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DOI
10.1016/0960-0779(93)E0019-8

Publication date
1995

Published in
Chaos, Solitons and Fractals

Pergamon

Chaos, Solitons & Fractals Vol. 5, No. 2, pp. 213–228, 1995

0960–0779/95$9.50 + .00

0960–0779(93)E0019–8

Externally Induced Changes in Chaotic Behaviour: a Phase Transition

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Abstract—We present a particular, control-variable-tuned chaotic map which shows discontinuities in macroscopic observables. This phenomenon can be mapped on an exactly solvable phase transition in a one-dimensional Ising model.

1. INTRODUCTION

In this contribution we report on an exactly solvable phase transition in one dimension arising in a chaotic iterated map. Details about this phase transition have been published by use elsewhere [1, 2, 3, 5], but here we want to highlight the main results with emphasis on the physical aspects rather than on the mathematical subtleties. The main objective is to make clear that we discovered a genuine phase transition in one dimension within the framework of the statistical mechanics of many-body systems. Here we want to stress that this phase transition is of a different nature from those usually discussed in the dynamical systems literature [4, 6–9] where discontinuities in the spectrum of generalized dimensions and generalized entropies [10] are studied.

The model presented by us depends on an external control variable which tunes the shape of the map. For a number of critical values of this control parameter, discontinuities in what can be regarded as macroscopic characteristics appear. We focus attention on one of those critical values and show that the discontinuity is triggered by the occurrence of long laminar intervals which disappear as the control variable is tuned past the critical value. We show that the height of the discontinuity can be exactly calculated. An exact calculation of the correlation function on both sides of the critical value can be performed, if one moves to a special, tailor-cut reduced symbolic description of the dynamical process. This choice for the reduced description is crucial, and illustrates at the same time that the usual type of symbolic description is not always well suited to describe a particular phenomenon. We show that there is long-range order on the critical value from below, which is absent on the critical value from above. Moreover, we can indicate quantitatively that there is critical slowing down in the correlation function, if the critical value is approached from below.

The model can be mapped on an Ising chain. The Hamiltonian of this Ising chain is not explicitly known, but the probability of each chain configuration is. This makes it possible to develop the equivalence between the correlation function in terms of the reduced symbolic description and the spin-pair correlation function.

2. THE BUNGALOW-TENT MAP

An iterated map on the interval

$$x_{n+1} = f(x_n), \qquad x \in I \tag{1}$$

is called *fully developed chaotic* (FDC) [11] if the map is chaotic (i.e. shows sensitive dependence on the starting point x_0) and ergodic (i.e. the chaotic trajectory $x_0, x_1, x_2, \ldots$ densely fills the entire interval I). For these iterated maps the probability density $\rho(x)$ is almost nowhere zero on I. A well-known example of such a map is the iterated (classical) tent map

$$x_{n+1} = 1 - 2|x_n| \tag{2}$$

on the interval $I = [-1, 1]$, see Fig. 1. This iterated map has a probability density

$$\rho(x) = \tfrac{1}{2}, \tag{3}$$

i.e. it is ergodic, and a Lyapunov exponent

$$\lambda = \int_{-1}^{1} \rho(x) \log|f'(x)| \, \mathrm{d}x = \log 2 > 0, \tag{4}$$

i.e. the system is chaotic.

Of special interest are FDC maps for which the degree of chaos is tuned by an external control parameter. An example is the z-map [11]

$$x_{n+1} = f_z(x_n) \tag{5}$$

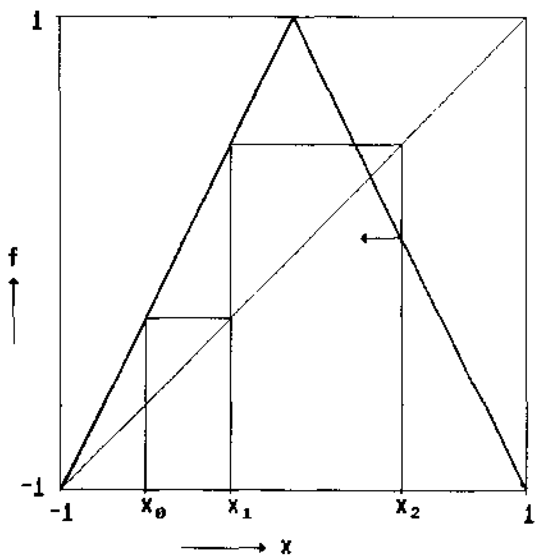

Fig. 1. The classical tent map.

with

$$f_z(x) = 1 - 2|x|^z \tag{6}$$

on the interval $I = [-1, 1]$, see Fig. 2. This iterated map is FDC for $z \geqslant \frac{1}{2}$. It was shown [11] that the Lyapunov exponent is a continuous function of z. Other examples of parameter-controlled FDC maps, e.g. the asymmetric tent map [3], show the same behaviour: a continuous Lyapunov spectrum reflecting the fact that the chaoticity of the map changes smoothly if the control parameter is varied. However, this is not true for any parameter-controlled FDC map. An exciting example of this is provided by the so-called bungalow-tent map [1]. This map is defined on the interval $[-1, 1]$ as follows:

$$f_a(x) = \begin{cases} g_1(x) = 1 + 2a + 2(a+1)x & \text{for } -1 \leqslant x \leqslant -\frac{1}{2} \\ g_2(x) = 1 + 2(1-a)x & \text{for } -\frac{1}{2} \leqslant x \leqslant 0 \\ g_3(x) = 1 + 2(a-1)x & \text{for } \ 0 \leqslant x \leqslant \frac{1}{2} \\ g_4(x) = 1 + 2a - 2(a+1)x & \text{for } \ \frac{1}{2} \leqslant x \leqslant 1. \end{cases} \tag{7}$$

