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Natural deduction for intuitionistic linear logic

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Dedicated to Dirk van Dalen on occasion of his 60th birthday

Abstract

The paper deals with two versions of the fragment with unit, tensor, linear implication and storage operator (the exponential $!$) of intuitionistic linear logic. The first version, \mathbf{ILL} , appears in a paper by Benton, Bierman, Hyland and de Paiva; the second one, \mathbf{ILL}^+ , is described in this paper. \mathbf{ILL} has a contraction rule and an introduction rule $!I$ for the exponential; in \mathbf{ILL}^+ , instead of a contraction rule, multiple occurrences of labels for assumptions are permitted under certain conditions; moreover, there is a different introduction rule for the exponential, $!I^+$, which is closer in spirit to the necessitation rule for the normalizable version of $\mathbf{S4}$ discussed by Prawitz in his monograph “Natural Deduction”.

It is relatively easy to adapt Prawitz’s treatment of natural deduction for intuitionistic logic to \mathbf{ILL}^+ ; in particular one can formulate a notion of strong validity (as in Prawitz’s “Ideas and Results in Proof Theory”) permitting a proof of strong normalization.

The conversion rules for \mathbf{ILL} explicitly mentioned in the paper by Benton et al. do not suffice for normal forms with subformula property, but we can show that this can be remedied by addition of a special permutation conversion plus some “satellite” permutation conversions.

Some discussion of the categorical models which might correspond to \mathbf{ILL}^+ is given.

1. Introduction

In this paper we shall assume familiarity with the proof-theoretic treatment of intuitionistic logic \mathbf{IL} as presented e.g. in [9, 13].

We discuss natural deduction versions of the *multiplicative-exponential* fragment of intuitionistic linear logic, \mathbf{ILL}_{me} (usually shortened to \mathbf{ILL} below, since we shall not deal with the full system \mathbf{ILL} here). The operators and constants of \mathbf{ILL}_{me} are \star (tensor), $\mathbf{1}$ (unit), \multimap (linear implication), and $!$ (storage operator, exponential).

$!$ behaves more or less like the modal necessity operator in the well-known system $\mathbf{S4}$ of modal logic; in particular, the first natural deduction formulations proposed for \mathbf{ILL} (e.g. in [1], in the form of a system of terms assigned to sequent calculus

deductions for **ILL**) had the following introduction rule for !

$$\frac{! \Gamma \vdash A}{! \Gamma \vdash ! A}$$

(we use $\Gamma, \Gamma', \dots, \Delta, \Delta', \dots$ for multisets of assumptions) or in tree form

$$\frac{\begin{array}{c} [! \Gamma] \\ \mathcal{D} \\ A \\ \hline ! A \end{array}}{}$$

where the brackets $[$ and $]$ in $[! \Gamma]$ serve to indicate that $[! \Gamma]$ is a *complete* multiset of open assumptions in \mathcal{D} , discharged at the application of !-introduction. This version has the disadvantage, as noted by several researchers, that the proof trees are not closed under substitution of deductions for open assumptions (substituting deductions for the assumptions $! \Gamma$ in an application of !-introduction leads to a deduction which ends in general not with a correct application of !-introduction). In [3] it was proposed to generalize the !I-rule to

$$\frac{\Delta_1 \vdash ! A_1, \dots, \Delta_n \vdash ! A_n \quad ! A_1, \dots, ! A_n \vdash B}{\Delta_1, \dots, \Delta_n \vdash ! B}$$

In the sequel we shall reserve the designation **ILL** for this version from [3]. Closure under substitution is now taken care of, but for a proof-theoretic treatment the new version of the !I-rule turns out to be somewhat awkward; in a sense, the rule both introduces and eliminates !-formulas, and there is no direct relation in complexity between $! B$ and the formulas $! A_i$; the latter may be much more complex than the conclusion. Prawitz's treatment of **S4** in [9] suggests another possibility, which we shall call $! I^+$: a correct application of $! I^+$ has the form

$$\frac{\begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \\ [! A_1, \dots, ! A_n] \\ \mathcal{D} \\ B \\ \hline ! B \end{array}}{}$$

(no assumptions open in \mathcal{D}_i become bound in \mathcal{D} ; $[! A_1, \dots, ! A_n]$ is a complete list of the open assumptions in \mathcal{D}). However, this does not combine very well with the contraction rule for the exponential. Therefore we study another version of **ILL**_{me}, called **ILL**⁺, in which !I is replaced by !I⁺, and contraction is eliminated by considering proof trees where multiple labels of variables are permitted, if they arise by substituting isomorphic copies of a deduction \mathcal{D}' for a collection of open assumptions of the form $! A$ in another deduction \mathcal{D} . Thus we suppress the *dynamic* aspect of contraction (i.e. the separate operation of replacing two distinctly labelled occurrences of a formula $! A$ by a single occurrence); a precise statement of the conditions permitting multiple occurrences of the same label will be given later on.

It appears that \mathbf{ILL}^+ permits a proof-theoretic treatment closely parallel to Prawitz's treatment of intuitionistic logic in [9, 10]. In particular, we can formulate a notion of strong validity giving rise to a proof of strong normalization for \mathbf{ILL}^+ ; normal forms of deductions in \mathbf{ILL}^+ have the subformula property and can be analysed in terms of the structure of tracks (track = path in [9]), which in normal deductions always consist of an elimination part, followed by a minimal part, followed by an introduction part. Applications of the kind given in [9] follow.

Returning to \mathbf{ILL} itself, the obvious "direct conversions" contracting an E-rule application with the conclusion of an I-rule as main premise, and the "permutative conversions" permitting to permute E-rule applications upward past minor premises of certain E-rules, do not suffice to give a normal form with subformula property. But some extra permutation conversions, corresponding to one of the equalities in the notion of categorical model of \mathbf{ILL} described in [3], and motivated by steps in the cut-elimination process for \mathbf{ILL} , suffice for this; it is consideration of \mathbf{ILL}^+ which suggests a suitable normalization strategy for \mathbf{ILL} relative to this set of conversion rules.

Finally, one may ask what notion of categorical model corresponds to \mathbf{ILL}^+ ? For \mathbf{ILL}^+ as such, the question does not make immediate sense, since the restrictions one has to impose on conversions in \mathbf{ILL}^+ are non-standard for a term-calculus. But the question does suggest investigating a notion of categorical model obtained by imposing one extra equation on the set of equations listed for the models of [3], to the effect that any map from $!I$ to $!A$ can be obtained as the result of an $!I$ -introduction to a map from $!I$ to A . The extra equation is very restrictive; it is true in algebraic models (trivially), but we do not know of a non-trivial type-theoretic or categorical model where it holds.

2. Notational representation of natural deductions

We recall that deductions in the system \mathbf{IL} of natural deduction for intuitionistic propositional logic can be presented, in a highly redundant way, as trees where the nodes are labelled by sequents of the form

$$(*) \quad x_1 : A_1, \dots, x_n : A_n \vdash t : B$$

where t is a rigidly typed term of type B , and the free variables of t occur among x_1, \dots, x_n . Such a representation obviously contains redundancies, since if t is rigidly typed, the variables x_i in t occur also typed as $x_i : A_i$; moreover, t reflects in its construction the complete proof-tree up to this node, so the conclusion label at the bottom of the tree contains in fact all relevant information concerning the tree.

Several isomorphic forms of presentation of deductions in \mathbf{IL} are obtained by stripping certain types of redundant information from the tree. Thus, for example, we obtain the usual *formula-tree presentation* by (1) stripping the terms and the context $x_1 : A_1, \dots, x_n : A_n$ of each label, retaining only B in $(*)$ above, (2) retaining the variable labels of assumptions appearing at the top nodes (the leaves) of the tree; (3) indicating

the rules used (when needed to avoid ambiguity) and (4) indicating, by repeating the labels, where assumptions are discharged.

The *term-presentation* is obtained by retaining only the rigidly typed term at the root of the tree, etc.

Each of these styles has its own merits; the formula-tree style has a certain “geometric flavour” and permits an appealing formulation of the structure of normalized proofs (as built from tracks with an elimination part, minimal part, and introduction part, cf. [9]) from which we can neatly derive a number of corollaries (the subformula property, a generalized form of the disjunction rule, etc.) It is true that for **IL** the \vee E-rule and the corresponding conversions (normalizing steps) are nastier than the other rules (a fact strongly emphasized in [8]) – but really not too nasty, I think – it is still manageable.

The term presentation is very compact and precise, and makes the isomorphism between typed-term calculi and deduction systems fully explicit. It also suggests further normalization steps, which serve as a stepping stone towards a category-theoretic formulation of the logic.

The preceding remarks apply, *mutatis mutandis*, also to natural deduction formulations of intuitionistic linear logic.

In exhibiting deductions as formula trees, we use some standard conventions. We use calligraphic $\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$, possibly sub- or superscripted, for formula prooftrees.

$$\begin{array}{c} x \\ [A] \\ \mathcal{D} \\ B \end{array}$$

is a prooftree \mathcal{D} with $[A]$ the set of all open assumptions of the form A with the label x . The label is often dropped. Several assumption classes may appear as:

$$[\overset{x}{A}, \overset{y}{B}] \quad \text{or} \quad [\overset{x}{A}][\overset{y}{B}]$$

whenever an open hypothesis $\overset{x}{A}$ is discharged by a rule application, *all* occurrences of A with label x above the application of the rule are discharged (closed) simultaneously. It is usually convenient to assume that any label x discharged by rule application α occurs *only* above α ; this can be achieved by relabelling closed assumptions if necessary (in term-notation this is just renaming bound variables).

