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Dekker, H.; de Leeuw, G.; Maassen van den Brin, A.

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Boundary-layer turbulence as a kangaroo process

H. Dekker
Netherlands Organization for Applied Scientific Research, TNO Physics & Electronics Laboratory, P.O. Box 96864, 2509 JG Den Haag, The Netherlands
and Institute for Theoretical Physics, University of Amsterdam, Amsterdam, The Netherlands

G. de Leeuw
Netherlands Organization for Applied Scientific Research, TNO Physics & Electronics Laboratory, P.O. Box 96864, 2509 JG Den Haag, The Netherlands

A. Maassen van den Brink
Institute for Theoretical Physics, University of Amsterdam, Amsterdam, The Netherlands
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A nonlocal mixing-length theory of turbulence transport by finite size eddies is developed by means of a novel evaluation of the Reynolds stress. The analysis involves the contruct of a sample path space and a stochastic closure hypothesis. The simplifying property of exhange (strong eddies) is satisfied by an analytical sampling rate model. A nonlinear scaling relation maps the path space onto the semi-infinite boundary layer. The underlying near-wall behavior of *fluctuating* velocities perfectly agrees with recent direct numerical simulations. The resulting integro-differential equation for the mixing of scalar densities represents fully developed boundary-layer turbulence as a *nondiffusive* (Kubo-Anderson or kangaroo) type of stochastic process. The model involves a scaling exponent ε (with $\varepsilon\to\infty$ in the diffusion limit). For the (partly analytical) solution for the *mean* velocity profile, excellent agreement with the experimental data yields $\varepsilon\approx0.58$.

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I. INTRODUCTION

The logarithmic velocity profile is one of the most famous results in the study of turbulent flows. It was first given by Von Kármán [1] and independently by Prandtl [2]. A purely dimensional derivation is due to Millikan [3] (and Landau [4]) and clearly shows the generality (i.e., model independence) of the result. Namely, the nondimensionalized mean velocity $\overline{U}_+=\overline{U}/u_*$ for Newtonian shear flow (in the x direction) along an infinitely extended smooth surface can only be a function of the nondimensionalized distance $y_+=u_*y/\nu$ to that surface, where ν is the kinematic viscosity and the friction velocity u_* is defined by means of the total stress $\tau=\rho u_*^2$ in the boundary layer (ρ being the fluid mass density). However, in the so-called inertial sublayer of the flow (the *outer* part), stress due to molecular viscosity will be unimportant. This is obviously possible only if $d\overline{U}_+/dy_+=1/\kappa y_+$, which implicitly defines the Von Kármán constant κ. Hence, for $y_+\to\infty$ one has

$$\overline{U}_+=\frac{1}{\kappa}[\ln(y_+)+\gamma]\ , \tag{1.1}$$

where γ arises as an integration constant. On the other hand, in the so-called viscous sublayer (the *inner* part), one has $\overline{U}_+=y_+$ as $y_+\to0$. In the crossover region the purely dimensional argument breaks down and one must resort to more specific turbulence modeling (see, e.g., [5]).

Such models describe turbulence mixing of scalar densities (e.g., particles, temperature, humidity). In the boundary layer one is particularly interested in the lateral transport of longitudinal momentum $P_x=\rho\overline{U}$. Under zero pressure gradient conditions (fully developed turbulence) the Reynolds-Navier-Stokes equation for this scalar density reads

$$\frac{\partial\overline{U}}{\partial t}=\frac{\partial}{\partial y}(\tau_R/\rho)+\nu\frac{\partial^2\overline{U}}{\partial y^2}\ , \tag{1.2}$$

where τ_R is the Reynolds stress

$$\tau_R/\rho=-\overline{uv}\ , \tag{1.3}$$

with $u=U-\overline{U}$ ($v=V-\overline{V}$) being the fluctuating velocity component in the x (y) direction. Equation (1.2) is a generic transport equation—with (1.3) revealing the hierarchy closure problem—and will *au fond* be handled as such. However, the actual solution of the transport problem in the specific case of $P_x=\rho\overline{U}$ allows for an easy comparison of the theory with a large body of existing data on the mean velocity profile (e.g., for pipe and channel flow).

A *unified* treatment of the transport process in the viscous and inertial sublayer (including the *crossover* region) should produce a value for the integration constant γ in the logarithmic profile (1.1) for which the experimental data yield $\gamma/\kappa\approx4-6$ (along with $\kappa\approx0.39\pm0.02$). In fact, two widely used turbulence closure models do allow for an analytic solution of the mean velocity profile in the entire boundary layer. As a result one obtains a $\gamma(\kappa)$.

However, both models yield manifestly incorrect values for γ/κ for *any* value of κ. For instance, the eddy-viscosity model (which takes $\tau_R/\rho=\nu_R d\overline{U}/dy$ for the Reynolds stress, with $\nu_R=\kappa u_* y$) gives $\gamma=\ln\kappa$. Prandtl's model (where $\nu_R=l^2|d\overline{U}/dy|$, with a mixing length $l=\kappa y$) is only marginally different, leading to $\gamma=\ln(4\kappa/e)$. For $\kappa=0.39$ this yields $\gamma/\kappa\approx-2.4$ and -1.4, respectively.

Both the eddy-viscosity (or K closure) and Prandtl's hypothesis are so-called *local* first-order closures [5–7]. That is, these models only account for turbulence mixing by infinitesimally small eddies. However, in contrast with molecular diffusion, such a local description (on the hydrodynamical scale) of turbulence transport is well known to be inadequate [8]. In the present article it will be argued that nonlocal effects (i.e., finite size eddies) are crucial for a theoretical modeling of the turbulent boundary layer.

Therefore, in Sec. II a *nonlocal* first-order closure theory will be developed by means of an analysis of turbulence sample paths and a well-defined stochastic hypothesis. The ensuing model will be shown to be intimately related to the general theory of continuous (but *not* necessarily *diffusive*) Markov processes [9–11]. The pertinent transition rates can be computed on the basis of (experimental) sampling rates. The model involves *both* an eddy viscosity *and* a correlation length (or mixing-length) concept.

In Sec. III the analysis will be specialized to the semi-infinite (i.e., constant-stress) boundary-layer situation. Explicit results are obtained for the case of exponential Eulerian sampling rates, which, under *boundary-layer scaling*, transform into algebraic rates for Lagrangian trajectories. The associated transition rates generate a nondiffusive stochastic process of the so-called *kangaroo* type (or strong-collision type [11–14]), which underscores the inadequacy of a gradient (diffusive or Fokker-Planck) type of turbulence modeling. The boundary-layer scaling function is established on the basis of a self-consistent fluctuating velocity field analysis.

The model involves only two intrinsic parameters, viz., a scaling exponent ε (local turbulence transport amounting to $\varepsilon\to\infty$) and the viscous sublayer correlation length a_+ (which is known to be $a_+\approx15$ from data concerning the normal velocity fluctuations near the wall, from both experiment [15–19] and direct numerical simulations [20–22]). For that matter, the perfect agreement between the latter and the present (analytical) predictions for near-wall fluctuations is noteworthy. The exponent ε is suggested to bear significance as a fractal dimension of a turbulence Cantor set [4,23,24].

