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A careful critical analysis of the complexity, at the annealed level, of the Sherrington-Kirkpatrick model has been performed. The complexity functional is proved to be always invariant under the Becchi-Rouet-Stora-Tyutin supersymmetry, disregarding the formulation used to define it. We consider two different saddle points of such functional, one satisfying the supersymmetry [A. Cavagna *et al.*, J. Phys. A **36**, 1175 (2003)] and the other one breaking it [A. J. Bray and M. A. Moore, J. Phys. C **13**, L469 (1980)]. We review the previews studies on the subject, linking different perspectives and pointing out some inadequacies and even inconsistencies in both solutions.

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I. INTRODUCTION

The organization of thermodynamic states in complex systems, fundamental to the understanding of dynamic properties, is a rather difficult task to deal with. The quenched disorder characterizing these systems and the consequent frustration are such that a huge number of stable and metastable states arises, even growing exponentially with the number of elements composing the system. In order to throw light on the structure of the landscape of the thermodynamic potential, a very important theoretical tool is the logarithm of the number of states, either called *complexity* or *configurational entropy*.

In structural glasses, seen as disordered amorphous solids and, thus, treated by the techniques of complex systems, at the glass transition the entropy drops to a (often much) smaller value, going from the liquid to the solid state and the states, as opposed, e.g., to the crystal states do not display any specific symmetry. The condensed system has lost the ability of visiting different states (at least on the observation time scale considered) and this codifies into the loss of entropy. All the other possible states, not selected at the moment of the transition of the liquid to a glass, are, anyway, still there from a statistical point of view and could, in principle, still be reached on much larger time scales. The complexity counts the many equivalent states that could have been chosen at the moment of the quench.

In disordered mean-field models for glasses, e.g., the *p*-spin interaction spin glass models with stable one-step replica symmetry breaking (1RSB) frozen phase,[1–3] the presence of many metastable states can be detected looking at the relaxation that displays a dynamical transition, with diverging relaxation time scales, at a temperature (the *dynamical temperature*) above the point where a true thermodynamic transition takes place, called *static temperature*. In the range between the static and dynamic temperatures the complexity displayed turns out to be extensive.

What happens moving from 1RSB-stable spin-glass models to spin-glass mean-field models whose frozen phase is described by means of a full replica symmetry breaking (FRSB) solution?

In the last 20 years basically two different proposals have been put forward for the nature and behavior of the complexity of the spin-glass mean-field Sherrington-Kirkpatrick (SK) model,[4] the prototype of mean-field spin-glass models. The first one was originally presented by Bray and Moore[5] (BM); there the complexity was analyzed both in the *annealed* approximation—i.e., as the logarithm of the disordered average of the number of states—and as quenched average—i.e., the average of the logarithm—performed making use of the replica trick in the case of replica symmetry. The second one was initially proposed in Ref. 6, where Parisi and Potters showed that the complexity could be obtained by calculating the partition function of *m* distinct real replicas of the system[7] and provided the connection with the previous formalism by means of the generalization of the *two-group ansatz*.[8] In that context the annealed solution was called ''unbroken'' two-group.

Over the years it has become more and more evident that an important role in the study of the complexity of disordered systems is played by the so-called Becchi-Rouet-Stora-Tyutin (BRST) symmetry. Such a property, first discovered in the quantization of gauge theories,[9] is a *supersymmetry* (susy), in the sense that it transforms bosons into fermions and vice versa. In the context of stochastic field equations, it can be shown that the integration of the generating functional of correlation functions over disorder leads to an action presenting BRST symmetry[10] (for the random field Ising model case see also Ref. 11). The integrated generating functional formally coincides with the average over the quenched random couplings of the number of metastable states of mean-field spin-glass models. In such a context, the property of BRST-SUSY invariance has recently been analyzed in Refs. 12 and 13. In this approach, imposing the invariance helps in simplifying the computation and it is equivalent to setting the due interdependence between the physical objects deriving from the Thouless-Anderson-Palmer (TAP) free energy functional and composing the action—namely, the TAP functional itself, its first derivatives with respect to the average site magnetization (i.e., the TAP equations), and its Hessian. In Ref. 13 a BRST symmetric annealed complexity is built, while the BM complexity functional appears to violate such symmetry.

Generally speaking, the fact that the solutions of a set of equations are BRST invariant is connected with some robustness of the equations under linear perturbation. Let us take

into account the equations $\partial_{m_i} F = 0$, where F is the thermodynamic functional and the derivative is taken with respect to the microscopic variable m_i, $i = 1,\ldots,N$. If we perturb the equations by means of small external fields, $\partial_{m_i} F = 0 \rightarrow \partial_{m_i} F = h_i$, the whole set of solutions could, in principle, drastically change. In general, solutions will appear or disappear at different values of F with no given prescription for the relative transformation of the complexity function and its dominion. The BRST SUSY can be recovered assuming that the number of solutions at a given value of F does not change. In Sec. II D we present an argument to explain such correspondence.

Both the approach of Ref. 5 and the one of Ref. 13 identify an extensive complexity, computing the number of solutions of TAP equations[14] in the annealed approximation and with the further simplification of neglecting the modulus of the determinant of the Hessian of the TAP free energy functional. However, the two complexities display many differences. They have a different lower band-edge value of the complexity curve versus free energy and a different dominion in free energy, with different thresholds. Their magnitude is different, BM complexity being much larger than the one satisfying the BRST SUSY. Performing the integration over the whole interval of possible free energy values yields a finite result in the BM case, but zero in the BRST-SUSY one.[15] At the threshold, the behavior is once again qualitatively different: the BM complexity goes to zero smoothly, whereas the BRST-SUSY one drops discontinuously to zero.

What do the two different complexities represent and which one of the two is the ''right'' one, yielding proper information over the organization of the states? Moreover, what do these quantities become in an exact FRSB quenched computation?

We study the complexity of the SK model, mean-field spin model with quenched disorder, critically reviewing the analysis made at the annealed level in the far and near past, linking apparently different approaches and discussing the role of BRST supersymmetry in this contest. We will carefully look at the limitations of the annealed approaches, in particular from the point of view of physical stability, including the incompleteness of the one-step RSB approximation for the SK model.

Our aim is to present here a comprehensive picture of the annealed level, leaving for elsewhere the study of the quenched (FRSB) complexity.[16]

Before dealing with details and derivations we now very briefly anticipate the main results of this paper and we outline the scheme of their presentation.

The BM formalism[5,17]*is equivalent to the supersymmetric one* as presented in Refs. 12 and 13. Both at the ''microscopic'' level (site commuting and anticommuting variables) and at the ''macroscopic'' level (order parameters) the actions in the two formalisms are related to each other by a simple change of variables. The form of BRST transformations and of the Ward identities are also obtained in the BM notation. We thus can consider just one (supersymmetric) functional generating one set of saddle point equations.

Both the BRST-SUSY and BM solutions (which we will often refer to as the BRST-SUSY-breaking solution) for the annealed complexity *are solutions of the same set of saddle point equations*.

In Secs. II B and II C we derive such properties, comparing the two approaches in all essential details.

Afterwards, we give some hints in order to understand the physical meaning of the BRST SUSY, showing, in Sec. II D, the connection between such property and the property of *nonbifurcation* of TAP solutions of linearly perturbed TAP equations with respect to the unperturbed solutions and the BRST SUSY. In Sec. II E the identification between the annealed BRST-SUSY solution and the static 1RSB solution at zero magnetic field, the main topic of Ref. 13, is rederived in the case of the BM formulation. Moreover, in Sec. II F, we recast everything in the formulation of the generalized *two-group ansatz* of Ref. 6 and we show that breaking the supersymmetry amounts to consider a nontrivial ansatz in the replica calculation.

In the second part of the paper, Sec. III, we discuss several issues connected with the problem of selecting the right solution between the BM and BRST supersymmetric one. In particular, we show that *the sign of the determinant of the Hessian cannot be determined by the saddle point solution at the leading order in N* and, applying Plefka's criterion to the analysis of the physical meaning of TAP solutions, we explain why the parameter B entering the determinant has to be set equal to zero, as in Refs. 5, 6, 12 and 13, and why, as opposed to what is stated in Refs. 5 and 18, such a choice *does not guarantee the positivity of the determinant* of the Hessian of the TAP free energy functional.

We also recall the results of Kurchan[19] about the spontaneous BRST-SUSY breaking taking place on the BM saddle point: the BRST SUSY is recovered by the analysis of the prefactor of the exponent of $N\Sigma_{BM}$ showing it to be zero on the BM saddle point. This result, while ensuring mathematical consistency, poses the problem of which is the correct prefactor of the saddle point when the modulus of the determinant is not dropped.

As a by-product, dealing with the problem that the true thermodynamic description of the low-temperature phase of the SK model is yielded by a FRSB solution instead of a 1RSB one, in Sec. III D we show that *the complexity proposed by Bray, Moore, and Young*,[20] as the quenched analog of the above broadly mentioned annealed complexity, *is computed over a BRST-SUSY saddle point*.

