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### Quantum optics and multiple scattering in dielectrics

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## Appendix C

# Dyadic Green and delta functions

The (full) Green tensor  $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$  of an inhomogeneous medium characterized by the dielectric function  $\varepsilon(\mathbf{r})$  is the solution of the wave equation

$$-\nabla \times \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) + \varepsilon(\mathbf{r})(\omega/c)^2 \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}')\mathbf{I}, \quad (\text{C.1})$$

where the right-hand side is the ordinary Dirac delta function times the unit tensor. Another useful Green function (which actually is a tensor as well) can be found by projecting out the left and right-hand sides of this equation with the generalized transverse delta function (4.12). In doing so, the transverse double-curl term is projected onto itself, see Eq. (4.15a). The (full) Green function can therefore be uniquely projected onto its generalized transverse part  $\mathbf{G}^T$  that is the solution of

$$-\nabla \times \nabla \times \mathbf{G}^T(\mathbf{r}, \mathbf{r}', \omega) + \varepsilon(\mathbf{r})(\omega/c)^2 \mathbf{G}^T(\mathbf{r}, \mathbf{r}', \omega) = \bar{\delta}_\varepsilon^T(\mathbf{r}', \mathbf{r}). \quad (\text{C.2})$$

The bar in  $\bar{\delta}_\varepsilon^T$  denotes the transpose. The longitudinal Green function  $\mathbf{G}^L$  is now defined as the difference between  $\mathbf{G}$  and  $\mathbf{G}^T$ , and has the form

$$\mathbf{G}^L(\mathbf{r}, \mathbf{r}') \equiv \frac{1}{(\omega/c)^2 \varepsilon(\mathbf{r})} [\delta(\mathbf{r} - \mathbf{r}')\mathbf{I} - \bar{\delta}_\varepsilon^T(\mathbf{r}', \mathbf{r})] \equiv \frac{1}{(\omega/c)^2 \varepsilon(\mathbf{r})} \bar{\delta}_\varepsilon^L(\mathbf{r}', \mathbf{r}). \quad (\text{C.3})$$

In the last equality of (C.3) the generalized longitudinal delta function was defined as the difference between the ordinary Dirac and the generalized transverse delta, so that  $\bar{\delta}_\varepsilon^T + \bar{\delta}_\varepsilon^L = \delta\mathbf{I}$ . It is not immediately clear that  $\mathbf{G}^L$  is longitudinal. To prove that indeed it has zero curl, one can apply the unique decomposition of generalized transverse and longitudinal vector fields (as explained in appendix B) to  $\int d\mathbf{r}' \mathbf{G}^L \cdot \mathbf{X}$ , where  $\mathbf{X}$  is a general vector field.

The Green tensor  $\mathbf{G}^T$  of the dielectric can be expanded either in terms of the real mode functions  $\mathbf{h}_\lambda$  that were introduced in section 4.2.2, or in terms of complex mode functions  $\mathbf{f}_\lambda$  which are used in chapter 5:

$$\mathbf{G}^T(\mathbf{r}, \mathbf{r}', \omega) = c^2 \sum_\lambda \frac{\mathbf{h}_\lambda(\mathbf{r}) \mathbf{h}_\lambda(\mathbf{r}')}{(\omega + i\eta)^2 - \omega_\lambda^2} = c^2 \sum_\lambda \frac{\mathbf{f}_\lambda(\mathbf{r}) \mathbf{f}_\lambda^*(\mathbf{r}')}{(\omega + i\eta)^2 - \omega_\lambda^2}. \quad (\text{C.4})$$

Both these expansions are equivalent by Eq. (4.23), be it that the real and imaginary parts can be read off more easily in the former form. The term  $i\eta$  in (C.4) makes explicit the positive and infinitesimally small imaginary part of the frequency  $\omega$ . With the positive sign, (C.4) is the causal Green function which transformed back to the time-domain gives a Green function  $\mathbf{G}^T(\mathbf{r}, \mathbf{r}', t - t_0)$  which is nonzero only for positive time differences ( $t - t_0$ ): a cause at time  $t_0$  can only have an effect at later times.

For free space, the transverse and longitudinal delta functions appearing in Eqs. (C.2) and (C.3) are [85]

$$\delta^T(\mathbf{r}) = \frac{2}{3}\delta(\mathbf{r})\mathbf{I} - \frac{1}{4\pi r^3}(\mathbf{I} - 3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) \quad (\text{C.5})$$

$$\delta^L(\mathbf{r}) = \frac{1}{3}\delta(\mathbf{r})\mathbf{I} + \frac{1}{4\pi r^3}(\mathbf{I} - 3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}), \quad (\text{C.6})$$

where  $\hat{\mathbf{r}}$  is defined as  $\mathbf{r}/|\mathbf{r}|$ , the unit vector in the direction of  $\mathbf{r}$ . The sum of the transverse and the longitudinal delta function is simply  $\delta(\mathbf{r})\mathbf{I}$ , since their "dipole" parts cancel. The dyadic Green function  $\mathbf{G}_0$  for free space is the sum of a transverse and a longitudinal part. The transverse part is [69]

$$\mathbf{G}_0^T(\mathbf{r}, \omega) = -\frac{\mathbf{I} - 3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}}{4\pi(\omega/c)^2 r^3} - \frac{e^{i\omega r/c}}{4\pi r} [P(i\omega r/c)\mathbf{I} + Q(i\omega r/c)\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}], \quad (\text{C.7})$$

with the function  $P(z)$  defined as  $\equiv (1 - z^{-1} + z^{-2})$  and  $Q(z)$  as  $(-1 + 3z^{-1} - 3z^{-2})$ . Using the definition (C.3) of the longitudinal Green function and the free-space transverse delta function (C.5), the longitudinal Green function is found to be

$$\mathbf{G}_0^L(\mathbf{r}, \omega) = \frac{\mathbf{I} - 3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}}{4\pi(\omega/c)^2 r^3} + \frac{\delta(\mathbf{r})}{3(\omega/c)^2} \mathbf{I}. \quad (\text{C.8})$$

The important delta-function term in  $\mathbf{G}_0^L$  appears naturally and there was no need to add it "by hand" as is done elsewhere [4, 69]. Both  $\mathbf{G}_0^T$  and  $\mathbf{G}_0^L$  have nonretarded dipole terms, meaning that a change in a source term changes instantaneously the longitudinal and transverse fields elsewhere. It is only their sum that is fully retarded [85]. The same is true for the Green functions  $\mathbf{G}^T$  and  $\mathbf{G}^L$  of inhomogeneous dielectrics.

In some cases, it is physical to replace the Green functions (C.7) and (C.8) by their angle-averaged values. The angle-averaged value of  $(\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{r}})^2$  is simply  $\frac{1}{3}$ . Effectively, the averaging amounts to replacing  $\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}$  by  $\frac{1}{3}\mathbf{I}$  both in Eq. (C.7) and in (C.8). The angle-averaged free-space Green function is simply

$$\langle \mathbf{G}_0(\mathbf{r}, \omega) \rangle_{\text{av.}} = -\frac{e^{i\omega r/c}}{6\pi r} \mathbf{I} + \frac{c^2}{3\omega^2} \delta(\mathbf{r}) \mathbf{I}. \quad (\text{C.9})$$

In the limit  $r \rightarrow 0$ , the imaginary part of this angle-averaged Green function still leads to the correct free-space spontaneous-emission rates.