



## UvA-DARE (Digital Academic Repository)

### Mass, entropy and holography in asymptotically de Sitter spaces

Balasubramanian, V.; de Boer, J.; Minic, D.

**Publication date**  
2002

**Published in**  
Physical Review B

[Link to publication](#)

**Citation for published version (APA):**

Balasubramanian, V., de Boer, J., & Minic, D. (2002). Mass, entropy and holography in asymptotically de Sitter spaces. *Physical Review B*, 65, 123508.

**General rights**

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

**Disclaimer/Complaints regulations**

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

**Supersymmetric conical defects: Towards a string theoretic description of black hole formation**Vijay Balasubramanian,<sup>1,\*</sup> Jan de Boer,<sup>2,†</sup> Esko Keski-Vakkuri,<sup>3,‡</sup> and Simon F. Ross<sup>4,§</sup><sup>1</sup>*David Rittenhouse Laboratories, University of Pennsylvania, Philadelphia, Pennsylvania 19103*<sup>2</sup>*Instituut voor Theoretische Fysica, Valckenierstraat 65, 1018XE Amsterdam, The Netherlands*<sup>3</sup>*Fysiikan tutkimuslaitos, Helsingin Yliopisto, PL 9, FIN-00014 Helsingin Yliopisto, Finland*<sup>4</sup>*Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, South Road, Durham DH1 3LE, United Kingdom*

(Received 8 March 2001; published 27 August 2001)

Conical defects, or point particles, in  $\text{AdS}_3$  are one of the simplest nontrivial gravitating systems, and are particularly interesting because black holes can form from their collision. We embed the BPS conical defects of three dimensions into the  $\mathcal{N}=4b$  supergravity in six dimensions, which arises from the IIB string theory compactified on K3. The required Kaluza-Klein reduction of the six dimensional theory on a sphere is analyzed in detail, including the relation to the Chern-Simons supergravities in three dimensions. We show that the six dimensional spaces obtained by embedding the 3D conical defects arise in the near-horizon limit of rotating black strings. Various properties of these solutions are analyzed and we propose a representation of our defects in the CFT dual to asymptotically  $\text{AdS}_3 \times \text{S}^3$  spaces. Our work is intended as a first step towards analyzing colliding defects that form black holes.

DOI: 10.1103/PhysRevD.64.064011

PACS number(s): 04.65.+e, 04.50.+h, 11.25.Hf

**I. INTRODUCTION**

Twenty-five years after Hawking showed that black holes emit thermal radiation [1], the apparent loss of quantum mechanical unitarity in the presence of a black hole remains an outstanding problem for theoretical physics. We expect that this “information puzzle,” which represents a fundamental tension between general relativity and quantum mechanics, should either be erased or explained in a quantum theory of gravity. In recent years string theory has explained microscopically the huge degeneracy required to account for the entropy of certain extremal black holes. However, there has been no insight into why this degeneracy of states is related to something geometric such as the area of a horizon. More fundamentally, the information puzzle remains exactly that—a puzzle.

This paper is the first in a series investigating the black hole information puzzle in the context of string theory. In general relativity, the simplest context for black hole formation is gravity in three dimensions where there are no local dynamics. In the presence of a negative cosmological constant, 3D gravity possesses black hole solutions [2]. There is also a family of conical defects, the so-called point particles [3]. These solutions interpolate between the vacuum solution ( $\text{AdS}_3$  with mass  $M = -1$  in conventional units) and the black hole spectrum which starts at  $M = 0$ . Exact solutions of 3D gravity are known in which the collision of conical defects forms a black hole [4]. We would like to use these simple classical processes to study the formation of higher dimensional black holes in string theory. To this end, we must first embed the conical defects supersymmetrically in a higher dimensional gravity arising from string theory. Pre-

serving supersymmetry is important because the controlled quantization of black holes and solitons in string theory usually requires supersymmetry. The presence of the negative cosmological constant in three dimensions suggests that there should be a dual description of such spaces in terms of a two-dimensional conformal field theory [5]. Our goal is to find such a dual picture and describe in it the process of black hole formation from collision of conical defects. In [6] it was shown that the 3D conical defects and their collisions can be detected in correlation functions of the dual CFT. Here we are interested in the direct description of the defects as objects in the dual.<sup>1</sup>

Type IIB supergravity compactified on K3 yields the chiral  $\mathcal{N}=4b$  supergravity in six dimensions, coupled to 21 tensor multiplets. This theory has classical solutions with the geometry of  $\text{AdS}_3 \times \text{S}^3$ . In Sec. II we will construct supersymmetric solutions where the sphere is fibered over  $\text{AdS}_3$  so that a minimum length circuit around the  $\text{AdS}_3$  base leads to a rotation of the sphere around an axis. Since  $\text{AdS}_3$  is simply connected, the fiber must break down at a point. Upon dimensional reduction to the base this produces supersymmetric conical defects in three dimensions. In fact, the identical objects have been obtained previously as solutions to extended 2+1 supergravity in the Chern-Simons formulation [8,9]. The  $U(1)$  Wilson lines used in these constructions to obtain a Bogomol’nyi-Prasad-Sommerfield (BPS) solution arise in our case from the Kaluza-Klein gauge field associated with the fibration. Our Kaluza-Klein ansatz for reducing the action and equations of motion of 6D gravity to the 3D base does not yield precisely a Chern-Simons theory. Nevertheless, the dimensionally reduced system admits solutions with vanishing field strength, for which the analysis of

\*Electronic address: vijay@endive.hep.upenn.edu

†Electronic address: jdeboer@wins.uva.nl

‡Electronic address: keski@rock.helsinki.fi

§Electronic address: S.F.Ross@durham.ac.uk

<sup>1</sup>It would also be interesting to make contact with the investigations of spherical shells in [7].

supersymmetry remains unchanged—the holonomy of Killing spinors under the spin connection is canceled by the holonomy under the gauge connection. Various details of sphere compactifications of 6D,  $\mathcal{N}=4b$  supergravity are reviewed in the main text and the Appendixes.<sup>2</sup>

It is well known that a horospheric patch of the  $\text{AdS}_3 \times S^3$  geometry can be obtained as a near-horizon limit of the black string soliton of 6D supergravity [12]. Compactifying the extremal string solution on a circle yields the black holes of five dimensional string theory whose states were counted in the classic paper [13].<sup>3</sup> The near-horizon limit of these solutions yields the BTZ black holes times  $S^3$  [12]. In Sec. III we show that the fibered  $S^3$  solutions described above arise as the near-horizon geometries of an extremal limit of spinning 6-dimensional strings compactified on a circle. Interestingly, when the angular momentum is suitably chosen, *global*  $\text{AdS}_3 \times S^3$  is recovered as a solution. We discuss various properties of the solution, including the nature of the conical singularity and potential Gregory-Laflamme instabilities in the approach to extremality.

The near-horizon limit of the six dimensional black string is also a decoupling limit for the worldvolume conformal field theory (CFT) description of the soliton. Following the reasoning of [5] we conclude that the BPS conical defects described above should enjoy a non-perturbative dual description in the worldvolume CFT of the black string—i.e., a deformation of the orbifold sigma model  $(K3)^{N/S^N}$  [15]. When reduced to the AdS base, the fibered geometries appearing in our solutions carry a  $U(1)$  charge measured by the Wilson line holonomy. Within the AdS-CFT duality, this spacetime  $U(1)$  charge translates into an  $R$  charge of the dual system. In Sec. IV, we propose that the conical defects are described in the dual as an ensemble of the chiral primaries carrying the same  $R$  charge. In subsequent papers we will test this proposal and then use it to analyze the spacetime scattering of conical defects.

## II. CONICAL DEFECTS FROM KALUZA-KLEIN REDUCTION

In this section, we obtain the supersymmetric conical defects in 3D via Kaluza-Klein reduction of the six-dimensional  $\mathcal{N}=4b$  supergravity. Defects in three dimensions that involve just the metric and gauge fields with a Chern-Simons action have been obtained previously [8]. We will construct a Kaluza-Klein ansatz for six dimensional gravity which reproduces these defects upon dimensional reduction.

We begin by reviewing the structure of the 3D conical defects. The action with a negative cosmological constant is

<sup>2</sup>Sphere compactifications have been extensively studied in the literature. See [10] for a review, and the recent work [11] for references.

<sup>3</sup>The  $\text{AdS}_3$  and Bañados-Teitelboim-Zanelli (BTZ) geometries can also be related to the near-horizon limit of extremal four dimensional black holes [14], by constructing the black holes as the near horizon limit of intersecting 5-branes in M theory.

$$S = \frac{1}{16\pi G_3} \int_{\mathcal{M}} d^3x \sqrt{-g} \left( R + \frac{2}{l^2} \right) - \frac{1}{8\pi G_3} \int_{\partial\mathcal{M}} \sqrt{-h} \left( \theta + \frac{1}{l} \right), \quad (1)$$

where  $\theta$  is the trace of the extrinsic curvature of the boundary. The boundary term  $\sqrt{-h}/l$  renders the equations of motion well-defined, leading to the solutions

$$ds^2 = - \left( \frac{r^2}{l^2} - M_3 \right) dt^2 + \left( \frac{r^2}{l^2} - M_3 \right)^{-1} dr^2 + r^2 d\phi^2, \quad (2)$$

where  $\phi \sim \phi + 2\pi$ .  $M_3 = -1$  is the vacuum, global anti-de Sitter space ( $\text{AdS}_3$ ). The boundary term  $\sqrt{-h}/l$  renders the action finite for any solution that approaches the vacuum sufficiently rapidly at infinity [16]. The mass of these solutions can then be computed following [16,17] to be  $M = M_3/8G_3$ . The  $M_3 \geq 0$  solutions are the non-rotating BTZ black holes [2] while the spacetimes in the range  $-1 < M_3 \leq 0$  are conical defects [3]. To display the defect, let  $\gamma^2 \equiv -M_3$  and rescale the coordinates:  $\hat{t} \equiv t\gamma$ ,  $\hat{r} \equiv r/\gamma$ , and  $\hat{\phi} = \phi\gamma$ . Then

$$ds^2 = - \left( 1 + \frac{\hat{r}^2}{l^2} \right) d\hat{t}^2 + \left( 1 + \frac{\hat{r}^2}{l^2} \right)^{-1} d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2, \quad (3)$$

where  $\hat{\phi} \sim \hat{\phi} + 2\pi\gamma$ , manifestly exhibiting a deficit angle of

$$\delta\hat{\phi} = 2\pi(1 - \gamma). \quad (4)$$

In these coordinates the mass measured with respect to translations in  $\hat{t}$  is  $M = -\sqrt{-M_3}/8G_3$ .

We are looking for an embedding of these solutions in the  $\mathcal{N}=4b$  chiral supergravity in six dimensions [18], coupled to tensor multiplets. The theory has self-dual tensor fields, so it has solutions where three directions are spontaneously compactified on  $S^3$ ; the vacuum for this sector is  $\text{AdS}_3$ , and the spectrum of fluctuations around this vacuum solution has been computed [19–21]. We seek a supersymmetric solution where  $\text{AdS}_3$  is replaced by a conical defect.

In extended three dimensional supergravity, the conical defects can be made supersymmetric [8]. These BPS defects achieve supersymmetry by canceling the holonomy of spinors under the spin connection by the holonomy under a Wilson line of a flat gauge field appended to the solution. Thus, we will consider a Kaluza-Klein ansatz which involves non-trivial Kaluza-Klein gauge fields (leading to a fibered  $S^3$  in the 6D geometry) and the three dimensional metric, since these were the only fields present in the extended three-dimensional supergravities.

Famously, three-dimensional gravity can be written as a sum two  $\text{SL}(2, R)$  Chern-Simons theories. The sphere reduction of six-dimensional,  $\mathcal{N}=4b$  gravity has symmetries appropriate to the  $\text{SU}(1,1|2) \times \text{SU}(1,1|2)$  Chern-Simons supergravity (see [22,23,21,24,25,9] and references therein). We will show that the three-dimensional equations of motion obtained from our Kaluza-Klein ansatz contain the (bosonic)

solutions of this theory. However, the six-dimensional action does not reduce to Chern-Simons in three dimensions. In fact, the equations of motion obtained from our ansatz are not obtainable from a three-dimensional action; we would have to include some non-trivial scalars in our general ansatz to obtain a consistent truncation to a three-dimensional action. That is, while our ansatz shows that we can construct solutions of the six-dimensional theory using all the solutions of the  $SU(1,1|2) \times SU(1,1|2)$  supergravity, asking that the ansatz solve the six-dimensional equations does not in general give the equations of motion of a three-dimensional theory.

The minimal  $\mathcal{N}=4b$  theory contains a graviton  $e_M^A$ , four left-handed gravitini  $\psi_{Mr}$ , and five antisymmetric tensor fields  $B_{MN}^i$ . The latter transform under the vector representation of  $Spin(5)$ . We adopt a notation where curved space-time indices are  $M, N=0, \dots, 5$  for the full six-dimensional geometry;  $\mu, \nu=0, \dots, 2$  in the AdS base;  $m, n=1, \dots, 3$  on the sphere. The flat tangent space indices are:  $A, B=0, \dots, 5$ , which parametrize six-dimensional  $[SO(1,5)]$  tangent vectors;  $\alpha, \beta=0, \dots, 2$ , which index  $AdS_3$   $[SO(1,2)]$  tangent vector indices;  $a, b=1, \dots, 3$ , indexing  $S^3$   $[SO(3)]$  tangent vectors. The Kaluza-Klein gauge symmetry arising from the isometries of  $S^3$  is  $SO(4) = SU(2) \times SU(2)$ . In our conventions,  $I, J=1, \dots, 6$  index  $SO(4)$ , while  $i, j=1, \dots, 3$  index  $SU(2)$ , as do  $i', j'$ . For  $Spin(5)$ ,  $i, j=1, \dots, 5$  labels the vector representation, while  $r, s=1, \dots, 4$  labels the spinors.