II. COUNTING THE TAP STATES: THE COMPLEXITY IN THE ANNEALED APPROXIMATION AND THE BRST SUSY

Before discussing the properties of the number of solutions of the mean-field equations for the SK model, we very shortly summarize the basic, widely known, features of the Thouless-Anderson-Palmer[14] formulation. The TAP equations for the local average magnetization m_i of the ith spin are

$$m_i = \tanh[\beta(\tilde{h}_i - \beta(1-q)m_i)] \quad \forall i = 1,\ldots,N, \qquad (1)$$

$$\tilde{h}_i = \sum_j J_{ij} m_j , \tag{2}$$

where q is the self-overlap of TAP configurations,

$$q \equiv \frac{1}{N} \sum_i m_i^2 , \tag{3}$$

and J_{ij} are distributed as

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi}} \exp\left(-N \frac{J_{ij}^2}{2} \right). \tag{4}$$

The associated free energy functional is

$$F_{\mathrm{tap}}(\{m\}) = E_{\mathrm{tap}}(\{m\}) - T S_{\mathrm{tap}}(\{m\}), \tag{5}$$

$$E_{\mathrm{tap}}(\{m\}) = -\sum_{i<j} J_{ij} m_i m_j - \frac{N\beta}{2}(1-q)^2, \tag{6}$$

$$\begin{aligned} S_{\mathrm{tap}} &= \sum_i s_{\mathrm{tap}}(m_i;q) \\ &= \sum_i \left\{ \ln 2 - \frac{\beta^2}{4}(1-q)^2 \right. \\ &\quad \left. - \frac{1}{2}\ln(1-m_i^2) + m_i \tanh^{-1} m_i \right\}, \end{aligned} \tag{7}$$

where E_{tap} and S_{tap} are, respectively, the expressions for the internal energy and the entropy of a $\{m_i\}$ configuration and Eq. (1) is yielded by differentiating Eq. (5) (Ref. 14) $\partial_{m_i} F_{\mathrm{tap}} = 0$. Furthermore, one defines the disorder-independent "field"

$$g(m_i;q) \equiv \frac{1}{\beta} \tanh^{-1} m_i + \beta(1-q) m_i \tag{8}$$

so that Eq. (1) can be reformulated as

$$\partial_{m_i} F_{\mathrm{tap}} = g(m_i;q) - \sum_j J_{ij} m_j = 0. \tag{9}$$

Using Eq. (9), the expression for the energy of a TAP solution $\{m_i^{\mathrm{sol}}\}$ can be written without making use of the disorder J_{ij} as

$$\begin{aligned} E(\{m^{\mathrm{sol}}\}) &= -\frac{1}{2\beta} \sum_i g(m_i;q) m_i - \frac{\beta}{2}(1-q)^2 \\ &= -\frac{1}{2\beta} \sum_i m_i \tanh^{-1} m_i - \frac{\beta}{2}(1-q^2). \end{aligned} \tag{10}$$

Combining this last result with the expression (7) for the entropy one is able to formally rewrite the TAP free energy as a sum of single site-free energies of TAP solution as

$$\begin{aligned} F(\{m^{\mathrm{sol}}\}) &= \sum_i f(m_i;q) \\ &= \frac{1}{N} \sum_i \left\{ -\ln 2 - \frac{\beta^2}{4}(1-q^2) \right. \\ &\quad \left. + \frac{1}{2}[m_i \tanh^{-1} m_i + \ln(1-m_i^2)] \right\}, \end{aligned} \tag{11}$$

where $f(m_i,q)$ is the site free energy computed on a solution of Eq. (9).

A. Complexity in the annealed approximation

What we are interested in is the number of solutions of the TAP equations at different free energy levels, which can provide substantial information in order to understand the free energy landscape and, therefore, the organization of the states. If we label each of the $\mathcal{N}$ solutions with the index $\alpha = 1,\ldots,\mathcal{N}$, the number of solutions having free energy density equal to f is

$$\rho_s(f) = \sum_{\alpha=1}^{\mathcal{N}} \delta[F_{\mathrm{tap}}(\{m^\alpha\}) - Nf]. \tag{12}$$

This can be formally transformed as

$$\begin{aligned} \rho_s(f) &= \sum_{\alpha=1}^{\mathcal{N}} \int_{-1}^{1} \prod_{i=1}^{N} \mathrm{d}m_i \, \delta(m_i - m_i^\alpha) \, \delta[F_{\mathrm{tap}}(\{m\}) - Nf] \\ &= \int_{-1}^{1} \prod_{i=1}^{N} \mathrm{d}m_i \, \delta(\partial_{m_i} F_{\mathrm{tap}}(\{m\})) \left| \det(\partial_{m_i} \partial_{m_j} F_{\mathrm{tap}}(\{m\}) \right| \\ &\quad \times \delta[F_{\mathrm{tap}}(\{m\}) - Nf], \end{aligned} \tag{13}$$

where the Hessian takes the form

$$\begin{aligned} &\partial_{m_i} \partial_{m_j} F_{\mathrm{tap}}(\{m\}) \\ &\quad = -J_{ij} + \left[\frac{1}{\beta} \frac{1}{1-m_i^2} + \beta(1-q) \right] \delta_{ij} + O\left(\frac{1}{N} \right). \end{aligned} \tag{14}$$

Terms of order $1/N$ will be neglected since they are not relevant for the present discussion.

At this point, one can compute the annealed complexity—i.e., the logarithm of the average over the disorder of the density of TAP solutions:

$$\Sigma_a = \frac{1}{N} \ln \overline{\rho_s(f)}, \tag{15}$$

where the overbar represents the average over the distribution of the J_{ij}'s. Details of such computation for the SK model can be found both in the original paper of Bray and Moore (Ref. 5) and in the supersymmetric formulation of Ref. 13, as well as in many other works: e.g., Refs. 17, 21 and 22. Here, we only stress the basic fact that, in both procedures, the determinant of the Hessian is taken without absolute value, thus implying that the quantity actually com-

puted would coincide with the ''true'' complexity (even though in the annealed approximation) only if the determinant of all the solutions we are counting were positive. Such an assumption is hard to verify in the SK model.

B. BRST invariance of the density of TAP solutions

The density of TAP solutions can be written making use both of anticommuting $\{\psi\},\{\bar{\psi}\}$ and commuting variables $\{m\},\{x\}$ as[12,13]

$$\rho_s(f)=\int \mathcal{D}x\mathcal{D}m\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\beta\mathcal{S}(m,x,\bar{\psi},\psi)} \tag{16}$$

$$\begin{aligned}\mathcal{S}(m,x,\bar{\psi},\psi)=&\sum_i x_i\partial_{m_i}F_{\rm tap}(\{m\})\\&+\sum_{ij}\bar{\psi}_i\psi_j\partial_{m_i}\partial_{m_j}F_{\rm tap}(\{m\})\\&+u[F_{\rm tap}(\{m\})-Nf],\end{aligned} \tag{17}$$

where $\mathcal{D}a\equiv$prefactor$\times\Pi_i da_i$. The transformation under which such an action is invariant is[10,12,13]

$$\delta m_i=\epsilon\psi_i \quad \delta x_i=\epsilon u\psi_i \quad \delta\bar{\psi}_i=-\epsilon x_i \quad \delta\psi_i=0. \tag{18}$$

The Ward identities generated by imposing invariance with respect to the above transformations of observables are, e.g.,

$$\langle\bar{\psi}_i\psi_i\rangle=-\langle m_i x_i\rangle, \tag{19}$$

$$u\langle\bar{\psi}_i\psi_i\rangle=\langle x_i^2\rangle, \tag{20}$$

where the average $\langle(\cdots)\rangle$ is computed over the measure $e^{\mathcal{S}}$.

Bray and Moore[5] used explicitly, in their computation, the TAP equation (9) in order to simplify the procedure. In Ref. 13 it was claimed that such a substitution led to an action no longer BRST invariant. Such an inconvenience is, however, only apparent. Indeed, shifting the integration variables $\{x\}$ the BRST-invariant form is readily restored. The action yielded by the BM procedure is different from the so-called SUSY one because of the insertion of Eq. (9), which means

$$\begin{aligned}\mathcal{S}(x,m,\bar{\psi},\psi)=&\mathcal{S}_{\rm BM}(x,m,\bar{\psi},\psi)\\&+u\left[\frac{1}{2}\sum_{ij}J_{ij}m_im_j-\frac{1}{2\beta}\sum_i g(m_i;q)m_i\right].\end{aligned} \tag{21}$$

Since the quenched disorder J_{ij} enters the action as $J_{ij}(x_im_j+\bar{\psi}_i\psi_j+u/2m_im_j)$, a shift in the integration variable

$$x_i\to x_i'=x_i+\frac{u}{2}m_i \tag{22}$$

is enough to make both actions coincide:

$$\mathcal{S}(x,m,\psi,\bar{\psi})=\mathcal{S}_{\rm BM}\left(x+\frac{u}{2}m,m,\psi,\bar{\psi}\right). \tag{23}$$

In the new set of variables $\{x',m,\psi,\bar{\psi}\}$ the transformation (18) keeping a BSRT-SUSY function invariant remains unchanged. The shift of x_i is performed over an integration variable, thus without affecting the final result, yet the transformations given in Eq. (18) and the Ward identities computed with the BM measure in the original set of variables are changed.

In the old set of variables the BRST transformation reads (as discussed also in Ref. 23)

$$\delta m_i=\epsilon\psi_i \quad \delta x_i=\epsilon\frac{u}{2}\psi_i \quad \delta\bar{\psi}_i=-\epsilon x_i \quad \delta\psi_i=0, \tag{24}$$

and the averages over the two actions are connected by

$$\langle a(x)\rangle=\left\langle a\left(x-\frac{u}{2}m\right)\right\rangle_{\rm BM}, \tag{25}$$

where the average $\langle(\cdots)\rangle_{\rm BM}$ is computed over the measure $e^{\mathcal{S}_{\rm BM}}$ so that Ward identities computed with the BM action become

$$\langle\bar{\psi}_i\psi_i\rangle_{\rm BM}=-\langle m_ix_i\rangle_{\rm BM}+\frac{u}{2}q, \tag{26}$$

$$u\langle\bar{\psi}_i\psi_i\rangle_{\rm BM}=\langle x_i^2\rangle_{\rm BM}-u\langle m_ix_i\rangle_{\rm BM}+\frac{u^2}{4}q. \tag{27}$$

Inserting Eq. (26) into Eq. (27) one gets

$$\langle x_i^2\rangle_{\rm BM}=\frac{u^2}{4}q. \tag{28}$$

Even if in the notation of Ref. 13 it seemed that the action $S_{\rm BM}$ was not satisfying the BRST relations, this was exclusively due to the fact that such relations in the BM notation read differently.

