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DOI
10.1016/j.entcs.2008.12.014
Publication date
2008

Published in
Electronic Notes in Theoretical Computer Science

Kolmogorov Complexity Theory over the Reals

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Abstract. Kolmogorov Complexity constitutes an integral part of computability theory, information theory, and computational complexity theory—in the discrete setting of bits and Turing machines. Over real numbers, on the other hand, the BSS-machine (aka real-RAM) has been established as a major model of computation. This real realm has turned out to exhibit natural counterparts to many notions and results in classical complexity and recursion theory; although usually with considerably different proofs. The present work investigates similarities and differences between discrete and real Kolmogorov Complexity as introduced by Montaña and Pardo (1998).

1 Introduction

It is fair to call Andrey Kolmogorov one of the founders of Algorithmic Information Theory. Central to this field is a formal notion of information content of a fixed finite binary string $\bar{x} \in \{0,1\}^*$: For a (not necessarily prefix) universal machine U let $K_U(\bar{x})$ denote the minimum length$(\langle M \rangle)$ of a binary encoded Turing machine M such that $U(\langle M \rangle)$, on empty input, outputs $\bar{x}$ and terminates. Among the properties of this important concept and the quantity K_U, we mention [LiVi97]:

Fact 1. *a) Its independence, up to additive constants, of the universal machine U under consideration.*
b) The existence and even prevalence of incompressible instances $\bar{x}$, that is with $K_U(\bar{x}) \approx \text{length}(\bar{x})$.
c) The incomputability (and even Turing-completeness) of the function $\bar{x} \mapsto K_U(\bar{x})$; which is, however, approximable from above.
d) Applications in the analysis of algorithms and the proof of (lower and average) running time bounds.

We are interested in counterparts to these properties in the theory of

1.1 Real Number Computation

Concerning problems over bits, the Turing machine is widely agreed to be the appropriate model of computation: it has tape cells to hold one bit each, receives as input and produces as output finite strings over $\{0,1\}$, can store finitely many of them in its 'program code', and execution basically amounts to the application of a finite sequence of Boolean operations. A somewhat more convenient model, yet equivalent with respect to computability, the Random Access Machine (RAM) operates on integers as entities. Both are thus examples of a model of computation on an algebra: $(\{0,1\}, \vee, \wedge, \neg)$ in the first case and $(\mathbb{Z}, +, -, \times, <)$ in the second. Among the natural class of such general machines [TuZu00], we are interested in that corresponding to the algebra of real numbers $(\mathbb{R}, +, -, \times, \div, <)$: this is known as the real-RAM and popular for instance in Computational Geometry [PrSh85,BKOS97]. In [BSS89,BCSS98], it has been re-discovered and promoted as an idealized abstraction of fixed-precision floating-point computation. The latter publication(s) led to the name "BSS model" which we also adopt in the present work:

⋆ supported by the German Research Foundation (DFG) with project Zi 1009/1-1

Definition 2. *A BSS machine* $\mathbb{M}$ *consists of*

i) An unbounded (input, work, and output) tape capable of holding a real number in each cell.
ii) A reading and a writing head to move independently.
iii) A finite set Q *of states.*
iv) A finite, numbered sequence $(c_1, \ldots, c_J)$ *of real constants.*
v) And a finite control δ *describing, when in state* q *and depending on the* sign *of the real* x *contained in the cell at the reading head's current position, which of the following actions to take:*
- *Copy, add, or multiply* x *to the real* y *under the writing head.*
- *Subtract* x *from* y *or divide* y *by* x *(the latter under the provision that* $x \neq 0$*).*
- *Copy some* c_j *to* y*.*
- *Move the reading or writing head one cell to the left or to the right.*
- *Halt.*

Let $\mathbb{R}^* := \bigcup_{n\in\mathbb{N}} \mathbb{R}^n$ *denote the set of finite sequences of real numbers and* $\mathrm{size}(\vec{x}) = n$ *for* $\vec{x} \in \mathbb{R}^n$*.* $\mathbb{M}$ *realizes a partial real function on* $\mathbb{R}^*$ *(by abuse of notation also called* $\mathbb{M} :\subseteq \mathbb{R}^* \to \mathbb{R}^*$*,* $\vec{x} \mapsto \mathbb{M}(\vec{x})$*) according to the following semantics:*
For $\vec{x} \in \mathbb{R}^n$*, execution starts with the tape containing* $(n, x_1, \ldots, x_n)$*. If* $\mathbb{M}$ *eventually terminates* and *the tape contents is of the form* $(m, y_1, \ldots)$ *with* $m \in \mathbb{N}$*, then* $\mathbb{M}(\vec{x}) := (y_1, \ldots, y_m)$*; otherwise* $\mathbb{M}(\vec{x}) := \bot$ *(i.e.* $\vec{x} \notin \mathrm{dom}(\mathbb{M})$*).*

A subset $\mathbb{L} \subseteq \mathbb{R}^*$ *is called a (real)* language*. It is (BSS)* semi-decidable *if* $\mathbb{L} = \mathrm{dom}(\mathbb{M})$ *for some BSS machine* $\mathbb{M}$*.* $\mathbb{L}$ *is (BSS)* decidable *if its characteristic function is realized by some* $\mathbb{M}$*.* $\mathbb{L}$ *being (BSS)* enumerable *means that* $\mathbb{L} = \mathrm{range}(\mathbb{M})$ *for some total (!)* $\mathbb{M}$*.*

The above definition refers to the BSS equivalent of a one-tape two-head Turing machine. It generalizes to k tapes: as usual without significantly increasing the power of this model. In [BSS89,BCSS98], the authors transfer several important concepts and results from the classical (i.e. discrete) theory of computation to the real setting, such as

- The existence of a universal BSS machine, capable of simulating any given machine and satisfying SMN and UTM-like properties.
- The undecidability of the termination of a given (encoding of another) BSS machine, i.e. of the *real* Halting problem $\mathbb{H}$.
- A real language decidable in polynomial time by a *non*-deterministic BSS machine can also be decided in exponential time by a *deterministic* one:

$$\mathsf{P}_\mathbb{R} \subseteq \mathsf{NP}_\mathbb{R} \subseteq \mathsf{EXP}_\mathbb{R} . \tag{1}$$

- There exist decision problems *complete* for $\mathsf{NP}_\mathbb{R}$; and, relatedly, an important open question asks whether and which of the inclusions in Equation (1) are strict.

Here, running times and asymptotics are considered in terms of the *size* n of the input $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^*$: a natural algebraic counterpart to the (bit-) *length* of binary Turing machine inputs $\bar{x} = (x_1, \ldots, x_n) \in \{0,1\}^*$.

In fact the last two items above have spurred the development of a rich theory of computational complexity over the reals with classes like $\#\mathsf{P}_\mathbb{R}$ [Meer00,BuCu06], $\mathsf{PSPACE}_\mathbb{R}$ [CuKo95,KoPe07], $\mathsf{BPP}_\mathbb{R}$ [CKK*95], or $\mathsf{PCP}_\mathbb{R}$ [Meer05] and their relations to the discrete realm [Bue00a,Bue00b,FoKo00,Buer07]. It is in a certain sense quite surprising (and usually rather involved to establish) that this theory of real computation exhibits so many properties similar to its classical counterpart, because proofs of the latter generally do *not* carry over. For instance, Hilbert's Tenth Problem (i.e. the question whether a system of polynomial equations over field F admits a solution in F) is undecidable over $F = \{0,1\}$ [Mati70] but for $F = \mathbb{R}$ becomes decidable due to Quantifier Elimination.

1.2 Pure Algebra

This section recalls some well-known mathematical notions and facts; see for instance [Cohn91,Lang93].

Definition 3. *Let $E \subseteq F$ denote fields.*

a) Call $x \in F$ algebraic over E *if $p(x) = 0$ for some non-zero $p \in E[X]$. Otherwise x is* transcendental *(over E).*
b) We say that $\{x_1, \ldots, x_n\} \subseteq F$ is algebraically dependent over E *if $p(x_1, \ldots, x_n) = 0$ for some non-zero $p \in E[X_1, \ldots, X_n]$.*
A set $X \subseteq F$ is algebraically dependent over E *if some finite subset of it is. Otherwise X is called algebraically* in*dependent.*
c) The transcendence degree *of $X \subseteq F$ (over E), $\mathrm{trdeg}_E(X)$, is the maximum cardinality of a subset Y of X algebraically independent (over E).*
d) A transcendence basis *of F (over E) is a maximal algebraically independent subset of F.*
e) F is purely transcendental *over E if $F = E(S)$ for some $S \subseteq F$ that is algebraically independent over E.*

Fact 4. *a) Let $a_1, \ldots, a_n \in F$ be algebraic over E. Then there exists some $a \in F$, called a* primitive *element, such that $E(a_1, \ldots, a_n) = E(a)$.*
b) If $Y \subseteq X$ is algebraically independent over E and $\mathrm{Card}(Y) = \mathrm{trdeg}_E(X)$, then every element of X is algebraic over $E(Y)$.
c) Two transcendence bases have equal cardinality.
d) For a chain $E \subseteq F \subseteq G$ of fields, it holds $\mathrm{trdeg}_E(G) = \mathrm{trdeg}_E(F) + \mathrm{trdeg}_F(G)$.
e) In $\mathbb{R}$, e and π are transcendental over $\mathbb{Q}$.
f) Let $a_1, \ldots, a_n$ be algebraic yet linearly independent over $\mathbb{Q}$. Then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent over $\mathbb{Q}$

Claim f) is the Lindemann-Weierstraß Theorem, cf. e.g. [Bake75, Theorem 1.4].

