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On the Height of Calabi-Yau Varieties in Positive Characteristic

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Abstract. We study invariants of Calabi-Yau varieties in positive characteristic, especially the height of the Artin-Mazur formal group. We illustrate these results by Calabi-Yau varieties of Fermat and Kummer type.

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1. Introduction

The large measure of attention that complex Calabi-Yau varieties drew in recent years stands in marked contrast to the limited attention for their counterparts in positive characteristic. Nevertheless, we think these varieties deserve a greater interest, especially since the special nature of these varieties lends itself well for excursions into the largely unexplored territory of varieties in positive characteristic. In this paper we mean by a Calabi-Yau variety a smooth complete variety of dimension n over a field with $\dim H^i(X, O_X) = 0$ for $i = 1, \ldots, n-1$ and with trivial canonical bundle. We study some invariants of Calabi-Yau varieties in characteristic $p > 0$, especially the height h of the Artin-Mazur formal group for which we prove the estimate $h \leq h^{1,n-1} + 1$ if $h \neq \infty$. We show how this invariant is related to the cohomology of sheaves of closed forms.

It is well-known that K3 surfaces do not possess non-zero global 1-forms. The analogous statement about the existence of global i-forms with $i = 1$ and $i = n-1$ on a n-dimensional Calabi-Yau variety is not known and might well be false in positive characteristic. We show that for a Calabi-Yau variety of dimension ≥ 3 over an algebraically closed field k of characteristic $p > 0$ with no non-zero global 1-forms there is no p-torsion in the Picard variety and $\mathrm{Pic}/p\mathrm{Pic}$ is isomorphic to NS/pNS with NS the Néron-Severi group of X. If in addition X does not have a non-zero global 2-form then $NS/pNS \otimes_{\mathbf{F}_p} k$ maps injectively into $H^1(X, \Omega^1_X)$. This yields the estimate $\rho \leq h^{1,1}$ for the Picard number. We also study Calabi-Yau varieties of Fermat type and of Kummer type to illustrate the results.

2. The Height of a Calabi-Yau Variety

The most conspicuous invariant of a Calabi-Yau variety X of dimension n in characteristic $p > 0$ is its *height*. There are several ways to define it, using crystalline cohomology or formal groups. In the latter setting one considers the functor $F_X^r : \mathrm{Art} \to \mathrm{Ab}$ defined on the category of local Artinian k-algebras with residue field k by

$$F_X^r(S) := \mathrm{Ker}\{H^r_{\mathrm{et}}(X \times S, \mathbb{G}_m) \longrightarrow H^r_{\mathrm{et}}(X, \mathbb{G}_m)\}.$$

According to a theorem of Artin and Mazur [2], for a Calabi-Yau variety X and $r = n$ this functor is representable by a smooth formal group Φ_X of dimension 1 with tangent space $H^n(X, O_X)$. Formal groups of dimension 1 in characteristic $p > 0$ are classified up to isomorphism by their height h which is a natural number ≥ 1 or ∞. In the former case ($h \neq \infty$) the formal group is p-divisible, while in the latter case the formal group is isomorphic to the additive formal group $\hat{\mathbb{G}}_a$.

For a non-singular complete variety X over an algebraically closed field k of characteristic $p > 0$ we let $W_m O_X$ be the sheaf of Witt rings of length m, which is coherent as a sheaf of rings. It has three operators F, V and R given by $F(a_0, \ldots, a_m) = (a_0^p, \ldots, a_m^p)$, $V(a_0, \ldots, a_m) = (0, a_0, \ldots, a_m)$ and $R(a_0, \ldots, a_m) = (a_0, \ldots, a_{m-1})$ satisfying the relations $RVF = FRV = RFV = p$. The cohomology groups $H^i(X, W_m O_X)$ with the maps induced by R form a projective system of finitely generated $W_m(k)$-modules. The projective limit is the cohomology group $H^i(X, WO_X)$. Note that this need not be a finitely generated $W(k)$-module. It has semi-linear operators F and V.

Let X be a Calabi-Yau manifold of dimension n. The vanishing of the groups $H^i(X, O_X)$ for $i \neq 0, n$ and the exact sequence

$$0 \to W_{m-1}O_X \to W_m O_X \to O_X \to 0$$

imply that $H^i(X, W_m O_X)$ vanishes for $i = 1, \ldots, n-1$ and all $m > 0$, hence $H^i(X, WO_X) = 0$. We also see that restriction $R : W_m O_X \to W_{m-1} O_X$ induces a surjective map $H^n(X, W_m O_X) \to H^n(X, W_{m-1} O_X)$ with kernel $H^n(X, O_X)$. The fact that F and R commute implies that if the induced map $F : H^n(X, W_m O_X) \to H^n(X, W_m O_X)$ vanishes then $F : H^n(X, W_i O_X) \to H^n(X, W_i O_X)$ vanishes for $i < m$ too. It also follows that $H^n(X, W_i O_X)$ is a k-vector space for $i < m$.

It is known by Artin-Mazur [2] that the Dieudonné module of the formal group Φ_X is $H^n(X, WO_X)$ with WO_X the sheaf of Witt vectors of O_X. This implies the following result, cf. [3] where we proved this for K3-surfaces. We omit the proof which is similar to that for K3 surfaces.

Theorem 2.1. *For a Calabi-Yau manifold X of dimension n we have the following characterization of the height:*

$$h(\Phi_X) = \min\{i \geq 1\colon [F : H^n(W_iO_X) \to H^n(W_iO_X)] \neq 0\}.$$

We now connect this with de Rham cohomology. Serre introduced in [14] a map $D_i : W_i(O_X) \to \Omega^1_X$ of sheaves in the following way:

$$D_i(a_0, a_1, \ldots, a_{i-1}) = a_0^{p^{i-1}-1} da_0 + \ldots + a_{i-2}^{p-1} da_{i-2} + da_{i-1}.$$

It satisfies $D_{i+1}V = D_i$, and Serre showed that this induces an injective map of sheaves of additive groups

$$D_i : W_iO_X/FW_iO_X \to \Omega^1_X. \tag{1}$$

The exact sequence $0 \to W_iO_X \xrightarrow{F} W_iO_X \longrightarrow W_iO_X/FW_iO_X \to 0$ gives rise to an isomorphism

$$H^{n-1}(W_iO_X/FW_iO_X) \cong \mathrm{Ker}[F : H^n(W_iO_X) \to H^n(W_iO_X)]. \tag{2}$$

Proposition 2.2. *If $h \neq \infty$ then the induced map*

$$D_i : H^{n-1}(X, W_iO_X/FW_iO_X) \to H^{n-1}(X, \Omega^1_X)$$

is injective, and $\dim \mathrm{Im}\, D_i = \min\{i, h-1\}$.