Moving to the macroscopic level, where the average number of solution is expressed as a function of the parameters q, Δ, λ, and B, the two approaches continue to be linked and equivalent to each other. The final expressions for the complexity in Refs. 5 (BM) and 13 [Cavagna, Giardina, Parisi, and Mezard (CGPM)] are

$$\begin{aligned}&\text{BM }\overline{\rho_s(f;u,B,q,\Delta,\lambda)}\\&=\text{ext}_{\{uBq\Delta\lambda\}}\exp\left\{N\left[-\lambda q-\beta uf\right.\right.\\&\left.\left.-(B+\Delta)(1-q)+\frac{(B^2-\Delta^2)}{2\beta^2}+\ln I\right]\right\},\end{aligned} \tag{29}$$

$$I=\int_{-1}^{1}\frac{\mathrm{d}m}{\sqrt{2\pi\beta^2 q}}\left(\frac{1}{1-m^2}+B\right)\times\exp\left\{-\frac{(\Delta m-\tanh^{-1}m)^2}{2\beta^2 q}+\lambda m^2+uf(m;q)\right\}, \tag{30}$$

$$f(m;q)=-\ln 2-\frac{\beta^2}{4}(1-q^2)+\frac{1}{2}m\tanh^{-1}m+\frac{1}{2}\ln(1-m^2), \tag{31}$$

$$\text{CGPM}\ \overline{\rho_s(f;u,B,q,\Delta,\lambda)}=\mathrm{ext}_{\{uBq\Delta\lambda\}}\exp\left\{N\left[-\lambda q-\beta uf-(B+\Delta)(1-q)+\frac{(B^2-\Delta^2)}{2\beta^2}-\frac{\beta^2}{4}u^2q^2-\Delta uq-\beta^2uq(1-q)+\ln I\right]\right\}, \tag{32}$$

$$I=\int_{-1}^{1}\frac{\mathrm{d}m}{\sqrt{2\pi\beta^2 q}}\left(\frac{1}{1-m^2}+B\right)\times\exp\left\{-\frac{(\Delta m-\tanh^{-1}m)^2}{2\beta^2 q}+\lambda m^2+u\ \ \Phi_0(m;q)\right\}, \tag{33}$$

$$\Phi_0(m;q)=-\ln 2-\frac{\beta^2}{4}(1-q)^2+m\tanh^{-1}m+\frac{1}{2}\ln(1-m^2). \tag{34}$$

The first expression is obtained making explicit use of Eq. (10) for the TAP energy, while the second one is computed without ever using such a relation. Apparently the two expressions differ. Since, however, both actions describe the evolution in the parameter space of the same system, the two formulations must coincide (exactly as the ''microscopic'' description and the final results do), introducing a proper change of variables. Indeed, to link the two approaches the following transformation can be set:

$$\Delta_{BM}=\Delta_{CGPM}+\frac{\beta^2}{2}uq, \tag{35}$$

$$\lambda_{BM}=\lambda_{CGPM}+\frac{1}{2}u\Delta_{CGPM}+\frac{\beta^2}{8}u^2q. \tag{36}$$

Notice that the two functions $f(m;q)$ (in the BM formalism) and $\Phi_0(m;q)$ (in the formalism of Ref. 13) are related to each other by

$$f(m;q)=\Phi_0(m;q)-\frac{1}{2}m\tanh^{-1}m-\frac{\beta^2}{2}q(1-q). \tag{37}$$

Since the formulations are equivalent, the notation to adopt is no longer important. We will use the original notation of Ref. 5, not simply because it is the oldest one, but rather to have a more direct comparison with other works: e.g., the one of Parisi and Potters,[6] which showed that the BM action can be obtained from the Legendre transform approach of Ref. 7 making a two-group ansatz on the matrix Q_{ab} entering the computation of the free energy of the coupled replicas (in this way they were able to find new solutions of the BM equations called ''unbroken two-group'') or the quenched computation of the complexity of Refs. 20 and 24. For simplicity we drop the subscript BM in the following.

C. Saddle point equations

The variational equations, yielding the saddle point(s) values of the parameters for the annealed complexity, take the form

$$\frac{\partial\Sigma_a}{\partial u}=0\rightarrow f=\langle f(m;q)\rangle, \tag{38}$$

$$\frac{\partial\Sigma_a}{\partial B}=0\rightarrow B=\beta^2\left(1-q-\left\langle\frac{1-m^2}{1+B(1-m^2)}\right\rangle\right), \tag{39}$$

$$\frac{\partial\Sigma_a}{\partial q}=0\rightarrow\lambda=B+\Delta-\frac{1}{2q}+\frac{\langle(\Delta m-\tanh^{-1}m)^2\rangle}{2\beta^2q^2}+\frac{\beta^2}{2}uq, \tag{40}$$

$$\frac{\partial\Sigma_a}{\partial\Delta}=0\rightarrow\Delta=-\frac{\beta^2}{2}(1-q)+\frac{1}{2q}\langle m\tanh^{-1}m\rangle, \tag{41}$$

$$\frac{\partial\Sigma_a}{\partial\lambda}=0\rightarrow q=\langle m^2\rangle, \tag{42}$$

where the average

$$\langle O(m)\rangle=\frac{1}{I}\int_{-1}^{1}\mathrm{d}mO(m)e^{\mathcal{L}(m;u,q,\Delta,\lambda)}, \tag{43}$$

$$I=\int_{-1}^{1}\mathrm{d}m e^{\mathcal{L}(m;u,q,\Delta,\lambda)}, \tag{44}$$

is taken over the action

$$\mathcal{L}(m;u,q,\Delta,\lambda)=\ln\left(\frac{1}{1-m^2}+B\right)-\frac{1}{2}\ln(\beta^2 2\pi q)-\frac{(\Delta m-\tanh^{-1}m)^2}{2\beta^2q}+\lambda m^2+uf(m;q). \tag{45}$$

As noted in Ref. 13 the last term of Eq. (40) was missing in Ref. 5. The free energy of the TAP solutions $f(m;q)$ is the one expressed in Eq. (11) or (31).

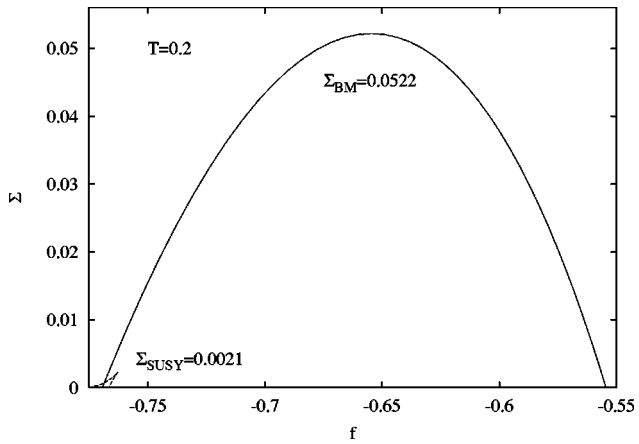

FIG. 1. Comparison of complexities yielded, respectively, by the BRST-SUSY solution (dashed line) and the BRST-SUSY-breaking solution (solid line) of the annealed saddle point equations at $T=0.2$. The range of free energy values of the solution breaking the supersymmetry of the TAP action (45) is much larger than the f range of the SUSY one. The maximum complexity for this solution is also much higher than the one for the BRST-invariant solution (the value at the cusp of the dashed curve). Such a difference increases by lowering T.

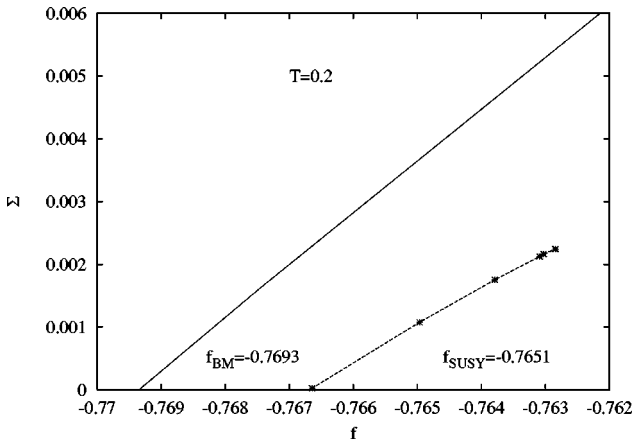

FIG. 2. Detail of the complexity curves for low free energy values. The asterisks stand for the complexity of the static one-step RSB solution at zero magnetic field, coinciding with the BRST-SUSY saddle point. The values of free energy at which the complexity reaches zero (subexponential growth of solutions of TAP equations with increasing size) are $f_{\mathrm{BM}}=-0.7693$, $f_{\mathrm{SUSY}}=-0.7651$, both below the true equilibrium value at $T=0.2$: $f_{\mathrm{eq}}=-0.7594$ (as computed, e.g., in the FRSB scheme). Notice that the branch on the left-hand side of the BRST-SUSY solution in not shown.

Fixing u, therefore leaving f as a free parameter, the above equations have at least two different solutions.[25] One solution satisfies the two relations of BRST supersymmetry as stated in Refs. 12 and 13, which we rewrite here in the present notation as

$$B+\Delta=-\frac{\beta^2}{2}uq, \tag{46}$$

$$\lambda=\frac{\beta^2}{8}u^2q. \tag{47}$$

The saddle point Equation (39), substituting Eq. (42) into it, admits a solution for $B\geqslant 0$ (see Appendix A for details). In Sec. III A we recall that a general criterion, formulated by Plefka,[36] can be applied as a *necessary* condition to select physically relevant solutions. We anticipate that this criterion requires $B=0$, so that Eq. (46) becomes a condition for Δ alone.

