



UvA-DARE (Digital Academic Repository)

Galois modular invariants of WZW models

Fuchs, J.; Schellekens, A.N.J.J.; Schweigert, C.

DOI

[10.1016/0550-3213\(94\)00577-2](https://doi.org/10.1016/0550-3213(94)00577-2)

Publication date

1995

Published in

Nuclear Physics B

[Link to publication](#)

Citation for published version (APA):

Fuchs, J., Schellekens, A. N. J. J., & Schweigert, C. (1995). Galois modular invariants of WZW models. *Nuclear Physics B*, 437, 667. [https://doi.org/10.1016/0550-3213\(94\)00577-2](https://doi.org/10.1016/0550-3213(94)00577-2)

General rights

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.



ELSEVIER

Nuclear Physics B 437 (1995) 667–694

NUCLEAR
PHYSICS B

Galois modular invariants of WZW models

J. Fuchs¹, A.N. Schellekens, C. Schweigert

NIKHEF-H / FOM, Postbus 41882, 1009 DB Amsterdam, The Netherlands

Abstract

The set of modular invariants that can be obtained from Galois transformations is investigated systematically for WZW models. It is shown that a large subset of Galois modular invariants coincides with simple current invariants. For algebras of type B and D infinite series of previously unknown exceptional automorphism invariants are found.

1. Introduction

The problem of finding all modular invariant partition functions of rational conformal field theories (RCFT's) remains to a large extent unsolved. This problem is part of the programme of classifying all rational conformal field theories, which in turn is part of the even more ambitious programme of classifying all string theories.

The aim is to find a matrix P that commutes with the generators S and T of the modular group, and that furthermore is integer-valued, non-negative and has $P_{00} = 1$, where 0 represents the identity primary field. The partition function of the theory has then the form $\sum_{ij} \mathcal{X}_i P_{ij} \bar{\mathcal{X}}_j^*$, where \mathcal{X}_i are the characters of the left chiral algebra and $\bar{\mathcal{X}}_j$ those of the right one (the left and right algebras need not necessarily coincide).

At present the classification is complete only for the simplest RCFT's, whose chiral algebra consists only of the Virasoro algebra [1,2]. The next simplest case is that of WZW models, whose chiral algebra has in addition to the Virasoro algebra further currents of spin 1. In general such a theory can be "heterotic" (i.e. it may have different left and right Kac–Moody algebras) and both the left and right chiral algebra may have more than one affine factor, but even in the simplest case—equal left and right simple affine algebras—the classification is complete at arbitrary level only for the cases

¹ Heisenberg fellow.

A_1 [2] and A_2 [3]. Several other partial classification results have been presented, see for example Refs. [4–6].

Although there is no complete classification, many methods are known for finding at least a substantial number of solutions, for example simple currents [7] (see also Refs. [8–12]), conformal embeddings [13], rank–level duality [14–19], supersymmetric index arguments [20], selfdual lattice methods [21], orbifold constructions using discrete subgroups of Lie groups [22], and the elliptic genus [23]. In a previous paper [24], we introduced an additional method based on Galois symmetry of the matrix S of a RCFT, a symmetry that was discovered by de Boer and Goeree [25] and further investigated by Coste and Gannon [26]. This work will be reviewed briefly in the next section. The main purpose of this paper is to study in more detail the application of this new method to WZW models.

Galois symmetry organizes the fields of a CFT into orbits, and along these orbits the matrix elements of S are algebraically conjugate numbers. Based on this knowledge we are able to write down a number of integer-valued matrices P that commute with S , but do not necessarily commute with T and are not necessarily positive. These matrices span what we call the “Galois-commutant” of S . This commutant can be constructed in a straightforward manner from the Galois orbits, which in turn can be obtained by scaling vectors in weight space by certain integers, and mapping them back into the fundamental affine Weyl chamber (for a more precise formulation we refer to Section 2 and Appendix A). This is a simple algorithm that can be carried out easily with the help of a computer. The time required for this computation increases linearly with the number of primary fields, and for each primary the number of calculational steps is bounded from above by the order of the Weyl group. This should be compared with the computation of the modular matrix S , which grows quadratically with the number of primaries, and which requires a sum over the full Weyl group (although several shortcuts exist, for example simple currents and of course Galois symmetry).

Our second task is then to find the positive T -invariants within the Galois commutant. In some cases this can be done analytically. This class, which contains only simple current invariants, is discussed in Section 3. In general however one has to solve a set of equations for a number of integer coefficients. The number of unknowns can grow rather rapidly with increasing level of the underlying affine Kac–Moody algebra—Galois symmetry is a huge and very powerful symmetry—which is another limitation on the scope of our investigations.

In practice we have considered algebras with rank ≤ 8 and up to 2500 primary fields, but this range was extended when there was reason to expect something interesting. Although a lot of exploratory work has already been done on the classification of modular invariants, only fairly recently new invariants were found [23] for E_6 and E_7 at rather low levels (namely 4 and 3), showing that there are still chances for finding something new. Indeed, we did find new invariants, namely an infinite series of exceptional automorphism invariants for algebras of type B at level 2, starting at rank 7, as well as for algebras of type D at level 2. In addition we find for the same algebras some clearly unphysical extensions by spin-1 currents. This is explained in Section 4. Other exceptional invariants that can be explained in terms of Galois symmetry are presented in Section 5.

We have also considered the possibility of combining Galois orbits with simple current orbits. In Section 6 we discuss two ways of doing that, one of which is to apply Galois symmetry to simple current extensions of the chiral algebra.

To conclude this introduction we fix some notations. If $P_{0i} = P_{i0} = 0$ for all $i \neq 0$, the matrix P defines a permutation of the fields in the theory that leaves the fusion rules invariant. We will refer to this as an *automorphism invariant*. Under multiplication such matrices form a group which is a subgroup of the group of *fusion rule automorphisms*. These are all permutations of the fields that leave the fusion rules invariant, but which do not necessarily commute with S or T . Finally there is a third group of automorphisms we will encounter, namely that of *Galois automorphisms*. They act as a permutation combined with sign flips, and may act non-trivially on the identity. It is important not to confuse these three kinds of automorphisms.

If a matrix P does not have the form of an automorphism invariant, and if the partition function is a sum of squares of linear combinations of characters, we will refer to it as a (*chiral algebra*) *extension*. If it is not a sum of squares it can be viewed as an automorphism invariant of an extended algebra [27,28] (at least if an associated CFT exists).

A matrix P corresponding to a chiral algebra extension may contain squared terms appearing with a multiplicity higher than 1. Such terms will be referred to as “fixed points”, a terminology which up to now was appropriate only for extensions by simple currents. Galois automorphisms provide us with a second rationale for using this name. Usually such fixed points correspond to more than one field in the extended CFT, and they have to be “resolved”. The procedure for doing this is available only in some cases, and then only for S, T , the fusion rules and in a few cases also the characters [29].

2. Galois symmetry in conformal field theory

As is well known, a rational conformal field theory gives rise to a finite-dimensional unitary representation of $SL_2(\mathbb{Z})$, the double cover of the modular group. Namely, given a rational fusion ring with generators $\phi_i, i \in I$ (I some finite index set), and relations

$$\phi_i \times \phi_j = \sum_{k \in I} \mathcal{N}_{ij}^k \phi_k,$$

there is a unitary and symmetric matrix S that diagonalizes the fusion matrices, i.e. the matrices \mathcal{N}_i with entries $(\mathcal{N}_i)_j^k := \mathcal{N}_{ij}^k$. S and the matrix T with entries $T_{ij} := e^{2\pi i(h_i - c/24)} \delta_{ij}$ (with h_i the conformal weights and c the conformal central charge), generate a representation of $SL_2(\mathbb{Z})$. In particular, $S^2 = C = (ST)^3$ where C , the charge conjugation matrix, is a permutation of order two, which we write as $C_{ij} = \delta_{i,j^+}$. By the Verlinde formula

$$\mathcal{N}_{ij}^k = \sum_{\ell \in I} \frac{S_{i\ell} S_{j\ell} S_{k\ell}^*}{S_{0\ell}},$$

the eigenvalues of the fusion matrices \mathcal{N}_i are the generalized *quantum dimensions* S_{ij}/S_{0j} ; the label 0 refers to the identity primary field; it satisfies $0 = 0^+$ and corresponds to the unit of the fusion ring. The quantum dimensions realize all inequivalent irreducible representations of the fusion ring (which are one-dimensional), i.e. we have

$$\frac{S_{i\ell}}{S_{0\ell}} \frac{S_{j\ell}}{S_{0\ell}} = \sum_{k \in I} \mathcal{N}_{ij}^k \frac{S_{k\ell}}{S_{0\ell}} \tag{2.1}$$

for all $\ell \in I$.

The quantum dimensions are the roots of the characteristic polynomial

$$\det(\lambda \mathbf{1} - \mathcal{N}_i).$$

This polynomial has integral coefficients and is normalized, i.e. its leading coefficient is equal to 1. As a consequence, the quantum dimensions are algebraically integer numbers in some algebraic number field L over the rational numbers \mathbb{Q} . The extension L/\mathbb{Q} is normal [25]; since the field \mathbb{Q} has characteristic zero, this implies that it is a *Galois* extension; its *Galois group*, denoted by $\text{Gal}(L/\mathbb{Q})$, is abelian. Invoking the theorem of Kronecker and Weber, this shows that L is contained in some cyclotomic field $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n th root of unity.

By applying an element $\sigma_L \in \text{Gal}(L/\mathbb{Q})$ on equation (2.1) it follows that the numbers $\sigma_L(S_{ij}/S_{0j})$, $i \in I$, again realize a one-dimensional representation of the fusion ring. As the (generalized) quantum dimensions exhaust all inequivalent one-dimensional representations of the fusion ring, it follows that there exists some permutation σ of the index set I such that

$$\sigma_L\left(\frac{S_{ij}}{S_{0j}}\right) = \frac{S_{i,\sigma j}}{S_{0,\sigma j}}.$$

To obtain an action of a Galois group on the entries S_{ij} of the S -matrix, rather than just on the quantum dimensions, one has to consider also the field M which is the extension of \mathbb{Q} that is generated by all S -matrix elements. M extends L as well, and the extension M/\mathbb{Q} is again normal and has abelian Galois group. It follows that $\text{Gal}(M/L)$ is a normal subgroup of $\text{Gal}(M/\mathbb{Q})$ and that the sequence $0 \rightarrow \text{Gal}(M/L) \rightarrow \text{Gal}(M/\mathbb{Q}) \rightarrow \text{Gal}(L/\mathbb{Q}) \rightarrow 0$, where the second map is the canonical inclusion and the third one the restriction map, is exact. Therefore

$$\text{Gal}(L/\mathbb{Q}) \cong \text{Gal}(M/\mathbb{Q}) / \text{Gal}(M/L).$$

In particular, upon restriction from M to L any $\sigma_M \in \text{Gal}(M/\mathbb{Q})$ maps L onto itself and coincides with some element $\sigma_L \in \text{Gal}(L/\mathbb{Q})$, and conversely, any $\sigma_L \in \text{Gal}(L/\mathbb{Q})$ can be obtained this way. Correspondingly, as in Ref. [24] we will frequently use the abbreviation σ for both σ_M and its restriction σ_L .