1.3 Real Kolmogorov Complexity

The similarities between the discrete theory of Turing computation and the real one of BSS machines (Section 1.1) have led Montaña and Pardo to introduce and study in [MoPa98] the following real counterpart to classical Kolmogorov complexity:

Definition 5. *For a universal BSS machine $\mathbb{U}$ and for $\vec{x} \in \mathbb{R}^*$ let $\mathbb{K}_{\mathbb{U}}(\vec{x}) \in \mathbb{N}$ denote the minimum $\mathrm{size}(\vec{p})$, $\vec{p} \in \mathbb{R}^*$, such that $\mathbb{U}(\vec{p})$, on empty input, outputs $\vec{x}$ and terminates.*

Based on Item a) in Section 1.1, they conclude in [MoPa98, Theorem 2] that Fact 1a) carries over from the discrete to the real setting:

Observation 6 *For another universal machine $\mathbb{U}'$, $\mathbb{K}_{\mathbb{U}}(\vec{x})$ differs from $\mathbb{K}_{\mathbb{U}'}(\vec{x})$ only by an additive constant independent of $\vec{x}$.*

Moreover for the special case of the constant-free universal BSS machine $\mathbb{U}_0$ introduced in [BSS89, Section 8], [MoPa98, Theorems 3 and 6] establish the real Kolmogorov complexity to be bounded from below, and up to an additive constant from above, by the transcendence degree:

Fact 7. *There exists some $c \in \mathbb{Z}$ such that, for any $\vec{x} \in \mathbb{R}^*$, it holds*

$$\mathrm{trdeg}_{\mathbb{Q}}(\vec{x}) \;\leq\; \mathbb{K}_{\mathbb{U}_0}(\vec{x}) \;\leq\; \mathrm{trdeg}_{\mathbb{Q}}(\vec{x}) + c \;. \tag{2}$$

As an application, [MoPa98, Corollary 4] presents an alternative proof to a known lower bound in the algebraic complexity theory of polynomials, thus exemplifying the real incompressibility method as a natural counterpart to Fact 1d). We will give another application in Observation 28.

A further consequence of Fact 7: Since a 'random' n-element real vector has transcendence degree equal to n, incompressible strings are prevalent—a counterpart to Fact 1b), however based on entirely different arguments; see also Corollary 13 below. Moreover, as opposed to the discrete case, one can explicitly write down such instances, compare [MoPa98, Theorem 8] and Example 14a) below.

1.4 Overview

We focus on a natural variant of the universal machine $\mathbb{U}_0$ which leads to particularly compact BSS programs: all discrete code information (i.e. anything except for the real constants) is encoded into the first real number. For this Gödelization, we extend the results in [MoPa98] in five directions.

First, Fact 7 can be improved in that the constant c may be chosen as 1; and we show that this is generally best possible. Second, in Section 2.3, we consider the mathematical question in which cases the first inequality of Equation (2) is tight and in which cases the second one; the answer turns out to be related to deep issues in algebraic geometry. Then we investigate the computational properties of the real Kolmogorov complexity function $\mathbb{K}$: The classical incomputability argument, being based on exhaustively searching for an incompressible string, does not carry over to this continuous setting. Our third contribution features an entirely different proof establishing, as a partial analogue to Fact 1c), the BSS incomputability of $\mathbb{K}$ (Section 3). Fourth, we show that $\mathbb{K}$ can (as in the discrete case but again by different arguments) be approximated from above. And finally in Section 3.2, $\mathbb{K}$ is proven *not* BSS-*complete*.

2 Compact BSS Gödelization

While Observation 6 asserts a certain invariance of the Kolmogorov complexity of all strings, a fixed $\vec{x}$'s complexity on the other hand may change dramatically when proceeding from $\mathbb{U}$ to $\mathbb{U}'$: simply by constructing $\mathbb{U}'$ to give this particular $\vec{x}$ a special short code treated separately. Nevertheless, and as opposed to the classical case, we will now introduce a particular class of universal real machines $\mathbb{U}$ and show them to give rise to relatively 'minimal' $\mathbb{K}_{\mathbb{U}}$:

Definition 8. *Fix a finite choice $\vec{z} := (z_1, \ldots, z_D)$ of reals and let $\mathbb{U}_{\vec{z}}$ denote a universal BSS machine with constants $z_1, \ldots, z_D$ to simulate, upon input of 'program' $\langle \mathbb{M} \rangle_{\vec{z}}$ and of $\vec{x} \in \mathbb{R}^*$, $\mathbb{M}$ on $\vec{x}$. (The empty program produces no output and terminates precisely on the empty input.) Here, $\langle \mathbb{M} \rangle_{\vec{z}}$ is defined as follows:*

Consider a BSS-computable integer/real pairing function $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ with computable inverse; for instance something like

$$(n, x) \;\mapsto\; \mathrm{sign}(x) \cdot \big(2^n \cdot (2\lfloor |x| \rfloor + 1) \;+\; (|x| - \lfloor |x| \rfloor)\big) \;.$$

Encode some machine $\mathbb{M}$, with constants $c_1, \ldots, c_J, z_1, \ldots, z_D$ and control δ according to Definition 2, as $\langle \mathbb{M} \rangle_{\vec{z}} := (\langle \delta, c_1 \rangle, c_2, \ldots, c_J)$.

Finally abbreviate $\mathbb{K}_{\vec{z}} := \mathbb{K}_{\mathbb{U}_{\vec{z}}}$ and $\mathbb{K}_0 := \mathbb{K}_{()}$.

Here we have exploited that the *control* of $\mathbb{M}$ contains no real constants by itself but just *references* to them: to c_j by virtue of an index $j \in \{1, \ldots, J\}$; or to z_d provided by its 'host' machine $\mathbb{U}_{\vec{z}}$ by virtue of an index $d \in \{1, \ldots, D\}$. That δ thus being a purely discrete object permits to combine it with one other real, thus saving 1 element in size.

Remark 9. More precisely, *any* finite information (like, e.g. the number J of real constants following or the length of the input $\vec{x}$ to simulate $\mathbb{M}$ on) can be incorporated in this way without increasing the size of the encoding. This simplifies several putative pitfalls from classical Kolmogorov Complexity like [LiVi97, Example 2.1.4]

$$\mathbb{K}_{\vec{z}}(\vec{x},\vec{y}) \;\leq\; \mathbb{K}_{\vec{z}}(\vec{x}) + \mathbb{K}_{\vec{z}}(\vec{y})$$

and, for instance, lifts the need for a real counterpart to classical *prefix* complexity [LiVi97, Section 3].

Also note that a fully real/real pairing function cannot be BSS computable: For instance it follows from the *invariance of domain* principle in Algebraic Topology that a BSS computable function from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ cannot be injective. Alternatively, Observation 28 below shows that a BSS-computable function from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$ cannot be surjective: with a simple proof based on real Kolmogorov Complexity Theory!

2.1 Real Kolmogorov Complexity and Transcendence Degree

Intuitively, the encoding introduced in Definition 8 is as 'compact' as possible. Indeed, we have the following

Observation 10 *For any universal real machine $\mathbb{U}'$ with constants $\subseteq \{z_1,\ldots,z_D\}$, it holds $\mathbb{K}_{(z_1,\ldots,z_D)} \leq \mathbb{K}_{\mathbb{U}'}$.*

Proof. Since $\mathbb{U}_{\vec{z}}$ already contains all real constants of $\mathbb{U}'$, $\langle\mathbb{U}'\rangle_{\vec{z}}$ is purely discrete; now apply Remark 9. □

Since we are aiming for bounds on BSS Kolmogorov Complexity that are as tight as possibly, it turns out beneficial to refine Definition 5 to distinguish between the following closely related quantities corresponding to enumerability, decidability, and semi-decidability:

Definition 11. *a) For $\bar{x}\in\{0,1\}^*$ let $K^o_U(\bar{x})$ denote the minimum* length$(\bar{p})$*, $p\in\{0,1\}^*$, such that $U(\bar{p})$, on empty input,* outputs $\bar{x}$ *and terminates.*

b) $K^s_U(\bar{x})$ and $K^d_U(\bar{x})$ are defined similarly by the condition that $U(\bar{p})$ semi-/decides *the single-word language $\{\vec{x}\}$.*

c) For $\vec{x}\in\mathbb{R}^*$ let $\mathbb{K}^o_{\mathbb{U}}(\vec{x})$ denote the minimum size$(\vec{p})$*, $\vec{p}\in\mathbb{R}^*$, such that $\mathbb{U}(\vec{p})$, on empty input,* outputs $\vec{x}$ *and terminates.*

d) $\mathbb{K}^s_{\mathbb{U}}(\vec{x})$ and $\mathbb{K}^d_{\mathbb{U}}(\vec{x})$ are defined similarly by the condition that $\mathbb{U}(\vec{p})$ semi-/decides *the single-word language $\{\bar{x}\}$.*