3. Intuitionistic linear logic

In presenting intuitionistic linear logic **ILL** care has to be taken in handling assumptions. For the purely multiplicative fragment with $\star, \multimap, \mathbf{1}$, this is simple: in the formula-tree style, the assumptions are treated as a multiset, or more precisely, as a set

of occurrences, each occurrence with a distinct label; each \rightarrow I-application discharges precisely one occurrence, each \star E-application precisely two occurrences.

If we add the exponential !, however, we must build into the rules that multiple use is equivalent to single use of the assumption !A.

We can stick to the convention that distinct occurrences of assumptions always have distinct labels by having a contraction rule. The effect of this rule is to replace two distinct labels (x, y say) of a formula occurrence !A with a new single occurrence with a new label (z say). In the formula-tree style an application of the contraction rule looks like

$$\frac{[\!A^x, \!A^y] \quad \mathcal{D}'}{\frac{!A^z \quad B}{B} x, y} \quad \text{or more generally} \quad \frac{\mathcal{D} \quad \mathcal{D}'}{\frac{!A \quad B}{B} x, y}$$

Similarly with weakening; the possibility of “vacuously” depending on assumption of the form !A (labelled x) is expressed by a weakening rule:

$$\frac{!A^x \quad \mathcal{D}'}{B} \quad \text{or more generally} \quad \frac{\mathcal{D} \quad \mathcal{D}'}{!A \quad B}$$

3.1. Definition. In an application of the *promotion rule* !I

$$\frac{[\!A_1^{x_1}, \dots, \!A_n^{x_n}] \quad \mathcal{D}_1 \quad \mathcal{D}_n \quad \mathcal{E} \quad B}{!B} x_1, \dots, x_n$$

the conclusions !A_{*i*} of the \mathcal{D}_i are the *side premises* of the !I-application, and *B* is the *main premise*.

3.2. Notation. If Γ, Δ are used for collections of formulas in versions of **ILL**, the Γ, Δ are treated as multisets; for sequences of formulas and derivations we use vector notation $\vec{B}, \vec{\mathcal{D}}$, etc.

3.3. Definition (*The system ILL*). For reference, we give a version of a natural deduction calculus for **ILL** (restricted to !, \rightarrow , \star , **I**), presented as a termcalculus.

Axiom $x : A \Rightarrow x : A$

$$\star I \frac{\Gamma \Rightarrow s : A \quad \Delta \Rightarrow t : B}{\Gamma, \Delta \Rightarrow s \star t : A \star B} \quad \star E \frac{\Gamma \Rightarrow s : A \star B \quad \Delta, x : A, y : B \Rightarrow t : C}{\Gamma, \Delta \Rightarrow E_{x,y}^*(s, t) : C}$$

$$\neg I \frac{\Gamma, x : A \Rightarrow t : B}{\Gamma \Rightarrow \lambda x. t : A \neg B} \quad \neg E \frac{\Gamma \Rightarrow s : A \neg B \quad \Delta \Rightarrow t : A}{\Gamma, \Delta \Rightarrow st : B}$$

$$II \Rightarrow * : I \quad IE \frac{\Gamma \Rightarrow s : I \quad \Delta \Rightarrow t : A}{\Gamma, \Delta \Rightarrow E^I(s, t) : A}$$

$$!I \frac{\Gamma_1 \vdash t_1 : !A_1, \dots, \Gamma_n \vdash t_n : !A_n \quad x_1 : !A_1, \dots, x_n : !A_n \vdash s : B}{\Gamma_1, \dots, \Gamma_n \vdash !_{x_1, \dots, x_n}(t_1, \dots, t_n; s)}$$

$$W \frac{\Gamma \Rightarrow s : !B \quad \Delta \Rightarrow t : A}{\Gamma, \Delta \Rightarrow E^w(s, t) : A} \quad D \frac{\Gamma \Rightarrow s : !B}{\Gamma \Rightarrow E^d(s) : B}$$

$$C \frac{\Gamma \Rightarrow s : !B \quad x : !B, y : !B, \Delta \Rightarrow t : A}{\Gamma, \Delta \Rightarrow E_{x,y}^c(s; t) : A}$$

Γ, Δ are sets of statements $x_i : A_i$ with the x_i all distinct; Γ, Δ disjoint. In $!_{\vec{x}}(\vec{t}; s)$ the operator $!_{\vec{x}}$ binds \vec{x} in s ; in $E_{x,y}^c(s, t)$ $E_{x,y}^c$ binds x, y in t . W = weakening, D = dereliction, C = contraction.

In discussing ILL it is often advantageous to generalize both weakening and contraction. Weakening is generalized to:

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_n \quad \mathcal{D}'}{!A_1 \dots !A_n \quad B} B$$

and contraction to

$$\frac{[(!A_1)^{k_1}, \dots, (!A_n)^{k_n}]}{\mathcal{D}_1 \quad \mathcal{D}^n \quad \mathcal{D}'}{!A_1 \dots !A_n \quad B} B$$

where $(!A_i)^{k_i}$ refers to k_i ($k_i > 1$) assumptions of the form $!A_i$ in \mathcal{D}' . This form of contraction is a combination of n applications of

$$\frac{[(!A_i)^{k_i}]}{\mathcal{D}_i \quad \mathcal{D}'}{!A_i \quad B} B$$

which in turn is a mild generalization of the original contraction rule.

3.4. Definition. In the applications of the general forms of W and C, the $!A_i$ appearing as conclusions of the \mathcal{D}_i are called the *major premises* (plural!) of the application, and B the *minor premise*.

3.5. In [3] normalization for natural deduction is not discussed, but some conversions are listed, in particular,

(1) “detour-conversions”, i.e. the removal of a formula occurrence introduced by an I-rule, only to be immediately eliminated as major premise of an E-rule.

(2) Permutation conversions of the following general form: a subdeduction of the form

$$\frac{\frac{\mathcal{D}_{00} \quad \mathcal{D}_{01}}{\frac{A \quad B}{B} \quad \mathcal{D}_1} \quad C}{C} \quad \text{converts to} \quad \frac{\mathcal{D}_{00} \quad \frac{\mathcal{D}_{01} \quad \mathcal{D}_1}{\frac{B \quad C}{A} \quad C}}{C}$$

where the final rule is an E-rule with B as major premise (and similarly with more premises in the rule).

Normalization becomes rather complicated in **ILL**, due to the complicated form of the promotion rule, as illustrated by the conversion of an !-introduction followed by a contraction. The dotted line in the second proof tree serves to make it visually clear that both formulas above it enter as assumptions in the deduction \mathcal{F} .

$$\frac{\frac{\mathcal{D}_1 \quad \mathcal{D}_n}{!A_1 \dots !A_n} \quad \frac{\frac{[[A_1^{x_1} \dots A_n^{x_n}]] \quad \mathcal{E}}{B} \quad x_1, \dots, x_n \quad \frac{[!B^u \ !B^v]}{\mathcal{F}} \quad C}{!B} \quad C}{C} \quad u, v$$

is transformed into

$$\frac{\frac{\mathcal{D}_1 \quad \mathcal{D}_n}{!A_1 \dots !A_n} \quad \frac{\frac{[[!A_1, \dots, !A_n]] \quad \mathcal{E}}{!B} \quad B \quad \frac{[[!A_1, \dots, !A_n]] \quad \mathcal{E}}{!B} \quad B}{\mathcal{F}} \quad C}{C} \quad \text{(contractions)} \quad x_1, y_1, \dots, x_n, y_n$$

Detour conversions and permutative conversions are not sufficient to guarantee the subformula property for normal proofs, as we shall see.

3.6. Example. Here is an example of a deduction in the fragment $\rightarrow, !$ which is normal w.r.t. detour- and permutation conversions, but which does not have the subformula property. (In particular, $!(B_1 \rightarrow B_2)$ does not occur as subformula in the conclusion.)

$$\begin{array}{c}
\frac{!(C \rightarrow (B_1 \rightarrow B_2))^{(1)} \quad !C^{(2)}}{!(B_1 \rightarrow B_2)} \quad \frac{!(C \rightarrow (B_1 \rightarrow B_2))^{(3)}}{!C \rightarrow (B_1 \rightarrow B_2)} \quad !C^{(4)}}{B_1 \rightarrow B_2} \quad 3, 4 \quad \frac{!(B_1 \rightarrow B_2)^{(6)} \quad !B_1^{(7)}}{B_1 \rightarrow B_2} \quad \frac{!B_1^{(5)}}{B_2} \quad 6, 7}{!B_1^{(5)}} \\
\frac{!B_2}{!B_1 \rightarrow !B_2} \quad 5 \\
\frac{!(B_1 \rightarrow !B_2)}{!C \rightarrow !(B_1 \rightarrow !B_2)} \quad 2 \\
\frac{!(C \rightarrow (B_1 \rightarrow B_2)) \rightarrow !(C \rightarrow !(B_1 \rightarrow !B_2))}{!C \rightarrow (B_1 \rightarrow B_2)} \quad 1
\end{array}$$

Formulating an analogue of the notion of path (*track* in our terminology) as used by Prawitz, it seems natural to let in an application of !I the occurrences of !A_i as conclusion of \mathcal{D} be followed by the assumption !A_i in \mathcal{E} .

We then see that in our counterexample the introduction of $!(B_1 \rightarrow B_2)$ on the left is followed by a dereliction from the assumption $!(B_1 \rightarrow B_2)$ on the right; but the detour-conversions and permutation conversions mentioned before do not permit contracting the !I followed by !E (dereliction) in the path.