For any $\sigma \in \text{Gal}(L/\mathbb{Q})$ there exist [26] signs $\epsilon_\sigma(i) \in \{\pm 1\}$ such that the relation

$$\sigma(S_{ij}) = \epsilon_\sigma(i) \cdot S_{\sigma i, j} \tag{2.2}$$

holds for all $i, j \in I$. While the order N of the Galois group element σ and the order \tilde{N} of the permutation $\tilde{\sigma}$ of the labels that is induced by σ need not necessarily coincide,

only an extra factor of 2 can appear, and the elements with $N = 2\dot{N}$ turn out to be quite uninteresting [24].

Let us now describe a few elementary facts about Galois theory of cyclotomic fields. Denote by \mathbb{Z}_n^* the multiplicative group of all elements of $\mathbb{Z}_n \equiv \mathbb{Z}/n\mathbb{Z}$ that are coprime with n . Note that precisely these elements have an inverse with respect to multiplication. (For example, the group $(\mathbb{Z}_{10}^*, \cdot) \cong (\{\pm 1, \pm 3\}, \cdot \text{ mod } 10)$ is isomorphic to the additive group $(\mathbb{Z}_4, +)$.) The number $\varphi(n)$ of elements of \mathbb{Z}_n^* is given by Euler’s φ function, which can be computed as follows. If $n = \prod_i p_i^{n_i}$ is a decomposition of n into distinct primes p_i , then one has

$$\varphi(n) = \varphi\left(\prod_i p_i^{n_i}\right) = \prod_i \varphi(p_i^{n_i}) = \prod_i p_i^{n_i-1}(p_i - 1).$$

The Galois automorphisms (relative to \mathbb{Q}) of the cyclotomic field $\mathbb{Q}(\zeta_n)$ in which $\text{Gal}(L/\mathbb{Q})$ is contained are in one-to-one correspondence with the elements $\ell \in \mathbb{Z}_n^*$. The automorphism associated to each such ℓ simply acts as

$$\sigma_{(\ell)} : \zeta_n \mapsto (\zeta_n)^\ell.$$

This implies in particular that $\ell = -1$ corresponds to complex conjugation. Thus if the fusion ring is self-conjugate in the sense that $i^+ = i$ for all $i \in I$, so that the S -matrix is real, then the automorphism $\sigma_{(-1)}$ acts trivially. In this case the relevant field L is already contained in the maximal real subfield $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ of the cyclotomic field $\mathbb{Q}(\zeta_n)$, which is the field that is fixed under complex conjugation.

In the special case where the fusion ring describes the fusion rules of a WZW theory based on an affine Lie algebra \mathfrak{g} at level k , the Galois group is a subgroup of $\mathbb{Z}_{M(k+g)}^*$, where g is the dual Coxeter number of the horizontal subalgebra of \mathfrak{g} and M is the smallest positive integer for which the numbers MG_{ij} , with G_{ij} the entries of the metric on the weight space of the horizontal subalgebra, are all integral. A Galois transformation labelled by $\ell \in \mathbb{Z}_{M(k+g)}^*$ then induces the permutation

$$\dot{\sigma}_{(\ell)}(\Lambda) = \hat{w}(\ell \cdot (\Lambda + \rho) - \rho), \tag{2.3}$$

where Λ , the horizontal part of an integrable highest weight of \mathfrak{g} at level k , labels the primary fields, ρ is the Weyl vector of the horizontal subalgebra, and where \hat{w} is the horizontal projection of a suitable affine Weyl transformation. The sign $\epsilon_{\sigma_{(\ell)}}$ is just given by the sign of the Weyl transformation \hat{w} , up to an overall sign η that only depends on $\sigma_{(\ell)}$, but not on Λ . (For more details, see Appendix A.)

2.1. Fusion rule automorphisms

As has been shown in Ref. [24], the properties of Galois transformations can be employed to construct automorphisms of the fusion rules as well as S -invariants. Consider first the case where the permutation $\dot{\sigma}$ induced by the Galois group element σ leaves the identity fixed,

$$\dot{\sigma} 0 = 0;$$

then $\hat{\sigma}$ is an automorphism of the fusion rules, and the sign $\epsilon_{\sigma}(i)$ is the same for all $i \in I$,

$$\epsilon_{\sigma}(i) = \epsilon_{\sigma}(0) =: \epsilon_{\sigma} = \text{const.}$$

The presence of such automorphisms of the fusion rules can be understood as follows. The “main” quantum dimensions

$$\frac{S_{i0}}{S_{00}}$$

all lie in a real field $L_{(0)}$ that is contained in the field L generated by all (generalized) quantum dimensions S_{ij}/S_{0j} . The elements of the group $\text{Gal}(L/L_{(0)})$ leave the main quantum dimensions invariant, and hence the associated permutations $\hat{\sigma}$ are fusion rule automorphisms.

2.2. *S*-invariants

To deduce *S*-invariants from these considerations it is convenient to act on (2.2) with σ^{-1} , and permuting the second label of *S* on the right-hand side. Then we obtain the relation

$$S_{ij} = \epsilon_{\sigma}(i)\epsilon_{\sigma^{-1}}(j) S_{\hat{\sigma}i, \hat{\sigma}^{-1}j}. \tag{2.4}$$

Now for any Galois transformation σ we define the orthogonal matrix

$$(\Pi_{\sigma})_{ij} := \epsilon_{\sigma}(i) \delta_{j, \hat{\sigma}i} = \epsilon_{\sigma^{-1}}(j) \delta_{i, \hat{\sigma}^{-1}j},$$

where in the second equality we used the relation

$$\epsilon_{\sigma}(\hat{\sigma}^{-1}(i)) = \epsilon_{\sigma^{-1}}(i)$$

which is obtained from the identity $\sigma\sigma^{-1}S_{ij} = S_{ij}$ when acting twice on the first label of *S*. These orthogonal matrices can easily be shown to satisfy the identities

$$(\Pi_{\sigma})^{-1} = \Pi_{\sigma^{-1}} = (\Pi_{\sigma})^T,$$

and they implement the Galois transformations (2.2) in the following way:

$$\sigma S = \Pi_{\sigma} \cdot S = S \cdot \Pi_{\sigma}^{-1}.$$

Now we can write (2.4) in matrix notation as (omitting the subscript σ of Π_{σ})

$$S = \Pi S \Pi, \tag{2.5}$$

or $\Pi^{-1}S = S\Pi$. Obviously the same identity holds with Π replaced by its inverse, and by adding these two relations we see that the matrix $\Pi + \Pi^{-1} = \Pi + \Pi^T$ commutes with *S*. If Π is equal to its own inverse one can take half this matrix, i.e. Π itself.

The full Galois commutant is obtained by considering all sums and products of these matrices. Because the matrices Π form an abelian group (isomorphic to the Galois group $\text{Gal}(L/\mathbb{Q})$) it is easy to see that the product of any two matrices of the form

$\Pi + \Pi^{-1}$ is a linear combination of such matrices with integral coefficients. Hence the most general integer-valued S -invariant that can be obtained in this way is

$$P = \sum_{(\sigma, \sigma^{-1}) \in G} f_{\sigma}(\Pi_{\sigma} + \Pi_{\sigma^{-1}}), \tag{2.6}$$

where the sum is over all elements of the Galois group G modulo inversion, and $f_{\sigma} \in \mathbb{Z}$. This result was obtained before in Ref. [24] in a slightly different formulation.

Note that this derivation of S -invariants goes through for any matrix Π that satisfies (2.5), even if it did not originate from Galois symmetry. If such a new matrix Π commutes with all matrices Π_G that represent Galois symmetries, one may extend the Galois group G to a larger group $\tilde{G} \supset G$ by including all matrices $\Pi \cdot \Pi_G$. The most general S -invariant related to \tilde{G} is then obtained by extending the sum in (2.6) to \tilde{G} .

As was observed in Ref. [26], Galois symmetry implies a relation that any modular invariant P , irrespective of whether it is itself a Galois invariant, should satisfy. Indeed, using $\sigma P = P$ and $\sigma S^{-1} = (\sigma S)^{-1}$, one derives $P = \sigma P = \sigma(S P S^{-1}) = \Pi_{\sigma} P \Pi_{\sigma}^{-1}$, i.e. P commutes with Π . If P is an automorphism of order 2, then we have in addition the relation $S = P S P$, and hence P is a ‘‘Galois-like’’ automorphism that can be used to extend the Galois group as described above. If P is an automorphism of higher order or corresponds to an extension of the chiral algebra, then it has different commutation properties with S , and it cannot be used to extend the Galois group, but one can still enlarge the commutant by multiplying all matrices (2.6) with the new invariant P and its higher powers. In this case the full commutant is considerably harder to describe, however.

It must be noted that even if the matrix (2.6) contains negative entries, or does not commute with T , it can still be relevant for the construction of physical modular invariants, because the prescription may be combined with other procedures in such a manner that the unwanted contributions cancel out. For example one may use simple currents to extend the chiral algebra before employing the Galois transformation. This will be discussed in Section 6.

3. Infinite series of invariants

In this Section we will discuss an infinite class of WZW modular invariants that can be obtained both by a Galois scaling as well as by means of simple currents. Both Galois transformations and simple currents organize the fields of a CFT into orbits. In general, the respective orbits are not identical. In the special case of WZW models which we focus on in this paper, these orbits are in fact never identical, except for a few theories with too few primary fields to make the difference noticeable. However, since the orbits are used in quite different ways to derive modular invariants, it can nevertheless happen that these invariants are the same.

The Galois scalings we consider are motivated by the following argument. As already mentioned, Galois automorphisms of the fusion rules arise if the field $L_{(0)}$ is strictly smaller than the field L . In the case of WZW theories L is contained in the cyclotomic

field $\mathbb{Q}(\zeta_{M(k+g)})$ where M is the denominator of the metric on weight space, while the quantum Weyl formula [30]

$$\frac{S_{a,\rho}}{S_{\rho,\rho}} = \prod_{\alpha>0} \frac{\sin[\pi a \cdot \alpha^\vee / (k+g)]}{\sin[\pi \rho \cdot \alpha^\vee / (k+g)]}$$

shows that $L_{(0)}$ is already contained in $\mathbb{Q}(\zeta_{2(k+g)})$. Now as any element of $\text{Gal}(L/\mathbb{Q})$ can be described by at least one element of $\text{Gal}(\mathbb{Q}(\zeta_{M(k+g)})/\mathbb{Q})$, we do not lose anything by working with the latter Galois group. Any Galois automorphism of the fusion rules can now be described by at least one element of $\text{Gal}(\mathbb{Q}(\zeta_{M(k+g)})/L_{(0)})$. Unfortunately, $L_{(0)}$ is not explicitly known in practice; therefore we would like to replace $L_{(0)}$ by the field $\mathbb{Q}(\zeta_{2(k+g)})$ in which it is contained. However, M is not always even, and hence we consider instead of $\mathbb{Q}(\zeta_{2(k+g)})$ the smaller field $\mathbb{Q}(\zeta_{k+g})$ and the corresponding Galois group $\text{Gal}(\mathbb{Q}(\zeta_{M(k+g)})/\mathbb{Q}(\zeta_{k+g}))$. The elements of this group are precisely covered by scalings by a factor $m(k+g)+1$. This way we recover at least part of the automorphisms, but due to the difference between $\mathbb{Q}(\zeta_{2(k+g)})$ and $\mathbb{Q}(\zeta_{k+g})$, generically some of these scalings do not describe automorphisms, but rather correspond to an extension of the chiral algebra.