One usually focuses on K^{o} (and we on $\mathbb{K}^{\mathrm{o}}$). Indeed, K^{o}_U, K^{d}_U, and K^{s}_U differ at most by an additive constant independent of $\bar{x}$: a machine M outputting $\bar{x}$ can be turned (with a fixed increase in complexity) into one which, given $\bar{y}$, simulates M and compares its output to the input in order to semi-/decide $\{\bar{x}\}$; conversely, M semi-deciding $\{\bar{x}\}$ may be used by M' generating *all* binary strings $\bar{y}$ to output the one that M terminates on. In the BSS realm, the inequality "$\mathbb{K}^{\mathrm{s}}_{\mathbb{U}}(\vec{x}) \leq \mathbb{K}^{\mathrm{d}}_{\mathbb{U}}(\vec{x}) \leq \mathbb{K}^{\mathrm{o}}_{\mathbb{U}}(\vec{x}) + \mathcal{O}(1)$" can be proven similarly; whereas "$\mathbb{K}^{\mathrm{o}}_{\mathbb{U}}(\vec{x}) \leq \mathbb{K}^{\mathrm{s}}_{\mathbb{U}}(\vec{x}) + \mathcal{O}(1)$" requires some more work, because one cannot generate *all* real strings. In fact, it is a consequence of Observation 6 and the following, already announced

Theorem 12. *For every $\vec{x}\in\mathbb{R}^+$ and $\vec{z}\in\mathbb{R}^*$ it holds*

a) $\mathbb{K}^s_{\vec{z}}(\vec{x}) = \mathbb{K}^d_{\vec{z}}(\vec{x}) = \max\{1, \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x})\}$.

b) $\max\{1, \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x})\} \leq \mathbb{K}^o_{\vec{z}}(\vec{x}) \leq \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x}) + 1$;

c) If $\mathbb{Q}(\vec{z},\vec{x})$ is purely transcendental over $\mathbb{Q}(\vec{z})$, then $\mathbb{K}^o_{\vec{z}}(\vec{x}) = \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x})$.

Section 2.2 contains the proof of this theorem.

Corollary 13. *Incompressible strings exist; they are in fact prevalent.*

Proof. For fixed $z_1, \ldots, z_D, x_1, \ldots, x_n \in \mathbb{R}$, the set $\{x \in \mathbb{R} : x \text{ algebraic over } \mathbb{Q}(\vec{z}, \vec{x})\}$ is countable. Therefore, guessing $x_1, \ldots, x_n \in [0,1]$ inductively independently uniformly at random yields with certainty $\mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x}) = n$. □

Example 14. a) $\mathbb{K}_0^{\mathrm{o}}(e^{\sqrt{2}}, e^{\sqrt{3}}, e^{\sqrt{5}}, e^{\sqrt{7}}, e^{\sqrt{11}}, \ldots, e^{\sqrt{p_n}}) = n$, where $p_n \in \mathbb{N}$ denotes the n-th prime number.

b) For $t \in \mathbb{R}$, it holds $\mathbb{K}_0^{\mathrm{o}}(t, \sqrt{2}) = 1$ in case t is algebraic and $\mathbb{K}_0^{\mathrm{o}}(t, \sqrt{2}) = 2$ if t is transcendental.

Proof. Indeed $\sqrt{2}, \sqrt{3}, \ldots, \sqrt{p_n}$ are square roots of distinct square-free numbers and therefore [Rick00] linearly independent over $\mathbb{Q}$; from which it follows by Fact 4f) that their exponentials are algebraically independent over $\mathbb{Q}$. Now apply Theorem 12c).

The first part of Claim b) follows immediately from Theorem 12b); similarly for the inequality "≤ 2" of the second part. The reverse inequality is a consequence of Proposition 15a) below since $\sqrt{2} \notin \mathbb{Q}(t)$ for t transcendental. Indeed the presumption $\sqrt{2} = p(t)/q(t)$ with polynomials $p, q \in \mathbb{Q}[T]$ would imply $p^2(t) = 2q^2(t)$, hence $p^2 - 2q^2$ vanishes identically: in contradiction to the (classical proof of the) irrationality of $\sqrt{2}$. □

2.2 Proof of Theorem 12

a) A machine deciding $\mathbb{L}$ is easily turned into one *semi*-deciding $\mathbb{L}$ without introducing any further constant: this shows $\mathbb{K}^{\mathrm{s}} \leq \mathbb{K}^{\mathrm{d}}$.
In [Mich90] it has been shown that a language $\mathbb{L}$ semi-decided by some BSS machine $\mathbb{M}$ is a countable union of sets basic semi-algebraic (i.e. solutions of a system of polynomial in-/equalities) over the rational field extension $\mathbb{Q}(y_1, \ldots, y_N)$ generated by the real constants $y_1, \ldots, y_N$ of $\mathbb{M}$; see also [Cuck92, Theorem 2.4]. Since in our case $\mathbb{L} = \{\vec{x}\}$ is a singleton, it must even be basic semi-algebraic. In fact, semi-algebraic sets being closed under projection [BPR03, Section 2.4], each single component $x_1, \ldots, x_n$ is a solution of some polynomial in-/equalities over $\mathbb{Q}(y_1, \ldots, y_N)$. It cannot be inequalities only, otherwise the solution would be open. Thus $x_1, \ldots, x_n$ are all algebraic over $\mathbb{Q}(y_1, \ldots, y_N)$. Applied to the BSS machine $\mathbb{U}_{\vec{z}}(\vec{p})$ with constants $\{y_1, \ldots, y_N\} = \{\vec{z}, \vec{p}\}$ shows that $\{\vec{x}\}$ is algebraic over $\mathbb{Q}(\vec{z})(\vec{p})$. Therefore, according to Fact 4, $\mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x}) \leq \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{p}) \leq \mathrm{size}(\vec{p})$ shows $\mathbb{K}^{\mathrm{s}}_{\vec{z}}(\vec{x}) \geq \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x})$. $\mathbb{K}^{\mathrm{s}}_{\vec{z}}(\vec{x}) \geq 1$ holds because $\vec{x} \neq ()$ requires some coding.
Finally to see $\mathbb{K}^{\mathrm{d}}_{\vec{z}}(\vec{x}) \leq \max\{1, \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x})\}$, first consider the case $\mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x}) = 0$. By Fact 4b), $x_1, \ldots, x_n$ are all algebraic over $\mathbb{Q}(\vec{z})$. For each $i = 1, \ldots, n$ let $0 \neq p_i(\vec{z}, X) \in \mathbb{Q}(\vec{z})[X]$ denote some polynomial having x_i as unique root within the interval (a_i, b_i), $a_i, b_i \in \mathbb{Q}$. These finitely many rationals a_i, b_i constitute discrete information only; and so do the coefficients of p_i described in terms of rational functions over $\vec{z}$. Since $\vec{z}$ itself is provided by the universal host machine $\mathbb{U}_{\vec{z}}$, the remaining data about $p_1, \ldots, p_n$ can be combined into one number which admits an effective evaluation of $y_i \mapsto p_i(y_i)$ and tests "$p_i(y_i) = 0, a_i < y_i < b_i$" to decide whether a given input $\vec{y}$ belongs to $\{\vec{x}\}$: $\mathbb{K}^{\mathrm{d}}_{\vec{z}}(\vec{x}) \leq 1$.
In remaining case $d := \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x}) > 0$, let $\{p_1, \ldots, p_d\}$ denote some transcendence basis of $\mathbb{Q}(\vec{z}, \vec{x})$ over $\mathbb{Q}(\vec{z})$. Again by virtue of Fact 4, all x_i are algebraic over $\mathbb{Q}(\vec{z}, \vec{p})$ and describable by rational bounds and polynomials $p_i(\vec{z}, \vec{p}, X) \in \mathbb{Q}(\vec{z}, \vec{p})[X]$. By virtue of Remark 9, this data can be combined with the d reals $p_1, \ldots, p_d$ to show $\mathbb{K}^{\mathrm{d}}_{\vec{z}}(\vec{x}) \leq d$.

b1) The first inequality of b) follows from a) by observing $\mathbb{K}^{\mathrm{d}}_{\vec{z}} \leq \mathbb{K}^{\mathrm{o}}_{\vec{z}}$: a machine to output $\vec{x}$ can be transformed into one deciding $\{\vec{x}\}$ incurring only discrete additional cost; now apply Remark 9.

c) Let $p_1, \ldots, p_d$ denote a transcendence basis of $\mathbb{Q}(\vec{z}, \vec{x})$ over $\mathbb{Q}(\vec{z})$. By prerequisite, $x_1, \ldots, x_n$ are not only algebraic over (Fact 4b), but even belong to, $\mathbb{Q}(\vec{z}, \vec{p})$. They can thus be described and computed using, in addition to $\vec{z}$ and $\vec{p}$, only discrete information. In view of Remark 9, this shows $\mathbb{K}^{\mathrm{o}}_{\vec{z}}(\vec{x}) \leq \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(\vec{x})$.

b2) For the second inequality of b), we proceed similarly to the proof of Claim c), however taking into account that now $x_1, \ldots, x_n$ need not belong to, but are only algebraic over, $\mathbb{Q}(\vec{z}, \vec{p})$. On the other hand, by Fact 4a), there exists some primitive element $a \in \mathbb{R}$ such that $x_1, \ldots, x_n \in \mathbb{Q}(\vec{z}, \vec{p}, a)$. Now $\vec{x}$ can be described and computed as above, using $\vec{z}$, $\vec{p}$, *and* a. □

2.3 Non-Purely Transcendental Extensions

Unless $\vec{x}$ is purely transcendental, Theorem 12b) leaves a gap of 1 between lower and upper bound. This turns out very difficult to close and leads to deep questions in algebraic geometry:

Proposition 15. *a) Let $t \in \mathbb{R}$ be transcendental over $\mathbb{Q}(\vec{z})$ and $a \notin \mathbb{Q}(\vec{z}, t)$ algebraic over $\mathbb{Q}(\vec{z}, t)$. Then $\mathbb{K}^{o}_{\vec{z}}(t, a) = 2 > 1 = \mathrm{trdeg}_{\mathbb{Q}(\vec{z})}(t, a)$.*

b) To any $s, t \in \mathbb{R}$ algebraically independent over $\mathbb{Q}$ there exist $x, y, a \in \mathbb{R}$ such that $s, t, a \in \mathbb{Q}(x, y)$ and $a \notin \mathbb{Q}(s, t)$.
In particular, it holds $\mathbb{K}^{o}_{0}(s, t, a) = 2 = \mathrm{trdeg}_{\mathbb{Q}}(s, t, a)$ although a is not algebraic over $\mathbb{Q}(s, t)$.

