1}} e^{-a_r^j(T_{i-1}-u)} dW_{r_j}^{T_i}(u),
\tag{159}
$$

$$
V^j(T_{i-1}) = O(dt) + \frac{\sigma_r^j \rho_{I,v} \tau}{(a_r^j - \kappa)} \int_t^{T_{i-1}} \left[e^{-\kappa(T_{i-1}-u)} - e^{-a_r^j(T_{i-1}-u)} \right] dW_v^{T_i}(u).
\tag{160}
$$

All of the above processes can be written in the form

$$
y_m(T_{i-1}) = O(dt) + c_m \int_t^{T_{i-1}} a_m(u) dW_m(u),
$$

38

hence by Ito's isometry the correlation can be easily calculated; in general, we have that the correlation between, say $y_1(T_{i-1})$ and $y_2(T_{i-1})$, is given by

$$\rho_{y_1,y_2}(t, T_{i-1}) \;=\; \frac{\text{Cov}\big(y_1(T_{i-1}), y_2(T_{i-1})\big)}{\sqrt{\text{Var}\big(y_1(T_{i-1})\big) \cdot \text{Var}\big(y_2(T_{i-1})\big)}} \tag{161}$$

$$= \;\; \rho_{y_1,y_2} \int_t^{T_{i-1}} a_1(u) a_2(u)\, du \bigg/ \sqrt{\int_t^{T_{i-1}} [a_1(u)]^2 du \cdot \int_t^{T_{i-1}} [a_2(u)]^2 du}\,.$$

After identification in (157)-(160), one has that $a_m(u)$ takes two particular forms

$$a_m(u) = \begin{cases} e^{-b_m(T_{i-1}-u)} & \begin{aligned} &\text{for } \nu, x_n^1, \ldots, x_n^K, x_r^1, \ldots, x_r^K, \\ &b_m \in \{\kappa, a_n^1, \ldots, a_n^k, a_r^1, \ldots, a_r^M\}, \end{aligned} \\[2mm] e^{-\kappa(T_{i-1}-u)} - e^{-b_m(T_{i-1}-u)} & \begin{aligned} &\text{for } V^1, \ldots, V^M, \\ &b_m \in \{a_r^1, \ldots, a_r^M\} \end{aligned} \end{cases}$$

Hence by combining the above two forms and using formula (161), one has that the resulting correlations take one of the three forms below; to ease notation, we first define the following two integral expressions:

$$I_1(b_m) \;=\; \int_t^{T_{i-1}} \Big[e^{-b_m(T_{i-1}-u)}\Big]^2 du$$

$$= \;\; \frac{1 - e^{-2b_m(T_{i-1}-t)}}{2b_m}$$

$$I_2(b_m) \;=\; \int_t^{T_{i-1}} \Big[e^{-\kappa(T_{i-1}-u)} - e^{-b_m(T_{i-1}-u)}\Big]^2 du$$

$$= \;\; \frac{1}{2\kappa} + \frac{1}{2b_m} - \frac{2}{(\kappa + b_m)} - \frac{e^{-2\kappa(T_{i-1}-t)}}{2\kappa} - \frac{e^{-2b_m(T_{i-1}-t)}}{2b_m} + \frac{2e^{-(\kappa+b_m)(T_{i-1}-t)}}{(\kappa + b_m)}$$

if $a_1(u)$ and $a_2(u)$ are both of the first form, then the correlation between $y_1(T_{i-1})$ and $y_2(T_{i-1})$ is given by

$$\frac{\rho_{y_1,y_2}}{\sqrt{I_1(b_1)I_1(b_2)}} \frac{1 - e^{-(b_1+b_2)(T_{i-1}-t)}}{(b_1 + b_2)}, \tag{162}$$

if $a_1(u)$ is of the first form and $a_2(u)$ of the second, then the correlation between $y_1(T_{i-1})$ and $y_2(T_{i-1})$ is given by

$$\frac{\rho_{y_1,y_2}}{\sqrt{I_1(b_1)I_2(b_2)}}\left[\frac{1 - e^{-(b_1+\kappa)(T_{i-1}-t)}}{(b_1 + \kappa)} - \frac{1 - e^{-(b_1+b_2)(T_{i-1}-t)}}{(b_1 + b_2)}\right], \tag{163}$$