The following type of conversion permits us to contract the promotion/dereliction in our example.

$$\frac{
\begin{array}{c}
\mathcal{F} \quad \mathcal{D} \quad [\dots !A_i \dots] \\
!B \quad A_i \quad \mathcal{D}' \\
\hline
\mathcal{E} \quad !A_i \quad \mathcal{E}' \quad C \\
!C
\end{array}
}{
\begin{array}{c}
[!B] \\
\mathcal{F} \quad \mathcal{D} \\
!B \quad A_i \\
\hline
[\dots !A_i \dots] \\
\mathcal{E} \quad \mathcal{E}' \quad C \\
!C
\end{array}
}$$

($!B$ is a sequence of deductions of the form $!B_i$) is replaced by

$$\frac{
\begin{array}{c}
[!B] \\
\mathcal{D} \\
!B \quad A_i \\
\hline
[\dots !A_i \dots] \\
\mathcal{E}' \\
\mathcal{E} \quad !B \quad \mathcal{E}' \quad C \\
!C
\end{array}
}{
!C
}$$

This additional conversion (together with some satellite permutation conversions) permits normalization with subformula property for normal deductions, as we shall see later; an appropriate normalization strategy will be suggested by the system ILL^+ , to be discussed next. This conversion is not listed in [3] for the natural deduction system, but as an equality it occurs in their list of categorical equalities and is

motivated by certain steps in the cut elimination process for the sequent calculus for **ILL**. It corresponds to X1 in 5.3.

For the reasons given above it seems worthwhile to explore the possibility of an alternative formula-tree presentation which is geometrically more manageable, at the expense of a slightly more complicated treatment of labelling of assumptions. Our solution (system **ILL**⁺) is closer in spirit to Prawitz's treatment of natural deduction for **S4** (cf. [9]) and permits a satisfactory normalization theorem, with the subformula property for normal proofs, and a structure of paths in deductions similar to the case of intuitionistic logic.

4. The system **ILL**⁺

In comparing the two systems we shall stick to the convention that in a proof tree $[A]$ always refers to a single assumption occurrence of the form A . The principal features in which **ILL**⁺ differs from **ILL** are the following.

4.1. The promotion rule. In **ILL**⁺ the rule, now called $!I^+$, takes the form

$$\frac{\begin{array}{c} \mathcal{E}_1 \quad \mathcal{E}_n \\ \llbracket !A_1 \cdots !A_n \rrbracket \\ \mathcal{D} \\ B \\ \hline !B \end{array}}{!B}$$

that is to say, in deduction \mathcal{D} with conclusion B and complete set of open assumptions $!A_1, \dots, !A_n$, deductions $\mathcal{E}_1, \dots, \mathcal{E}_n$ have been substituted; this premise permits deriving $!B$ from B . In term style this becomes

$$\text{If } t[x_1, \dots, x_n/s_1, \dots, s_n]:B, \quad \text{FV}(t) = \{x_1: !A_1, \dots, x_n: !A_n\}$$

$$s_1: !A_1, \dots, s_n: !A_n, \quad \text{then } !t[\bar{x}/\bar{s}]:!B.$$

So in **ILL**⁺ the operator $!$ does not bind variables; we may assume $(\text{FV}(s_1) \cup \dots \cup \text{FV}(s_n)) \cap \text{FV}(t) = \emptyset$.

Definition. An application of $!I^+$ as exhibited is said to be *based* on $\mathcal{E}_1, \dots, \mathcal{E}_n$.

Remark. It is *not* intended that an instance of the promotion rule in **ILL**⁺ is labelled with a basis; the existence of some basis is simply a (decidable) condition for correctness of the instance. In a deduction an instance of the promotion rule may have several possible bases, and a refinement of our term system might consist in adding labels for the basis of an application of the promotion rule, but this precisely introduces the sort of complication we try to avoid in **ILL**⁺.

4.2. The contraction rule and multiple label occurrences. The weakening rule is not changed. The contraction rule does not appear explicitly, but is built into the system by permitting multiple occurrences of the same free variable for assumptions.

Let us formulate the condition for multiple label occurrences more precisely. Whenever a label x for an open assumption A in deduction \mathcal{D} is used precisely k times ($k > 1$), then there are k isomorphic copies $\mathcal{E}_1, \dots, \mathcal{E}_k$ of the same deduction \mathcal{F} with conclusion of the form $!B$, such that in each \mathcal{E}_i there is a single occurrence of x , and \mathcal{D} is of the form

$$\begin{array}{c} \mathcal{E}_1 \quad \mathcal{E}_k \quad \mathcal{F} \quad \mathcal{F} \\ [!B \dots !B] \equiv [!B \dots !B] \quad \text{or} \quad t[x_1 : !B, \dots, x_k : !B / s, \dots, s] \\ \mathcal{E} \quad \mathcal{E} \end{array}$$

A special case is where the \mathcal{E}_i consist of $!B$ alone. Labels y_1, \dots, y_k in $\mathcal{E}_1, \dots, \mathcal{E}_k$, respectively, corresponding to a label y bound in \mathcal{F} , are all distinct. (This is necessary to guarantee that identical labels are always discharged simultaneously.) Intuitively we may think of the multiple occurrence as representing a generalized contraction rule application. The set of occurrences of $!B$ is called a *substitution location*.

The weakening rule is generalized as already indicated for **ILL**.

4.3. Definition (*The termsystem for ILL^+*).

$$\begin{array}{l} \star\text{I} \frac{t:A \quad s:B}{t \star s : A \star B} \quad \star\text{E} \frac{t:A \star B \quad s[x:A, y:B]:C}{E_{x,y}^*(t, s):C} \\ \neg\text{I} \frac{t[x:A]:B}{\lambda y. t[x/y]:A \neg B} \quad \neg\text{E} \frac{t:A \neg B \quad s:A}{ts:B} \\ \mathbf{1I} \star : \mathbf{1} \quad \mathbf{1E} \frac{t[x:\mathbf{1}]:A}{E_x^1(t):A} \\ \mathbf{!I}^+ \frac{t[\bar{x}:\bar{A}/\bar{s}:\bar{A}]:B}{!t[\bar{x}/\bar{s}]:!B} \quad \mathbf{D} \frac{t:!B}{E^d(t):B} \quad \mathbf{C} \frac{\bar{t}:\bar{A} \quad s:B}{E^*(\bar{t}, s):B} \end{array}$$

with restrictions on variables as indicated above.

4.4. Proposition. *There is a map \circ from the deductions in ILL^+ to the deductions in ILL , and a map $^+$ in the opposite direction, such that if \mathcal{D} in ILL^+ (\mathcal{E} in ILL) proves $\Gamma \vdash A$, then \mathcal{D}° in ILL (\mathcal{E}^+ in ILL^+) proves $\Gamma \vdash A$.*

Proof. We introduce an auxiliary system ILL^{++} , containing all the rules of ILL^+ , having the same conditions on labels, and in addition has the (derivable) rules **!I** and

the generalized contraction rule of **ILL**:

$$\frac{\mathcal{D} \quad \mathcal{E}}{\frac{!A \quad C}{C} x_1, \dots, x_n} \begin{matrix} x_1 & x_n \\ [!A, \dots, !A] \end{matrix}$$

The map $+$ may be defined on the deductions of **ILL**, inductively on the length of derivations:

- (1) replace any application of C as above by

$$\frac{\mathcal{D} \quad \mathcal{D}}{[!A, \dots, !A]} \frac{\mathcal{E}}{C}$$

(Nothing in any copy of \mathcal{D} is bound in \mathcal{E} .)

- (2) Replace an $!I$ -application as on the left by the $!I^+$ -application on the right.

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_n \quad \mathcal{E}}{\frac{!A_1 \dots !A_n}{!B} x_1, \dots, x_n} \begin{matrix} x_1 & x_n \\ [!A_1, \dots, !A_n] \end{matrix} \quad \frac{\mathcal{D}_1 \quad \mathcal{D}_n}{[!A_1, \dots, !A_n]} \frac{\mathcal{E}}{B} \frac{B}{!B}$$

(Nothing in any \mathcal{D}_i becomes bound in \mathcal{E} .) For the converse map $^\circ$, we proceed as follows. Let \mathcal{D} be a deduction in **ILL** $^{++}$; suppose a k -fold occurrence of a label results from the substitution of copies of \mathcal{D}' at k ($k > 1$) assumptions of the form $!A$ in \mathcal{D}'' , i.e.

$$\mathcal{D} \equiv \frac{\mathcal{D}' \quad \mathcal{D}'}{[!A, \dots, !A]} \mathcal{D}''$$

Then the k assumptions $!A$ form a substitution location (“subl”) and the \mathcal{D}' is the corresponding substitution deduction (“sded”). Define the *multiplicity degree* md of a deduction \mathcal{D} as the sum of the lengths of its sded’s. (Note: If we encounter *nested* subdeductions

$$\mathcal{D}''' \equiv \frac{\mathcal{D}'}{[!B]} \frac{\mathcal{D}''}{!A}$$

with both $!B$ and $!A$ elements of substitution sets, then this contributes *at least* $\text{length}(\mathcal{D}') + \text{length}(\mathcal{D}'')$ (i.e. the elements of \mathcal{D}' count at least twice!).

Any replacement of a subdeduction \mathcal{D}'' of the form

$$\begin{array}{c} \mathcal{D}' \quad \mathcal{D}' \\ [!A, \dots, !A] \\ \mathcal{D}'' \\ B \end{array} \quad \text{by} \quad \begin{array}{c} \mathcal{D}' \quad \mathcal{D}'' \\ \frac{!A \quad B}{B} x_1, \dots, x_n \end{array}$$

lowers the md of the deduction. We successively remove multiple occurrences of labels from a given deduction \mathcal{D} as follows. Given a multiple label x , arising from substitution of deduction \mathcal{D}' at k occurrences of $!A$ in \mathcal{D} , we distinguish two cases:

- (1) x is open in the whole deduction \mathcal{D} , and we replace the multiple substitution of \mathcal{D}' by a contraction applied after the last rule application.
- (2) If x is bound, there must be a rule application where all occurrences of x become bound simultaneously, so \mathcal{D} contains a subdeduction \mathcal{D}^*

$$\begin{array}{c} \mathcal{D}' \quad \mathcal{D}' \\ [!A, \dots, !A] \\ \mathcal{D}'' \\ B \\ \hline B' \end{array} \quad \mathcal{E} \quad x, y, \dots \quad \text{or} \quad \begin{array}{c} \mathcal{D}' \quad \mathcal{D}' \\ [!A, \dots, !A] \\ \mathcal{D}'' \\ B \\ \hline B' \end{array} \quad \mathcal{E} \quad x, y, \dots$$

In this case we introduce the contraction after the conclusion B of \mathcal{D}'' .