In Fig. 3 this map is visualized. One sees that it consists of four straight lines, which are connected in such a way that the impression of a symmetric bungalow-tent is made. The tent has three kinks, one at the centre, one at the location $x = \frac{1}{2}$ and one at the location $x = -\frac{1}{2}$. The kink at the centre has a fixed height. The two outer kinks have the same height, but this height is adjustable. This adjustable height is equal to a, the control parameter. The iterated system

$$x_{n+1} = f_a(x_n) \tag{8}$$

is FDC for $-\frac{1}{2} < a < 1$, while for $-1 < a < -\frac{1}{2}$ the fixed point $\hat{x} = -1$ is stable. (For $a = -\frac{1}{2}$ the trajectory falls on a point of the subinterval $[-1, -\frac{1}{2}]$.)

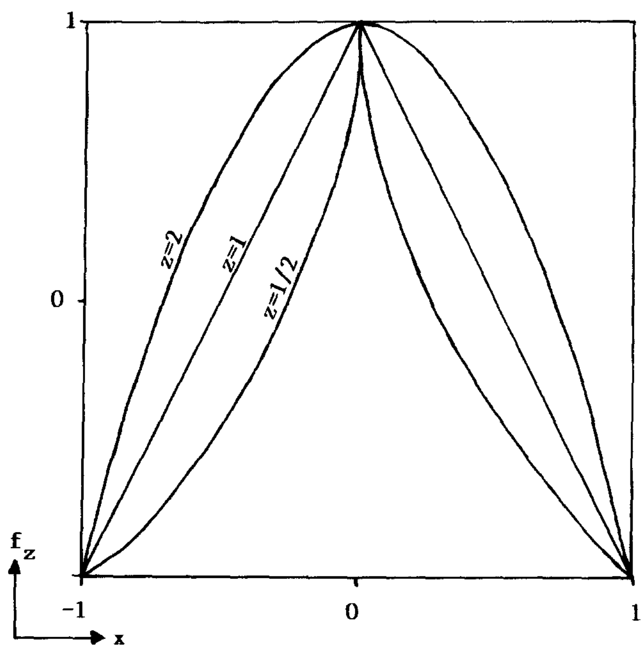

Fig. 2. The z-map for three values of the control parameter z.

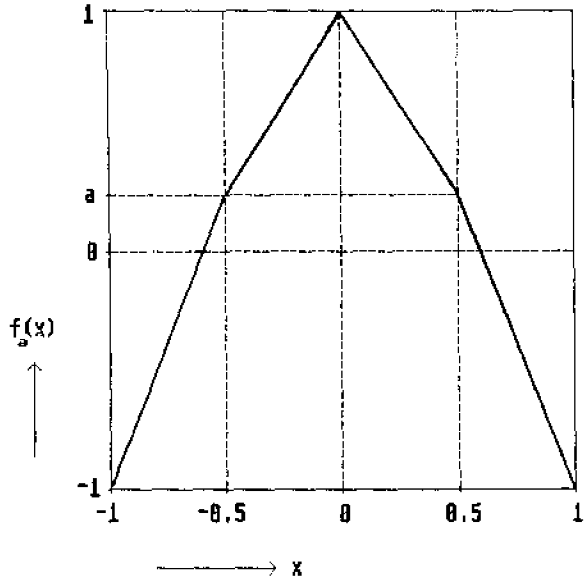

Fig. 3. The bungalow-tent map.

For 640 values of a, equally distributed over the range $-\frac{1}{2} < a < 1$, we have numerically determined the Lyapunov exponent $\lambda(a)$ using the formula:

$$\lambda(a) = \frac{1}{N}\sum_{i=0}^{N-1} \log|f'_a(x_i| \tag{9}$$

$$(N \to \infty)$$

with $N = 10^6$, the probability density $\rho_a(x)$ not being known as a function of $a \in [-\frac{1}{2}, 1[$. In Fig. 4 $\lambda(a)$ is depicted as a function of a. An unusual phenomenon is visible in this picture: one observes many discontinuities in the Lyapunov spectrum, indicating *abrupt* changes in the chaotic behaviour of the orbit $x_0, x_1, x_2, \ldots$ when the control parameter is tuned past some critical values of a. In the next section we will look more closely at the different chaotic phenomena on both sides of such a critical a-value.

3. PHENOMENOLOGY

From Fig. 4 one sees that the first discontinuity appears at $a = \frac{1}{2}$. In order to get an impression of the two different chaotic states, we have plotted a part of the chaotic trajectory for $a = 0.495$ (Fig. 5(a)) and for $a = \frac{1}{2}$ (Fig. 5(b)). For $a = 0.495$ an interesting phenomenon is observed. The trajectory shows an intermittent hopping between order and chaos: periods of regular behaviour ('laminar intervals') alternate with periods of irregular behaviour ('chaotic bursts'). From numerical experiments it became clear that the closer a is set near $\frac{1}{2}$ from below, the longer is the mean duration of the laminar intervals as well as the duration of the chaotic bursts. In fact, in the limit that a has reached $\frac{1}{2}$ from below, the mean duration of laminar intervals and chaotic bursts has become infinite. But if a is set *exactly on the value* $\frac{1}{2}$, the intermittent phenomenon has suddenly disappeared, and we have only chaos. This is also the case for a-values slightly larger than $\frac{1}{2}$.