Finally, in Sec. IV the exponent ε is determined by (partly analytically) solving the integro-differential kangaroo process transport equation in the steady state for the *mean velocity profile.* While the Von Kármán constant is *not* an intrinsic parameter of the theory, it relates the theoretical variables $\overline{\mathcal{U}}=f(\mathcal{y})$ to the experimental ones, as defined in Eq. (1.1), by the simple rescaling $\overline{\mathcal{U}}=\kappa\overline{U}_+$ and $\mathcal{y}=\kappa y_+$. The theoretical result is in excellent agreement with the experimental data for $\varepsilon\approx0.58$ (with $a_+\approx16$ and $\kappa\approx0.39$, corresponding to $\gamma/\kappa\approx4.6$). Some final remarks are made in Sec. V. An exhaustive account will be available in Ref. [25].

II. NONLOCAL CLOSURE

A. Sample paths

Using the definition of time averaging, the Reynolds stress (1.3) may be written as

$$\tau_R/\rho=-\frac{1}{T}\int_0^T u(y,t+\tau)v(y,t+\tau)d\tau\ , \tag{2.1}$$

with T sufficiently large being tacitly understood. A sample path $\eta(y,t)$ will be defined on the basis of the fluctuation velocity $v(y,t)$ through $v(t+\tau)=d\eta(\tau)/d\tau$. A typical example of both $v(t)$ and the corresponding trajectory

$$\eta(y,\tau)=\int_0^\tau v(y,t+s)ds \tag{2.2}$$

has been sketched in Fig. 1. Thus (2.1) becomes

$$\tau_R/\rho=-\frac{1}{T}\int_{R(y,T)}u[y,t+\tau(\eta)]d\eta\ , \tag{2.3}$$

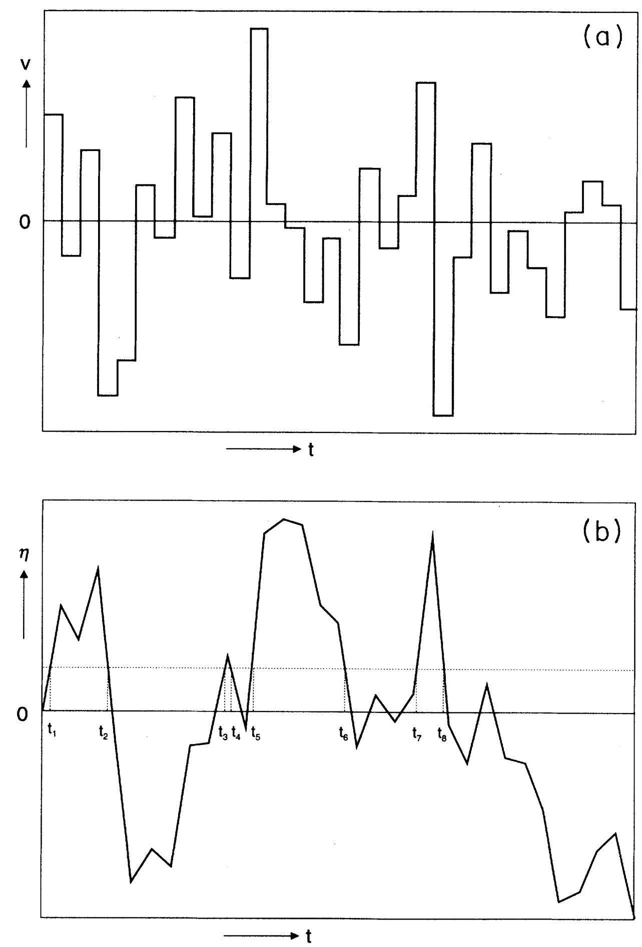

FIG. 1. Sketch of (a) a typical trace of the fluctuating velocity $v(t)$ at a fixed point in space and (b) the ensuing sample path $\eta(t)$ with its indicated visits at a particular value of the coordinate η.

where $R(y,T)$ is the range of values covered by η during $\tau\in(0,T)$. Since $\tau(\eta)$ is a multivalued function, (2.3) will be rewritten as

$$\tau_R/\rho = -\frac{1}{T}\int_{-\infty}^{\infty}\sum_{n=1}^{N(\eta,y,T)} u[y,t+\tau_n(\eta)]d\eta \;, \qquad (2.4)$$

where $R(y,T)$ is accounted for by the number of crossings (or visits) of the sample path $\eta(y,\tau)$ with a fixed value of the coordinate η during $\tau\in(0,T)$.

Now let $N_+(\eta,y,T)$ be the number of times the sample path visits the value η with $v_n>0$. Similarly, let $N_-(\eta,y,T)$ be the number of times the sample path visits the value η with $v_n<0$ (note that N_+ and N_- differ at most by one). Then (2.4) reads

$$\tau_R/\rho = -\frac{1}{T}\int_{-\infty}^{\infty}\left[\sum_{n=1}^{N_+(\eta,y,T)} u[y,t+\tau_n(\eta|+)] - \sum_{n=1}^{N_-(\eta,y,T)} u[y,t+\tau_n(\eta|-)]\right]d\eta \;, \qquad (2.5)$$

where the conditional functions $\tau_n(\eta|\pm)$ have been defined as the nth $\tau(\eta)$ for which $v>0$ (upper sign) or $v<0$ (lower sign).

Let us now introduce the mean visiting rates

$$\lambda_\pm(y,\eta) = \frac{1}{T}N_\pm(y,\eta,T) \qquad (2.6)$$

as the limiting values of the right-hand side, so that upon defining the conditional averages

$$\bar{u}(y,\eta,t|\pm) = \frac{1}{N_\pm(y,\eta,T)}\sum_{n=1}^{N_\pm(y,\eta,T)} u[y,t+\tau_n(\eta|\pm)] \;, \qquad (2.7)$$

the expression (2.5) may be written as

$$\tau_R/\rho = -\int_{-\infty}^{\infty}[\lambda_+(y,\eta)\bar{u}(y,\eta,t|+) - \lambda_-(y,\eta)\bar{u}(y,\eta,t|-)]d\eta \;. \qquad (2.8)$$

Notice that (2.8) is merely a reorganized, but otherwise exactly equivalent, version of (2.1).

B. Closure hypothesis

The expression (2.8) is convenient for the implementation of the time-reversal symmetry-breaking, stochastic closure hypothesis. For the purpose of the present subsection, let the statistical properties of the turbulence be spatially homogeneous, so that the coordinate η defined, for fixed y, by the sample path (2.2) can be mapped onto the coordinate y without further ado. The inhomogeneity of the boundary layer will be considered in Sec. III.

Since $u=U-\bar{U}$ in (1.3), one has $\bar{u}(y,\eta,t|\pm)=\bar{U}(y,\eta,t|\pm)-\bar{U}(y,t)$ in (2.8). Now let the transport (of longitudinal momentum, in this case) over $\Delta y=\eta$ be effectively instantaneous (within the averaging-time window T). Transport *to* y then amounts to events with $v_n<0$ while $\eta<0$ (downward) or with $v_n>0$ while $\eta>0$ (upward), which are therefore counted with $\bar{U}(y,\eta,t|\pm)=\bar{U}(y-\eta,t)$. On the other hand, events with $v_n<0$ and $\eta>0$ (downward) or $v_n>0$ and $\eta<0$ (upward) represent transport *from* y and thus amount to $\bar{U}(y,\eta,t|\pm)=\bar{U}(y,t)$. Compactly written, one has $\bar{U}(y,\eta,t|\pm)=\bar{U}[y-\eta\theta(\pm\eta),t]$. Hence, setting

$$\bar{U}(y,\eta,t|+)=\bar{U}(y-\eta,t) \quad \text{if } \eta>0 \;,$$
$$\bar{U}(y,\eta,t|-)=\bar{U}(y-\eta,t) \quad \text{if } \eta<0 \;, \qquad (2.9)$$

and $\bar{U}(y,\eta,t|\pm)=\bar{U}(y,t)$ in all other cases, (2.8) becomes

$$\tau_R/\rho = \int_{-\infty}^{\infty}\Lambda(y,\eta)[U(y+\eta,t)-\bar{U}(y,t)]d\eta \;, \quad (2.10)$$

where $\Lambda(y,\eta<0)=-\lambda_+(y,-\eta)$ and $\Lambda(y,\eta>0)=\lambda_-(y,-\eta)$. This result is in line with a recent Lagrangian analysis of the Reynolds stress by Bernard and Handler [8], although their more specific modeling does not account for nondiffusive correlations.