The latter shows that there is no "only if" in Theorem 12c).

Proof (Proposition 15).

a) Suppose toward contradiction that some BSS machine $\mathbb{M}$ with one real constants $\vec{z}, x$ can output t, a. By induction on the number of steps performed by $\mathbb{M}$, it is easy to see that any intermediate result and in particular its output constitutes a rational function of $\vec{z}, x$, that is, belongs to $\mathbb{Q}(\vec{z}, x)$. Since $t \in \mathbb{Q}(\vec{z}, x)$ is transcendental over $\mathbb{Q}(\vec{z})$, so must be x itself. Lüroth's Theorem asserts every subfield between $\mathbb{Q}(\vec{z})$ and its simple transcendental extension $\mathbb{Q}(\vec{z}, x)$ to be simple again; cf. e.g. [Cohn91, Theorem 5.2.4]. However $\mathbb{Q}(\vec{z}, t, a)$ by prerequisite is not simple over $\mathbb{Q}(\vec{z})$: a contradiction.

b) Lüroth's Theorem has been extended by Castelnuovo to the case of transcendence degree 2—however over algebraically *closed* fields. It is now known to fail from transcendence degree 3 on, and also for 2 over an algebraically *non*-closed field. See for instance to [GiSz06, Remarks 6.6.2] for a historical account of these results.
In particular for the field $\mathbb{Q}$, we refer to a classical counter-example [Segr51] due to Beniamino Segre showing the $\mathbb{Q}$-variety V defined by the cubic $b^3 + 3a^3 + 5s^3 + 7t^3$ on the $\mathbb{Q}$-sphere $\mathcal{S}^3 = \{(a, b, s, t) \in \mathbb{Q}^4 : a^2 + b^2 + s^2 + t^2 = q^2\}$, $q \in \mathbb{Q}$, to be unirational but not rational. In other words (cmp. Lemma 26a below): For arbitrary s, t transcendental over $\mathbb{Q}$ and sufficiently large q, a (thus real) solution a to $q^2 - a^2 - s^2 - t^2 = (3a^3 + 5s^3 + 7t^3)^2$ is algebraic over (but not contained in) $\mathbb{Q}(s, t)$; whereas unirationality of V means that $\mathbb{Q}(s, t, a)$ be in turn contained in some purely transcendental extension $\mathbb{Q}(x, y)$. A BSS machine storing x, y can therefore output s, t, a as rational functions thereof, showing $\mathbb{K}^{o}_{0}(s, t, a) \leq 2$. □

3 Incomputability

A folklore property of classical Kolmogorov Complexity is its incomputability: No Turing machine can evaluate the function $\{0,1\}^* \ni \bar{x} \mapsto K(\bar{x})$. This follows from a formal argument related to the Richard-Berry Paradox which involves a contradiction arising from searching for some $\bar{x} \in \{0,1\}^*$ of minimum length n such that $K(\bar{x})$ exceeds a given bound; cf. e.g. [More98, Theorem 5.5].

Remark 16. Over the reals, as opposed to $\{0,1\}^n$, $\mathbb{R}^n$ is too 'large' to be searched. As a consequence, concerning the simulation of a nondeterministic BSS machine by deterministic one, based on Tarski's Quantifier Elimination as in [BPR03, Section 2.5.1] the *existence* of a successful real guess can be decided, but a *witness* can in general not be found. More precisely, a BSS machine with constants $c_1, \ldots, c_J$ is limited to generate numbers in $\mathbb{Q}(c_1, \ldots, c_J)$ (compare the proof of Proposition 15a) and thus cannot *output*, even with the help of oracle access to $\mathbb{K}^{\mathrm{o}}$, any real vector of Kolmogorov Complexity exceeding J in order to raise a contradiction to the presumed computability of $\mathbb{K}^{\mathrm{o}}$.

Similarly, the classical proof does not carry over to show the incomputability of the *decision* version $\mathbb{K}^{\mathrm{d}}$, either: Given $\vec{x}$ as *input* one can, relative to $\mathbb{K}^{\mathrm{d}}$, detect (and terminate, provided) that $\vec{x}$ has sufficiently high Kolmogorov Complexity; however this approach accepts a large, not a one-element real language. □

Nevertheless we succeed in establishing

Theorem 17. *For each $\vec{z} \in \mathbb{R}^*$, both $\mathbb{K}^o_{\vec{z}}$ and $\mathbb{K}^d_{\vec{z}}$ are BSS–incomputable, even when restricted to $\mathbb{R}^2$.*

The proof is based on Claim c) of the following

Lemma 18. *a) The set $\mathbb{T} \subseteq \mathbb{R}$ of transcendental reals (over $\mathbb{Q}$) is not BSS semi-decidable.*
b) $\mathbb{T}$ is not even semi-decidable relative to oracle $\mathbb{Q}$.
c) For $\vec{y}, \vec{z} \in \mathbb{R}^*$, the real language $\mathbb{T}_{\vec{z}} := \{x \in \mathbb{R} : x \text{ transcendental over } \mathbb{Q}(\vec{z})\}$ is not BSS semi-decidable relative to oracle $\mathbb{Q}(\vec{y})$.
d) For $\vec{z} \in \mathbb{R}^*$, the real language $\mathbb{R} \setminus \mathbb{T}_{\vec{z}} = \{x \in \mathbb{R} : x \text{ algebraic over } \mathbb{Q}(\vec{z})\}$ is BSS semi-decidable.

Claim a) is folklore. Its extension b) has been established as [MeZi05, Theorem 4] and generalizes straight-forwardly to yield Claim c). Here we implicitly refer to the concept of BSS *oracle* machines $\mathbb{M}^{\mathbb{O}}$ whose transition function δ may, in addition to Definition 2v), enter a query state corresponding to the question whether the contents of the dedicated query tape belongs to $\mathbb{O} \subseteq \mathbb{R}^*$, and proceed according to the (Boolean) answer.

Regarding Claim d) it suffices to enumerate all non-zero $p \in \mathbb{Q}(\vec{z})[X]$ and test "$p(x) = 0$".

Proof (Theorem 17). Concerning $\mathbb{K}^{\mathrm{d}}_{\vec{z}}$, fix some $s \in \mathbb{R}$ transcendental over $\mathbb{Q}(\vec{z})$. Then, according to Theorem 12a), $\mathbb{K}^{\mathrm{d}}_{\vec{z}}(s,t) = 2$ if $t \in \mathbb{T}_{\vec{z},s}$, and $\mathbb{K}^{\mathrm{d}}_{\vec{z}}(s,t) = 1$ otherwise; that is BSS-computability of $\mathbb{K}^{\mathrm{d}}_{\vec{z}}(s,\cdot)$ contradicts Lemma 18c).

Similarly, according to Example 14b), $\mathbb{K}^{\mathrm{o}}_{\vec{z}}(t,\sqrt{2}) = 2$ if $t \in \mathbb{T}_{\vec{z}}$, and $\mathbb{K}^{\mathrm{o}}_{\vec{z}}(t,\sqrt{2}) = 1$ otherwise. □

3.1 Approximability

Although the function $\bar{x} \mapsto K(\bar{x})$ is not Turing-computable, it can be approximated [LiVi97, Theorem 2.3.3]: from above, in the point-wise limit without error bounds.

Fact 19. *The set* $\{(\bar{x},k) : K(\bar{x}) \leq k\} \subseteq \{0,1\}^* \times \mathbb{N}$ *is semi-decidable.*

In particular K becomes computable given oracle access to the Halting problem H.

Fact 20 (Shoenfield's Limit Lemma). *A function* $f :\subseteq \{0,1\}^* \to \mathbb{N}$ *is computable* relative *to* H *iff* $f(\bar{x}) = \lim_{m\to\infty} g(\bar{x},m)$ *for some* ordinarily *computable* $g : \mathrm{dom}(f) \times \mathbb{N} \to \mathbb{N}$.

See for instance [Soar87, §III.3.3]...

Remark 21. Concerning a real counterpart of Fact 20, only the domain but not the range extends from discrete to $\mathbb{R}$:

a) A function $f : \mathbb{R}^* \to \mathbb{N}$ is BSS computable relative to the *real* Halting Problem

$$\mathbb{H} \;=\; \big\{\langle \mathbb{M}\rangle : \mathbb{M} \text{ terminates on input } ()\big\}$$

iff $f(\vec{x}) = \lim_{m\to\infty} g(\vec{x},m)$ for some BSS computable $g : \mathrm{dom}(f) \times \mathbb{N} \to \mathbb{N}$.

b) The function $\exp : \mathbb{R} \ni x \mapsto e^x \in \mathbb{R}$ is the point-wise limit of BSS-computable $g(x,m) := \sum_{n=0}^{m} x^n/n! \in \mathbb{R}$; exp is, however, not BSS-computable relative to any oracle $\mathbb{O} \subseteq \mathbb{R}^*$.