Proof. We give a proof for the reader's convenience. Take an affine open covering $\{U_i\}$ of X. Assuming some D_ℓ is not injective, we let ℓ be the smallest natural number such that D_ℓ is not injective on $H^{n-1}(W_\ell O_X/FW_\ell O_X)$. Let $\alpha = \{f_I\}$ with $f_I = (f_I^{(0)}, \ldots, f_I^{(\ell-1)}) \in \Gamma(U_{i_0,\ldots,i_{n-1}}, W_\ell O_X)$ represent a non-zero element of $H^{n-1}(W_\ell O_X/FW_\ell O_X)$ such that $D_\ell(\alpha)$ is zero in $H^{n-1}(\Omega^1_X)$. Then there exists elements $\omega_J = \omega_{j_0 j_1 \ldots j_{n-2}}$ in $\Gamma(U_{j_0} \cap \ldots \cap U_{j_{n-2}}, \Omega^1_X)$ such that

$$\sum_{j=0}^{\ell-1} (f_I^{(j)})^{p^{\ell-j-1}} d\log f_I^{(j)} = \sum_j \omega_{I_j},$$

where the multi-index $I_j = \{i_0, \ldots, i_{n-1}\}$ is obtained from I by omitting i_j. By applying the inverse Cartier operator we get an equation

$$\sum_{j=0}^{\ell-1} (f_I^{(j)})^{p^{\ell-j}} d\log f_I^{(j)} + df_I^{(\ell)} = \sum_j \tilde{\omega}_{I_j}$$

for certain functions $f_I^{(\ell)}$ and differential forms $\tilde{\omega}_{I_j}$ with $C(\tilde{\omega}_{I_j}) = \omega_{I_j}$. Since α is an non-zero element of $H^{n-1}(W_\ell O_X/FW_\ell O_X)$, the element $\beta = (f_I^{(0)}, \ldots, f_I^{(\ell-1)}, f_I^{(\ell)})$ gives a non-zero element of $H^{n-1}(W_{\ell+1}O_X/FW_{\ell+1}O_X)$. In view of (2) for $i = \ell + 1$ the element β gives a non-zero element $\tilde{\beta}$ of $H^n(W_{\ell+1}O_X)$ such that $F(\tilde{\beta}) = 0$ in $H^n(W_{\ell+1}O_X)$. Take the element $\tilde{\alpha}$ in $H^n(W_\ell O_X)$ which corresponds to the element α under the isomorphism (2) for $i = \ell$. Then we have $F(\tilde{\alpha}) = 0$ in $H^n(W_\ell O_X)$, and $R^\ell(\tilde{\alpha}) \neq 0$ in $H^n(X, O_X)$ by the assumption on ℓ. Therefore, we have $R^\ell(\tilde{\beta}) \neq 0$ in $H^n(X, O_X)$, and the elements $V^jR^j(\tilde{\beta})$ for $j = 0, \dots, \ell$ generate $H^n(W_{\ell+1}O_X)$. Hence the Frobenius map is zero on $H^n(W_{\ell+1}O_X)$. Repeating this argument, we conclude that the Frobenius map is zero on $H^n(W_iO_X)$ for any $i > 0$ and this contradicts the assumption $h \neq \infty$.

Corollary 2.3. *If the height h of an n-dimensional Calabi-Yau variety X is not ∞ then $h \leq \dim H^{n-1}(\Omega^1_X) + 1$.*

Definition 2.4. A Calabi-Yau manifold X is called *rigid* if $\dim H^{n-1}(X, \Omega^1_X) = 0$.

Please note that the tangent sheaf Θ_X is the dual of Ω^1_X, hence by the triviality of the canonical bundle it is isomorphic to Ω^{n-1}_X. Therefore, by Serre duality the space of infinitesimal deformations $H^1(X, \Theta_X)$ is isomorphic to the dual of $H^{n-1}(X, \Omega^1_X)$.

Corollary 2.5. *The height of a rigid Calabi-Yau manifold X is either 1 or ∞.*

3. Cohomology Groups of Calabi-Yau Varieties

Let X be a Calabi-Yau variety of dimension n over k. The existence of Frobenius provides the de Rham cohomology with a very rich structure from which we can read off characteristic p properties. If $F : X \to X^{(p)}$ is the relative Frobenius operator then the Cartier operator C gives an isomorphism

$$\mathcal{H}^j(F_*\Omega^\bullet_{X/k}) = \Omega^j_{X,d\text{-closed}}/d\Omega^{j-1}_X \xrightarrow{\sim} \Omega^j_{X^{(p)}}$$

of sheaves on $X^{(p)}$. We generalize the sheaves $d\Omega^{j-1}_X$ and $\Omega^j_{X,d\text{-closed}}$ by setting (cf. [6])

$$B_0\Omega^j_X = (0), \quad B_1\Omega^j_X = d\Omega^{j-1}_X, \quad B_{m+1}\Omega^j_X = C^{-1}(B_m\Omega^j_X).$$

and

$$Z_0\Omega^j_X = \Omega^j_X, \quad Z_1\Omega^j_X = \Omega^j_{X,d\text{-closed}}, \quad Z_{m+1}\Omega^j_X = \mathrm{Ker}(dC^m).$$

Note that we have the inclusions

$$\begin{gathered} 0 = B_0\Omega^j_X \subset B_1\Omega^j_X \subset \ldots \subset B_m\Omega^j_X \subset \ldots \\ \ldots \subset Z_m\Omega^j_X \subset \ldots \subset Z_1\Omega^j_X \subset Z_0\Omega^j_X = \Omega^j_X \end{gathered}$$

and that we have an exact sequence

$$0 \to Z_{m+1}\Omega^j_X \longrightarrow Z_m\Omega^j_X \xrightarrow{dC^m} d\Omega^j_X \to 0.$$

Alternatively, the sheaves $B_m\Omega^j_X$ and $Z_m\Omega^j_X$ can be viewed as locally free subsheaves of $(F^m)_*\Omega^j_X$ on $X^{(p^m)}$. Duality for the finite morphism F^m implies that for every $j \geq 0$ there is a perfect pairing of $O_{X^{(p^m)}}$-modules $F^m_*\Omega^j_X \otimes F^m_*\Omega^{n-j}_X \longrightarrow \Omega^n_{X^{(p^m)}}$ given by $(\alpha, \beta) \mapsto C^m(\alpha \wedge \beta)$. This induces perfect pairings of $O_{X^{(p^m)}}$-modules

$$B_m\Omega^j_X \otimes F^m_*\Omega^{n-j}_X/Z_m\Omega^{n-j}_X \to \Omega_{X^{(p^m)}}$$

and

$$Z_m\Omega_X^j \otimes F_*^m\Omega_X^{n-j}/B_m\Omega_X^{n-j} \to \Omega_{X^{(p^m)}}$$

Now we have an isomorphism $F_*^m\Omega_X^j/Z_m\Omega_X^j \cong B_m\Omega_X^{j+1}$ induced by the map d. Going back to the interpretation of the $B_m\Omega_X^j$ as sheaves on X we find in this way for $1 \leq j \leq n$ and $m > 0$ perfect pairings

$$B_m\Omega_X^j \otimes B_m\Omega_X^{n+1-j} \to \Omega_X^n \qquad (\omega_1 \otimes \omega_2) \mapsto C^m(\omega_1 \wedge \omega_2).$$

We first note another interpretation for $B_m\Omega_X^1$: the injective map of sheaves of additive groups $D_m : W_m(\mathcal{O}_X)/FW_m(\mathcal{O}_X) \to \Omega_X^1$ induces an isomorphism

$$D_m : W_m(\mathcal{O}_X)/FW_m(\mathcal{O}_X) \xrightarrow{\sim} B_m\Omega_X^1. \tag{3}$$

We write $h^i(X,-)$ for $\dim_k H^i(X,-)$. Note that duality implies $h^i(B_m\Omega_X^n) = h^{n-i}(B_m\Omega_X^1)$.

Proposition 3.1. *We have $h^i(X, B_m\Omega_X^1) = 0$ unless $i = n$ or $i = n-1$. If $i = n-1$ or $i = n$ we have*

$$h^i(B_m\Omega_X^1) = \begin{cases} \min\{m, h-1\} & \text{if } h \neq \infty \\ m & \text{if } h = \infty. \end{cases}$$

Proof. The statement about $h^{n-1}(B_m\Omega_X^1)$ follows from (3) and the characterization of the height given in Section 2. The other statements follow from the long exact sequence associated with the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \xrightarrow{d} d\mathcal{O}_X \longrightarrow 0$$

and the exact sequence

$$0 \to B_m \to B_{m+1} \xrightarrow{C^m} B_1 \to 0. \tag{4}$$

The details can safely be left to the reader. This concludes the proof.