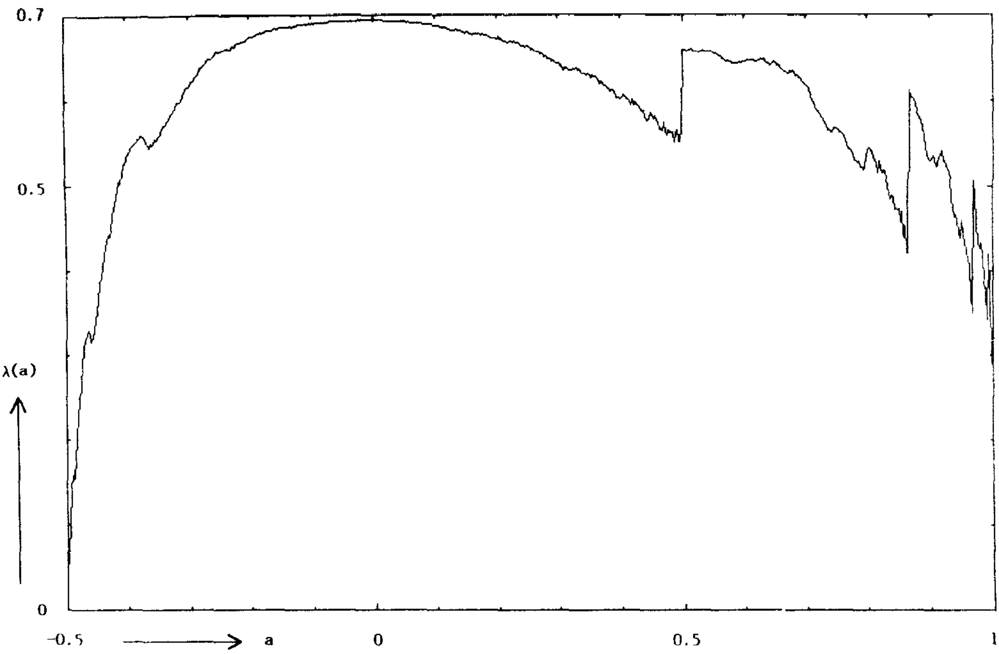

Fig. 4. The Lyapunov spectrum of the bungalow-tent map.

A qualitative explanation for the occurrence of long laminar intervals can be given as follows. In Fig. 6 we have zoomed in on the (unstable) right-hand fixed point x_F for $a = \frac{1}{2} - \varepsilon$. For this value of a the slope of the line connecting the top of the tent with the kink at the right is $-1 - 2\varepsilon$. Since for $0 < \varepsilon \ll 1$ this slope is only a fraction smaller than -1, it will take a considerable amount of iterations before an iterate, which has landed in the neighbourhood of the fixed point, will have spiralled away. This spiralling behaviour is a laminar interval. The trajectory is 'trapped' in the subinterval $[x_L, x_R]$ (see Fig. 6), which has a size of order ε. Eventually, the trajectory will spiral out of this interval and will explore chaotically other parts of the interval $[-1, 1]$. But since the iterated system is FDC, and hence ergodic, the trajectory will certainly 'hit' the interval $[x_L, x_R]$ again, i.e. the chaotic burst will end and a new laminar interval will be started up.

If ε is made smaller (i.e. if a approaches $\frac{1}{2}$ from below), the size of the interval $[x_L, x_R]$ will also become smaller. Therefore, the probability that the chaotic trajectory will hit this interval will also become smaller. This explains why the mean duration of the chaotic bursts becomes larger when a is set closer to $\frac{1}{2}$ from below. On the other hand, for decreasing values of ε, the slope of the line connecting the top of the tent with the right-hand kink will be closer to -1, as a consequence of which it takes longer before the trajectory has spiralled out of the interval $[x_L, x_R]$. This explains why the mean duration of the laminar intervals increases when a is set closer to $\frac{1}{2}$ from below. In the same spirit it can be argued why the mean duration of the chaotic bursts as well as the laminar intervals goes to infinity in the limit that a has reached $\frac{1}{2}$ from below.

But if a is exactly equal to $\frac{1}{2}$, the size of the interval $[x_L, x_R]$ is zero and the right-hand kink is exactly on the location of the fixed point (see Fig. 7). Laminar intervals cannot exist anymore, and there remains only an 'eternal' chaotic 'burst' on the entire interval $[-1, 1]$.