If the visiting rates obey mirror symmetry, $\lambda_-(y,\eta)=\lambda_+(y,-\eta)$, so that Λ is odd in η, (2.10) can be shown to imply Fieldler's [26] version of Stull's [7] transilient turbulence theory. If, in addition, the sampling rates λ depend only on η, (2.10) further agrees with spectral diffusivity theory [27]. In the latter case the rates have been modeled as $\lambda(\eta)=\int_\eta^\infty p(l)\omega(l)dl$, where $p(l)$ is the probability density of occurrence of an eddy of length l with a typical transport frequency $\omega(l)$. Indeed, mixing over a distance $\Delta y=\eta$ can only be due to eddies of size $l\geq\eta$. Neither transilient theory nor spectral diffusivity has been developed under boundary-layer scaling.

C. Transport equation: Transition rates

Substituting (2.10) for τ_R/ρ into the transport equation (1.2) (for the time being neglecting molecular diffusion), differentiating the first term in the integrand with respect to y, and partially integrating it, one obtains

$$\frac{\partial\bar{U}}{\partial t} = -\frac{\partial}{\partial y}[\Lambda(y)\bar{U}(y,t)] + \int_{-\infty}^{\infty}\mathbb{W}(y,\eta)\bar{U}(y+\eta,t)d\eta \;, \qquad (2.11)$$

$\Lambda(y)=\int_{-\infty}^{\infty}\Lambda(y,\eta)d\eta$, and

$$\mathbb{W}(y,\eta) = \frac{\partial\Lambda(y,\eta)}{\partial y} - \frac{\partial\Lambda(y,\eta)}{\partial\eta} \;. \qquad (2.12)$$

Since $\Lambda(y,\eta)$ is discontinuous at $\eta=0$ with $\Lambda(y,+0)-\Lambda(y,-0)=2\lambda(y,0)$, the transport kernel $\mathbb{W}(y,\eta)$ defines a transition rate $W(y,\eta)$ according to [11]

$$\mathbb{W}(y,\eta) = W(y,\eta) - 2\lambda(y,0)\delta(\eta) \;. \qquad (2.13)$$

Note that $\lambda=\lambda_\pm$ at $\eta=0$. The inverse of (2.12) is found by considering it as an ordinary first-order differential equation for the $\lambda_\pm(y,\eta)$. Setting $\eta=0$ in the resulting line integral yields

$$2\lambda(y,0) = \int_{-\infty}^{\infty}W(y-\eta,\eta)d\eta \;. \qquad (2.14)$$

Using (2.13) and (2.14), the transport equation (2.11) may be written as

$$\frac{\partial \bar{U}}{\partial t}+\frac{\partial}{\partial y}[\Lambda(y)\bar{U}(y,t)] = \int_{-\infty}^{\infty}[W(y|y')\bar{U}(y',t)-W(y'|y)\bar{U}(y,t)]dy' , \tag{2.15}$$

where we have let $\eta=y'-y$. Equation (2.15) describes a stochastic process with transition rate $W(y|y') = W(y,y'-y)$. The corresponding transport kernel $\mathbb{W}(y|y')$ obviously satisfies $\int_{-\infty}^{\infty}\mathbb{W}(y|y')dy=0$, as it should. Of course, $\mathbb{W}(y|y')$ has significance as a statistical matrix only if its off-diagonal elements are nonnegative, i.e., if $W(y|y')\geq 0$. It is further worth noting that the existence of a master equation does not rigorously imply Markovian properties on all levels of description [11,28].

D. Exchange

Turbulent flows typically have a strong eddy structure. As a consequence, the mixing process may involve the property of strong exchange. For the transition rates this implies

$$W(y|y')=W(y'|y) . \tag{2.16}$$

Exchange expresses a strong correlation between the mixing from y to y' and back to y, for any pair of points. By virtue of (2.12), exchange is related to a symmetry property for (the spatial rate of change of) the visiting rates. If, for given y, the sampling rate for the point y' is denoted by $\lambda(y|y')=|\Lambda(y,\eta)|$, one finds

$$\frac{\partial}{\partial y}\lambda(y|y')=-\frac{\partial}{\partial y'}\lambda(y'|y) , \tag{2.17}$$

where $\partial/\partial y$ ($\partial/\partial y'$) now denotes differentiation for constant y' (y). For instance, assuming $y'>y$, increasing y, as in $\lambda(y|y')$, decreases the distance $\eta=y'-y$ and hence typically increases the rate. On the other hand, increasing y', as in $\lambda(y'|y)$, one instead increases η, leading to a lowering of the rate. Equation (2.17) shows how these changes for upward and downward visiting rates are related under exchange.

If λ only depends on η and if $\lambda(\eta)$ is continuous at $\eta=0$, (2.17) implies that $\lambda(\eta)=\lambda(-\eta)$. In fact, this property is implicit in the derivation of the spectral diffusivity equation [27] for homogeneous turbulence. In that case the explicit flow term in the transport equation (2.11) and (2.15) vanishes because $\Lambda(y)=0$ by symmetry. This latter feature is a more general consequence of exchange. Namely, by means of (2.17) one readily shows that $d\Lambda/dy=0$ if $\lambda(\pm\infty|y)=0$. Hence $\Lambda(y)=0$ if $\Lambda(\pm\infty)=0$. These conditions are typically satisfied in a well-posed physical problem. The boundary-layer model of Sec. III provides a particular instance. Of course, in that case $y,y'\in(0,\infty)$.

III. BOUNDARY LAYER

A. Scaling

While the linear mapping in Sec. II of the sample path coordinate η (constructed for fixed y) onto the coordinate y may be correct for homogeneous turbulence, it is certainly not valid in the boundary layer (where the size of a typical eddy scales with the distance y from the surface). The basic scaling hypothesis for fully developed boundary-layer turbulence asserts the existence of an invariant characteristic time scale τ_0 (defined at some y_0) such that $dt/d\tau_0=y/y_0$ scales with y. Since the flow in the viscous sublayer ($y\to 0$) is also fully turbulent (despite the fact that it is sometimes misleadingly denoted as the laminar sublayer) [17], this linear time scaling $\ell(y)=y$ is expected to hold down to molecular distances from the surface. In addition, the velocity fluctuations themselves may depend on y. The velocity scaling function $\upsilon(y)\sim(\overline{v^2})^{1/2}$ will be discussed in more detail in Sec. III C. In the inertial sublayer ($y\to\infty$) one has $\upsilon(y)=1$. Finally, the eddy scaling function is given by $\mathcal{s}(y)=\upsilon\ell$.