The natural inclusions $B_i\Omega_X^j \hookrightarrow \Omega_X^j$ and $Z_i\Omega_X^j \hookrightarrow \Omega_X^j$ of sheaves of groups on X induce homomorphisms

$$H^1(B_i\Omega_X^j) \to H^1(\Omega_X^j) \quad \text{and} \quad H^1(Z_i\Omega_X^j) \to H^1(\Omega_X^j)$$

whose images are denoted by $\operatorname{Im} H^1(B_i\Omega_X^j)$ and $\operatorname{Im} H^1(Z_i\Omega_X^j)$. Note that we have a non-degenerate cup product pairing

$$\langle\, ,\, \rangle : H^{n-1}(X, \Omega_X^1) \otimes H^1(X, \Omega_X^{n-1}) \to H^n(X, \Omega_X^n) \cong k.$$

Lemma 3.2. *The images $\operatorname{Im} H^{n-1}(B_i\Omega_X^1)$ and $\operatorname{Im} H^1(Z_i\Omega_X^{n-1})$ are orthogonal to each other for the pairing $\langle\, ,\, \rangle$.*

Proof. From the definitions it follows that for elements $\alpha \in H^{n-1}(B_i\Omega_X^1)$ and $\beta \in H^1(Z_i\Omega_X^{n-1})$ we have $C^i(\alpha \wedge \beta) = 0$. The long exact sequence associated to

$$0 \to B_1\Omega_X^n \to Z_1\Omega_X^n \to \Omega_X^n \to 0 \tag{5}$$

together with the fact that $H^n(Z_i\Omega_X^n) = H^n(\Omega_X^n)$ for $i \geq 0$ implies that C acts without kernel on $H^n(\Omega_X^n)$. This proves the required orthogonality.

Lemma 3.3. *If $h \neq \infty$ we have* $\dim \mathrm{Im} H^1(X, Z_i\Omega_X^{n-1}) = \dim H^1(\Omega_X^{n-1}) - i$ *for* $0 \leq i \leq h-1$.

Proof. If the height $h = 1$ then we have $H^{n-1}(B_i\Omega_X^1) = 0$ by (4) and moreover the vanishing of $H^i(X, d\Omega_X^{n-1})$ and the exact sequence

$$0 \to Z_{i+1}\Omega_X^{n-1} \longrightarrow Z_i\Omega_X^{n-1} \xrightarrow{dC^i} d\Omega_X^{n-1} \to 0 \tag{6}$$

imply that $\mathrm{Im}\, H^1(X, Z_i\Omega_X^{n-1}) = H^1(X, \Omega_X^{n-1})$ for $i \geq 1$. For $2 \leq h < \infty$, we know by Proposition 2.2 that $\mathrm{Im}\, H^{n-1}(X, B_i\Omega_X^1) \subset H^{n-1}(X, \Omega_X^1)$ is of dimension $\min\{i, h-1\}$. The exact sequence (6) gives an exact sequence

$$k \longrightarrow H^1(Z_{i+1}\Omega_X^{n-1}) \xrightarrow{\psi_{i+1}} H^1(Z_i\Omega_X^{n-1}) \longrightarrow k$$

from which we deduce that either $\dim \psi_{i+1}(H^1(Z_{i+1}\Omega_X^{n-1})) = \dim H^1(Z_i\Omega_X^{n-1}) + 1$ or $\dim \psi_{i+1}(H^1(Z_{i+1}\Omega_X^{n-1})) = \dim H^1(Z_i\Omega_X^{n-1})$. By induction $\dim \mathrm{Im} H^1(Z_i\Omega_X^{n-1})$ is at least $\dim H^1(\Omega_X^{n-1}) - i$. On the other hand, by Proposition 3.1 we have $\dim \mathrm{Im} H^1(Z_i\Omega_X^{n-1}) \leq \dim H^1(\Omega_X^{n-1}) - i$ for $i \leq h-1$.

Lemma 3.4. *If X is a Calabi-Yau manifold of dimension n with $h = \infty$ then*

$$(\mathrm{Im}\, H^{n-1}(X, B_i\Omega_X^1))^{\perp} = \mathrm{Im}\, H^1(Z_i\Omega_X^{n-1}).$$

Proof. We prove this by induction on i. By the exact sequence (5) we have $\dim H^i(X, d\Omega_X^{n-1}) = 1$ for $i = 0, 1$. Thus, by the exact sequence (6) we see that the difference $\dim \mathrm{Im} H^1(Z_i\Omega^{n-1}) - \dim \mathrm{Im} H^1(Z_{i+1}\Omega^{n-1})$ is equal to 0 or 1, and we have an exact sequence

$$H^1(Z_{i+1}\Omega^{n-1}) \xrightarrow{\phi} H^1(Z_i\Omega^{n-1}) \xrightarrow{dC^i} H^1(d\Omega_X^{n-1}).$$

Assume that $\mathrm{Im}\, H^{n-1}(B_{j-1}\Omega^1) \neq \mathrm{Im}\, H^{n-1}(B_j\Omega^1)$ for $j \leq i$ and $\mathrm{Im}\, H^{n-1}(B_i\Omega^1) = \mathrm{Im}\, H^{n-1}(B_{i+1}\Omega^1)$. By Lemma 3.2,

$$\mathrm{Im} H^1(Z_{i-1}\Omega^{n-1}) \supset \mathrm{Im} H^1(Z_i\Omega^{n-1})$$

and $\mathrm{Im} H^1(Z_{i-1}\Omega^{n-1}) \neq \mathrm{Im} H^1(Z_i\Omega^{n-1})$ for $j \leq i$. Suppose $\mathrm{Im} H^1(Z_i\Omega^{n-1}) \neq \mathrm{Im} H^1(Z_{i+1}\Omega^{n-1})$. The natural homomorphism $\phi : H^1(Z_{i+1}\Omega^{n-1}) \to H^1(Z_i\Omega^{n-1})$ is not surjective. Since $H^1(d\Omega_X^{n-1}) \cong k$, we see that $dC^i : H^1(Z_i\Omega^{n-1}) \to H^1(d\Omega_X^{n-1})$ is surjective and we factor it as

$$H^1(Z_i\Omega^{n-1}) \xrightarrow{C^i} H^1(\Omega^{n-1}) \xrightarrow{d} H^1(d\Omega_X^{n-1}).$$

Since dC^i is surjective, d is not the zero map on $C^i(H^1(Z_i\Omega^{n-1}))$. Therefore, we have

$$C^i(H^1(Z_i\Omega^{n-1})) \not\subset \mathrm{Im}\, H^1(Z_1\Omega^{n-1}).$$

Take an affine open covering of X, and take any Čech cocycle $C^i(\eta) = \{C^i(\eta_{jk})\}$ of $C^i(H^1(Z_i\Omega^{n-1}))$ with respect to this affine open covering. Take any element $\zeta \in H^{n-1}(B_i\Omega^1)$. Then there exists an element $\tilde{\zeta}$ such that $C^i(\tilde{\zeta}) = \zeta$. We consider the image of the element $C^i(\eta) \wedge \zeta$ in $H^n(X, \Omega_X^n)$. Then, we have

$$C^i(\tilde{\eta}) \wedge \zeta = C^i(\tilde{\eta}) \wedge C^i(\tilde{\zeta}) = C^i(\eta \wedge \tilde{\zeta})$$