We will not discuss the field content of the tensor multiplets to which the minimal  $\mathcal{N}=4b$  theory is coupled in detail. The only piece of information that we need in the remainder is that tensor multiplets contain two-form fields with anti-self-dual three-form field strengths.

### A. Kaluza-Klein reduction reexamined

Considerable work has been carried out on the topic of sphere compactifications (see the review [10] and the recent works [11] for further references). The discussion below should serve as a review in a simplified setting.

*The metric.* A general compactification of six-dimensional gravity on a three dimensional compact space takes the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{mn} D x^m D x^n, \quad (5)$$

$$D x^m = dx^m - A_\mu^I K_I^m dx^\mu. \quad (6)$$

The Kaluza-Klein gauge fields  $A_\mu^I$  are associated with the Killing vectors  $K_I^m$  of the compact space. (Note that the indices  $I$  can be raised and lowered by the metric  $\delta_{IJ}$ .)

We choose  $g_{mn}$  to be the round metric on  $S^3$ . Thus, we do not include any scalars in our ansatz; as stated earlier, this is motivated by the absence of scalar fields in the 3D Chern-Simons supergravities with which we seek to make contact. Then there are six Killing vectors arising from the  $SO(4)$  isometry group, and it is manifest that the metric is invariant under  $SO(4)$  gauge transformations:

$$\delta x^m = \epsilon^I K_I^m, \quad (7)$$

$$\delta x^\mu = 0, \quad (8)$$

$$\delta A_\mu^I = \partial_\mu \epsilon^I + f_{JK}^I A_\mu^J \epsilon^K. \quad (9)$$

Here  $f_{JK}^I$  are the  $SO(4)$  structure constants, expressed in terms of the Killing vectors as

$$f_{IJ}^K K_K^m = K_I^n \partial_n K_J^m - K_J^n \partial_n K_I^m. \quad (10)$$

The  $SO(4)$  gauge invariance of Eq. (5) follows from the transformations of  $g_{mn}$  and  $D x^m$ :

$$\delta D x^m = \epsilon^I \partial_n K_I^m D x^n, \quad (11)$$

$$\begin{aligned} \delta g_{mn} &\equiv \epsilon^I K_I^r \partial_r g_{mn} \\ &= -g_{rn} \epsilon^I \partial_m K_I^r - g_{mr} \epsilon^I \partial_n K_I^r. \end{aligned} \quad (12)$$

Observe that  $D x^m$  transforms under a local gauge transformation in the same way as  $dx^m$  under a global gauge transformation— $D$  is like a covariant exterior derivative.

*The 3-form.* We must have a non-zero 3-form to satisfy the equations of motion. We will consider turning on just one of the five three-form fields  $H_{MNP}^i$ . We require an  $SO(4)$  gauge invariant ansatz for this 3-form field. Let

$$V(x^m) \epsilon_{mnr} dx^m \wedge dx^n \wedge dx^r, \quad W(x^\mu) \epsilon_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \quad (13)$$

be the volume forms on  $S^3$  and on the non-compact factor in Eq. (5) respectively. In terms of these forms, the six-dimensional equations of motion have an  $AdS_3 \times S^3$  solution of the form (5) with vanishing Kaluza-Klein gauge fields and a 3-form background

$$\begin{aligned} H &= \frac{1}{l} [W(x^\mu) \epsilon_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &\quad + V(x^m) \epsilon_{mnr} dx^m \wedge dx^n \wedge dx^r], \end{aligned} \quad (14)$$

where  $l$  is the radius of the  $S^3$ . This cannot be quite right when the gauge fields are turned on, because it is not gauge invariant. A candidate gauge invariant generalization is

$$\begin{aligned} H &= \frac{1}{l} [W(x^\mu) \epsilon_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &\quad + V(x^m) \epsilon_{mnr} D x^m \wedge D x^n \wedge D x^r]. \end{aligned} \quad (15)$$

Since the  $S^3$  volume form is  $SO(4)$  invariant,  $\partial_m (K_I^m V(x^m)) = 0$ , (15) is gauge invariant. However, we should find a proposal for the 2-form potential  $B_{MN}$ , rather than the field strength  $H$ , which is only possible if  $dH = 0$ . The exterior derivative of Eq. (15) is computed using

$$dD x^m = -F^I K_I^m - A_\mu^I \partial_n K_I^m D x^n \wedge dx^\mu, \quad (16)$$

where  $F^I = \frac{1}{2} F_{\mu\nu}^I dx^\mu \wedge dx^\nu$ . We obtain



$$dH = -\frac{3}{l} V \epsilon_{mnr} K_I^m F^I \wedge D x^n \wedge D x^r, \quad (17)$$

using the SO(4) invariance of the  $S^3$  volume form and the fact that one cannot anti-symmetrize over more than three indices.

When the gauge field is flat (which is typically our interest)  $dH=0$ , as desired. Nevertheless, it is worth seeking a more generally valid ansatz. We wish to add a contribution to  $H$  that cancels the term on the right hand side of Eq. (17). To find this, it is helpful to consider the 2-form  $\omega_I = V \epsilon_{mnr} K_I^m dx^n \wedge dx^r$  which appears as part of Eq. (17). In terms of  $\Omega$ , the volume form on  $S^3$ , this 2-form can also be written as  $\iota_{K_I} \Omega$ . It is a standard fact that  $d \iota_{K_I} \Omega + \iota_{K_I} d\Omega = \mathcal{L}_{K_I} \Omega$ . Since the volume form is SO(4) invariant, and annihilated by  $d$ , it follows that  $\omega$  is closed. Therefore, since we are on the three sphere there must be globally well defined one-form  $N_{I_r} dx^r$  such that  $d(N_{I_r} dx^r) = \omega$ . Assembling these facts, a candidate Kaluza-Klein ansatz for a closed 3-form is

$$H_{KK} = H + \frac{3}{l} F^I \wedge N_{I_r} D x^r. \quad (18)$$

The 1-forms  $N_{I_r} dx^r$  for  $S^3$  are related to the Killing one-forms and are derived explicitly in Appendix B. The choice of  $N_{I_r}$  given there satisfy the relation

$$K_J^m \partial_m N_{I_r} + N_{I_r} \partial_r K_J^I = f_{JI}^K N_{K_r}. \quad (19)$$

Using this relation it can be checked that  $H_{KK}$  is still gauge invariant, and that

$$d(F^I N_{I_r} D x^r) = V \epsilon_{mnr} K_I^m F^I \wedge D x^n \wedge D x^r. \quad (20)$$

Combining this with Eq. (17) shows that  $H_{KK}$  is a closed form, as desired. Thus, we have a consistent SO(4) invariant ansatz for Kaluza-Klein reduction of six dimensional gravity on a sphere, with gauge field vacuum expectation values (VEVs).

Notice that the three-form  $H_{KK}$  is not self-dual. Therefore, this ansatz cannot be given for the minimal  $\mathcal{N}=4b$  theory, but we need at least one tensor multiplet as well. The self-dual part of  $H_{KK}$  then lives in the gravity multiplet, the anti-self-dual part lives in the tensor multiplet. Together, one self-dual and one anti-self-dual tensor combine into an unconstrained two-form field. We can think of such a two-form field as originating in either the Neveu-Schwarz (NS) or Ramond-Ramond (RR) two-form in type IIB string theory in ten dimensions. In particular, for the equations of motion we can use the equations of motion of string theory, rather than the more complicated ones of  $\mathcal{N}=4b$  supergravity.

*Equations of motion.* Using the results collected in [10] and the above remarks, it is now a straightforward, if lengthy, exercise to compute the six-dimensional equations of motion for our Kaluza-Klein ansatz. As in [10], it is easier to work out the equations of motion using the vielbein formalism. It is convenient to display the  $\text{SO}(4) = \text{SU}(2) \times \text{SU}(2)$  gauge symmetry inherited from isometries of the

sphere explicitly by picking a basis of Killing vectors such that the left ( $F_L^i$ ,  $i=1,2,3$ ) and right ( $F_R^{i'}$ ,  $i'=1,2,3$ ) SU(2) field strengths are

$$F_{\alpha\beta}^I = F_{L\alpha\beta}^I \quad I=1,2,3 \quad (21)$$

$$= F_{R\alpha\beta}^{I-3} \quad I=4,5,6. \quad (22)$$

Such a basis is explicitly constructed in Appendix B. In simplifying the equations of motion, the following identities are useful. First, one can show that

$$K_I^m g_{mn} K_J^n + \frac{1}{l^2} N_{Im} g^{mn} N_{Jn} = \frac{l^2}{2} \delta_{IJ}. \quad (23)$$

Second, there is a simple map from SO(4) to itself, that acts as  $+1$  on  $\text{SU}(2)_L$  and as  $-1$  on  $\text{SU}(2)_R$ , which we will denote by  $A_I^J$ . In other words, it sends  $K_I^m$  to  $A_I^J K_J^m$ . Then we have

$$g_{mn} K_I^n = \frac{1}{l} A_I^J N_{Jm}. \quad (24)$$

Then, if we take the metric  $g_{\mu\nu}$  and the Kaluza-Klein gauge fields  $A_\mu^I$  to only depend on the coordinates  $x^\mu$  of the three-dimensional non-compact space, the ansatz will satisfy all the equations of motion of the six-dimensional theory if the metric and gauge field satisfy the following three-dimensional equations :

$$R_{\alpha\beta} + \frac{2}{l^2} \delta_{\alpha\beta} - \frac{1}{2} \delta_{IJ} F_{\alpha\gamma}^I F_{\beta}^J{}^\gamma = 0, \quad (25)$$

$$D^* F^{(L)} + F^{(L)} + g(D^* F^{(R)} - F^{(R)}) g^{-1} = 0, \quad (26)$$

$$\text{tr}(F_{\beta\gamma}^{(L)} g \partial_m g^{-1}) \text{tr}(F^{(R)\beta\gamma} g^{-1} \partial_n g) = 0 \quad (27)$$

$$\text{tr}(F^{(L)} - g F^{(R)} g^{-1})^2 = 0. \quad (28)$$

Here, we used a group element  $g \in \text{SU}(2)$  to parametrize the  $S^3$ , and  $\text{SU}(2)_{L,R}$  correspond to the left and right actions on the three-sphere. The last equation of motion (28) has its origin in the dilaton equation of motion. It is clear that the equations of motion are gauge invariant, and that any solution to three dimensional cosmological gravity with flat gauge fields solves these equations. These are the solutions of the bosonic part of the  $\text{SU}(1,1|2) \times \text{SU}(1,1|2)$  Chern-Simons supergravity, and include the conical defects:

$$ds^2 = -\left(\frac{r^2}{l^2} - M_3\right) dt^2 + \left(\frac{r^2}{l^2} - M_3\right)^{-1} dr^2 + r^2 d\phi^2, \quad (29)$$

$$F_L^i = 0; \quad F_R^{i'} = 0. \quad (30)$$

However, although Eqs. (25)–(28) allow  $F^{(L)} = F^{(R)} = 0$  they do not obviously *imply* this. If they did, we would have found a consistent truncation of the six-dimensional theory to three-dimensional Chern-Simons theory. Notice that the first two equations of motion (25) and (26) can naturally be

obtained from a three-dimensional theory consisting of the Einstein-Hilbert term, a Yang-Mills term and a Chern-Simons term. The other two equations (27) and (28) do not have such a clear interpretation. It has been shown in [11] that consistent Kaluza-Klein reductions with general  $SO(4)$  gauge fields can be achieved by also turning on scalar fields that parametrize the shape of the compact manifold.

Thus, although the  $SU(1,1|2) \times SU(1,1|2)$  Chern-Simons supergravity in 3 dimensions has the symmetries of the six-dimensional theory reduced on a sphere, our ansatz does not produce this theory. The Chern-Simons formulation of  $AdS_3$  supergravity has been an important tool in investigations of the AdS-CFT correspondence (see, e.g., [21,9,25] amongst many other references). While many of these works relied primarily on symmetries, it remains desirable to explain precisely how and whether the six-dimensional,  $\mathcal{N}=4b$  gravity reduces to the three-dimensional  $SU(1,1|2) \times SU(1,1|2)$  theory. Once we include scalars, we can obtain consistent truncations to a three-dimensional action. Although these theories have more than just a Chern-Simons term, at low energies they can be approximated by a Chern-Simons theory—the  $F^2$  terms in the action can be ignored at low energy. A more precise argument is given in [26], where it is shown that wave functions in the Yang-Mills Chern-Simons theory can be decomposed in a natural way in a Yang-Mills piece and a Chern-Simons piece.

We should also comment on the relation between our Kaluza-Klein ansatz and the results in Sec. 7 of [19], where a Chern-Simons like structure is found for the field equations for a certain set of gauge fields. The computation in [19] differs from ours in several ways. First of all, the gauge fields appearing in the three form and the metric of their Kaluza-Klein ansatz are different. Thus, the dimensionally reduced theory has two different “gauge fields,” but only one gauge invariance. Secondly, they only consider the self-dual three-form, whereas our KK ansatz contains both a self-dual and an anti-self-dual three-forms. In particular, Eq. (152) in [19] depends explicitly on the gauge fields, and is a consequence of the self-duality equation for the three-form. In our case we do not impose such a self-duality relation, and as a consequence, we do not find a field equation of the form (152). The field equation (27) is not obtained in [19], because they only consider the linearized system.

The results of [19] were extended in [27] where not only quadratic but also cubic couplings in the six-dimensional theory were considered. It was found that, to that order, there exists a gauge field whose field equation becomes the Chern-Simons field equation and that massive fields can be consistently put to zero. The gauge field in question is a linear combination of the gauge fields appearing in the metric and in a self-dual two-form. If we were to insist that our three-form is self-dual, we would also find the Chern-Simons field equation, and in this sense the results agree with each other.

*Summary.* We have found an  $SO(4)$  invariant Kaluza-Klein ansatz for the  $S^3$  compactification of six dimensional supergravity, involving just the KK gauge fields and no scalars. Upon dimensional reduction, however, we do not find equations of motion that could arise from a three dimensional effective action. In any case, if  $F=0$ , our ansatz for

the metric and  $H_{KK}$  provide solutions to the 6D equations of motion. The effective 3-dimensional equations are solved by any solution to three dimensional cosmological gravity with a flat gauge field. This spectrum of solutions includes the supersymmetric conical defects we are interested in. Below we will show how the gauge fields are chosen to make the solutions supersymmetric.