Consider now the Kac–Peterson [31] formula

$$S_{ab} = \mathcal{N} \sum_w \varepsilon(w) \exp \left[-2\pi i \frac{w(a) \cdot b}{k+g} \right] \tag{3.1}$$

for the modular matrix S . Here \mathcal{N} is a normalization factor which follows by the unitarity of S and is irrelevant for our purposes, and the summation is over the Weyl group of the horizontal subalgebra of the relevant affine Lie algebra; a and b are integrable weights, shifted by adding the Weyl vector ρ . In the following we will denote such shifted weights by roman characters a, b, \dots , while for the Lie algebra weights $a - \rho, b - \rho, \dots$ we will use Greek characters.

The scaling by a factor $\ell = m(k+g)+1$ is an allowed Galois scaling if the following condition is fulfilled (note that m is defined modulo M):

- (a) $m(k+g)+1$ is prime relative to $M(k+g)$.

We will return to this condition later. (Let us mention that even if condition (a) is not met, the scaling by ℓ can still be used to define an S -invariant. We will describe the implications of such “quasi-Galois” scalings elsewhere.)

Under such a scaling one has

$$\begin{aligned} S_{ab} \mapsto \sigma S_{ab} &= \mathcal{N} \sum_w \varepsilon(w) \exp \left[-2\pi i \frac{w(a) \cdot b}{k+g} (m(k+g)+1) \right] \\ &= e^{-2\pi i m a \cdot b} S_{ab} \end{aligned} \tag{3.2}$$

where the last equality holds if $mw(a) \cdot b = ma \cdot b \pmod 1$ for all Weyl group elements w . To analyze when this condition is fulfilled, first note that any Weyl transformation can be written as a product of reflections with respect to the planes orthogonal to the simple

roots. For a Weyl reflection r_i with respect to a simple root α_i ($i \in \{1, 2, \dots, \text{rank}\}$) one has in general

$$\begin{aligned} r_i(a) \cdot b &= a \cdot b - \left(\frac{2}{\alpha_i \cdot \alpha_i} \right) \alpha_i \cdot a \alpha_i \cdot b \\ &= a \cdot b - \frac{1}{2} \alpha_i \cdot \alpha_i a_i b_i, \end{aligned} \tag{3.3}$$

where a_i and b_i are Dynkin labels. Thus $r_i(a) \cdot b$ equals $a \cdot b$ modulo integers if and only if all simple roots have norm 2 (which is for all algebras our normalization of the longest root), i.e. iff the algebra is simply laced. However, the derivation depends on this relation with an extra factor m . This yields one more non-trivial solution, namely $m = 2$ for B_n , n odd. Note that for B_n with n even, one has $M = 2$ so that the only allowed scaling, $m = 2$, yields a trivial solution. This is also true for all other non-simply laced algebras.

As is easily checked, the quantity $a \cdot b \pmod 1$ is closely related to the product of the simple current charges; we find

$$\begin{aligned} A_n : & \quad a \cdot b = -(n + 1) Q(a) Q(b) \\ B_n : & \quad 2a \cdot b = 2n Q(a) Q(b) \\ D_n \text{ (} n \text{ odd)} : & \quad a \cdot b = 4n Q(a) Q(b) \\ D_n \text{ (} n \text{ even)} : & \quad a \cdot b = 2Q_s(a) Q_s(b) + 2Q_c(a) Q_c(a) + (n - 2) Q_v(a) Q_v(b) \\ E_6 : & \quad a \cdot b = 3Q(a) Q(b) \\ E_7 : & \quad a \cdot b = 2Q(a) Q(b). \end{aligned} \tag{3.4}$$

Here $Q(a)$ is the monodromy charge with respect to the simple current J of a WZW representation with highest weight a (which is at level $k + g$). This should not be confused with the simple current charge of the field labelled by a , which we denote by $Q(a)$. The relation between these two quantities is

$$Q(a) = Q(a - \rho) = Q(a) - Q(\rho), \tag{3.5}$$

since the field labelled by a has highest weight $a - \rho$ (which is at level k). The charge Q (as well as Q) depends only on the conjugacy class of the weight. The WZW theory with algebra D_n , n even, has a center $\mathbb{Z}_2 \times \mathbb{Z}_2$ and simple currents J_s, J_v and $J_c = J_v \times J_s$. It has thus two independent charges, for which one may take Q_v and Q_s .

If ρ is on the root lattice, then $Q(\rho) = 0$ and the shift in (3.5) is irrelevant, i.e. $Q = Q \pmod 1$. In general, either ρ is a vector on the root lattice, or it is a weight with the property that 2ρ is on the root lattice. In the cases of interest here, ρ is on the root lattice for A_n , n even, D_n with $n = 0 \pmod 4$ or $1 \pmod 4$, and for E_6 . In all other cases $Q = Q + \frac{1}{2} \pmod 1$ (if the algebra is D_n , $n = 2 \pmod 4$, the charges affected by this shift are Q_s and Q_c).

Note that the left-hand sides of (3.4) are always of the form $lNQ(a)Q(b)$ or a sum of such terms, where N is the order of the simple current and l is an integer. The relation for B_n has an essential factor of 2 in the left-hand side. Since the relations are defined modulo integers we cannot simply divide this factor out. The most convenient

way to deal with it is to rewrite m in this case as $m = 2\tilde{m}$ (we have already seen above that m has to be even for B_n). After substituting (3.4) into (3.2) we get generically

$$\sigma S_{ab} = e^{-2\pi i l m N Q(a) Q(b)} S_{ab} . \quad (3.6)$$

This formula holds for B_n if one replaces m by \tilde{m} , and for D_n , n even, if one replaces the exponent by the appropriate sum, as in (3.4). We will postpone the discussion of the latter case until later, and consider for the moment only theories with a center \mathbb{Z}_N .

Now we wish to make use of the simple current relation

$$S_{J^n a, b} = e^{2\pi i n Q(b)} S_{ab} .$$

This is simplest if we can replace Q by \tilde{Q} , and this is the case we consider first. This replacement is allowed if ρ is on the root lattice, but this is not a necessary condition because of the extra factor lmN . Suppose $Q = \tilde{Q} + \frac{1}{2}$. Then we see from the foregoing that N is even and l odd. Replacing Q by \tilde{Q} in the exponent of (3.6) yields the extra terms

$$\frac{1}{2} l m N Q(a) + \frac{1}{2} l m N Q(b) + \frac{1}{4} l m N , \quad (3.7)$$

which should be an integer. Now $NQ(a)$ (or $NQ(b)$) is an integer, which as a function of a (or b) takes all values modulo N . Hence each of the three terms must separately be an integer. The first two terms are integers if and only if m is even. Then the last one is an integer as well, since N is even. Thus the condition that Q can be replaced by \tilde{Q} is equivalent to

(b) $m\rho$ is an element of the root lattice.

We remind the reader that for B_n this is valid with m replaced by $\tilde{m} = \frac{1}{2}m$. Hence condition (b) is in fact not satisfied for B_n for any non-trivial value of m . In all remaining algebras M (the denominator of the inverse symmetrized Cartan matrix) is equal to N .

If conditions (a) and (b) hold we can derive

$$\sigma S_{ab} = S_{J^{-mNQ(a)} a, b} = S_{a, J^{-mNQ(b)} b} . \quad (3.8)$$

On the other hand according to (2.2) Galois invariance implies

$$\sigma S_{ab} = \epsilon_\sigma(a) S_{\sigma a, b} = \epsilon_\sigma(b) S_{a, \sigma b} . \quad (3.9)$$

Furthermore if $m\rho$ is an element of the root lattice, it is easy to see that the scale transformation fixes the identity field: the identity is labelled by ρ , and transforms into $\rho' = \rho + m(k+g)\rho$. The second term is a Weyl translation if $m\rho$ is on the root lattice. In these cases ρ' is mapped to ρ by the transformations described in Appendix A, which implies that the identity primary field is fixed. Then, according to Ref. [24], it follows that $\epsilon \equiv 1$, and hence we find

$$S_{J^{-mNQ(a)} a, b} = S_{\sigma a, b} ,$$

or

$$S_{a,b} = S_{\tau a,b} ,$$

where $\tau a = J^{mlNQ(a)} \dot{\sigma} a$. Then unitarity of S implies $\delta_{a,\tau a} = \sum_b S_{\tau a,b} S_{ba}^* = \sum_b S_{ab} S_{ba}^* = 1$, so that $a = \tau a$, and hence $\dot{\sigma} a = J^{-mlNQ(a)} a$.

As described in Section 2, any Galois transform that fixes the identity generates an automorphism of the fusion rules, and in this case we see that it connects fields on the same simple current orbit. It is a positive S -invariant, but so far it was not required to respect T -invariance. Thus the last condition we will now impose is

(c) T -invariance.

In general for simple currents of order N one has

$$h(J^n a) = h(a) + h(J^n) - nQ(a) \pmod 1$$

and

$$h(J^n) = \frac{rn(N - n)}{2N} \pmod 1 ,$$

where r is the monodromy parameter, which is equal to k for A_n at level k , to $3nk \pmod 8$ for D_n , n odd, to $2k$ for E_6 , and to $3k$ for E_7 . Condition (c) amounts to the requirement that the difference $h(J^{-mlNQ(a)} a) - h(a)$ of conformal weights be an integer. We have

$$\begin{aligned} h(J^{-mlNQ(a)} a) &= h(a) + h(J^{-mlNQ(a)}) + mlNQ(a)Q(a) \\ &= h(a) - \frac{r}{2}mlNQ(a) - \left(\frac{r}{2}(ml)^2 - ml\right)NQ(a)Q(a) . \end{aligned} \quad (3.10)$$

For algebras of type A or E , the second term on the right-hand side is always an integer, or can be chosen integer: if N is odd, r is defined modulo N and hence can always be chosen even (provided one makes the same choice also in the third term), whereas if N is even by inspection one sees that m must be even as well in order for $m\rho$ to be an element of the root lattice, and hence $mr/2 \in \mathbb{Z}$. Then the only threat to T -invariance is the last term, $(\frac{1}{2}rml - 1)mlNQ(a)Q(a)$. This is an integer for any a if and only if $(\frac{1}{2}rml - 1)ml = 0 \pmod N$.

Now we will determine the solutions to the three conditions (a), (b) and (c) formulated above. Any solution to these conditions will be a positive modular invariant of automorphism type, that can be obtained both from Galois symmetry as well as from simple currents.

Consider first E_6 . Condition (b) is trivial, so that m has to satisfy (a) $m(k+12)+1 \neq 0 \pmod 3$, i.e. $km+1 \neq 0 \pmod 3$, and (c) $(km-1)m = 0 \pmod 3$. We may assume that $m \neq 0$ to avoid the trivial Galois scaling. Then both conditions are satisfied if and only if $km = 1 \pmod 3$. There is always a solution for m , namely $m = k \pmod 3$, unless k is a multiple of 3.

Next consider E_7 . Now m has to be even in order that $m\rho$ is a root, and this only allows the trivial solution $m = 0$.