Since $\mathrm{Im}H^{n-1}(B_{2i}\Omega^1) = \mathrm{Im}H^{n-1}(B_i\Omega^1)$, the image of $\tilde{\zeta}$ in $H^{n-1}(\Omega_X^1)$ is contained in $\mathrm{Im}H^{n-1}(B_i\Omega^1)$. As $\mathrm{Im}H^1(Z_i\Omega^{n-1}))$ is orthogonal to $\mathrm{Im}H^{n-1}(B_i\Omega^1)$, we see that $\eta\wedge\tilde{\zeta}$ is zero in $H^n(X,\Omega_X^n)$, and we have $C^i(\eta\wedge\tilde{\zeta}) = 0$ in $H^n(X,\Omega_X^n)$. Therefore, we see that the image of $C^i(H^1(Z_i\Omega^{n-1}))$ in $H^1(\Omega_X^{n-1})$ is orthogonal to $\mathrm{Im}H^{n-1}(B_i\Omega^1)$ and we have

$$C^i(H^1(Z_i\Omega^{n-1})) \subset \mathrm{Im}H^{n-1}(B_i\Omega^1)^\perp \subset \mathrm{Im}H^1(Z_i\Omega^{n-1}) \subset \mathrm{Im}H^1(Z_1\Omega^{n-1}),$$

a contradiction. Hence, we have $\mathrm{Im}H^1(Z_i\Omega^{n-1}) = \mathrm{Im}H^1(Z_{i+1}\Omega^{n-1})$.

Collecting results we get the following theorem.

Theorem 3.5. *If X is a Calabi-Yau variety of dimension n and height h then for $i \leq h-1$ we have*

$$\mathrm{Im}H^{n-1}(X, B_i\Omega_X^1)^\perp = \mathrm{Im}H^1(X, Z_i\Omega_X^{n-1}).$$

One reason for our interest in the spaces $\mathrm{Im}H^1(X, Z_i\Omega_X^{n-1})$ comes from the fact that they play a role as tangent spaces to strata in the moduli space as in the analogous case of K3 surfaces, cf. [3]. We intend to come back to this in a later paper.

4. Picard groups

We suppose that X is a Calabi-Yau variety of dimension $n \geq 3$. We have the following result for the space of regular 1-forms.

Proposition 4.1. *All global 1-forms are indefinitely closed: for $i \geq 0$ we have $H^0(X, Z_i\Omega_X^1) = H^0(X, \Omega_X^1)$. The action of the Cartier operator on this space is semi-simple.*

Proof. Since the sheaves $B_i\Omega_X^1$ have non non-zero cohomology in degree 0 and 1 the exact sequence

$$0 \to B_i\Omega_X^1 \longrightarrow Z_i\Omega_X^1 \xrightarrow{C^i} \Omega_X^1 \to 0.$$

implies $\dim H^0(Z_i\Omega_X^1) = \dim H^0(\Omega_X^1)$. Since the natural map $H^0(Z_i\Omega_X^1) \to H^0(\Omega_X^1)$ is injective, we have $H^0(Z_i\Omega_X^1) = H^0(\Omega_X^1)$. The second assertion follows from $H^0(B_i\Omega_X^1) = 0$.

It is well known that for a p^{-1}-linear semi-simple homomorphism λ on a finite-dimensional vector space V the map $\lambda - \mathrm{id}_V$ is surjective. This means that we have a basis of logarithmic differential forms $C\omega = \omega$.

Corollary 4.2. *If* id *denotes the identity homomorphism on $H^0(X, \Omega_X^1)$ the map $C - \mathrm{id} : H^0(X, \Omega_X^1) \to H^0(X, \Omega_X^1)$ is surjective.*

Proposition 4.3. *Suppose that X is a smooth complete variety for which all global 1-forms are closed and such that C gives a bijection $H^0(X, Z_1\Omega_X^1) \longrightarrow H^0(X, \Omega_X^1)$. Then we have an isomorphism*

$$H^0(X, \Omega_X^1) \cong \mathrm{Pic}(X)[p] \otimes_{\mathbf{Z}} k.$$

Proof. Let L be a line bundle representing an element $[L]$ of order p in $\mathrm{Pic}(X)$. Then there exists a rational function $g \in k(X)^*$ such that $(g) = pD$, where D is a divisor corresponding to L. One observes now by a local calculation that dg/g is a regular 1-form and thus defines an element of $H^0(X, \Omega_X^1)$. Conversely, if ω is a global regular 1-form with $C\omega = \omega$ then ω can be represented locally as df_i/f_i with respect to some open cover $\{U_i\}$. From the relation $df_i/f_i = df_j/f_j$ we see $d\log(f_i/f_j) = 0$ and this implies $d(f_i/f_j) = 0$. Hence we see that $f_i/f_j = \phi_{ij}^p$ form some 1-cocycle $\{\phi_{ij}\}$. This cocycle defines a torsion element of order p of $\mathrm{Pic}(X)$. These two maps are each others inverse and the result follows.

We are using the notation $\mathrm{Pic}(X)$ (resp. $NS(X)$) for the Picard group (resp. Néron-Severi group) of X. If L is a line bundle with transition functions $\{f_{ij}\}$ then $d\log f_{ij}$ represents the first Chern class of L. In this way we can define a homomorphism

$$\varphi_1 : \mathrm{Pic}(X) \longrightarrow H^1(Z_1\Omega_X^1), \quad [L] \mapsto c_1(L) = \{df_{ij}/f_{ij}\}$$

which obviously factors through $\mathrm{Pic}(X)/p\mathrm{Pic}(X)$.

Proposition 4.4. *The homomorphism* $\varphi_1 : \mathrm{Pic}(X)/p\mathrm{Pic}(X) \longrightarrow H^1(X, Z_1\Omega_X^1)$ *is injective.*

Proof. We take an affine open covering $\{U_i\}$. Suppose that there exists an element $[L]$ such that $\varphi_1([L]) = 0$. Then there exists a d-closed regular 1-form ω_i on an affine open set U_i such that $df_{ij}/f_{ij} = \omega_j - \omega_i$ on $U_i \cap U_j$ and we have $df_{ij}/f_{ij} = C(\omega_j) - C(\omega_i)$. Therefore, we have $\omega_j - C(\omega_j) = \omega_i - C(\omega_i)$ on $U_i \cap U_j$. This shows that there exists an regular 1-form ω on X such that $\omega = \omega_i - C(\omega_i)$ on U_i. By Corollary 4.2, there exists an element $\omega' \in H^0(\Omega_X^1)$ such that $(C - \mathrm{id})\omega' = \omega$. Replacing $\omega_i + \omega'$ by ω_i, we have

$$df_{ij}/f_{ij} = \omega_j - \omega_i$$

with $C(\omega_i) = \omega_i$. Then, there exists an regular function f_i on U_i such that $\omega_i = df_i/f_i$. So we have $d\log f_{ij} = d\log(f_j/f_i)$. Therefore, there exists a regular function φ_{ij} on $U_i \cap U_j$ such that $f_{ij} = (f_j/f_i)\varphi_{ij}^p$. Thus $[L]$ is a p-th power. We conclude that $\varphi_1 : \mathrm{Pic}(X)/p\mathrm{Pic}(X) \to H^1(Z_1\Omega_X^1)$ is injective.