## B. Supersymmetry

Having found an appropriate Kaluza-Klein ansatz, we investigate the supersymmetry of the solutions incorporating conical defects. By examining the Killing spinor equations, with a flat KK gauge field, we recognize the effective 3D equations as the Killing spinor equations of the  $SU(1,1|2) \times SU(1,1|2)$  Chern-Simons supergravity. This allows us to use the work of [8,9] to choose a Wilson line for which the 3D conical defects lift to supersymmetric solutions of the six-dimensional theory.

### 1. 6D Killing spinor equations

First, the 10D type IIB supergravity has 32 supersymmetries. Half of them are broken by the reduction on  $K3$ , so we are left with 16 supersymmetries in six dimensions. The resulting theory is the  $N=4b$  supergravity in six dimensions. As long as we consider flat gauge fields, the three-form is self-dual, and we can ignore the tensor multiplets.  $N=4b$  supergravity is a chiral theory, with four chiral, symplectic-Majorana supercharges (labeled by  $r=1, \dots, 4$ ), each having four real components. Following Romans [18], the  $\mathcal{N}=4b$  algebra can be viewed as an extension of an  $N=2$  algebra. The  $N=2$  algebra is generated by a doublet of chiral spinorial charges, and it has an  $USp(2)=SU(2)$   $R$ -symmetry. The charges are doublets under the  $SU(2)$ . The  $\mathcal{N}=4b$  algebra can be viewed as an extension of  $N=2$  to  $N=4$ , where one takes two copies of the  $N=2$  charges of the same chirality. The resulting algebra has an  $USp(4)=Spin(5)$   $R$ -symmetry, and the four supersymmetry parameters  $\epsilon_r$  transform in the fundamental representation of  $Spin(5)$ .

$Spin(5)$  is represented by the  $4 \times 4$  Gamma matrices  $\Gamma^i$ :

$$\{\Gamma^k, \Gamma^l\} = \delta^{kl}, \quad k, l = 1, \dots, 5. \quad (31)$$

$\Gamma^5$  has two  $+1$  eigenvalues, and two  $-1$  eigenvalues. Hence, by taking suitable linear combinations of the supersymmetry parameters  $\epsilon_r$ , we can organize things so that

$$(\Gamma^5)_{rs} \epsilon_s = \begin{cases} +\epsilon_r & \text{for } r=1,3 \\ -\epsilon_r & \text{for } r=2,4. \end{cases} \quad (32)$$

The 6D Killing spinor equation is

$$D_M \epsilon_r - \frac{1}{4} H_{MNP}^k \Gamma^{NP} (\Gamma^k)_{rs} \epsilon_s = 0. \quad (33)$$

In our solutions only one of the five three form fields is turned on, and by U-duality, we can choose  $H_{MNP}^k \sim \delta^{k5}$ . When the field strengths  $F^I$  vanish, the gauge invariant definition of  $H$  in Eq. (18) reduces to Eq. (15). For the  $M = \mu$

components of the Killing spinor equation, the relevant components of the three form field are thus

$$\begin{aligned} H_{\alpha\beta\gamma}^5 &= l^{-1} \epsilon_{\alpha\beta\gamma}; & H_{abc}^5 &= l^{-1} \epsilon_{abc}; \\ H_{\mu ab}^5 &= -l^{-1} K_I^m A_\mu^I e_m^c \epsilon_{abc}. \end{aligned} \quad (34)$$

$\Gamma^5$  can be dropped from the Killing spinor equation with the help of Eq. (32). For the purposes of Kaluza-Klein reduction, we also decompose the SO(1,5) gamma matrices  $\Gamma^A$  as direct products of SO(3) and SO(1,2) matrices ( $\gamma^a$  and  $\gamma^\alpha$ ) as follows:

$$\Gamma^\alpha = \sigma^1 \otimes 1 \otimes \gamma^\alpha; \quad \Gamma^a = \sigma^2 \otimes \gamma^a \otimes 1, \quad (35)$$

$$\gamma^0 = -i\sigma_2; \quad \gamma^1 = \sigma_1; \quad \gamma^2 = \sigma_3;$$

$$\gamma^a = \sigma^a, \quad a = 1, 2, 3. \quad (36)$$

Then, for example, we get  $\Gamma^{\alpha\beta} = 1 \otimes 1 \otimes \epsilon^{\alpha\beta\delta} \gamma_\delta$ ; and  $\Gamma^{ab} = 1 \otimes i \epsilon^{abc} \gamma_c \otimes 1$ .

Note that the 6D gamma matrices are 8x8, but the chiral spinors in 6D have 4 components. Chiral spinors  $\Psi^{(\pm)}$  satisfy

$$\Psi^{(\pm)} = \frac{1}{2} (1 \pm \Gamma^7) \Psi \quad (37)$$

where  $\Gamma^7 = \Gamma^0 \Gamma^1 \dots \Gamma^5 = \sigma_3 \otimes 1 \otimes 1$ . We let the  $\mathcal{N}=4b$  spinors be of positive chirality ( $\Psi^{(+)}$ ). Then, in the Killing spinor equation (33), all the supersymmetry parameters  $\epsilon_r$  are of the form

$$\epsilon_r = \begin{pmatrix} \varepsilon_r \\ 0 \end{pmatrix}, \quad (38)$$

where  $\varepsilon_r$  is a doublet of two-component spinors. We can additionally impose a symplectic Majorana condition on these spinors [18]. It then follows, as is shown in detail in Appendix A, that  $\varepsilon_r$  can be written as an SU(2) doublet of complex conjugate two-component spinors:

$$\varepsilon_r = \begin{pmatrix} \varepsilon_r^{(2)} \\ \varepsilon_r^{(2)*} \end{pmatrix}. \quad (39)$$

Consider first the  $M=m$  internal component of the Killing spinor equation:

$$\left( D_m \mp \frac{1}{4} H_{mNP}^5 \Gamma^{NP} \right) \epsilon_r = 0. \quad (40)$$

The upper signs and lower signs ( $-$  and  $+$ ) correspond to  $r=1,3$  and  $r=2,4$  respectively. This split will relate to the  $SU(2)_L$  and  $SU(2)_R$  sectors. We assume that the Killing spinor is in a zero mode on the sphere, in accord with our Kaluza-Klein approach. That is,  $\epsilon_r$  is independent of the sphere coordinates, so that

$$\begin{aligned} D_m \epsilon_r &= \left( \partial_m + \frac{1}{4} \hat{\omega}_m^{AB} \Gamma_{AB} \right) \epsilon_r = \frac{1}{4} \hat{\omega}_m^{AB} \Gamma_{AB} \epsilon_r \\ &= \frac{i}{4} \epsilon_{abc} \omega_m^{ab} 1 \otimes \gamma^c \otimes 1 \epsilon_r. \end{aligned} \quad (41)$$

The three-form contribution is

$$\begin{aligned} \mp \frac{1}{4} H_{mNP}^5 \Gamma^{NP} &= \mp \frac{1}{4} H_{mnp}^5 \Gamma^{np} \\ &= \mp \frac{i}{4l} e_m^a \epsilon_{abc} \epsilon^{bcd} 1 \otimes \sigma_d \otimes 1 \\ &= \mp \frac{i}{2l} e_m^a 1 \otimes \sigma_a \otimes 1. \end{aligned} \quad (42)$$

Thus, the internal Killing spinor equation is

$$\frac{i}{4} \left( \epsilon_{abc} \omega_m^{bc} \mp \frac{2}{l} e_{ma} \right) (1 \otimes \sigma^a \otimes 1) \epsilon_r = 0. \quad (43)$$

Now,  $g_{mn}$  is by assumption the metric of a round three-sphere. We can show by explicit calculation, using the bases for  $S^3$  in Appendix B, that

$$\epsilon^{abc} \omega_{bc} = \frac{2}{l} e^a \quad (44)$$

when we use the basis  $e_a = (2/l)L_a$ , and

$$\epsilon^{abc} \omega_{bc} = -\frac{2}{l} e^a \quad (45)$$

when we use the basis  $e_a = -(2/l)R_a$ . Thus, the internal Killing spinor equation can be trivially satisfied. This is as we might have expected; since our Kaluza-Klein ansatz leaves the form of the metric  $g_{mn}$  fixed, the internal Killing spinor equation is always the same, and we know it is satisfied in the  $\text{AdS}_3 \times S^3$  vacuum.

Consider now the  $M=\mu$  component of the 6D Killing spinor equation:

$$\begin{aligned} \left( D_\mu \mp \frac{1}{4} H_{\mu NP}^5 \Gamma^{NP} \right) \epsilon_r &= \left( D_\mu \mp \frac{1}{2l} e_{\mu\alpha} (1 \otimes 1 \otimes \gamma^\alpha) \right. \\ &\quad \left. \pm \frac{i}{2l} A_\mu^I K_I^m (1 \otimes \sigma_m \otimes 1) \right) \epsilon_r = 0. \end{aligned} \quad (46)$$

As before, the upper signs and lower signs correspond to  $r=1,3$  and  $r=2,4$  respectively. The gauge covariant derivative is [18]

$$D_\mu \epsilon_r = \partial_\mu \epsilon_r + \frac{1}{4} \hat{\omega}_\mu^{AB} \Gamma_{AB} \epsilon_r \quad (47)$$

$$\hat{\omega}_\mu^{AB} \Gamma_{AB} = \omega_\mu^{\alpha\beta} \Gamma_{\alpha\beta} - A_\mu^I \nabla_a K_{Ib} \Gamma^{ab}. \quad (48)$$

Using the definition of the gamma matrices, the last term of Eq. (48) becomes

$$A^I_\mu \nabla_a K_{Ib} \Gamma^{ab} = A^I \nabla_a K_{Ib} i \epsilon^{abc} (1 \otimes \sigma_c \otimes 1), \quad (49)$$

$$\nabla_a K_{Ib} = \frac{1}{l^2} \epsilon_{abc} N_I^c, \quad (50)$$

where we have used the relation between the Lorentz covariant derivative of  $K$  and the components of a one-form  $N$  (see Appendix B). Folding these facts into the last term in Eq. (47) yields the Killing spinor equation (46) as

$$\left( \partial_\mu 1 \otimes 1 + \frac{1}{4} \epsilon_{\alpha\beta\delta} \omega_\mu^{\alpha\beta} 1 \otimes \gamma^\delta + \frac{i}{2l} A_\mu^I \left( -\frac{1}{l} N_I^c \mp K_I^c \right) \sigma_c \otimes 1 \pm \frac{1}{2l} e_{\mu\alpha} 1 \otimes \gamma^\alpha \right) \varepsilon_r = 0, \quad (51)$$

where we used Eq. (38) for the chiral spinors. Now, according to Appendix B, the combinations  $l^{-1} N_c^I \pm K_c^I$  are projectors to the left and right  $SU(2)$  sectors,

$$L_c^I = -\frac{1}{l} N_c^I + K_c^I = \begin{cases} l \delta_c^I & \text{for } I=1,2,3 \\ 0 & \text{for } I=4,5,6, \end{cases} \quad (52)$$

$$R_c^I = -\frac{1}{l} N_c^I - K_c^I = \begin{cases} 0 & \text{for } I=1,2,3 \\ l \delta_c^{I-3} & \text{for } I=4,5,6. \end{cases} \quad (53)$$

Then, the two Killing spinor equations labeled by  $r=1,3$  ( $r=2,4$ ) give the  $SU(2)_L$  [ $SU(2)_R$ ] sector equations:

$$\left( \partial_\mu 1 \otimes 1 + \frac{1}{4} \epsilon_{\alpha\beta\delta} \omega_\mu^{\alpha\beta} 1 \otimes \gamma^\delta + \frac{i}{2} A_\mu^c \sigma_c \otimes 1 - \frac{1}{2l} e_{\mu\alpha} 1 \otimes \gamma^\alpha \right) \varepsilon_r = 0 \quad (54)$$

for  $r=1,3$  and

$$\left( \partial_\mu 1 \otimes 1 + \frac{1}{4} \epsilon_{\alpha\beta\delta} \omega_\mu^{\alpha\beta} 1 \otimes \gamma^\delta + \frac{i}{2} A_\mu^c \sigma_c \otimes 1 + \frac{1}{2l} e_{\mu\alpha} 1 \otimes \gamma^\alpha \right) \varepsilon'_r = 0 \quad (55)$$

for  $r=2,4$ . Because of the doublet structure (39), each spinor  $\varepsilon_r$  has four real degrees of freedom. Since we have two equations in the  $SU(2)_L$  sector and two in the  $SU(2)_R$  sector, in total we have  $8+8=16$  supersymmetry parameters, in agreement with the 16 supersymmetries of the 6D theory. From the three dimensional point of view of the  $\text{AdS}_3$  base of our fibered compactification, this is the  $N=(4,4)$  supersymmetry, since  $N$  counts the number of supercharges, which in 3D are real two-component spinors. Below, we will use the results of [8,9] to choose a Kaluza-Klein Wilson line for our 6D solutions that makes them supersymmetric.

