On the mass spectrum of the two-dimensional O(3) sigma model with theta term

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Mass Spectrum of the Two-Dimensional O(3) Sigma Model with a θ Term

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Form factor perturbation theory is applied to study the spectrum of the O(3) nonlinear sigma model with the topological term in the vicinity of $\theta = \pi$. Its effective action near this value is given by the nonintegrable double sine-Gordon model. Using previous results by Affleck and the explicit expressions of the form factors of the exponential operators $e^{\pm i\sqrt{8\pi}\varphi(x)}$, we show that the spectrum consists of a stable triplet of massive particles for all values of θ and a singlet state of higher mass. The singlet is a stable particle only in an interval of values of θ close to π, whereas it becomes a resonance below a critical value θ_c.

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The O(3) nonlinear sigma model is a two-dimensional quantum field theory for a three-component, unit-vector field n_α ($\alpha = 1, 2, 3; n_\alpha^2 = 1$), with the Euclidean action given by

$$\mathcal{A}_\theta = \frac{1}{2f^2}\int d^2x(\partial_\mu n_\alpha)^2 + i\theta T, \qquad (1)$$

where f and θ are dimensionless coupling constants and

$$T = \frac{1}{8\pi}\int d^2x \epsilon_{\mu\nu}\epsilon^{\alpha\beta\gamma} n_\alpha \partial_\mu n_\beta \partial_\nu n_\gamma \qquad (2)$$

is the integer-valued topological term related to the instanton solutions of the model. This model has been the subject of a huge amount of study for its theoretical properties and for the large variety of its application to condensed matter systems [1–6](see also [7]). For a generic value of θ, the topological term T breaks both the Z_2 invariance $n_a \to -n_a$ and the parity space symmetry of the action $\mathcal{A}_0$, symmetries which are, however, restored at $\theta = \pi$. As a matter of fact, the two values $\theta = (0, \pi)$ are the only ones for which the action (1) is known to be integrable. The physical properties, however, are completely different in the two cases. At $\theta = 0$, the model consists of a O(3) triplet of massive particles, with an exact S matrix given in [8]. At $\theta = \pi$, the theory is instead massless [3,9–11] and corresponds to the renormalization group flow between the $c = 2$ conformal field theory (CFT) and the $SU(2)_1$ Wess-Zumino-Witten (WZW) model at level 1, with central charge $c = 1$. In this case, the spectrum of the excitations consists of massless particles which transform according to the $s = 1/2$ representation of SU(2), the so-called spinons. The exact massless scattering amplitudes for the right and left moving doublets are given in [10].

Nonintegrable quantum field theories.—One may wonder how the spectrum of the theory (1) evolves by moving the coupling θ, in particular, how the two doublets of the massless spinons at $\theta = \pi$ are transformed into the triplet of massive states at $\theta = 0$. It is, of course, difficult, if not impossible, to provide an exact answer to this question since the model is nonintegrable for generic values of θ. However, one can gain a significant insight about this question by using the form factor perturbation theory (FFPT) proposed in Refs. [12,13]. This method allows one to study with a certain accuracy those nonintegrable models obtained as deformation of an integrable quantum field theory. For the theory (1) we have two possibilities; i.e., we can either apply the FFPT in the vicinity of $\theta = 0$ or use it to analyze the nonintegrable theory defined near $\theta = \pi$. For reasons that will become clear later, it is simpler to follow the evolution of the particle content starting from the value $\theta = \pi$. Let us discuss then in more details the model in the vicinity of this point.

