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Staggered fermions for chiral gauge theories: Test on a two-dimensional axial-vector model

Wolfgang Bock ^{a,1}, Jan Smit ^{a,2} and Jeroen C. Vink ^{b,3}

^a *Institute of Theoretical Physics, University of Amsterdam, Valckenierstraat 65, NL-1018 XE Amsterdam, The Netherlands*

^b *University of California, San Diego, Department of Physics, 9500 Gilman Drive 0319, La Jolla, CA 92093-0319, USA*

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As a first step towards constructing chiral models on the lattice with staggered fermions, we study a U(1) model with axial-vector coupling to an external gauge field in two dimensions. In our approach gauge invariance is broken, but it is restored in the classical continuum limit. We find that the continuum divergence relations for the vector and axial-vector currents are reproduced, up to contact terms, which we determine analytically. The current divergence relations are also studied numerically for smooth external gauge fields with topological charge zero. We furthermore investigate the effect of fluctuating gauge transformations and of gauge configurations with non-trivial topological charge.

1. Introduction

A consistent non-perturbative formulation of a chiral gauge theory is much desired. Within perturbation theory the quantization of chiral gauge theories has been studied and it is claimed to be solved satisfactorily [1], although a gauge invariant regulator does not appear to exist. Various proposals have been made for a non-perturbative formulation of chiral gauge theories on the lattice. A problem is the doubling phenomenon: a gauge invariant model on a regular lattice is necessarily non-chiral in the sense that each fermion is accompanied by extra degrees of freedom, the so-called species doublers, which couple with opposite chiral charge to the gauge fields and render the theory vector-like. Most of the currently existing lattice proposals try to eliminate the unwanted species doublers either by making them very heavy or by tuning their interactions to zero (for an overview see ref. [2]). One can also use the doublers as physical degrees of

¹ E-mail: bock@phys.uva.nl.

² E-mail: jsmit@phys.uva.nl.

³ E-mail: vink@yukawa.ucsd.edu.

freedom, with the staggered fermion method [3–5], which is the strategy followed in this paper.

As in perturbation theory in the continuum, the regulated lattice model violates gauge invariance, but it is restored in the classical continuum limit. The question is how to restore it in the quantum theory. One can mimic the continuum methods closely, by attempting non-perturbative gauge fixing and adding counterterms to restore gauge invariance [6]. An alternative approach focuses on a possible dynamical restoration of gauge invariance [3,5]. The question of dynamical gauge symmetry restoration will be investigated in an other publication [7]. In this paper we shall test the staggered method in a two-dimensional axial-vector model, where the staggered fermion fields are coupled to external gauge fields. A preliminary account of this work has already been presented in ref. [8].

2. The target model

Our continuum target model is given by the following euclidean action in two dimensions:

$$S = - \int d^2x \left[\bar{\psi} \gamma_\mu (\partial_\mu + i \gamma_5 A_\mu) \psi + m \bar{\psi} \psi \right], \quad (2.1)$$

where A_μ is an external gauge field and $\gamma_5 = -i\gamma_1\gamma_2$. For the purpose of numerical simulations and also for the use in tests we have added a bare mass term for the fermions with mass parameter m . In two dimensions the axial-vector model (2.1) can be rewritten in vector form with a Majorana mass term, e.g. by a charge conjugation transformation on the right-handed fermion fields. So it is not really a chiral gauge theory, and for $m = 0$ it is equivalent to the Schwinger model. However, the technical aspects of “ γ_5 ” in its lattice version are very similar to truly chiral gauge theories. The reason for choosing this target model and not e.g. a left-handed model, is that the staggered fermion version of the axial model has a larger lattice symmetry group.

For $m = 0$ the action (2.1) is invariant under the local gauge transformations $A_\mu(x) \rightarrow A_\mu(x) + i\Omega(x)\partial_\mu\Omega^*(x)$, $\psi(x) \rightarrow [\Omega(x)P_L + \Omega^*(x)P_R]\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x)[\Omega(x)P_L + \Omega^*(x)P_R]$, with $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$ and $\Omega(x) \in U(1)$. The action is furthermore invariant under the global vector symmetry $\psi(x) \rightarrow \Omega\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x)\Omega^*$. The vector and axial-vector currents, $J_\mu^V = i\bar{\psi}\gamma_\mu\psi$ and $J_\mu^A = i\bar{\psi}\gamma_\mu\gamma_5\psi$, satisfy the classical divergence equations

$$\partial_\mu J_\mu^V = 0, \quad (2.2)$$

$$\partial_\mu J_\mu^A = 2mJ^P, \quad (2.3)$$

where $J^P = i\bar{\psi}\gamma_5\psi$ is the pseudoscalar density.

As is well known these classical relations (2.2), (2.3) may be invalidated by quantum effects. Requiring gauge invariance for $m = 0$, eq. (2.3) has to remain valid but (2.2) becomes anomalous and takes the form

$$\partial_\mu J_\mu^V = 2iq, \quad q = \frac{1}{2\pi} F_{12}, \quad (2.4)$$

where $F_{12}(x) = \partial_1 A_2(x) - \partial_2 A_1(x)$ is the field strength and $q(x)$ is the topological charge density. The topological charge Q is defined by $Q = \int d^2x q(x)$. Our aim is to study the divergence equations (2.2) and (2.4) in the lattice version of the model.

Let us briefly review the anomaly structure of the current divergences in the quantum theory. We evaluate $\langle \partial_\mu J_\mu^{V,A} \rangle$ and $\langle J^P \rangle$ in perturbation theory, by expansion in A_μ , which leads to a series of diagrams in which external A_μ -lines are attached to a fermion loop. We concentrate on the diagrams shown in fig. 1a, since diagrams with one external line vanish and diagrams with more than two external lines are convergent by power counting. We shall evaluate these diagrams using a spherical cutoff in momentum space, because this regularization has analogies to our staggered fermion formulation introduced later. Such a cutoff violates gauge invariance, but the desired result is easily obtained by adding suitable contact terms. For a discussion (including e.g. the gauge invariant point splitting method, which gives rise to the additional diagrams in fig. 1b) see ref. [9].

Although the diagrams in fig. 1a seem to be logarithmically divergent, the momentum cutoff gives a finite answer when it is removed. Fig. 1a leads to

$$\begin{aligned} T_{\mu\nu}^{VA}(p) &\equiv \int d^4x \exp(-ix \cdot p) \langle J_\mu^V(x) J_\nu^A(0) \rangle_{A=0} \\ &= \frac{i}{2\pi} \int_0^1 dz \frac{m^2 \epsilon_{\mu\nu} - z(1-z)(\epsilon_{\mu\alpha} p_\alpha p_\nu + \epsilon_{\nu\alpha} p_\alpha p_\mu)}{m^2 + z(1-z)p^2} + C_{\mu\nu}^{VA}, \end{aligned} \quad (2.5)$$

where $C_{\mu\nu}^{VA}$ is a contact term to be determined shortly. From this expression we get the two Ward identities

$$\begin{aligned} ip_\mu T_{\mu\nu}^{VA}(p) &= \frac{1}{2\pi} \epsilon_{\nu\alpha} p_\alpha + ip_\mu C_{\mu\nu}^{VA}, \\ -ip_\nu T_{\mu\nu}^{VA}(p) &= 2mT_\mu^{VP}(p) - \frac{1}{2\pi} \epsilon_{\mu\alpha} p_\alpha - ip_\nu C_{\mu\nu}^{VA}, \end{aligned} \quad (2.6)$$

where the amplitude

$$T_\mu^{VP}(p) = \frac{1}{2\pi} \int_0^1 dz \frac{m \epsilon_{\mu\alpha} p_\alpha}{m^2 + z(1-z)p^2}, \quad (2.7)$$