Let $\eta(y,t)$ be the sample path attached to the point y and let $v(y,t)$ be the fluctuating velocity at y (as in Sec. II), so that $d\eta=v(y,t)dt$. Now let $\eta_*(y,t)$ denote the actual trajectory of a fluid particle with fluctuation velocity $v(y_*,t)$, where $y_*=y+\eta_*$. That is, $d\eta_*=v(y+\eta_*,t)dt$. The corresponding trajectory is given by

$$\eta_*(y,t)=\int_0^t v[y+\eta_*(\tau),\tau]d\tau . \tag{3.1}$$

Boundary-layer scaling then implies that $d\eta=\mathcal{s}(y)d\varphi_0$, where $\varphi_0(\tau_0)$ is a nondimensional invariant function. Similarly, $d\eta_*=\mathcal{s}(y_*)d\varphi_0$. Hence the Jacobian $J(\eta,\eta_*) =|d\eta/d\eta_*|$ of the mapping $\eta(\eta_*)$ reads

$$J(\eta,\eta_*)=\frac{\mathcal{s}(y)}{\mathcal{s}(y+\eta_*)} \tag{3.2}$$

so that

$$\frac{\eta}{\mathcal{s}(y)}=\int_0^{\eta_*}\frac{1}{\mathcal{s}(y+\eta_*')}d\eta_*' . \tag{3.3}$$

With $\mathcal{s}(y)\geq 0$ and $\mathcal{s}(0)=0$, (3.3) maps the fictitious path space $\eta\in(-\infty,\infty)$ onto the actual path space $\eta_*\in(-y,\infty)$, as it should be for the boundary layer. For example under pure time scaling ($\upsilon=1$) one has $\mathcal{s}(y)=y$ and (3.3) explicitly yields the inverse mapping $\eta_*=y(e^{\eta/y}-1)$.

B. Sampling rates: Analytical model

Using (3.2) one may rewrite (2.9) in terms of η_*. The closure hypothesis of Sec. II now implies the linear mapping of η_* onto y. Letting $\eta_*=y'-y$ and defining $\Lambda_*(y|y')=J(\eta,\eta_*)\Lambda(y,\eta)$, the Reynolds stress formula (2.10) thus becomes

$$\tau_R/\rho=\int_0^{\infty}\Lambda_*(y|y')\bar{U}(y',t)dy' , \tag{3.4}$$

where, in view of the visiting rates model to be presented in Eq. (3.5), we have already accounted for exchange.

Since by (2.7) the visiting rates are defined in terms of velocity fluctuations at a fixed point y, one expects $\lambda_+(y,-\eta)=\lambda_-(y,\eta)$. Moreover, due to scaling one should have $\lambda=\lambda[y,|\eta|/\mathcal{s}(y)]$. Therefore, let us consider

$$\lambda(y,\eta)=\frac{D}{\mathcal{s}(y)}e^{-\varepsilon|\eta|/\mathcal{s}(y)} , \tag{3.5}$$

as suggested by (3.3) in the inertial sublayer. The parameters D and ε will be determined further on. Using (3.2) and (3.3), one obtains

$$\Lambda_*(y|y') = -\frac{D}{\varepsilon}\frac{\partial}{\partial y'}\left[\frac{k(y)}{k(y')}\right]^{\pm\varepsilon}, \tag{3.6}$$

with

$$k(y) = k_0 \exp\left[\int_{y_0}^{y}\frac{dy'}{s(y')}\right], \tag{3.7}$$

the upper (lower) sign in the exponent applying if $y' > y$ ($y' < y$) and k_0 and y_0 being constants [of integration of $\partial k/\partial y = k/s(y)$] that are immaterial for the rates. For example, let $k_0 = k(y_0)$ be such that $k(y)/y \to 1$ if $y \to \infty$.

The rates (3.6) satisfy exchange according to (2.17). Hence the transport equation (2.15), i.e., (1.2), now becomes

$$\frac{\partial \overline{U}}{\partial t} = \int_0^\infty [W_*(y|y')\overline{U}(y',t) - W_*(y'|y)\overline{U}(y,t)]dy' + \nu\frac{\partial^2 \overline{U}}{\partial y^2}, \tag{3.8}$$

with $W_*(y|y') = \partial\Lambda_*(y|y')/\partial y$. Equation (3.6) yields

$$W_*(y|y') = \frac{\varepsilon D}{s(y)s(y')}\left[\frac{k(y)}{k(y')}\right]^{\pm\varepsilon}, \tag{3.9}$$

which is always non-negative and therefore indeed generates a proper stochastic process (see Sec. II C). Note that (3.9) factorizes as $W_*(y|y') = p(y)q(y')$. Such a stochastic process is known as a "kangaroo process" (see, e.g., [11,14]).

A kangaroo process [in particular, with $p(y)=1$ or $q(y)=1$] is also known as a Kubo-Anderson process. It has *inter alia* been used to describe motional narrowing in spin systems [11–13,29]. An application to Mössbauer spectroscopy is due to Blume and co-workers [30–32]. In atomic collision theory [33–35] and in laser line-width calculations [14,36,37] it has been applied in the strong-collision limit. This limit is the extreme opposite of diffusive motion (as described by local gradient models, i.e., Fokker-Planck-type equations). In fact, since boundary-layer scaling amounts to both $s(y) \to y$ and $k(y) \to y$ if $y \to \infty$, a gradient expansion in (3.8) and (3.9) is not rigorously possible for any finite value of $\varepsilon < \infty$ (see also Ref. [8]).

C. Scaling function

The scaling function is given by $s(y) = \upsilon\ell$ with $\ell(y) = y$. Since $s(y) = y$ for $y \to \infty$ implies the logarithmic velocity profile (1.1), one should have $\upsilon(y) = 1$ in the inertial sublayer. Indeed, there exists ample experimental evidence that the root mean square of the normal velocity fluctuations $(\overline{v^2})^{1/2} \sim \upsilon(y)$ does not depend on y in that region [15–17,38,39]. Of course, $\upsilon(y) = 1$ cannot hold down to the surface. On an as yet undetermined length scale (a) near the surface ($y = 0$), the normal velocity goes to zero as $\upsilon(y) = (y/a)^2$ by virtue of continuity (see, e.g., [4,15–22] and Appendix B).

The alternative case [17] $\upsilon(y) \sim y^3$ if $y \to 0$ (i.e., $a = \infty$) would imply mixing length scaling $s(y) \sim y^4$ in the viscous sublayer. However, in Appendices B and C it is shown, by means of a systematic analysis of the fully three-dimensional Navier-Stokes equation for the fluctuating velocity fields, that $s(y) \sim y^n$ (with $n > 1$) implies $\tau_R/\rho \sim y^n$ (if $y \to 0$) for the Reynolds stress and that in the power series expansion of the Reynolds stress the term with $n = 4$ is always absent. Hence $\upsilon(y) \sim y^2$ if $y \to 0$.

Consider the inverse scaling function $s^{-1} = 1/s(y)$ as a function of y^{-1}. It has the properties $s^{-1} \to y^{-1}$ if $y^{-1} \to 0$ and $s^{-1} \to y^{-n}$ if $y^{-1} \to \infty$. Since it should be a continuous and differentiable function on $y^{-1} \in (0,\infty)$, it may in general have the polynomial representation $s^{-1} = \sum_{k=1}^{n} c_k y^{-k}$. In particular, one has $n = 3$ while $c_1 = 1$ and $c_3 = a^2$. The inverse velocity scaling function $\upsilon^{-1} = 1/\upsilon(y)$ therefore reads $\upsilon^{-1} = 1 + c_2 y^{-1} + (y/a)^{-2}$. Consequently, one has $\upsilon(y) = (y/a)^2[1 - c_2 y/a^2 + \cdots]$ if $y \to 0$. However, in Appendix B it is shown that in the Taylor expansion of $\upsilon(y)$ the cubic term $\sim y^3$ is always absent. Therefore, $c_2 = 0$ so that

$$\upsilon(y) = \frac{(y/a)^2}{1+(y/a)^2}. \tag{3.10}$$

A comparison of $(\overline{v^2})^{1/2} \approx u_* \upsilon(y)$ with the available data from experiments as well as from numerical simulations [15–22] confirms that $c_2 \approx 0$ and yields $a_+ \approx 10$–20 (where $a_+ = u_*^a/\nu$), with an average $a_+ \approx 15$. Finally, (3.10) implies

$$\frac{1}{s(y)} = \frac{1}{y}\left[1 + \left[\frac{a}{y}\right]^2\right] \tag{3.11}$$

for the inverse mixing-length scaling function.