Double sine-Gordon model.—At $\theta = \pi$, the O(3) sigma model corresponds to a massless flow from the CFT with $c = 2$ to an infrared fixed point described by the $SU(2)_1$ WZW model. In the vicinity of this point, it is appropriate to use the conformal fields of the WZW model to write an effective action of the sigma model (1). This was done in [2,3], and the results can be summarized as follows. Near its infrared fixed point, the action $\mathcal{A}_\pi$ corresponds to the $SU(2)_1$ WZW model perturbed by the marginally irrelevant perturbation $(\mathrm{Tr}\, g)^2$ (i.e., with $\tilde{\gamma} > 0$)

$$\mathcal{A}_\pi^{\mathrm{eff}} = \mathcal{A}_{SU(2)_1} + \tilde{\gamma}\int d^2x (\mathrm{Tr}\, g)^2, \qquad (3)$$

where g is the SU(2) matrix field with conformal dimension $\Delta = \overline{\Delta} = \frac{1}{4}$. Clearly $\tilde{\gamma}$ is a function of f, although it is well known that the explicit relation between these two coupling constants is difficult to find due to the presence of the topological term (see, for instance, [2,3] and also [14]). Beside the symmetry SU(2), the action (3) has also a Z_2 invariance related to the transformation $g \to -g$. In terms of the WZW fields, the perturbation that moves the topological term away from the value $\theta = \pi$ is proportional to $\mathrm{Tr}\, g$ [3]. This is the only relevant field of the WZW model and, moreover, the only SU(2) invariant operator in the theory that breaks parity. Thus, for a generic value of θ in the vicinity of $\theta = \pi$, we have an effective action given by the following nonintegrable perturbation of (3):

$$\mathcal{A}^{\text{eff}} = \mathcal{A}_{\pi}^{\text{eff}} + \tilde{\eta} \int d^2x \,\text{Tr}\, g, \tag{4}$$

with $\tilde{\eta} \simeq |\theta - \pi|$. At $\tilde{\eta} = 0$ the massless particles of the theory are the spinons, which can also be viewed as the fundamental excitations of the IR point [15]. However, the operator $\text{Tr}\, g$ is nonlocal with respect to them. As shown in [13] for the case of massive theories, this is the crucial property responsible for the confinement of the particles. The same also happens in the massless cases [16]. Hence, in the presence of $\text{Tr}\, g$, i.e., as soon as we move away from the point $\theta = \pi$, the spinons are confined and the model has no longer spin 1/2 excitations. To recover its actual spectrum near the value $\theta = \pi$, it is convenient to write Eqs. (3) and (4) in terms of a scalar bosonic field, φ, as $\mathcal{A}_{\pi}^{\text{eff}} = \int d^2x[\frac{1}{2}(\partial\varphi)^2 + \gamma \cos\sqrt{8\pi}]$ and $\tilde{\eta}\,\text{Tr}\, g = \eta \cos\sqrt{2\pi}\varphi$. In this new formulation, the effective action of the O(3) sigma model in the vicinity of $\theta = \pi$ is thus given by the double sine-Gordon model

$$\mathcal{A}^{\text{eff}} = \int d^2x[\tfrac{1}{2}(\partial\varphi)^2 + \eta \cos\sqrt{2\pi}\varphi + \gamma \cos\sqrt{8\pi}\varphi]. \tag{5}$$

This is a nonintegrable quantum field theory which has been studied in detail in [13]. In this model, the two periodic interactions play a symmetric role and each of the cosine term can be regarded as a deformation of the integrable theory defined by the other [17].

Affleck's result.—Because of the particular values of the cosine frequencies, the quantum field theory (5) presents a series of remarkable peculiarities which, as we show, have far-reaching consequences on its spectrum. The first important peculiarity, noticed by Affleck [18], is the special pattern of the integrable sine-Gordon model at $\beta^2 = 2\pi$, obtained for $\gamma = 0$ in Eq. (5). In fact, the spectrum of this integrable model consists of a soliton s and an antisoliton $\bar{s}$ of mass m, which are *degenerate* with a breather state b_1. Moreover, all these particles have the same S matrix. In addition, there is another breather state b_2 of higher mass, given by $m_2 = \sqrt{3}m$. The excitations can then be organized into a triplet $(s, \bar{s}, b_1)$ of bosonic states of mass m_t and a singlet of mass $m_s = \sqrt{3}m_t$, explicitly showing the hidden SU(2) symmetry of the model at this specific point [19,20]. Their exact S matrix can be found in [18] and is not given here.