We continue till we have found a deduction of md zero. The result is almost an ILL-proof, except for the possible occurrences of $!I^+$ -applications

$$\begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \\ [!A_1, \dots, !A_n] \\ \mathcal{E} \\ B \\ \hline !B \end{array}$$

We may now assume that all assumption occurrences have distinct labels in the whole deduction. We replace such an $!I^+$ -application by an $!I$ -application

$$\begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \quad \mathcal{E} \\ [!A_1 \dots !A_n] \quad B \\ \hline !B \end{array} x_1, \dots, x_n$$

(x_1, \dots, x_n fresh labels). In finitely many steps we reach an ILL-deduction. While there are several possibilities for transformation, as e.g. in the case of $!I^+$ -applications, a unique choice is easily stipulated (e.g. choose highest occurrences $!A_1, \dots, !A_n$ which can serve as a basis for the $!I^+$ -application). \square

5. Conversions of ILL⁺

5.1. New conversions. Permutative conversions are defined as usual, and involve $\star E$, **1E** and (multiple) weakening followed by some elimination rule. Also standard are the $\star-$, $\neg-$ and **1**-conversions.

5.2. Notation. We write $\mathcal{D} \triangleright \mathcal{D}'$ if \mathcal{D} reduces to \mathcal{D}' , and $\mathcal{D} \triangleright_1 \mathcal{D}'$ or $\mathcal{D}' \triangleleft_1 \mathcal{D}$, if \mathcal{D}' is obtained from \mathcal{D} by a single conversion.

$!I^+$ followed by dereliction contracts according to

$$\frac{\frac{\mathcal{D}_i}{[\dots !A_i \dots]} \quad \frac{\mathcal{D}}{B} \quad \frac{!B}{B}}{\mathcal{D}} \triangleright_1 \frac{\mathcal{D}_i}{[\dots !A_i \dots]} \quad \frac{\mathcal{D}}{B} \quad B$$

i.e.

$$D4^+ \quad E^d(!t[\vec{x}/\vec{s}]) = t[\vec{x}/\vec{s}]$$

where $!t$ is based on the \vec{s} . $!I^+$ followed by weakening contracts as follows:

$$\frac{\frac{\mathcal{D}_i}{[\dots !A_i \dots]} \quad \frac{\mathcal{D}}{B} \quad \frac{!B}{B} \quad \mathcal{C}}{\mathcal{C}} \triangleright_1 \frac{\mathcal{D}_i \quad \mathcal{E}}{\dots !A_i \dots \quad C} \quad C$$

an instance of the generalized rule of weakening (which may be replaced by n successive weakenings). In term notation:

$$E^w(!t[\vec{x}/\vec{s}]; t') = E^w(\vec{s}, t').$$

With n -fold weakening as primitive, the $!I^+$ -W contraction may be formulated accordingly:

$$W1^+ \quad E^w(\vec{r}_1, !t[\vec{x}/\vec{s}], \vec{r}_2; t') = E^w(\vec{r}_1, \vec{s}, \vec{r}_2; t').$$

Here the \vec{x} indicates the set of occurrences $[[!A_1, \dots, !A_n]]$ in a promotion application, i.e. t is based on \vec{s} .

It is to be noted that application of a single conversion in a subtree belonging to a set of isomorphic subtrees inserted at a substitution location, may fall outside our

class of proofrees for \mathbf{ILL}^+ ; but finitely many “isomorphic” conversions will then bring us back into the class of \mathbf{ILL}^+ -proofrees.

5.3. The term equations for \mathbf{ILL} . We shall briefly compare the term-equivalences of [3] with the equivalences generated by our conversions. For brevity, we state the term equivalences of [3] in our notation; it is instructive to write them out as operations on proofrees. We arrange the equations in groups.

(1) *Equalities corresponding to detour-conversions for $\mathbf{1}, \star, \multimap$ and $!I\text{--}!E$.*

$$\text{D1} \quad E^1(\star, s) = s,$$

$$\text{D2} \quad E_{x,y}^\star(t \star t', s) = s[x, y/t, t'],$$

$$\text{D3} \quad (\lambda x.t)s = t[x/s],$$

$$\text{D4} \quad E^d(!_{\bar{x}}(\vec{s}, t)) = t[\bar{x}/\vec{s}].$$

In \mathbf{ILL}^+ , D1–3 also hold as conversions, to D4 corresponds the conversion of dereliction following promotion in the form D4^+ mentioned before.

(2) *Extensionalities (analogues of η -conversion).*

$$\text{E1} \quad E^1(t, f[z/\star]) = f[z/t],$$

$$\text{E2} \quad E_{x,y}^\star(t, f[z/x \star y]) = f[z/t],$$

$$\text{E3} \quad \lambda x.tx = t,$$

$$\text{E4} \quad !_x(t, E^d(x)) = t.$$

To E1–E4 correspond in \mathbf{ILL}^+ E1–E3 and E4 in modified form

$$\text{E4}^+ \quad !(E^d(t)) = t.$$

(3) *Equalities involving weakening.* $E^w(\vec{s}; t)$ is short for

$$E^w(s_1, E^w(s_2, \dots, E^w(s_n, t) \dots))$$

The equations are

$$\text{W1} \quad E^w(!_{\bar{x}}(\vec{s}; t), t') = E^w(\vec{s}; t'),$$

$$\text{W2} \quad !_x, \bar{x}'(s, \vec{s}'; E^w(x, t)) = E^w(s, !_x, \bar{x}'(\vec{s}', t)),$$

$$\text{W3} \quad E_{x,y}^c(s, E^w(x, t)) = t[y/s], \quad E_{x,y}^c(s, E^w(y, t)) = t[x/s],$$

$$\text{W4} \quad f[E^w(z, s)] = E^w(z, f[s]).$$

W1 corresponds to the conversion of promotion followed by weakening and corresponds in \mathbf{ILL}^+ to W1^+ mentioned above.

W2 expresses that for a promotion following a weakening the weakening may be pushed “downward” past the promotion. In \mathbf{ILL}^+ it corresponds to (in term notation)

$$W2^+ \quad !E^w(x; t[\bar{x}/\bar{s}]) = E^w(x; !t[\bar{x}/\bar{s}]).$$

W3 corresponds to

$$W3^+ \quad E^w(t, s[x/t]) = s[x/t],$$

which is also not among our conversions. W4 permits us to push weakening up/down as long as no binding of hypotheses is involved, and contains our permutation conversions for weakening as a special case.

(4) *Equalities with contraction.* We use an abbreviation

$$E_{\bar{y}, \bar{z}}^c(\vec{t}; s) := E_{y_1, z_1}^c(t_1, E_{y_2, z_2}^c(t_2, \dots, E_{y_n, z_n}^c(t_n, s) \dots)),$$

where $\bar{y} \equiv y_1, \dots, y_n$, $\bar{z} \equiv z_1, \dots, z_n$. The equalities are

$$C1 \quad E_{\bar{y}, \bar{z}}^c(!_{\bar{x}}(\vec{s}; t); s') = E_{\bar{x}', \bar{x}''}^c(\vec{s}; s'[y, z / !_{\bar{x}}(\vec{x}'; t), !_{\bar{x}}(\vec{x}''; t)]),$$

$$C2 \quad !_{\bar{z}, \bar{z}'}^c(s, \vec{s}'; E_{\bar{x}, \bar{y}}^c(z, t)) = E_{\bar{x}', \bar{y}'}^c(s, !_{\bar{x}, \bar{y}, \bar{z}'}(x', y', \vec{s}'; t)),$$

$$C3 \quad E_{\bar{x}, \bar{y}}^c(s, t) = E_{\bar{y}, \bar{x}}^c(s, t),$$

$$C4 \quad E_{\bar{x}, \bar{w}}^c(s, E_{\bar{y}, \bar{z}}^c(w, t)) = E_{\bar{w}, \bar{z}}^c(s, E_{\bar{x}, \bar{y}}^c(w, t)),$$

$$C5 \quad f[z/E_{\bar{x}, \bar{y}}^c(s, t)] = E_{\bar{x}, \bar{y}}^c(s, f[z/t]).$$

C1–C5 disappear (i.e. left- and right-hand side of the equation translate into identical terms) in \mathbf{ILL}^+ .

The *generalized* form of contraction requires a much more involved term operator.

(5) *Other rules.*

$$P1 \quad f[w/E^1(s, t)] = E^1(s, f[w/t]),$$

$$P2 \quad f[w/E_{\bar{x}, \bar{y}}^*(s, t)] = E_{\bar{x}, \bar{y}}^*(t, f[w/s]).$$

The same equations can be adopted in \mathbf{ILL}^+ ; these equalities contain the permutation conversions for E^* and E^1 as special cases.

$$X1 \quad !_{\bar{y}', \bar{y}, \bar{y}''}(\vec{t}', !_{\bar{x}}(\vec{t}; f), \vec{t}''; g) = !_{\bar{y}', \bar{x}', \bar{y}''}(\vec{t}', \vec{t}, \vec{t}''; g[y / !_{\bar{x}}(\vec{x}'; f)]).$$

In \mathbf{ILL}^+ X1 disappears.

