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Quantum Hall States and Boson Triplet Condensate for Rotating Spin-1 Bosons

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We propose and analyze two series of clustered quantum Hall states for rotating systems of spin-1 bosons. The first series [labeled $SU(4)_k$] includes the exact ground states of a model Hamiltonian at large angular momentum L, and also for $N = 3k$ particles at $L = N$. The latter is a spin-singlet boson-triplet condensate. The second series, labeled $SO(5)_k$, includes exact ground states at large L for different parameter values.

In the study of the phenomenon of Bose-Einstein condensation (BEC) of cold atoms, the possibility of using bosons with internal degrees of freedom (spin 1 in particular) and of rotating the condensate has led to fascinating new possibilities for the states of matter that can be formed and observed. A multicomponent BEC can be realized by trapping higher spin atoms such as ^{87}Rb [1] and ^{23}Na [2] in optical traps, which do not affect the spin degeneracy, and spin-1 or "vector" BEC's have been realized and studied [2,3]. In a model description, the two-body interactions of the spin-1 atoms are written as $\delta(\mathbf{r}_1 - \mathbf{r}_2)[c_0 + c_2 \mathbf{S}_1 \cdot \mathbf{S}_2]$, with $c_0 = (g_0 + 2g_2)/3$, $c_2 = (g_2 - g_0)/3$, and $g_S = 4\pi\hbar^2 a_S/M_b$ with M_b the boson mass and a_S the s-wave scattering length in the total spin S channel [4,5]. In the case of ^{23}Na one has $c_2 > 0$ and, hence, "antiferromagnetic" spin correlations. The "polar" ground state [4,5] has $\langle \mathbf{S} \rangle = 0$; it supports many interesting collective excitations (see [6] for a review). The case of ^{87}Rb has $c_2 < 0$, leading to a spin-polarized (ferromagnetic) ground state.

The effect of rotating a "scalar" BEC of spinless or spin-polarized atoms in an isotropic harmonic trap of frequency ω_0 can be studied at mean field level (the Gross-Pitaevskii equation), and a vortex lattice is found at sufficiently high rotation frequency ω, in agreement with experimental observations [7]. At still higher frequencies, quantum fluctuations become important and it has been argued that the vortex lattice is replaced by a sequence of distinct quantum fluids, which can be understood using a mapping to the fractional quantum Hall (qH) effect [8]. In this Letter, we explore the corresponding possibilities in a rotating system of spin-1 bosons.

In the regime $\omega_0 - \omega \ll \omega_0$ and very weak interaction, only the lowest-energy single-particle states in the trap are of interest. These have non-negative angular momentum $m = 0, 1, \ldots$ about the rotation axis, and no excitation of the motion along the axis, so the motion is effectively in the xy plane. The x, y dependence of these wave functions is $\psi_m(z) \propto z^m e^{-|z|^2/2}$ [where $z = x + iy$, and the quantum

length $(\hbar/M_b\omega_0)^{1/2}$ of the trap, and $\hbar$, have been set to 1]; they correspond to the lowest Landau level (LLL) [8].

We consider the interaction Hamiltonian for bosons i, $j = 1, \ldots, N$, restricted to the LLL,

$$H_{\text{int}} = \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)[c_0 + c_2 \mathbf{S}_i \cdot \mathbf{S}_j]. \qquad (1)$$

The ground state (GS) for the system rotating at frequency ω is found by minimizing $H_{\text{tot}} = H_{\text{int}} + (\omega_0 - \omega)L$ [8], where $L = \sum_i m_i$ is the total angular momentum. For N particles in the LLL with zero center-of-mass angular momentum, the average "filling factor" of the occupied states can be defined as $\nu = N/(N_V + 1)$, where $N_V = 2L/N$ is the number of vortices there would be if there were a Bose condensate (we assume here a uniform average occupation of states up to $m = N_V$). The use of only the LLL states is physically reasonable when $\omega_0 - \omega \ll \omega_0/N_V$ (so that particles in the LLL with $m = N_V$ cannot lower their energy by moving to a non-LLL state with $m = 0$) and $\nu c_S \ll \omega_0$, $S = 0, 2$ (so that perturbative corrections from mixing non-LLL states into the GS are negligible).