The above pattern for the particles, in particular, the triplet of massive bosonic states, strongly reminds one of the spectrum of the original O(3) sigma model at $\theta = 0$. However, one may wonder and even cast doubt on whether this was just a fortunate coincidence that would no longer persist in the presence of the second interaction in the action (5). The analysis of the double sine-Gordon model shows, in fact, that an additional cosine term has generally a drastic impact on the spectrum of the solitonic sector of the unperturbed theory producing, in particular, their confinement [13]. However, this circumstance does not occur for the theory (5), and this is the second remarkable peculiarity of the action (5). To show that, we apply the form factor perturbation theory.

Form factor perturbation theory.—The form factor perturbation theory [12] allows one to estimate the variation of the spectrum of an integrable theory, once it has been perturbed by an additional term in the action $\gamma \int d^2x \Psi(x)$. At first order in γ one has

$$\delta m_i^2 \simeq 2\gamma F_{ii}^{\Psi}(i\pi), \tag{6}$$

where $F_{ii}^{\Psi}(\lambda_1 - \lambda_2) = \langle 0|\Psi(0)|A_i(\lambda_1)A_i(\lambda_2)\rangle$ is the two-particle form factor of the operator $\Psi(x)$ as a function of their rapidities parametrizing the dispersion relation $E_i = m_i \cosh\lambda_i$, $p_i = m_i \sinh\lambda_i$. For a generic sine-Gordon model

$$\mathcal{A} = \int d^2x[\tfrac{1}{2}(\partial\varphi)^2 + g\cos\beta\varphi], \tag{7}$$

perturbed by another cosine term $\Psi(x) = \cos\alpha\varphi(x)$, the evaluation of (6) may, however, be problematic. As shown in [13], the matrix element of $\cos\alpha\varphi(x)$ on the soliton states has in general a pole at $\lambda = i\pi$, with a residue ruled by the nonlocality index of this operator with respect to the soliton. (The presence of this pole signals the confinement of the soliton states in the perturbed theory.) Explicitly

$$-i\,\text{Res}_{\lambda=i\pi} F_{s\bar{s}}^{\Psi}(\lambda) = [1 - \cos(2\pi\alpha/\beta)]\langle 0|\cos\alpha\varphi(0)|0\rangle. \tag{8}$$

However, for the double sine-Gordon (5), considered as a deformation of the sine-Gordon model (7) with $\beta = \sqrt{2\pi}$, we have $\Psi(x) = \cos\sqrt{8\pi}\varphi(x)$, i.e., $\alpha/\beta = 2$, and the matrix element $F_{s\bar{s}}^{\Psi}(i\pi)$ is instead finite. Moreover, since this form factor is determined by the S matrix (which is the same for all the particles of the triplet), all of them get the same mass correction. In other words, the initial triplet identified by Affleck in the theory (5) at $\gamma = 0$ is going to stay degenerate even at $\gamma \neq 0$, a result which can be proved to hold at any order in the FFPT. It remains, then, to compute its actual correction and to compare it with the mass correction of the second breather. Here we present only the basic results of this calculation, while their complete derivation and the relative discussion will be presented somewhere else [16].