In Figs. 1, 2, and 3 we show the behavior, at $T=0.2$, of both solutions, both versus f and u. The annealed complexity computed over the supersymmetric solution goes to zero smoothly as $u\to 0^-$; it displays a maximum at some $u_{\max}$ (or f_{th}, if the behavior versus free energy is considered) and crosses the u-ax at some u_0 such that $u_0<u_{\max}<0$. Unlike $\Sigma_a(u)$, the curve $\Sigma_a(f)$ is not univocal: it displays a cusp at f_{th} (see Fig. 1) and then turns back.

The other solution of the saddle point equations, which will refer to as the BM or BRST-SUSY-breaking solution, is the one introduced in Ref. 5. Such a solution does not satisfy relations (46) and (47), thus spontaneously breaking BRST SUSY; its dominion in u (or f) is broader than the one of the symmetric solution, the maximum value of Σ_a is for $u=0$, and, at any temperature, it is larger than the correspondent value on the BRST-invariant solution. The behavior of the annealed complexity is univocal both in u and in f.

D. Physical meaning of BRST SUSY in the context of TAP solutions

In this section we will follow a reasonment allowing for an intuitive explanation of the physical meaning of the BRST supersymmetry in terms of a particular behavior of the solutions of the TAP equations.

The objects of interest of the present approach are sums over TAP states, multiplied by some weight function, typically of the form $e^{\beta uF(m)}$. We make the hypothesis that the

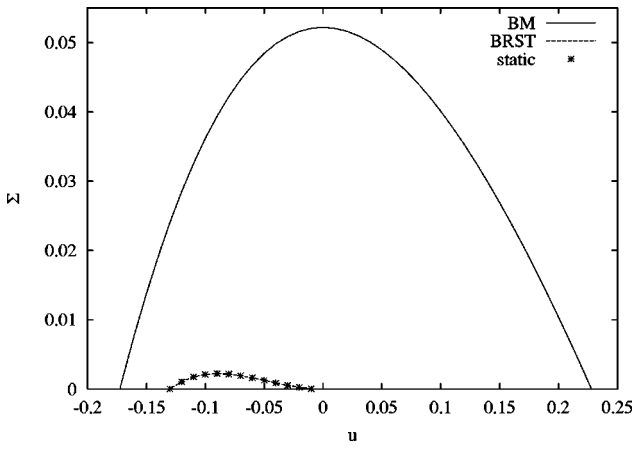

FIG. 3. Complexity vs u (conjugated to f) for both saddle points. The BRST-SUSY complexity displays a maximum below zero [corresponding to the cusp in the f dominion (see Fig. 1)]. It goes to zero as $u\to 0$. On the contrary the solution with broken SUSY has a maximum for $u=0$.

set of solutions over which we perform the sums does not qualitatively change adding a small magnetic field to a single spin k. This means that each solution α with associated magnetizations $\{m_i^\alpha\}$ goes continuously to a new solution in presence of the field h_k, defining the functions $\{m_i^\alpha(h_k)\}$.

Given a generic observable depending on the magnetizations, its white average—i.e., the average value that the observable g takes over each TAP solution—is given by

$$[g(m)]_{\rm sol} \equiv \frac{1}{\mathcal{N}_s} \sum_\alpha g(m_i^\alpha), \tag{48}$$

where $\mathcal{N}_s$ is the number of solutions we are summing over.

We now assume that (a) no solution bifurcates and (b) no new solution appears. This can take place either as a global property of all TAP states or as a property of a restricted subset of dominant states, selected according to their weights. Under these hypotheses this means that we may write

$$[g(m(h))]_{\rm sol} = \frac{1}{\mathcal{N}_s} \sum_\alpha g(m_i^\alpha(h)). \tag{49}$$

As a consequence, for any n, the following relation holds:

$$\frac{\partial^n [g(m(h))]_{\rm sol}}{\partial h^n} = \frac{1}{\mathcal{N}_s} \sum_\alpha \frac{\partial^n g(m_i^\alpha(h))}{\partial h^n}. \tag{50}$$

We start considering $g(m) = m_k e^{\beta u F(m)}$, which is the average of the magnetization with the weight $e^{\beta u F}$, used to count solutions of a given energy. In general we have

$$\frac{\partial [m_l e^{\beta u F(m)}]_{\rm sol}}{\partial h_k} = -\langle x_k m_l \rangle, \tag{51}$$

where the average on the right-hand side (RHS) is computed with respect to the action $S(x, m, \bar{\psi}, \psi)$ [Eq. (17)]. However, under the above hypothesis, this must be equal to the average on each solution of the derivative of $m_k e^{\beta u F(m)}$ with respect to h_k, which is

$$\sum_\alpha \left(\frac{\partial m_l^\alpha}{\partial h_k} e^{\beta u F(m^\alpha)} + m_l \beta u \frac{\partial F(m^\alpha)}{\partial m_j^\alpha} \frac{\partial m_j^\alpha}{\partial h_k} \right). \tag{52}$$

We can drop the second term since, by definition, it is zero on the TAP solutions, and we are left with

$$\frac{1}{\mathcal{N}_s} \sum_\alpha \chi_{kl} e^{\beta u F(m^\alpha)} = \frac{1}{\mathcal{N}_s} \sum_\alpha \left(\frac{\partial^2 F}{\partial m^\alpha \partial m^\alpha} \right)^{-1}_{kl} e^{\beta u F(m^\alpha)} = \langle \bar{\psi}_l \psi_k \rangle. \tag{53}$$

Thus we have

$$-\langle x_k m_l \rangle = \langle \bar{\psi}_l \psi_k \rangle, \tag{54}$$

i.e., for $k = l$, the first Ward identity for BRST-SUSY systems, Eq. (19). To obtain the second Ward identity, Eq. (20), we must apply the same procedure to the second derivative of the quantity $g(m) = e^{\beta u F(m)}$, proportional to the complexity. From Eq. (17) we have

$$\frac{\partial^2 [e^{\beta u F(m)}]_{\rm sol}}{\partial h_i \partial h_j} = \beta \langle x_i x_j \rangle. \tag{55}$$

Under the hypothesis of Eq. (49), this must be equal to the following quantity:

$$\frac{1}{\mathcal{N}_s} \sum_\alpha \frac{\partial^2 e^{\beta u F(m^\alpha)}}{\partial h_i \partial h_j} = \frac{1}{\mathcal{N}_s} \sum_\alpha \left(\beta u \frac{\partial^2 F(m^\alpha)}{\partial h_i \partial h_j} + \beta^2 u^2 \frac{\partial F}{\partial m_l} \frac{\partial m_l}{\partial h_i} \frac{\partial F}{\partial m_s} \frac{\partial m_s}{\partial h_j} \right) e^{\beta u F(m^\alpha)}, \tag{56}$$

where the second term is zero over any TAP solution and we are left with

$$\sum_\alpha \beta u \frac{\partial^2 F(m^\alpha)}{\partial h_i \partial h_j} e^{\beta u F(m^\alpha)} = \beta u \langle \bar{\psi}_i \psi_j \rangle. \tag{57}$$

Thus we obtain

$$\langle x_i x_j \rangle = u \langle \bar{\psi}_i \psi_j \rangle. \tag{58}$$

The previous BRST relations can be obtained under weaker conditions than those considered initially; in particular the above derivation still holds if we allow the onset of new solutions, provided that they appear *only outside* the ensemble of dominant solutions: e.g., at the threshold. Indeed, in presence of a weight $e^{\beta u F(m)}$, only solutions of a certain free energy count, thus supporting our assumption that, in the dominion of interest, the total number of states, $\mathcal{N}_s$, does not change. Instead, the condition of no bifurcation seems unavoidable to obtain the BRST relations by means of this argument.

E. Identification of the BRST-SUSY solution with the static solution: Link with replica computation

In this section we very shortly recall the connection between the static solution at the one-step RSB approximation and the BRST-SUSY solution of Σ_a (Ref. 13). We compare results at given values of the breaking parameter m in replica formalism and at given values of the variable u, Legendre conjugated of f, in the framework of the annealed computation of the complexity.

The total replica free energy at one-step RSB is

$$f^{\rm rep} = -\frac{\beta}{4}(1-2q_1) - \frac{\beta}{4}[(1-m)q_1^2 + m q_0^2] - \frac{1}{\beta m} \int \mathcal{D}z_0 \ln \int \mathcal{D}z_1 \; p_1(m, z_0, z_1), \tag{59}$$

with

$$p_1(m, z_0, z_1) \equiv [2 \cosh(\beta z_0 \sqrt{q_0} + \beta z_1 \sqrt{q_1 - q_0})]^m. \tag{60}$$

Self-consistency equations take the form

$$q_0 = \int \mathcal{D}z_0 \langle \tanh(\beta z_0 \sqrt{q_0} + \beta z_1 \sqrt{q_1 - q_0}) \rangle^2, \quad (61)$$

$$q_1 = \int \mathcal{D}z_0 \langle \tanh^2(\beta z_0 \sqrt{q_0} + \beta z_1 \sqrt{q_1 - q_0}) \rangle, \quad (62)$$

$$\frac{\beta}{4}(q_1^2 - q_0^2) + \frac{1}{\beta m^2} \int \mathcal{D}z_0 \left[\ln \int \mathcal{D}z_1 p_1(m, z_0, z_1) - \langle \ln p_1(m, z_0, z_1) \rangle \right] = 0, \quad (63)$$

with

$$\langle (\cdots) \rangle \equiv \frac{\int \mathcal{D}z_1 (\cdots) p_1(m, z_0, z_1)}{\int \mathcal{D}z_1 p_1(m, z_0, z_1)}. \quad (64)$$

Leaving m as a free parameter [thus ignoring Eq. (63)], at a given temperature we can define the complexity of the system as the Legendre transform of $\beta m f^{\mathrm{rep}}$:

$$\Sigma_1(f) = \max_m [\beta m f - \beta m f^{\mathrm{rep}}(m)], \quad (65)$$

with conjugated variables βm and f,

$$f = \frac{\partial m f^{\mathrm{rep}}}{\partial m}, \quad (66)$$

$$\beta m = \frac{\partial \Sigma_1}{\partial f}. \quad (67)$$

Equations (66) and (67) yield the relation between f and m. Introducing Eq. (66) in the Legendre transformation of $\beta m f^{\mathrm{rep}}$ [Eq. (65)] one can obtain the following relation:

$$\Sigma_1(f) = \beta m^2 \left. \frac{\partial f^{\mathrm{rep}}}{\partial m} \right|_{m(f)} = \frac{\beta^2}{4} m^2 (q_1^2 - q_0^2) + \int \mathcal{D}z_0 \left[\ln \int \mathcal{D}z_1 p_1(m, z_0, z_1) - \langle \ln p_1(m, z_0, z_1) \rangle \right], \quad (68)$$

where $m = m(f)$.