For A_n the problem is a bit more complicated. As T -invariance must hold for any charge $Q(a)$ it is clearly sufficient to consider $Q(a) = 1/N$. Several cases have to

be distinguished. We start with odd $N = n + 1$. Then condition (b) is automatically satisfied. For even level $k = 2j$ the other two conditions read

$$\begin{aligned} \text{(a)} \quad & \text{GCD}(2jm + 1, N) = 1, \\ \text{(c)} \quad & (jm + 1)m = 0 \pmod{N}. \end{aligned} \tag{3.11}$$

The solution of the second equation depends crucially on the common factors of j and N . It is easy to see that if j and N have a common factor p , then m is divisible by p as many times as N . In particular, if $N = p^\ell$ and j contains a factor p , then the only solution is the trivial one. To remove common factors, write $j = j'q_a$, $m = m'q_b$ and $N = N'q_b$, where q_a is the greatest common divisor of j and N , and q_b consists of all the prime factors of q_a to the power with which they appear in N . Now the second equation becomes

$$(j'q_a m'q_b + 1)m' = 0 \pmod{N'}. \tag{3.12}$$

Now we know that N' has no factors in common with j' , q_a or q_b , and hence we can find a m' for which the first factor vanishes $\pmod{N'}$. This solution m' is non-trivial provided $N' \neq 1$; if $N' = 1$ the solution is $m' = 1$ (or 0), i.e. $m = 0 \pmod{N}$.

The solution m' has no factors in common with N' . Hence we may write $2jm + 1 = jm + (jm + 1) = jm \pmod{N'} = j'q_a m'q_b \pmod{N'}$, so that we see that $2jm + 1$ and N' have no common factors. Furthermore $2jm + 1$ and q_b have no common factors, since m has a factor q_b . Hence $2jm + 1$ has no common factors with $N = N'q_b$, and therefore the first equation is also satisfied.

In addition to the solution described here, (3.12) may have additional solutions with m' and N' having a common factor. It is again easy to see that if m' contains any such prime factor, it must contain it with the same power with which it occurs in N' . Let us denote the total common factor as p_b , which is in general a product of several prime factors. Then the second equation reads

$$(j'q_a m''p_b q_b + 1)m'' = 0 \pmod{N''}, \tag{3.13}$$

where $m' = m''p_b$ and $N' = N''p_b$. We now look for solutions where m'' and N'' have no further common factors. Such a solution does indeed exist, since the coefficient of m'' has no factors in common with N'' . To show that the first condition is also satisfied one proceeds exactly as in the foregoing paragraph.

When N is odd and k is also odd, we choose the even monodromy parameter $r = k + N$, and define $j = (k + N)/2$. The rest of the discussion is then exactly as before.

If N is even condition (b) implies that m must be even as well, and condition (c) becomes $(km/2 + 1)m = 0 \pmod{N}$, or, writing $m = 2t$, $N = 2p$, $(kt + 1)t = 0 \pmod{p}$. Condition (a) reads $\text{GCD}(km + 1, N) = 1$, which is equivalent to $\text{GCD}(2kt + 1, p) = 1$. Now we have succeeded in bringing the conditions in exactly the same form as (3.11), and we can read off the solutions almost directly. The only slight difference is that above N was odd, whereas here p can be odd or even. However, the value of N did not play any rôle anywhere in the discussion following (3.11) (it was used to derive (3.11), though), and hence everything does indeed go through.

If the algebra is D_n , n odd, then we have to distinguish two cases. If $n = 1 \pmod 4$, then condition (b) is trivially satisfied, and condition (a) reads

$$(a) \quad \text{GCD}(m(k + 2n - 2) + 1, 4) = 1 ,$$

from which we conclude that mk (and hence $mr = 3mk$) must be even, so that just as for A_n and E_n the second term on the right-hand side of (3.10) plays no rôle. Condition (c) thus reduces to

$$(c) \quad - \left(\frac{r}{2} (mn)^2 - mn \right) = 0 \pmod 4 ,$$

with k satisfying $3nk = r \pmod 8$, or what is the same, $nk = 3r \pmod 8$. To substitute this we multiply the first argument of (a) with n , which does not affect this condition. Afterwards we use that $n = 1 \pmod 4$, and then the conditions simplify to

$$\begin{aligned} (a) \quad & \text{GCD}(3mr + 1, 4) = 1 , \\ (c) \quad & - \left(\frac{r}{2} m^2 - m \right) = 0 \pmod 4 . \end{aligned} \tag{3.14}$$

If r is even, $r = 2j$, condition (c) then reduces to $jm^2 - m = 0 \pmod 4$. This clearly has a non-trivial solution if j is odd (then m is odd), but only trivial solutions if j is even. If r is odd the only solution to both equations is $m = 2$.

If $n = 3 \pmod 4$ this argument goes through in much the same way, but now solutions for odd m are eliminated by condition (b).

3.1. Automorphisms from fractional spin simple currents

Nearly all these results can be summarized as follows. Define $\tilde{N} = N$ if N is odd, $\tilde{N} = N/2$ if N is even. Decompose \tilde{N} into prime factors, $\tilde{N} = p_1^{n_1} \dots p_l^{n_l}$. Then the set of solutions m consists of all integers of the form $m = m'' \frac{N}{\tilde{N}} p_1^{k_1} \dots p_l^{k_l}$, where $k_i = n_i$ if the monodromy parameter r is divisible by p_i , and $k_i = 0$ or $k_i = n_i$ otherwise. The solutions are thus labelled by all combinations of distinct prime factors of \tilde{N} that are not factors of r . The parameter m'' for each solution in this set is the unique solution of the equation

$$\frac{1}{2} r l m'' (p_1^{k_1} \dots p_l^{k_l}) = 1 \pmod{N''} ,$$

where $N'' = \tilde{N} / p_1^{k_1} \dots p_l^{k_l}$, and r chosen even if N is odd. These automorphism invariants have both a Galois interpretation and a simple current interpretation: they can be generated by the Galois scaling $m(k + g) + 1$ or alternatively by the fractional spin simple current J^m .

These are precisely all the pure automorphisms generated by single simple currents $K = J^m$ of fractional spin which have a “square root”, i.e. for which there exists a simple current K' such that $(K')^2 = K$. Such a square root exists always if K has odd order, but if K has even order it must be an even power m of the basic simple current J . The condition on the common factors of r and N has a simple interpretation in terms of

simple currents: If it is not satisfied, then there are integral spin currents on the orbit of J . If one constructs the simple current invariant associated with J these currents extend the chiral algebra, so that one does not get a pure automorphism invariant.

The condition that K must have a square root is a familiar one: in Ref. [7] the same condition appeared as a requirement that an invariant can be obtained by a simple left-right symmetric orbifold-like construction with “twist operator” $L\bar{L}^c$. If K does not have a square root and r is even, then there are additional invariants, which were described in Ref. [7] and derived in Ref. [32]. Recently in Ref. [33] it was observed that these invariants could be described as orbifolds with discrete torsion. It is quite interesting that precisely these discrete torsion invariants are missing from the list of Galois invariants.

There is one exception, namely the automorphism invariants of D_{4l+1} at level $2j$, which are Galois invariants even though they violate the foregoing empirical rule: In this case $\tilde{N} = 2$, which is a factor of r . Indeed, they are generated by the current J_s (or J_c) which does not have a square root. Technically the reason for the existence of this extra solution is that this is the only simply laced algebra with ρ lying on the root lattice but N even.

3.2. Automorphisms from integer spin simple currents

Finally, we have to return to the case D_n , n even. Since $M = 2$ in this case, the only potentially interesting solution is $m = 1$. Hence \mathcal{Q} is equivalent to Q if and only if ρ is on the root lattice, which is true if and only if $n = 0 \pmod 4$. It is straightforward to derive the analogue of (3.8):

$$\sigma S_{ab} = S_{j_s^{2mQ_s(a)} j_c^{2mQ_c(a)} j_v^{(n-2)mQ_v(a)} a, b} .$$

(Since the three currents and charges are dependent this is a somewhat redundant notation.) The solution $m = 1$ satisfies condition (a) if and only if the level is even. This implies immediately that all three currents J_s , J_v and J_c have integer spin, and we can write the transformation of S in the following symmetric way:

$$\sigma S_{ab} = S_{j_s^{2Q_s(a)} j_c^{2Q_c(a)} j_v^{2Q_v(a)} a, b} .$$

Since $Q_s + Q_c + Q_v = 0 \pmod 1$ for any weight a , at least one of the charges, say Q_v , must vanish. Then $Q_s = Q_c \pmod 1$, and the field a is transformed to $j_s^{2Q_s(a)} j_c^{2Q_c(a)} a = j_v^{2Q_s(a)} a$. Since J_v has integral spin and $Q_v(a) = 0$, this field has the same conformal weight as a , and hence T -invariance is respected. Due to the symmetry in s, c and v the same is true for any other field as well. Thus we do find an infinite series of modular invariant partition functions. These are automorphism invariants, again with both a Galois and a simple current interpretation, although this time they are due to simple currents of integer spin. Invariants of this type have been described before in Ref. [34].

3.3. Chiral algebra extensions

Now we will examine what happens if we relax condition (b), i.e. we will consider the case that the replacement of \mathcal{Q} by Q leads to a different answer. This obviously

requires that ρ is not on the root lattice, and that the extra terms (3.7) are non-integral for some values of Q . The latter is true if m is odd, or if the algebra is B_n, n odd, and $m = 2$ ($\tilde{m} = 1$). Now we can write (omitting for the moment the case D_n, n even)

$$\begin{aligned} \sigma S_{ab} &= e^{-2\pi i l m N [Q(a)+1/2][Q(b)+1/2]} S_{ab} \\ &= e^{-\pi i l m N [Q(a)+1/2]} S_{J^{-lmN[Q(a)+1/2]} a, b} \end{aligned} \tag{3.15}$$

instead of (3.6). As before, a similar formula holds also for B_n, n odd, with m replaced by $\tilde{m} = \frac{1}{2}m$.

Since $mlNQ(a)$ is always integral and N is even, the exponential prefactor is in fact a sign, and the result may be written as

$$\sigma S_{ab} = \eta(a) S_{J^{-lmN[Q(a)+1/2]} a, b} . \tag{3.16}$$

Comparing this with (3.9) we find now that $S_{a,b} = \omega(a) S_{\tau a, b}$, where ω is the product of the overall signs η and ϵ , and $\tau a = J^{-lmN[Q(a)+1/2]} \sigma a$. Unitarity of S now gives $\delta_{a, \tau a} = \sum_b S_{\tau a, b} S_{ba}^* = \sum_b \omega(a) S_{ab} S_{ba}^* = \omega(a)$, which implies that $\omega = 1$, i.e. $\eta = \epsilon$, and that τ is the trivial map.

Also in this case the Galois transformation generates an automorphism that lies within simple current orbits, and hence if it generates a positive modular invariant, it must be a simple current invariant. The identity is not fixed in this case: it must thus be mapped to a simple current. The candidate modular invariant has the form $P = 1 + \eta(0) \Pi$, where Π is the matrix representing the transformation (3.16).