The qualitative arguments presented above make clear that the chaotic behaviour of the iterated bungalow-tent for $a = \frac{1}{2}$ differs from the chaotic behaviour for $a = \lim_{\varepsilon \to 0} \frac{1}{2} - \varepsilon$. The

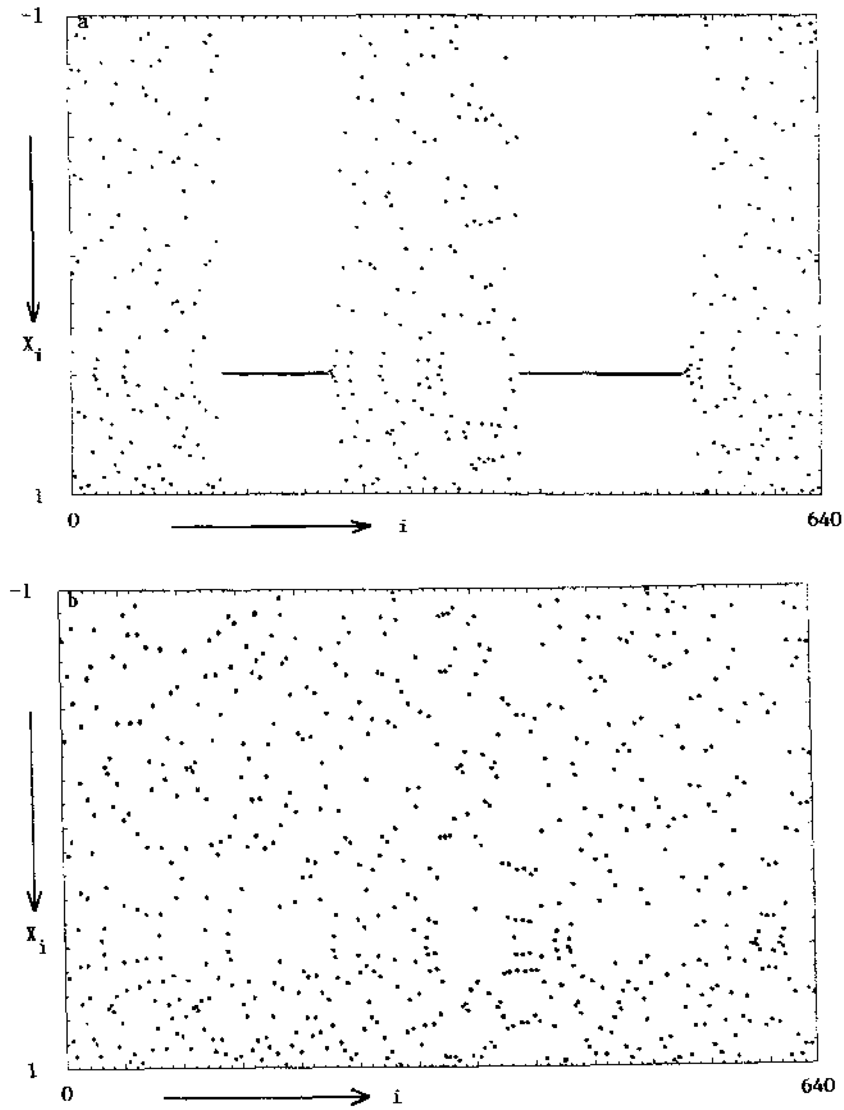

Fig. 5. Iterates $x_i = f_a^{(i)}(x_0)$ of the bungalow-tent map for $a = 0.495$ (a) and $a = \frac{1}{2}$ (b).

insight provided by this pictorial treatment can be put on a more rigorous basis using an analysis in terms of the probability density $\rho_a(x)$: solving the Frobenius–Perron equation [1] associated with the bungalow-tent map for $a = \frac{1}{2}$ and $a = \frac{1}{2} - \varepsilon$ yields the following expressions for the probability density:

$$\rho_{1/2}(x) = \tfrac{2}{5}\theta(\tfrac{1}{2} - x) + \tfrac{4}{5}\theta(x - \tfrac{1}{2}), \tag{10}$$

and

$$\rho_{1/2-\varepsilon}(x) = \begin{cases} \frac{1}{3} + O(\varepsilon) & \text{if } -1 < x < \frac{1}{2} - \varepsilon \\ \frac{1}{6}\varepsilon^{-1} + O(1) & \text{if } \frac{1}{2} - \varepsilon < x < \frac{1}{2} \\ \frac{2}{3} + O(\varepsilon) & \text{if } \frac{1}{2} < x < 1, \end{cases} \tag{11}$$

see Fig. 8.

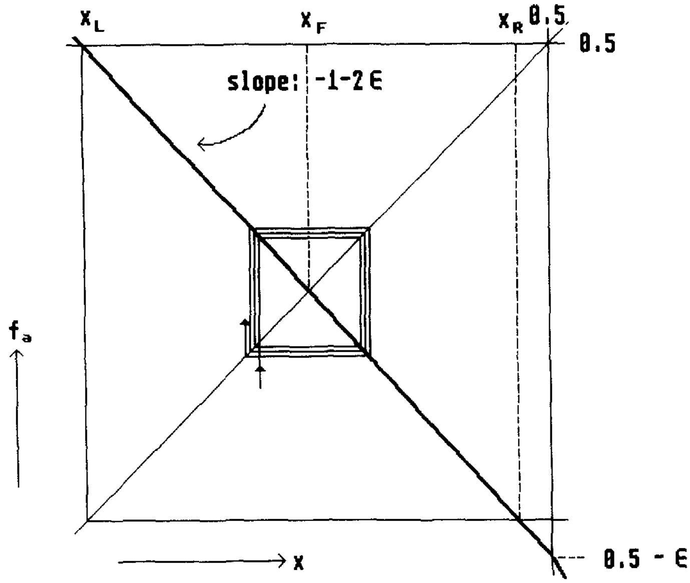

Fig. 6. Magnification of the region near the right-hand fixed point x_F for $a = \frac{1}{2} - \varepsilon$. The trajectory slowly spirals away from the fixed point (laminar interval). Once the trajectory is out of the interval $[x_L, x_R]$, the laminar interval has ended, and a chaotic burst begins. ($f_a(x_L) = \frac{1}{2}$, $f(x_R) = x_L$).