In zero external magnetic field $q_0 = 0$. In this particular case Σ_1 represents the entropy of *hidden states* that we would get with the method of the m coupled real replicas[7] and a formal connection can be established between the complexity of the BRST-SUSY saddle point of the annealed computation and the one-step RSB replica free energy. This was shown in Ref. 13 [Eqs. (32)–(34)] but the formal connection also holds in the original notation of Ref. 5.

Indeed, the expression of the annealed complexity obtained simplifying Δ and λ in the logarithm of Eq. (29) by means of the BRST relations, Eqs. (46) and (47), comes out to be

$$\Sigma_a(f; u, q) = \frac{1}{N} \ln \rho_s(f; u, q, \Delta(u, q), \lambda(u, q)) = \beta u \left\{ -\frac{\beta}{4}[1 - 2q + (1 + u)q^2] + \frac{1}{\beta u} \ln \int \mathcal{D}z [2 \cosh \beta z \sqrt{q}]^{-u} - f \right\}. \quad (69)$$

It can be easily seen that, putting $q_0 = 0$ in the replica free energy, Eq. (59), this is bounded to Eq. (69) by

$$\Sigma_a(f; u, q) = \beta u [f^{\mathrm{rep}}(q, -u) - f] \quad (70)$$

provided that one makes the identification

$$m = -u, \quad q_1 = q. \quad (71)$$

Furthermore, the saddle points of such a function Σ_a coincide with those of f^{rep}, if the correspondence $f^{\mathrm{rep}}(q, -u = m^\star) = f$ holds at the static value of the breaking parameter m; i.e., $m^\star$ satisfies Eq. (63).

The identification in Eq. (71) is evident for the overlap: q_1 is the self-overlap of states in the replica 1RSB framework and q is the self-overlap of TAP states; i.e., they are just different representations of the same thermodynamic observable. For what concerns the connection between the breaking parameter m and the variable u conjugated with the TAP free energy, we can observe that the derivative with respect to f of Eq. (69) gives

$$\frac{\partial \Sigma_a}{\partial f} = -\beta u. \quad (72)$$

Simply comparing this with Eq. (67) one can identify $-\beta u$ as βm, the slope of the complexity as a function of the free energy. As a matter of fact we are just saying that Eq. (69) can be seen as the Legendre transform of $(-\beta u f^{rep})$ with conjugated variables $-\beta u$ and f.

Notice that, as a consequence of the link shown in Eqs. (70) and (71), the average defined in Eq. (43) coincides with the one defined in Eq. (64). Therefore we used the same symbol.

The one-step RSB static solution is not thermodynamically stable, as can be shown computing the Hessian of the replica thermodynamic potential with respect to the order parameters variations around their self-consistently derived values, given by Eqs. (61) and (62). Indeed, the eigenvalue associated with the overlap-overlap fluctuations—i.e., the replicon—is

$$\Lambda_R = 1 - \beta^2 \langle \cosh^{-4}(\beta \sqrt{q_1} z_1) \rangle. \quad (73)$$

Using the identity $\cosh^{-4} = (1 - \tanh^2)^2$, the stability condition $\Lambda_R > 0$ can be written as

$$T^2 > 1 - 2q_1 + \langle \tanh^4(\beta \sqrt{q_1} z_1) \rangle. \quad (74)$$

This is the analogous in the replica formalism of Plefka's criterion for the physical relevance of TAP solutions[36] (see Sec. III A).

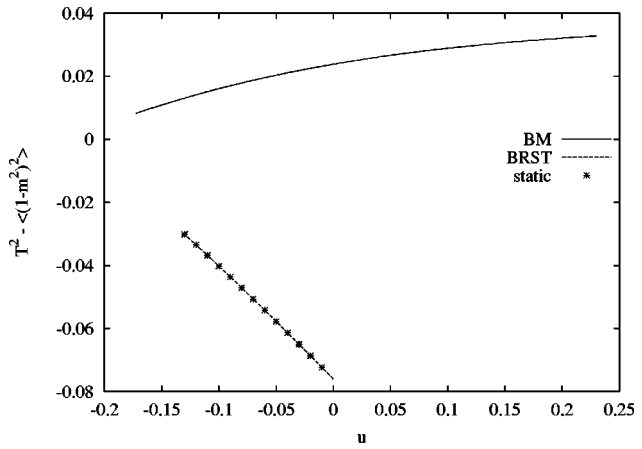

FIG. 4. The quantity $T^2-\langle(1-m^2)^2\rangle$ is plotted vs u. Following the Plefka's criterion, if such a quantity is positive, the solution is physically meaningful (see text for details); otherwise it is not. In the case of the BRST solution, it coincides with the replicon (stability eigenvalue of the q_1 fluctuations in the one-step RSB static solution with no external field), which we plot with asterisks. From the plot one can see that Plefka's criterion is satisfied only by the BM solution. This means that if the annealed approximation in the complexity computation would be a reliable one, and if no further inconsistency would arise, the physically relevant saddle point over which it should be computed would be the one breaking the BRST SUSY.

As we can see from Fig. 4, the replicon of the one-step RSB solution at zero magnetic field, as a function of $u=-m$, is always negative in the dominion where the complexity is positive, even on the "static" point $f_0\equiv f(m^*)$. From the same figure, though, we observe that the equivalent quantity computed over the BM saddle point of the annealed complexity is such that Plefka's criterion is always satisfied. This is a necessary condition supporting the possibility that the BM annealed complexity could represent the number of states of the SK model, yet not a sufficient one, as we will see in Sec. III.

F. Two-group ansatz breaks the BRST SUSY

In Ref. 8 Bray and Moore introduced the *two-group ansatz* in order to solve the instability problem of the replica symmetric solution of the SK model. In Ref. 7 Monasson showed how the formalism of Legendre transforms can be applied to mean-field disordered models through the *pinning of real replicas* in a configuration space extended to m copies of the system.

Parisi and Potters[6] explained how the BM action can be obtained through the method of Monasson provided that the symmetry between real replicas is broken according to a generalized two-group ansatz. n being the number of replicas introduced to compute the quenched average (following the standard scheme[4,26]) and m the number of real copies, they analyzed

$$\lim_{n\to 0}\frac{1}{n}\ln\overline{Z^{mn}}=\mathrm{ext}_f[\overline{\ln\rho_s(f)}-m\beta Nf], \tag{75}$$

where Z^{mn} is the partition function of $n\times m$ copies of the system. In terms of the replicated matrix parameter $\hat{Q}$ its average is

$$\overline{Z^{mn}}=\mathrm{ext}_{\hat{Q}}\exp\left\{N\left[\frac{\beta^2}{4}(mn-\mathrm{Tr}\hat{Q}^2)+\ln\sum_{\{\sigma_a^c\}}\exp\left(\frac{\beta^2}{2}\sum_{ab,cd}\hat{Q}_{ab}^{cd}\sigma_a^c\sigma_b^d\right)\right]\right\}, \tag{76}$$

where the indexes $a,b=1,\ldots,n$ while $c,d=1,\ldots,m$. The four-index matrix $\hat{Q}_{ab}^{cd}$ can be expressed as the composition of n^2 submatrices $\mathbf{Q}_{ab}$ of dimension $m\times m$ of the form

$$\mathbf{Q}_{ab}=\begin{pmatrix}\overbrace{Q_{ab}^{+}}^{y} & \overbrace{Q_{ab}}^{m-y}\\ Q_{ab} & Q_{ab}^{-}\end{pmatrix}. \tag{77}$$

The matrices $Q_{ab}^{\pm}$ are further parametrized as

$$Q_{ab}^{\pm}=Q_{ab}\pm\frac{A_{ab}}{y}+\frac{C_{ab}}{2y^2}. \tag{78}$$

Furthermore is $\hat{Q}_{aa}^{cc}\equiv 0$. In Ref. 6 the last term was B_{ab}/y^2. We write $C_{ab}=2B_{ab}$ both in order to avoid confusion with the parameter B in the Σ_a expression of Sec. II and to obtain more symmetric expressions in following works.

Equation (75) is, then, computed making use of the above ansatz, getting a complexity that can be formally connected to the one of Bray, Moore, and Young, Ref. 20 through a given change of variables. We will analyze it more explicitly elsewhere,[16] where we will use such transformation.

For the time being we are mostly interested in the annealed case, which can be obtained putting in matrix (77) all Q_{ab} with $a\neq b$ equal to zero. The ansatz reduces, then, to consider n diagonal blocks of $m\times m$ matrices (77), for $b=a$, where the elements are built with $Q_{aa}=Q$, $A_{aa}=A$, $C_{aa}=C$. In this case, the change of variables we need to connect the BM formulation of the annealed complexity to the Parisi-Potters (PP) one is[27]

BM PP

$$q=Q, \tag{79}$$

$$\Delta=\left(A+\frac{m}{2}Q\right), \tag{80}$$

$$\lambda=\frac{\beta^2}{2}\left(C+mA+\frac{m^2}{4}Q\right), \tag{81}$$

$$u=-m.$$

Writing the equations with the substitutions (79)–(81) allows for an immediate connection between breaking the matrices structure into two groups and breaking the BRST SUSY. Indeed, Eqs. (80) and (81) transform into the BRST relations (46) and (47) if we set $A=C=0$—i.e., if we do not

break the matrix structure at all (*unbroken two-group*). On the contrary, setting values of A and C different from zero amounts to breaking the BRST symmetry and leads to values of q, Δ, and λ independent among each other: i.e., to the BM solution.