IV. VELOCITY PROFILE

A. Steady state

Let us now consider (3.8) and (3.9), for the scalar momentum density $\rho\overline{U}$, in the steady state $\partial\overline{U}/\partial t = 0$. In that case it is more convenient to return to (1.2) and to invoke (3.4) for τ_R/ρ, which yields

$$\nu\frac{\partial \overline{U}}{\partial y} + \int_0^\infty \Lambda_*(y|y')\overline{U}(y')dy' = u_*^2. \tag{4.1}$$

Substituting (3.6) for the rates and doing a partial integration, one obtains

$$\nu\frac{\partial \overline{U}}{\partial y} + \frac{D}{\varepsilon}\left[\int_y^\infty\left[\frac{k(y)}{k(y')}\right]^{\varepsilon}\frac{\partial\overline{U}}{\partial y'}dy' + \int_0^y\left[\frac{k(y)}{k(y')}\right]^{-\varepsilon}\frac{\partial\overline{U}}{\partial y'}dy'\right] = u_*^2. \tag{4.2}$$

By (3.7) and (3.11) one has

$$\mathscr{k}(y)=y\exp\left[-\frac{1}{2}\left[\frac{a}{y}\right]^2\right] . \tag{4.3}$$

The Fredholm integral equation (4.2) can be mapped onto an equivalent differential equation. First let $z=\ln[\mathscr{k}(y)/\mathscr{k}_0]$, so that

$$\nu\frac{\partial\overline{U}}{\partial y}+\frac{D}{\varepsilon}\int_{-\infty}^{\infty}e^{-\varepsilon|z-z'|}\frac{\partial\overline{U}}{\partial z'}dz'=u_*^2 , \tag{4.4}$$

with $\partial/\partial z=\mathscr{s}(y)\partial/\partial y$. Note that if $z\to\infty$ (i.e., $y\to\infty$) one has $\partial\overline{U}/\partial z\to u_*/\kappa$ according to (1.1), so that $D=\frac{1}{2}\varepsilon^2\kappa u_*$. Considering now $\mathscr{J}=\int_{-\infty}^{\infty}dz'K(z-z')$ with $K(z)=e^{-\varepsilon|z|}$ as a linear operator [40], which maps the function $\varphi=\partial\overline{U}/\partial z$ onto another function, say, $\psi=\mathscr{J}\varphi$, one obtains $\mathscr{J}^{-1}=\frac{1}{2}\varepsilon[1-\varepsilon^{-2}(\partial/\partial z)^2]$ for the inverse operation (i.e., $\mathscr{J}^{-1}\mathscr{J}=I$). Therefore, operating with $\mathscr{J}^{-1}$ from the left on (4.4) reduces it to an inhomogeneous second-order differential equation for the velocity gradient. In terms of $\mathscr{y}=\kappa y_+$ (where $y_+=u_*y/\nu$) and $\mathscr{U}=\kappa\overline{U}_+$ (where $\overline{U}_+=\overline{U}/u_*$) it becomes a Sturm-Liouville equation for $\mathscr{f}(\mathscr{y})=\mathscr{U}'$, viz.,

$$-\varepsilon^{-2}\mathscr{s}(\mathscr{y})[\mathscr{s}(\mathscr{y})\mathscr{f}']'+[1+\mathscr{s}(\mathscr{y})]\mathscr{f}(\mathscr{y})=1 , \tag{4.5}$$

where a prime denotes differentiation with respect to $\mathscr{y}$, where $\mathscr{s}(\mathscr{y})=\mathscr{y}^3/(\mathscr{a}^2+\mathscr{y}^2)$ with $\mathscr{a}=\kappa a_+$ (and $a_+=u_*a/\nu$) so that $\mathscr{a}\approx 6$. The ensuing mean velocity profile $\mathscr{U}(\mathscr{y})$ is universal in the sense that it is independent of the Von Kármán constant κ.

B. Profile crossover: Scaling exponent

The exponent ε may be computed in two ways. First, on the "input side" it can be determined by processing fluctuating normal velocity data as described in Sec. II A and comparing the resulting sampling rates $\lambda(y,\eta)$ with (3.5). Second, on the "output side" it can be found by comparing the measured mean velocity distribution over a smooth surface with the solution of (4.5). The present estimate of ε will be obtained using the output method.

Since (4.5) allows for an exact solution in two limiting cases, a rough estimate for ε can be made analytically. First, for $\varepsilon\to\infty$, (4.5) reduces to its local limit $\mathscr{f}(\mathscr{y})=1/(1+\mathscr{s})$. In particular, for the logarithmic profile (1.1) it leads to $\gamma=\gamma_a+\ln\kappa$, γ_a given in (A2) and (A3). This yields $\gamma\approx 1.5$ for $\mathscr{a}\approx 6$. Second, for $a\to 0$, (4.5) reduces to its pure time-scaling limit where $\mathscr{s}(\mathscr{y})=\mathscr{y}$, which allows for a solution in terms of Lommel functions. The resulting logarithmic profile constant is $\gamma=\gamma_\varepsilon+\ln\kappa$, γ_ε given in (A10). With $\gamma\approx\gamma_a+\gamma_\varepsilon+\ln\kappa$, the experimental value $\gamma\approx 2$ implies $\gamma_\varepsilon\approx 0.5$. This corresponds to $\varepsilon\approx 0.5$.

A more accurate value for ε can be obtained by a best fit to the entire measured profile, i.e., including the crossover region between viscous and inertial sublayers. For that purpose, Eq. (4.5) has been integrated by means of a simple (matrix inversion) routine, starting at $\mathscr{y}=0$ (the surface) with $\mathscr{f}(0)=1$. A final integration then yields the universal profile $\mathscr{U}(\mathscr{y})$. An excellent fit to the experimental data $\overline{U}_+=\mathscr{U}/\kappa$ as a function of $y_+=\mathscr{y}/\kappa$ for $1<y_+<10^4$ [taken from standard references for both pipe and channel flow (e.g., [41]) and processed for their residual Reynolds number dependence as $\mathrm{Re}\to\infty$; see Ref. [25]] is shown in Fig. 2. It corresponds to $\kappa\approx 0.39$ and $a_+\approx 16$ and yields the value $\varepsilon\approx 0.58$.

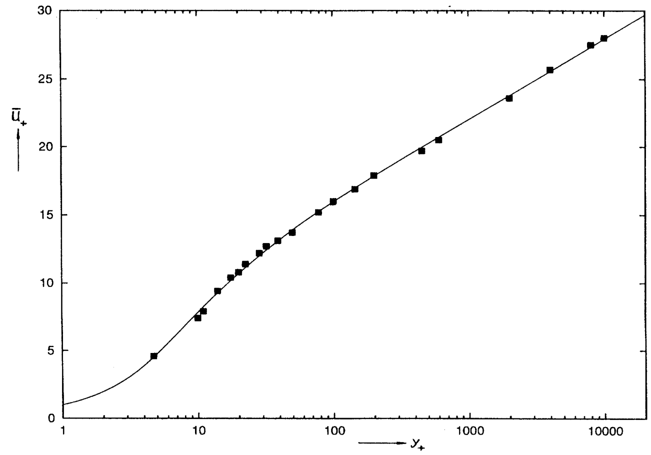

FIG. 2. Plot of the best unconditional least-squares fit of the theoretical profile to the experimental data, corresponding to $\kappa=0.39$, $a_+=16$, and $\varepsilon=0.58$.