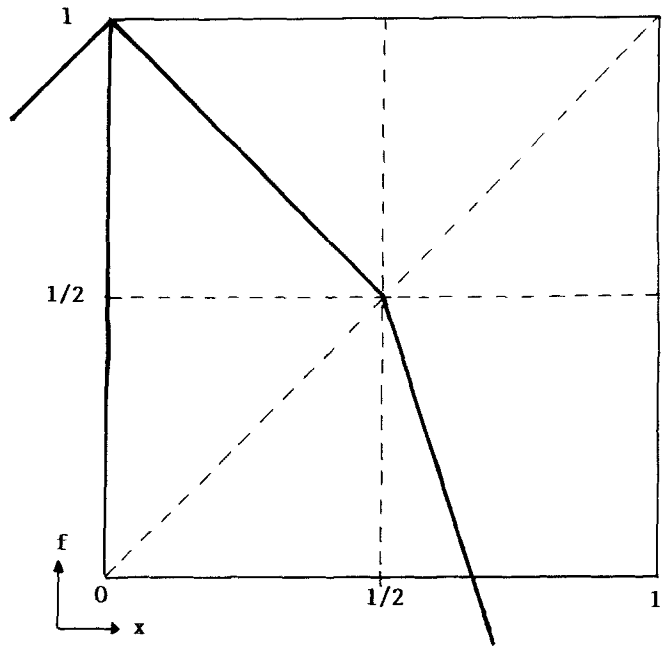

Fig. 7. If $a = \frac{1}{2}$ the kink has merged with the fixed point. Laminar intervals are absent.

One sees that for $a = \frac{1}{2}$ the probability density consists of two plateaux. But for $a = \frac{1}{2} - \varepsilon$ the two plateaux are separated from each other by a third plateau with a height of order $1/\varepsilon$. This third, high plateau extends over a tiny interval $[\frac{1}{2} - \varepsilon, \frac{1}{2}]$ and suggests the occurrence of long laminar intervals in this tiny interval. (One can show that $[\frac{1}{2} - \varepsilon, \frac{1}{2}]$ is equal to $[x_L, x_R]$ up to order ε^2.) In the limit $\varepsilon \to 0$ equation (11) becomes:

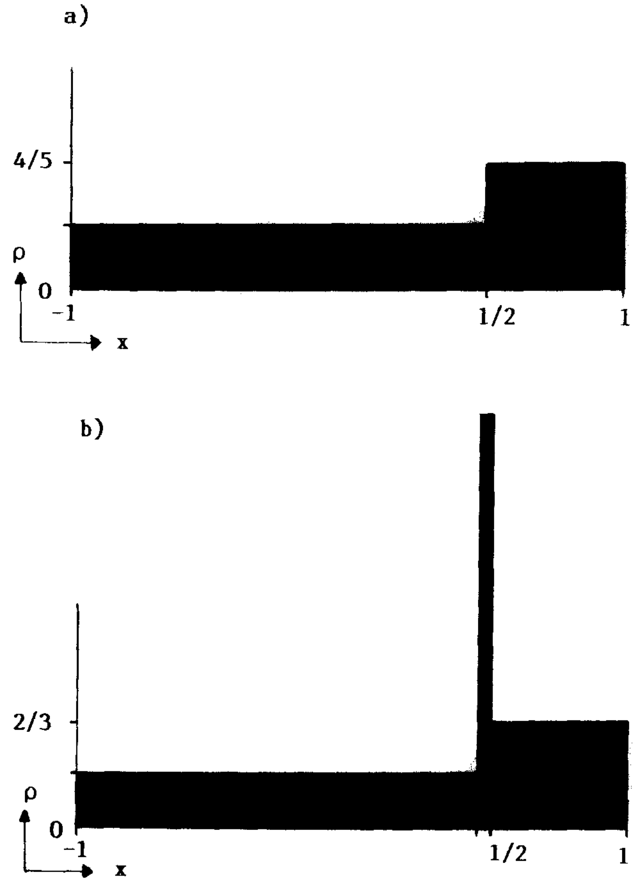

Fig. 8. The probability density of the iterated bungalow-tent map for (a) $a = \frac{1}{2}$ and (b) $a = \frac{1}{2} - \varepsilon$.

$$\lim_{\varepsilon \downarrow 0} \rho_{1/2-\varepsilon}(x) = \tfrac{1}{3}\theta(\tfrac{1}{2} - x) + \tfrac{2}{3}\theta(x - \tfrac{1}{2}) + \tfrac{1}{6}\delta(x - \tfrac{1}{2}), \tag{12}$$

which differs from $\rho_{1/2}(x)$ as given by (10). This difference provides the analytic explanation for the jump in the Lyapunov spectrum at $a = \frac{1}{2}$. In fact, it is easily shown that

$$\lambda_{a \uparrow 1/2} = \lim_{\varepsilon \downarrow 0} \int_{-1}^{1} \rho_{1/2-\varepsilon}(x) \log |f'_{1/2-\varepsilon}(x)| \, \mathrm{d}x$$
$$= \tfrac{1}{2}\log 3 = 0.5493061 \ldots, \tag{13}$$

and

$$\lambda_{1/2} = \int_{-1}^{1} \rho_{1/2}(x) \log |f'_{1/2}(x)| \, \mathrm{d}x = \tfrac{3}{5}\log 3 = 0.6591673 \ldots. \tag{14}$$

