Quantum optics and multiple scattering in dielectrics

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Appendix C

Dyadic Green and delta functions

The (full) Green tensor $\mathbf{G}(\mathbf{r},\mathbf{r}',\omega)$ of an inhomogeneous medium characterized by the dielectric function $\varepsilon(\mathbf{r})$ is the solution of the wave equation

$$-\nabla\times\nabla\times\mathbf{G}(\mathbf{r},\mathbf{r}',\omega)+\varepsilon(\mathbf{r})(\omega/c)^2\mathbf{G}(\mathbf{r},\mathbf{r}',\omega)=\delta(\mathbf{r}-\mathbf{r}')\mathbf{I}, \qquad \text{(C.1)}$$

where the right-hand side is the ordinary Dirac delta function times the unit tensor. Another useful Green function (which actually is a tensor as well) can be found by projecting out the left and right-hand sides of this equation with the generalized transverse delta function (4.12). In doing so, the transverse double-curl term is projected onto itself, see Eq. (4.15a). The (full) Green function can therefore be uniquely projected onto its generalized transverse part $\mathbf{G}^{\mathrm{T}}$ that is the solution of

$$-\nabla\times\nabla\times\mathbf{G}^{\mathrm{T}}(\mathbf{r},\mathbf{r}',\omega)+\varepsilon(\mathbf{r})(\omega/c)^2\mathbf{G}^{\mathrm{T}}(\mathbf{r},\mathbf{r}',\omega)=\bar{\boldsymbol{\delta}}_{\varepsilon}^{\mathrm{T}}(\mathbf{r}',\mathbf{r}). \qquad \text{(C.2)}$$

The bar in $\bar{\boldsymbol{\delta}}_{\varepsilon}^{\mathrm{T}}$ denotes the transpose. The longitudinal Green function $\mathbf{G}^{\mathrm{L}}$ is now defined as the difference between $\mathbf{G}$ and $\mathbf{G}^{\mathrm{T}}$, and has the form

$$\mathbf{G}^{\mathrm{L}}(\mathbf{r},\mathbf{r}')\equiv\frac{1}{(\omega/c)^2\varepsilon(\mathbf{r})}\left[\delta(\mathbf{r}-\mathbf{r}')\mathbf{I}-\bar{\boldsymbol{\delta}}_{\varepsilon}^{\mathrm{T}}(\mathbf{r}',\mathbf{r})\right]\equiv\frac{1}{(\omega/c)^2\varepsilon(\mathbf{r})}\bar{\boldsymbol{\delta}}_{\varepsilon}^{\mathrm{L}}(\mathbf{r}',\mathbf{r}). \qquad \text{(C.3)}$$

In the last equality of (C.3) the generalized longitudinal delta function was defined as the difference between the ordinary Dirac and the generalized transverse delta, so that $\boldsymbol{\delta}_{\varepsilon}^{\mathrm{T}}+\boldsymbol{\delta}_{\varepsilon}^{\mathrm{L}}=\delta\mathbf{I}$. It is not immediately clear that $\mathbf{G}^{\mathrm{L}}$ is longitudinal. To prove that indeed it has zero curl, one can apply the unique decomposition of generalized transverse and longitudinal vector fields (as explained in appendix B) to $\int \mathrm{d}\mathbf{r}\mathbf{G}^{\mathrm{L}}\cdot\mathbf{X}$, where $\mathbf{X}$ is a general vector field.

The Green tensor $\mathbf{G}^{\mathrm{T}}$ of the dielectric can be expanded either in terms of the real mode functions $\mathbf{h}_\lambda$ that were introduced in section 4.2.2, or in terms of complex mode functions $\mathbf{f}_\lambda$ which are used in chapter 5:

$$\mathbf{G}^{\mathrm{T}}(\mathbf{r},\mathbf{r}',\omega)=c^2\sum_\lambda\frac{\mathbf{h}_\lambda(\mathbf{r})\,\mathbf{h}_\lambda(\mathbf{r}')}{(\omega+i\eta)^2-\omega_\lambda^2}=c^2\sum_\lambda\frac{\mathbf{f}_\lambda(\mathbf{r})\,\mathbf{f}_\lambda^*(\mathbf{r}')}{(\omega+i\eta)^2-\omega_\lambda^2}. \qquad \text{(C.4)}$$

Both these expansions are equivalent by Eq. (4.23), be it that the real and imaginary parts can be read off more easily in the former form. The term $i\eta$ in (C.4) makes explicit the positive and infinitesimally small imaginary part of the frequency ω. With the positive sign, (C.4) is the causal Green function which transformed back to the time-domain gives a Green function $\mathbf{G}^{\mathrm{T}}(\mathbf{r}, \mathbf{r}', t - t_0)$ which is nonzero only for positive time differences $(t - t_0)$: a cause at time t_0 can only have an effect at later times.