For the scalar case (in the LLL, without c_2), a study [9] using periodic boundary conditions found that the vortex lattice melts at a critical $\nu = \nu_c \sim 10$, that incompressible fluid states occur at $\nu = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots < \nu_c$, and that the corresponding GS wave functions have large overlaps with the Read-Rezayi (RR) [10] series of fractional qH states.

Here, for the case of vector bosons, we first propose two series of spin-singlet generalizations of the RR states [labeled $SU(4)_k$ and $SO(5)_k$, respectively, and $k = 1, 2, \ldots$] and explain their structure. In a thermodynamic limit $N \rightarrow \infty$ with ν fixed, they each represent incompressible liquid phases of the system, with $\nu = 3k/4$ for the $SU(4)_k$ states and $\nu = k$ for the $SO(5)_k$ series. Then we present results for the GS's of H_{int} as a function of L, c_0, and c_2. We identify a region in the c_0-c_2 plane where the

proposed qH states with $k = 1$ give the exact GS's at high values of L. We demonstrate that the $k = N/3$ member of the $SU(4)_k$ series is an exact eigenstate and, apparently, the GS of H_{int} with $c_2 = 0$, $c_0 > 0$, at angular momentum $L = N$. In second quantization, this state, a spin-singlet boson-triplet condensate (BTC), takes the form

$$|BTC\rangle = [\epsilon^{\mu_1\mu_2\mu_3} b^\dagger_{0\mu_1} b^\dagger_{1\mu_2} b^\dagger_{2\mu_3}]^{N/3}|0\rangle, \qquad (2)$$

where $b^\dagger_{m\mu}$ is a creation operator for the single-particle state $\psi_m(z)$ with spin μ, where $\mu = x, y, z$. Finally, we show the GS's of H_{tot} as a function of ω.

We now explain our analysis, starting with the construction of spin-singlet qH states for spin-1 bosons. This construction is a generalization of similar constructions for scalar and spin-1/2 particles (either of which could be electrons or bosons) [10–12]. These papers all employed a correspondence [13] between qH states and specific conformal field theories (CFT's). This connection allows one to obtain trial qH wave functions as chiral correlators in CFT's that are associated to a Lie algebra G and integers $k \geq 1$ and $M \geq 0$, where M is even (odd) for bosons (fermions). These states generalize the pairing familiar from the theory of superconductivity to the formation of "clusters." The resulting qH wave functions have a property that guarantees that they are exact zero-energy eigenstates of certain model Hamiltonians. For example, starting from $G = SU(n + 1)$ and putting $M = 0$, one can find completely symmetric wave functions that obey

$$\tilde\Psi(z_1, \ldots, z_N) \neq 0 \quad \text{for } z_1 = \cdots = z_l \quad (l \leq k),$$
$$\tilde\Psi(z_1, \ldots, z_N) = 0 \quad \text{for } z_1 = \cdots = z_{k+1}, \qquad (3)$$

independent of the spins of the particles involved. Hence, these are zero-energy eigenstates of a Hamiltonian with a repulsive $k + 1$-body δ-function interaction [10] and are the unique states of the lowest L with this property. The existence of such a Hamiltonian and the fact that it has a gap in its energy spectrum ensures that a corresponding incompressible liquid phase of matter exists (when $N \to \infty$ at fixed k, M) over a range of interaction parameters, not just for the $k + 1$-body model. The filling factors of these $SU(n + 1)_k$ states are