The results (13) and (14) are in excellent agreement with Fig. 4, especially when one takes into account that for $a = \frac{1}{2} - \varepsilon$ the Lyapunov exponent shows anomalously large fluctuations. In fact, any spectrum

$$A_a \equiv \int_{-1}^{1} \rho_a(x) A(x) \, \mathrm{d}x \tag{15}$$

with $A(x)$ an arbitrary (but integrable) function of x, shows a discontinuity at $a = \frac{1}{2}$ due to the shockwise change of the probability density at the transition $(a \uparrow \frac{1}{2}) \to (a = \frac{1}{2})$. The value $a = \frac{1}{2}$ is the first of an infinite series of critical values which lay in the interval $[\frac{1}{2}, 1]$, see Fig. 4. These critical values can be evaluated using an exact renormalization procedure [1].

4. CORRELATION FUNCTIONS AND THE PHASE TRANSITION

4.1. *Introduction*

The analysis in terms of the probability density exposed in the previous section is incomplete. Though the occurrence of jumps in a spectrum A_a is quantitatively explained, other aspects of the jump phenomenon are waiting for a more profound understanding. For example, one would like to know with which power the mean length of the laminar intervals goes to infinity as a function of ε. But more importantly, since the jump phenomenon is caused by the variation of an external control parameter, one would like to know to which extent this phenomenon resembles a phase transition within the context of statistical mechanics. In concreto, we would like to know if there is *long-range order* at one side of the critical value $a = \frac{1}{2}$ which is absent at the other side. Therefore, we would like to derive exact expressions for the correlation functions at both sides of the critical value, rather than drawing conclusions on the basis of numerical studies.

An obvious way to try to attain the goals mentioned in the previous paragraph is to give a description in terms of the chaotic trajectory

$$\vec{x} \equiv [x_0, x_1, x_2, \ldots]. \tag{16}$$

Of course mean values along this trajectory depend on the control variable a. A correlation function $C(n)$ can be defined by

$$C(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} (x_i - \langle x \rangle)(x_{i+n} - \langle x \rangle) \tag{17}$$

$$= \int_{-1}^{1} \rho_a(x) x f_a^{(n)}(x)\, \mathrm{d}x - \left[\int_{-1}^{1} \rho_a(x) x\, \mathrm{d}x\right]^2. \tag{18}$$

Unfortunately, even though we have the analytical expressions (10) and (11) for the probability density $\rho_a(x)$ at $a = \frac{1}{2}$ and $a = \frac{1}{2} - \varepsilon$, it is extremely complicated if not impossible to derive closed analytical expressions for $C(n)$ for these a-values for general n. This approach does not work in practice.

Another way of attacking the problem is to use a symbolic description of the chaotic process. An obvious choice [12] for such a symbolic description is the following: if a point of the trajectory is on the left of $x = 0$ (which is the location of the top of the bungalow-tent map), the symbol L is attributed to it, and if a point of the trajectory is on the right of $x = 0$, the symbol R is attributed to it. So instead of the trajectory $\vec{x}$ we consider the half-infinite sequence

$$\vec{t} = [t_0, t_1, t_2, \ldots], \tag{19}$$

with

$$t_j = \begin{cases} L & \text{if } x_j < 0 \\ r & \text{if } x_j > 0. \end{cases} \tag{20}$$

The sequence $\vec{t}$ describes the chaotic process in terms of locations left (L) or right (R) of the point $x = 0$, see Fig. 9. One can show [3] that there is a one-to-one correspondence

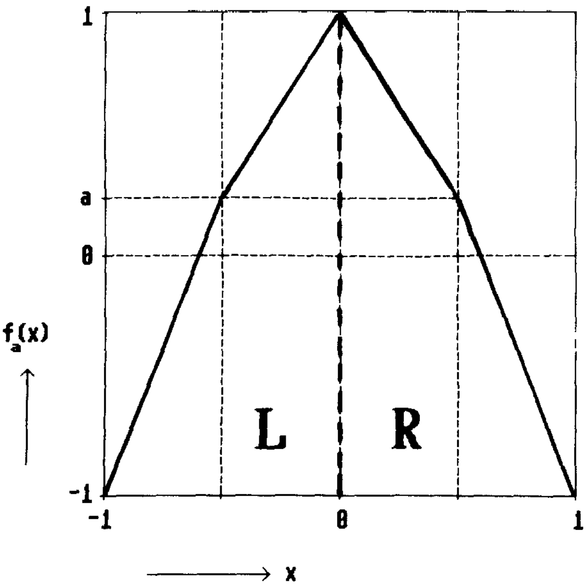

Fig. 9. The LR-partition of the bungalow-tent map.

between $\vec{x}$ and $\vec{l}$. Again, using this symbolic description, one can define [2] four correlation function $\gamma_{L,L}(n)$, $\gamma_{L,R}(n)$, $\gamma_{R,L}(n)$, $\gamma_{R,R}(n)$ measuring the correlation between two symbols over a distance n. And again, it turns out that it is extremely difficult if not impossible to find closed analytical expressions or an (exact) asymptotic approximation for these correlation functions at both sides of the critical value, i.e. for $a = \frac{1}{2} - \varepsilon$ and $a = \frac{1}{2}$. So this approach also fails.