## 2. $SU(1,1|2) \times SU(1,1|2)$ supergravity

We now compare the three-dimensional spinor equations (54), (55) to the Killing spinor equations for the three-dimensional  $SU(1,1|2) \times SU(1,1|2)$  supergravity. The latter is described by the action [24,25]

$$S = \frac{1}{16\pi G} \int d^3x \left[ eR + \frac{2}{l^2} e + i \varepsilon^{\mu\nu\rho} \bar{\psi}_{\mu r} \mathcal{D}_\nu \psi_{\rho r} - l \varepsilon^{\mu\nu\rho} \text{Tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) + i \varepsilon^{\mu\nu\rho} \bar{\psi}'_{\mu r} \mathcal{D}'_\nu \psi'_{\rho r} + l \varepsilon^{\mu\nu\rho} \text{Tr} \left( A'_\mu \partial_\nu A'_\rho + \frac{2}{3} A'_\mu A'_\nu A'_\rho \right) \right], \quad (56)$$

where  $e_\mu^\alpha$  is the dreibein,  $A_\mu$  and  $A'_\mu$  are the  $SU(2)_L$  and  $SU(2)_R$  gauge fields

$$A_\mu = A_\mu^a \frac{i\sigma_a}{2}, \quad A'_\mu = A'^a_\mu \frac{i\sigma_a}{2}, \quad (57)$$

and  $\psi_{\mu r}$  ( $\psi'_{\mu r}$ ) with  $r=1,2$  are the  $SU(2)_L$  [ $SU(2)_R$ ] doublet two-component spinors of Appendix A. The covariant derivatives are

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4} \omega_{\mu\alpha\beta} \gamma^{\alpha\beta} + A_\mu - \frac{1}{2l} e_{\mu\alpha} \gamma^\alpha \quad (58)$$

$$\mathcal{D}'_\mu = \partial_\mu + \frac{1}{4} \omega_{\mu\alpha\beta} \gamma^{\alpha\beta} + A'_\mu + \frac{1}{2l} e_{\mu\alpha} \gamma^\alpha. \quad (59)$$

Recall that  $\gamma^{\alpha\beta} = (1/2)[\gamma_\alpha, \gamma_\beta] = \varepsilon^{\alpha\beta\delta} \gamma_\delta$ . Recall that in three spacetime dimensions there are two inequivalent two-dimensional irreducible representations for the  $\gamma$ -matrices ( $\gamma$  and  $-\gamma$ ) (see [22,28]). The two sectors in the action (56) are related to the two inequivalent representations. Therefore, the two covariant derivatives  $\mathcal{D}$  differ by a minus sign in the  $\gamma$ -matrices.

The supersymmetry transformation of the spinors gives the Killing spinor equations

$$\delta\psi_{\mu r} = \mathcal{D}_\mu \varepsilon_r = 0; \quad \delta\psi'_{\mu r} = \mathcal{D}'_\mu \varepsilon'_r = 0. \quad (60)$$

One can readily see that the equations (60) are identical to Eqs. (54), (55). The solution of these equations for the point particle spacetimes was already considered in the context of the  $SU(1,1|2) \times SU(1,1|2)$  supergravity in [9]. However, [9] presents a rather brief discussion of the actual embedding of the solutions of [8], leaving out many issues that are relevant to us. We therefore give a complete discussion of the solution of Eqs. (54), (55), using the results of [8], in the next two subsections.

## 3. Conical defects as BPS solutions in (2,0) supergravity

We have reduced the problem of finding the Killing spinors in 6D supergravity to solving Eqs. (54), (55) in  $2+1$  dimensions. Then the task has been made much easier, since



a related problem has already been solved in [8]. We only need a minor generalization of the solutions of [8] to construct solutions for our equations. In this and the following section we will show in detail how to do the embedding. In particular, we are interested in keeping track of the number of supersymmetries that are preserved as the conical deficit parameter increases from 0 to its extreme value.

Extended AdS<sub>3</sub> supergravity theories were first constructed based on the  $Osp(p|2,R) \otimes Osp(q|2,R)$  supergroups [22], and are referred to as  $(p,q)$  supergravities. The number of supercharges is  $N=p+q$ , and each of them is a two-component real spinor. The action also contains  $O(p) \times O(q)$  gauge fields. Izquierdo and Townsend [8] embedded the 3D conical defects into (2,0) supergravity and investigated their supersymmetry. In [8], the two-component real spinors have been combined into a single complex spinor, so the  $O(2)$  gauge group has been interpreted as a  $U(1)$ . Then there is a single complex vector-spinor gravitino field, with a supersymmetry transformation parametrized by a single complex two-component spinor parameter. The corresponding Killing spinor equation is

$$\mathcal{D}_\mu \epsilon = 0 \quad (61)$$

with the covariant derivative<sup>4</sup>

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4} \epsilon_{\alpha\beta\delta} \omega_\mu^{\alpha\beta} \gamma^\delta + \frac{i}{l} A_\mu - \frac{1}{2l} e_{\mu\alpha} \gamma^\alpha. \quad (62)$$

Izquierdo and Townsend find *two* Killing spinors (out of the maximum of four, counting the real degrees of freedom) for conical defects with Wilson lines. The three-dimensional metric we are interested in is (2) with  $M_3 = -\gamma^2$ . The  $U(1)$  gauge potential producing to the Wilson line is

$$A = -\frac{l}{2} (\gamma + n) d\phi, \quad (63)$$

where  $n$  is an integer related to the periodicity of the Killing spinors. If  $\gamma = -n$ , the gauge field is zero. If, in addition,  $\gamma = \pm 1$  we recover a global adS<sub>3</sub> metric. The case  $n=0$ ,  $0 < |\gamma| < 1$  corresponds to the point mass spacetimes in which we are interested. These have charge

$$Q = \frac{1}{2\pi l} \oint A = -\frac{\gamma}{2}, \quad (64)$$

so that  $M = -4Q^2$ . The deficit angle is  $\Delta\phi = 2\pi(1 - |\gamma|)$ , as we saw at the beginning of this section. The origin  $r=0$  is a conical singularity and is excised from the spacetime.

The Killing spinor solution is [8]

$$\begin{aligned} \epsilon = & e^{in\phi/2 + i\gamma t/2l} [k_- \sqrt{f+\gamma} - k_+ \sqrt{f-\gamma}] \\ & \times \left\{ \left[ 1 - \frac{1}{f} (i\gamma\gamma^0 + \sqrt{f^2 - \gamma^2} \gamma^1) \right] \right. \\ & \left. - ib_2 \gamma^2 \left[ 1 + \frac{1}{f} (i\gamma\gamma^0 + \sqrt{f^2 - \gamma^2} \gamma^1) \right] \right\} \zeta_0, \quad (65) \end{aligned}$$

where  $k_\pm$  are arbitrary constants,

$$b_2 = \frac{k_+ \sqrt{f+\gamma} + k_- \sqrt{f-\gamma}}{k_- \sqrt{f+\gamma} - k_+ \sqrt{f-\gamma}}, \quad (66)$$

and  $\zeta_0$  is a constant spinor. It satisfies a projection condition  $P\zeta_0 = \zeta_0$  with the projection matrix

$$P = \frac{-1}{(k_+^2 + k_-^2)} [i(k_-^2 - k_+^2) \gamma^0 - 2k_+ k_- \gamma^1]. \quad (67)$$

For fixed  $k_\pm$ , the projection removes two of the four real spinor degrees of freedom, so the space of Killing spinors  $\epsilon$  has two real dimensions. Note that Izquierdo and Townsend find Killing spinors for arbitrary  $\gamma, n$ . Apparently this leads to BPS solutions of arbitrarily negative mass. We will comment briefly on their meaning in Sec. III.

The Killing spinors may be singular at  $r=0$ . Near the origin,  $\epsilon$  behaves as

$$\epsilon \sim r^{\sigma/2} e^{in\phi/2} \epsilon_0 \quad (68)$$

where  $\epsilon_0$  is some constant spinor and  $\sigma$  depends on  $\gamma, n$ . If  $\sigma$  is a positive integer,  $\epsilon$  will be regular at the origin. If  $\sigma=0$ , the spinor will be regular if  $|n|=1$ , but otherwise it is singular. For  $\sigma < 0$  the spinor is singular.

When  $n=0$ ,  $0 < |\gamma| < 1$ , corresponding to the conical defects,  $\sigma=0$  in Eq. (68) but  $n=0$ , the Killing spinors are periodic, and, since we are working in a polar frame, singular at the origin. However, the origin is in any case a singular point, and removed from the spacetime. That is to say, the spacetime has noncontractible loops so  $Q \neq 0$  is possible. There are then two Killing spinors.

Let us consider the case of global AdS<sub>3</sub> in greater detail. AdS<sub>3</sub> in global coordinates with zero gauge fields is obtained when  $\gamma = -n = \pm 1$ . In this case, the origin becomes regular. The corresponding Killing spinors have  $\sigma=0$  and are regular at the origin, as required. They are antiperiodic in  $\phi$ , as expected since the space is now contractible. We get two Killing spinors with  $\gamma = -n = 1$ , and two with  $\gamma = -n = -1$ . Since both these choices give the AdS<sub>3</sub> geometry, we see it has four Killing spinors, that is, it preserves the full supersymmetry of (2,0) supergravity.

What is the relation between global AdS<sub>3</sub> and the conical defects with Wilson lines? There are two limits of the point particles. The limit  $n = \gamma = 0$  corresponds to the  $M = J = Q = 0$  black hole vacuum, and it has two Killing spinors. One can move away from this limit in either the  $\gamma > 0$  direction or

<sup>4</sup>In converting from the  $(+--)$  signature of [8] to our  $(-++)$  signature, we have replaced  $\gamma^\alpha$  by  $-i\gamma^\alpha$ . Note that [8] uses a different notation, with  $1/2l = m$ .

the  $\gamma < 0$  direction. The limit  $\gamma = \pm 1$ ,  $n = 0$  corresponds to  $\text{AdS}_3$  with non-zero gauge fields of charge  $Q = \mp \frac{1}{2}$ . Now note that the integer  $n$  can be changed by a large gauge transformation [8] (from the six-dimensional point of view, this corresponds to a coordinate transformation on  $S^3$ ; see Sec. III for details). In Sec. IV, we will see that such large gauge transformations correspond to a spectral flow in the boundary CFT. For  $\gamma = \pm 1$ , we can make a gauge transformation to make  $n = \mp 1$ ; this turns the periodic spinors associated with the point particle geometries into the antiperiodic spinors associated with  $\text{AdS}_3$ . Again,  $\text{AdS}_3$  has twice as many supersymmetries, because there are two ways to reach the  $\text{AdS}_3$  limit.

#### 4. Embedding into 6D $N=4b$ supergravity

It is quite simple to promote Izquierdo's and Townsend's solutions for (2,0) Killing spinors to solutions of the Killing spinor Eqs. (54), (55). To relate the Killing spinor equation (61) to (54), we replace the  $U(1)$  gauge potential by a  $SU(2)_L$  gauge potential,

$$\frac{1}{l}A_\mu^{U(1)} \rightarrow \frac{1}{2}A_\mu^{SU(2),c} \sigma_c, \quad (69)$$

and the spinor by the  $SU(2)_L$  doublet of spinors,

$$\epsilon \rightarrow \epsilon_r = \begin{pmatrix} \epsilon_r \\ \epsilon_r^* \end{pmatrix}. \quad (70)$$

Recall that the label  $r=1,3$  is needed, since Eq. (54) contains two identical Killing spinor equations. The  $U(1)$  Wilson line is embedded into the  $SU(2)$  by

$$\frac{1}{l}A_\phi^{U(1)} = -\frac{\gamma}{2} \rightarrow \frac{1}{2}A_\phi^{SU(2),3} \sigma_3 = -\frac{\gamma}{2} \sigma_3. \quad (71)$$

Thus the  $SU(2)_L$  gauge field has a non-zero component  $A_\phi^3$ ,

$$A_\phi^3 = -\gamma. \quad (72)$$

Then the solutions to the Killing spinor equations (54) are the two  $SU(2)_L$  doublet  $(\epsilon_r, \epsilon_r^*)^T$ , where  $\epsilon_r$  is the solution (65) and  $\epsilon_r^*$  is its complex conjugate. Note that the complex conjugate structure is consistent with the  $\sigma_3$  having opposite sign diagonal entries. Note also that the number of Killing spinors is doubled in each sector, because of the label  $r$ .

Similar manipulations are done on the  $SU(2)_R$  sector. However, there is a subtlety when the  $L$  and  $R$  sector Killing spinor equations are combined. The two sectors each have their own  $SU(2)$  gauge fields  $A, A'$  and Killing spinor equations (54), (55). For the charged point mass spacetimes, the two background gauge fields need not be equal. In general,

$$A_\phi^3 = -\gamma \neq A_\phi'^3 = -\gamma'. \quad (73)$$

For the point masses, the maximum supersymmetry is obtained by setting  $A = \pm A'$ . The point mass and zero mass black hole spacetimes then have four Killing spinors in each sector, and the pure  $\text{AdS}_3$  background without a Wilson line has the maximum, eight, in each sector. Thus, as in the (2,0) supergravity, the point masses break half of the supersymmetry.

In summary, the supersymmetric solutions are given by a three-dimensional metric

$$ds^2 = -\left(\frac{r^2}{l^2} + \gamma^2\right) dt^2 + \left(\frac{r^2}{l^2} + \gamma^2\right)^{-1} dr^2 + r^2 d\phi^2 \quad (74)$$

and gauge fields

$$A_\phi^3 = \pm A_\phi'^3 = -\gamma. \quad (75)$$

This gives a six-dimensional metric by the Kaluza-Klein ansatz (5), which satisfies the six-dimensional equations of motion and preserves half the supersymmetry. In the next section, we will discuss how this metric arises in the near-horizon limit of the rotating black string.

### III. CONICAL DEFECTS FROM THE SPINNING BLACK STRING

In the previous section, we saw how the three-dimensional solutions in which we are interested arose by spontaneous compactification of the six-dimensional  $\mathcal{N} = 4b$  theory. Interest in the six-dimensional theory is often focused on its black string solutions, so we would like to see if we can relate the point particles to these black strings. The presence of non-trivial Kaluza-Klein gauge fields in the supersymmetric point particle solutions suggests we should consider a rotating black string, as the gauge field arises from off-diagonal components of the higher-dimensional metric and  $B$ -field, which we would associate with rotation.