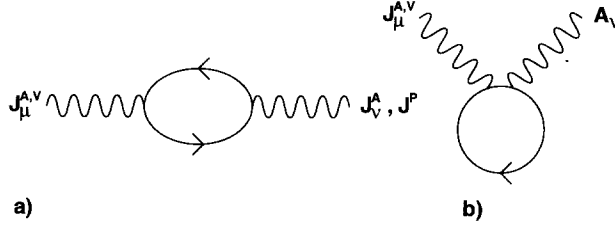


Fig. 1. Feynman diagrams for $T_{\mu\nu}^{\text{VA}}(p)$, $T_{\mu}^{\text{VP}}(p)$, $T_{\mu\nu}^{\text{AA}}(p)$, and $T_{\mu}^{\text{AP}}(p)$.

results from the evaluation of the $J_{\mu}^{\text{V}}-J^{\text{P}}$ correlation function at $A_{\mu} = 0$ (see fig. 1a). These relations show anomalous terms in both the vector and axial-vector Ward identities. By choosing the contact terms $C_{\mu\nu}^{\text{VA}} = \pm(i/2\pi)\epsilon_{\mu\nu}$ one may shift the anomaly either to the vector (+) or to the axial-vector Ward identity (-). Since we insist on axial gauge invariance the contact term is determined as

$$C_{\mu\nu}^{\text{VA}} = + \frac{i}{2\pi} \epsilon_{\mu\nu}, \quad (2.8)$$

giving

$$ip_{\mu} T_{\mu\nu}^{\text{VA}}(p) = \frac{1}{\pi} \epsilon_{\nu\alpha} p_{\alpha}, \quad -ip_{\nu} T_{\mu\nu}^{\text{VA}}(p) = 2m T_{\nu}^{\text{VP}}(p). \quad (2.9)$$

For $\langle J_{\mu}^{\text{A}} \rangle$ fig. 1a gives

$$\begin{aligned} T_{\mu\nu}^{\text{AA}}(p) &\equiv \int d^4x \exp(-ix \cdot p) \langle J_{\mu}^{\text{A}}(x) J_{\nu}^{\text{A}}(0) \rangle_{A=0} \\ &= -\frac{1}{\pi} \int_0^1 dz \frac{(\delta_{\mu\nu} p^2 - p_{\mu} p_{\nu})z(1-z) + m^2 \delta_{\mu\nu}}{m^2 + z(1-z)p^2} + \frac{1}{2\pi} \delta_{\mu\nu} + C_{\mu\nu}^{\text{AA}}. \end{aligned} \quad (2.10)$$

With the choice

$$C_{\mu\nu}^{\text{AA}} = -\frac{1}{2\pi} \delta_{\mu\nu}, \quad (2.11)$$

we get from this relation the desired Ward identity

$$ip_{\nu} T_{\mu\nu}^{\text{AA}}(p) = 2m T_{\mu}^{\text{AP}}(p), \quad T_{\mu}^{\text{AP}}(p) = -\frac{i}{2\pi} \int_0^1 dz \frac{mp_{\mu}}{m^2 + z(1-z)p^2}, \quad (2.12)$$

with $T_{\mu}^{\text{AP}}(p)$ the $J_{\mu}^{\text{A}}-J^{\text{P}}$ correlation function at $A_{\mu} = 0$. From the above results for the current correlation functions one can derive the divergence equations (2.3) and (2.4).

We shall show in the following that the symmetry properties of the model, as they manifest themselves in the current divergence relations (2.3) and (2.4), can be recovered in a staggered fermion version of the model on the lattice.

3. The staggered fermion model

We generalize our target model to two flavors, which makes it somewhat simpler to describe in the staggered fermion formalism. We introduce 2×2 matrix fermion fields $\Psi_x^{\alpha\kappa}$ and $\bar{\Psi}_x^{\kappa\alpha}$ on a two-dimensional square lattice, where α and κ are Dirac and flavor indices, respectively. Using these matrix fields we find after the naive lattice transcription of the two-flavor version of the target model in eq. (2.1) the following action:

$$S = - \sum_{x\mu} \frac{1}{2} \text{Tr} \left[\bar{\Psi}_x \gamma_\mu (U_{\mu x}^L P_L + U_{\mu x}^R P_R) \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} \gamma_\mu (U_{\mu x}^{L*} P_L + U_{\mu x}^{R*} P_R) \Psi_x \right] - m \sum_x \text{Tr} (\bar{\Psi}_x \Psi_x), \quad (3.1)$$

$$U_{\mu x}^L = \exp(-iaA_{\mu x}), \quad U_{\mu x}^R = \exp(iaA_{\mu x}), \quad (3.2)$$

with a the lattice spacing. We shall use lattice units, $a = 1$. The action above would be gauge invariant if $\Psi_x^{\alpha\kappa}$ and $\bar{\Psi}_x^{\kappa\alpha}$ would be independent degrees of freedom, and this would lead to fermion doublers with opposite chiralities.

The fermion doublers are situated at the boundary of the Brillouin zone in momentum space, i.e. at momenta $p_\mu = \pm\pi$. Consider restricting the momenta of the matrix fields such that $-\frac{1}{2}\pi < p_\mu \leq +\frac{1}{2}\pi$. Then we loose the fermion doublers of the matrix fields, but also gauge invariance. It is clear, however, that in the classical continuum limit, where the field momenta go to zero (in lattice units), eq. (3.1) goes over into eq. (2.1) with two flavors. Hence gauge invariance gets restored in this limit. One might think that the cutoff in momentum space has to result in a non-local action. However, it is possible to express the action in a form that is local, using staggered fermions.

Staggered fermion fields on the lattice, denoted by the one-component fields χ_x and $\bar{\chi}_x$, do not carry explicit flavor and Dirac labels. These labels are supplied through the doubling phenomenon. In the classical continuum limit one recovers the usual Dirac and flavor structure. We make the connection with the $\Psi_x^{\alpha\kappa}$ and $\bar{\Psi}_x^{\kappa\alpha}$ by writing [3,5]

$$\Psi_x = \frac{1}{2\sqrt{2}} \sum_b \gamma^{x+b} \chi_{x+b}, \quad \bar{\Psi}_x = \frac{1}{2\sqrt{2}} \sum_b (\gamma^{x+b})^\dagger \bar{\chi}_{x+b}, \quad (3.3)$$

where $\gamma^x \equiv \gamma_1^{x_1} \gamma_2^{x_2}$ and the sum runs over the corners of an elementary lattice square, $b_\mu = 0, 1$. In momentum space we have the relation [4]

$$\Psi_{\alpha\kappa}(p) = z(p) \sum_b T_{\alpha\kappa,b} \chi(p + \pi b), \quad -\frac{1}{2}\pi < p \leq \frac{1}{2}\pi, \quad (3.4)$$

where $T_{\alpha\kappa,b} = \sum_c \frac{1}{4} \exp(ibc\pi) \gamma_{\alpha\kappa}^c$ is a unitary matrix and $z(p)$ is a non-vanishing function in the restricted momentum interval. This shows clearly that in this restricted interval the Fourier components of the matrix fields are independent.