Computing real limits is the distinct feature of so-called *Analytic* Machines [ChHo99].

Proof. a1) Since $g(\vec{x},\cdot)$ has discrete range, the sequence $\big(g(\vec{x},m)\big)_m$ must eventually stabilize to its limit $f(\vec{x})$. Now the real UTM and SMN theorems make it easy to construct from $\vec{x} \in \mathbb{R}^*$ and $M \in \mathbb{N}$ a BSS machine $\mathbb{M}$ which terminates iff $\big(g(\vec{x},m)\big)_{m\geq M}$ is not constant. Repeatedly querying $\mathbb{H}$ thus allows to determine $\lim_{m\to\infty} g(\vec{x},m) = f(\vec{x})$.

a2) Let f be computable relative to $\mathbb{H}$ by BSS oracle machine $\mathbb{M}^{\mathbb{H}}$. Given $\vec{x} \in \mathrm{dom}(f)$, $\mathbb{M}^{\mathbb{H}}$ thus makes a finite number (say N) of steps and oracle queries; let $\vec{u}_1,\ldots,\vec{u}_N \in \mathbb{H}$ denote those answered positively and $\vec{v}_1,\ldots,\vec{v}_N \notin \mathbb{H}$ those answered negatively. Now define $g(\vec{x},m)$ as the output of the following computation: Simulate $\mathbb{M}$ for at most m steps and, for each oracle query "$\vec{w} \in \mathbb{H}$?", perform the first m steps of a semi-decision procedure: if it succeeds, answer positively, otherwise negatively.
Now although the latter answer may in general be wrong, the finitely many queries $\vec{u}_1,\ldots,u_N \in \mathbb{H}$ admit a common M beyond which all are reported correctly; and so are the negative ones $\vec{v}_j \notin \mathbb{H}$ anyway. Hence for $m \geq M,N$, $g(\vec{x},m) = f(\vec{x})$.

b) The proof of Proposition 15a) has already exploited that all intermediate results (and in particular the output y), computed by a BSS machine with constants $\vec{c}$ upon input $\vec{x}$, belong to $\mathbb{Q}(\vec{c},\vec{x})$ and in particular satisfy $\mathrm{trdeg}_{\mathbb{Q}}(\vec{y}) \leq \mathrm{trdeg}_{\mathbb{Q}}(\vec{c},\vec{x}) \leq \mathrm{size}(\vec{c}) + \mathrm{trdeg}_{\mathbb{Q}(\vec{c})}(\vec{x})$ according to Fact 4d); whereas, for $(x_n) := (\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7},\sqrt{11},\ldots)$ denoting the sequence of square roots of prime integers, the corresponding values $y_n := \exp(x_n)$ have according to Fact 4f) transcendence degree unbounded compared to $\mathrm{trdeg}(x_n) = 0$. □

We now establish a real version of Fact 19.

Proposition 22. *Fix* $\vec{z} \in \mathbb{R}^*$.

a) The real Kolmogorov set $\mathbb{S}^d_{\vec{z}} := \{(\vec{x},k) : \mathbb{K}^d_{\vec{z}}(\vec{x}) \leq k\} \subseteq \mathbb{R}^* \times \mathbb{N}$ *is BSS semi-decidable.*

b) $\mathbb{K}^d_{\vec{z}} : \mathbb{R}^* \to \mathbb{N}$ *is BSS-computable* relative *to* $\mathbb{H}$.

By virtue of Remark 21a), Claim b) follows from a); which in turn is based on Lemma 18d) in combination with Part b) of the following

Lemma 23. *a) Let U denote a vector space and $V = \mathrm{lspan}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n) \subseteq U$ the subspace spanned by $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n \in U$. Then*

$$\dim(V) \;=\; n - \max\big\{k \;\big|\; \exists 1 \leq i_1 < \ldots < i_k \leq n : \\ \forall j \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_k\} : \boldsymbol{y}_j \in \mathrm{lspan}(\boldsymbol{y}_{i_1},\ldots,\boldsymbol{y}_{i_k})\big\}$$

b) Let $F = E(y_1, \ldots, y_n)$ denote a finitely generated field extension. Then

$$\mathrm{trdeg}_E(F) \;=\; n - \max\big\{k \;\big|\; \exists 1 \leq i_1 < \ldots < i_k \leq n : \\ \forall j \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_k\} : y_j \text{ algebraic over } E(y_{i_1},\ldots,y_{i_k})\big\}$$

Part a) is of course the rank-nullity theorem from highschool linear algebra and mentioned only in order to point out the similarity to b).

Proof. Any y_j algebraic over $E(y_{i_1}, \ldots, y_{i_k})$ cannot be part of a transcendence basis; hence $\mathrm{trdeg}_E(F) \leq n-k$. Conversely, choosing $(y_{i_1}, \ldots, y_{i_k})$ as a transcendence basis yields $\mathrm{trdeg}_E(F) \geq n - k$ according to Fact 4. □

3.2 (Lack of) Completeness

Classically, undecidable problems are 'usually' also Turing-complete in the sense of admitting a (Turing-) reduction to the discrete Halting problem H. This holds in particular for the Kolmogorov Complexity function; cf. e.g. [LiVi97, Exercise 2.7.7]. Over the reals on the other hand, $\mathbb{Q}$ has been identified in [MeZi05] as a decision problem BSS undecidable but *not* complete. Similarly, BSS incomputability of $\mathbb{K}^{\mathrm{d}}$ according to Theorem 17 turns out to *not* extend to BSS completeness:

Theorem 24. *Fix $\vec{z} \in \mathbb{R}^*$.*

a) Let

$$\mathbb{I}_{\vec{z}} \;:=\; \big\{\vec{x} \in \mathbb{R}^* : \vec{x} \text{ algebraically independent over } \mathbb{Q}(\vec{z})\big\} \;.$$

Then $\mathbb{S}^d_{\vec{z}}$ is decidable relative to $\mathbb{I}_{\vec{z}}$ and vice versa.

b) Let $C \subseteq [0,1]$ denote Cantor's Excluded Middle Third, that is the set of all $x = \sum_{n=1}^{\infty} t_n 3^{-n}$ with $t_n \in \{0,2\}$. Then C's complement is BSS semi-decidable

c) but C itself is not semi-decidable even relative to $\mathbb{I}_{\vec{z}}$.

d) $\mathbb{H}$ is not decidable relative to $\mathbb{S}^d_{\vec{z}}$ or to $\mathbb{K}^d_{\vec{z}}$.

Lemma 25. *Fix $\vec{w} \in \mathbb{R}^*$.*

a) To $x \in C$ and $\epsilon > 0$, there exists $y \in \mathbb{T}_{\vec{w}} \setminus C$ with $|x - y| \leq \epsilon$.

b) The set $C \cap \mathbb{T}_{\vec{w}}$ is uncountable and perfect (i.e. to $\epsilon > 0$ and $x \in C \cap \mathbb{T}_{\vec{w}}$ there exists $y \in C \cap \mathbb{T}_{\vec{w}}$ with $0 < |x - y| \leq \epsilon$).

Proof. Notice that $\mathbb{R} \setminus \mathbb{T}_{\vec{w}}$ is only countable.

a) Let $x = \sum_{n=1}^{\infty} t_n 3^{-n}$ with $t_n \in \{0,2\}$ and $\epsilon = 3^{-N}$. The open interval $I_{x,N} := \sum_{n=1}^{N-1} t_n 3^{-n} + 3^{-N} \cdot (\frac{1}{3}, \frac{2}{3})$ is disjoint from C and uncountable; hence so is $I_{x,N} \setminus (\mathbb{R} \setminus \mathbb{T}_{\vec{w}})$. From the latter, choose any y: done.

b) Since C is uncountable, so must be $C \setminus (\mathbb{R} \setminus \mathbb{T}_{\vec{w}})$.
Let $x = \sum_{n=1}^{\infty} s_n 3^{-n}$ with $s_n \in \{0,2\}$ and $\epsilon = 3^{-N}$. Already knowing that $C \cap \mathbb{T}_{\vec{w}}$ is infinite, we conclude that there exists some $y' = \sum_{n=1}^{\infty} t_n 3^{-n} \in C \cap \mathbb{T}_{\vec{w}}$ distinct from x with $t_n \in \{0,2\}$. Now let $y := \sum_{n=1}^{N} s_n 3^{-n} + \sum_{n=N+1}^{\infty} t_{n-N} 3^{-n}$: It satisfies $|x - y| \leq \epsilon$, belongs to C (having ternary expansion consisting only of 0s and 2s) and to $\mathbb{T}_{\vec{w}}$ (since it differs from $y \in \mathbb{T}_{\vec{w}}$ by a rational scaling and rational offset). □

Proof (Theorem 24).

d) Since C is decidable relative to $\mathbb{H}$ (b), $\mathbb{H}$ cannot be decidable relative to $\mathbb{I}_{\vec{z}}$ (by b) or (by a) to $\mathbb{S}^{\mathrm{d}}_{\vec{z}}$ or to $\mathbb{K}^{\mathrm{d}}_{\vec{z}}$.