Galois automorphisms of this type always have orbits with positive and negative signs. A positive invariant can only be obtained if the negative sign orbits are in fact fixed points of the Galois automorphism (these should not be confused with fixed points of the simple current!). One sees immediately from (3.15) that the sign $\eta(a)$ is opposite for fields of charge $Q(a) = 0$ and $Q(a) = 1/N$. Since the former includes the identity we fix that sign to be positive. Hence the orbits of charge $1/N$ must be fixed points. This leads to the condition

$$-lmN \left[\frac{1}{N} + \frac{1}{2} \right] = 0 \pmod{N} ,$$

or, writing $N = 2N'$, $lm(N' + 1) = 0 \pmod{2N'}$. From this we conclude that N' must be odd and lm must be a multiple of $N' = N/2$.

We are now in the familiar situation of an extension by a simple current of order 2, and clearly T -invariance will then require this current to have integral spin. The solutions can now easily be listed:

$$\begin{aligned} A_{4l+1}, & \quad \text{level } 4j \quad (l, j \in \mathbb{Z}) , \\ B_{2l+1}, & \quad \text{level } 2j \quad (l, j \in \mathbb{Z}) , \\ E_7, & \quad \text{level } 4j \quad (j \in \mathbb{Z}) . \end{aligned}$$

Now consider D_n for even n . Then ρ is not an element of the root lattice, but a vector weight if $n = 2 \pmod{4}$. Hence $Q_s(\rho) = Q_c(\rho) = \frac{1}{2}$ and $Q_v(\rho) = 0$. The transformation of S is now

$$\sigma S_{ab} = e^{2\pi i [Q_s(a)+Q_c(a)]} S_{J_s^{2[Q_s(a)+1/2]} J_c^{2[Q_c(a)+1/2]} a, b} ,$$

where we set $m = 1$, the only acceptable value. It is not hard to see that the resulting S -invariant cannot be a positive one, since there do exist wrong-sign Galois orbits that are not fixed points.

There are several simple current extensions that cannot be obtained from Galois symmetry, at least not in the way described here. Since we considered here only a single Galois scaling, only Galois automorphisms of order 2 can give us a positive modular invariant [24] (this is also true for the automorphism invariants discussed earlier in this Section, as one may verify explicitly). Hence there is a priori no chance to obtain extensions by more than one simple current. However, some simple currents of order 2 are missing as well, namely those generated by the current J^{2l} of A_{4l-1} , the current J of B_l , l even, and the currents J_l of D_l and J_s, J_c of D_{2l} , with levels chosen so that these currents have integer spin. Note that the existence of a modular invariant of order two implies the existence of a “Galois-like” automorphism. This may suggest the existence of some generalization of Galois symmetry that would also explain those invariants.

4. New infinite series

In this Section we will describe several infinite series of exceptional invariants that we obtained from Galois symmetry. They occur for algebras of type B and D at level 2 and certain values of the rank. Let us start the discussion with type B , which is slightly simpler.

The new invariants occur for the algebras $B_7, B_{10}, B_{16}, B_{17}, B_{19}, B_{22}$ etc., always at level 2. The pattern of the relevant ranks n becomes clear when we consider the number $2n + 1$, corresponding to the identity $B_n = \text{so}(2n + 1)$; namely, $2n + 1$ must have at least two distinct prime factors. For example, for $\text{so}(15)$ at level 2 we find the following three non-diagonal modular invariants:

$$\begin{aligned} \mathcal{P}_1 &= |\mathcal{X}_0 + \mathcal{X}_1|^2 + 2 (|\mathcal{X}_4|^2 + |\mathcal{X}_5|^2 + |\mathcal{X}_6|^2 + |\mathcal{X}_7|^2 + |\mathcal{X}_8|^2 + |\mathcal{X}_9|^2 + |\mathcal{X}_{10}|^2) , \\ \mathcal{P}_2 &= |\mathcal{X}_0|^2 + |\mathcal{X}_1|^2 + |\mathcal{X}_2|^2 + |\mathcal{X}_3|^2 + |\mathcal{X}_5|^2 + |\mathcal{X}_6|^2 + |\mathcal{X}_8|^2 + (\mathcal{X}_4\mathcal{X}_9^c + \mathcal{X}_7\mathcal{X}_{10}^c + \text{c.c.}) , \\ \mathcal{P}_3 &= |\mathcal{X}_0 + \mathcal{X}_1|^2 + |\mathcal{X}_4 + \mathcal{X}_9|^2 + |\mathcal{X}_7 + \mathcal{X}_{10}|^2 + 2 (|\mathcal{X}_5|^2 + |\mathcal{X}_6|^2 + |\mathcal{X}_8|^2) . \end{aligned}$$

Here the labels $i = 1, 2 \dots 10$ of \mathcal{X}_i denote the following representations:

$$\begin{array}{ll} 0 : & (0, 0, 0, 0, 0, 0, 0) & 6 : & (0, 0, 0, 0, 1, 0, 0) \\ 1 : & (2, 0, 0, 0, 0, 0, 0) & 7 : & (0, 0, 0, 1, 0, 0, 0) \\ 2 : & (0, 0, 0, 0, 0, 0, 1) & 8 : & (0, 0, 1, 0, 0, 0, 0) \\ 3 : & (1, 0, 0, 0, 0, 0, 1) & 9 : & (0, 1, 0, 0, 0, 0, 0) \\ 4 : & (0, 0, 0, 0, 0, 0, 2) & 10 : & (1, 0, 0, 0, 0, 0, 0) \\ 5 : & (0, 0, 0, 0, 0, 1, 0) \end{array}$$

The first of these invariants is not new: it corresponds to the conformal embedding $\text{so}(15) \subset \text{su}(15)$. The fields $i = 4 \dots 10$ are fixed points, each of which is resolved into two distinct complex conjugate fields in the extended algebra. In $\text{su}(15)$ the two fields originating from the $\text{so}(15)$ field i are the antisymmetric tensor representations $[4 + i]$

and $[11 - i]$. The invariant \mathcal{P}_1 is in fact an integer spin simple current invariant. The other two B_7 invariants are manifestly not simple current invariants.

The second B_7 invariant is new, as far as we know, and can be explained in the following way. The algebra A_{14} at level 1 has three distinct automorphism invariants which are generated by the simple currents J , J^3 and J^5 . They read

$$\sum_{i=0}^{14} \mathcal{X}_i \mathcal{X}_{-i}^*, \quad \sum_{i=0}^{14} \mathcal{X}_i \mathcal{X}_{-11i}^*, \quad \sum_{i=0}^{14} \mathcal{X}_i \mathcal{X}_{-4i}^*,$$

respectively, where the labels are defined modulo 15. The first one is equal to the charge conjugation invariant, and the last one is the “product” of the first two. The existence of an $A_{14,1}$ automorphism implies relations among the matrix elements of the modular matrix S of that algebra. Owing to the existence of the conformal embedding $B_{7,2} \subset A_{14,1}$, these matrix elements are related to those of $B_{7,2}$. The precise relation is

$$\begin{aligned} S_{00}[A_{14,1}] &= 2S_{00}[B_{7,2}] \text{ ,} \\ S_{0,4+i}[A_{14,1}] &= S_{0,11-i}[A_{14,1}] = S_{0,i}[B_{7,2}] \text{ ,} \\ S_{4+i,4+j}[A_{14,1}] &= S_{11-i,11-j}[A_{14,1}] \\ &= S_{4+i,11-j}^*[A_{14,1}] = S_{11-i,4+j}^*[A_{14,1}] = \frac{1}{2}S_{ij}[B_{7,2}] + i\Sigma_{ij} \text{ .} \end{aligned}$$

Here Σ denotes the fixed point resolution matrix. The first automorphism, charge conjugation, just sends i to $-i$ and hence acts trivially on the $B_{7,2}$ fields. The other two $su(15)$ automorphisms interchange the $B_{7,2}$ fields (4, 9) and (7, 10), leaving 5, 6 and 8 fixed (in addition one gets relations from the imaginary part on the matrix elements of Σ). This implies relations like $S_{0,4} = S_{0,9}$ and $S_{4,7} = S_{9,10}$ for the $B_{7,2}$ matrix elements. All these relations hold also if the label 0 is replaced by 1, but we do not get any relations for matrix elements involving the fields that are projected out, i.e. the fields 2 and 3. In the general case, the absence of relations involving fields that get projected out implies that the automorphisms of an algebra \mathfrak{g} do not lead to automorphisms for a conformal subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The present case is an exception, since all the fields on which the automorphism acts (and in fact all the fields with labels 4, . . . , 10) are fixed points of the $B_{7,2}$ simple current that extends the algebra. Then the matrix elements $S_{2,i}$ and $S_{3,i}$ vanish for $i = 4, \dots, 10$ and we need no further relations among them.

This explains the presence of the second invariant listed above. The third one is a linear combination of the foregoing ones and the diagonal invariant: $\mathcal{P}_3 = \mathcal{P}_1 + \mathcal{P}_2 - 1$. This is a remarkable invariant: it looks like a normal extension by a spin 1 current, but it does not follow from any conformal embedding. The only conformal embedding of B_7 at level 2 is in $su(15)$, and the corresponding invariant is \mathcal{P}_1 , not \mathcal{P}_3 . This implies in particular that there cannot exist any conformal field theory corresponding to the modular invariant \mathcal{P}_3 ! In fact, it is not even possible to write down a fusion algebra for this invariant, because there does not exist a fixed point resolution matrix. In Ref. [29] another example of this kind was described, although that theory was unphysical for a somewhat different reason.

The existence of \mathcal{P}_3 can also be seen as a consequence of the closure of the set of Galois automorphisms. Each Galois modular invariant, automorphism invariants as well as chiral algebra extensions, originates from a Galois symmetry of S , which acts on the fields as a permutation accompanied by sign flips. For the “chiral extension” \mathcal{P}_3 this Galois automorphism is represented by the matrix $\mathcal{P}_3 - 1$. This set of Galois automorphisms will always close as a group. Indeed, the automorphism underlying \mathcal{P}_3 is simply the product of that of \mathcal{P}_1 and \mathcal{P}_2 .

By the same arguments there will be pure automorphism invariants for $B_{n,2}$ whenever $2n+1$ contains at least two different prime factors. The spin-1 extension always involves an identity block plus n fixed points that yield each two $su(2n+1)$ level 1 fields (this is true since all non-trivial representations of $su(2n+1)$ are complex). If there is only one prime factor the only automorphism is charge conjugation, which acts trivially. When there are K different prime factors there are 2^K distinct pure Galois automorphisms for $su(2n+1)$ at level 1, including the identity and the charge conjugation invariant. When “projected down” to $B_{n,2}$ these are related in pairs by charge conjugation, and we expect therefore 2^{K-1} distinct $B_{n,2}$ modular invariants of automorphism type. In addition there is of course the invariant corresponding to the conformal embedding in $su(2n+1)$ itself. In combination with the $2^{K-1} - 1$ non-trivial automorphisms this extension gives rise to as many other invariants that look like conformal embeddings, but actually do not correspond to a consistent conformal field theory.

How does this come out in terms of Galois symmetry? First of all the spin-1 extension of the conformal embedding is in fact a simple current extension, and we have seen in the previous Section that it follows from Galois symmetry only for B_n with n odd. If n is odd the Galois periodicity is $4(2n+1)$ for $B_{n,2}$ and $2(n+1)(2n+1)$ for $A_{2n,1}$. Hence the cyclotomic field of the former is contained in that of the latter, so that all Galois transformations of $A_{2n,1}$ have a well-defined action on the modular matrix S of $B_{n,2}$. In this case we may thus expect 2^K distinct Galois modular invariants, including the identity and the unphysical invariants described above. If n is even the Galois periodicities are respectively $2(2n+1)$ and $2(n+1)(2n+1)$, so that also in this case all Galois transformations are well-defined on B_n . But due to the fact that the simple current invariant is not a Galois invariant, we get only half the number of invariants now, namely 2^{K-1} .