Having expressed the matrix fermion fields in terms of the independent χ_x and $\bar{\chi}_x$ fields, substitution into the action (3.1) leads to a *local* action. The indices α and κ on Ψ and $\bar{\Psi}$ act like Dirac and flavor indices and one can construct staggered fermion models involving arbitrary spin–flavor couplings to other fields in a straightforward manner such that the target models are recovered in the classical continuum limit. For the Standard Model and Grand Unified Theories like SO(10) and SU(5) this can be done such that the staggered fermion symmetry group is preserved [5]. This invariance is important for reducing the number of counterterms needed to get a satisfactory continuum limit [10]. This strategy of coupling the staggered fermion spin–flavors has recently been successfully applied to a fermion–Higgs model [11].

By working out the trace in (3.1) one obtains the action in terms of the staggered fermion fields,

$$\begin{aligned} S = & -\frac{1}{2} \sum_{x\mu} \left(c_{\mu x} \cdot \frac{1}{4} \sum_b \eta_{\mu x+b} (\bar{\chi}_{x+b} \chi_{x+b+\hat{\mu}} - \bar{\chi}_{x+b+\hat{\mu}} \chi_{x+b}) \right. \\ & \left. - s_{\mu x} \cdot \frac{1}{4} \sum_{b+c=n} \eta_{12x+c} (\eta_{\mu x+c} \bar{\chi}_{x+b} \chi_{x+c+\hat{\mu}} - \eta_{\mu x+b} \bar{\chi}_{x+b+\hat{\mu}} \chi_{x+c}) \right) - m \sum_x \bar{\chi}_x \chi_x, \end{aligned} \quad (3.5)$$

with the abbreviations $c_{\mu x} = \text{Re } U_{\mu x}$, $s_{\mu x} = \text{Im } U_{\mu x}$ and $n = (1, 1)$. The sign factors $\eta_{1x} = 1$ and $\eta_{2x} = (-1)^{x_1}$ represent the Dirac matrices γ_1 and γ_2 and $\eta_{12x} = \eta_{2x} \eta_{1x+\hat{2}} = (-1)^{x_1}$ the $i\gamma_5 = \gamma_1 \gamma_2$. In the classical continuum limit this action describes two flavors of axially coupled Dirac fermions.

For a one-flavor staggered fermion model we would have used only one Grassmann variable per site (so-called “reduced” or “real” staggered fermions). One then defines the $\bar{\chi}_x$ fields usually on the even, and the χ_x fields on the odd lattice sites. This would require a one-link mass term instead of the simple one-site mass term in (3.5). The continuum interpretation is in this case somewhat more involved [4,12].

The couplings in the $s_{\mu x}$ term in the action (3.5) are not confined within a plaquette. For example, for $c = 0$ and $b = n$ we have three-link couplings. The

action is of course not unique. According to standard staggered fermion properties, the three-link couplings may be replaced by one-link couplings by shifting the χ or $\bar{\chi}$ field over two lattice spacings in the same direction (even shifts). Such actions are equivalent in the sense that they lead to the same classical continuum limit, and in the quantum theory they are expected to be in the same universality class. It is instructive to give here a particularly simple alternative to (3.5),

$$S = - \sum_{x\mu} \left[\bar{c}_{\mu x} \eta_{\mu x} \cdot \frac{1}{2} (\bar{\chi}_x \chi_{x+\hat{\mu}} - \bar{\chi}_{x+\hat{\mu}} \chi_x) - \bar{s}_{\mu x} \varepsilon_{\mu\nu} \eta_{\nu x} \cdot \frac{1}{2} (\bar{\chi}_x \chi_{x+\hat{\nu}} + \bar{\chi}_{x+\hat{\nu}} \chi_x) \right] - \sum_x m \bar{\chi}_x \chi_x, \quad (3.6)$$

$$\bar{c}_{\mu x} = \frac{1}{4} \sum_b c_{\mu x-b}, \quad \bar{s}_{\mu x} = \frac{1}{4} \sum_b s_{\mu x-b}, \quad (3.7)$$

obtained by using $c = n - b$, the identities $\eta_{12x+n} \eta_{\mu x+n} = \varepsilon_{\mu\nu} \eta_{\nu x}$, $n + \hat{\mu} = \hat{\nu} + 2\hat{\mu}$, and the equivalence of e.g. $\bar{\chi}_x \chi_{x+n+\hat{\nu}+2\hat{\mu}-2b}$ with $\bar{\chi}_x \chi_{x+n+\hat{\nu}}$ which differ by even shifts. A further reduction could be achieved by the replacements $\bar{c}_{\mu x} \rightarrow 1$ and $\bar{s}_{\mu x} \rightarrow s_\mu$ in the above expression, as the resulting model still has the same classical continuum limit.

A model with all the couplings confined within a plaquette may be called a canonical model, as it allows for a canonical construction of the transfer operator [12,13]. In the following we shall use, however, the original version as written in eq. (3.5).

For $m = 0$ the action (3.1) appears to be invariant under the local gauge transformations,

$$U_{\mu x} \rightarrow \Omega_x U_{\mu x} \Omega_{x+\hat{\mu}}^*, \quad (3.8)$$

$$\Psi_x \rightarrow (\Omega_x P_L + \Omega_x^* P_R) \Psi_x, \quad \bar{\Psi}_x \rightarrow \bar{\Psi}_x (\Omega_x P_L + \Omega_x^* P_R). \quad (3.9)$$

However, this gauge invariance is broken because of the momentum space cutoff on the matrix fields. The four components of the Ψ_x and $\bar{\Psi}_x$ matrix fields are not independent, as is evident from their expression in terms of the independent χ_x and $\bar{\chi}_x$ fields. Therefore we cannot translate the gauge transformations on Ψ and $\bar{\Psi}$ to local transformations on χ and $\bar{\chi}$, and the action (3.5) lacks gauge invariance.

The global vector U(1) transformation

$$\chi_x \rightarrow \exp(i\omega) \chi_x, \quad \bar{\chi}_x \rightarrow \bar{\chi}_x \exp(-i\omega) \quad (3.10)$$

is an exact symmetry of the actions (3.5) and (3.6). For $m = 0$ there is furthermore a second exact global U(1) invariance, the ‘‘U(1)_e’’ invariance $\chi_x \rightarrow \exp(i\omega \varepsilon_x) \chi_x$, $\bar{\chi}_x \rightarrow \bar{\chi}_x \exp(i\omega \varepsilon_x)$, $\varepsilon_x = (-1)^{x_1+x_2}$, which corresponds to a flavor non-singlet chiral

transformation $\Psi_x \rightarrow \cos \omega \Psi_x + i \sin \omega \gamma_5 \Psi_x \gamma_5$, $\bar{\Psi}_x \rightarrow \cos \omega \bar{\Psi}_x + i \sin \omega \gamma_5 \bar{\Psi}_x \gamma_5$. These $U(1) \times U(1)$ transformations are part of the $U(2) \times U(2)$ global invariance of the classical two-flavor continuum action. In a one-flavor staggered fermion model $U(1)_\epsilon$ would be the only global $U(1)$ symmetry.