These optimal values are obtained by means of a least-squares computation [42] *without a priori* constraints on the triple (κ,a_+,ε). Assuming a constant relative data error over the entire y_+ range, the theoretical curve fits the data with an accuracy of 2%. Statistical tests on the optimum triple have been performed as well. The scatter plots show that the 95% confidence intervals amount to $0.37\lesssim\kappa\lesssim 0.41$, $14\lesssim a_+\lesssim 18$, and $0.4\lesssim\varepsilon\lesssim 0.8$. A detailed discussion can be found in [25].

V. FINAL REMARKS

In this article we have shown that mixing of scalar quantities in the turbulent boundary layer can be described as a *kangaroo process.* This extremely *nondiffusive* stochastic process is found by means of (i) the construct of a local sample path space $\eta(y)$ at each point y in the fluid, (ii) a nonlinear scaling $\mathscr{s}(y)$ that maps η onto a global path space $\eta_*(\eta)$ with $\eta_*=y'-y$, (iii) a stochastic, time-reversal symmetry-breaking, closure hypothesis, and (iv) an analytical sampling rate model satisfying exchange.

The closure hypothesis (2.9) implicitly assumes the transported quantity to be a tracer, which is trivially true for the properties of the fluid itself (for instance, for the Reynolds stress). The hypothesis can also be shown [25] to be rigorously correct for a Rayleigh particle in the Smoluchovski limit (i.e., for diffusion). The sampling rate model (3.5) is suggested by the nature of the mapping (3.3) of Eulerian onto Lagrangian sample paths in the inertial sublayer [i.e., where $\mathscr{s}(y)=y$]. The general expression (3.11) for the scaling function in the semi-infinite boundary layer arises from an analysis (in Appendix B) of fully developed three-dimensional turbulence over an infinite flat plate and an assumption concerning its simplest analytical form.

The sampling rates $\Lambda_*(y|y')$, which are defined in terms of the *fluctuating* velocity field at fixed y, define transition probability rates $W_*(y|y')$ in a Chapman-Kolmogorov (master) type of equation that describes the migration of a *mean* scalar density $P(y,t)$. In particular, in that manner the Navier-Stokes equation leads to an integro-differential equation for the longitudinal momentum $P_x=\rho\overline{U}$, i.e., for the mean velocity $\overline{U}(y,t)$ at a distance y from a flat surface. The theory involves both an eddy viscosity (defining the Von Kármán constant $\kappa\approx0.39$) and an exponent ε. The latter defines a Prandtl-type mixing length ($l\approx y/\varepsilon$ if $\varepsilon\to\infty$, $l\approx ye^{1/\varepsilon}$ if $\varepsilon\to0$). In the, analytically, highly nonuniform limit $\varepsilon\to\infty$ the model reduces to local (diffusive) K theory. While in the inertial sublayer ($y\to\infty$) turbulence mixing is self-similar by perfect time scaling $\mathscr{s}(y)\sim y$, it involves a characteristic length ($a_+\approx16$) to yield viscous sublayer scaling $\mathscr{s}(y)\sim y^3$.

In the steady state the *integral* equation (3.8) is easily solved (analytically for $a_+=0$ and numerically for $a_+\neq0$) since it can be transformed into the *differential* equation (4.5). The theoretical result has been compared with experimental data for the mean velocity profile in Fig. 2. This yields $\varepsilon\approx0.58$, thereby showing the nonlocality of turbulent transport. As a consequence, the turbulence kangaroo process (3.8) and (3.9) has fractal features. Namely, let $\mathcal{P}(Y>y)$ denote the probability that (during a short period of time τ) the system, e.g., a tracer particle, *jumps* from y_0 to *beyond* $y>y_0$. In terms of the sampling rates: $\mathcal{P}(Y>y)=-\tau\Lambda_*(y_0|y)$. Hence, using (3.6) and (3.7) one has $\mathcal{P}(Y>y)=C_0y^{-\varepsilon}$ (for all y in the inertial sublayer) with $C_0=\tau D\mathscr{k}_0^{\varepsilon}/\mathscr{s}_0$, which, with $\varepsilon=d$, may be compared with the number of *gaps* $\mathcal{N}(L>l)=N_0l^{-d}$ of length $L>l$ in a Cantor set $\mathcal{C}_\infty$ with fractal dimension $0<d<1$.

While the uncertainty in ε is as yet rather large, it is almost surely (for more than 95%) less than unity. It is gratifying that the value $\varepsilon\approx0.58$ has been determined within the triple (κ,a_+,ε) with $\kappa\approx0.39$, which confirms its by-now accepted value, and $a_+\approx16$. This result for a_+ agrees with its value estimated on the basis of *both* the root mean square $\mathscr{v}(y)$ of the normal velocity fluctuations, according to (3.10), *and* their longitudinal correlation function $R_{22}(x)$. The latter involves a correlation length $\overline{x}_+=\kappa a_+^2/2$, for which the experimental data indeed indicate $\overline{x}_+\approx45-50$, which is discussed in more detail in Appendix C. An exhaustive account of the present work will be available in Ref. [25].

APPENDIX A: THE LOGARITHMIC PROFILE CONSTANT

1. The case $\varepsilon=\infty$

By (4.5) strict locality amounts to $\overline{\mathcal{U}}'(\mathscr{y})=1/[1+\mathscr{s}(\mathscr{y})]$, which is readily integrated to yield the velocity profile

$$\overline{\mathcal{U}}=N_0\ln\left[1+\frac{\mathscr{y}}{A}\right]+\frac{1}{2}N_1\ln\left[1-\frac{\mathscr{y}}{A}+\frac{\mathscr{y}^2}{C}\right]+\frac{1}{\Delta}N_2\left[\arctan\left[\frac{1}{\Delta}\right]-\arctan\left[\frac{1-2\mathscr{y}/B}{\Delta}\right]\right], \tag{A1}$$

where A is the real root of the cubic equation $A^3-A^2-a^2=0$ (which has a positive discriminant), $B=A-1$, $C=a^2/A$ [$AB=C$ so that $B=(a/A)^2$], $N_0=A^2/(1+3B)$, $N_1=1-N_0$, $N_2=1-(1-3B)N_0/A$, and $\Delta=(3+4/B)^{1/2}$. If $\mathscr{y}\to\infty$, (A1) yields the logarithmic profile (1.1) as $\overline{\mathcal{U}}\to\ln(\mathscr{y})+\gamma_a$, with

$$\gamma_a=-N_0\ln(A)-\frac{1}{2}N_1\ln(C)+\frac{1}{\Delta}N_2\operatorname{arccot}\left[-\frac{1}{\Delta}\right]. \tag{A2}$$

If $a\gg1$, (A2) leads to

$$\gamma_a\approx(2\pi/3^{3/2})a^{2/3}-\ln(a^{2/3})-\tfrac{1}{3}, \tag{A3}$$

where terms of order $a^{-2/3}\ln a$ have been disregarded. For example, $\gamma_a\approx2.5$ if $a\approx6$.