Remark. If there is a notion of categorical model corresponding to \mathbf{ILL}^+ , which might be seen as strengthening of the conversion rules for \mathbf{ILL}^+ as well as the categorical identities for \mathbf{ILL} as stipulated in [3], it should be based on the \mathbf{ILL}^+ -conversion rules plus $E1$ – $E3$, $E4^+$, $W2^+$, $W3^+$, $W4$, $P1$ – $P2$.

However, the term calculus of \mathbf{ILL}^+ does not behave in the standard way, as we already pointed out: isomorphic subterms of type $!A$ giving rise to multiple

occurrences of the same variable (multiple labels) ought always to be converted simultaneously in order to stay within the same class of \mathbf{ILL}^+ -terms.

It does make immediate sense, however, to ask for the notion of categorical model corresponding to a system based on contraction as for \mathbf{ILL} , but with the rule $!I^+$. In this intermediate system a very natural conversion rule suggests itself, namely

$$E4^* \quad !_{\bar{x}}(\bar{y}; E^d(t)) = t[\bar{x}/\bar{y}],$$

from which it follows that

$$!_{\bar{x}}(\bar{z}; E^d(t)) = t[\bar{x}/\bar{z}].$$

This conversion has in \mathbf{ILL} the effect that

$$\frac{\begin{array}{c} [!\bar{B}] \\ \mathcal{E}_i \\ \dots !A_{i-1} !A_i !A_{i+1} \dots \end{array} \quad \begin{array}{c} [!A_1, \dots, !A_n] \\ \mathcal{D} \\ C \end{array}}{!C}$$

is equivalent to

$$\frac{\begin{array}{c} [!\bar{B}] \\ \mathcal{E}_i \\ [!A_1 \dots, !A_i, \dots, !A_n] \\ \mathcal{D} \\ !A_1 \dots !A_{i-1} !\bar{B} !A_{i+1} \dots !A_n \end{array} \quad C}{!C}$$

(Replace \mathcal{E}_i by

$$\frac{\begin{array}{c} [!\bar{B}] \\ \mathcal{E}_i \\ !A_i \\ !\bar{B} \quad A_i \end{array}}{!A_i}$$

and apply the rule X1, etc.)

The rule $E4^*$ holds in algebraic models for linear logic (intuitionistic linear logic with storage, in the terminology of [T2]), but it appears to be very restrictive: we do not know of a non-trivial type-theoretic model were $E4^*$ is fulfilled.¹ On the other

¹ Gavin Bierman noted, in correspondence, that $E4^*$ follows from idempotency of the comonad, i.e. in the notation of [T2] $r_{!A} = !r_A, s_A \circ r_{!A} = \text{id}_{!A}$. Idempotency permits to write any $f: !A \rightarrow !B$ as $!g \circ s_A$ by taking $g \equiv r_B \circ f$. $E4^*$ is implied by $!r_A \circ !f \circ s_A = f$. Now $!r_A \circ !f \circ s_A = !r_B \circ !!g \circ !s_A \circ s_A = !r_B \circ !!g \circ s_{!A} \circ s_A = (!r_B \circ s_B) \circ (!g \circ s_A) = f$.

hand, not all models of the set of equations with E4* collapse to algebraic models: \mathbf{ILL}^+ contains the $\wedge, \rightarrow, \top$ -fragment of intuitionistic logic embedded, which corresponds at least to a non-algebraic cartesian closed category. E4* is in fact *equivalent* to the requirement that a “change of basis” for the promotion rule leaves the proof term the same. Specifically,

$$\begin{array}{c} \llbracket !\bar{B} \rrbracket \\ \mathcal{D} \\ !A \end{array} \equiv \begin{array}{c} \llbracket !\bar{B} \rrbracket \\ \mathcal{D} \\ !A \quad A \\ \hline !A \end{array} \equiv \begin{array}{c} \llbracket !\bar{B} \rrbracket \\ \mathcal{D} \\ !A \quad A \\ \hline !\bar{B} \quad A \\ \hline !A \end{array}$$

where the first instance of !I has basis !A, the second the basis $\llbracket !\bar{B} \rrbracket$; the first equivalence is the usual E4, the second is the “change of basis” equivalence, the combination yields E4*. We now turn to the description of a strategy for normalizing.

5.4. Definition. A *segment* in a deduction is a set of formula occurrences A_1, \dots, A_n of the same formula, such that A_{i+1} is immediately below A_i for $1 \leq i \leq n$, A_i for $i < n$ is minor premise of W, IE or $\star E$, A_1 is not conclusion of such a rule, and A_n is not minor premise of such a rule.

A segment is *maximal* if either $n = 1$ and $A_1 = A_n$ is conclusion of an I-rule and major premise of an E-rule, or $n > 1$ and A_n is major premise of an E-rule.

A *terminal* segment of \mathcal{D} is a segment where A_n is the conclusion of \mathcal{D} . (In our fragment of \mathbf{ILL}^+ the terminal segment is unique.)

A segment is *critical* if it is a maximal segment of maximal degree (degree of a segment = complexity of the formula of the segment).

5.5. Proposition (Normalization for \mathbf{ILL}^+). *Each deduction \mathcal{D} in \mathbf{ILL}^+ can be brought into normal form by a finite sequence of reduction steps.*

Proof. We may normalize deductions by making conversions at the leftmost-opmost critical segment. If this is done in the leftmost subdeduction \mathcal{D}' of a finite set of copies of \mathcal{D}' inserted for several occurrences of a formula !A, then the result might fall outside our class of deductions; but if we next successively make the same conversion in each of the copies, we are back at a deduction of \mathbf{ILL}^+ . Each step in this procedure according to the strategy just described results in a diminishing of the total length of all maximal segments of maximal complexity.

One case requires attention: what if for a substituted deduction \mathcal{D}' , at two occurrences o_1 and o_2 of the substitution location consisting of occurrences of !A say, at o_1 a conversion cutting out !A is possible, and at o_2 not? But in this case Rule (\mathcal{D}') is promotion, and it is easy to see that in this case we can take the basis of the promotion (a set of occurrences of a formula !B say) in all copies of the original \mathcal{D}' as new substitution locations.

The form of normal deductions may now be analyzed as done in [9]; more details are given in Section 6.

6. Strong normalization in ILL^+

We may prove strong normalization for ILL^+ using Prawitz's concept of strong validity, adapted to the present system (for an exposition of the intuitionistic case see e.g. [11]).

Since single conversions do sometimes lead outside ILL^+ , we consider a wider class of proof trees, where multiple labels are permitted (as usual, open assumptions with the same label always have to be discharged simultaneously); for the rest the rules have the same form as for ILL^+ .

If we can prove strong normalization for this wider class of deductions, we have SN for ILL^+ , with respect to those normalization strategies where, if one of a series of copies of \mathcal{D} substituted at a set of occurrences of $!A$ is converted, then all the others are converted in the same way at the next steps until the whole group has again become isomorphic.

6.1. Definition. \mathcal{L} is a *conversion candidate* of \mathcal{D} if the terminal segment of \mathcal{D} is of the form $A \star B$ and begins with an \star -introduction with deductions $\mathcal{F}, \mathcal{F}'$ of the premises, and $\mathcal{L} \equiv (\mathcal{F}, \mathcal{F}')$.

If $\mathcal{D} \triangleright \mathcal{D}^*$ and \mathcal{D}^* has a conversion candidate \mathcal{L} , we say that \mathcal{L} is a *derivate* of \mathcal{D} .

Notation. Below we shall (unless indicated otherwise) stick to the convention that given a derivation \mathcal{D} not ending with W, the subdeductions of the premises from left to right are $\mathcal{D}_0, \mathcal{D}_1, \dots$; in the case of weakening \mathcal{D}_1 is the minor premise and \mathcal{D}_0 stands for one of the major premises. This may be iterated giving rise to notations \mathcal{D}_{0_1} etc. Similarly for \mathcal{D}' .

We write $\mathcal{D} \triangleright \mathcal{D}'$ or $\mathcal{D}' \triangleleft \mathcal{D}$ if \mathcal{D} reduces to \mathcal{D}' , i.e. \mathcal{D}' is obtained from \mathcal{D} by a series of conversion steps. We write $\mathcal{D} \triangleright_1 \mathcal{D}'$ or $\mathcal{D}' \triangleleft_1 \mathcal{D}$ if \mathcal{D}' is obtained from \mathcal{D} by a single conversion applied to some subdeduction of \mathcal{D} .

Rule (\mathcal{D}) is the last rule applied in \mathcal{D} .

“SN” abbreviates “strongly normalizable”.

6.2. Definition. A deduction \mathcal{D} is said to be *strongly valid* (SV) if one of the following clauses applies:

- (1) \mathcal{D} consists of an assumption, or the axiom II.
- (2) Rule (\mathcal{D}) $\in \{\star I, !I^+\}$ and the subdeductions of the premises are SV.

(3) Rule $(\mathcal{D}) = \neg\text{I}$, i.e. \mathcal{D} is of the form

$$\frac{\begin{array}{c} [A] \\ \mathcal{D}' \\ B \end{array}}{A \rightarrow B}$$

then $\mathcal{D} \in \text{SV}$ if for all $\mathcal{D}^* \in \text{SV}$ with conclusion A

$$\begin{array}{c} \mathcal{D}^* \\ [A] \\ \mathcal{D} \\ B \end{array}$$

is SV.

(4) Rule $(\mathcal{D}) \in \{\neg\text{E}, !\text{E}\}$, \mathcal{D} is normal or for all $\mathcal{D}' \triangleleft_1 \mathcal{D}$, \mathcal{D}' is SV.

(5) Rule $(\mathcal{D}) = \text{W}$, and \mathcal{D} is normal or for all $\mathcal{D}' \triangleleft_1 \mathcal{D}$, \mathcal{D}' is SV, $\vec{\mathcal{D}}_0 \in \text{SN}$, $\mathcal{D}_1 \in \text{SV}$.