We are not concerned here with the full aspects of flavor symmetry restoration, but concentrate on the flavor singlet vector and axial-vector currents, $J_{\mu x}^V$ and $J_{\mu x}^A$. The latter is the gauge current which should be exactly conserved. The former is the $U(1)$ vector current which should have the anomaly discussed in the previous section.

4. Divergence equations on the lattice

In this section we determine the contact terms in the divergence equations for the gauge current J_μ^A and a flavor singlet $U(1)$ current J_μ^V in our lattice model (3.5).

To introduce these currents we generalize (3.1), (3.2) to include also an external vector field $V_{\mu x}$, writing

$$U_{\mu x}^L = \exp(-iA_{\mu x} - iV_{\mu x}), \quad U_{\mu x}^R = \exp(+iA_{\mu x} - iV_{\mu x}). \quad (4.1)$$

Since gauge invariance is broken we expect to have to add counterterms to the action to restore it in the continuum limit. From sect. 2 we expect these to have the form

$$S_{\text{ct}} = \sum_x \left(\frac{1}{2} C_{\mu\nu}^{\text{AA}} A_{\mu x} A_{\nu x} + C_{\mu\nu}^{\text{VA}} V_{\mu x} A_{\nu x} + \frac{1}{2} C_{\mu\nu}^{\text{VV}} V_{\mu x} V_{\nu x} \right). \quad (4.2)$$

The coefficients C^{AA} and C^{VA} have to be determined such that in the scaling region the effective action obtained by integrating out the fermion fields is invariant for $m=0$ under the gauge transformations (3.8) on $A_{\mu x}$. We have included a C^{VV} term which may be determined such that under gauge transformations on $V_{\mu x}$ the effective action in the scaling region suffers only the anomaly and no further symmetry breaking. There is no reason, of course, for the numerical values of C^{AA} and C^{VA} to be the same as in sect. 2, where we used the spherical cutoff as a regulator.

The above counterterms may be extended to periodic functions in A_μ and V_μ . E.g. writing $C_{\mu\nu}^{\text{AA}} = \tau \delta_{\mu\nu}$ the C^{AA} term may be replaced by the standard lattice form for a gauge field mass term $\tau \sum_{x\mu} (1 - \cos A_{\mu x})$. This replacement could help to reduce the scaling violations. In this paper we shall however stay with the form (4.2).

The currents $J_\mu^{V,A}$ are identified from the total action $S + S_{\text{ct}}$ by letting $A_\mu \rightarrow A_\mu + \delta A_\mu$, $V_\mu \rightarrow V_\mu + \delta V_\mu$, and collecting terms linear in δA_μ and δV_μ . The

current correlation functions are obtained by differentiating the effective action with respect to A_μ and V_μ . As we shall study $\langle J_\mu^{\text{V,A}} \rangle$ only for zero V_μ (but for arbitrary A_μ), we give the currents here for $V_\mu = 0$,

$$J_{\mu x}^{\text{V}} = \frac{1}{2}i \text{Tr} \left[\bar{\Psi}_x \gamma_\mu (U_{\mu x} P_L + U_{\mu x}^* P_R) \Psi_{x+\hat{\mu}} + \bar{\Psi}_{x+\hat{\mu}} \gamma_\mu (U_{\mu x}^* P_L + U_{\mu x} P_R) \Psi_x \right] + C_{\mu\nu}^{\text{VA}} A_{\nu x}, \quad (4.3)$$

$$J_{\mu x}^{\text{A}} = -\frac{1}{2}i \text{Tr} \left[\bar{\Psi}_x \gamma_\mu (U_{\mu x} P_L - U_{\mu x}^* P_R) \Psi_{x+\hat{\mu}} + \bar{\Psi}_{x+\hat{\mu}} \gamma_\mu (U_{\mu x}^* P_L - U_{\mu x} P_R) \Psi_x \right] + C_{\mu\nu}^{\text{AA}} A_{\nu x}, \quad (4.4)$$

where $U_{\mu x} = U_{\mu x}^{\text{L}} = \exp(-iA_{\mu x})$. A natural choice for the pseudoscalar density is given by

$$J_x^{\text{P}} = i \text{Tr} (\bar{\Psi}_x \gamma_5 \Psi_x). \quad (4.5)$$

The currents in terms of the staggered field χ are obtained by inserting the relations (3.3) into (4.3)–(4.5),

$$J_{\mu x}^{\text{V}} = \frac{1}{8}i \left(c_{\mu x} \sum_b \eta_{\mu x+b} (\bar{\chi}_{x+b} \chi_{x+b+\hat{\mu}} + \bar{\chi}_{x+b+\hat{\mu}} \chi_{x+b}) - s_{\mu x} \sum_{b+c=n} \eta_{12x+c} (\eta_{\mu x+c} \bar{\chi}_{x+b} \chi_{x+c+\hat{\mu}} + \eta_{\mu x+b} \bar{\chi}_{x+b+\hat{\mu}} \chi_{x+c}) \right) + C_{\mu\nu}^{\text{VA}} A_{\nu x}, \quad (4.6)$$

$$J_{\mu x}^{\text{A}} = \frac{1}{8} \left(c_{\mu x} \sum_{b+c=n} \eta_{12x+c} (\eta_{\mu x+c} \bar{\chi}_{x+b} \chi_{x+c+\hat{\mu}} - \eta_{\mu x+b} \bar{\chi}_{x+b+\hat{\mu}} \chi_{x+c}) - s_{\mu x} \sum_b \eta_{\mu x+b} (\bar{\chi}_{x+b} \chi_{x+b+\hat{\mu}} - \bar{\chi}_{x+b+\hat{\mu}} \chi_{x+b}) \right) + C_{\mu\nu}^{\text{AA}} A_{\nu x}, \quad (4.7)$$

$$J_x^{\text{P}} = \frac{1}{4} \sum_{b+c=n} \eta_{12x+b} \bar{\chi}_{x+b} \chi_{x+c}. \quad (4.8)$$

The above vector current $J_{\mu x}^{\text{V}}$ is not directly related to the exact global U(1) invariance (3.10) which we mentioned in the previous section, because the prescription (4.1) does not make this symmetry an exact local symmetry.

To obtain the divergence relation of the conserved current $j_{\mu x}^{\text{V}}$ which is associated with the exact U(1) symmetry (3.10) we replace $\chi_x \rightarrow \exp(i\omega_x) \chi_x$ and $\bar{\chi}_x \rightarrow \bar{\chi}_x \exp(-i\omega_x)$ in the action (3.5) and collect terms linear in ω_x . But then the three-link couplings in (3.5) lead to an awkward looking divergence equation. The

usual form of the divergence equation is obtained in the canonical model (3.6). We find

$$\begin{aligned} \partial'_\mu j_\mu^V &= 0, \\ j_{\mu x}^V &= i \sum_\mu \left[\bar{c}_{\mu x} \eta_{\mu x} \cdot \frac{1}{2} (\bar{\chi}_x \chi_{x+\hat{\mu}} + \bar{\chi}_{x+\hat{\mu}} \chi_x) - \bar{s}_{\nu x} \epsilon_{\mu\nu} \eta_{\mu x} \cdot \frac{1}{2} (\bar{\chi}_x \chi_{x+\hat{\mu}} - \bar{\chi}_{x+\hat{\mu}} \chi_x) \right]. \end{aligned} \quad (4.9)$$

The potential problem of the appearance of additional global symmetries on the lattice has been emphasized in particular in ref. [14], in the context of fermion number nonconservation in the Standard Model. We shall assume here the following resolution, which is, as we believe, in accordance with current lore [15]. We can construct many currents, each of which reduces in the scaling region to a linear combination of the gauge invariant J_μ^V and the gauge variant $\epsilon_{\mu\nu} A_\nu$. In particular, the exactly conserved but gauge non-invariant current $j_{\mu x}^V$ will reduce to the divergence-free combination $J_\mu^V - (i/\pi) \epsilon_{\mu\nu} A_\nu$. Because this current is not gauge invariant, the corresponding conserved charge is unphysical. It may have a physical (gauge invariant) component, but there is no reason why this should be conserved (for an exposition of the physics of the equivalent Schwinger model, see for example ref. [16]).