2. The case $a=0$

For pure time scaling $\mathscr{s}(\mathscr{y})=\mathscr{y}$. Rewriting (4.5) in terms of $\xi=2\varepsilon\mathscr{y}^{1/2}$ it becomes the Lommel equation [43]

$$\xi^2\Phi''+\xi\Phi'-(\xi^2+\sigma^2)\Phi=k \tag{A4}$$

for $\Phi=\partial\overline{U}/\partial y$. A prime denotes $\partial/\partial\xi$, $\sigma=2\varepsilon$, and $k=-(2\varepsilon u_*)^2/\nu$. Hence [43] $\Phi=ks_{-1,\sigma}(i\xi)+BI_\sigma(\xi)$, with $B=(k\pi/2\varepsilon)\csc(\varepsilon\pi)$. This is conveniently written as

$$\Phi=-k\left[K_\sigma(\xi)\int_0^\xi I_\sigma(\xi')\frac{d\xi'}{\xi'}+I_\sigma(\xi)\int_\xi^\infty K_\sigma(\xi')\frac{d\xi'}{\xi'}\right], \tag{A5}$$

where $I_\sigma(\xi)$ and $K_\sigma(\xi)$ are modified Bessel functions. Once more using (A4), the logarithmic part of the velocity profile $\overline{\mathcal{U}}$ becomes

$$\overline{\mathcal{U}}=\lim_{a\to0}\left[2\ln(\xi/a)+(8\varepsilon^2/k)\int_a^\infty\Phi(\xi')\frac{d\xi'}{\xi'}\right], \tag{A6}$$

where terms that vanish if $\xi\to\infty$ have been omitted. Equation (A6) leads to $\overline{\mathcal{U}}\to\ln(\mathscr{y})+\gamma_\varepsilon$ with

$$\gamma_\varepsilon=\lim_{a\to0}[\ln(2\varepsilon/a)^2-8\varepsilon^2A], \tag{A7}$$

$A=A_0+A_1$, and

$$A_0=2\int_a^\infty K_\sigma(\xi)\frac{d\xi}{\xi}\int_0^\xi I_\sigma(\xi')\frac{d\xi'}{\xi'},\qquad A_1=-\int_a^\infty K_\sigma(\xi)\frac{d\xi}{\xi}\int_0^a I_\sigma(\xi')\frac{d\xi'}{\xi'}. \tag{A8}$$

Consider A_0. Using the series expansion for $I_\sigma(\xi')$, the integrations can be done for $a=0$, except in the leading term ($n=0$). After using form. 6.561.16 from [44], the sum over $n=1,2,\ldots$ can be related to a combination of two Euler ψ functions (form. 6.3.16 from [45]). The remaining integral ($n=0$) over ξ can be done using the Fourier cosine integral representation (form. 9.6.25 from [45]) of $\xi^\sigma K_\sigma(\xi)$. In the ensuing double integral the order of integrations may be interchanged and, using $\ln t=\partial t^\mu/\partial\mu$ at $\mu=0$ to introduce a partial derivative of a beta function, one obtains

$$A_0=\left[\frac{1}{\sigma}\right]^2\left[\psi_{\mathrm{E}}\left[1+\frac{1}{2}\sigma\right]-\frac{1}{2\sigma}+\ln(2/a)\right], \tag{A9}$$

omitting terms that vanish if $a\to 0$.

For A_1 one only needs $I_\sigma(\xi)$ and $K_\sigma(\xi)$ for $\xi\to 0$. One obtains $A_1=-\sigma^{-3}/2$. Substitution of the resulting $A=A_0+A_1$ into (A7) yields

$$\gamma_\varepsilon=\frac{1}{\varepsilon}-2[\psi_{\mathrm{E}}(1+\varepsilon)-\ln\varepsilon]\ . \tag{A10}$$

If $\varepsilon\to\infty$, one finds $\gamma_\varepsilon\approx B_2/\varepsilon^2\to 0$. $B_2=\frac{1}{6}$ being a Bernoulli number. For example, $\gamma_\varepsilon\approx 0.5$ if $\varepsilon\approx 0.5$. If $\varepsilon\to 0$, γ_ε tends to infinity as $1/\varepsilon$.

APPENDIX B: TIME-SCALING ANALYSIS OF VISCOUS SUBLAYER FLUCTUATIONS

Fully developed boundary-layer turbulence is defined by all *mean* quantities being stationary and translation invariant in the (x,z) plane parallel to the surface ($y=0$). For convenience, let $\overline{U}=\overline{U}_1$ (i.e., $\overline{U}_2=\overline{U}_3=0$). For the mean values the Navier-Stokes equation then yields (1.2), with $\partial\overline{U}/\partial t=0$, and $\overline{P}=-\rho(\overline{v^2})$ subject to the (often overlooked) constraint

$$\overline{vw}=0\ . \tag{B1}$$

To discuss the eddy-scaling function $\mathscr{s}(y)=\mathscr{v}t$ it is required to study the actual velocity *fluctuations* near the wall under time scaling $\mathscr{t}(y)=y$. For (u,v,w), where $u=U-\overline{U}$, etc., one obtains [6,15–17,38,46]

$$\begin{aligned}
&\frac{\partial u}{\partial t}+v\frac{\partial\overline{U}}{\partial y}+\overline{U}\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta u-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}\\
&\qquad\qquad-w\frac{\partial u}{\partial z}+\frac{\partial}{\partial y}\overline{uv}\ ,\\
&\frac{\partial v}{\partial t}+\overline{U}\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\Delta v-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}\\
&\qquad\qquad-w\frac{\partial v}{\partial z}+\frac{\partial}{\partial y}\overline{v^2}\ ,\\
&\frac{\partial w}{\partial t}+\overline{U}\frac{\partial w}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\Delta w-u\frac{\partial w}{\partial x}-v\frac{\partial w}{\partial y}-w\frac{\partial w}{\partial z}\ ,
\end{aligned} \tag{B2}$$

where Δ is the three-dimensional Cartesian Laplacian. The pressure fluctuations follow from

$$\begin{aligned}
\frac{1}{\rho}\Delta p=&-2\frac{\partial v}{\partial x}\frac{\partial\overline{U}}{\partial y}-2\frac{\partial^2 uv}{\partial x\,\partial y}-2\frac{\partial^2 uw}{\partial x\,\partial z}-2\frac{\partial^2 vw}{\partial y\,\partial z}\\
&-\frac{\partial^2}{\partial x^2}(u^2)-\frac{\partial^2}{\partial y^2}(v^2-\overline{v^2})-\frac{\partial^2}{\partial z^2}(w^2)\ .
\end{aligned} \tag{B3}$$

The scaling hypothesis is introduced by letting both $t\to\sigma t$ and $y\to\sigma y$. In addition, let $u=\sigma\mathscr{u}$, $v=\sigma\mathscr{v}$, $w=\sigma\mathscr{w}$, and $\overline{U}=\sigma\overline{\mathscr{U}}$, expand all fields in powers of σ (e.g., $\mathscr{u}=\mathscr{u}^{(0)}+\sigma\mathscr{u}^{(1)}+\sigma^2\mathscr{u}^{(2)}+\cdots$) and collect terms of equal power in σ ($\sigma=1$ in the end). The continuity equation ($\mathrm{div}\,\mathscr{u}=0$) at once implies $\mathscr{v}^{(0)}=0$, while the leading contributions from (B2) yield $\partial^2\mathscr{u}^{(0)}/\partial y^2=\partial\mathscr{p}^{(0)}/\partial y=\partial^2\mathscr{w}^{(0)}/\partial y^2=0$ so that

$$\begin{aligned}
&\mathscr{u}^{(0)}=y\mathscr{f}(x,z,t)\ ,\\
&\mathscr{p}^{(0)}=\phi(x,z,t)\ ,\\
&\mathscr{w}^{(0)}=y\mathscr{g}(x,z,t)\ ,
\end{aligned} \tag{B4}$$