The solution describing a non-extremal spinning black string in six dimensions is [29,30]<sup>5</sup>

<sup>5</sup>Notice that in [29], there is also a nontrivial three-form field in the solution. We expect that this three-form reduces, in the near-horizon limit, to our KK ansatz for the three-form, but we have not checked this explicitly.

$$\begin{aligned}
ds_6^2 = & \frac{1}{\sqrt{H_1 H_2}} \left[ - \left( 1 - \frac{2mf_D}{r^2} \right) d\tilde{t}^2 + d\tilde{y}^2 + H_1 H_2 f_D^{-1} \frac{r^4}{(r^2 + l_1^2)(r^2 + l_2^2) - 2mr^2} dr^2 \right. \\
& - \frac{4mf_D}{r^2} \cosh \delta_1 \cosh \delta_2 (l_2 \cos^2 \theta d\psi + l_1 \sin^2 \theta d\phi) d\tilde{t} \\
& - \frac{4mf_D}{r^2} \sinh \delta_1 \sinh \delta_2 (l_1 \cos^2 \theta d\psi + l_2 \sin^2 \theta d\phi) d\tilde{y} \left( (r^2 + l_2^2) H_1 H_2 + (l_1^2 - l_2^2) \cos^2 \theta \left( \frac{2mf_D}{r^2} \right)^2 \sinh^2 \delta_1 \sinh^2 \delta_2 \right) \\
& \times \cos^2 \theta d\psi^2 \left( (r^2 + l_1^2) H_1 H_2 + (l_2^2 - l_1^2) \sin^2 \theta \left( \frac{2mf_D}{r^2} \right)^2 \sinh^2 \delta_1 \sinh^2 \delta_2 \right) \\
& \left. \times \sin^2 \theta d\phi^2 \frac{2mf_D}{r^2} (l_2 \cos^2 \theta d\psi + l_1 \sin^2 \theta d\phi)^2 + H_1 H_2 r^2 f_D^{-1} d\theta^2 \right], \tag{76}
\end{aligned}$$

where

$$H_i = 1 + \frac{2mf_D \sinh^2 \delta_i}{r^2} \tag{77}$$

for  $i = 1, 2$ ,

$$\frac{r^2}{f_D} = r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta, \tag{78}$$

and  $\tilde{t}$  and  $\tilde{y}$  are boosted coordinates,

$$\tilde{t} = t \cosh \delta_0 - y \sinh \delta_0, \quad \tilde{y} = y \cosh \delta_0 - t \sinh \delta_0. \tag{79}$$

For this metric, the asymptotic charges are

$$M = m \sum_{i=0}^2 \cosh 2\delta_i, \tag{80}$$

$$Q_i = m \sinh 2\delta_i; \quad i = 0, 1, 2, \tag{81}$$

$$J_{L,R} = m(l_1 \mp l_2) \left( \prod_{i=0}^2 \cosh \delta_i \pm \prod_{i=0}^2 \sinh \delta_i \right). \tag{82}$$

### A. Near-horizon limit

Cvetič and Larsen [30] showed that this metric has a near-horizon limit of the form  $\text{BTZ} \times S^3$ . To reach this limit, we take  $\alpha' \rightarrow 0$  while holding

$$\frac{r}{\alpha'}, \quad \frac{m}{\alpha'^2}, \quad \frac{l_{1,2}}{\alpha'}, \quad \frac{Q_{1,2}}{\alpha'}, \quad \text{and } \delta_0 \tag{83}$$

fixed. The resulting metric (after removing an overall factor of  $\alpha'$ ) can be written as

$$ds_6^2 = -N^2 d\tau^2 + N^{-2} d\rho^2 + \rho^2 (d\varphi - N^\phi d\tau)^2 + l^2 d\tilde{\Omega}_3^2, \tag{84}$$

$$d\tilde{\Omega}_3^2 = d\theta^2 + \cos^2 \theta d\tilde{\psi}^2 + \sin^2 \theta d\tilde{\phi}^2 \tag{85}$$

where

$$N^2 = \frac{\rho^2}{l^2} - M_3 + \frac{16G_3^2 J_3^2}{\rho^2}, \tag{86}$$

$$N^\phi = \frac{4G_3 J_3}{\rho^2}, \tag{87}$$

and there is a non-trivial transformation between the coordinates  $(\theta, \tilde{\phi}, \tilde{\psi})$  on the near-horizon  $S^3$  and the asymptotic coordinates,

$$\begin{aligned}
d\tilde{\phi} &= d\phi - \frac{R_y}{l^2} (l_2 \cosh \delta_0 - l_1 \sinh \delta_0) d\varphi \\
&\quad - \frac{R_y}{l^3} (l_1 \cosh \delta_0 - l_2 \sinh \delta_0) d\tau \\
d\tilde{\psi} &= d\psi - \frac{R_y}{l^2} (l_1 \cosh \delta_0 - l_2 \sinh \delta_0) d\varphi \\
&\quad - \frac{R_y}{l^3} (l_2 \cosh \delta_0 - l_1 \sinh \delta_0) d\tau. \tag{88}
\end{aligned}$$

The parameters of this near-horizon metric are related to the parameters of the full metric by

$$M_3 = \frac{R_y^2}{l^4} [(2m - l_1^2 - l_2^2) \cosh 2\delta_0 + 2l_1 l_2 \sinh 2\delta_0], \tag{89}$$

$$8G_3 J_3 = \frac{R_y^2}{l^3} [(2m - l_1^2 - l_2^2) \sinh 2\delta_0 + 2l_1 l_2 \cosh 2\delta_0], \tag{90}$$

and  $l = (Q_1 Q_2)^{1/4}$ . The BTZ coordinates are given by

$$\tau = \frac{tl}{R_y}, \quad \varphi = \frac{y}{R_y}, \tag{91}$$

and

$$\rho^2 = \frac{R_y^2}{l^2} [r^2 + (2m - l_1^2 - l_2^2) \sinh^2 \delta_0 + 2l_1 l_2 \sinh \delta_0 \cosh \delta_0]. \quad (92)$$

The near-horizon metric looks like the direct product of a rotating BTZ metric and an  $S^3$ . However, in the original spacetime, we identified  $\varphi \sim \varphi + 2\pi$  at fixed  $\psi, \phi$ , which is not in general the same as  $\varphi \sim \varphi + 2\pi$  at fixed  $\tilde{\psi}, \tilde{\phi}$ . Thus, the coordinate transformation (88) is not globally well-defined; that is, there are still off-diagonal terms in the near-horizon metric, which give rise to gauge fields in the three-dimensional solution. [The part of the transformation (88) involving  $\tau$  is well-defined, as  $\tau$  is not identified.]

It is convenient to trade the  $l_{1,2}$  for parameters  $a_{1,2}$  which are related to the strength of the Kaluza-Klein gauge field:

$$a_1 = l_1 \cosh \delta_0 - l_2 \sinh \delta_0, \quad a_2 = l_2 \cosh \delta_0 - l_1 \sinh \delta_0. \quad (93)$$

Then we can write

$$\tilde{\phi} = \phi - \frac{R_y}{l^2} a_2 \varphi - \frac{R_y}{l^3} a_1 \tau, \quad \tilde{\psi} = \psi - \frac{R_y}{l^2} a_1 \varphi - \frac{R_y}{l^3} a_2 \tau, \quad (94)$$

and the relations between the near-horizon and full metric parameters become

$$8GJ_3 = \frac{R_y^2}{l^3} (2m \sinh 2\delta_0 + 2a_1 a_2) \quad (95)$$

and

$$M_3 = \frac{R_y^2}{l^4} (2m \cosh 2\delta_0 - a_1^2 - a_2^2). \quad (96)$$

It is more convenient to keep some  $l_2$  dependence in  $\rho$ , and write it as

$$\rho^2 = \frac{R_y^2}{l} (r^2 + 2m \sinh^2 \delta_0 + l_2^2 - a_2^2). \quad (97)$$

To extract the Kaluza-Klein gauge fields, we need to write the metric on the 3-sphere in the coordinates used in Sec. II. This coordinate transformation is given in Appendix B. The result is

$$A^3 = \frac{R_y}{l^2} (a_1 - a_2) d\varphi, \quad A^{3'} = -\frac{R_y}{l^2} (a_1 + a_2) d\varphi, \quad (98)$$

where the indices 3, 3' refer to  $SU(2)_L$  and  $SU(2)_R$  respectively. The near-horizon limit of the spinning black string thus gives a three-dimensional metric of BTZ form coupled to gauge fields. Furthermore, the BTZ mass  $M_3$  (96) can be negative for suitable choices of the parameters (in particular, it is possible to make  $M_3$  negative while  $m \geq 0$ ).

We can now choose the parameters so that we recover the supersymmetric point particle solutions of the preceding sec-

tion. For simplicity, we have only considered non-rotating conical defects, so we require  $J_3 = 0$ . Since we seek a supersymmetric solution, it is reasonable to set  $m = 0$ . Then  $J_3 = 0$  implies  $a_1 a_2 = 0$ ; without loss of generality, take  $a_2 = 0$ . Note that for this choice of parameters, all dependence on  $\delta_0$  disappears from the metric. The mass and gauge field are now

$$M_3 = -\frac{R_y^2}{l^4} a_1^2 \equiv -\gamma^2 \quad (99)$$

and

$$A^3 = -A^{3'} = \frac{R_y}{l^2} a_1 d\varphi = \gamma d\phi. \quad (100)$$

Therefore, we recover the conical defects of the previous section.

The near-horizon limit of strings with physically reasonable choices for the parameters can thus give rise to point particle spacetimes, with negative values for  $M_3$ . Remarkably, this shows that *global* AdS<sub>3</sub> appears as the near-horizon limit of a suitable compactified black string.<sup>6</sup> To explore the consequences of this, it will be useful to also consider a family of non-extremal solutions with the same parameters. A convenient choice is to take  $\delta_0 = 0$ ,  $a_2 = 0$  (which is equivalent to  $\delta_0 = 0$ ,  $l_2 = 0$ ). In this case,  $J_3 = 0$  and  $M_3 = R_y^2(2m - a_1^2)/l^4$ .

## B. The full metric

Having seen that point particles can arise in the near-horizon limit of spinning black strings, we would like to be able to say something about the geometry of the full string solution. The near-horizon limit is also a near-extreme limit of the full black string. The extremal limit involved is<sup>7</sup>

$$m \rightarrow 0, \quad Q_{1,2} \text{ and } \delta_0 \text{ fixed.} \quad (101)$$

Initially, we will leave the value of  $a_2$  unspecified. In this limit,

$$M = Q_1 + Q_2, \quad (102)$$

$$\begin{aligned} J_{L,R} &= \frac{\sqrt{Q_1 Q_2}}{2} (l_1 \mp l_2) (\cosh \delta_0 \pm \sinh \delta_0) \\ &= \frac{\sqrt{Q_1 Q_2}}{2} (a_1 \mp a_2). \end{aligned} \quad (103)$$

The coordinate transformation  $\bar{\rho}^2 = r^2 + l_2^2 - a_2^2$  results in an extremal metric in the extremal metric of the form

<sup>6</sup>The Wilson line that appears in this limit of our solutions can be removed by a coordinate transformation from the 6D point of view.

<sup>7</sup>Note that this implies  $Q_0 \rightarrow 0$ , and is hence not the same as the limit  $m \rightarrow 0$  with  $Q_{0,1,2}$  fixed that is usually considered in the context of studies of extremal black strings [31].



$$\begin{aligned}
ds_6^2 = & \frac{1}{\sqrt{H_1 H_2}} \left[ -dt^2 + dy^2 + H_1 H_2 g_D^{-1} \frac{\bar{\rho}^4}{(\bar{\rho}^2 + a_1^2)(\bar{\rho}^2 + a_2^2)} d\bar{\rho}^2 \right. \\
& - \frac{2\sqrt{Q_1 Q_2} g_D}{\bar{\rho}^2} [\cos^2 \theta d\psi (a_2 dt + a_1 dy) + \sin^2 \theta d\phi (a_1 dt + a_2 dy)] \\
& + \left( (\bar{\rho}^2 + a_2^2) H_1 H_2 + (a_1^2 - a_2^2) \cos^2 \theta \left( \frac{g_D}{\bar{\rho}^2} \right)^2 Q_1 Q_2 \right) \cos^2 \theta d\psi^2 \\
& \left. + \left( (\bar{\rho}^2 + a_1^2) H_1 H_2 + (a_2^2 - a_1^2) \sin^2 \theta \left( \frac{g_D}{\bar{\rho}^2} \right)^2 Q_1 Q_2 \right) \sin^2 \theta d\phi^2 + H_1 H_2 \bar{\rho}^2 g_D^{-1} d\theta^2 \right], \quad (104)
\end{aligned}$$

where

$$H_i = 1 + \frac{g_D Q_i}{\bar{\rho}^2} \quad (105)$$

for  $i = 1, 2$ , and

$$\frac{\bar{\rho}^2}{g_D} = \bar{\rho}^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta. \quad (106)$$

The metric is now independent of  $\delta_0$ . That is, when we take the extremal limit with  $\delta_0$  fixed, we find that it becomes just a coordinate freedom in the limit. This is presumably a form of the usual restoration of boost-invariance at extremality. Thus, the fact that the near-horizon extremal metric did not depend on this parameter is a property of the extremal limit, not the near-horizon limit. If we take  $a_2 = 0$ , we find that  $J_L = J_R$ .

We can also consider the non-extremal metric with  $\delta_0 = 0$ ,  $l_2 = 0$  (corresponding to the simple family of non-extremal generalizations we considered in the previous section). The form of the metric is not substantially simplified relative to Eq. (5), so we will not write it out again here. We merely note that this metric has a single horizon at  $r^2 = 2m - l_1^2$ , of area

$$A = 8\pi^3 m R_y \cosh \delta_1 \cosh \delta_2 \sqrt{2m - l_1^2}. \quad (107)$$

In the near-horizon limit, this reduces to  $2\pi l \sqrt{M_3} \times 4\pi^2 l^3$ , which we recognize as the product of the area of the BTZ black hole horizon and the volume of the  $S^3$ , as expected.