In this way the non-gauge invariance of our lattice model provides presumably a possible way out of the embarrassing exact global U(1) invariance. Another possibility to avoid difficulties with undesired global invariances is to construct the lattice models such that additional extra symmetries do not emerge [4,14,17]. The potential problem of having a larger global symmetry group on the lattice than in the continuum target model, requires further detailed investigation. However, in this paper we shall restrict ourselves to a study of the currents $J_{\mu x}^V$ and $J_{\mu x}^A$ as given in the eqs. (4.6) and (4.7).

With the above definitions of the currents and using the staggered fermion formalism outlined in refs. [10,12] we derived the lattice analogues of the Ward identities in sect. 2. Let us first concentrate on the vector current. The diagram in fig. 1a gives a non-zero contribution, whereas the contribution from the typical lattice tadpole diagram in fig. 1b happens to vanish. The amplitude then reads

$$\begin{aligned} T_{\mu\nu}^{VA}(p) &= i \exp\left[\frac{1}{2}i(p_\nu - p_\mu)\right] \\ &\times \int_q \frac{m^2 \epsilon_{\mu\nu} + \epsilon_{\mu\alpha} s(q_\alpha) s(q_\nu + p_\nu) + \epsilon_{\nu\alpha} s(q_\alpha + p_\alpha) s(q_\mu)}{D(q) D(q+p)} \\ &\times c(q_\mu + \frac{1}{2}p_\mu) c(q_\nu + \frac{1}{2}p_\nu) \prod_j c(\frac{1}{2}p_j) c(q_j + \frac{1}{2}p_j), \end{aligned} \quad (4.10)$$

where $s(q_j) = \sin q_j$, $c(q_j) = \cos q_j$, $D(q) = \sum_\alpha \sin^2 q_\alpha + m^2$ and $f_q = \int_{-\pi/2}^{+\pi/2} d^2q/\pi^2$. To calculate the continuum limit of (4.10) we let p and m approach zero, and separate the integration region into a small ball around the origin, $|q| < \delta$, with radius $\delta \ll \frac{1}{2}\pi$ and the outer region, $|q| \geq \delta$ (see, for example, ref. [10]). We let $\delta \rightarrow 0$, $m/\delta \rightarrow 0$ and $p/\delta \rightarrow 0$, with p/m fixed. In the inner region we can replace the integrand by its covariant form, while in the outer region the integrand is expanded in powers of m , p and only the non-vanishing terms are kept. For the integral (4.10) the contribution from the outer region vanishes and only the contribution from the inner region remains, which is exactly a continuum loop integral with a spherical cutoff δ , and coincides with eq. (2.5), except for a factor of two corresponding to the two flavors. Consequently, after normalizing to one flavor we have the same contact term (2.8) as in the cutoff regulated continuum theory. So we find the following vector current divergence relation on the lattice:

$$\partial'_\mu \langle J_{\mu x}^V \rangle_\chi = 2iF_x/2\pi + O(a), \quad (4.11)$$

where the contact term $C_{\mu\nu}^{VA}$ in the definition (4.6) is given in eq. (2.8) and F_x is a suitable form for the field strength, cf. eq. (5.2) below. Here $\partial'_\mu f_{\mu x} \equiv \sum_\mu (f_{\mu x} - f_{x-\hat{\mu}})$ is the divergence on the lattice and $\langle \cdot \rangle_\chi$ denotes the integration over the χ fields and includes here and in the following also a normalization to one staggered flavor. The $O(a)$ indicates terms which arise due to the discretization and vanish when $a \rightarrow 0$.

For the amplitude $T_{\mu\nu}^{AA}$ we obtained the expression

$$\begin{aligned} T_{\mu\nu}^{AA}(p) &= T_{\mu\nu}^{AA(a)}(p) + T_{\mu\nu}^{AA(b)}(p) + C_{\mu\nu}^{AA}, \quad (4.12) \\ T_{\mu\nu}^{AA(a)}(p) &= -\exp\left[\frac{1}{2}i(p_\nu - p_\mu)\right] \\ &\times \int_q \frac{m^2 \delta_{\mu\nu} + s(q_\mu)s(q_\nu - p_\nu) - \epsilon_{\mu\alpha}\epsilon_{\nu\beta}s(q_\alpha)s(q_\beta + p_\beta)}{D(q)D(q+p)} \\ &\times c\left(q_\mu + \frac{1}{2}p_\mu\right)c\left(q_\nu + \frac{1}{2}p_\nu\right) \prod_j c\left(q_j + \frac{1}{2}p_j\right)^2, \quad (4.13) \end{aligned}$$

$$T_{\mu\nu}^{AA(b)}(p) = -\delta_{\mu\nu} \int_q \frac{s(q_\mu)^2}{D(q)}, \quad (4.14)$$

where $T_{\mu\nu}^{AA(a)}(p)$ and $T_{\mu\nu}^{AA(b)}(p)$ denote the contributions from the diagrams in figs. 1a and 1b. The amplitude of the tadpole diagram is independent of the external momentum p . After a normalization to one flavor, we find the lattice divergence relation for the axial-vector current,

$$\partial'_\mu \langle J_{\mu x}^A \rangle_\chi = 2m \langle J_x^P \rangle_\chi + O(a), \quad (4.15)$$

where the contact term coefficient $C_{\mu\nu}^{\text{AA}}$ in (4.7) has to be chosen as

$$C_{\mu\nu}^{\text{AA}} = \tau \delta_{\mu\nu}, \quad \tau = -\left(\frac{1}{2\pi} + I_1 + I_2\right) \approx 0.0625, \quad (4.16)$$

$$I_1 = \frac{1}{2} \int_q \frac{[s(q_2)^2 - s(q_1)^2] c(q_1)^4 c(q_2)^2}{[s(q_1)^2 + s(q_2)^2]^2} \approx 0.0283,$$

$$I_2 = -\frac{1}{2} \int_q \frac{s(q_1)^2}{[s(q_1)^2 + s(q_2)^2]^2} = -\frac{1}{4}. \quad (4.17)$$

The first two terms, $1/2\pi$ and I_1 , in (4.16) come from inner and outer region parts of the integral (4.13) and I_2 from the lattice integral (4.14).