a) By Theorem 12a) for $\vec{x} \in \mathbb{R}^n$, $\vec{x} \in \mathbb{I}_{\vec{z}} \Leftrightarrow (\vec{x}, n) \in \mathbb{S}^{\mathrm{d}}_{\vec{z}}$. Conversely, $\mathbb{K}^{\mathrm{d}}_{\vec{z}}(\vec{x})$ can be computed (and "$(\vec{x}, k) \in \mathbb{S}^{\mathrm{d}}_{\vec{z}}$" thus decided) by finding the maximal k such that there exist integers $1 \le n_1 < \ldots < n_k \le n$ with $(x_{n_1}, \ldots, x_{n_k}) \in \mathbb{I}_{\vec{z}}$.

b) $[0,1]\backslash C$ is semi-decidable as the union of countably many open intervals $\sum_{n=1}^{N} t_n 3^{-n} + 3^{-N} \cdot (\frac{1}{3}, \frac{2}{3})$, $N \in \mathbb{N}$, $t_1, \ldots, t_N \in \{0, 2\}$.

c) Suppose machine $\mathbb{M}$ with constants $c_1, \ldots, c_J$ and supported by oracle $\mathbb{I}_{\vec{z}}$ semi-decides C. Unrolling its computations on all inputs $x \in \mathbb{R}$ leads to an infinite 6-ary tree whose nodes u are labelled with (vectors of) rational functions $f_u \in \mathbb{Q}(\vec{c}, X)$ meaning that $\mathbb{M}$ branches on the sign of $f_u(\vec{c}, x)$ and depending on whether $\vec{f_u}(\vec{c}, x) \in \mathbb{I}_{\vec{z}}$. Moreover, by hypothesis, the path in this tree taken by input x ends in a leaf iff $x \in C$.
Fix some $x \in C$ transcendent over $\mathbb{Q}(\vec{c}, \vec{z})$ according to Lemma 25b). Then $f_u(\vec{c}, x) \neq 0$ for all u on the finite path $(u_1, \ldots, u_I)$ taken by x. Therefore the set

$$\{y \in \mathbb{R} : \operatorname{sign} f_{u_i}(\vec{c}, y) = \operatorname{sign} f_{u_i}(\vec{c}, x), i = 1, \ldots, I\}$$

is open (and non-empty). Hence, by Lemma 25a), there are (plenty of) $y \in \mathbb{T}_{(\vec{c},\vec{z})} \setminus C$ belonging to this set. Moreover, for any such y it holds $\vec{f_u}(\vec{c}, x) \in \mathbb{I}_{\vec{z}} \Leftrightarrow \vec{f_u}(\vec{c}, y) \in \mathbb{I}_{\vec{z}}$ according to Lemma 26a) below. We conclude that y takes the very same path (i.e. follows the same computation of $\mathbb{M}$) as x: although $x \in C$ and $y \notin C$, a contradiction. □

Lemma 26. *Let $E \subseteq F$ denote infinite fields.*

a) Fix $x \in F$ transcendental over E and $p_1, \ldots, p_n \in E[X]$. Then the vector of 'numbers' $\big(p_1(x), \ldots, p_n(x)\big) \in E(x)^n$ is algebraically independent over E iff the vector of 'functions' $(p_1, \ldots, p_n) \in E(X)^n$ is.

b) Fix $\mathcal{X}, \mathcal{Y} \subseteq F$, $\mathcal{X}$ algebraically independent over E. Then $\mathcal{X} \cup \mathcal{Y}$ is algebraically in-/dependent over E iff Y is algebraically in-/dependent over $E(\mathcal{X})$.

c) Let $p \in E[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ and $x_1, \ldots, x_n \in F$ be algebraically independent over E. Then p is irreducible (in $E[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$) iff $p(x_1, \ldots, x_n, \cdots)$ is irreducible in $E(x_1, \ldots, x_n)[Y_1, \ldots, Y_m]$.

d) Let $p \in E[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z]$ be irreducible and $x_1, \ldots, x_n, y_1, \ldots, y_m \in F$ algebraically independent over E but $y_1, \ldots, y_m, z \in F$ algebraically dependent over E and $p(x_1, \ldots, x_n, y_1, \ldots, y_m, z) = 0$. Then p does not 'depend' on $X_1, \ldots, X_n$, i.e. belongs to $E[Y_1, \ldots, Y_m, Z]$.

Proof. a) If $(p_1, \ldots, p_n)$ are algebraically dependent, say $q(p_1, \ldots, p_n) = 0$ for $0 \neq q \in E[X_1, \ldots, X_n]$, then *a fortiori* $q\big(p_1(x), \ldots, p_n(x)\big) = 0$.
Conversely let $q\big(p_1(x), \ldots, p_n(x)\big) = 0$ for some non-zero $q \in E[X_1, \ldots, X_n]$. Then $q(p_1, \ldots, p_n) \in E[X]$ vanishes on x. Since x is by hypothesis transcendental over E, this implies $q(p_1, \ldots, p_n) = 0$.

b) Let $\mathcal{Y}$ be algebraically dependent over $E(\mathcal{X})$, $0 = p(y_1, \ldots, y_m)$ for $0 \neq p \in E(\mathcal{X})[Y_1, \ldots, Y_m]$ where

$$n \in \mathbb{N}, \quad p = \sum_{\bar{\imath}} \frac{q_{\bar{\imath}}(x_1, \ldots, x_n)}{r_{\bar{\imath}}(x_1, \ldots, x_n)} \cdot Y^{\bar{\imath}}, \quad x_1, \ldots, x_n \in \mathcal{X},$$

$$\text{and} \quad q_{\bar{\imath}}, r_{\bar{\imath}} \in E[X_1, \ldots, X_n], r_{\bar{\imath}}(x_1, \ldots, x_n) \neq 0 \ .$$

Proceed to $\tilde{p} := \prod_{\bar{\jmath}} r_{\bar{\jmath}} \cdot \sum_{\bar{\imath}} \frac{q_{\bar{\imath}}}{r_{\bar{\imath}}} \cdot Y^{\bar{\imath}}$: This polynomial in $E[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ is non-zero (e.g. on $x_1, \ldots, x_n$) and vanishes on $x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathcal{X} \cup \mathcal{Y}$.

Conversely let $\mathcal{X} \cup \mathcal{Y}$ be algebraically dependent over E. Then it holds $p(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$ for some $n, m \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $y_1, \ldots, y_m \in Y$, and non-zero $p \in E[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$. A fortiori, $q := p(x_1, \ldots, x_n, \cdots) \in E(X)[Y_1, \ldots, Y_m]$ satisfies $q(y_1, \ldots, y_m) = 0$. To conclude algebraic independence of $y_1, \ldots, y_m$ over $E(X)$, it remains to show $q \neq 0$. $0 \neq p \in E[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ implies that there exist $z_1, \ldots, z_m \in E$ such that $0 \neq p(X_1, \ldots, X_n, z_1, \ldots, z_m) = r(X_1, \ldots, X_n) \in E[X_1, \ldots, X_n]$. Then $q(z_1, \ldots, z_m) = r(x_1, \ldots, x_n) \neq 0$ holds because $x_1, \ldots, x_n \in \mathcal{X}$ are algebraically independent by hypothesis.

c) Take some hypothetical non-trivial factorization $p = q_1 \cdot q_2$ in $E[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$. A fortiori, $p(\vec{x}, \vec{Y}) = q_1(\vec{x}, \vec{Y}) \cdot q_2(\vec{x}, \vec{Y})$ constitutes a factorization in $E(\vec{x})[\vec{Y}]$; a non-trivial one: because if for instance $q_1(\vec{x}, \vec{Y})$ were the constant polynomial, say $q_1(\vec{x}, \vec{Y}) = c \in E$, then $q_1(\vec{X}, \vec{y}) - c \neq 0$ for some $y_1, \ldots, y_m \in E$ (since q_1 is by presumption a non-trivial factor of p) constitutes a non-zero polynomial in $E[\vec{X}]$ vanishing on $x_1, \ldots, x_n$: contradicting that the latter are algebraically independent over E.
Conversely suppose $p(\vec{x}, \vec{Y}) = q_1(\vec{x}, \vec{Y}) \cdot q_2(\vec{x}, \vec{Y})$ in $E(\vec{x})[\vec{Y}]$ and consider the polynomial $r := p - q_1 \cdot q_2 \in E[\vec{X}, \vec{Y}]$. Although vanishing on $(\vec{x}, \vec{Y})$, it cannot be identically zero because that would mean a non-trivial factorization of irreducible p. On the other hand $r(\vec{X}, y_1, \ldots, y_m) \neq 0$ for some $y_1, \ldots, y_m \in E$ would constitute a non-zero polynomial in $E[\vec{X}]$ vanishing on $x_1, \ldots, x_n$: contradicting that the latter are algebraically independent over E.

d) Since $(\vec{x}, \vec{y})$ are algebraically independent over E, $p(\vec{x}, \vec{y}, Z)$ is irreducible in $E(\vec{x}, \vec{y})[Z]$ by c). Since $(\vec{y}, z)$ are algebraically dependent over E, $q(\vec{y}, z) = 0$ for some non-zero $q \in E[\vec{Y}, Z]$; w.l.o.g., q is irreducible: and so is $q(\vec{y}, Z)$ in $E(\vec{x}, \vec{y})[Z]$, again by c). Each $p(\vec{x}, \vec{y}, Z)$ and $q(\vec{y}, Z)$ vanishes on z, hence they share a common factor $r \in E(\vec{x}, \vec{y})[Z]$; but both being irreducible requires that they all coincide. □

Proposition 27. *For any fixed $\vec{z} \in \mathbb{R}^*$, $\mathbb{T}$ is BSS decidable relative to $\mathbb{I}_{\vec{z}}$; which is in turn decidable relative to $\mathbb{I} := \mathbb{I}_{()}$. In formula: $\mathbb{T} \leqslant \mathbb{I}_{\vec{z}} \leqslant \mathbb{I}$.*

Proof. Suppose we are given oracle access to $\mathbb{I}$. Since $\vec{z}$ is fixed, a BSS machine may store as constants a transcendence basis $\vec{y}$ for $\mathbb{Q}(\vec{z})$ over $\mathbb{Q}$. Given $\vec{x} \in \mathbb{R}^*$, it can then decide membership to $\mathbb{I}_{\vec{z}}$ by querying "$(\vec{y}, \vec{x}) \in \mathbb{I}$?": Since $\vec{y}$ is algebraically independent over $\mathbb{Q}$ by construction, $(\vec{y}, \vec{x})$ is iff $\vec{x}$ is over $\mathbb{Q}(\vec{y})$ (Lemma 26b) or, equivalently, over $\mathbb{Q}(\vec{z})$.