### C. Properties of the solution: Instabilities and singularities

From the three-dimensional point of view, there is a conical singularity at  $\rho = 0$ , for both the non-rotating BTZ black holes and for the point particle spacetimes. In the full six-dimensional solution, we need to check the nature of this singularity. The curvature invariants are everywhere finite, so there is no curvature singularity. Consider a small neighborhood of the point  $\rho = 0$ ,  $\theta = 0$  in a constant time slice. The metric near this point can be approximated by

$$ds^2 \approx \frac{d\rho^2}{\gamma^2} + \rho^2 d\varphi^2 + d\theta^2 + d\phi^2 + \theta^2 (d\psi + \gamma d\varphi)^2. \quad (108)$$

This suggests a further coordinate transformation

$$\rho = \gamma R \cos u, \quad \theta = R \sin u, \quad (109)$$

which brings the metric to the form

$$ds^2 \approx dR^2 + R^2 [du^2 + \gamma^2 \cos^2 u d\varphi^2 + \sin^2 u (d\psi + \gamma d\varphi)^2]. \quad (110)$$

Thus, the area of a surface at  $\epsilon$  proper distance from the point  $\rho = 0, \theta = 0$  is  $\epsilon^3 \gamma^2 \pi^2$ . The difference between this area and the standard  $S^3$  area  $\epsilon^3 2\pi^2$  indicates that there is a conical defect at this point. Note that the choices of parameters for which we get negative  $M_3$ , and hence a point particle solution, are precisely those for which the full six-dimensional solution does not have an event horizon. Hence this is a naked conical singularity.

For a given value of  $a_1$ , we can obtain point particle solutions with all values of  $M_3$  by varying  $R_y$ . There is no obvious bound associated with the value  $M_3 = -1$  corresponding to pure AdS space. It was already noted by Izquierdo and Townsend in [8] that there exist supersymmetric solutions to 3D gravity for arbitrarily negative values of  $M_3$ . These solutions are all singular, and the singularities which occur for  $M_3 < -1$  are not essentially different from those which occur for  $M_3 > -1$ . From a three-dimensional point of view, one simply asserts that while the singular solutions with  $M_3 > -1$  are physically relevant, as they can arise from the collapse of matter, those with  $M_3 < -1$  are physically irrelevant. We similarly expect that only the solutions with  $M_3 > -1$  will have a physical interpretation in the dual CFT, as AdS space corresponds to the NS vacuum of the CFT, and we do not expect to find excitations with lower energy. It is therefore surprising that the six-dimensional string metric makes no distinction between  $M_3 < -1$  and  $M_3 > -1$ . It is clear that it does not, as the nature of the singularity in the six-dimensional solution is independent of the value of  $R_y$ .

However, we should still ask whether this solution is stable for all values of  $R_y$ . In [32], it was argued the BTZ

$\times S^3$  solution (for all masses) would be stable against localization on  $S^3$  so long as global  $\text{AdS}_3$  did not appear in the spectrum of the compactified string. Here we have argued that for certain parameters, the rotating, compactified string does include global  $\text{AdS}_3$ . Therefore, it is doubly worthwhile to consider the question of instabilities for near-extremal solutions with angular momentum.<sup>8</sup>

In fact, the full asymptotically flat rotating black string solution has a more familiar instability: localization on the circle ( $y$ ) along which the string is compactified. Such an instability typically sets in when the entropy of the localized solution is greater than that of the extended one [33]. Since the present solution carries a charge, a simple model for the localized solution is the extreme black string carrying the same charge, along with a six-dimensional Schwarzschild black hole carrying the energy above extremality of the original solution. Consider, for definiteness, the non-extremal solutions discussed above, with  $\delta_0=0$ ,  $l_2=0$ . From Eq. (80),  $M - M_{ext} \approx mR_y$  for near-extremal solutions, so the entropy of the Schwarzschild black hole in the candidate localized solution is

$$S_{BH} \sim (mR_y)^{4/3}. \quad (111)$$

Thus, for  $R_y > R_{crit}$ , we expect the solution to be unstable, where  $R_{crit}$  is given by  $S_{BS} = S_{BH}$ . That is,

$$R_{crit}^{2/3} \sim \frac{Q_1 Q_2 (2m - l_1^2)}{m^{8/3}} \quad (112)$$

for near-extremal solutions. Thus, as we approach extremality,  $R_{crit}$  may grow, but it will eventually decline and reach zero at  $m = l_1^2/2$ . For fixed  $R_y$ , all the near-extremal solutions with  $m$  small enough are unstable to localization.<sup>9</sup> This instability sets in at a finite distance from extremality; so we will always encounter it before reaching the instability to localization on  $S^3$  that is suggested by the physics of the near-horizon limit.

There is hence an  $R_y$ -dependent instability. Does this allow us to exclude the undesirable singularities (those with  $M_3 < -1$ )? We have argued for this instability by comparing the entropy of a near-extreme string to that of the extreme string plus a localized black hole. Thus we have *assumed* that the extreme string, which corresponds to a supersymmetric point particle solutions, is stable, and we cannot use this approach to argue that the extremal solutions are unstable. The assumption of stability of the extremal solutions is consistent, since, as we approach extremality, the entropy gain in the localization (111) is going to zero. Furthermore,

<sup>8</sup>It was argued in [32] that such a localization instability should not occur for the full asymptotically flat black string solutions, as it would break spherical symmetry. In our case, the spherical symmetry is already broken by the rotation; so it is not obvious that this argument applies.

<sup>9</sup>This is quite different from the usual behavior near extremality: for a non-rotating black string,  $R_{crit} \rightarrow \infty$  as  $m \rightarrow 0$ , as we can see from Eq. (112) with  $l_1 = 0$ .

there is no lower-energy system than the extreme string that carries the same angular momentum and charges. Together with experience in other examples, this suggests the extreme string is stable for all values of  $R_y$ , and hence instabilities do not serve to rule out the cases corresponding to  $M_3 < -1$ .

#### IV. A PROPOSAL FOR A DUAL DESCRIPTION

In [9], an interpretation of the point mass geometries in terms of spectral flow operators was given. Here, we propose a somewhat different model in terms of density matrices in the RR sector of the boundary CFT. It may seem surprising to propose that a gravitational system without a horizon, and hence no Bekenstein-Hawking entropy, would be described by a density matrix. However, the classical formulas only register a sufficiently large degeneracy. The ensemble of supersymmetric states that we are proposing contains fewer states than the number that enter the ensemble describing the  $M=0$  black hole. As is well known, the latter system has vanishing entropy in the semiclassical limit. Below, we briefly summarize the main idea of our proposal. Details and various tests will be presented in a future publication [34].

All geometries we have considered are either singular or have a horizon. Once we remove the singular region, we are left with a space with topology  $\mathbb{R}^2 \times S^1$ . This is true even for pure  $\text{AdS}_3$  with nonzero  $SU(2)$  Wilson lines. The singularity in those cases is not a curvature singularity, but one where the  $SU(2)$  gauge fields are ill-defined. The only exception is pure  $\text{AdS}_3$  without Wilson lines, whose topology is that of  $\mathbb{R}^3$ . We will first ignore pure  $\text{AdS}_3$ , but as we will see a bit later it fits in quite naturally.

On a space with topology  $\mathbb{R}^2 \times S^1$ , there are two topological choices for the spin bundle, corresponding to periodic and anti-periodic boundary conditions along the  $S^1$ . By periodic and anti-periodic we refer to spinors expressed in terms of a Cartesian frame on the boundary cylinder, which correspond to a radial frame in the AdS geometry. Thus, periodic boundary conditions correspond to the RR sector, anti-periodic boundary conditions to the NS sector. The proposed dual description of the point mass geometries will be valid assuming periodic boundary conditions, but as we will see, one can derive an equivalent description using anti-periodic boundary conditions.

It may be confusing that we impose periodic boundary conditions on the spinor and fermion fields, because if we use the field equations to parallel transport a spinor along the circle, we can pick up arbitrary phases, depending on the choice of point mass geometry, and also on the choice of  $SU(2)$  Wilson lines. These phases are the holonomies of the flat  $SL(2)$  and  $SU(2)$  connections that define the geometry and Wilson lines, but they are still connections on the same topological spinor bundle. In other words, given a bundle with a given topology, there are still many flat connections on that bundle, which are parametrized by its holonomies. In our case we choose the (periodic) spinor bundle, and view the gauge fields as connections on this bundle. Whether there exist global covariantly constant sections of the spinor bundle is a question that does depend crucially on the choices of flat connections, and is precisely the question

whose answer tells us whether or not a given solution preserves some supersymmetries.

The near horizon geometries in Sec. III, that include the BTZ and spinning point particle solutions, depend on five quantities, namely  $l = (Q_1 Q_2)^{1/4}$ ,  $M_3$ ,  $J_3$ ,  $A^3$ ,  $A^{3'}$ . In order to give the dual conformal field theory description, we define

$$c = \frac{3l}{2G_3} \quad (113)$$

$$l_0 = \frac{lM_3 + 8G_3J_3}{16G_3} \quad (114)$$

$$\bar{l}_0 = \frac{lM_3 - G_3J_3}{16G_3} \quad (115)$$

$$j_0 = \frac{c}{12}A^3 \quad (116)$$

$$\bar{j}_0 = \frac{c}{12}A^{3'}. \quad (117)$$

Our proposal is that the geometry corresponds in the boundary theory to a density matrix of (equally weighted) states in the RR sector with quantum numbers

$$J_0 = j_0 \quad (118)$$

$$\bar{J}_0 = \bar{j}_0 \quad (119)$$

$$L_0 = l_0 + \frac{c}{24} + \frac{6(j_0)^2}{c} \quad (120)$$

$$\bar{L}_0 = \bar{l}_0 + \frac{c}{24} + \frac{6(\bar{j}_0)^2}{c}. \quad (121)$$

The quadratic terms in  $L_0$  and  $\bar{L}_0$  may appear surprising, but there are several ways to justify them. First of all, in this way  $l_0$  and  $\bar{l}_0$  are spectral flow invariants, and the asymptotic density of RR states with the quantum numbers (118)–(121) is a function of  $l_0, \bar{l}_0$  only. This is in nice agreement with the fact that the area of the horizon and therefore the entropy of BTZ black holes also depends on  $l_0, \bar{l}_0$  only.

The quadratic terms in Eqs. (120) and (121) are also natural if we use the relation between the Hamiltonian reduction of  $SU(1,1|2)$  current algebra and the boundary superconformal algebra [35,23,36,21,37]. The stress tensor obtained in this Hamiltonian reduction procedure contains the Sugawara stress tensor of the  $SU(2) \subset SU(1,1|2)$  current algebra, and this extra contribution yields the quadratic terms in (120), (121).

Spectral flow in the boundary theory corresponds in the bulk to the following procedure. In the bulk, we can remove part of the  $SU(2)$  Wilson lines by a singular field redefinition. Namely, if a field  $\psi(x)$  has charge  $q$  under the  $U(1) \subset SU(2)$  subgroup, we can introduce new fields

$$\tilde{\psi}(x) = P \exp \left( q \xi \int_{x_0}^x A \cdot dx \right) \psi(x) \quad (122)$$

and at the same time replace the gauge field by

$$\tilde{A}(x) = (1 - \xi)A(x). \quad (123)$$

This is a (singular) gauge transformation and does not affect the physics. The only consequence of this transformation is that it gives twisted boundary conditions to all fields charged under the  $U(1)$ . If we compute the new quantum numbers according to Eqs. (118)–(121), we find

$$J'_0 = J_0(1 - \xi) \quad (124)$$

$$L'_0 = L_0 - \frac{12}{c} \xi J_0^2 + \frac{6}{c} \xi^2 J_0^2 \quad (125)$$

which is precisely the behavior of these quantum numbers under spectral flow with parameter  $\eta = (12/c)\xi j_0$  [38]. In other words, we can set up the AdS-CFT correspondence with arbitrary twisted boundary conditions. The twisted boundary conditions in the bulk match the twisted boundary conditions of the CFT, and the relations (118)–(121) are valid independently of the twist. Spectral flow corresponds to a field redefinition both in the bulk and in the boundary theory, and does not affect the physics. For other discussions of the role of spectral flow, see [39,40,9,37].

We can now understand how pure AdS arises in this picture. We start with pure AdS with a flat gauge field with holonomy  $-1$  in the fundamental representation. According to the above proposal, this corresponds to states in the RR sector with  $L_0 = c/24$  and  $J_0 = c/12$ . If we remove the gauge field completely by a field redefinition, this changes the boundary conditions of the fermions, and they become anti-periodic instead of periodic. Therefore, the field redefinition brings us from the R to the NS sector. In addition, the quantum numbers after the field redefinition become  $L_0 = J_0 = 0$ . We see that pure AdS with anti-periodic boundary conditions (the only boundary conditions that are well-defined on pure AdS) corresponds to the vacuum in the NS sector, as expected.

As a final check of our proposal, we will rederive the results of Izquierdo and Townsend [8] regarding the supersymmetries in point mass geometries with non-trivial gauge fields turned on. Consider again the point mass geometries with  $M_3 = -\gamma^2$ , and  $J_3 = 0$ , and only look at the left moving sector. The equation for  $L_0$  reads

$$L_0 = (1 - \gamma^2) \frac{c}{24} + \frac{c}{24} (A^3)^2 \quad (126)$$

where  $A$  is the value of the  $U(1)_L$  gauge field. The two choices of spin bundle give two inequivalent situations. If we take periodic boundary conditions for the fermions, we find a state with

$$J_0 = \frac{c}{12}A^3, \quad L_0 = (1 - \gamma^2) \frac{c}{24} + \frac{c}{24} (A^3)^2 \quad (127)$$

in the RR sector. If we start with anti-periodic boundary conditions for the fermions we find a state with quantum numbers (127), but now in the NS sector. Using the spectral flow procedure outlined above, this can be mapped to a state in the RR sector with

$$J_0 = \frac{c}{12}(A^3 + 1), \quad L_0 = (1 - \gamma^2)\frac{c}{24} + \frac{c}{24}(A^3 + 1)^2. \quad (128)$$

There are also spectral flows that map the RR sector to itself, and these are labeled by an integer  $n$ . Applying these spectral flows to Eq. (127) we obtain states in the RR sector with

$$J_0 = \frac{c}{12}(A^3 + 2n), \quad L_0 = (1 - \gamma^2)\frac{c}{24} + \frac{c}{24}(A^3 + 2n)^2 \quad (129)$$

and from Eq. (128) we obtain states with

$$J_0 = \frac{c}{12}(A^3 + 2n + 1),$$

$$L_0 = (1 - \gamma^2)\frac{c}{24} + \frac{c}{24}(A^3 + 2n + 1)^2. \quad (130)$$

The quantum numbers in Eqs. (129) and (130) can be summarized by the equations

$$J_0 = \frac{c}{12}(A^3 + n), \quad L_0 = (1 - \gamma^2)\frac{c}{24} + \frac{c}{24}(A^3 + n)^2 \quad (131)$$

where  $n$  is an arbitrary integer. In the RR sector, supersymmetry is preserved for RR ground states with  $L_0 = c/24$  only. Thus, we need that

$$A^3 = \pm \gamma + n \quad (132)$$

for some integer  $n$ . This is precisely the same condition as found in [8]; see Eqs. (63) and (71).