An alternative form of the staggered fermion action is given by eq. (3.5), but with non-compact gauge fields, i.e. with the replacements $c_{\mu x} \rightarrow 1$ and $s_{\mu x} \rightarrow A_{\mu x}$. Of course, this form also reduces in the classical continuum limit to the target model with two flavors. Using this modified form of the action we would have to drop the U fields from eq. (4.4). Then the tadpole diagram in fig. 1b would not have given a contribution and we would instead have obtained $C_{\mu\nu}^{\text{AA}} \approx -0.1875\delta_{\mu\nu}$ which is larger than the result in (4.16). This shows that the form (3.5) is more appropriate for restoring gauge invariance. However, unlike with the continuum point split current [9], a contact term is still needed because the staggered theory is not gauge invariant.

5. Numerical results

In this section we numerically compute the current divergence relations (4.11) and (4.15) for smooth external gauge field configurations with variable amplitude. A similar test [18] has been recently applied to a proposal using domain wall fermions [19]. Since gauge invariance is violated in the staggered model it is interesting to investigate the effect of fluctuating gauge transformations on the divergence relations. As in an earlier work [20] we shall also investigate the effects of external gauge fields with $Q \neq 1$.

5.1. SMOOTH EXTERNAL GAUGE FIELD CONFIGURATIONS

To test the relation (4.11) for the vector current, we use the same external fields as in ref. [18] which are spatially constant,

$$A_{1x} = A_0 \sin(2\pi t/T), \quad A_{2x} = 0, \quad (5.1)$$

with $t \equiv x_2$. We shall use here and in the following a lattice, with extents T and L in the time and spatial directions. The gauge fields $U_{\mu x}$ are close to one everywhere provided the amplitude A_0 is sufficiently small. The topological charge Q is equal to zero for this class of configurations.

To reduce the discretization error, which arises naturally when transcribing a continuum gauge field configuration to the lattice, we use for F_x in (4.11) the average over the four lattice plaquettes adjoined to the point x :

$$F_x = \frac{1}{4} \sum_b F_{12x-b}, \quad (5.2)$$

with $F_{12x} = \partial_1 A_{2x} - \partial_2 A_{1x}$ the plaquette field strength ($\partial_\mu f_x \equiv f_{x+\hat{\mu}} - f_x$). Fig. 2a shows the divergence $-i\partial'_\mu \langle J_{\mu x}^V \rangle_\chi$ as a function of t for various values of A_0 . The quantity F_x/π is represented in this plot by the full lines which were obtained by connecting the points at $t = 1, \dots, T$ by a cubic spline. We use a lattice with $T = L = 64$ and periodic boundary conditions for the χ fields. To regulate the near zero mode we used a small non-zero mass $m = 0.01$. In fig. 2a and in the following graphs we have multiplied the currents and the anomaly by the lattice volume $L \times T$. For the smaller amplitudes the agreement between the anomaly and the current divergence is almost perfect, showing that the $O(a)$ effects in eq. (4.11) are very small. We find the maximal relative error to increase from 0.1% to 5% when A_0 is raised from 0.1 to 0.25.

To test the divergence relation (4.15) for the axial-vector current we use the field

$$A_{1x} = 0, \quad A_{2x} = A_0 \sin(2\pi t/T), \quad (5.3)$$

which in contrast to the previous case is longitudinal, i.e. $\epsilon_{\mu\nu} \partial_\mu A_\nu = 0$ and $\partial'_\mu A_\mu \neq 0$. In fig. 2b, $\partial'_\mu \langle J_{\mu x}^A \rangle_\chi$ is represented by the various symbols, which as in fig. 2a correspond to different values of the amplitude A_0 . The quantity $2m \langle J_x^P \rangle_\chi$ is represented by the solid lines which were obtained by connecting the data points at $t = 1, \dots, T$ by cubic splines. The plot shows that the relation (4.15) holds nicely within the given range of A_0 values. The relative error is smaller than 6%. The complicated looking A_0 -dependence of $\partial'_\mu \langle J_{\mu x}^A \rangle_\chi$ shows that for $m \neq 0$ the effective action $S_{\text{eff}}(A)$, which results after carrying out the χ integration in the path integral, and thereto also $\langle J_{\mu x}^A \rangle = \delta S_{\text{eff}}(A)/\delta A_{\mu x}$, are non-linear functionals of the vector potential $A_{\mu x}$.

5.2. FLUCTUATING GAUGE DEGREES OF FREEDOM

Since the model lacks gauge invariance it is instructive to investigate how relation (4.11) changes if we perform gauge transformations

$$U_{\mu x} \rightarrow \Omega_x U_{\mu x} \Omega_{x+\hat{\mu}}^*, \quad (5.4)$$

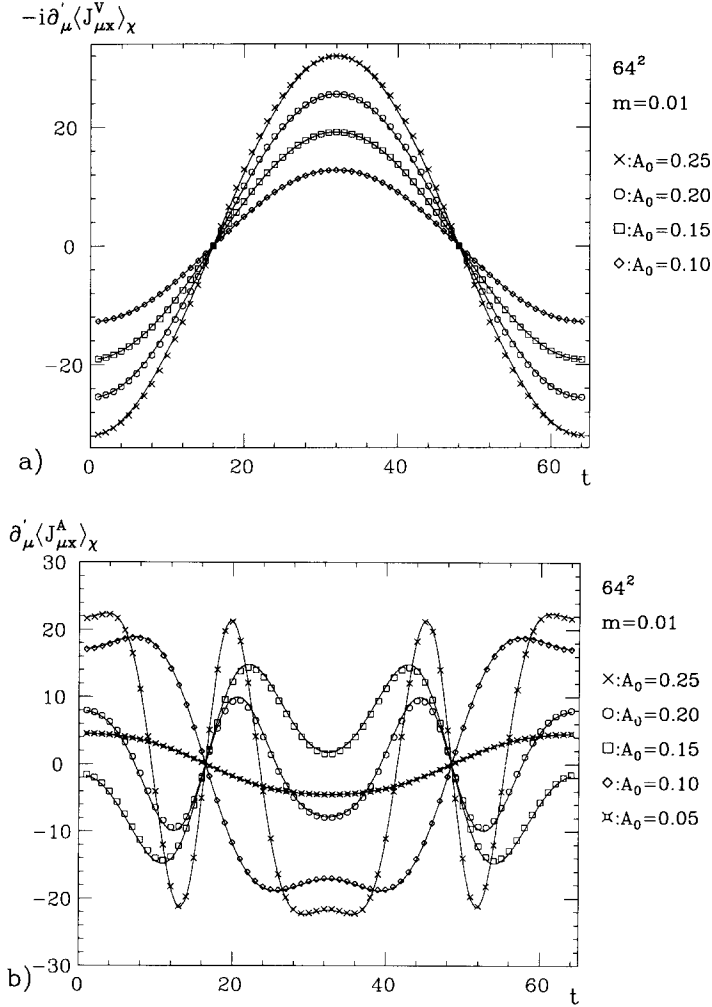


Fig. 2. (a) The divergence $-i\partial'_\mu \langle J_{\mu x}^V \rangle_\chi$ as a function of t for several values of A_0 and $m = 0.01$. The anomaly F_x/π is given by the full lines. (b) $\partial'_\mu \langle J_{\mu x}^A \rangle_\chi$ as a function of t for several values of A_0 and $m = 0.01$. The numerical results for $2m \langle J_x^P \rangle_\chi$ are represented by the full lines which were obtained by connecting the data points at various t by cubic splines.

on a smooth link configuration. To investigate the effect of Ω field fluctuations, we can again compute the current divergence, but now averaged over an Ω field ensemble,

$$\langle \partial'_\mu J_{\mu x}^V \rangle_\Omega \equiv \frac{1}{Z} \int \mathcal{D}\Omega \partial'_\mu \langle J_{\mu x}^V \rangle_\chi e^{S(\Omega)}, \quad Z = \int \mathcal{D}\Omega e^{S(\Omega)}, \quad (5.5)$$

with Boltzmann weight $\exp S(\Omega)$. We will generate here the configurations of the gauge degrees of freedom in two different ways, depending on which formulation of the full model with dynamical gauge fields we are aiming for.