Proposition 4.5. *The natural homomorphism* $H^1(Z_1\Omega_X^1) \to H^2_{DR}(X)$ *is injective.*

Proof. Let $\{U_i\}$ be an affine open covering of X. A Čech cocycle $\{\omega_{ij}\}$ in $H^1(Z_1\Omega_X^1)$ is mapped to $\{(0, \omega_{ij}, 0)\}$ in $H^2_{DR}(X)$. Suppose this element is zero in $H^2_{DR}(X)$. Then there exist elements $(\{f_{ij}\}, \{\omega_i\})$ with $f_{ij} \in \Gamma(U_i \cap U_j, O_X)$ and $\omega_i \in \Gamma(U_i, \Omega_X^1)$ such that

$$f_{jk} - f_{ik} + f_{ij} = 0 \quad \omega_{ij} = df_{ij} + \omega_j - \omega_i \quad d\omega_i = 0.$$

Since $\{f_{ij}\}$ gives an element of $H^1(O_X)$ and $H^1(O_X) = 0$, there exists an element $\{f_i\}$ such that $f_{ij} = f_j - f_i$ on $U_i \cap U_j$. Therefore, we have $\omega_{ij} = (df_j + \omega_j) - (df_i + \omega_i)$. Since $d(df_i + \omega_i) = 0$, we conclude that $\{\omega_{ij}\}$ is zero in $H^1(Z_1\Omega_X^1)$.

The results above imply the following theorem.

Theorem 4.6. *The natural homomorphism* $\mathrm{Pic}(X)/p\mathrm{Pic}(X) \longrightarrow H^2_{DR}(X)$ *is injective.*

Let us point out at this point that for a Calabi-Yau manifold X the Picard group $\mathrm{Pic}(X)$ is reduced and coincides with the Néron-Severi group $NS(X)$ because $NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(X)$ vanishes because of $H^1(X, O_X) = 0$.

Lemma 4.7. *For a Calabi-Yau manifold X of dimension $n \geq 3$ with non non-zero global 1-forms $\mathrm{Pic}(X)$ has no p-torsion.*

Proof. Take an affine open covering $\{U_i\}$ of X. Assume $\{f_{ij}\}$ represents an element $[L] \in \mathrm{Pic}(X)$ which is p-torsion. Then, there exist regular functions $f_i \in H^0(U_i, O_X^*)$ such that $f_{ij}^p = f_i/f_j$. The df_i/f_i on U_i glue together to yield a regular 1-form ω on X. Since $H^0(X, \Omega_X^1) = 0$, we see $\omega = 0$, i.e., $df_i = 0$. Therefore, there exist regular functions $g_i \in H^0(U_i, O_X^*)$ such that $f_i = g_i^p$. Hence, we have $\{f_{ij}\} \sim 0$ and we see that $\mathrm{Pic}(X)$ has no p-torsion.

Lemma 4.8. *Let X be a Calabi-Yau manifold X of dimension $n \geq 3$ with no non-zero global 2-forms. Then, the homomorphism*

$$\mathrm{Pic}(X)/p\mathrm{Pic}(X) \longrightarrow H^1(\Omega_X^1)$$

defined by $\{f_{ij}\} \mapsto \{df_{ij}/f_{ij}\}$ is injective.

Proof. By the assumption $H^0(X, \Omega_X^2) = 0$ we have $H^0(X, d\Omega_X^1) = 0$. Therefore, from the exact sequence

$$0 \to Z_1\Omega_X^1 \longrightarrow \Omega_X^1 \xrightarrow{d} d\Omega_X^1 \to 0,$$

we deduce a natural injection $H^1(Z_1\Omega_X^1) \longrightarrow H^1(\Omega_X^1)$. So the result follows from Lemma 4.4.

Theorem 4.9. *Let X be a Calabi-Yau manifold X of dimension $n \geq 3$ with $H^0(X, \Omega_X^i) = 0$ for $i = 1, 2$. Then the natural homomorphism*

$$NS(X)/pNS(X) \otimes_{\mathbf{F}_p} k \longrightarrow H^1(\Omega_X^1) = H^2_{dR}(X)$$

is injective and the Picard number satisfies $\rho \leq \dim_k H^1(\Omega_X^1)$.

Proof. Suppose that this homomorphism is not injective. Then with respect to a suitable affine open covering $\{U_i\}$ there exist elements $\{f_{ij}^{(\nu)}\}$ representing non-zero elements in $NS(X)/pNS(X)$, such that

$$\sum_{\nu=1}^{\ell} a_\nu df_{ij}^{(\nu)}/f_{ij}^{(\nu)} = 0 \quad \text{in } H^1(\Omega_X^1)$$

for suitable $a_\nu \in k$. We take such elements with the minimal ℓ. We may assume $a_1 = 1$ and we have $a_i/a_j \notin \mathbf{F}_p$ for $i \neq j$. By Lemma 4.8, we have $\ell \geq 2$. There exists $\omega_i \in H^1(U_i, \Omega_X^1)$ such that

$$\sum_{\nu=1}^{\ell} a_\nu df_{ij}^{(\nu)}/f_{ij}^{(\nu)} = \omega_j - \omega_i \tag{1}$$

on $U_i \cap U_j$. There exists an element $\tilde{\omega}_i \in H^1(U_i, \Omega^1_X)$ such that $C(\tilde{\omega}_i) = \omega_i$. Therefore, taking the Cartier inverse, we have

$$\sum_{\nu=1}^{\ell} \tilde{a}_\nu df_{ij}^{(\nu)}/f_{ij}^{(\nu)} + dg_{ij} = \tilde{\omega}_j - \tilde{\omega}_i$$

with $\tilde{a}_\nu \in k$, $\tilde{a}_\nu^p = a_\nu$, and suitable $dg_{ij} \in H^0(U_i \cap U_j, dO_X)$. Since $\{df_{ij}^{(\nu)}/f_{ij}^{(\nu)}\}$ is a cocycle, we see that $\{dg_{ij}\} \in H^1(X, dO_X)$. Since $H^1(X, dO_X) = 0$, there exists an element $dg_i \in H^0(U_i, dO_X)$ such that $dg_{ij} = dg_j - dg_i$. Therefore, we have

$$\sum_{\nu=1}^{\ell} \tilde{a}_\nu df_{ij}^{(\nu)}/f_{ij}^{(\nu)} = (\tilde{\omega}_j - dg_j) - (\tilde{\omega}_i - dg_i). \tag{2}$$

Subtracting (2) from (1) we get a non-trivial linear relation with a smaller ℓ in $H^1(\Omega^1_X)$, a contradiction.

Remark. In the case of a K3 surface X the natural homomorphism

$$NS(X)/pNS(X) \otimes_{\mathbf{F}_p} k \longrightarrow H^2_{DR}(X)$$

is not injective if X is supersingular in the sense of Shioda. Ogus showed that the kernel can be used for describing the moduli of supersingular K3 surfaces, cf. Ogus[10]. So the situation is completely different in dimension ≥ 3.