(6) Rule $(\mathcal{D}) = 1\text{E}$, and \mathcal{D} is normal, or for all $\mathcal{D}' \triangleleft_1 \mathcal{D}$, \mathcal{D}' is SV, $\text{SN}(\mathcal{D}_0)$ and $\text{SV}(\mathcal{D}_1)$.

(7) Rule $(\mathcal{D}) = \star\text{E}$, and \mathcal{D} is normal or for all $\mathcal{D}' \triangleleft_1 \mathcal{D}$, \mathcal{D}' is SV, and condition (\star) holds, that is to say $\text{SN}(\mathcal{D}_0)$, $\text{SV}(\mathcal{D}_1)$, and whenever the deduction \mathcal{D}_0 of the main premise $A \star B$ has a derivate $(\mathcal{F}, \mathcal{F}')$, and the minor premise has deduction

$$\begin{array}{c} [A, B] \\ \mathcal{D}_1, \text{ then } \begin{array}{c} \mathcal{F} \quad \mathcal{F}' \\ [A, B] \\ \mathcal{D}_1 \\ C \end{array} \text{ is SV.} \\ C \end{array}$$

6.3. Lemma. Let $\text{Rule}(\mathcal{D}) \in \{\star\text{I}, !\text{I}^+, \neg\text{I}\}$. Then, if $\mathcal{D} \triangleright_1 \mathcal{D}^{(1)} \triangleright_1 \mathcal{D}^{(2)} \triangleright_1 \dots$, where

$$\mathcal{D}^{(n)} = \frac{\mathcal{D}_0^{(n)}}{A} \quad \text{or} \quad \frac{\mathcal{D}_0^{(n)} \quad \mathcal{D}_1^{(n)}}{A}$$

then $\dots \triangleright_1 \mathcal{D}_i^{(n)} \triangleright_1 \mathcal{D}_i^{(n+1)} \triangleright_1 \mathcal{D}_i^{(n+2)} \triangleright_1 \dots$ become reduction sequences after deletion of repetitions.

6.4. Lemma. If $\mathcal{D} \triangleright \mathcal{D}'$ and $\text{SV}(\mathcal{D})$ then $\text{SV}(\mathcal{D}')$.

Proof. By induction over the inductively defined class of SV deductions. Obviously it suffices to show that if $\mathcal{D} \triangleright_1 \mathcal{D}'$, and $\text{SV}(\mathcal{D})$, then $\text{SV}(\mathcal{D}')$. \square

6.5. Lemma. $\mathcal{D} \in \text{SV} \Rightarrow \mathcal{D} \in \text{SN}$.

Proof. By induction over the class of SV deductions. The induction step is immediate if $\mathcal{D} \in \text{SV}$ by clauses 3–6, since then, if \mathcal{D} is not normal, all $\mathcal{D}' \triangleleft_1 \mathcal{D}$ are SV, and so by

the induction hypothesis SN. If $\mathcal{D} \in \text{SV}$ by clauses 1–2, strong normalizability is also immediate from the induction hypothesis. \square

6.6. Lemma. *Let $\text{Rule}(\mathcal{D}) \in \{\star E, \neg E, !E, W, !E\}$. Then $\mathcal{D} \in \text{SV}$ if*

- (i) $\text{SN}(\mathcal{D}_i)$ for all immediate subdeductions \mathcal{D}_i .
- (ii) If $\text{Rule}(\mathcal{D}) = \neg E$ or $!E$, then $\text{SV}(\mathcal{D}_i)$.
- (iii) If $\text{Rule}(\mathcal{D}) = W$, then $\text{SN}(\vec{\mathcal{D}}_0)$ and $\text{SV}(\mathcal{D}_1)$.
- (iv) If $\text{Rule}(\mathcal{D}) = !E$, then $\text{SN}(\mathcal{D}_0)$ and $\text{SV}(\mathcal{D}_1)$.
- (v) If $\text{Rule}(\mathcal{D}) = \star E$, clause (\star) in the definition of SV applies.

Proof. In order to prove this lemma, we assign to each \mathcal{D} satisfying the conditions of the lemma with conclusion of given complexity an induction value $\text{IV}(\mathcal{D}) = (\beta, \gamma, \delta)$ where

- $\beta =$ (sum of) length(s) of reduction tree(s) of \mathcal{D}_0 (resp. $\vec{\mathcal{D}}_0$);
- $\gamma =$ (sum of) length(s) of \mathcal{D}_0 (resp. $\vec{\mathcal{D}}_0$);
- $\delta =$ sum of lengths of reduction trees of the deductions of the premises.

The ordering is lexicographic: $(\beta, \gamma, \delta) < (\beta', \gamma', \delta') := (\beta < \beta') \vee (\beta = \beta' \wedge \gamma < \gamma') \vee (\beta = \beta' \wedge \gamma = \gamma' \wedge \delta < \delta')$. We prove the lemma by induction on $\text{IV}(\mathcal{D})$. It suffices to prove $\forall \mathcal{D}' \triangleleft_1 \mathcal{D} (\text{SV}(\mathcal{D}'))$, since the other conditions imposed on SV by the definition hold automatically if the assumptions of the lemma are satisfied.

Case 1: \mathcal{D} normal: we are done.

In all other cases, let $\mathcal{D}' \triangleleft_1 \mathcal{D}$; let $\text{IV}(\mathcal{D}) = (\beta, \gamma, \delta)$ and $\text{IV}(\mathcal{D}') = (\beta', \gamma', \delta')$ (if defined, which has to be shown).

Case 2: \mathcal{D}' is obtained by a conversion step applied to the deduction of one of the premises of the last rule application in \mathcal{D} . Then \mathcal{D}' falls under the conditions of the lemma, and has a well-defined lower IV.

Case 3: $\mathcal{D}' \triangleleft_1 \mathcal{D}$ by a detour-conversion involving the final rule-application. Then the major premise of the last rule in \mathcal{D} is obtained by an I-rule. For example,

$$\mathcal{D} \equiv \frac{\frac{[A] \quad \mathcal{D}_1}{\mathcal{D}_{00}} \quad \mathcal{D}_1}{\frac{B \quad \mathcal{D}_1}{A \multimap B} \quad A} \quad B$$

$$\mathcal{D}' \equiv \frac{[A] \quad \mathcal{D}_{00}}{\mathcal{D}_{00}} \quad B$$

By clause (ii), $\text{SV}(\mathcal{D}_1), \text{SV}(\mathcal{D}_0)$, and hence by the definition of SV, it follows that $\text{SV}(\mathcal{D}')$. If

$$\mathcal{D} \equiv \frac{B}{\frac{!B}{B}} \quad \mathcal{D}' \equiv \frac{\mathcal{D}_{00}}{B}$$

then $\text{SV}(\mathcal{D}_0)$, hence $\text{SV}(\mathcal{D}_{00})$, where $\mathcal{D}_{00} \equiv \mathcal{D}'$.

Case 4: $\mathcal{D}' \triangleleft_1 \mathcal{D}$ by a permutative reduction involving the final rule application.

Subcase 4.1: $\rightarrow E$ or $!E$ is permuted over $\star E$. Let

$$\mathcal{D} \equiv \frac{\frac{[A, B] \quad \mathcal{D}_{00} \quad \mathcal{D}_{0,1}}{A \star B \quad C \rightarrow D} \mathcal{D}_1}{C \rightarrow D} C}{D}$$

$$\mathcal{D}' \equiv \frac{\mathcal{D}_{00} \quad \frac{[A, B] \quad \mathcal{D}_1}{\mathcal{D}_{0,1} \quad C}}{A \star B \quad D} D}{D}$$

SN(\mathcal{D}_0), hence also SN(\mathcal{D}'_0). The induction value of \mathcal{D}' , if defined, is clearly lower: $\beta < \beta'$ or $\beta = \beta' \wedge \gamma' < \gamma$. We must show that \mathcal{D}' again satisfies the conditions of the lemma.

Note: SV(\mathcal{D}_0), SV(\mathcal{D}_1), SV(\mathcal{D}_{01}), hence SV(\mathcal{D}'_1) by the lemma, since IV(\mathcal{D}'_1) = $(\beta'', \gamma'', \delta'')$ with $\beta'' < \beta \vee (\beta'' = \beta \wedge \gamma'' < \gamma)$, hence SV(\mathcal{D}'_1).

Also, if \mathcal{D}_{00} has a derivative $(\mathcal{F}, \mathcal{F}')$, we must show that

$$\mathcal{F}'' \equiv \frac{[A \quad B] \quad \mathcal{D}_{01} \quad \mathcal{D}_1}{C \rightarrow D \quad C} D$$

is SV. For this we need that the left subdeduction of \mathcal{F}'' is SV. But SV(\mathcal{D}_0), hence this follows by condition (*) in the definition of SV.

As a result, $\mathcal{D}'_0, \mathcal{D}'_1$ are SV, hence SN, and IV(\mathcal{D}') is defined and the IH applies.

The treatment of $!E$ over $\star E$ is completely similar.

Subcase 4.2: $\rightarrow E$ or $!E$ over \vee or $!E$. The arguments are quite similar to, but slightly simpler than in the preceding case.

Subcase 4.3: $\star E$ or \vee or $!E$ over $\star E$ or \vee or $!E$. Let us consider the most complicated case of $\star E$ over $\star E$.

$$\mathcal{D} \equiv \frac{\frac{[A, B] \quad \mathcal{D}_{00} \quad \mathcal{D}_{01} \quad [C, D]}{A \star B \quad C \star D} \mathcal{D}_1}{C \star D} E}{E}$$

$$\mathcal{D}' \equiv \frac{\mathcal{D}_{00} \quad \frac{[A, B] \quad [C, D]}{\mathcal{D}_{01} \quad \mathcal{D}_1}}{A \star B \quad E} E}{E}$$

IV(\mathcal{D}) = (β, γ, δ) . We have

- (a) $\mathcal{D}_0 \in \text{SN}, \mathcal{D}_1 \in \text{SV}$;
- (b) If \mathcal{D}_0 has a derivate $(\mathcal{F}, \mathcal{F}')$, then

$$\mathcal{H} \equiv \frac{[C \quad D] \quad \mathcal{F} \quad \mathcal{F}'}{\mathcal{D}_1} E$$

is also SV.