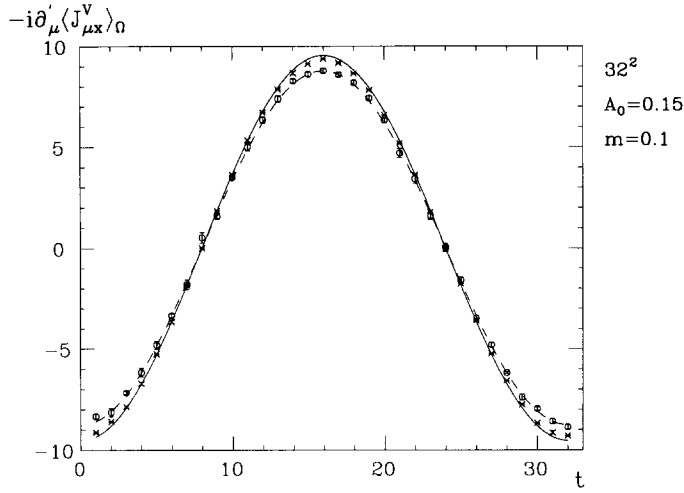


Fig. 3. The divergence $-i\partial'_\mu \langle J_{\mu x}^V \rangle_\Omega$ as a function of t with the gauge degrees of freedom Ω generated by the gauge fixing action (5.6) at $\zeta = 10$ (crosses) and by the XY-model action (5.7) at $\kappa = 1.0$ (circles). The anomaly F_x/π is represented by the full line. The dashed curve is obtained by a fit.

As we mentioned in sect. 1, one possibility is to use non-perturbative gauge fixing and add counterterms to restore gauge invariance [6]. With non-compact U(1) gauge fields (but still coupled compactly to the fermions), we may use a gauge fixing action $-\frac{1}{2}\zeta \sum_x (\partial'_\mu A_{\mu x})^2$. The corresponding Faddeev–Popov determinant is independent of A_μ . Since $A_{\mu x}$ transforms into $A_{\mu x} + \partial_\mu \omega_x$ under a gauge transformation, this suggests to use the gauge fixing action

$$S(\Omega) = -\frac{1}{2}\zeta \sum_x (\square \omega_x)^2, \quad \Omega_x = \exp(i\omega_x), \quad (5.6)$$

to generate the gauge degrees of freedom. Here \square denotes the lattice laplacian operator $\partial'_\mu \partial'_\mu$ and ζ is the gauge fixing parameter ($\zeta = \infty$ corresponds to the Landau gauge). The non-compact phases ω_x are coupled through $\Omega_x = \exp(i\omega_x)$ and the replacement (5.4) to the fermions. In fig. 3 the crosses represent the results for $-i\langle \partial'_\mu J_{\mu x}^V \rangle_\Omega$ after averaging over 1400 independent Ω configurations at $\zeta = 10$. The numerical result lies slightly below the solid line which represents here again the anomaly F_x/π for the given external gauge field configuration (5.1). This shows that the anomaly relation remains valid after multiplying the vector current by a factor which is slightly larger than one. Such a current renormalization is expected because there is no protection by symmetry. This result shows that a further numerical investigation of the gauge fixing approach may be technically feasible, at least for U(1) with non-compact gauge potentials where one has not to worry about the Faddeev–Popov factor.

We find that at smaller values of ζ the statistical fluctuations increase tremendously. Recall, however, that in the convention we are using the gauge coupling g is absorbed in $A_{\mu x}$, implying that $\zeta \propto 1/g^2$; e.g. $\zeta = 1/g^2$ corresponds to the Feynman gauge. Since $g^2 \rightarrow 0$ in lattice units, large ζ 's are not unnatural.

An alternative approach to regain a gauge invariant quantum model aims at a dynamical restoration of gauge invariance, by integration over all gauge transformations [3,5]. In this approach the expected mass counterterm for the gauge field $2\kappa \sum_{x\mu} \text{Re} U_{\mu x}$, which transforms into $2\kappa \sum_{x\mu} \text{Re}(\Omega_x U_{\mu x} \Omega_{x+\hat{\mu}}^*)$, suggests to use the action

$$S(\Omega) = \kappa \sum_{x\mu} (\Omega_x^* \Omega_{x+\hat{\mu}} + \Omega_{x+\hat{\mu}}^* \Omega_x), \quad (5.7)$$

to generate the Ω configurations in (5.5). This action is identical to the action for the XY-model in two dimensions with hopping parameter κ . The most ambitious scenario [3,5] corresponds to choosing κ in the vortex phase of the XY-model (in ref. [5] denoted as ‘‘scenario C’’). In other scenarios the models would have to allow an interpretation of Ω_x as a (radially frozen) Higgs field, for which κ would have to be close to the Kosterlitz–Thouless phase transition at $\kappa_c \approx 0.5$.

For large values of κ , deep in the spin wave phase, the fluctuations of Ω are small and we expect that $-i \langle \partial'_\mu J_{\mu x}^V \rangle_\Omega = F_x/\pi$ within statistical errors. This is confirmed by a simulation at $\kappa = 2$. For smaller values of κ , when approaching the phase transition at $\kappa_c \approx 0.5$, the fluctuations increase dramatically. The result at $\kappa = 1.0$ (indicated by circles in fig. 3) shows that the values for $-i \langle \partial'_\mu J_{\mu x}^V \rangle_\Omega$ now lie significantly below F_x/π (solid line). This result was obtained after averaging over 6400 independent Ω configurations. The dashed line was obtained by fitting the numerical data to the ansatz $-c(A_0 2\pi/T) \cos(2\pi t/T)$ with free parameter c . The good quality of the fit shows that the relation (4.11) remains valid after renormalizing the vector current by a factor $1/c \approx 1.20$. In the vortex phase the fluctuations of $-i \partial'_\mu J_{\mu x}^V$ were so strong that even after an excessive increase of the statistics we were not able to get a meaningful estimate of the average current divergence. The danger is that the nice scaling behavior of the staggered fermions, which we could demonstrate for smooth external fields, is washed out when the gauge degrees of freedom fluctuate too strongly.

An interesting difference between the actions (5.6) and (5.7) is that in the latter case the small fluctuations in the spin wave phase were weighted by $\exp(\kappa \omega \square \omega)$, whereas in the former case a \square^2 appears instead of the \square , which leads to much smoother ω configurations.