For n odd the $su(2n+1)$ simple current automorphisms are mapped to two B_n modular invariants: one physical automorphism and one chiral extension, which (except for the one originating from the diagonal invariant, i.e. the conformal embedding invariant) is unphysical. For n even each $su(2n+1)$ automorphism is mapped to just one B_n invariant. The diagonal invariant is mapped to the diagonal one of B_n , but it turns out that the non-trivial automorphisms are mapped to either a pure automorphism or an unphysical chiral extension, in such a way that the closure of the set of Galois automorphisms is respected.

Now consider algebras of type D . Again the crucial ingredient is the conformal embedding $so(2n)_2 \subset su(2n)_1$. In terms of D_n fields the $su(2n)$ characters are built as follows: The identity character is the combination $\mathcal{X}_0 + \mathcal{X}_v$ and the antisymmetric tensor $[n]$ has a character equal to $\mathcal{X}_s + \mathcal{X}_c$. All other $su(2n)$ representations are complex, and each pair of complex conjugate representations arises from a resolved fixed point of

the vector current of D_n . Even though D_n has complex representations itself for n odd, these get projected out, and all the non-real contributions to the $\mathfrak{su}(n)$ modular matrix S arise from fixed point resolution.

The center of the $\mathfrak{su}(2n)$ WZW theory is \mathbb{Z}_{2n} , but the “effective center” (in the terminology of Ref. [35]) is \mathbb{Z}_n . This means that only the simple current J^2 of the $\mathfrak{su}(2n)$ theory yields non-trivial modular invariants, and that the order $2n$ current J may be ignored. It is easy to see that the field $[n]$ has zero charge with respect to J^2 , so that it is mapped onto itself by any automorphism generated by powers of J^2 . This implies that, just as before, all $\mathfrak{su}(2n)$ simple current automorphisms act non-trivially only on resolved fixed points, and hence can be “projected down” to D_n . If n is prime, then the only automorphism is equivalent to charge conjugation, and hence it projects down to the trivial invariant. Hence just as before we will get non-trivial D_n automorphisms whenever n contains at least two distinct prime factors, where the prime is now allowed to be two. The counting of invariants is the same as for $B_{(n-1)/2}$ above. Again they come in pairs: an automorphism and an unphysical extension by a spin-1 current.

All these invariants exist, but not all of them follow from Galois theory. Just as for B_n , the automorphism invariants do, but the conformal embedding invariant does not always follow. In fact, it never comes out as a result of the scalings discussed in the previous Section. However, if $n = 3 \pmod{4}$ the simple current extension by the current J_v is an exceptional Galois invariant only at level 2 (see Table 1). In that case all the expected invariants are Galois invariants. For all other values of n only half of the expected invariants are Galois invariants, and from each pair only one member appears, either the automorphism or the unphysical extension.

There is still one interesting observation to be made here. If there are just two distinct prime factors, and $n = 6 \pmod{8}$, then the extra invariant is an unphysical extension. Remarkably, however, that extension is a simple current invariant. It is equal to the extension by J_v , but it has additional terms of the form $|\mathcal{X}_a + \mathcal{X}_b|^2$, where a and b are fields that appear diagonally, as fixed points of order 2, in the normal simple current invariant. The fields a and b are however on the same orbit with respect to the current J_s , which makes this a simple current invariant by definition. Nevertheless, it is not part of the classification presented in Ref. [35], because that classification was obtained under a specific regularity condition on the matrix S that is not satisfied here (indeed, D_{2n} at level 2 was explicitly mentioned as an exception in the appendix of Ref. [35]; the reason for it being an exception is that all orbits except for the identity field are fixed points of one or all currents). It also follows that this simple current invariant cannot be obtained using orbifolds with discrete torsion, unlike the simple current invariants within the classification [33]. Hence the fact that it is unphysical is not in contradiction with the expectation that simple current invariants should normally be physical.

In the previous case the automorphism would be obtained by subtracting the normal spin-1 extension, and adding the identity matrix. Clearly the resulting automorphism is not really exceptional, but is simply the automorphism generated by the spinor simple current J_s (or J_c , which at level 2 gives the same result). The same happens if the rank is $2 \pmod{8}$, except that in that case the automorphism comes out directly as a Galois invariant. It is listed in Table 2. To get really new automorphisms that are not simple current invariants for $n = 2 \pmod{8}$ or $n = 6 \pmod{8}$ one has to consider cases

where n contains three or more distinct prime factors. Finally, if the rank is divisible by 4 the spinor currents have integer spin, and do not interfere with the exceptional automorphisms discussed in this section.

5. Pure Galois invariants

In Table 1 we list all the remaining Galois invariants of simple WZW models, i.e. not including those described in the previous Sections. All these invariants are positive and result directly from a single Galois automorphism of order 2. Although the full Galois commutant was investigated, in all but one case there is only a single non-trivial orbit contributing (in terms of the formula (2.6) this means that f_0 is used to get $P_{00} = 1$, and apart from f_0 only one other coefficient f_σ is non-zero.) The exception is the E_8 -type invariant of A_1 at level 28, which can also be interpreted as a combined simple current/Galois invariant, and which is therefore included in Table 2 in Section 6. The results are listed in Table 1. The notation is as follows:

CE: Conformal embedding.

$S(J)$: Simple current invariant. The argument of S is the simple current responsible for the invariant.

RLD: Rank-Level Dual. The S -matrices of $\mathfrak{su}(N)_k$, $\mathfrak{so}(N)_k$ and $C_{n,k}$ are related to those of respectively $\mathfrak{su}(k)_N$, $\mathfrak{so}(k)_N$ and $C_{k,n}$ by rank-level duality. One might expect that Galois transformations of one matrix are mapped to similar transformations of the other. The relation is not quite that straightforward however, and we will not examine the details here. The results clearly respect this duality.

EA: Exceptional Automorphism. These are modular invariants of pure automorphism type that are not due to simple currents. The only invariants of this type known so far were found in Ref. [18], and appear also in Table 1.

HSE: Higher Spin Extension, an extension of the chiral algebra by currents of spin larger than 1 that are not simple currents. Some of these invariants can be predicted using rank-level duality; all other known ones are related to meromorphic $c = 24$ theories [23].

Note that there are some simple current invariants in this list. This is not in conflict with the results of Section 3, as we did not claim that the list given there was complete. The scales of the Galois transformations for which these simple current invariants are obtained are interesting. For A_{4m-1} and D_{4m+3} these scales are equal respectively to $(2m+1)(k+g) - 1$ and $3(k+g) - 1$. If the contribution -1 were replaced by $+1$, they would be of the kind discussed in Section 3. In fact we can write these scales as $(-1)[(2m-1)(k+g) + 1] \bmod 4m(k+g)$ and $(-1)[(k+g) + 1] \bmod 4(k+g)$, respectively, which shows that these Galois automorphisms are nothing but the product of a scaling of the type discussed in Section 3 and charge conjugation. It can be checked that without the charge conjugation one does not get a positive invariant: certain fields are transformed to their charge conjugate with a sign flip. After multiplying with the charge conjugation automorphism these fields become fixed points. The scale factor for C_{4m} , $4m+3$, is of the form $(k+g) + 1$, but for C_n the arguments of Section 3 break

Table 1

Algebra	Level	Galois scaling	Type	Interpretation
A_2	5	19	Extension	CE $\subset A_5$
A_{4m-1}	2	$8m^2 + 8m + 1$	Extension	$S(J^{2m})$; RLD of $A_{1,4m}$
A_4	3	11	Extension	CE $\subset A_9$
A_9	2	31	Extension	RLD of $A_{1,10}$
C_{4m}	1	$4m + 3$	Extension	$S(J)$; RLD of $C_{1,4m} = A_{1,4m}$
D_{8m+2}	2	$8m + 1$	Automorphism	$S(J_s)$
D_{4m+3}	2	$24m + 17$	Extension	$S(J_u)$
D_7	3	49	Extension	HSE; RLD of $so(3)_{14} = A_{1,28}$
G_2	3	8	Extension	CE $\subset E_6$
G_2	4	5	Automorphism	EA
G_2	4	11	Extension	CE $\subset D_7$
F_4	3	5	Extension	CE $\subset D_{13}$
F_4	3	11	Automorphism	EA
E_6	4	7	Extension	HSE
E_7	3	13	Extension	HSE

down right from the start, so that no conclusions can be drawn for this case. For the other simple current invariants the scale factor does not have the right form, and hence the arguments of Section 3 simply do not apply.

6. Combination of Galois and simple current symmetries

In Section 3 we have discussed a large set of invariants for which the Galois and simple current methods overlap. If they do not overlap, it may be fruitful to combine them. To do so we first have to understand how the orbit structures of both symmetries are interfering with each other. This can be seen by computing $\sigma S_{Ja,b}$. On the one hand, this is equal to

$$\sigma S_{Ja,b} = \epsilon_\sigma(Ja) S_{\sigma Ja,b} .$$

On the other hand, it is equal to

$$\begin{aligned} \sigma [e^{2\pi i Q(b)} S_{ab}] &= e^{2\pi i l Q(b)} \epsilon_\sigma(a) S_{\sigma a,b} \\ &= \epsilon_\sigma(a) S_{J^l \sigma a,b} . \end{aligned} \tag{6.1}$$

Here l is the power to which σ raises the generator of the cyclotomic field. In the first step we used that the simple current phase factor is contained in the field M , which follows from $e^{2\pi i Q(b)} = S_{Ja,b}/S_{ab} \in M$. Using unitarity of S we then find that

$$\begin{aligned} \epsilon_\sigma(Ja) &= \epsilon_\sigma(a) , \\ \sigma J &= J^l \sigma . \end{aligned} \tag{6.2}$$

Here J denotes the permutation of the fields that is generated by the simple current J . Since l is prime with respect to the order of the cyclotomic field, it is—at least in the

case of WZW models—also prime with respect to the order N of the simple current. If $N = 2$ this means that l must be odd so that $J^l = J$, and hence we conclude that σ and J commute. For all other values of N they do not commute unless $l = 1 \pmod N$, but at least it is true that σ maps simple current orbits to simple current orbits, and furthermore it respects the orbit length.

If $N = 2$ the simple currents yield the relation

$$S_{Ja,Jb} = e^{2\pi i(Q(a)+Q(b)+r/2)} S_{ab}$$

among matrix elements of S , where r is the monodromy parameter. If r is even (which is the case for simple currents of integer or half-integer spin) this relation takes the form

$$S_{ab} = \epsilon(a)\epsilon(b)S_{Ja,Jb} ,$$

since the phase factors are in fact signs. This is precisely the form of a Galois symmetry, as expressed in (2.4). We can represent this symmetry in matrix notation as

$$\Pi_J S \Pi_J = S ,$$

where $\Pi_J = (\Pi_J)^{-1}$ is an orthogonal matrix that commutes with the analogous matrices representing the Galois group. Hence we can extend the Galois group by this transformation as explained in Section 2. Furthermore if $r = 2 \pmod 4$ the simple current invariant produced by J is a fusion rule automorphism that can also be used to extend the Galois group.