The two-particle form factors of the field $\Psi(x) = \cos\sqrt{8\pi}\varphi(x)$ on the particles of the triplet and on the higher breather b_2 can be computed by an analytic continuation of the matrix elements of the cluster operators of the sine-Gordon model [21] (see also [22]). They can be written, up to their vacuum normalization, as

$$F^{\Psi}_{ii}(\lambda) = \mu^2 \frac{\sinh^2 \frac{1}{2}\lambda}{\sinh\frac{1}{2}(\lambda + i\frac{\pi}{3}) \sinh\frac{1}{2}(\lambda - i\frac{\pi}{3})} \frac{1}{\mathcal{F}(\lambda)}, \qquad i = s, \bar{s}, 1;$$
$$F^{\Psi}_{22}(\lambda) = -\mu^4 \frac{1}{6\sqrt{3}\mathcal{F}^2(i\frac{\pi}{3})} \frac{1}{\mathcal{F}^3(\lambda)} \left(1 + \frac{1}{2\cosh\frac{1}{2}(\lambda + i\frac{\pi}{3})\cosh\frac{1}{2}(\lambda - i\frac{\pi}{3})}\right) \times \frac{\sinh^4 \frac{1}{2}\lambda}{\cosh\frac{1}{2}(\lambda + i\frac{2\pi}{3})\cosh\frac{1}{2}(\lambda - i\frac{2\pi}{3})\cosh\frac{1}{2}(\lambda + i\frac{\pi}{3})\cosh\frac{1}{2}(\lambda - i\frac{\pi}{3})}, \tag{9}$$

where $\mu^2 = \frac{3}{2}\sqrt{3}\,\mathcal{F}(i\pi)$ and the analytic function $\mathcal{F}(\lambda)$ can be expressed as

$$\mathcal{F}(\lambda) = \prod_{k=0}^{\infty} \left| \frac{\Gamma(k + \frac{3}{2} + \frac{i\hat{\lambda}}{2\pi})\Gamma(k + \frac{2}{3} + \frac{i\hat{\lambda}}{2\pi})\Gamma(k + \frac{5}{6} + \frac{i\hat{\lambda}}{2\pi})}{\Gamma(k + \frac{1}{2} + \frac{i\hat{\lambda}}{2\pi})\Gamma(k + \frac{4}{3} + \frac{i\hat{\lambda}}{2\pi})\Gamma(k + \frac{7}{6} + \frac{i\hat{\lambda}}{2\pi})} \right|^2 \tag{10}$$

($\hat{\lambda} = i\pi - \lambda$). This function does not have either poles or zero in the physical strip $0 < \mathrm{Im}\,\lambda < \pi$ and satisfies the functional equations

$$\mathcal{F}(\lambda + i\pi)\mathcal{F}(\lambda) = \frac{\sinh\lambda}{\sinh\lambda + \sinh\frac{i\pi}{3}}; \qquad \mathcal{F}\left(\lambda + i\frac{\pi}{3}\right)\mathcal{F}\left(\lambda - i\frac{\pi}{3}\right) = \frac{\cosh\frac{1}{2}(\lambda + i\frac{\pi}{3})\cosh\frac{1}{2}(\lambda - i\frac{\pi}{3})}{\sinh^2\frac{\lambda}{2}}\mathcal{F}(\lambda).$$

We have, moreover, the following identity $\mathcal{F}(i\pi) \times \mathcal{F}^2(i\frac{\pi}{3}) = \frac{1}{3}$. Using the expressions (9) we can now evaluate the mass correction (6), which is given by

$$\delta m_t^2 = 2\sqrt{3}\gamma; \qquad \delta m_s^2 = 6\sqrt{3}\gamma, \tag{11}$$

i.e., the first order correction to the mass of the singlet state is *three times larger* than the one relative to the mass of the particles of the triplet (Fig. 1). On the basis of the considerations in [17], this implies the same qualitative behavior of the masses as a function of θ. Note that the actual values of these corrections depends on the normalization of the operator $\Psi(x)$. One can get rid of this problem by considering the universal quantity given by their ratio.