5.3. CONFIGURATIONS WITH $Q \neq 0$

One would like to promote the divergence equation (2.4) to gauge fields with arbitrary topological charge. At first sight this leads to an apparent contradiction:

On a torus with periodic boundary conditions for gauge invariant quantities, integration over the left-hand side of eq. (2.4) gives zero, while the right-hand side gives $2iQ \neq 0$. This paradox makes it interesting to see if the *local* divergence equation (2.4) is valid also for topologically non-trivial gauge fields.

In the massive Schwinger model the divergence equation for the axial-vector current reads $\partial'_\mu J_{\mu x}^A = 2mJ_x^P + 2iq$. After integration over the torus with periodic boundary conditions we find $Q = m \text{Tr}[\gamma_5(\not{D} + m)^{-1}]$ and the Atiyah–Singer index theorem is effectively valid [20]. A paradox is therefore avoided by the fermion mass term in the massive Schwinger model. The paradox in the axial-vector model would not emerge if we would replace the usual mass term in eq. (2.1) by a Majorana mass term which makes the model equivalent to the massive Schwinger model.

However, it is not as easy to deal with $Q \neq 0$ in the axial-vector model as previously in the Schwinger model (see ref. [20]). The reason is that we recover gauge invariance only if the gauge potentials are smooth (and, of course, $m = 0$). For example, the perturbative derivation of the divergence relations is valid only when the momenta of the external gauge field A_μ are negligible compared to the cutoff. On a periodic lattice a typical lattice gauge field with non-zero topological charge is not smooth. As an example we investigate the effect of configurations with constant F_{12x} used in the vector case in ref. [20].

We consider a configuration with a constant field strength $F \equiv F_{12x} = 2\pi Q/TL$.

$$\begin{aligned} U_{1x} &= \exp(iFt), \quad t = 1, \dots, T, \\ U_{2x} &= 1, \quad t = 1, \dots, T-1, \\ U_{2x} &= \exp(iFTx_1), \quad t = T. \end{aligned} \tag{5.8}$$

For small Q the link field $U_{\mu x}$ is close to one and smooth everywhere except for $t = T$ where U_2 contains a transition function [20]. When shifting t from $T-1$ to T and then to $T+1 = 1 \pmod{T}$ U_2 makes a jump as a function of t (e.g. for $Q = 1$ and $x_1 = \frac{1}{2}L$ U_2 jumps from $+1$ to -1 and then to $+1$ again). Also, U_1 jumps when shifting t from T to $t = T+1 = 1 \pmod{T}$. In gauge invariant models these transition functions are invisible.

We expect the divergence equation to hold with small $O(a)$ corrections in the region of the lattice where the gauge fields are sufficiently smooth, but expect deviations in the region near the transition function. As in fig. 2 we have plotted in fig. 4 the divergence $-i\partial'_\mu \langle J_{\mu x}^V \rangle_\chi$ (squares) for the space slice $x_1 = \frac{1}{2}L$ as a function of t/T with $L = T = 64$ (diamonds) and $L = T = 32$ (crosses), but now for the gauge field configuration (5.8). We used here antiperiodic boundary conditions for the χ fields and $m = 0$. For other space slices we got similar plots. The solid line represents again F_x/π , which is now independent of x . Far way from the time slice which carries the transition function the agreement of $-i\partial'_\mu \langle J_{\mu x}^V \rangle_\chi$ with F_x/π

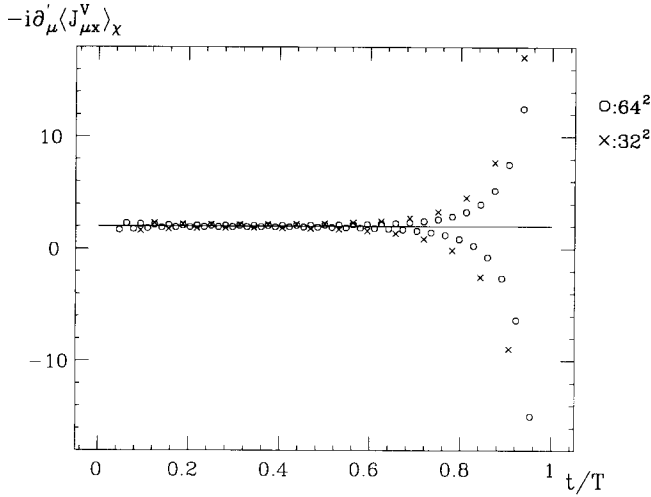


Fig. 4. The divergence $-i\partial'_\mu \langle J_{\mu x}^V \rangle_\chi$ as a function of t/T for a configuration with $Q = 1$ on two different lattices ($T = L = 32, 64$). We have used $m = 0$, and antiperiodic boundary conditions for the χ field. The solid line represents the anomaly F_χ/π .

is satisfactory, however in the vicinity of this time slice the deviations become huge (the values at $t = 31, 32$ ($T = 32$) and $t = 62, 63, 64$ ($T = 64$) were dropped from the graph since they are much larger than 20). The figure indicates also that the region of disturbance shrinks for increasing lattice size. The strong deviations induced by the transition function is analogous to the enormous fluctuations we observed in $-i\partial'_\mu \langle J_{\mu x}^V \rangle_\Omega$ when lowering κ in the action (5.7). The increase of the vorticity renders the effective gauge field configuration $\Omega_x U_{\mu x} \Omega_{x+\hat{\mu}}^*$ very rough, similar to the above gauge field at $t = T$.

The above results for the $Q = 1$ configuration are not entirely satisfactory. To avoid the large errors with topological non-trivial gauge fields we will have to follow the mathematicians and introduce charts in which the gauge potentials are smooth, such that the Dirac operator has negligible discretization errors in any open region (see e.g. ref. [21]). A proper treatment of this will have to be done in the future.

6. Discussion

We have showed that the staggered fermion model (3.5) can reproduce continuum Ward identities after incorporating the appropriate counterterms. The coefficients of these counterterms have been computed in this paper within lattice perturbation theory. Using smooth external gauge fields with zero topological charge we have numerically verified the validity of the divergence relations for the

vector and axial-vector currents and found good accuracy, for gauge field configurations with $|U_{\mu x} - 1| \lesssim 0.2$.

We tested the sensitivity of the divergence relation for the vector current to fluctuating gauge degrees of freedom which were generated either with the gauge fixing action (5.6) or the scalar field action (5.7). After renormalizing the currents with a finite factor > 1 , the divergence relation remains valid in both cases, provided that the fluctuations of the gauge modes Ω_x are not too strong. This is an encouraging result for the gauge fixing approach to chiral theories. When the gauge modes becomes less constrained, the induced fluctuations are very severe and might wash out the anomaly signal completely.

The disconcerting effect of rough gauge transformations is also seen when the current divergence is measured for a non-smooth gauge field with topological charge one, used in earlier tests in QED₂. We found the divergence relation for the vector current to be strongly violated near the region of the lattice where the gauge potentials lack smoothness. This means that in a description which violates gauge invariance at the cutoff level, topologically non-trivial gauge fields have presumably to be dealt with using the full apparatus of charts and transition functions that are smooth in space and time.

The results of this paper show that our staggered fermion approach passes a first test in reproducing the current divergence relations. It is not clear yet at this stage whether it is possible to obtain a valid quantum model after the integration over all gauge field configurations has been carried out in the path integral. This question shall be addressed in a separate publication [7]. Further clarification is also needed of the question how (if indeed) the additional global U(1) invariance on the lattice does not give rise to a local conservation law.

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