5. Fermat Calabi-Yau manifolds

Again p is a prime number and m a positive integer which is prime to p. Let f be a smallest power of p such that $p^f \equiv 1 \bmod m$ and put $q = p^f$. We denote by $\mathbf{F}_q$ a finite field of cardinality q. We consider the Fermat variety $X^r_m(p)$ over $\mathbf{F}_q$ defined by

$$X_0^m + X_1^m + \ldots + X_{r+1}^m = 0$$

in projective space $\mathbf{P}^{r+1}$ of dimension $r+1$. The zeta function of X^r_m over $\mathbf{F}_q$ was calculated by A. Weil (cf. [18]). The result is:

$$Z(X^r_m/\mathbf{F}_q, T) = \frac{P(T)^{(-1)^{r-1}}}{(1-T)(1-qT)\ldots(1-q^rT)},$$

where $P(T) = \prod_\alpha (1 - j(\alpha)T)$ with the product taken over a set of vectors α and $j(\alpha)$ is a Jacobi sum defined as follows. Consider the set

$$A_{m,r} = \{(a_0, a_1, \ldots, a_{r+1}) \in \mathbf{Z}^{r+2} \ \mid \ 0 < a_i < m, \textstyle\sum_{j=0}^{r+1} a_j \equiv 0 (\operatorname{mod} m)\},$$

and choose a character $\chi : \mathbf{F}_q^* \to \mathbf{C}^*$ of order m. For $\alpha = (a_0, a_1, \ldots, a_{r+1}) \in A_{m,r}$ we define

$$j(\alpha) = (-1)^r \sum \chi(v_1^{a_1}) \ldots \chi(v_{r+1}^{a_{r+1}}),$$

where the summation runs over $v_i \in \mathbf{F}_q^*$ with $1 + v_1 + \ldots + v_{r+1} = 0$. Thus the $j(\alpha)$'s are eigenvalues of the Frobenius map over $\mathbf{F}_q$ on the ℓ-adic étale cohomology group $H^r_{et}(X^r_m, \mathbf{Q}_\ell)$.

Now, let $\zeta = \exp(2\pi i/m)$ be a primitive m-th root of unity, and $K = \mathbf{Q}(\zeta)$ the corresponding cyclotomic field with Galois group $G = \mathrm{Gal}(K/\mathbf{Q})$. For an

element $t \in (\mathbf{Z}/m\mathbf{Z})^*$ we let σ_t be the automorphism of K defined by $\zeta \mapsto \zeta^t$. The correspondence $t \leftrightarrow \sigma_t$ defines an isomorphism $(\mathbf{Z}/m\mathbf{Z})^* \cong G$ and we shall identify G with $(\mathbf{Z}/m\mathbf{Z})^*$ by this isomorphism. We define a subgroup H of order f of G by $H = \{p^j \bmod m \mid 0 \le j < f\}$
Let $\{t_1, \dots, t_g\}$ with $t_i \in \mathbf{Z}/m\mathbf{Z}^*$, be a complete system of representatives of G/H with $g = |G/H|$, and put

$$A_H(\alpha) = \sum_{t \in H} [\sum_{j=1}^{r+1} \langle ta_j/m \rangle],$$

where $[a]$ (resp. $\langle a \rangle$) means the integral part (resp. the fractional part) of a rational number a.
Choose a prime ideal $\mathcal{P}$ in K lying over p; it has norm $N(\mathcal{P}) = p^f = q$. If $\mathcal{P}_i$ denotes the prime ideal $\mathcal{P}^{\sigma^{-1}_{-t_i}}$ we have the prime decomposition $(p) = \mathcal{P}_1 \cdots \mathcal{P}_g$ in K and Stickelberger's theorem tells us that

$$(j(\alpha)) = \prod_{i=1}^{g} \mathcal{P}_i^{A_H(t_i \alpha)},$$

where $t_i\alpha = (t_i a_0, \dots, t_i a_{r+1})$. For the details we refer to Lang[7] or Shioda-Katsura[16].
Now we restrict our attention to Fermat Calabi-Yau manifolds $X^r_m(p)$ with $m = r+2$.

Theorem 5.1. *Assume $r \ge 2$. Let Φ^r be the Artin-Mazur formal group of the r-dimensional Calabi-Yau variety $X = X^r_{r+2}(p)$. The height h of Φ^r is equal to either 1 or ∞. Moreover, $h = 1$ if and only if $p \equiv 1 \pmod{r+2}$.*

Before we give the proof of this theorem we state a technical lemma.

Lemma 5.2. *Under the notation above, assume $[\sum_{j=1}^{r+1} \langle ta_j/(r+2) \rangle] = 0$ with $t \in (\mathbf{Z}/(r+2)\mathbf{Z})^*$. Then $a_j = t^{-1}$ in $(\mathbf{Z}/(r+2)\mathbf{Z})^*$ for all $j = 0, 1, \dots, r+1$.*

Proof. Since $t \in (\mathbf{Z}/(r+2)\mathbf{Z})^*$, we have $\langle ta_j/(r+2) \rangle \ge 1/(r+2)$. Suppose there exists an index i such that $ta_i \not\equiv 1 \pmod{r+2}$. Then we have the inequality $\langle ta_i/(r+2) \rangle \ge 2/(r+2)$ and thus $\sum_{j=1}^{r+1} \langle ta_\rho/(r+2) \rangle \ge 1$, which contradicts the assumption. So we have $ta_i \equiv 1 \pmod{r+2}$ and $a_j \equiv t^{-1}$ for $j = 1, \dots, r+1$. Since $a_0 + a_1 + \ldots + a_{r+1} \equiv 0 \pmod{r+2}$, we conclude $a_0 \equiv t^{-1}$.
Proof of the theorem. The Dieudonné module $D(\Phi^r)$ of Φ^r is isomorphic to $H^r(X, WO_X)$. We denote by Q(W) the quotient field of the Witt ring $W(k)$ of k. Then, if $h < \infty$, we have

$$h = \dim_{Q(W)} H^r(X, WO_X) \otimes_{W(k)} Q(W).$$

and by Illusie [6] we know we have

$$H^r(X, WO_X) \otimes_{W(k)} Q(W) \cong H^r_{\text{cris}}(X) \otimes Q(W)_{[0,1[}.$$

According to Artin-Mazur [2], the slopes of $H^r_{\text{cris}}(X) \otimes Q(W)$ are given by $(\text{ord}_{\mathcal{P}} q)/f$ and the $(\text{ord}_{\mathcal{P}} j(\alpha))/f$. Hence, the height h is equal to the number of $j(\alpha)$ such that $A_H(\alpha) < f$.

First, assume $p \equiv 1 \pmod{r+2}$, i.e. $f = 1$. Then $H = \langle 1 \rangle$ and $A_H(\alpha) < f = 1$ implies $A_H(\alpha) = 0$. Therefore, by Lemma 5.2, we have $a_j = 1$ for all $j = 0, 1, \ldots, r+1$ and there is only one α, namely $\alpha = (1, 1, \ldots, 1)$, such that $\mathrm{ord}_{\mathcal{P}} j(\alpha) = 0$. So we conclude $h = 1$ in this case.
Secondly, assume $p \not\equiv 1 \pmod{r+2}$. By definition, we have $f \geq 2$. We now prove that there exists no α such that $A_H(\alpha) < f$. Suppose $A_H(\alpha) = \sum_{t \in H}[\sum_{j=1}^{r+1}\langle ta_j/(r+2)\rangle] < f$. Then there exists an element $t \in H$ such that $[\sum_{j=1}^{r+1}\langle ta_j/(r+2)\rangle] = 0$. By Lemma 5.2 we have $\alpha = (t^{-1}, t^{-1}, \ldots, t^{-1})$. For $t' \in H$ with $t' \neq t$ we have $[\sum_{\rho=1}^{r+1}\langle t't^{-1}/(r+2)\rangle] \neq 0$. Therefore the inequality yields $[\sum_{j=1}^{r+1}\langle t't^{-1}/(r+2)\rangle] = 1$ for $t' \in H$, $t' \neq t$. Since $A_H(t\alpha) = A_H(\alpha)$ for any $t \in H$, by a translation by t, we may assume $\alpha = (1, 1, \ldots, 1)$, i.e., $t = 1$. Moreover, we can take a representative of $t' \in H$ such that $0 < t' < r+2$. Then,

$$\begin{aligned} 1 &= [\sum_{j=1}^{r+1}\langle t't^{-1}/(r+2)\rangle] = [\sum_{j=1}^{r+1}\langle t'/(r+2)\rangle] = [\sum_{j=1}^{r+1} t'/(r+2)] \\ &= [(r+1)t'/(r+2)] \end{aligned}$$

and we get $1 \leq (r+1)t'/(r+2) < 2$. By this inequality, we see $t' = 2$. Therefore, we have $H = \{1, 2\}$. Since H is a subgroup of $(\mathbf{Z}/(r+2)\mathbf{Z})^*$, we see that $2^2 \equiv 1 \bmod r+2$. Therefore, we have $r = 1$, which contradicts our assumption.
Hence there exists no α such that $\mathrm{ord}_{\mathcal{P}} j(\alpha) < 1$ and we conclude $h = \infty$ in this case. This completes the proof of the theorem.