and finally, if $a_1(u)$ and $a_2(u)$ are both of the second form then the correlation between $y_1(T_{i-1})$ and $y_2(T_{i-1})$ is given by

$$\frac{\rho_{y_1,y_2}}{\sqrt{I_2(b_1)I_2(b_2)}}\left[\frac{1 - e^{-2\kappa(T_{i-1}-t)}}{2\kappa} + \frac{1 - e^{-(b_1+b_2)(T_{i-1}-t)}}{(b_1 + b_2)}\right.$$
$$\left. - \frac{1 - e^{-(b_1+\kappa)(T_{i-1}-t)}}{(b_1 + \kappa)} - \frac{1 - e^{-(b_2+\kappa)(T_{i-1}-t)}}{(b_2 + \kappa)}\right] \tag{164}$$

B.4 Constants in the Quadratic form ([85](#))

The constants a_0, b_0 and vector a of the quadratic form ([85](#)) can be directly extracted from equation ([84](#)) and are given by

$$
\begin{aligned}
a_0 \quad := \quad & iu\Big[A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i)\Big] + A(u, T_{i-1}, T_i) \\
& + C(T_{i-1})\mu_\nu(t, T_{i-1}) + \frac{1}{2}D(T_{i-1})\mu_\nu^2(t, T_{i-1}) \\
& + iu\sum_{k=1}^{K} B_n^k(T_{i-1}, T_i)\mu_n^k(t, T_{i-1}) \\
& - iu\sum_{j=1}^{M} B_r^j(T_{i-1}, T_i)\Big[\mu_V^j(t, T_{i-1}) + \mu_r^j(t, T_{i-1})\Big],
\end{aligned}
\tag{165}
$$

$$
b_0 \quad := \quad \frac{1}{2}D(u, T_{i-1}, T_i)\sigma_\nu^2(t, T_{i-1}),
\tag{166}
$$

$$
a \quad := \quad iu
\begin{bmatrix}
\sigma_\nu(t, T_{i-1})\Big[C(T_{i-1}) + D(T_{i-1})\mu_\nu(t, T_{i-1})\Big] \\
\sigma_n^1(t, T_{i-1})B_n^1(T_{i-1}, T_i) \\
\vdots \\
\sigma_n^K(t, T_{i-1})B_n^K(T_{i-1}, T_i) \\
-\sigma_r^1(t, T_{i-1})B_r^1(T_{i-1}, T_i) \\
\vdots \\
-\sigma_r^M(t, T_{i-1})B_r^M(T_{i-1}, T_i) \\
\sigma_V^1(t, T_{i-1})B_r^1(T_{i-1}, T_i) \\
\vdots \\
\sigma_V^M(t, T_{i-1})B_r^M(T_{i-1}, T_i)
\end{bmatrix},
\tag{167}
$$

and with the $(1 + K + 2M) \times (1 + K + 2M)$ correlation matrix S given by

$$
S :=
\begin{pmatrix}
1 & \rho_{x_n^1,\nu}(t, T_{i-1}) & \ldots & \rho_{V^M,\nu}(t, T_{i-1}) \\
\rho_{x_n^1,\nu}(t, T_{i-1}) & 1 & \ldots & \rho_{x_n^1,V^M}(t, T_{i-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{V^M,\nu}(t, T_{i-1}) & \rho_{x_n^1,V^M}(t, T_{i-1}) & \ldots & 1
\end{pmatrix}.
\tag{168}
$$

The moments of the Gaussian factors ν, x_n^k, x_r^j, V^j are given by simple analytical expressions, see appendix [B.2](#). Where the correlations between all instantaneous quantities are fixed input parameters, the (terminal) correlations $\rho(t, T_{i-1})$ between the driving processes are model/parameter dependent, however these are also given by a simple analytical expressions, see appendix [B.3](#).