where $\mathscr{f}$, ϕ, and $\mathscr{g}$ are as yet arbitrary, random functions. From (B3) one finds $\partial^2\mathscr{p}^{(0)}/\partial y^2=0$, which is already satisfied by (B4). The next order from (B3) yields $\partial^2\mathscr{p}^{(1)}/\partial y^2=0$, so that

$$\mathscr{p}^{(1)}=y\psi(x,z,t)\ , \tag{B5}$$

with ψ being another arbitrary function. Using (B5), the terms of order σ^0 from (B2) give expressions for $\mathscr{u}^{(1)}$, $\mathscr{v}^{(1)}$, and $\mathscr{w}^{(1)}$ in terms of ϕ, ψ, $\mathscr{f}$, and $\mathscr{g}$. In particular, $\mathscr{v}^{(1)}=(1/2!\mu)y^2\psi$ (where $\mu=\rho\nu$). Continuity then implies

$$\frac{\partial\mathscr{f}}{\partial x}+\frac{1}{\mu}\psi+\frac{\partial\mathscr{g}}{\partial z}=0\ . \tag{B6}$$

Hence, in addition to the pressure fields ϕ and ψ, *only* $\mathscr{f}$ or $\mathscr{g}$ can be an independent source function. Consider then the constraint (B1), which in leading order (σ^3) requires that $\overline{\psi\mathscr{g}}=0$. This fixes the *independent triple* $(\phi,\psi,\mathscr{g})$.

By virtue of (B6), continuity is satisfied through all higher orders in σ [25]. Further, the constraint (B1) is satisfied at least through order σ^5 due to (i) the mutual independence of $(\phi,\psi,\mathscr{g})$ and (ii) statistical homogeneity (in x, z, and t).

Let us finally collect the result for the normal velocity fluctuations $v=v_1y+(1/2!)v_2y^2+(1/3!)v_3y^3+\cdots$ [17–21]. One finds

$$\begin{aligned}
&v_1=0,\quad v_2=\frac{1}{\mu}\psi,\quad v_3=-\frac{1}{\mu}\left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}\right],\\
&v_4=\frac{1}{\mu\nu}\frac{\partial\psi}{\partial t}-\frac{2}{\mu}\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right],
\end{aligned} \tag{B7}$$

which implies that $\overline{v_2v_3}=0$. Hence $(\overline{v^2})=\frac{1}{4}y^4\mathscr{K}_{22}(0)+O(y^6)$, where $\mathscr{K}_{22}(x)=\overline{v_2(x)v_2(0)}$, so that in the expansion of the root mean square $(\overline{v^2})^{1/2}$ the term of order y^3 is always *absent* (the case $\psi=0$ is ruled out in Appendix C). This feature is used to arrive at (3.10) for $\mathscr{v}(y)$, which, with $\mathscr{t}(y)=y$, yields (3.11) for $\mathscr{s}(y)$.

APPENDIX C: REYNOLDS STRESS NEAR THE SURFACE

Consider $\tau_{\mathrm{R}}/\rho=-\overline{uv}$ near the wall. From Appendix B one finds $u=u_1y+(1/2!)u_2y^2+(1/3!)u_3y^3+\cdots$ with

$$u_1=\mathscr{f},\quad u_2=\frac{1}{\mu}\frac{\partial\phi}{\partial x},$$
$$u_3=\frac{1}{\nu}\frac{\partial\mathscr{f}}{\partial t}+\frac{1}{\mu}\frac{\partial\psi}{\partial x}-\left[\frac{\partial^2\mathscr{f}}{\partial x^2}+\frac{\partial^2\mathscr{f}}{\partial z^2}\right]. \tag{C1}$$

Hence, using (C1) and (B7) one obtains

$$\tau_R/\rho=-\tfrac{1}{2}(\overline{u_1v_2})y^3+(\tfrac{1}{24}\overline{u_1v_4}+\tfrac{1}{12}\overline{u_3v_2})y^5+\cdots, \tag{C2}$$

where terms of order y^4 are always *absent* because $\overline{u_1v_3}=\overline{u_2v_2}=0$. Also $\overline{u_2v_3}=0$. Integrating (B6) over $x'\in(-\infty,x)$ and using $\overline{\psi\mathscr{y}}=0$, (C2) yields

$$\tau_R/\rho=(y^3/2)\int_0^\infty\mathscr{K}_{22}(x)dx+O(y^5), \tag{C3}$$

$\mathscr{K}_{22}(x)$ being defined below (B7). The absence of a term of order y^4 in the Reynolds stress is worth noticing in view of a longstanding controversy [17,25]. For example, Hinze [16] does not decide between $n=3$ and 4 in $\tau_R/\rho\sim y^n$. Reichardt [47] advocated $n=3$, but noted that this would not apply if certain correlations were absent, viz., if $\overline{\mathscr{f}\psi}=0$. Since according to Appendix B $(\mathscr{f},\psi)$ is *not* a pair of independent random functions, let us consider the case $\psi=0$. By (B7) one then has $v_2=v_4=0$, so that in (C2) terms $\sim y^3$ as well as $\sim y^5$ would vanish. But still $v_3\neq0$. Hence one would have $\mathscr{v}(y)\sim y^3$ and $\tau_R/\rho\sim y^6$. These functions, however, are shown to be mutually incompatible in the following model calculation.

Let $\mathscr{v}(\mathscr{y})=\mathscr{A}\mathscr{y}^k[1+O(\mathscr{y})]$ so that $\mathscr{s}(\mathscr{y})=\mathscr{A}\mathscr{y}^{k+1}[1+O(\mathscr{y})]$ with $k\geq2$. Substitution in (4.5) and letting $\mathscr{f}(\mathscr{y})=1+c_1\mathscr{y}+c_2\mathscr{y}^2+\cdots$ readily yields $\mathscr{f}(\mathscr{y})=1-\mathscr{A}\mathscr{y}^{k+1}[1+O(\mathscr{y})]$. Noticing that $\tau_R/\rho u_*^2=1-\mathscr{f}(\mathscr{y})$, one thus obtains $\tau_R/\rho u_*^2=\mathscr{A}\mathscr{y}^n[1+O(\mathscr{y})]$ with $n=k+1$. Clearly, the case $\psi=0$ (i.e., $k=3$ and $n=6$) is incompatible with this *fluctuation-dissipation* relation. Hence $k=2$ and $n=3$. This behavior is perfectly confirmed in a recent comparison [48] of both experimental data [17–19] and results from direct numerical simulations [20–22].

The analysis with $k=2$ ($n=3$) assumes the length scale $\mathscr{a}=\kappa a_+$ to be nonzero. With $\mathscr{A}=1/\mathscr{a}^2$ [see below (4.5)] and $\mathscr{y}=\kappa y_+$, recalling $\mathscr{v}^2\approx\frac{1}{4}(y^4/u_*^2)\mathscr{K}_{22}(0)$ from Appendix B, one obtains $\mathscr{K}_{22}(0)=4u_*^2/\mathscr{a}^4$. Finally, using $\tau_R/\rho\approx u_*^2\mathscr{y}^3/\mathscr{a}^2$ along with (C3) and defining $R_{22}(x)=\mathscr{K}_{22}(x)/\mathscr{K}_{22}(0)$, one finds $a_+=(2\bar{x}_+/\kappa)^{1/2}$ with $\bar{x}_+=\int_0^\infty R_{22}(x)dx_+$. See, e.g., [38] and Sec. V.

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