$$\nu(n, k, M) = \frac{nk}{nkM + n + 1}. \qquad (4)$$

The parameter n corresponds to the number of components of an internal degree of freedom of the particles, and these models have $U(n)$ symmetry; the clusters contain nk particles. For $n = 1$ this construction gives the RR states, while for $n = 2$ it produces spin-singlet states for spin-1/2 particles [11]. Finally, clustered qH states admit excitations (quasiparticles) of effectively fractional "charge" (i.e., particle number). The simplest of these can be viewed as the result of the adiabatic insertion of a fraction of a quantum of magnetic flux (or vorticity), where the allowed fraction is $1/nk$ in the present cases. States with more than

three quasiparticles at well-separated fixed positions display large degeneracies, which give rise to "non-Abelian statistics" [13] and are understood for $n = 1, 2$ [11,14].

Our first series of clustered spin-singlet qH states of spin-1 bosons consists of the $SU(4)_k$ states ($n = 3$ above), where we put $M = 0$. The filling factor is $\nu = 3k/4$. The CFT construction guarantees that, for N divisible by $3k$, this state is a singlet under $SU(3)$, and hence also under its $SO(3)$ subgroup, the usual spin-rotation group. The wave function for $N = 3kp$ particles can be written in terms of its components for particular spin states $\mu = x, y, z$ specified for each particle (similar to [15]), as

$$\tilde\Psi_k^{SU(4)} = S_{groups}[P_{groups} \tilde\Psi^{2,2,2,1,1,1}], \qquad (5)$$

where

$$\tilde\Psi^{2,2,2,1,1,1}(z_1^x, \ldots, z_p^x; z_1^y, \ldots, z_p^y; z_1^z, \ldots, z_p^z) =$$
$$\prod_{\mu=x,y,z} \prod_{i<j} (z_i^\mu - z_j^\mu)^2 \prod_{\mu'<\mu''} \prod_{i,j} (z_i^{\mu'} - z_j^{\mu''}). \qquad (6)$$

In words, the operations P_{groups} and S_{groups} in Eq. (5) tell one to divide the $3pk$ bosons into k groups of $3p$ bosons each (p of each spin polarization), to write a factor $\tilde\Psi^{2,2,2,1,1,1}$ as in Eq. (6) for each group, and, finally, to symmetrize over all ways the particles can be divided over the k groups. For $k = 1$ there is a single group, and one finds a state of total degree $L = 3p(2p - 1)$ that generalizes the Laughlin and Halperin states for scalar and spin-1/2 particles, respectively, and which is a zero-energy eigenstate of H_{int} for all c_0, c_2, and the unique GS when both g_0, $g_2 > 0$ since H_{int} is then positive. Putting instead $k = N/3$ gives k groups of three particles each, the resulting state being the $L = N$ BTC, Eq. (2). The fundamental quasiparticles over the $SU(4)_k$ qH liquids have fractional charge equal to $\pm 1/4$, and spin 1, for all k. We note, however, that such assignments, while meaningful for $N \gg k$, may be meaningless when $N \sim 3k$, where the quasiparticle size is comparable to the size of the fluid drop.

An important point is that for $c_2 = 0$, the Hamiltonian (1) has $SU(3)$, not just $SO(3)$, spin-rotation symmetry. Therefore, we expect the $SU(4)_k$ states to be relevant near this line when $c_0 > 0$, for sufficiently large L.

Our second series of qH states for spin-1 bosons is obtained from a CFT with $G = SO(5)$ and generalizes the construction in Ref. [12]. The GS's possess a symmetry under an $SO(3)$ subgroup of $SO(5)$. The general construction, which involves spin-singlet clusters of $2k$ particles, gives states of filling factor $\nu = k/(kM + 1)$. Putting $M = 0$, we obtain qH states with $\nu = k$. For $k = 1$, the wave function with values in the spin space of N spin-1 particles can be written as

$$\tilde\Psi_{k=1}^{SO(5)}(z_i) = Pf\left[\frac{\rho_i\rho_j + \sigma_i\sigma_j + \tau_i\tau_j}{(z_i - z_j)}\right]\tilde\Psi_L^1(z_i). \qquad (7)$$