III. PROPERTIES OF THE ANNEALED COMPLEXITIES

In the previous part of the paper, we have studied the so-called BM complexity[5] (the one breaking the supersymmetry) and the BRST-symmetric complexity[13] as two different solutions of the same saddle point equations, derived from the same BRST-SUSY functional (see Sec. II B).

In this section we will discuss several issues connected to the problem of selecting a proper, physically meaningful, complexity and to see whether any of the two candidates fulfill the minimal requirements.

We start noticing that not all TAP solutions can be associated to stable thermodynamic states. They have first to satisfy the Plefka's criterion (see below), which guarantees the right expression for the linear susceptibility of the states under consideration, and, then, to be minima of the TAP free energy functional.

Operationally speaking, what happens is that, even if on physical grounds we are willing to count only the minima of the TAP free energy, in practice, we also count other kinds of solutions of the TAP equations. As a consequence, it may happen that we have a mathematically correct computation which, however, has no physical meaning at all or a rather obscure one. In Sec. III A we will see that, at the present stage, we have no way to state that any of the two solutions of the complexity saddle point equations correspond to counting only minima of the TAP free energy. On the other hand, though it is just a necessary condition, a partial discrimination can be done on the basis of Plefka's criterion, which is satisfied by the BRST-SUSY-breaking solution but not by the BRST-SUSY solution.

Since the BRST solution can be linked to the saddle point solution obtained in a 1RSB replica computation in zero magnetic field, it is natural to check its stability according to the usual criteria in that framework—e.g., the positivity of the replicon eigenvalue; however, as we anticipated in Sec. II E, this condition is formally identical to the Plefka criterion, which is, therefore violated by the BRST-SUSY solution. This violation leads to a mathematical inconsistency, as we show in Sec. III A.

The BRST supersymmetry is a property of the action and, therefore, its violation poses a consistency problem to the BM saddle point, too, due to the lack of control on what we are counting. The only possibility for the BRST-SUSY-breaking saddle point to be valid is to guess that it counts only the minima, as it is claimed in Ref. 18. It would describe only a physically stable portion of the set of TAP solution and we would not expect that it satisfies global properties valid on the whole set, such as the Ward identities derived from the BRST SUSY [see Eqs. (19) and (20)]. However, this assumption is hard to justify, as we will discuss in the Secs. III B and III C, recalling the properties of the Hessian of the TAP free energy as derived in Ref. 28.

The effects of neglecting the sign of the determinant of the TAP solutions, which is at the origin of the BRST susy, are particularly dramatic when we set $u=0$. In this case we are summing over each extremum of the TAP functional with a weight given by the sign of its determinant. The Morse theorem tells us that this quantity is a topological invariant and it is equal to 1 in this specific case. While the BRST-SUSY solution does satisfy the theorem, the BRST-SUSY-breaking saddle point yields a quantity of order $e^{N\Sigma_{\rm BM}} \gg 1$, if $\Sigma_{\rm BM} > 0$.

A first way to recover mathematical consistency in the $u=0$ case is to guess that BRST SUSY is restored—e.g., because the prefactor of the exponential is $e^{-N\Sigma_{\rm BM}}$. In this case the expansion of the prefactor in powers of $1/N$ will be zero at all orders (such as the expansion of $e^{-1/x}$ in powers of x). This series was considered by Kurchan in Ref. 19 where it is shown that its coefficients are indeed all null. This proves that the BM saddle point is possibly mathematically consistent but raises another difficulty. We already stressed that at the order N it is not possible to prove whether the BM complexity is counting only minima or not. When computing the corrections without a modulus of the determinant of the Hessian one seems to have a vanishing prefactor, though. Indeed, at this stage, in order to accept the BM prediction, we must, then, assume that the effect of taking the modulus of the determinant into account is to change the value of the prefactor from exponentially small to finite, a completely unjustified assumption at the present state.

A. Plefka's criterion

In this subsection we recall the results of Ref. 28 on the properties of the Hessian of the TAP free energy on a generic configuration $\{m_i\}$, which is also the inverse of the susceptibility matrix:

$$\chi_{ij}^{-1} = \frac{\partial^2 F_{\rm tap}(m)}{\partial m_i \partial m_j}. \tag{82}$$

The problem is studied considering the resolvent of the Hessian. As a by-product of the computation one obtains the magnetic susceptibility of a solution. On physical grounds this must be equal to $\beta(1-q)$; however, it turns out that not all TAP configurations satisfy this relation. Instead the condition for a TAP solution $\{m_i\}$ to yield the right physical susceptibility is

$$x_{\rm P} \equiv 1 - \beta^2 \frac{1}{N} \sum_i (1-m_i^2)^2 \geqslant 0. \tag{83}$$

The TAP solutions which do not satisfy the previous relation turn out to have the following nonphysical susceptibility:

$$\chi_l = \beta(1-q) + \frac{x_{\rm P}}{p}, \tag{84}$$

where p is defined as

$$p \equiv \beta^3 \frac{1}{N} \sum_i (1-m^2)^3. \tag{85}$$

Therefore, after having computed the complexity, we must check that the Plefka relation is verified to be sure that we are counting physical solutions.

For $N\to\infty$, the site average becomes, in the notation of the annealed computation of Sec. II C, the average of Eq. (83) over the action $\mathcal{L}$ [Eq. (45)] and the above inequality reads

$$1-\beta^2(1-2q+\langle m^4\rangle)\geqslant 0. \tag{86}$$

As we can see from Fig. 4 such a condition is satisfied by the saddle point breaking the BRST SUSY, but is violated by the BRST-invariant one. In the replica language Eq. (86) is the replicon. The identification of the measures over which the average is performed is given by Eqs. (69)–(71).

As we show in Appendix B, the computation of the determinant of the Hessian, which is a crucial step in the derivation of the action, is deeply connected to the Plefka computation of the resolvent.[28] As a consequence, one sees that, in order to set the parameter B of the determinant equal to zero, as was chosen in both solutions, one must check that the Plefka criterion is satisfied. Therefore, the fact that the BRST-SUSY solution violates it turns out to be not only a problem of physical meaning, but also a problem both for the replica computation and for the mathematical consistency of the solution.

We conclude stressing that although the annealed computation is well defined on mathematical grounds, what really matters for physics is the quenched computation where this problem can possibly be cured considering a full-RSB BRST-SUSY solution. As we will see, this certainly happens at the lower band edge of the quenched complexity (the free energy value at which the complexity vanishes) which, as expected, coincides with the equilibrium free energy given by the Parisi solution and it is BRST symmetric.

B. Spectrum of the Hessian of $F_{\rm tap}(\{m\})$

The spectrum of the eigenvalues of the inverse susceptibility matrix $\boldsymbol{\chi}^{-1}(m)$ on a generic configuration $\{m_i\}$ is determined in Ref. 28 for small eigenvalues and in the region of $\{m\}$ configurations such that $x_{\rm P}\simeq 0$. The eigenvalue distribution is written, at the leading order in N, as[29]

$$\rho(\lambda)=\frac{1}{\pi\sqrt{p}}\sqrt{\lambda-\frac{x_{\rm P}^2}{4p}},\quad |x_{\rm P}|\ll 1,\quad \lambda\ll 1. \tag{87}$$

The minimum of the spectrum is therefore positive, irrespective of the value of $x_{\rm P}$, implying that the TAP free energy is semiconvex everywhere at the extensive level. In the same paper this result was proved rigorously also for generic configurations with a finite $x_{\rm P}$, either positive or negative. We remark that this result, however, *does not exclude the existence of solutions with a subextensive number of negative eigenvalues*. Actually, the Morse theorem implies that these solutions do exist and, indeed, there must be an equal number of solutions with positive and negative determinants. Plefka's result just states that we cannot distinguish them at an extensive level.

C. Determinant of the Hessian

As we have previously discussed, the fact that no negative eigenvalue exists in an extensive quantity does not mean that saddles and maxima do not exist, which is indeed impossible on topological grounds. Therefore, the condition $x_{\rm P}\geqslant 0$ does not bypass the problem of the sign of the determinant of χ^{-1}.

In Appendix A the determinant is computed making use of Grassmann variables and using the saddle point method. If one evaluates the saddle point over B before evaluating the integrals over the m_i, it turns out that there are two solutions $B=0$ and $B>0$ (B is the parameter originally used by Bray and Moore[30]). If $x_{\rm P}\geqslant 0$, the solution $B=0$ satisfies the condition for the saddle point of being a maximum of the integrand on the integration path and is the correct one, while if $x_{\rm P}<0$, the correct solution is the one with $B>0$.

Both the BRST-SUSY and the BRST-SUSY-breaking solutions assume that $B=0$; consequently the expression of the determinant of the Hessian of the TAP free energy becomes

$$\overline{\det\chi^{-1}}=\prod_i\left(\frac{1}{\beta}\frac{1}{1-m_i^2}\right)\exp\left\{N\frac{\beta^2}{2}(1-q)^2\right\}. \tag{88}$$

Since the prefactor is positive for any $\{m_i\}$ configuration, this choice of B *would provide* an *a posteriori* justification for neglecting the absolute value of the determinant *if the calculation were exact* at all orders in N. Anyway, we cannot neglect the fact that all computations are performed at the leading term in N, as $N\to\infty$, thus ignoring all subextensive contributions.