## V. SUMMARY AND DISCUSSION

We have embedded the 3D BPS conical defects into a higher dimensional supergravity arising from string theory. The defects in three dimensions provide particularly simple laboratories for the AdS-CFT correspondence. They are examples of systems that are neither perturbations of the AdS vacuum, nor semiclassical thermal states like black holes. Understanding the detailed representation of such objects in a dual CFT is bound to be instructive. Furthermore, the conical defects which we have constructed in six dimensions can be collided to yield the (near horizon limit) of the classic 5D black holes whose entropy was explained by Strominger and Vafa [13].

To recap, we have given a detailed analysis of the Kaluza-Klein reduction of the  $\mathcal{N}=4b$  chiral supergravity in six dimensions coupled to tensor multiplets. Our KK ansatz gives solutions to the 6D equations of motion which correspond from the dimensionally reduced point of view to 3D conical

defects with Wilson lines. Supersymmetry is preserved by a judicious choice of the gauge potential. From the 6D point of view, our solutions are spheres fibered over an  $\text{AdS}_3$  base, and the conical defect arises at a point where the fibration breaks down. Although we thereby embed all the solutions of the 3D Chern-Simons supergravities into the six dimensional theory, our ansatz does not in general produce a consistent truncation to a Chern-Simons theory. (Solutions with  $F=0$  are admitted, but the six dimensional equations of motion do not impose this.)<sup>10</sup>

Our solutions can also be understood as near-horizon limits of rotating string solutions in six dimensions compactified on a circle. Surprisingly, global  $\text{AdS}_3 \times S^3$  appears in one corner of the parameter space. Although our solutions contain conical singularities, they remain interesting because we expect them to be resolved by string theory. In particular, we have a proposal for a non-singular dual description in a conformal field theory. If our solutions are admissible, they appear to imply a Gregory-Laflamme instability for the near-extremal rotating black strings.

We have suggested a concrete representation of our conical defects as ensembles of chiral primaries in a dual CFT. Subsequent articles will test our proposal.

## ACKNOWLEDGMENTS

We have benefited from conversations with Sander Bais, Mirjam Cvetič, Sumit Das, Robbert Dijkgraaf, Per Kraus, Don Marolf, Samir Mathur, Andy Strominger, and Arkady Tseytlin. During the course of this work V.B. was supported first by the Harvard Society of Fellows and the Milton Fund of Harvard University, and then by DOE grant DOE-FG02-95ER40893. The authors thank the Aspen Center for Physics, Harvard University and University of Amsterdam for hospitality at various points in this project.

## APPENDIX A: FROM 6D SYMPLECTIC MAJORANA SPINORS TO 3D SPINORS

In this appendix, we discuss the symplectic Majorana condition on 6D chiral spinors. In particular, we show in detail how the 6D spinors can be chosen to be  $\text{SU}(2)$  doublets of complex conjugate two-component spinors

$$\epsilon_r = \begin{pmatrix} \epsilon_r^{(2)r} \\ \epsilon_r^{(2)*} \end{pmatrix}. \quad (A1)$$

The 6D Killing spinor equation in  $N=4b$  supergravity was

$$\left( D_M \mp \frac{1}{4} H_{MNP}^5 \Gamma^{NP} \right) \epsilon_r = 0, \quad (A2)$$

where the upper (lower) sign is for  $r=1,3$  ( $r=2,4$ ). The supersymmetry parameters  $\epsilon_r$  are positive chirality spinors

<sup>10</sup>While this paper was in the final stages of preparation we became aware that related investigations have been conducted by Samir Mathur.



$$\epsilon_r = \begin{pmatrix} \epsilon_r \\ 0 \end{pmatrix}. \quad (\text{A3})$$

Each of the four ( $r=1, \dots, 4$ ) spinors has four complex components. That gives 32 real degrees of freedom, of which we must remove half, since the  $N=4b$  supergravity has only 16 supersymmetries. This can be done by imposing a reality condition on the chiral spinors. In 6D, the appropriate reality condition is either the SU(2) or the symplectic Majorana condition, depending on the  $R$  symmetry of the supersymmetry algebra [41]. It can be consistently imposed along with the chirality projection. Literature on the subject includes [41,42,18,19,24]. Here we are mostly following [42].

Reference [42] first considers  $N=2$  susy in 6D. There is an SU(2) doublet of four-component complex spinors, satisfying the SU(2)-Majorana condition

$$(\psi_\alpha^i)^* \equiv \bar{\psi}_{\dot{\alpha}i} = \epsilon_{ij} B_\alpha^\beta \psi_\beta^j \quad (\text{A4})$$

where  $i, j=1,2$  label the doublet and  $\alpha, \dot{\alpha}$  are spinor indices. The matrix  $B$  must satisfy

$$BB^* = B^*B = -1. \quad (\text{A5})$$

One can see this by applying the SU(2)-Majorana condition twice and remembering that  $\epsilon_{21} = -\epsilon_{12} = -1$ .

For  $N=4$  supersymmetry, we have four complex four-component spinors, transforming as a fundamental of the USp(4)  $R$ -symmetry group. The four-component spinors can be understood as chiral 8-component complex spinors, with 4 components projected out by the chirality projection. Now the SU(2)-Majorana condition is promoted to a symplectic Majorana condition

$$\bar{\Psi}_{r\dot{\alpha}} = \Omega_{rs} B_\alpha^\beta \Psi_{s\beta} \quad (\text{A6})$$

where  $\Omega_{rs}$  is the symplectic metric of the USp(4) group, and  $\dot{\alpha}, \beta$  label the 8 components of the spinor.  $B$  is a  $4 \times 4$  matrix satisfying Eq. (A5). The symplectic metric is

$$\Omega = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \quad (\text{A7})$$

Let us take the spinors  $\Psi_r$  to be the chiral 8-component spinors  $\epsilon_r$ . Recall that we have chosen the spinors  $\epsilon_r$  with  $r=1,3$  to have opposite  $\Gamma^5$  eigenvalues from  $r=2,4$ . In this choice, we have ensured that the symplectic metric will not mix spinors with opposite eigenvalues.

For the supersymmetry parameters, the symplectic Majorana condition (A6) becomes

$$\bar{\epsilon}_1^T = B \epsilon_3, \quad (\text{A8})$$

and similarly for  $\bar{\epsilon}_2, \epsilon_4$ . The left hand side of Eq. (A8) is

$$\begin{aligned} \bar{\epsilon}_1^T &= (\epsilon_1^\dagger \Gamma_0)^T \\ &= \begin{pmatrix} 0 & 1 \otimes \gamma^{0,T} \\ 1 \otimes \gamma^{0,T} & 0 \end{pmatrix} \begin{pmatrix} \epsilon_1^* \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -(1 \otimes \gamma^0) \epsilon_1^* \end{pmatrix}, \end{aligned} \quad (\text{A9})$$

where in the last line we used  $\gamma^{0,T} = -\gamma^0$  (recall that  $\gamma^0 = -i\sigma_2$ ).

To evaluate the right hand side of Eq. (A8), we need the matrix  $B$ . We can assume it to be real, and of the form

$$B = \begin{pmatrix} & \hat{B} \\ \hat{B} & \end{pmatrix}, \quad (\text{A10})$$

where  $\hat{B}$  is real  $4 \times 4$ -matrix satisfying  $\hat{B}^2 = -1$ . A convenient choice turns out to be

$$\hat{B} = \sigma_1 \otimes \gamma^0. \quad (\text{A11})$$

The right hand side of Eq. (A8) becomes

$$B \epsilon_3 = \begin{pmatrix} 0 \\ \hat{B} \epsilon_3 \end{pmatrix}. \quad (\text{A12})$$

Thus Eq. (A8) reduces to the equation

$$-(1 \otimes \gamma^0) \epsilon_1^* = \hat{B} \epsilon_3 = (\sigma_1 \otimes \gamma^0) \epsilon_3. \quad (\text{A13})$$

Next, introduce the notation

$$\epsilon_r = \begin{pmatrix} \chi_r \\ \xi_r \end{pmatrix}, \quad r=1,3 \quad (\text{A14})$$

where  $\chi_r, \xi_r$  are 2-component complex spinors. Then Eq. (A13) is equivalent to

$$\begin{pmatrix} -\gamma^0 \chi_1^* \\ -\gamma^0 \xi_1^* \end{pmatrix} = \begin{pmatrix} \gamma^0 \xi_3 \\ \gamma^0 \chi_3 \end{pmatrix}. \quad (\text{A15})$$

Thus the two 4-component spinors  $\epsilon_{1,3}$  are

$$\epsilon_1 = \begin{pmatrix} \chi_1 \\ \xi_1 \end{pmatrix}; \quad \epsilon_3 = -\begin{pmatrix} \xi_1^* \\ \chi_1^* \end{pmatrix}. \quad (\text{A16})$$

Out of the 8 complex degrees of freedom, only 4 remain. Since the Killing spinor equations are linear, we can take linear combinations of  $\epsilon_1, \epsilon_3$ :

$$\begin{aligned} \tilde{\epsilon}_1 &= \epsilon_1 - \epsilon_3 \\ \tilde{\epsilon}_3 &= i(\epsilon_1 + \epsilon_3). \end{aligned} \quad (\text{A17})$$

Then, the  $\tilde{\epsilon}_r$  are of the complex conjugate doublet form (A1). The corresponding 8-component spinors are

$$\tilde{\epsilon}_r = \begin{pmatrix} \tilde{\epsilon}_r \\ 0 \end{pmatrix}. \quad (\text{A18})$$

The same can be done to the  $r=2,4$  spinors which had the opposite  $\Gamma^5$  eigenvalues. We can then drop the tildes, and assume that in the Killing spinor calculation the 6D spinors are such that the resulting 3D spinors will be of the form (A1).

### APPENDIX B: THE 3-SPHERE

The 3-sphere of radius  $l$  is explicitly described as

$$l^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad (\text{B1})$$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \quad (\text{B2})$$

One solution to the constraint is

$$x_1 = l \cos \theta, \quad (\text{B3})$$

$$x_2 = l \sin \theta \cos \phi, \quad (\text{B4})$$

$$x_3 = l \sin \theta \sin \phi \cos \psi, \quad (\text{B5})$$

$$x_4 = l \sin \theta \sin \phi \sin \psi, \quad (\text{B6})$$

which gives the metric

$$ds^2 = l^2(d\theta^2 + s_\theta^2 d\phi^2 + s_\theta^2 s_\phi^2 d\psi^2). \quad (\text{B7})$$

(We are using the notation  $s_\theta \equiv \sin \theta$  and  $c_\theta \equiv \cos \theta$ .) The generators of the  $\text{SO}(4)$  isometry group of  $S^3$  are  $\Lambda_j^i \sim x^i \partial_j - x^j \partial_i$ . We are actually interested in exposing the  $\text{SU}(2) \times \text{SU}(2)$  structure and so it is better to go to complex coordinates. Let  $z_1 = x_1 + ix_2$ ,  $z_2 = x_3 + ix_4$ . Then the sphere can also be written as

$$ds^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2; \quad l^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2. \quad (\text{B8})$$

Let us parametrize solutions to these equations as

$$z_1 = l \cos(\theta/2) e^{i(\phi + \psi)/2}, \quad (\text{B9})$$

$$z_2 = l \sin(\theta/2) e^{i(\phi - \psi)/2}. \quad (\text{B10})$$

(Note that exchanging  $\phi \leftrightarrow \psi$  complex conjugates  $z_2$ .) We arrive at the  $S^3$  metric

$$ds^2 = \frac{l^2}{4} [d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi]. \quad (\text{B11})$$

#### 1. $\text{SU}(2) \times \text{SU}(2)$

In the complex coordinates, it is clear that there are two  $\text{SU}(2)$  symmetries under which  $S^3$  is invariant:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow U_L \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}; \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow U_R \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \quad (\text{B12})$$

Here  $U_L \in \text{SU}(2)_L$  and  $U_R \in \text{SU}(2)_R$ . We go between these two transformations by exchanging  $\phi \leftrightarrow \psi$ .