We have to show that \mathcal{D}' falls under the IH. In the first place $\text{SN}(\mathcal{D}_{00})$ holds, since $\text{SN}(\mathcal{D}_0)$. Secondly, we must check that \mathcal{D}'_1 is SV. This requires (1) $\text{SN}(\mathcal{D}_{01})$, which follows from $\text{SN}(\mathcal{D}_0)$, (2) $\text{SV}(\mathcal{D}_1)$ which holds by (a), and (3) whenever \mathcal{D}_{01} has a derivate $(\mathcal{F}, \mathcal{F}')$, then \mathcal{H} as above is SV.

But if \mathcal{D}_{01} has a derivate $(\mathcal{F}, \mathcal{F}')$ then $(\mathcal{F}, \mathcal{F}')$ is also derivate of \mathcal{D}_0 , so (3) follows from (b). Hence $\text{SV}(\mathcal{D}'_1)$ follows by IH, since \mathcal{D}'_1 has a lower IV.

In order to get $\text{SV}(\mathcal{D}')$ it remains to be shown that if \mathcal{D}_{00} has a derivate $(\mathcal{G}, \mathcal{G}')$, then

$$\mathcal{G}'' \equiv \frac{\begin{array}{c} \mathcal{G} \quad \mathcal{G}' \\ [A, B] \quad [C, D] \\ \mathcal{D}_{01} \quad \mathcal{D}_1 \\ C \star D \quad E \\ \hline E \end{array}}$$

is also SV. This is similar to the preceding part of the argument; the crucial clause to be verified is now: if \mathcal{G}''_0 has a derivate $(\mathcal{F}, \mathcal{F}')$, then \mathcal{H} is SV.

However, if \mathcal{D}_{00} has a derivate $(\mathcal{G}, \mathcal{G}')$, this means that \mathcal{D}_0 reduces to something like

$$\mathcal{E}_0 \frac{\begin{array}{c} \mathcal{G} \quad \mathcal{G}' \\ A \quad B \\ \mathcal{E}_n \quad A \star B \\ \hline A \star B \\ \vdots \\ A \star B \end{array} \quad \begin{array}{c} [A, B] \\ \mathcal{D}_{01} \\ C \star D \end{array}}{C \star D}$$

and then \mathcal{D}_0 also reduces to

$$\mathcal{E}_0 \frac{\begin{array}{c} \mathcal{G} \quad \mathcal{G}' \quad [A, B] \\ A \quad B \quad \mathcal{D}_{01} \\ \hline A \star B \quad C \star D \\ \mathcal{E}_n \quad C \star D \\ \hline C \star D \\ \vdots \\ C \star D \end{array}}{C \star D}$$

so \mathcal{D}_0 reduces to a deduction

$$\begin{array}{c} \mathcal{G} \quad \mathcal{G}' \\ [A, B] \\ \mathcal{D}_{01} \\ \frac{\mathcal{E}_n \quad C \star D}{C \star D} \\ \vdots \\ \frac{\mathcal{E}_0 \quad C \star D}{C \star D} \end{array}$$

and it appears that $(\mathcal{F}, \mathcal{F}')$ is also a derivate of \mathcal{D}_0 , hence \mathcal{H} is indeed SV.

Subcase 4.4: Permutation of $\mathbf{1E}$ over $\star E$. This case is similar to earlier cases, but simpler. \square

6.7. Definition. \mathcal{D} is SVS (strongly valid under substitution) if every substitution of SV deductions for open assumptions in \mathcal{D} yields a SV deduction.

6.8. Proposition. All deductions in \mathbf{ILL}^+ are SVS.

Proof. By induction on the lengths of deductions. We consider two typical cases.

Case 1: Let Rule $(\mathcal{D}) = \mathbf{W}$. Then

$$\mathcal{D} \equiv \frac{\mathcal{D}_0 \quad \mathcal{D}_1}{\frac{!A \quad B}{B}}$$

$\mathcal{D}_0, \mathcal{D}_1$ are SVS by induction hypothesis. Let \mathcal{D}^* be a substitution instance of \mathcal{D} , so

$$\mathcal{D}^* \equiv \frac{\mathcal{D}_0^* \quad \mathcal{D}_1^*}{\frac{!A \quad B}{B}}$$

then $\mathcal{D}_0^*, \mathcal{D}_1^*$ are SV, hence \mathcal{D}_0^* is SN and so \mathcal{D}^* is SV by the preceding lemma.

Case 2: Rule (\mathcal{D}) is $\star E$, so

$$\mathcal{D} \equiv \frac{[A, B] \quad \frac{\mathcal{D}_0 \quad \mathcal{D}_1}{A \star B \quad C}}{C}$$

and let \mathcal{D}^* be a substitution instance

$$\mathcal{D}^* \equiv \frac{[A, B] \quad \frac{\mathcal{D}_0^* \quad \mathcal{D}_1^*}{A \star B \quad C}}{C}$$

By induction hypothesis $\mathcal{D}_0^*, \mathcal{D}_1^*$ are SV, so \mathcal{D}_0^* is SN. Suppose \mathcal{D}_0^* has a derivate $(\mathcal{F}, \mathcal{F}')$ occurring in a $\mathcal{D}^{**} \triangleleft \mathcal{D}^*$; \mathcal{D}^{**} is SV, and it follows that $\mathcal{F}, \mathcal{F}'$ are SV. Then

$$\begin{array}{c} \mathcal{F} \quad \mathcal{F}' \\ [A, B] \\ \mathcal{D}_1^* \\ C \end{array}$$

is SV, etc. \square

7. The structure of normal deductions in ILL^+

7.1. Definition. A *track* in a normal \mathcal{D} is a sequence of formula occurrences $A_0, A_1, A_2, \dots, A_n$ such that

- (1) A_0 is an axiom, open assumption or assumption closed by $\neg I$ in \mathcal{D} ;
- (2) A_{i+1} is immediately below A_i if A_{i+1} is conclusion of an I-rule, A_i a premise of the same rule;
- (3) A_{i+1} is immediately below A_i if A_i is major premise of an application of $!E = D, \neg E$, or minor premise of an application of $\star E, W$, or $1E$;
- (4) A_i is major premise of an application of $\star E$ and A_{i+1} is an assumption discharged by that application;
- (5) A_n is either conclusion of \mathcal{D} , or major premise of $1E$, or a major premise of W .

We can divide a track into segments as in the case of intuitionistic logic; in a track of a normal deduction we can then distinguish the elimination part, followed by the minimal part, followed by the introduction part.

7.2. Lemma. *Each formula occurrence in the proof tree of a normal deduction belongs to some track.*

Proof. By induction on the depth of deductions. \square

7.3. Proposition (Subformula property). *Let $\Gamma \vdash A$ by a normal deduction \mathcal{D} . Then all formulas in \mathcal{D} are subformulas of $\Gamma \cup \{A\}$.*

Proof. A *track of order 1* of a deduction \mathcal{D} ends in the conclusion of \mathcal{D} (i.e. is a terminal track). A *track of order $n + 1$* terminates either in a major premise of $1E$, or in a major premise $!B$ of W , or in a minor premise of $\neg E$, while the minor premise in the case of $1E, W$, and the major premise in the case of $\neg E$, belong to a track of order n .

We prove by induction on the order of tracks that all formulas in a track are subformulas of $\Gamma \cup \{A\}$.

For the track of any order we have that all formulas occurring in it are subformulas of the open assumptions of the deduction or of the final formula of the track. Let π be a track of \mathcal{D} (with conclusion A) of order $n + 1$.

If π terminates in a minor premise B of $\neg E$, then the major premise $B \rightarrow C$ belongs to a track of order n and so by induction hypothesis, $B \rightarrow C$ is subformula of $\Gamma \cup \{A\}$. Then B is also subformula of $\Gamma \cup \{A\}$, so π satisfies the subformula property.

If π terminates in a major premise of an $1E$ - or W -application, the last rule must be an elimination, and the whole track consists of subformulas of the first formula in the track. The first formula is either an open assumption of the deduction, or is discharged below the end of the track. If discharged by $\neg I$, or by $\star E$, this happens in a track of lower order, and the IH applies to this track. \square

As an example of an application we give the next proposition.

7.4. Definition. A formula (-occurrence) in a formula A is said to be a *strictly positive part (s.p.p.)* of A according to one of the following clauses:

- (1) A is s.p.p. of A ;
- (2) if $B \star C$ is s.p.p. of A , then B, C are s.p.p. of A ;
- (3) if $!B$ is s.p.p. of A , so is B ;
- (4) if $B \rightarrow C$ is s.p.p. of A , so is C .

7.5. Proposition. If $\Gamma \vdash A \star B$ in ILL , and Γ does not contain \star in a strictly positive position (\star not main operator of a s.p.p. subformula of Γ), then $\Gamma_1 \vdash A$ and $\Gamma_2 \vdash B$ with Γ_1 and Γ_2 sub-multisets of Γ .