The reason why the two previous approaches are not successful is roughly the following: the origin of the jump phenomenon lies in the occurrence of infinitely long laminar intervals in the close neighbourhood of the (unstable) right-hand fixed point x_F for $a = \lim_{\varepsilon \downarrow 0} \frac{1}{2} - \varepsilon$, which are absent for $a = \frac{1}{2}$. Since the $\vec{x}$ description as well as the symbolic $\vec{l}$ description *do not specify the point x_F, the point around which it all happens*, these two descriptions are unsuitable for an exact statistical analysis of the phenomenon. In fact, as we will see, a description which uses a partition of the interval $[-1, 1]$ into regions left and right of the fixed point x_F is extremely successful in obtaining exact results:

If

$$\vec{x} = x_0, x_1, x_2, \dots \tag{21}$$

is the trajectory of the iterated bungalow-tent map, then the half-infinite $(+-)$-sequence $\vec{s}$ is defined as follows

$$\vec{s} = [s_1, s_2, \dots], \qquad s_i = \begin{cases} + & \text{if } x_i > x_{i-1} \\ - & \text{if } x_i < x_{i-1}. \end{cases} \tag{22}$$

An equivalent way of forming $\vec{s}$ is:

$$\vec{s} = [s_1, s_2, \dots], \qquad s_i = \begin{cases} + & \text{if } x_{i-1} < x_F \\ - & \text{if } x_{i-1} > x_F, \end{cases} \tag{23}$$

where x_F is the extreme right fixed point of the one-hump map, see Fig. 10. The symbol sequence $\vec{s}$ gives a description of the trajectory $\vec{x}$ in terms of the ascending and descending

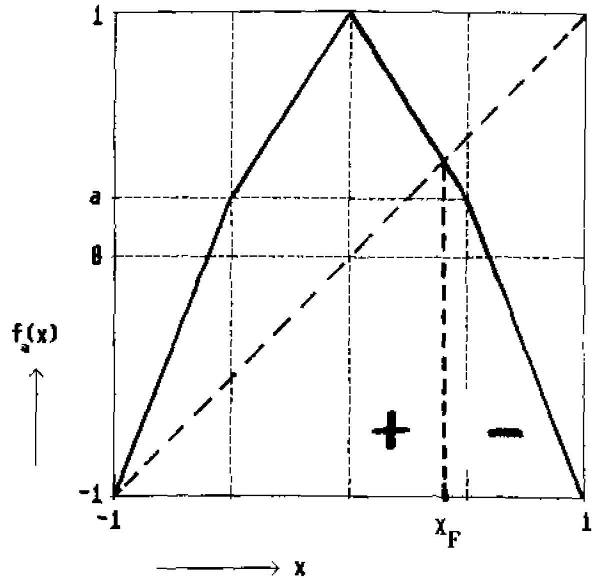

Fig. 10. The $(+-)$-partition of the bungalow-tent map.

character of the trajectory $\vec{x}$. There is no one-to-one correspondence between $\vec{s}$ and $\vec{x}$: the $\vec{s}$ description is a *local description* which only says whether the trajectory is left $(+)$ or right $(-)$ of the fixed point x_F. In spite of the loss of dynamical information in going from the $\vec{x}$ description to the $\vec{s}$ description, one achieves a better understanding of the jump phenomenon in the bungalow-tent map from the point of view of statistical mechanics. For an analysis of local and global aspects of iterated one-dimensional piecewise linear maps in terms of generalized Markov partitions and spectral decomposition of the Frobenius–Perron operator, we refer to [13].

The probability of finding a particular *word* or subsequence

$$\vec{w}^k = [w_1, w_2, \ldots, w_k],\ w_i = + \text{ or } -, \qquad k = 1, 2, \ldots \tag{24}$$

of k successive symbols $w_1, w_2, \ldots, w_k$ in $\vec{s}$ is given by

$$P_{\vec{s}}(\vec{w}^k) = \lim_{N\to\infty} \frac{1}{N-k+1} \sum_{i=1}^{N-k+1} \delta_{w_1,s_i}\delta_{w_2,s_{i+1}} \cdots \delta_{w_k,s_{i+k-1}}. \tag{25}$$

Due to the nature of a one-hump map, a minus in the $(+-)$-sequence $\vec{s}$ is always followed by a plus, or, equivalently, one cannot find words $\vec{w}^k$ in $\vec{s}$ which possess one or more pairs of neighbouring minuses.

The four symbolic correlation functions $\gamma_{+,+}(n)$, $\gamma_{+,-}(n)$, $\gamma_{-,+}(n)$ and $\gamma_{-,-}(n)$ of two symbols at distance n of a sequence $\vec{s}$ are defined in the following way [2]. If $n = 0$ then

$$\gamma_{w_1,w_2}(0) = \delta_{w_1,w_2} P_{\vec{s}}([w_1]) - P_{\vec{s}}([w_1]) P_{\vec{s}}([w_2]), \tag{26}$$

and if $n \geqslant 1$ then

$$\gamma_{w_1,w_2}(n) = P_{\vec{s}}([w_1, \underbrace{n-1}, w_2]) - P_{\vec{s}}([w_1]) P_{\vec{s}}([w_2]), \tag{27}$$

where the underbrace denotes the summation over the intermediate $n-1$ symbols:

$$P_{\vec{s}}([w_1, \underbrace{n-1}, w_2]) \equiv \sum_{v_1=+,-} \cdots \sum_{v_{n-1}=+,-} P_{\vec{s}}([w_1, v_1, v_2, \ldots, v_{n-1}, w_2]). \tag{28}$$

In [2] it was shown that

$$\gamma_{+,+}(n) = \gamma_{-,-}(n) = -\gamma_{+,-}(n) = -\gamma_{-,+}(n), \tag{29}$$

i.e. one needs only one of the four correlation functions between two symbols over a distance n.