We can compute the action of  $\text{SU}(2)_L$  explicitly. Write the group elements as  $U_L = e^{-\theta^i T_i}$  in terms of generators

$$T_1 = -\frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad T_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

$$T_3 = -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B13})$$

With a little labor one can show that the infinitesimal transformations are explicitly realized on  $(z_1, z_2)$  by the differential operators

$$\mathbf{L}_1 = c_\psi \partial_\theta + \frac{s_\psi}{s_\theta} \partial_\phi - s_\psi \cot \theta \partial_\psi, \quad (\text{B14})$$

$$\mathbf{L}_2 = -s_\psi \partial_\theta + \frac{c_\psi}{s_\theta} \partial_\phi - c_\psi \cot \theta \partial_\psi, \quad (\text{B15})$$

$$\mathbf{L}_3 = \partial_\psi. \quad (\text{B16})$$

Since the exchange ( $\phi \leftrightarrow \psi$ ) exchanges  $\text{SU}(2)_L$  and  $\text{SU}(2)_R$ , the  $\text{SU}(2)_R$  transformations are explicitly realized by the differential operators

$$\mathbf{R}_1 = c_\phi \partial_\theta + \frac{s_\phi}{s_\theta} \partial_\psi - s_\phi \cot \theta \partial_\phi, \quad (\text{B17})$$

$$\mathbf{R}_2 = -s_\phi \partial_\theta + \frac{c_\phi}{s_\theta} \partial_\psi - c_\phi \cot \theta \partial_\phi, \quad (\text{B18})$$

$$\mathbf{R}_3 = \partial_\phi. \quad (\text{B19})$$

It is also easy to check explicitly that these operators obey the Lie algebra of  $\text{SU}(2) \times \text{SU}(2)$ :

$$[\mathbf{L}_i, \mathbf{L}_j] = \epsilon_{ijk} \mathbf{L}_k; \quad [\mathbf{R}_{i'}, \mathbf{R}_{j'}] = \epsilon_{i'j'k'} \mathbf{R}_{k'}; \quad [\mathbf{L}_i, \mathbf{R}_{j'}] = 0. \quad (\text{B20})$$

The indices  $i$  and  $i'$  on  $\mathbf{L}_i$  and  $\mathbf{R}_{i'}$  can be raised and lowered freely.

#### 2. Killing vectors and vielbeins

$S^3$  has six Killing vectors, which can be taken to be the generators of the  $\text{SU}(2)_L$  and  $\text{SU}(2)_R$  symmetries above. That is,

$$\mathbf{K}_I^m = \mathbf{L}_I^m \quad I = 1, 2, 3, \quad (\text{B21})$$

$$= \mathbf{R}_{I-3}^m \quad I = 4, 5, 6. \quad (\text{B22})$$

The corresponding one-forms have components

$$\mathbf{L}_{1m} = \frac{l^2}{4} (c_\psi, s_\psi s_\theta, 0), \quad (\text{B23})$$

$$\mathbf{L}_{2m} = \frac{l^2}{4}(-s_\psi, c_\psi s_\theta, 0), \quad (\text{B24})$$

$$\mathbf{L}_{3m} = \frac{l^2}{4}(0, c_\theta, 1), \quad (\text{B25})$$

$$\mathbf{R}_{1m} = \frac{l^2}{4}(c_\phi, 0, s_\phi s_\theta), \quad (\text{B26})$$

$$\mathbf{R}_{2m} = \frac{l^2}{4}(-s_\phi, 0, c_\phi s_\theta), \quad (\text{B27})$$

$$\mathbf{R}_{3m} = \frac{l^2}{4}(0, 1, c_\theta). \quad (\text{B28})$$

There are also two choices of vielbein for  $S^3$  constructed from the  $SU(2)_L$  and  $SU(2)_R$  generators. A vielbein is defined by

$$e_m^a e_n^b \delta_{ab} = g_{mn}, \quad (\text{B29})$$

$$e_m^a e_n^b g^{mn} = \delta^{ab}. \quad (\text{B30})$$

The norm of the one-forms above is

$$\mathbf{L}_{im} \mathbf{L}_{jn} g^{mn} = \delta_{ij} \frac{l^2}{4}, \quad \mathbf{R}_{i'm} \mathbf{R}_{j'n} g^{mn} = \delta_{i'j'} \frac{l^2}{4}. \quad (\text{B31})$$

Since the sphere is 3-dimensional, the  $\mathbf{L}$  and  $\mathbf{R}$  cannot of course be mutually orthogonal as vectors. It is readily checked that

$$\mathbf{L}_{im} \mathbf{L}_{jn} \delta^{ij} = g_{mn} \frac{l^2}{4}, \quad (\text{B32})$$

and similarly for  $\mathbf{R}$ . Thus, we can construct a vielbein by identifying the group index  $i$  with a tangent index  $a$  and introducing an appropriate normalization factor. The left and right vielbeins defined in this manner are

$$e_{Lam} = \frac{2}{l} \mathbf{L}_{am}; \quad e_{Ra'm} = -\frac{2}{l} \mathbf{R}_{a'm}. \quad (\text{B33})$$

### 3. Volumes

In these Euler angle coordinates, the volume of the sphere is

$$\begin{aligned} Vol &= \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi \sqrt{\det g} \\ &= \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi \left(\frac{l}{2}\right)^3 \sin\theta = l^3 2\pi^2. \end{aligned} \quad (\text{B34})$$

Accordingly, the volume form for  $S^3$  is

$$\left(\frac{l}{2}\right)^3 \sin\theta d\theta \wedge d\phi \wedge d\psi \equiv V \epsilon_{mnr} dx^m dx^n dx^r. \quad (\text{B35})$$

### 4. Computing $N_{Ir}$ and the $SU(2)$ projectors

The discussion of the consistent ansatz for the three-form involved a two-form

$$\omega = V \epsilon_{mnr} K_I^m dx^n dx^r, \quad (\text{B36})$$

which is closed, and hence, on the sphere, an exact form. So we can write

$$\omega = d(N_{Ir} dx^r) = \partial_n N_{Ir} dx^n \wedge dx^r \quad (\text{B37})$$

for some  $N_{Ir}$ . That is,  $N_{Ir}$  are defined as the solutions of

$$\partial_n N_{Ir} - \partial_r N_{In} = 2V \epsilon_{mnr} K_I^m. \quad (\text{B38})$$

It is easy to show that a solution is<sup>11</sup>

$$I=1,2,3 \quad \Rightarrow N_{Im} = -l \mathbf{K}_{Im}, \quad (\text{B39})$$

$$I=4,5,6 \quad \Rightarrow N_{Im} = l \mathbf{K}_{Im}. \quad (\text{B40})$$

The defining Eq. (B38) then implies

$$\partial_n K_{Ir} - \partial_r K_{In} = \frac{2V}{l^2} \epsilon_{mnr} N_I^m. \quad (\text{B41})$$

We can rewrite this with tangent indices by contracting with the vielbein  $e_a^m$ , yielding

$$\partial_a K_{Ib} - \partial_b K_{Ia} = \frac{2}{l^2} \epsilon_{abc} N_I^c. \quad (\text{B42})$$

Taken together with the fact that  $K^I$  are Killing vectors, this implies

$$\nabla_a K_{Ib} = \frac{1}{l^2} \epsilon_{abc} N_I^c. \quad (\text{B43})$$

We can construct the combinations:

$$R_{Ir} = -\frac{N_{Ir}}{l} - \mathbf{K}_{Im}; \quad L_{Ir} = -\frac{N_{Ir}}{l} + \mathbf{K}_{Im}. \quad (\text{B44})$$

Clearly,

$$R_{Ir} = 0, \quad I=1,2,3, \quad (\text{B45})$$

$$= -2\mathbf{K}_{Im} = -2\mathbf{R}_{(I-3)m}, \quad I=4,5,6, \quad (\text{B46})$$

and

$$L_{Ir} = 2\mathbf{K}_{Im} = 2\mathbf{L}_{Im}, \quad I=1,2,3, \quad (\text{B47})$$

$$= 0, \quad I=1,2,3. \quad (\text{B48})$$

Thus, these combinations act as projectors onto  $SU(2)_L$  and  $SU(2)_R$  respectively. In the Killing spinor equations, these

<sup>11</sup>We can of course add any closed one-form to  $N_{Ir}$  and we will still have a solution; we will always choose to use the above solution.

projectors appear with flat *tangent* indices, i.e.,  $L_{Ia} = L_{Im} e_a^m$  and  $R_{Ia} = R_{Im} e_a^m$  where  $e_a^m$  is a left or right vielbein. Recalling the expressions for the vielbeins given in Eq. (B33),

$$R_{Ir} = 0, \quad I = 1, 2, 3, \quad (\text{B49})$$

$$= l e_{R(I-3)m}, \quad I = 4, 5, 6, \quad (\text{B50})$$

and

$$L_{Ir} = l e_{LIm}, \quad I = 1, 2, 3, \quad (\text{B51})$$

$$= 0, \quad I = 1, 2, 3. \quad (\text{B52})$$

Since the  $SU(2)_L$  and  $SU(2)_R$  equations decouple, we can go to a tangent frame using  $e_L$  and  $e_R$  separately in each case. So, choosing the left and right tangent frames in each case (call the indices  $a$  and  $a'$ ), we find

$$R_{Ia'} = 0, \quad I = 1, 2, 3, \quad (\text{B53})$$

$$= l \delta_{(I-3)a'}, \quad I = 4, 5, 6, \quad (\text{B54})$$

and

$$L_{Ia} = l \delta_{Ia}, \quad I = 1, 2, 3, \quad (\text{B55})$$

$$= 0, \quad I = 1, 2, 3. \quad (\text{B56})$$

- 
- [1] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).  
 [2] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **48**, 1506 (1993).  
 [3] S. Deser, R. Jackiw, and G. 't Hooft, *Ann. Phys. (N.Y.)* **152**, 220 (1984); S. Deser and R. Jackiw, *ibid.* **153**, 405 (1984).  
 [4] H. Martschull, *Class. Quantum Grav.* **16**, 1069 (1999); A. R. Steif, *Phys. Rev. D* **53**, 5527 (1996).  
 [5] J. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).  
 [6] V. Balasubramanian and S. F. Ross, *Phys. Rev. D* **61**, 044007 (2000).  
 [7] U. H. Danielsson, E. Keski-Vakkuri, and M. Kruczenski, *J. High Energy Phys.* **02**, 039 (2000); *Nucl. Phys.* **B563**, 279 (1999).  
 [8] J. M. Izquierdo and P. K. Townsend, *Class. Quantum Grav.* **12**, 895 (1995).  
 [9] J. R. David, G. Mandal, S. Vaidya, and S. R. Wadia, *Nucl. Phys.* **B564**, 128 (2000).  
 [10] M. J. Duff, B. E. Nilsson, and C. N. Pope, *Phys. Rep.* **130**, 1 (1986).  
 [11] M. Cvetič, H. Lu, and C. N. Pope, *Phys. Rev. D* **62**, 064028 (2000); M. Cvetič, H. Lu, C. N. Pope, A. Sadrzadeh, and T. A. Tran, *Nucl. Phys.* **B590**, 233 (2000).  
 [12] A. Strominger, *J. High Energy Phys.* **02**, 009 (1998); D. Birmingham, I. Sachs, and S. Sen, *Phys. Lett. B* **424**, 275 (1998).  
 [13] A. Strominger and C. Vafa, *Phys. Lett. B* **379**, 99 (1996).  
 [14] V. Balasubramanian and F. Larsen, *Nucl. Phys.* **B528**, 229 (1998).  
 [15] F. Larsen and E. Martinec, *J. High Energy Phys.* **06**, 019 (1999); R. Dijkgraaf, *Nucl. Phys.* **B543**, 545 (1999).  
 [16] V. Balasubramanian and P. Kraus, *Commun. Math. Phys.* **208**, 413 (1999).  
 [17] L. F. Abbott and S. Deser, *Nucl. Phys.* **B195**, 76 (1982); S. W. Hawking and G. T. Horowitz, *Class. Quantum Grav.* **13**, 1487 (1996).  
 [18] L. J. Romans, *Nucl. Phys.* **B276**, 71 (1986).  
 [19] S. Deger, A. Kaya, E. Sezgin, and P. Sundell, *Nucl. Phys.* **B536**, 110 (1998).  
 [20] F. Larsen, *Nucl. Phys.* **B536**, 258 (1998).  
 [21] J. de Boer, *Nucl. Phys.* **B548**, 139 (1999).  
 [22] A. Achucarro and P. K. Townsend, *Phys. Lett. B* **229**, 383 (1989).  
 [23] E. J. Martinec, “Matrix models of AdS gravity,” hep-th/9804111.  
 [24] M. Nishimura and Y. Tanii, *Int. J. Mod. Phys. A* **14**, 3731 (1999).  
 [25] J. R. David, *Mod. Phys. Lett. A* **14**, 1143 (1999).  
 [26] M. Asorey, F. Falceto, and S. Carlip, *Phys. Lett. B* **312**, 477 (1993).  
 [27] G. Arutyunov, A. Pankiewicz, and S. Theisen, *Phys. Rev. D* **63**, 044024 (2001).  
 [28] O. Coussaert and M. Henneaux, *Phys. Rev. Lett.* **72**, 183 (1994).  
 [29] M. Cvetič and D. Youm, *Nucl. Phys.* **B499**, 253 (1997).  
 [30] M. Cvetič and F. Larsen, *Nucl. Phys.* **B351**, 239 (1998).  
 [31] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa, *Phys. Lett. B* **391**, 93 (1997).  
 [32] A. W. Peet and S. F. Ross, *J. High Energy Phys.* **12**, 020 (1998).  
 [33] R. Gregory and R. Laflamme, *Phys. Rev. Lett.* **70**, 2837 (1993); *Nucl. Phys.* **B428**, 399 (1994).  
 [34] V. Balasubramanian, J. de Boer, E. Keski-Vakkuri, and S. F. Ross (in preparation).  
 [35] O. Coussaert, M. Henneaux, and P. van Driel, *Class. Quantum Grav.* **12**, 2961 (1995).  
 [36] M. Bañados, K. Bautier, O. Coussaert, M. Henneaux, and M. Ortiz, *Phys. Rev. D* **58**, 085020 (1998).  
 [37] M. Henneaux, L. Maoz, and A. Schwimmer, *Ann. Phys. (N.Y.)* **282**, 31 (2000).  
 [38] A. Schwimmer and N. Seiberg, *Phys. Lett. B* **184**, 191 (1987); M. Yu, *ibid.* **196**, 345 (1987).  
 [39] E. Martinec and V. Sahakian, *Phys. Rev. D* **60**, 064002 (1999).  
 [40] J. Cho and S. Nam, “AdS(3) black hole entropy and the spectral flow on the horizon,” hep-th/9903058.  
 [41] T. Kugo and P. Townsend, *Nucl. Phys.* **B221**, 357 (1983).  
 [42] P. S. Howe, G. Sierra, and P. K. Townsend, *Nucl. Phys.* **B221**, 331 (1983).