Here $\tilde\Psi_L^1(z_i) = \prod_{i<j}(z_i - z_j)$ is a spin-independent

Laughlin factor in all N particle coordinates; the Pfaffian Pf is defined by $\text{Pf}(M_{ij}) = \mathcal{A}(M_{12}M_{34} \ldots M_{N-1,N})$, where M_{ij} are the elements of an antisymmetric matrix and $\mathcal{A}$ denotes the operation of antisymmetrization; ρ_i, σ_i, τ_i are basis vectors in the spin space for particle i, which correspond to x, y, z, and the product is the tensor product. For $k = 1$, N must be even, and the clustering (or pairing) of particles is seen explicitly. Clearly, there must be even (but otherwise arbitrary) numbers of particles in each of the three spin states ρ, σ, τ. The $k = 1$ wave function for $N = 2q$ particles has total degree $L = 2q(q - 1)$. The function (7) can be viewed as spin-singlet p-wave pairing of composite fermions of spin 1 (for a review, see Ref. [16]). There are also excited states with unpaired charge-neutral spin-1 fermions. We note that, in the state (7), two bosons are found at the same point only if the total spin of the pair is zero, not if it is 2, and hence the state is an exact zero-energy eigenstate, of lowest L, for a δ-function interaction that includes a projection onto spin 2. In our parametrization, that is, $c_2 = c_0/2$ or $g_0 = 0$, and if also $c_0 > 0$, then H_{int} is positive, so our $k = 1$ state is the GS at this L. For general k, the $\text{SO}(5)_k$ wave function can be written as a CFT chiral correlator and is an exact zero-energy eigenstate of a certain $k + 1$-body δ-function interaction (details will appear elsewhere). In general, the quasiparticles over the $\text{SO}(5)_k$ qH state have charge $\pm 1/2$ and spin $1/2$, thus displaying a fractionalization of both charge and spin.

We next report on our study (numerical and analytic) of the GS phase diagram of the model H_{int}. The numerical work is restricted to small values of N (up to 12), but it indicates many features which we believe to hold for general values of N. To guide our discussion we have displayed in Fig. 1 various special directions (rays) and regions in the c_0-c_2 plane.

For $L = 0$, the GS has total spin $S = N$ for $c_2 < 0$ (ferro regime) and $S = 0$ (1) for N even (respectively, odd) for $c_2 > 0$ (antiferro regime). For $c_2 = 0$, there is a single
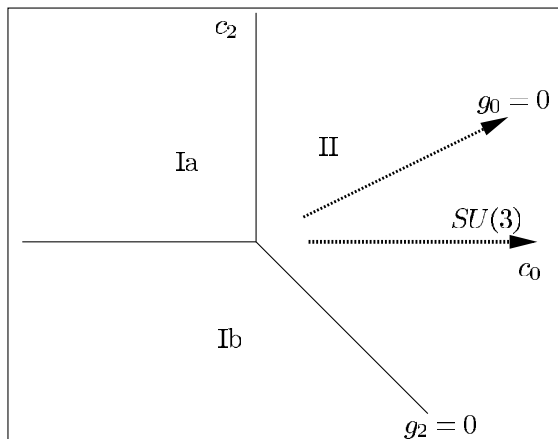

FIG. 1. Overview of c_0-c_2 plane, with special regions and directions marked.

SU(3) multiplet of spin states, decomposing into unique SO(3) multiplets of each spin $S = N, N - 2, \ldots$.