Conversely given x, query membership to $\mathbb{I}_{\vec{z}} \supseteq \mathbb{T}$ and accept if the answer is positive. Otherwise $(\vec{y}, x)$ is algebraically dependent over $\mathbb{Q}$, hence there exists for some non-zero polynomial $p \in \mathbb{Z}[\vec{Y}, X]$ irreducible over $\mathbb{Q}[\vec{Y}, X]$ and vanishing on $(\vec{y}, x)$. Moreover such p can be sought for (and hence found): By the Gauß Lemma [Lang93, Theorem IV.§2.3], $p \in \mathbb{Z}[\vec{Y}, X]$ is irreducible over $\mathbb{Q}[\vec{Y}, X]$ iff it is irreducible over $\mathbb{Z}[\vec{Y}, X]$; and the latter property is decidable by testing the finitely many candidate divisors $q \in \mathbb{Z}[\vec{Y}, X]$ of $\deg_i(q) \leq \deg_i(p)$ whose coefficients $q_i \in \mathbb{Z}$ divide p_i for all i. Now once such $p = p(\vec{Y}, X)$ is found, check whether it actually 'depends' on (i.e. has in dense representation a nonzero coefficient to) some Y_i: According to Lemma 26d), this is the case iff x is transcendental over $\mathbb{Q}$. □

4 Real Incompressibility Method

Discrete Kolmogorov Complexity Theory is a useful tool for establishing (lower and average) bounds on running times of specific algorithms as well as generally on the complexity of certain problems [LiVi97, Section 6]. The same can be said about its BSS counterpart [MoPa98, Corollary 4]. For instance we conclude from Example 14a) an entirely new proof of the following

Observation 28 *There exists no BSS-computable surjective (and in particular no fully real pairing) function $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$.*

Proof. Suppose that f is computable by machine $\mathbb{M}$ with constants $c_1, \ldots, c_J$. Iteration yields a surjection $f^{(n)} : \mathbb{R} \to \mathbb{R}^n$ for any fixed n, computable again by a machine with constants $c_1, \ldots, c_J$. Take $n \in \mathbb{N}$ and $\vec{z} \in \mathbb{R}^n$ of Kolmogorov Complexity much larger than J according to Example 14a). By surjectivity, there exists $\zeta \in \mathbb{R}$ with $f^{(n)}(\zeta) = \vec{z}$. Thus, $\vec{z}$ can be output by storing the single constant ζ and invoking the machine evaluating $f^{(n)}$: contradicting $\mathbb{K}(\zeta) \approx J \ll \mathbb{K}(\vec{z})$. □

5 Miscellaneous

This section handles off few further, related topics from classical computability theory [More98, Section 5.6] (see also [Moss06]) in the context of real number computation: Radó's Busy Beaver function, Quines, and Kleene's Recursion and Fixed Point Theorems.

5.1 Busy Beaver

Classically, the busy beaver function $\Sigma(n)$ amounts to the length of a longest string $\bar{x} \in \{0,1\}^*$ output by a terminating, input-free Turing machine M of length$(\langle M \rangle) \leq n$. It is well-known, as is the Kolmogorov complexity function, incomputable, approximable, and equivalent to the Halting problem.

Now every Turing machine M can be simulated by a BSS machine $\mathbb{M}$ of size$(\langle \mathbb{M} \rangle) = 1$ independent of length$(\langle M \rangle)$; hence it does *not* make sense to ask the following

Question 29 (unreasonable). What is the maximum size of a string $\vec{x} \in \mathbb{R}^*$ output by a terminating, input-free BSS machine $\mathbb{M}$ of size$(\langle \mathbb{M} \rangle) \leq n$?

The answer is, of course: infinite.

In view of Theorem 12, one might be tempted to instead consider

Question 30. What is the maximum transcendence degree of a string $\vec{x} \in \mathbb{R}^*$ output by a terminating, input-free BSS machine $\mathbb{M}$ of size$(\langle \mathbb{M} \rangle) \leq n$?

However, again, this question is easy to answer (namely "n") and to compute.

5.2 Quines, Fixed-point and Recursion Theorems

A quine is a program p which (upon empty input) outputs itself (e.g. its own source code) and terminates. More generally, one may demand that p performs some prescribed computable operation on its input x and on its own encoding which, however, is *not* passed as input. Solutions to both problems are well-known to exist in the discrete realm and amount to Kleene's first and second Recursion Theorem, respectively. Closely related is his Fixed Point Theorem, asserting that every recursive total function on Gödel indices has a (semantic) fixed point.

All of them immediately carry over to the real setting: Since a BSS machine $\mathbb{M}$ accesses its constants by reference, it suffices to consider only $\mathbb{M}$'s finite control δ — to which the discrete theorems apply. Alternatively, their classical proofs based on SMN and UTM properties translate literally to the real setting (recall Section 1.1a).

Observation 31 *Fix a universal BSS machine $\mathbb{U}$.*

a) To any BSS machine $\mathbb{M}$ (with constants $c_1, \ldots, c_J$), there exists another one $\mathbb{M}'$ (again with constants $c_1, \ldots, c_J$) such that $\mathbb{M}'$ on $\vec{x}$ behaves like $\mathbb{M}$ on $(\langle \mathbb{M} \rangle, \vec{x})$.

b) To every total BSS-computable function $f : \mathbb{R}^* \to \mathbb{R}^*$, there exists some $\vec{x} \in \mathbb{R}^*$ such that

$$\forall \vec{y} \in \mathbb{R}^* : \quad \mathbb{U}\big(\vec{x}, \vec{y}\big) \;=\; \mathbb{U}\big(f(\vec{x}), \vec{y}\big) \;. \tag{3}$$

Moreover, if f is realized by $\mathbb{M}$, the mapping $\langle \mathbb{M} \rangle \to \vec{x}$ is BSS-computable.

The equality in (3) is meant in the extended sense that either side is undefined iff the other is.

6 Conclusion

The present work has extended the work [MoPa98] and its real variant of Kolmogorov complexity theory. Some important properties have turned out to carry over, however with considerably different proofs. Specifically, 'most' real vectors have complexity equal to their length; and the complexity of a given string can be computationally approximated from above but not determined exactly. However opposed to the classical discrete case, real Kolmogorov Complexity is not reducible *from* the real Halting problem $\mathbb{H}$.

We close with some open

Question 32. a) Does Proposition 22 extend to $\mathbb{K}^{\mathrm{o}}_{\vec{z}}$?
Does Theorem 24 extend to $\mathbb{S}^{\mathrm{o}}_{\vec{z}} := \{(\vec{x}, k) : \mathbb{K}^{\mathrm{o}}_{\vec{z}}(\vec{x}) \leq k\} \subseteq \mathbb{R}^* \times \mathbb{N}$?

b) Theorem 17 is only concerned with BSS Gödelizations induced by machines of the form $\mathbb{U}_{\vec{z}}$. Does it extend to all universal machines $\mathbb{U}$?

c) How about the complex case, i.e. w.r.t. BSS-machines over $\mathbb{C}$ permitted tests only for equality?

Acknowledgments: The first author is grateful to his colleagues Dennis Amelunxen, Peter Scheiblechner, and Thorsten Wedhorn for discussions about algebraic varieties and the rationality questions arisen in Section 2.3. Moreover, Peter Scheiblechner has been a great help in finding a proof for Lemma 26c). Finally we owe to Klaus Meer for pointing us to the seminal work of Montaña and Pardo who first introduced real Kolmogorov Complexity.

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A Facsimile of [Segr51]

Con saluti cordiali

Dai « Rendiconti dell'Accademia Nazionale dei Lincei »
(Classe di Scienze fisiche, matematiche e naturali)
serie VIII, vol. X, fasc. 2

Geometria. — *Sull'esistenza, sia nel campo razionale che nel campo reale, di involuzioni piane non birazionali* (1). Nota (*) del Corrisp. BENIAMINO SEGRE.

1. Un teorema classico relativo ad un corpo γ commutativo (o campo) qualsiasi è quello di Lüroth, che può venir enunciato nel modo seguente (2). Detto t un elemento trascendente sopra γ, ogni campo δ che soddisfi alle

$$\gamma \subset \delta \subseteq \gamma(t), \tag{1}$$

e cioè che sia intermedio fra γ ed un'estensione trascendente semplice $\gamma(t)$ di γ, risulta a sua volta un'*estensione trascendente semplice di* γ. In altri termini, in forza delle (1) esiste in δ un elemento τ - trascendente sopra γ – tale che

$$\delta = \gamma(\tau). \tag{2}$$

Dal suddetto teorema segue, com'è ben noto, che ogni curva algebrica *unirazionale in* γ, e cioè rappresentabile parametricamente nella forma

$$x_1 = r_1(t), x_2 = r_2(t), \cdots, x_n = r_n(t) \tag{3}$$

(*) Presentata nella seduta del 10 febbraio 1951.