We have not examined these extended Galois-like symmetries systematically, but we will illustrate that new invariants can be found by giving one example. Consider A_1 at level 10. One of the Galois invariants (invariant under S as well as T) is

$$\mathcal{P}_1 = |\mathcal{X}_0 + \mathcal{X}_6|^2 + |\mathcal{X}_4 + \mathcal{X}_{10}|^2 + |\mathcal{X}_1 - \mathcal{X}_9|^2 + 2|\mathcal{X}_3|^2 + 2|\mathcal{X}_7|^2 ,$$

where the indices are the highest weights (in the Dynkin basis). The only problem with this invariant is that it is not positive. However, at level 10 we also have the D -type invariant

$$\begin{aligned} \mathcal{P}_2 = & |\mathcal{X}_0|^2 + (\mathcal{X}_1\mathcal{X}_9^* + \mathcal{X}_3\mathcal{X}_7^* + \text{c.c.}) + |\mathcal{X}_2|^2 + |\mathcal{X}_4|^2 + |\mathcal{X}_5|^2 \\ & + |\mathcal{X}_6|^2 + |\mathcal{X}_8|^2 + |\mathcal{X}_{10}|^2 , \end{aligned}$$

which is a simple current automorphism. If we now take the linear combination

$$\mathcal{P}_1 + \mathcal{P}_2 - \mathbf{1} ,$$

we get a positive modular invariant which is in fact the well-known E_6 -type invariant.

There is a second way of combining simple currents and Galois symmetries. One can extend the chiral algebra of the WZW model by integer spin simple currents. This projects out some of the fields, so that the negative sign Galois orbits of some Galois invariants are removed. It is essential that the Galois automorphisms respect the simple current orbits, and that the matrix elements of S are constant on these orbits for the

fields that are not projected out. The simple current extension has its own S -matrix which can be derived partly from that of the original theory. If N is prime this matrix has the form [29,36]

$$\tilde{S}_{a_i,b_j} = \frac{N_a N_b}{N} S_{ab} E_{ij} + \Sigma_{ab} F_{ij} , \tag{6.3}$$

where \tilde{S} is the new modular matrix and S the original one, N_a is the orbit length of the field a (it is a divisor of the simple current order N , and hence either 1 or N), and i labels the resolved fixed points for those orbits with $N_a < N$ (i.e. $i = 1, \dots, N/N_a$), and analogously for b and j . The matrix E_{ij} is equal to 1 independent of i and j , and $F_{ij} = \delta_{ij} - (1/N)E_{ij}$. Finally, the matrix Σ_{ab} is non-vanishing only for fixed point fields and cannot be expressed in terms of S , or at least not in any known way, but it is subject to severe constraints from the requirement of modular invariance.

All general considerations regarding Galois transformations can be applied directly to this new S -matrix. Clearly the matrix elements S_{ab} which correspond to the primary fields of the original theory that are not projected out belong to a number field M' which is contained in the number field M of the original theory. While E_{ij} , F_{ij} and $N_a N_b / N$ are all rational and hence transform trivially under $\text{Gal}(M'/\mathbb{Q})$, the presence of the matrix Σ_{ab} in (6.3) may require this number field to be extended to a field $\tilde{M}' \supset M'$ (a simple example is provided by the $A_{1,4}$ WZW theory, which has a real matrix S , whereas the S -matrix of the extended algebra $A_{2,1}$ is complex). Now because of the projections \tilde{M}' does not necessarily contain the original number field M ; however, at the possible price of redundancies we can consider an even larger number field \tilde{M} that contains both \tilde{M}' and M . When working with \tilde{M} , we do not lose any of the Galois transformations that act non-trivially on the surviving matrix elements S_{ab} . Note that any element of $\text{Gal}(\tilde{M}/M)$ acts trivially on S_{ab} and hence induces a permutation which leaves non-fixed points invariant and acts completely within the set of primary fields into which a fixed point gets resolved. Further, for any element of $\text{Gal}(\tilde{M}/\mathbb{Q})$ the associated permutation must act on the labels a, b in the same way in both terms on the right-hand side of (6.3). In particular, for any matrix element involving only non-fixed points the action of a Galois transformation on S already determines its action on \tilde{S} , since the two matrix elements are equal up to a rational factor. The same is true for all matrix elements between fixed points and full orbits, since in that case Σ is absent, too. This is often already enough information to determine the Galois orbits of the extended theory completely. The transformations of the fixed point–fixed point elements of \tilde{S} are more subtle, and in principle would require knowledge of the matrix Σ . However, as already pointed out any element of the Galois group must act on Σ exactly as it does on S . Although this still leaves undetermined the action within the set of primary fields into which the relevant fixed point is resolved, this limited information nevertheless can provide useful additional information on the matrix Σ , whose determination in general is a problem that is far from being solved.

Fortunately, as long as we are only interested in modular invariants of the original theory, we may in fact ignore fixed point resolution completely. By definition that issue is determined solely by S (and T), and the precise form of Σ should not matter.

Table 2

Algebra	level	Galois scaling	Simple current	Type	Interpretation
A_1	10	7	J^a	Extension	$CE \subset B_2$
A_1	28	11	J	Extension	$CE \subset G_2$
A_2	9	17	J	Extension	$CE \subset E_6$
A_2	21	35×53^b	J	Extension	$CE \subset E_7$
A_3	8	7	J	Extension	$CE \subset D_{10}$
A_7	4	7	J^2	Extension	HSE; RLD of $A_{3,8}$
A_7	$4m+2$	$4m+11$	J^4	Aut \times Ext	$S(J^2)$
A_{27}	2	71	J^{14}	Extension	HSE; RLD of $A_{1,28}$
C_3	4	7	J	Extension	$CE \subset B_{10}$
C_4	3	7	J	Extension	RLD of $C_{3,4}$
D_{4m+2}	$4l$	$8m+4l+3$	J_s	Extension	$S(J_v) \times S(J_s)$
D_4	6	5	J_s, J_v	Extension	$CE \subset D_{14}$

^a This is a simple current of half-integer spin; see the main text for details.

^b Invariant originating from a non-cyclic subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ of the Galois group.

We have performed a computer search for invariants of the type described above, and obtained the results as shown in Table 2.

Note that this table contains a few infinite series of simple current invariants. Since they were inferred from a finite computer scan, the statement that the series continues is a conjecture. Presumably these series can also be derived by arguments similar to those in Section 3, but we have not pursued this.

We have in principle just looked for invariants originating from single orbits, but there is one exception, namely the modular invariant of A_2 at level 21. This invariant is obtained as a sum over a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the Galois group that is generated by the two scalings indicated in Table 2. Separately each of these scalings yields an S, T invariant with a few minus signs.

7. Conclusions

To conclude, let us make a rough comparison between the various methods for constructing modular invariants that were mentioned in the introduction. We will compare them on the basis of the following aspects.

Generality. A common property of simple currents and Galois symmetry is that neither is a priori restricted to WZW models, unlike all other methods. (In practice this is less important than it may seem, since essentially all RCFT's we know are WZW models or WZW-related coset theories.)

Positivity. Most methods do not directly imply the existence of positive modular invariants, but rather they yield generating elements of the commutant of S and T that have to be linearly combined to get a positive invariant; the exceptions are simple currents, conformal embeddings and rank-level duality.

Existence of a CFT. It should be emphasized that a positive modular invariant partition function is only a necessary condition for a consistent conformal field theory. Most methods do not guarantee that a conformal field theory exists. Exceptions are conformal embeddings (the new CFT is itself a WZW model) and probably simple current invariants, since the construction of the new theory can be rephrased in orbifold language. Clearly any construction that may yield negative invariants cannot guarantee existence of the theory, and this includes Galois invariants. Indeed, we found examples of positive Galois modular invariants that cannot correspond to any sensible CFT.

Explicit construction. Simple current invariants can be constructed easily and straightforwardly. On the other hand, the explicit construction of an invariant corresponding to a conformal embedding is usually extremely tedious. Indeed, many of these invariants are not known explicitly. The other methods fall somewhere between these two extremes. The explicit construction of a Galois invariant is straightforward but requires long excursions through the Weyl group, as explained in Appendix A.

Classification. All simple current invariants have been classified in Refs. [37,35,33], under a mild regularity assumption for \mathcal{S} , which, as we have seen in Section 4, is not always satisfied. The simple currents of WZW models were classified in Ref. [38]. All conformal embeddings have been classified in Refs. [39,40]. All cases of rank-level duality are presumably known, but all other methods mentioned in the introduction have only been applied to a limited number of cases, without claims of completeness. Our results on Galois invariants are based partly on computer searches (inevitably restricted to low levels) and partly on rigorous derivations (Section 3). For the pure Galois invariants we expect our results to be complete, but we have no proof.

To summarize, we find that the Galois construction does not yield all solutions, but also that it is not contained in any of the previously known methods. It generates invariants of all known types. Most of the partition functions we found were already known in the literature, but we did find several new infinite series of pure automorphism invariants not due to simple currents.

In the course of this investigation we realized that the restriction that the scaling be prime with respect to $M(k+g)$ can in fact be dropped, at least for WZW models. This yields even more relations among elements of \mathcal{S} , which take the form of sum rules, and hence even more information about modular invariants. These transformations, which we call “Quasi-Galois” symmetries, will be discussed in a forthcoming paper.

Appendix A

Here we describe in detail how Galois scalings are implemented when the conformal field theory in question is a WZW theory based on an untwisted affine Lie algebra \mathfrak{g} at integral level k . Then the Galois group is a subgroup of $\mathbb{Z}_{M(k+g)}^*$, where g is the dual Coxeter number of the horizontal subalgebra $\bar{\mathfrak{g}}$ of \mathfrak{g} (i.e. the subalgebra generated by the zero modes of \mathfrak{g}) and M is the denominator of the metric on the weight space of $\bar{\mathfrak{g}}$.

We label the primary fields by the shifted highest weight a with respect to the

horizontal subalgebra \bar{g} , which differs from the ordinary highest weight by addition of the Weyl vector ρ of \bar{g} . Thus a is an integrable highest weight of g at level $k + g$, i.e. the components a^i of a in the Dynkin basis satisfy

$$a^i \in \mathbb{Z}_{\geq 0} \quad \text{for } i = 0, 1, \dots, \text{rank}(\bar{g}),$$

where $a^0 \equiv k + g - \sum_{i=1}^{\text{rank}(\bar{g})} \theta_i a^i$ with θ_i the dual Coxeter labels of g . However, because of the shift not all such integrable weights belong to primary fields, but only the strictly dominant integral weights, i.e. the primary fields of the WZW theory correspond precisely to those weights a which obey

$$a^i \in \mathbb{Z}_{> 0} \quad \text{for } i = 0, 1, \dots, \text{rank}(\bar{g}). \tag{A.1}$$

A Galois transformation labelled by $\ell \in \mathbb{Z}_{M(k+g)}^*$ acts as the permutation [26]

$$\hat{\sigma}_{(\ell)}(a) = \hat{w}(\ell a). \tag{A.2}$$

If we label the fields by the weights $a - \rho$ which are at level k , this is rewritten as in (2.3). That it is the shifted weight a rather than $a - \rho$ that is scaled is immediately clear from the formula (3.1) for the modular matrix S . In fact, it is possible to derive the formula (2.4) directly by scaling the row and column labels of S by ℓ and ℓ^{-1} , respectively, using (A.2). Galois symmetry is thus not required to derive this formula, nor is it required to show that (2.6) commutes with S . Galois symmetry has however a general validity and is not restricted to WZW models.