B.5 Constants in proposition 4.6

The constant h_0 and vector h and correlation matrix S_R be extracted from equation (105) and are given by:

$$h_0 := [A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i)] \tag{169}$$

$$+ \sum_{k=1}^{K} B_n^k(T_{i-1}, T_i)\mu_n^k(t, T_{i-1}) - \sum_{j=1}^{M} B_r^j(T_{i-1}, T_i)\mu_r^j(t, T_{i-1}),$$

$$h := \begin{bmatrix} \sigma_n^1(t, T_{i-1})B_n^1(T_{i-1}, T_i) \\ \dots \\ \sigma_n^K(t, T_{i-1})B_n^K(T_{i-1}, T_i) \\ -\sigma_r^1(t, T_{i-1})B_r^1(T_{i-1}, T_i) \\ \dots \\ -\sigma_r^M(t, T_{i-1})B_r^M(T_{i-1}, T_i) \end{bmatrix}, \tag{170}$$

with $(K + M) \times (K + M)$ correlation matrix S_R given by

$$S_R := \begin{pmatrix} 1 & \dots & \rho_{x_n^1, x_n^K}(t, T_{i-1}) & \rho_{x_n^1, x_r^1}(t, T_{i-1}) & \dots & \rho_{x_n^1, x_r^M}(t, T_{i-1}) \\ \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ \rho_{x_n^1, x_n^K}(t, T_{i-1}) & \dots & 1 & \rho_{x_n^K, x_r^1}(t, T_{i-1}) & \dots & \rho_{x_n^K, x_r^M}(t, T_{i-1}) \\ \rho_{x_n^1, x_r^1}(t, T_{i-1}) & \dots & \rho_{x_n^K, x_r^1}(t, T_{i-1}) & 1 & \dots & \rho_{x_r^1, x_r^M}(t, T_{i-1}) \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ \rho_{x_n^1, x_r^M}(t, T_{i-1}) & \dots & \rho_{x_n^K, x_r^M}(t, T_{i-1}) & \rho_{x_r^1, x_r^M}(t, T_{i-1}) & \dots & 1 \end{pmatrix}, \tag{171}$$

and where the above moments and correlations of the Gaussian factors x_n^k, x_r^j can be found in appendix B.2 and B.3.

C FX calibration data

For completeness, we provide here the description of the FX market data that was being used in Piter-barg (2005); Ten maturities, each with seven strikes are considered. The strikes are computed according to formula (113). These strikes and corresponding Black and Scholes (1973) implied volatilities can be found in table 6 and 7 below.

	strike 1	strike 2	strike 3	strike 4	strike 5	strike 6	strike 7
0.5	93.03	96.38	99.84	103.44	107.16	111.02	115.01
1	87.70	92.20	96.93	101.90	107.12	112.61	118.39
3	74.01	80.70	88.00	95.96	104.64	114.11	124.43
5	64.62	72.27	80.81	90.37	101.06	113.02	126.39
7	57.23	65.33	74.57	85.11	97.15	110.89	126.57
10	48.41	56.70	66.41	77.79	91.11	106.72	125.00
15	37.45	45.45	55.16	66.95	81.26	98.62	119.69
20	29.46	36.85	46.08	57.63	72.06	90.12	112.71
25	23.43	30.08	38.63	49.60	63.69	81.77	105.00
30	18.77	24.69	32.46	42.69	56.14	73.82	97.08

Table 6: Strikes.

	strike 1	strike 2	strike 3	strike 4	strike5	strike 6	strike 7
0.5	11.41%	10.49%	9.66%	9.02%	8.72%	8.66%	8.68%
1	12.23%	10.98%	9.82%	8.95%	8.59%	8.59%	8.65%
3	12.94%	11.35%	9.89%	8.78%	8.34%	8.36%	8.46%
5	13.44%	11.84%	10.38%	9.27%	8.76%	8.71%	8.83%
7	14.29%	12.68%	11.23%	10.12%	9.52%	9.37%	9.43%
10	16.43%	14.79%	13.34%	12.18%	11.43%	11.07%	10.99%
15	20.93%	19.13%	17.56%	16.27%	15.29%	14.65%	14.29%
20	22.96%	21.19%	19.68%	18.44%	17.50%	16.84%	16.46%
25	23.97%	22.31%	20.92%	19.80%	18.95%	18.37%	18.02%
30	25.09%	23.48%	22.17%	21.13%	20.35%	19.81%	19.48%

Table 7: Market implied vols.

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