For free space, the transverse and longitudinal delta functions appearing in Eqs. (C.2) and (C.3) are [85]

$$\boldsymbol{\delta}^{\mathrm{T}}(\mathbf{r}) = \frac{2}{3}\delta(\mathbf{r})\mathbf{I} - \frac{1}{4\pi r^3}(\mathbf{I} - 3\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}}) \tag{C.5}$$

$$\boldsymbol{\delta}^{\mathrm{L}}(\mathbf{r}) = \frac{1}{3}\delta(\mathbf{r})\mathbf{I} + \frac{1}{4\pi r^3}(\mathbf{I} - 3\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}}). \tag{C.6}$$

where $\hat{\boldsymbol{r}}$ is defined as $\mathbf{r}/|\mathbf{r}|$, the unit vector in the direction of $\mathbf{r}$. The sum of the transverse and the longitudinal delta function is simply $\delta(\mathbf{r})\mathbf{I}$, since their "dipole" parts cancel. The dyadic Green function $\mathbf{G}_0$ for free space is the sum of a transverse and a longitudinal part. The transverse part is [69]

$$\mathbf{G}_0^{\mathrm{T}}(\mathbf{r}, \omega) = -\frac{\mathbf{I} - 3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}}{4\pi(\omega/c)^2 r^3} - \frac{e^{i\omega r/c}}{4\pi r}\left[P(i\omega r/c)\mathbf{I} + Q(i\omega r/c)\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right], \tag{C.7}$$

with the function $P(z)$ defined as $\equiv (1 - z^{-1} + z^{-2})$ and $Q(z)$ as $(-1 + 3z^{-1} - 3z^{-2})$. Using the definition (C.3) of the longitudinal Green function and the free-space transverse delta function (C.5), the longitudinal Green function is found to be

$$\mathbf{G}_0^{\mathrm{L}}(\mathbf{r}, \omega) = \frac{\mathbf{I} - 3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}}{4\pi(\omega/c)^2 r^3} + \frac{\delta(\mathbf{r})}{3(\omega/c)^2}\mathbf{I}. \tag{C.8}$$

The important delta-function term in $\mathbf{G}_0^{\mathrm{L}}$ appears naturally and there was no need to add it "by hand" as is done elsewhere [4, 69]. Both $\mathbf{G}_0^{\mathrm{T}}$ and $\mathbf{G}_0^{\mathrm{L}}$ have nonretarded dipole terms, meaning that a change in a source term changes instantaneously the longitudinal and transverse fields elsewhere. It is only their sum that is fully retarded [85]. The same is true for the Green functions $\mathbf{G}^{\mathrm{T}}$ and $\mathbf{G}^{\mathrm{L}}$ of inhomogeneous dielectrics.

In some cases, it is physical to replace the Green functions (C.7) and (C.8) by their angle-averaged values. The angle-averaged value of $(\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{r}})^2$ is simply $\frac{1}{3}$. Effectively, the averaging amounts to replacing $\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}$ by $\frac{1}{3}\mathbf{I}$ both in Eq. (C.7) and in (C.8). The angle-averaged free-space Green function is simply

$$\langle\, \mathbf{G}_0(\mathbf{r}, \omega)\, \rangle_{\text{av.}} = -\frac{e^{i\omega r/c}}{6\pi r}\mathbf{I} + \frac{c^2}{3\omega^2}\delta(\mathbf{r})\mathbf{I}. \tag{C.9}$$

In the limit $\mathbf{r} \longrightarrow 0$, the imaginary part of this angle-averaged Green function still leads to the correct free-space spontaneous-emission rates.