Substituting (A.2) into the formula for WZW conformal weights one easily obtains a condition for T -invariance, namely $(\ell^2 - 1) = 0 \pmod{2M(k+g)}$ (or $\pmod{M(k+g)}$ if all integers $M a \cdot a$ are even). Since ℓ has an inverse $\pmod{M(k+g)}$, it follows that $\ell = \ell^{-1} \pmod{M(k+g)}$, i.e. the order of the transformation must be 2, as is also true [24] for arbitrary conformal field theories.

Let us explain the prescription (A.2) in more detail. First one performs a dilatation of the shifted weight $a = (a^1, a^2, \dots)$ by the factor $\ell \in \mathbb{Z}_{M(k+g)}^*$. Now the weight ℓa does not necessarily satisfy (A.1), i.e. does not necessarily correspond to a primary field. If it does not, then the dilatation has to be supplemented by the horizontal projection $\hat{w} \equiv \hat{w}_{(\ell, a)}$ of a suitable affine Weyl transformation. More precisely, to any arbitrary integral weight b one can associate an affine Weyl transformation \hat{w} such that either $\hat{w}(b)$ satisfies (A.1), and in this case \hat{w} is in fact unique, or else such that $\hat{w}(b)$ obeys $(\hat{w}(b))^i = 0$ for some $i \in \{0, 1, \dots, \text{rank}(\bar{g})\}$ (in the latter case $\hat{w}(b)$ lies on the boundary of the horizontal projection of the fundamental Weyl chamber of g at level $k + g$). To construct the relevant Weyl group element \hat{w} for a given weight b as a product of fundamental Weyl reflections $w_{(l)}$ (i.e. reflections with respect to the l th simple root of g), one may use the following algorithm. Denote by $j_1 \in \{0, 1, \dots, \text{rank}(\bar{g})\}$ the smallest integer such that $b^{j_1} < 0$, and consider instead of b the Weyl-transformed weight $\hat{w}_1(b)$ with $\hat{w}_1 := \hat{w}_{(j_1)}$; next denote by j_2 the smallest integer such that $(\hat{w}_1(b))^{j_2} < 0$, and consider instead of $\hat{w}_1(b)$ the weight $\hat{w}_2 \hat{w}_1(b)$ with $\hat{w}_2 := \hat{w}_{(j_2)}$, and so on, until one ends up with a weight $\hat{w}_n \dots \hat{w}_2 \hat{w}_1(b)$ obeying (A.1), and then $\hat{w} = \hat{w}_n \dots \hat{w}_2 \hat{w}_1$ is the unique Weyl group element which does the job. (The presentation of an element $\hat{w} \in \hat{W}$ as a product of fundamental reflections is however not unique; the present algorithm

provides one specific presentation of this type, which is not necessarily reduced in the sense that the number of fundamental reflections is minimal.)

It is worth noting that there is no guarantee that starting from an integral weight b one gets this way a weight satisfying (A.1), but in the case where b is of the form $b = \ell a$ with a integrable and ℓ coprime with $r(k + g)$, the algorithm does work. Here r denotes the maximal absolute value of the off-diagonal matrix elements of the Cartan matrix of \bar{g} , i.e. $r = 1$ if \bar{g} is simply laced, $r = 2$ for the algebras of type B and C and for F_4 , and $r = 3$ for $\bar{g} = G_2$. (The property that ℓ is coprime with $r(k + g)$ in particular holds whenever (A.2) corresponds to an element of the Galois group, and hence for Galois transformations the algorithm works simultaneously for all primary fields of the theory.) Namely, assume that for some choice of a there is no choice of $\hat{w} \in \hat{W}$ such that $\hat{w}(\ell a)$ obeys (A.1). This means that any $\hat{w}(\ell a)$ lies on the boundary of some affine Weyl chamber, and hence the same is already true for the weight ℓa . Then there must exist some non-trivial $\hat{v} \in \hat{W}$ which leaves ℓa fixed, $\hat{v}(\ell a) = \ell a$. Decomposing \hat{v} into its finite Weyl group part $v \in W$ and its translation part $(k + g)t$ (with t an element of the coroot lattice of \bar{g}), this means that we have $\ell v(a) + (k + g)t = \ell a$, or in other words,

$$\ell(a - v(a)) = (k + g)t. \tag{A.3}$$

Now assume that ℓ is coprime with $r(k + g)$. This implies that there exists integers m, n such that $m\ell = nr(k + g) + 1$. Multiplying (A.3) with m then yields

$$a = v(a) + (k + g)[mt - nr(a - v(a))]. \tag{A.4}$$

Since for any integral weight a the weight $r(a - v(a))$ is an element of the coroot lattice, the same is also true for the expression in square brackets, and hence (A.4) states that the weight a stays fixed under some affine Weyl transformation. But a satisfies (A.1), and hence the fact that \hat{W} acts freely on such weights implies that this Weyl transformation must be the identity. This implies that \hat{v} must be the identity as well. Thus for ℓ coprime with $r(k + g)$ the assumption that $\hat{w}(\ell a)$ is not integrable leads to a contradiction.

In the general case where b is not of the form ℓa with a subject to (A.1) and ℓ coprime with $r(k + g)$, the algorithm described above still works unless at one of the intermediate steps one of the Dynkin labels becomes zero, which means that the weight lies on the boundary of the fundamental affine Weyl chamber. In the latter case any Weyl image of this weight lies on the boundary of some affine Weyl chamber as well, and hence we can never end up with a weight that satisfies (A.1), i.e. in the interior of the fundamental affine Weyl chamber. It may also be remarked that one can speed up the algorithm considerably using not the weight b itself as a starting point, but rather the weight $\tilde{b} = b + (k + g)t$ that is obtained from b by such a Weyl translation $(k + g)t$ for which the length of \tilde{b} becomes minimal.

Finally, there is a general formula for the sign $\epsilon_{\sigma_{(\ell)}}$, namely

$$\epsilon_{\sigma_{(\ell)}}(a) = \eta_\ell \text{sign}(w_{(\ell,a)}),$$

i.e. the sign is just given by that of the Weyl transformation \hat{w} , up to an overall sign η_ℓ that only depends on $\sigma_{(\ell)}$ [26], but not on the individual highest weight a . (Actually the

cyclotomic field $\mathbb{Q}(\zeta_{M(k+g)})$ whose Galois group is $\mathbb{Z}_{M(k+g)}^*$ does not yet always contain the overall normalization \mathcal{N} that appears in the formula (3.1) for S , but rather sometimes a slightly larger cyclotomic field must be used [26]. However, the permutation σ of the primary fields that is induced by a Galois scaling can already be read off the generalized quantum dimensions, which do not depend on the normalization of S . The correct Galois treatment of the normalization of S just amounts to the overall sign factor η_ℓ , which is irrelevant for our purposes.)

References

- [1] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575.
- [2] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B 280 (1987) 445; Commun. Math. Phys. 113 (1987) 1.
- [3] T. Gannon, Commun. Math. Phys. 161 (1994) 233.
- [4] C. Itzykson, Nucl. Phys. (Proc. Suppl.) 5B (1988) 150.
- [5] T. Gannon, Nucl. Phys. B 396 (1993) 708.
- [6] T. Gannon, The Classification of $\widehat{SU}(m)_k$ automorphism invariants, hep-th/9408119.
- [7] A.N. Schellekens and S. Yankielowicz, Nucl. Phys. B 327 (1989) 673.
- [8] D. Bernard, Nucl. Phys. B 288 (1987) 628.
- [9] D. Altschuler, J. Lacki and P. Zaugg, Phys. Lett. B 205 (1988) 281.
- [10] C. Ahn and M. Walton, Phys. Lett. B 223 (1989) 343.
- [11] K. Intriligator, Nucl. Phys. B 332 (1990) 541.
- [12] G. Felder, K. Gawedzki and A. Kupiainen, Commun. Math. Phys. 117 (1988) 127.
- [13] P. Bouwknegt and W. Nahm, Phys. Lett. B 184 (1987) 359.
- [14] V.G. Kac and M. Wakimoto, Adv. Math. 70 (1988) 156.
- [15] M. Walton, Nucl. Phys. B 322 (1989) 775.
- [16] D. Altschuler, M. Bauer and C. Itzykson, Commun. Math. Phys. 132 (1990) 349.
- [17] S. Naculich and H. Schnitzer, Nucl. Phys. B 347 (1990) 687; Phys. Lett. B 244 (1990) 235.
- [18] D. Verstegen, Nucl. Phys. B 346 (1990) 349; Commun. Math. Phys. 137 (1991) 567.
- [19] J. Fuchs and C. Schweigert, Ann. Phys. 234 (1994) 102.
- [20] N. Warner, Commun. Math. Phys. 190 (1990) 205.
- [21] P. Roberts and H. Terao, Int. J. Mod. Phys. A7 (1992) 2207.
- [22] M. Abolhassani and F. Ardalan, Int. J. Mod. Phys. A9 (1994) 2707.
- [23] A.N. Schellekens, Commun. Math. Phys. 153 (1993) 159.
- [24] J. Fuchs, B. Gato-Rivera, A.N. Schellekens and C. Schweigert, Phys. Lett. B 334 (1994) 113.
- [25] J. de Boer and J. Goeree, Commun. Math. Phys. 139 (1991) 267.
- [26] A. Coste and T. Gannon, Phys. Lett. B 323 (1994) 316.
- [27] G. Moore and N. Seiberg, Nucl. Phys. B 313 (1988) 16.
- [28] R. Dijkgraaf and E. Verlinde, Nucl. Phys. (Proc. Suppl.) 5B (1988) 87.
- [29] A.N. Schellekens and S. Yankielowicz, Nucl. Phys. B 334 (1990) 67.
- [30] V. Pasquier and H. Saleur, Nucl. Phys. B 330 (1990) 523.
- [31] V.G. Kac and D.H. Peterson, Adv. Math. 53 (1984) 125.
- [32] A.N. Schellekens and S. Yankielowicz, Phys. Lett. B 227 (1989) 387.
- [33] M. Kreuzer and A.N. Schellekens, Nucl. Phys. B 411 (1994) 97.
- [34] A.N. Schellekens, Phys. Lett. B 244 (1990) 255.
- [35] B. Gato-Rivera and A.N. Schellekens, Commun. Math. Phys. 145 (1992) 85.
- [36] A.N. Schellekens and S. Yankielowicz, Int. J. Mod. Phys. A5 (1990) 2903.
- [37] B. Gato-Rivera and A.N. Schellekens, Nucl. Phys. B 353 (1991) 519.
- [38] J. Fuchs, Commun. Math. Phys. 136 (1991) 345.
- [39] F. Bais and P. Bouwknegt, Nucl. Phys. B 279 (1987) 561.
- [40] A.N. Schellekens and N.P. Warner, Phys. Rev. D 34 (1986) 3092.