Proof. Let \mathcal{D} be a normal deduction of $\Gamma \vdash A \star B$. If \mathcal{D} ends with an I-rule we are done. If the terminal segment starts with an introduction, the deduction takes the form

$$\begin{array}{c}
 \mathcal{F} \quad \mathcal{F}' \\
 \hline
 A \quad B \\
 \mathcal{E}_n \quad \hline
 A \star B \\
 \hline
 A \star B \\
 \vdots \\
 \mathcal{E}_2 \quad \hline
 A \star B \\
 \hline
 \mathcal{E}_1 \quad \hline
 A \star B \\
 \hline
 A \star B
 \end{array}$$

where the final segment passes through a number of $\star E$, $1E$, and W -applications. However, $\star E$ -applications are in fact excluded, since no strictly positive occurrence of

★ appears in Γ . But this means that in \mathcal{F} and \mathcal{F}' no assumptions are discharged, and

$$\begin{array}{ccc}
 \begin{array}{c} \mathcal{F} \\ \mathcal{E}_n \frac{A}{A} \\ \vdots \\ \mathcal{E}_2 \frac{A}{A} \\ \mathcal{E}_1 \frac{A}{A} \\ A \end{array} & \text{and} & \begin{array}{c} \mathcal{F}' \\ \mathcal{E}_n \frac{B}{B} \\ \vdots \\ \mathcal{E}_2 \frac{B}{B} \\ \mathcal{E}_1 \frac{B}{B} \\ B \end{array}
 \end{array}$$

are both correct deductions. \square

Remark. The statement of the proposition may be considerably refined, e.g. by noting that assumptions common to Γ_1, Γ_2 must permit to derive exponential formulas, etc.

Another application may be (almost) copied from [9, p. 57].

7.6. Proposition. Let C be without \neg , $\Gamma \equiv \{A_i \multimap B_i; 1 \leq i \leq n\}$, and assume $\Gamma \vdash C$. then $\Gamma' \vdash A_i$ for some $i \leq n$, Γ' a submultiset of Γ .

8. Normalization in ILL

We shall now show how the normalization strategy in ILL^+ suggests a corresponding strategy in ILL .

8.1. Definition. A *segment* in ILL is a sequence of occurrences A_1, \dots, A_n of the same formula such that

- (1) A_1 not conclusion of $W, 1E, \star E$, nor assumption discharged by C or $!I$;
- (2) A_n not minor premise of $W, 1E, \star E$, or side premise of $!I$, or major premise of C ;
- (3) for $1 \leq i < n$, either A_i is minor premise of an application α of $W, 1E$ or $\star E$, and A_i is the conclusion of α ; or A_i is major premise of C and A_{i+1} is one of the assumptions discharged by α ; or A_i is side premise of an instance α of $!I$, and A_{i+1} is an assumption discharged by α .

A segment is *maximal* if $\sigma \equiv A_1, \dots, A_n, A_1$ conclusion of I -rule, A_n major premise of E -rule. As before, we define a *critical* segment as a maximal segment of maximal degree.

8.2. Definition. A *track* of \mathcal{D} in ILL is a sequence of formula occurrences A_1, \dots, A_n in \mathcal{D} such that

- (1) A_1 is an open assumption or an axiom or an assumption discharged by $\neg I$;
- (2) A_i is major premise of $\neg E$ or D , or minor premise of $\star E, C, 1E, W$, or premise of $\neg I, \star I$, and A_{i+1} is the conclusion;

- (3) $A_i \equiv C \star D$ is major premise of $\star E$, and A_{i+1} is one of the assumptions discharged;
- (4) A_i is major premise of a contraction, and A_{i+1} one of the assumptions discharged;
- (5) A_i is a side premise of a promotion, and A_{i+1} is an occurrence discharged by the promotion.
- (6) A_n is major premise of a weakening or $1E$, or minor premise of $\rightarrow E$.

8.3. Description of a strategy for normalization. We look for an analogue of the strategy which works well in the intuitionistic case and for ILL^+ , namely: look for the rightmost branch in the formula tree containing a critical segment; apply a conversion in the topmost critical segment in this branch.

This strategy works for ILL^+ , because segments (in contrast to tracks) always belong to a unique branch of the tree. But this is not any longer the case for ILL . So in order to determine the proper place for a conversion, we construct, inspired by ILL^+ , an auxiliary partially ordered system with nodes labeled by formulas as follows. Given \mathcal{D} , the auxiliary structure $[\mathcal{D}]$ is obtained by systematically replacing

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_n \quad \mathcal{D}' \quad [!A_1, \dots, !A_n]}{!A_1 \dots !A_n \quad B} \quad !B \quad \text{by} \quad \frac{[\mathcal{D}_1] \dots [\mathcal{D}_n] \quad \frac{!A_1 \quad !A_n}{[!A_1 \dots !A_n]}}{[\mathcal{D}']} \quad \frac{B}{!B}$$

and

$$\frac{\mathcal{D} \quad \mathcal{D}' \quad [!A^{x_1}, \dots, !A^{x_n}]}{!A \quad B} \quad \text{by} \quad \frac{[\mathcal{D}] \quad !A}{/ \dots \backslash} \quad \frac{[\mathcal{D}']} {[!A \dots !A]} \quad \frac{B}{!B}$$

letting the map $[\]$ act as a homomorphism for all other rules. There is a bijective correspondence between the formula occurrences in \mathcal{D} and in $[\mathcal{D}]$.

A more formal description of the partial order of $[\mathcal{D}]$ is as follows: $[\mathcal{D}] = (\mathcal{D}, <)$ is a partial ordered set of the formula occurrences of \mathcal{D} , extending the tree order $(\mathcal{D}, <)$ of \mathcal{D} . Occurrence A is below occurrence B in $[\mathcal{D}]$ if

- (a) $A < B$ in \mathcal{D} , or
- (b) there is an $!I$ -application where B is in the deduction of the main premise below assumption $!A_i$, and A is in the deduction of the side premise $!A_i$, or

- (c) there is a C-application with A in the deduction of the major premise $!C$, and B in the deduction of the minor premise below one of the occurrences of $!C$ discharged by the C-application, or
- (d) A below B by an application of transitivity for the ordering $<$.

For an instance of $!I$, all occurrences in the derivations of the side premises are above the occurrences in the derivation of the main premise in $[D]$; and for an instance of contraction, all occurrences in the derivation of the major premise are above all occurrences in the derivation of the minor premise.

The strategy is now described as follows. Select in $[D]$ a rightmost branch containing a topmost critical segment; take the topmost critical segment in this branch and apply the conversion to this segment.

A crucial instance of conversion may serve to show that the strategy has the desired effect.

$$\begin{array}{c}
 \begin{array}{c}
 \overline{D} \quad D' \quad \overline{D'} \\
 \hline
 !\overline{E} \quad !A \quad !\overline{E'} \\
 \hline
 !B
 \end{array}
 \quad
 \begin{array}{c}
 \overline{D} \\
 \hline
 B
 \end{array}
 \quad
 \begin{array}{c}
 \overline{[!E, !A, !E']} \\
 \hline
 [!B, !B] \\
 \hline
 D'' \\
 \hline
 C
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \hline
 C
 \end{array}$$

converted to

$$\begin{array}{c}
 \begin{array}{c}
 \overline{D} \quad D' \quad \overline{D'} \\
 \hline
 !\overline{E} \quad !A \quad !\overline{E'} \\
 \hline
 !B
 \end{array}
 \quad
 \begin{array}{c}
 \overline{[!E, !A, !E']} \\
 \hline
 B \\
 \hline
 [!B]
 \end{array}
 \quad
 \begin{array}{c}
 \overline{[!E, !A, !E']} \\
 \hline
 B \\
 \hline
 [!B]
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \overline{D} \quad D' \quad \overline{D'} \\
 \hline
 !\overline{E} \quad !A \quad !\overline{E'} \\
 \hline
 C
 \end{array}
 \quad
 \begin{array}{c}
 \hline
 C
 \end{array}$$

has as effect that indeed an occurrence of $!B$ is removed, but on the other hand, there is on the right-hand side an extra occurrence of $!A$; but if we have chosen the critical segment according to our strategy, the occurrences of $!A$ cannot belong to a critical segment.

In the case where a promotion application α in a deduction of a side premise of a promotion application β is separated from β by some intervening contractions, weakenings, unit- and tensor eliminations, we use some permutation conversions (namely W4, C5, P1, P2 from left to right, where f is the operator of a promotion rule) to bring α closer to β ; these permutations do not increase the number of formula occurrences in critical segments.

9. Concluding remarks

The simpler version of promotion plus the *removal* of the “dynamical” aspect of contraction (by which we mean that identifying the labels of two distinct assumptions of a formula of the form $!A$ is made into a separate *operation*) result in a variant \mathbf{ILL}^+ with a relatively simple proof theory. In addition, it suggests the consideration of a special class of categorical models for \mathbf{ILL} .

There is a price to pay: the condition on the occurrence of multiple labels for \mathbf{ILL}^+ is not difficult to manage, but if we want to extend \mathbf{ILL}^+ by additive operators and constants, it becomes rather unwieldy.

On the other hand, the study of \mathbf{ILL}_{me}^+ suggested a suitable normalization strategy for \mathbf{ILL}_{me} as well; this strategy seems also suitable for a complete system \mathbf{ILL} . Although there seems no reason to doubt strong normalization for \mathbf{ILL} (presumably a variant of the method in [7] would do the job), it is not clear how to extend strong validity to \mathbf{ILL} .

We have not troubled ourselves with the Church–Rosser property (confluence), which holds for \mathbf{ILL} and \mathbf{ILL}^+ . The significance of confluence for these systems seems to be limited, as it is not likely that the conversion rules identify all intuitively equal deductions.

It would be interesting to extend the treatment of \mathbf{ILL}^+ to a multiple conclusion sequent calculus for a correspondence fragment for classical linear logic (cf. [6] for classical logic) and compare normal forms for this case with proofnets.

Strong normalization for the \mathbf{ILL} -term calculus w.r.t. detour conversions is proved in [2] by means of an embedding in the second-order lambda calculus. Can this method be extended to cover permutation conversions and X1 as well?

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