As a matter of fact, the sign also depends on neglected prefactors

$$\overline{\det\chi^{-1}}=\alpha(N)\overline{\det\chi^{-1}}|_{\rm sp}, \tag{89}$$

where $\alpha(N)$ cannot be determined at the leading order in N, but needs to be obtained from corrections of $O(1)$:

$$\frac{1}{N}\ln\overline{\det\chi^{-1}}=\frac{1}{N}\ln\overline{\det\chi^{-1}}\Big|_{\rm sp}+\frac{1}{N}\ln\alpha(N). \tag{90}$$

While the magnitude of the determinant is not qualitatively changed by $O(1)$ corrections (provided they are not zero, of course), its sign can well be determined by eigenvalues that are present in a subextensive quantity.

D. Bray-Moore-Young FRSB quenched complexity is BRST invariant

Both the BRST-SUSY and BRST-SUSY-breaking solutions give a lower band edge (the free energy value at which the complexity vanishes) different from the equilibrium free energy of the SK model. This is not surprising since we are performing annealed averages, while we expect that the physically correct computation is quenched.

In order to cure this deficiency it is rather obvious that one has to go on computing the quenched complexity in the Full RSB scheme, for which the SK model is known to be thermodynamically stable. The quenched case is formally

discussed in Ref. 23, and one instance of FRSB quenched complexity is presented in Ref. 16. The first step in this direction, however, was performed by Bray, Moore, and Young in Ref. 20 where the solution there analyzed had a lower band edge coinciding with the FRSB static one. Now, we notice that the assumptions under which they look for such a solution are exactly the BRST relations (46) and (47) generalized to the function order parameters one has to introduce in the quenched FRSB scheme of computation. Since we are using their very same notation, we can directly rewrite here Eq. (19) of Ref. 20, which we break into two lines

$$\Delta = -\rho(1), \quad \lambda = \frac{\beta^2}{8} u^2 q_{EA}, \tag{91}$$

for the diagonal part (the one surviving in the annealed case), and

$$\rho(x) = \frac{\beta^2}{2} u q(x), \quad \eta^\star(x) = \frac{\beta^2}{4} u^2 q(x), \quad \forall x \in [0,1], \tag{92}$$

for the off-diagonal elements. The further assumption was, then, made:

$$q(1) = q_{EA}. \tag{93}$$

The relations expressed in Eq. (91) [inserting assumption (93)] are exactly our Eqs. (46) and (47). The same holds for the off-diagonal terms if we recognize that, in Eq. (92), the off-equilibrium analog of Δ is $-\rho(x)$ and the analog of λ is $\eta^\star(x)/2$. What was found is, then, the quenched improvement of the annealed BRST-SUSY solution and not the quenched analogue of the BM annealed solution of Ref. 5, for which the above-mentioned relations do not hold. Moreover, the choice of identifying the elements $q(1)$ of the diagonal block of the FRSB matrix with the elements on the diagonal, q_{EA}, leads precisely to the self-consistent, stable, Parisi solution of the SK model.[31] In such a case, though, no parameter is left free to vary and, therefore, no analysis over the number of states at given f can be performed; i.e., no complexity can be built.

It is the subject of another paper[16] to go beyond this point and look for a generalized solution that allows a ''quenched probe'' in a free parameter (the state free energy f or the generalization of the m parameter in the Legendre transform approach), yet recovering the right equilibrium value as lower band edge of the complexity.

IV. CONCLUSIONS

In this paper we have shown that the BM action[5] and the action considered by Cavagna *et al.*[13] are completely equivalent and one can move from one to the other through a simple change of variables. This is also true at the microscopic level, thus implying that the BM action too is BRST-SUSY invariant. In particular, this equivalence implies that each solution of the BM saddle point equations is a solution of the equations of Ref. 13 as well and vice versa. As a consequence we are left with the problem of selecting the ''right'' solution.

In order to determine which one, if any, of the two proposals was the one actually representing the multiplicity of stable and metastable states of the Sherrington-Kirkpatrick mean-field spin-glass model we have been critically reviewing the properties of both, sometimes solving apparent incongruencies, other times pointing out substantial inadequacies.

We first summarize the case of the BM BRST-SUSY breaking annealed complexity. The BM saddle point is not BRST SUSY at any value of u, while we have shown in Sec. III that the BM action satisfies such a supersymmetry, although this is differently expressed in the BM notation with respect to the one of Ref. 13. As already noticed by Kurchan in Ref. 19, we have to be careful and we need some justification before adopting it, if we want to preserve mathematical consistency. In particular, one may show that the BRST SUSY is restored considering subextensive corrections to the saddle point and notice that they amount to a prefactor of $\exp(N\Sigma_{\mathrm{BM}})$. In the case $u=0$, it has been shown by Kurchan by means of a series expansion in power of $1/N$ that the prefactor of the non-BRST saddle point is zero at all orders of the expansion. This could imply a zero prefactor or it could leave the way open to a nonzero but exponentially small prefactor. Both outcomes, however, strongly change the BM prediction. Therefore, to save this prediction, one should show that the effect of keeping the modulus of the determinant amounts to changing the prefactor from an exponentially small value to a finite one. A step in this direction could be possibly done generalizing the technique of Ref. 32 for one-dimensional random systems.

Very recently, in Ref. 18, it was claimed that the BM saddle point counts only minima. This would imply that the complexity does not involve a sum over all solutions and, therefore, is not constrained to satisfy global relations, like those imposed by BRST SUSY or by the Morse theorem. However, there is no proof at all that the BM saddle point counts only minima, since Plefka's analysis of the Hessian shows that all solutions of the TAP equations have strictly positive eigenvalues only at leading order in N. In order to get information about the sign of the determinant one should be able to select the minima among all states satisfying Plefka's criterion. This can be explicitly seen in the spherical p-spin models where these solutions can be classified as minima and saddles, differing only for one negative eigenvalue. The Plefka criterion $x_{\mathrm{P}} \equiv 1-\beta^2(1-2q+1/N\Sigma_i m_i^4) > 0$ is, thus, not related to the fact that a given solution is a minimum or a maximum or a saddle but rather it guarantees that it yields the correct susceptibility.

The lower band edge of the quenched complexity, computed with the Parisi ansatz by Bray, Moore, and Young,[20] gives the correct equilibrium free energy but such solution is not a modification of the BRST-SUSY-breaking annealed solution of Bray and Moore.[5] On the contrary, it turns out to be BRST symmetric, as we have shown in Sec. III D. This also means that, up to now, no quenched extension of the BM annealed saddle point has been taken into account. We will show elsewhere what are the minimal assumptions to break the BRST SUSY in the quenched case.[16]

Furthermore, the non-BRST solution is not in agreement with the numerical results recently obtained by Plefka

through some modified TAP equations.[33] By means of its method he obtains all the minima of the standard TAP free energy satisfying $x_{\mathrm{P}} \geqslant 0$. The special set of solutions he collects—i.e., the minima with $x_{\mathrm{P}} \geqslant 0$—is precisely the set that is supposed to be counted by the BM complexity. Now, the BM solution predicts that the great majority of solutions has a value of x_p, e.g., at temperature $T=0.5$ given by $x_p = 0.132\,408 > 0$ ($\Sigma_{\mathrm{BM}} = 0.002\,775$) and at $T=0.2$ by $x_p = 0.595\,297\,5 > 0$ ($\Sigma_{\mathrm{BM}} = 0.052\,19$). Plefka, however, on the basis of its numerical data, hints that all the minima of the TAP functional in the thermodynamic limit have a zero x_p as $N \to \infty$. Moreover, the range of free energy values over which the complexity is nonzero does not sensitively change with increasing N. This finding is also in agreement with what has been found at the FRSB level of computation, as will be discussed in Ref. 16.

Notice also that the BM result for x_{P} cannot be changed considering the quenched average instead of the annealed one. In Ref. 5 also the replica-symmetric quenched complexity was considered and there the authors showed that the annealed and the quenched non-BRST saddle points coincide at $u > u_c (u_c < 0)$ and, in particular, at $u=0$ to which the highest number of solutions would correspond if the modulus of the determinant of the Hessian was taken into account.

It is worth mentioning, anyway, that the zero-temperature limit of the BM total complexity coincides with the computation of the number of solutions of the zero-temperature limit of the TAP equations, $m_i = \mathrm{sgn}(\Sigma_{ij} J_{ij} m_j)$, where no reaction term is present.[5,21,22] If the coincidence of the exactly zero-temperature behavior with the $T \to 0$ behavior would be a necessary condition, this would be a strong hint that the BM saddle point provides, indeed, the right complexity. However, we notice that this does not exclude the existence of other solutions displaying a $T \to 0$ limit of the complexity different from the value directly computed at $T=0$. For instance, in the p-spin spherical model, a whole branch of TAP solutions existing at zero temperature disappears as soon as we infinitesimally heat the system[34] (see also the Appendix of Ref. 35).

Looking at the other solution, we observed that the BRST-SUSY saddle point does not yield a proper result either. Indeed, it counts TAP solutions that do not satisfy the Plefka criterion—i.e., solutions not corresponding to physical states. Furthermore, even its mathematical consistency is doubtful, since it is obtained setting $B=0$ (see Appendixes A and B), and, according to Plefka's analysis of the resolvent,[28,36] this assumption is only justified if $x_{\mathrm{P}} > 0$, while for $x_{\mathrm{P}} < 0$ a solution with $B \neq 0$ must be considered. It may happen that, upon passing to the quenched computation, the violation of the Plefka criterion of the BRST solution may be cured. Certainly this happens at the full-RSB lower band edge, which, as we said above, is BRST supersymmetric.[20]

In order to solve the problem of selecting a meaningful complexity one can still try to generalize the proof of Kurchan[19] to arbitrary u, possibly using a procedure not involving a series expansion in $1/N$ and/or look for a quenched FRSB solution breaking the BRST SUSY, thus solving a Parisi equation like the one of coupled replicas investigated in Ref. 37 but with a different boundary condition. Both approaches are currently under investigation, together with the study of the quenched BRST solution.[16]

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APPENDIX A: THE DETERMINANT OF THE HESSIAN OF THE TAP FREE ENERGY

The inverse susceptibility matrix is

$$\chi_{ij}^{-1} = -J_{ij} + a_i \delta_{ij}, \tag{A1}$$

$$a_i = \frac{1}{\beta^2} \frac{1}{1-m_i^2} + \beta(1-q) + O\left(\frac{1}{N}\right), \tag{A2}$$

where the J_{ij} are distributed according to

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi}} \exp\left(-\frac{J_{ij}^2 N}{2} \right). \tag{A3}$$

As is usually done, we will not consider terms of order $1/N$ in the following computation.