As L increases, these two phases survive in part of the phase diagram, as compact drops of fluid, with the center of mass (CM) carrying all the angular momentum. Meanwhile, the positive c_0 axis gradually opens into a region that contains other phases. By the time L is $\gtrsim N$, the c_0-c_2 plane contains the three regions labeled Ia, Ib, and II in Fig. 1. The GS's in regions Ia and Ib are similar to the GS in the "attractive" regime in the scalar case [8]. The orbital part of the GS wave function is of the form $\tilde{\Psi}(z_i) \propto z_c^L$, with $z_c = \sum_i z_i/N$ the CM coordinate. In region Ia ($c_0 < 0$, $c_2 > 0$), the spin state is the same spin singlet as for the $L = 0$ GS, and the GS energy becomes $[c_0 N(N - 1)/2 - Nc_2](2\pi)^{-3/2}$ [17]. In region Ib ($c_2 < 0$, $c_0 < -c_2$), the spin state is ferro, $S = N$, giving energy $(c_0 + c_2)N(N - 1)(2\pi)^{-3/2}/2 < 0$ [8]. At $c_2 = 0$, $c_0 < 0$, the spin states again form the SU(3) multiplet. In the "repulsive" region II, the GS is in general not a common eigenstate of the c_0 and c_2 parts of the interaction, and the GS energy depends nonlinearly on the ratio c_2/c_0. The $L = N$ GS's all have $S = 0$ or $S = 1$ (depending on the value of N modulo 6, and on the ratio c_2/c_0). For $L > N$, larger values of $S < N$ do occur near the boundary at $c_2 = -c_0$ ($g_2 = 0$).

Turning to $L \gg N$ in region II, we have already pointed out that the $\text{SU}(4)_1$ state is the zero-energy GS for $N = 3p$ particles at $L = 3p(2p - 1)$ when g_0 and $g_2 > 0$, while the $\text{SO}(5)_1$ state is the zero-energy GS for $N = 2q$ particles at $L = 2q(q - 1)$ for $g_0 = 0$. (For $N = 3, 4$, these states occur at $L = N$). For even larger L, each of these model cases possesses many degenerate GS's of zero energy. This implies that, within our model, the $\text{SU}(4)_1$ and $\text{SO}(5)_1$ GS's are those found (for the parameters as stated) at the critical rotation frequency $\omega = \omega_0$, and so the lowest possible filling factor is $3/4$ in the region g_0, $g_2 > 0$, but is 1 when $g_0 = 0$.

At intermediate $L > N$ values in this region, we expect similar physics. Thus, for $c_2 = c_0/2 > 0$, ($g_0 = 0$), the system can lower its energy by forming SO(3) singlet pairs of bosons. For $L < 2q(q - 1)$ ($N = 2q > 4$) we do not have exact eigenstates, but we expect that, similar to Ref. [9], for $\nu = k$ less than some critical value $\nu'_c > 1$, the bulk of the fluid will be in the $\text{SO}(5)_k$ state. For $c_2 = 0$, $c_0 > 0$, the preferred behavior is singlet formation via triples of bosons of spin zero; each such three bosons must be in an antisymmetric orbital state as well. For $L < 3p(2p - 1)$ ($N = 3p > 3$), we do not generally have exact eigenstates, but we expect the bulk of the fluid to be the $\text{SU}(4)_k$ states for $\nu = 3k/4$ less than another critical $\nu''_c > 3/4$.

We also found that the exact GS for $N = 6$, $L = 6$ at $c_2 = 0$ is the $\text{SU}(4)_2$ or BTC state (2). Prompted by this, we have proved analytically that for $N = 3p$ particles the $\text{SU}(4)_p$ state is an exact eigenstate of H_{int} with $c_2 = 0$, with eigenvalue
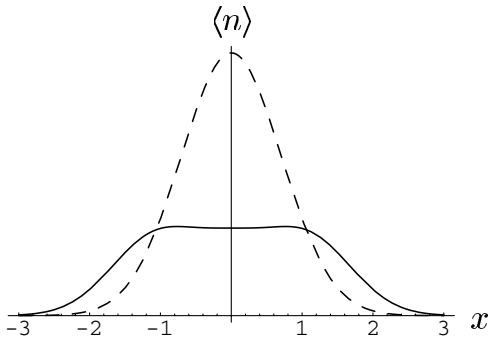
FIG. 2. Density profiles versus x at $y = 0$ in the LLL model: BTC (solid line); nonrotating BEC (broken line). Vertical axis is in units of the particle number $N = 3p$.