(1) La parola « razionale » vien usata di solito con diversi significati in algebra ed in geometria algebrica. Onde evitare confusioni, la sostituisco nella seconda accezione colla denominazione « birazionale », da attribuirsi dunque alle varietà rappresentabili birazionalmente sopra uno spazio lineare.

(2) Cfr. ad esempio B. L. VAN DER WAERDEN, *Moderne Algebra*, vol. I, 2ª ed. (Berlin, Springer, 1937), § 63.

colle r_i funzioni razionali a coefficienti in γ, è pure *birazionale in* γ [1]. All'uopo basta osservare che il campo

$$\delta = \gamma(x_1, x_2, \cdots, x_n)$$

soddisfa alle (1), qualora si definiscano le x colle (3); vi sarà dunque in δ un elemento τ per cui sussista la (2), onde varranno formule del tipo

$$x_1 = R_1(\tau), \quad x_2 = R_2(\tau), \cdots, \quad x_n = R_n(\tau)$$

assieme ad una della forma

$$\tau = R(x_1, x_2, \cdots, x_n),$$

colle R_i ed R funzioni razionali a coefficienti in γ, ciò che appunto prova l'asserto. Sotto forma equivalente si può anche dire che, *in un campo arbitrario, ogni involuzione* ∞^1 *sulla retta risulta birazionale.*

Il Castelnuovo, in un lavoro fondamentale [3], ha parzialmente esteso il citato teorema di Lüroth, stabilendo la *birazionalità di ogni involuzione piana* ∞^2 *nel campo complesso,* e quindi la *birazionalità* nel campo complesso, di ogni superficie che sia ivi *unirazionale* [4]. Rimaneva tuttavia da indagare se il teorema di Castelnuovo possa o meno venir esteso alle involuzioni ∞^2 in campi qualsiansi; ed il dubbio appariva specialmente fondato nel caso di campi non algebricamente chiusi, in cui resta senz'altro escluso che si possano trasportare le argomentazioni originali di Castelnuovo.

In questa breve Nota risolvo tale dubbio mostrando con semplici esempi, in sè non privi di interesse, che un risultato analogo a quello di Castelnuovo, dianzi richiamato, non sussiste nè nel campo razionale nè nel campo reale. Pertanto, mentre – a norma del teorema di Castelnuovo – ogni campo che abbia grado di trascendenza due sopra γ e sia intermedio fra γ e $\gamma(t_1, t_2)$ risulta un'*estensione trascendente doppia di* γ se γ è il campo complesso, ciò cessa dall'esser vero se γ è il campo razionale od il campo reale.

2. Denotiamo con F una superficie cubica non singolare di S_3, definita nel campo γ razionale, e rammentiamo le due seguenti proposizioni.

a) La superficie F è unirazionale in γ se, e soltanto se, essa contiene qualche punto razionale (ossia qualche punto a coordinate razionali) [5].

(3) G. CASTELNUOVO, *Sulla razionalità delle involuzioni piane.* «Math. Ann.», *44* (1894), 125–155, riprod. nelle *Memorie Scelte* (Bologna, Zanichelli, 1937), pp. 273–304.

(4) È però noto che questo risultato non può venir esteso alle varietà a più dimensioni nel campo complesso: ved. F. ENRIQUES, *Sopra una involuzione non razionale dello spazio.* «Rend. Acc. Naz. Lincei» (5), *21* (1912), 81–83.

(5) La *a*) è conseguenza immediata del risultato stabilito in B. SEGRE, *A parametric solution of the indeterminate cubic equation* $z^2 = f(x, y)$. «Journ. London Math. Soc.» *18* (1943), 24–31. Altre dimostrazioni di tale risultato sono state date successivamente da R. F. WHITEHEAD, *A rational parametric solution of the cubic indeterminate equation* $z^2 = f(x, y)$. «Journ. London Math. Soc.», *19* (1944), 68–71 e da B. SEGRE, *Arithmetic upon an algebraic surface.* «Bull. Amer. Math. Soc.», *51* (1945), 152–161.

b) Affinchè F sia birazionale in γ, è necessario (ma non sufficiente) che F soddisfi ivi ad almeno una delle condizioni $\rho_1, \rho_2, \rho_3, \rho_6$, e cioè che ammettano qualche radice razionale le equazioni algebriche ad un'incognita e coefficienti in γ, rispettivamente di gradi 27, 216, 720, 72, da cui dipendono la determinazione delle rette, delle coppie, delle terne, delle sestuple di rette di F a due a due sghembe [6].

Ciò premesso, mostriamo che *la superficie*

$$a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_4 x_4^3 = 0 \tag{4}$$

non è certamente birazionale nel campo γ *razionale*, se a_1, a_2, a_3, a_4 sono numeri razionali non nulli tali che nessuno dei rapporti di due di essi, e nessuno dei rapporti del prodotto di due di essi al prodotto dei rimanenti due, sia il cubo di un numero razionale. All'uopo osserviamo che le 27 rette della superficie (4) si distribuiscono nei tre sistemi coniugati di Steiner, consistenti di nove rette ciascuno, rappresentati dai sistemi di equazioni:

$$\begin{cases} a_1 x_1^3 + a_2 x_2^3 = 0 \\ a_3 x_3^3 + a_4 x_4^3 = 0 \end{cases} \qquad \begin{cases} a_1 x_1^3 + a_3 x_3^3 = 0 \\ a_2 x_2^3 + a_4 x_4^3 = 0 \end{cases} \qquad \begin{cases} a_1 x_1^3 + a_4 x_4^3 = 0 \\ a_2 x_2^3 + a_3 x_3^3 = 0. \end{cases}$$

Poichè, nelle ipotesi ammesse, ognuna di queste sei equazioni cubiche è irriducibile in γ e due qualunque di esse hanno radici indipendenti sopra γ, così il gruppo di Galois dell'equazione di 27° grado delle rette della superficie (4) ammette quei tre sistemi di rette come sistemi di imprimitività, su ciascuno dei quali opera transitivamente. Tali sistemi di nove rette sono dunque i soli sottoinsiemi razionali di rette della (4), sicchè questa superficie non soddisfa attualmente a nessuna delle anzidette condizioni $\rho_1, \rho_2, \rho_3, \rho_6$ [7]. In virtù della *b*), la (4) non può quindi essere birazionale in γ.

Rileviamo poi che *fra le superficie* (4), *i cui coefficienti soddisfino alle condizioni dianzi enunciate, ve ne sono infinite di unirazionali in* γ. A norma di *a*), tale è infatti ad esempio la superficie

$$x_1^3 + 3 x_2^3 + 5 x_3^3 + 7 x_4^3 = 0,$$

la quale contiene il punto razionale $(1, -1, -1, 1)$. Ciascuna di tali superficie è dunque *unirazionale e non birazionale* nel campo razionale.

3. Denotando ora con γ il campo reale, osserviamo che *ogni superficie cubica* F *non singolare, definita in* γ, *è ivi unirazionale.* Invero, fra le 27 rette

(6) La *b*) trovasi anzitutto enunciata in B. Segre, *A note on arithmetical properties of cubic surfaces.* « Journ. London Math. Soc. », *18* (1943), 24–31, teor. VIII, ed è quindi dimostrata in B. Segre, *On the rational solutions of homogeneous cubic equations in four variables.* « Mathematicae Notae », *11* (1951), p. 1, teor. 5, §§ 2, 29.

(7) Questo risultato segue anche subito dai §§ 31–34 del secondo lavoro citato in (6), dove le condizioni $\rho_1, \rho_2, \rho_3, \rho_6$ vengono espresse esplicitamente per la superficie (4), in funzione dei coefficienti a_1, a_2, a_3, a_4. Un risultato più preciso trovasi al principio del § 35 di detto lavoro.

appartenenti ad F nel campo complesso ve n'è sempre qualcuna reale [8]. È subito visto che, se l è una di queste, le rette tangenti ad F nei punti di l formano un insieme ∞^2 birazionale nel campo reale; l'asserto segue allora senz'altro notando che le rette di tale insieme sono riferite ai punti ov'esse incontrano F, fuori del loro punto d'appoggio colla l, in una corrispondenza algebrica (2, 1).

Se F è birazionale in γ, i suoi punti reali risultano in corrispondenza generalmente continua, perchè algebrica, e biunivoca (sia pure con eventuali eccezioni) coi punti reali di un piano reale, sicchè – nello spazio proiettivo reale a cui appartiene – la F deve constare di *un'unica falda reale*. È ben noto, d'altro canto, che nello spazio proiettivo reale esistono superficie cubiche reali non singolari consistenti di due disgiunte falde reali [9]. Ne consegue che ciascuna di tali superficie risulta *unirazionale e non birazionale* nel campo reale.

(8) Per lo Studio della configurazione delle rette di una superficie cubica in un campo qualsiasi, cfr. B. SEGRE, *Le rette delle superficie cubiche nei corpi commutativi.* «Bollettino Un. Mat. Ital.» (3), *4* (1949), 223–228.

(9) Ved. per esempio B. SEGRE, *The non-singular cubic surfaces.* (Oxford, The Clarendon Press, 1942), § 63.

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