For K3 surfaces we have two notions of supersingularity. We generalize these to higher dimensions.

Definition 5.3. A Calabi-Yau manifold X of dimension r is said to be of *additive Artin-Mazur type* ('supersingular in the sense of Artin') if the height of Artin-Mazur formal group associated with $H^r(X, O_X)$ is equal to ∞.

Definition 5.4. A non-singular complete algebraic variety X of dimension r is said to be *fully rigged* ('supersingular in the sense of Shioda') if all the even degree étale cohomology groups are spanned by algebraic cycles.

By the theorem above, we know that the Fermat Calabi-Yau manifolds are of additive Artin-Mazur type if and only if $p \not\equiv 1 \bmod m$ with $m = r+2$. As to being fully rigged we have the following theorem.

Theorem 5.5 (Shioda-Katsura [16]). *Assume $m \geq 4$, $(p, m) = 1$ and r is even. Then the Fermat variety $X_m^r(p)$ is fully rigged if and only if there exists a positive integer ν such that $p^\nu \equiv -1 \bmod m$.*

M. Artin conjectured that a K3 surface X is supersingular in the sense of Artin if and only if X is supersingular in the sense of Shioda. He also showed that "if part" holds. In the case of the Fermat K3 surface, i.e, $X_4^2(p)$, by the two theorems above, we see, as is well-known, that the Artin conjecture holds.

However, in the case of even $r \geq 4$, the above two theorems imply that this straightforward generalization of the Artin conjecture to higher dimension does not hold.

6. Kummer Calabi-Yau manifolds

Let A be an abelian variety of dimension $n \geq 2$ defined over an algebraically closed field of characteristic $p > 0$, and G be a finite group which acts on A faithfully. Assume that the order of G is prime to p, and that the quotient variety A/G has a resolution which is a Calabi-Yau manifold X. We call X a Kummer Calabi-Yau manifold. We denote by $\pi : A \longrightarrow A/G$ the projection, and by $\nu : X \longrightarrow A/G$ the resolution.

Theorem 6.1. *Under the assumptions above the Artin-Mazur formal group Φ^n_X is isomorphic to the Artin-Mazur formal group Φ^n_A.*

Proof. Since the order of G is prime to p, the singularities of A/G are rational, and we have $R^i\nu_* O_X = 0$ for $i \geq 1$. So by the Leray spectral sequence we have $H^n(A/G, O_{A/G}) \cong H^n(X, O_X) \cong k$ and $H^{n-1}(A/G, O_{A/G}) \cong H^{n-1}(X, O_X) \cong 0$. It follows that the Artin-Mazur formal group $\Phi^n_{A/G}$ is pro-representable by a formal Lie group of dimension 1 (cf. Artin-Mazur[2]). Since the tangent space $H^n(A/G, O_{A/G})$ of $\Phi^n_{A/G}$ is naturally isomorphic to the tangent space $H^n(X, O_X)$ of Φ^n_X as above, the natural homomorphism from $\Phi^n_{A/G}$ to Φ^n_X is non-trivial. One-dimensional formal groups are classified by their height and between formal groups of different height there are no non-trivial homomorphisms . So the height of $\Phi^n_{A/G}$ is equal to that of Φ^n_X and we thus see that $\Phi^n_{A/G}$ and Φ^n_X are isomorphic.

Since the order of G is prime to p, there is a non-trivial trace map from $H^n(A, O_A)$ to $H^n(A/G, O_{A/G})$. Therefore, $\pi^* : H^n(A/G, O_{A/G}) \longrightarrow H^n(A, O_A)$ is an isomorphism. Therefore, as above we see that the height of $\Phi^n_{A/G}$ is equal to the height of Φ^n_A, and that Φ^n_X is isomorphic to Φ^n_A. Q.e.d.

Though the following lemma might be well-known to specialists we give here a proof for the reader's convenience.

Lemma 6.2. *Let A be an abelian variety of dimension $n \geq 2$ and p-rank $f(A)$. The height h of the Artin-Mazur formal group Φ_A of A is as follows:*

(1) *$h = 1$ if A is ordinary, i.e., $f(A) = n$,*
(2) *$h = 2$ if $f(A) = n - 1$,*
(3) *$h = \infty$ if $f(A) \leq n - 2$.*

Proof. We denote by $H^i_{\mathrm{cris}}(A)$ the i-th cristalline cohomology of A and as usual by $H^i_{\mathrm{cris}}(A)_{[\ell,\ell+1[}$ the additive group of elements in $H^i_{\mathrm{cris}}(A)$ whose slopes are in the interval $[\ell, \ell + 1[$. By the general theory in Illusie [6], we have

$$H^n(A, W(O_A)) \otimes_W Q(W) \cong (H^n_{\mathrm{cris}}(A) \otimes_W Q(W))_{[0,1[}$$

with $Q(W)$ the quotient field of W. The theory of Dieudonné modules implies

$$h = \dim_{Q(W)} D(\Phi_A) = \dim_{Q(W)} H^n(A, W(O_A)) \otimes_W Q(W) \quad \text{if } h < \infty,$$

and $\dim_{Q(W)} D(\Phi_A) = 0$ if $h = \infty$. We know the slopes of $H^1_{\text{cris}}(A)$ for each case. Since we have

$$H^n_{\text{cris}}(A) \cong \wedge^n H^1_{\text{cris}}(A),$$

counting the number of slopes in $[0, 1[$ of $H^n_{\text{cris}}(A)$ gives the result.

Corollary 6.3. *Let X be a Kummer Calabi-Yau manifold of dimension n obtained from an abelian variety A as above. Then the height of the Artin-Mazur formal group Φ^n_X is equal to either 1, 2 or ∞.*

Example 6.4. Assume $p \geq 3$. Let A be an abelian surface and ι the map $A \to A$ sending $a \in A$ to its inverse $-a \in A$. We denote by $Km(A)$ the Kummer surface of A, i.e., the minimal resolution of $A/\langle\iota\rangle$. Then $\Phi^2_{Km(A)}$ is isomorphic to Φ^2_A.

Example 6.5. Assume $p \geq 5$, and let ω be a primitive third root of unity. Let E be a non-singular complete model of the elliptic curve defined by $y^2 = x^3 + 1$, and let σ be an automorphism of E defined by $x \mapsto \omega x$, $y \mapsto y$. We set $A = E^3$ and put $\tilde{\sigma} = \sigma \times \sigma \times \sigma$. The minimal resolution X of $A/\langle\tilde{\sigma}\rangle$ is a Calabi-Yau manifold, and the Artin-Mazur formal group Φ^3_X is isomorphic to Φ^3_A.