$$E = \frac{11}{48} N(N - 3)(2\pi)^{-3/2} c_0. \qquad (8)$$

We have numerically checked that for $N = 3, 6, 9, 12$ this state is the GS and believe this is true for all $N = 3p$.

The BTC state in Eq. (2) has a clear experimental signature, that is, the density profile given in Fig. 2, whose form is independent of N. We caution again that one should probably not think of the limit $N = 3k$ as a qH state, since the drop is so small. We identified the state as an extreme member of the $\mathrm{SU}(4)_k$ qH series, but it might be more useful to view it as a boson-triplet analog of the BCS paired electron states, or alternatively as an $\mathrm{SU}(3)$ analog of a Skyrmion spin texture (see Ref. [18]).

The response of the spin-1 boson system to a rotation frequency ω is found by minimizing H_{tot}. In Fig. 3 we display the result for $N = 6$ particles. The BTC is at $L = 6$, and the $\mathrm{SU}(4)_1$ state is at $L = 18$. The degenerate spin multiplets listed at the steps at $L < 6$ each form a single $\mathrm{SU}(3)$ multiplet. The wave functions of these GS's at $L \leq N$ are uniquely determined by their $\mathrm{SU}(3)$ and L quantum numbers.

The $\mathrm{SO}(5)_1$ state found here will survive for some distance off $g_0 = 0$. For N large at fixed ν, there will be several phases within region II, and, in particular, a boundary between the $\mathrm{SO}(5)_1$ and $\mathrm{SU}(4)_1$ phases that approaches $g_0 = 0$ as $\omega \to \omega_0$.

Our constructions for $c_2 = 0$ directly generalize to an n-component rotating Bose gas with repulsive spin-independent δ-function interactions, implying $\mathrm{SU}(n)$ symmetry. In particular, we expect a "boson n-plet condensate" at $L = (n - 1)N/2$, and $\mathrm{SU}(n + 1)_k$ states at sufficiently small $\nu = nk/(n + 1)$.

To conclude, we have found several interesting states of matter, not containing vortices, in rotating spin-1 bosons.
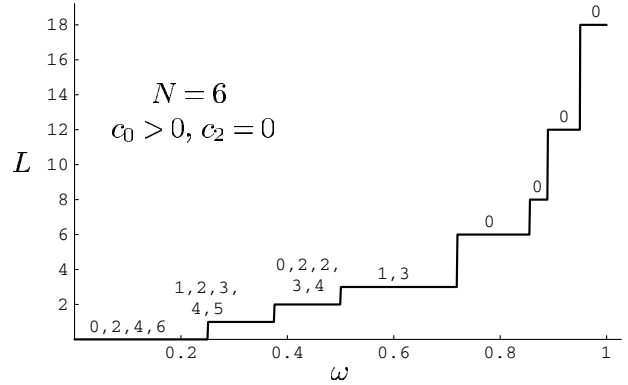
FIG. 3. GS values for L as a function of the rotation frequency ω for $N = 6$ particles at $c_2 = 0$. [We put $\omega_0 = 1$ and $c_0 = (2\pi)^{3/2}/4$.] The state at $L = 6$ is the BTC, while the $L = 18$ state is the $\mathrm{SU}(4)_1$ state. The spin values S are shown for each GS.

As this manuscript was being prepared, two e-prints appeared [19,20] that also address a rotating BEC. These papers point out the relevance of the $\mathrm{SU}(n + 1)_1$ states at large L but do not discuss the other qH states or the BTC state that we consider.

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