This result gives a strong indication that moving away from $\theta = \pi$ and going toward the value $\theta = 0$, the spectrum evolves as follows. The particles of the triplet remain degenerate and stable also for finite values of γ, alias for all the renormalization group trajectories of the $SU(2)_1$ WZW model which asymptotically reach those of the O(3) sigma model with θ in the interval $[0, \pi)$. Hence, these particles are those which become the triplet of the O(3) sigma model at $\theta = 0$ and their mass $M_t(\gamma)$ should always be finite. Concerning the singlet, the FFPT shows that its mass $M_s(\gamma)$ increases faster than $M_t(\gamma)$ by varying γ. Since at $\theta = 0$ there is no trace of this state, its mass should become unbounded moving toward this value, causing the complete decoupling of this particle from the theory. It is then easy to argue, by continuity, that this state corresponds to a stable particle of the theory only in an interval of θ near $\theta = \pi$. It becomes instead a resonance below a certain critical value θ_c, i.e., above γ_c determined by the threshold condition $M_s(\gamma_c) \geq 2M_t(\gamma_c)$. Notice, in particular, that at first order in γ this equation does not have a solution for $\gamma > 0$; i.e., the singlet is still a stable particle of the theory. The above considerations suggest that the masses should have, qualitatively, the behavior given in Fig. 2, with their cusp $M_i \simeq (\pi - \theta)^{2/3}$ at $\theta = \pi$ dictated by the anomalous dimension of the operator $\mathrm{Tr}\,g$ [3] (logarithmic corrections to the power law behavior were considered in Ref. [23]).

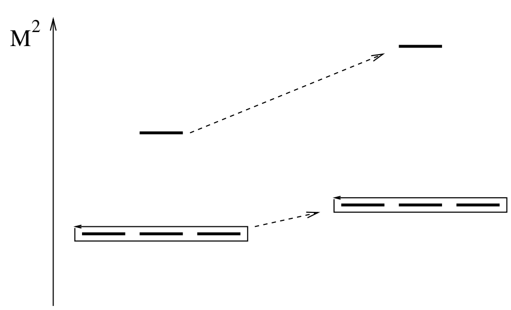

FIG. 1. Unperturbed and first order correction of the masses of the triplet and the singlet states.

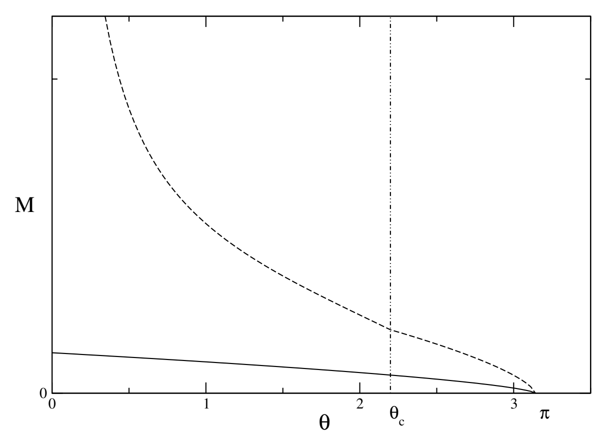

FIG. 2. Qualitative behavior of the masses of the triplet (continuum line) and the singlet (long-dashed line) as functions of θ in the interval $(0, \pi)$. The singlet particle is stable in the interval above θ_c.

In conclusion, the FFPT allowed us to gain new insights on the spectrum of the O(3) nonlinear sigma model with the θ term in the vicinity of $\theta = \pi$. As soon as θ is moved away from this value, the spinons are confined for the nonlocal properties of the associated perturbed operator. The spectrum can be obtained by analyzing the effective action of the model near its $SU(2)_1$ fixed point, given by the double sine-Gordon model (5). The pattern of triplet massive states identified by looking at the integrable model for $\gamma = 0$ turns out to be robust even in the nonintegrable field theory with $\gamma \neq 0$. The singlet state, on the contrary, belongs to the stable part of the spectrum only in an interval of values of θ close to π, whereas it becomes a resonance below a critical value θ_c. It would be interesting to reconfirm these predictions by a numerical study of the model, as in Refs. [11,24], and to determine correspondingly the critical value θ_c.

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