Let ω be a complex number with positive imaginary part, and $L = \mathbf{Z} + \mathbf{Z}\omega$ be a lattice in the complex numbers $\mathbf{C}$. From here on, we consider an elliptic curve $E = \mathbf{C}/L$, and we assume that E has a model defined over an algebraic number field K. Then $A = E \times E \times E$ is an abelian threefold, and we let $G \subseteq \mathrm{Aut}_K(E)$ be a finite group which faithfully acts on A. We assume that G has only isolated fixed points on A and that the quotient variety A/G has a crepant resolution $\nu : X \to A/G$ defined over K, [12]. We denote by π the projection $A \to A/G$. For a prime p of K, we denote by $\bar{X}$ the reduction modulo p of X.

Example 6.6. [K. Ueno [17]] Assume that E is an elliptic curve defined over $\mathbf{Q}$ having complex multiplication $\sigma : E \to E$ by a primitive third root of unity. Then $G = \mathbf{Z}/3\mathbf{Z} = \langle\sigma\rangle$ acts diagonally on $A = E^3$. A crepant resolution of A/G gives a rigid Calabi-Yau manifold defined over $\mathbf{Q}$. For a prime number $p \geq 5$ the reduction modulo p of A is the abelian threefold given in Example 6.5.

Theorem 6.7. *Let X be a Calabi-Yau obtained as crepant resolution of A/G as above. Assume moreover that X is rigid. Then the elliptic curve E has complex multiplication and the intermediate Jacobian of X is isogenous to E.*

Corollary 6.8. *Under the assumptions as in the theorem, we take a prime p of good reduction for X and let $\bar{X}$ be the reduction of X modulo p. Then the height of the formal group $\Phi_{\bar{X}}$ is either 1 or ∞. It is ∞ if and only if the reduction of the intermediate Jacobian variety of X at p is a supersingular elliptic curve.*

Example 6.9. We consider the reduction $\bar{X}$ modulo p of the variety X in the Example 6.6. We assume the characteristic of the residue field of p is not equal to 2 and 3. Then the height $h(\Phi_{\bar{X}}) = \infty$ if and only if the reduction modulo p

of the intermediate Jacobian of X is a supersingular elliptic curve, and this is the case if and only if $p \equiv 2 \pmod 3$.

Before we prove the theorem we introduce some notation. We have a natural identification $H_1(E, \mathbf{Z}) = \mathbf{Z} + \mathbf{Z}\omega$. Fixing a non-zero regular differential form η on E determines a regular three form $p_1^*\eta \wedge p_2^*\eta \wedge p_3^*\eta = \Omega_A$ on A. We have a natural homomorphism

$$H^3(X, \mathbf{Z}) \to H^3_{dR}(X) \to H^3(X, O_X).$$

If X is rigid, the corresponding quotient $H^3(X, O_X)/H^3(X, \mathbf{Z})$ gives the intermediate Jacobian of X. Since $\dim_{\mathbf{C}} H^3(X, O_X) = 1$, the intermediate Jacobian of X is isomorphic to an elliptic curve.
We can define the period map with respect to Ω_A:

$$\pi_A : H_3(X, \mathbf{Z}) \longrightarrow \mathbf{C} \qquad \gamma \mapsto \int_\gamma \Omega_A.$$

By Poincaré duality we can identify π_A with the natural projection $H^3(X, \mathbf{Z}) \to H^3(X, \mathbf{C}) \to H^3(X, O_X) = \mathbf{C}$ (cf. Shioda [15], for instance.) There exists a regular 3-form Ω_X on X such that $\Omega_A = (\nu^{-1} \circ \pi)^*\Omega_X$. We can define π_X with respect to Ω_X for the Calabi-Yau manifold X as well.
In order to describe the structure of the intermediate Jacobian of X we look at the period map of an abelian threefold, following the method in Shioda [15] (also see Mumford [8]). Choose a basis u_1, u_2 of $H_1(E, \mathbf{R})$. This determines a $\mathbf{C}$-basis $H_1(E, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = H_1(E, \mathbf{C})$. If e_i for $i = 1, 2, 3$ is the standard basis of $\mathbf{C}^3$ then $u_{2i-1} = e_i$ and $u_{2i} = \omega e_i$ for $i = 1, 2, 3$ form a basis of $H_1(A, \mathbf{Z})$ and $A = \mathbf{C}^3/M$ with M the lattice generated by $u_1, \ldots, u_6$. The dual basis is denoted by v_i. The basis of $H^1(A, \mathbf{Z})$ determines a canonical basis $v_i \wedge v_j \wedge v_k$ of $H^3(A, \mathbf{Z})$. The natural homomorphism

$$p_A : H^3(A, \mathbf{Z}) \longrightarrow H^3_{dR}(A) \longrightarrow H^3(A, O_A) \cong \mathbf{C}.$$

is an element of $\mathrm{Hom}_{\mathbf{C}}(H^3(A, \mathbf{C}), \mathbf{C})$ and can be considered as an element of $H^3(A, \mathbf{C})$ and is given by

$$p_A = \sum_{i<j<k} \det(u_i, u_j, u_k) v^i \wedge v^j \wedge v^k.$$

Therefore the image of p_A in $\mathbf{C}$ is spanned by the complex numbers 1, ω, ω^2 and ω^3 over $\mathbf{Z}$.
We now give the proof of Theorem 6.7. Let S be the set of non-free points of the action of G on A. Then the restriction of π to $A \setminus S$ is étale on $A/G \setminus \pi(S)$. Since S is of codimension 3 in A, we have the following diagram:

$$\begin{array}{ccccc}
H^3(A, \mathbf{Z}) & \cong & H^3(A \setminus S, \mathbf{Z}) & \xrightarrow{p_A} & H^3(A, O_A) \cong \mathbf{C} \\
\downarrow \pi_* & & \downarrow (\pi\,|_{A\setminus S})_* & & \downarrow \pi_* \\
H^3(A/G, \mathbf{Z}) & \cong & H^3(A/G \setminus \pi(S), \mathbf{Z}) & \xrightarrow{p_{A/G}} & H^3(A/G, O_{A/G}) \cong \mathbf{C} \\
\downarrow \cong & & & & \downarrow \nu^* \\
H^3(X, \mathbf{Z}) & & & \xrightarrow{p_X} & H^3(X, O_X) \cong \mathbf{C}.
\end{array}$$

The vertical arrows on the right hand side give an identification of $H^3(A, O_A)$ and $H^3(X, O_X)$. Because p does not divide the order of G we see that $H^3(A, \mathbb{Q})$ maps surjectively to $H^3(A/G, \mathbb{Q}) = H^3(A, \mathbb{Q})^G$, hence the image of $H^3(A, \mathbb{Z})$ is commensurable with $H^3(X, \mathbb{Z})$.
Now $\mathrm{Im}\, p_X$ is a lattice in $\mathbf{C}$, and $\mathrm{Im}\, p_A$ is a lattice in $\mathbf{C}$ as well. We know that $\mathrm{Im}\, p_A$ is generated by 1, ω, ω^2 and ω^3 and thus ω is a quadratic number and the intermediate Jacobian has complex multiplication by $\mathbf{Q}(\omega)$. Hence the intermediate Jacobian $\mathbf{C}/\mathrm{Im}\, p_X$ of X is isogenous to E.

7. Questions

We close with two natural basic questions that suggest themselves.
Is there a function $f(n)$ such that a Calabi-Yau variety in characteristic $p > 0$ of dimension n lifts to characteristic 0 if $p > f(n)$? Note that Hirokado constructed a non-liftable Calabi-Yau threefold in characteristic 3, see [5] (see also [13]).
Can a Calabi-Yau variety of dimension 3 in positive characteristic have non-zero regular 1-forms or regular 2-forms?

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