The determinant can be written with the help of Grassmann variables $(\eta, \overline{\eta})$ as

$$\begin{aligned} \det \chi^{-1} &= \int \prod_{i=1}^{N} \mathrm{d}\eta_i \ \mathrm{d}\overline{\eta}_i \exp\left\{ \sum_{ij} \overline{\eta}_i \chi_{ij}^{-1} \eta_j \right\} \\ &= \int \prod_{i=1}^{N} \mathrm{d}\eta_i \ \mathrm{d}\overline{\eta}_i \exp\left\{ -\sum_{i<j} J_{ij} (\overline{\eta}_i \eta_j + \overline{\eta}_j \eta_i) \right. \\ &\quad \left. + \sum_i \overline{\eta}_i a_i \eta_i \right\}. \end{aligned} \tag{A4}$$

Its average over the disordered interaction is

$$\begin{aligned} &\overline{\det \chi^{-1}} \\ &= \int \prod_{i=1}^{N} \mathrm{d}\eta_i \ \mathrm{d}\overline{\eta}_i \exp\left\{ -\frac{1}{2N} \left(\sum_i \overline{\eta}_i \eta_i \right)^2 + \sum_i \overline{\eta}_i a_i \eta_i \right\} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}w e^{-w^2 N/2}}{\sqrt{2\pi/N}} \int \prod_{i=1}^{N} \mathrm{d}\eta_i \ \mathrm{d}\overline{\eta}_i \\ &\quad \times \exp\left\{ \sum_i \overline{\eta}_i (iw + a_i) \eta_i \right\} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}w e^{-w^2 N/2}}{\sqrt{2\pi/N}} \exp\left\{ \sum_i \ln(iw + a_i) \right\} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{\sqrt{2\pi/N}} \exp[N H(w)], \end{aligned} \tag{A5}$$

$$H(w) \equiv -\frac{w^2}{2} + \frac{1}{N}\sum_i \ln(iw + a_i). \tag{A6}$$

To compute the integral for large N we make use of the saddle point approximation, thus evaluating the solution of

$$\frac{\partial H}{\partial w} = -w + \frac{1}{N}\sum_i \frac{i}{iw + a_i} = 0, \tag{A7}$$

$$\frac{\partial^2 H}{\partial w^2} = -1 + \frac{1}{N}\sum_i \frac{1}{(iw + a_i)^2} < 0. \tag{A8}$$

The second inequality is a condition that in most cases ensures that the integration path can be modified in order to cross the saddle point in the proper way and is analogous to the maximum condition of the Laplace method. Changing iw in

$$iv \equiv iw + \beta(1-q) = iw + a_i - \frac{1}{\beta}\frac{1}{1-m_i^2} \tag{A9}$$

the stationarity condition for the saddle point reads

$$v\left[1 - \frac{\beta^2}{N}\sum_i \frac{(1-m_i^2)^2}{1 + iv\beta(1-m_i^2)}\right] = 0 \tag{A10}$$

and the condition for H becomes

$$1 - \frac{\beta^2}{N}\sum_i \frac{(1-m_i^2)^2}{[1 + iv\beta(1-m_i^2)]^2} > 0. \tag{A11}$$

The saddle point equation (A10) has two solutions: $v=0$, $v = v^\star \neq 0$.

1. Solution $v=0$

In this case Eq. (A11) simplifies to

$$1 - \frac{\beta^2}{N}\sum_i (1-m_i^2)^2 > 0. \tag{A12}$$

This is exactly Plefka's criterion characterizing a physically relevant solution. The stationary value $v=0$ is a maximum of the exponent H and corresponds to TAP solutions yielding the physical expression of the linear susceptibility.

2. Solution $v=v^\star$

For such a saddle point Eq. (A11) can be written as

$$1 - \frac{\beta^2}{N}\sum_i \frac{(1-m_i^2)^2}{1 + iv\beta(1-m_i^2)} + iv\frac{\beta}{N}\sum_i \frac{(1-m_i^2)^3}{[1 + iv\beta(1-m_i^2)]^2} > 0, \tag{A13}$$

of which the first two terms cancel each other. Since $m_i^2 < 1$ always, this implies that in order to have

$$iv\frac{\beta}{N}\sum_i \frac{(1-m_i^2)^3}{[1 + iv\beta(1-m_i^2)]^2} > 0, \tag{A14}$$

iv must be real and positive. In order to go back to the BM notation we define the real variable $B = iv\beta$. If $B>0$, the stationary point $B = iv^\star\beta$ is a maximum of H. Inserting such a positive value into the saddle point, Eq. (A10), one gets the inequality

$$1 = \frac{\beta^2}{N}\sum_i \frac{(1-m_i^2)^2}{1 + B(1-m_i^2)} < \frac{\beta^2}{N}\sum_i (1-m_i^2)^2, \tag{A15}$$

thus violating Plefka's criterion.

Summarizing, if the Plefka criterion is satisfied, the correct solution is $B=0$, while if it is not satisfied, one must choose the solution with $B>0$. Therefore, if one sets *a priori* $B=0$, then one must self-consistently check that the Plefka criterion is verified. Therefore, from this point of view, the BRST solution is mathematically inconsistent. The Plefka criterion arises as a condition to determine which is the correct solution of the resolvent equation; the fact that it is also the condition for determine the correct saddle point for B, as we derived above, is not surprising; indeed the two computation are intimately related as we shown in the following appendix.

APPENDIX B: IDENTIFICATION OF $\det \chi^{-1}$ SADDLE POINT EQUATION AND RESOLVENT EQUATION FOR χ^{-1}

In this appendix we would like to stress the analogy between the saddle point equation for $w = -i[B/\beta + \beta(1-q)]$ (see Appendix A) and the equation for the resolvent of the inverse susceptibility. Using the notation of Ref. 28, the resolvent is

$$R(z) = \frac{1}{N}\mathrm{Tr}\frac{1}{z - \chi^{-1}} = \frac{1}{N}\mathrm{Tr}\frac{1}{z - \mathbf{J} - a_i}, \tag{B1}$$

where the resolvent equation is

$$R(z) = \frac{1}{N}\sum_i \frac{1}{z - R(z) - a_i} \tag{B2}$$

and the condition $\mathrm{Im}[R(z)] > 0$ must hold for $\mathrm{Im}(z) < 0$.

We notice that Eq. (B2) evaluated in $z=0$ is identical to Eq. (A7) in Appendix A provided the transformation $\mathrm{Re}[R(0)] = iw$ is performed. Conversely the condition $\mathrm{Im}[R(z)] > 0$ for $\mathrm{Im}(z) < 0$ is equivalent to Eq. (A8). Indeed the derivative of the resolvent comes out to be

$$\frac{\mathrm{d}R}{\partial z} = -\frac{Y(z)}{1 - Y(z)}, \tag{B3}$$

$$Y(z) \equiv \frac{1}{N}\sum_i \frac{1}{(-z + R(z) + a_i)^2}. \tag{B4}$$

The function $Y(z)$ is always positive, for any real z.

If we expand $R(z)$ around a given real value of $z=z_R$ for a small negative imaginary part $-i\epsilon$, we get

$$R(z_R-i\epsilon)=R(z_R)-i\epsilon\ Y(z_R)\left(1-\left.\frac{dR}{dz}\right|_{z_R}\right). \quad \text{(B5)}$$

Thus, the condition on the imaginary part of $R(z)$ for negative $\mathrm{Im}\,z$ (Pastur theorem[28,38]) reads

$$\lim_{\mathrm{Im}\,z\to 0^-}\mathrm{Im}\,R(z)=\epsilon\lim_{\mathrm{Im}\,z\to 0^-}Y(z)\left(1-\frac{dR}{dz}\right)>0, \quad \text{(B6)}$$

leading to the condition

$$\left.\frac{dR}{dz}\right|_{\mathrm{Im}\,z=0}<1. \quad \text{(B7)}$$

Equation (B3) evaluated at $z=0$ satisfies condition (B7) if $Y(0)<1$.

If we set $R(0)=iw$, Eqs. (B2) and (B7) evaluated at $z=0$ are equivalent respectively to Eqs. (A7) and (A8), thus legitimating this last equation as a validity condition for the saddle point of $H(w)$.

The resolvent equation has two roots, which, for small z and in the region where $\beta^2(1-2q+\langle m^4\rangle)\simeq 1$, were evaluated, e.g., in Ref. 28. They correspond to the $B=0$ and $B>0$ solutions of the previous section. The condition $\mathrm{Im}[R(z)]>0$ for $\mathrm{Im}(z)<0$ selects one or the other solution depending on the value of x_{P} inasmuch as the condition (A8) selects the correct solution $B=0$ or $B>0$ depending on the value of x_{P}.

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$$\overline{\det\chi^{-1}}=\mathrm{ext}_B\exp\left\{N\left[\frac{B^2}{2\beta^2}-B(1-q)+\frac{\beta^2}{2}(1-q)^2+\frac{1}{N}\sum_i\ln\left(\frac{1}{1-m_i^2}+B\right)\right]\right\},$$

where the initially integer positive number of replicas was set to -2 at the end of the computation.
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