4.2. *The case $a = \frac{1}{2}$ (the critical value from above)*

For $a = \frac{1}{2}$ it is not very difficult to find an analytical expression for the probabilities (25), expressing these probabilities in terms of integrals over the exactly known probability density (10). One finds the recursion relation [5]

$$P_{\vec{s}}([w_1, w_2, \ldots, w_k]) = (1 - \delta_{-,w_1}\delta_{-,w_2})\frac{2^{\delta_{-,w_1}}}{3^{\delta_{+,w_2}}} P_{\vec{s}}([w_2, w_3, \ldots, w_k]). \tag{30}$$

This means that the symbol sequence $\vec{s}$ is order-1 Markovian [2, 3] for this value of a. From equation (30) one derives an explicit formula for $P_{\vec{s}}(\vec{w}^k)$. Insertion of this expression in equation (27) leads to a simple formula for the correlation between two pluses over a distance n:

$$\gamma_{+,+}(n) = (-1)^n \tfrac{9}{25}(\tfrac{2}{3})^{n+1}, \tag{31}$$

see Fig. 11. (The other three correlation functions follow using equation (29).) Notice that the correlations die out exponentially. There is no long-range order.

4.3. *The case $a = \frac{1}{2} - \varepsilon$ (just before the critical value)*

For $a = \frac{1}{2} - \varepsilon$ one can derive the recursion relation [5]:

$$P_{\vec{s}}([w_1, w_2, \ldots, w_k]) = (1 - \delta_{-,w_1}\delta_{-,w_2})\left[\frac{(2-2\varepsilon)^{\delta_{-,w_1}}}{(3-2\varepsilon)^{\delta_{+,w_2}}} P_{\vec{s}}([w_2, w_3, \ldots, w_k]) - (-1)^{\delta_{-,w_1}}\delta_{+,w_2}\left(\prod_{i=3}^{k}\delta_{(-1)^i,w_i}\right)\frac{1}{36(1+2\varepsilon)^{k-2}}\right], \tag{32}$$

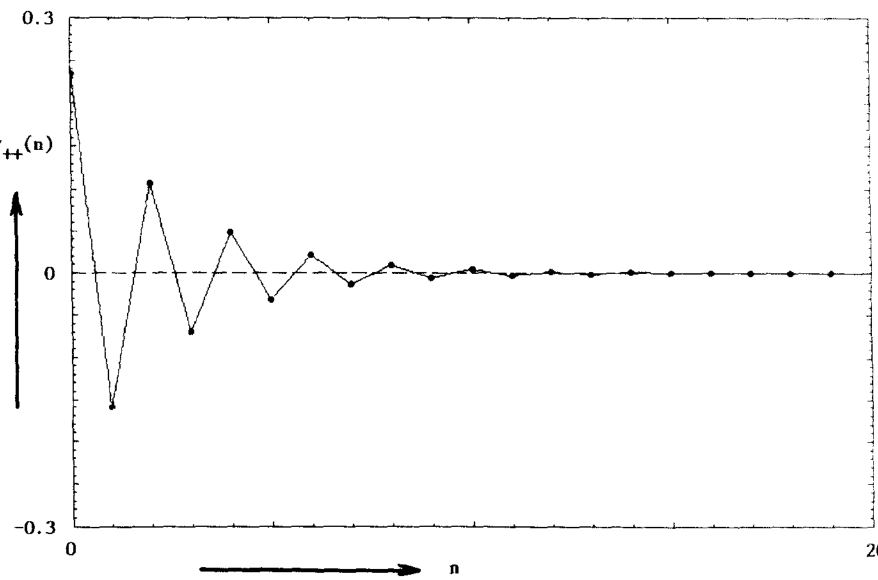

Fig. 11. The correlation function $\gamma_{++}(n)$ of the $(+-)$-sequence $\vec{s}$ associated with the $(a = \frac{1}{2})$-bungalow-tent map.

apart from quadratic terms in ε. The first term in the r.h.s. of equation (32) goes over in the r.h.s. of equation (30) for $\varepsilon \to 0$. The second term is due to the laminar intervals. As a consequence the symbol sequence $\vec{s}$ has an infinite memory [2] just before the critical value. From this formula we were able to prove:

$$\langle n_{\text{lam}} \rangle = \frac{1}{2\varepsilon} + O(1) \tag{33}$$

for the average length of the laminar intervals. Using the recursion relation (32) in combination with equation (27), one obtains [5]:

$$\begin{aligned} \gamma_{+,-}(n) &= \left(\frac{7}{720} + O(\varepsilon)\right) + \left(-\frac{36}{720} + O(\varepsilon)\right)\left(\frac{1}{1+2\varepsilon}\right)^{n-1} \\ &\quad + \left(-\frac{2}{15} + O(\varepsilon)\right)\left(\frac{4-8\varepsilon}{(3-2\varepsilon)^2}\right)^{(n-1)/2} + O(\varepsilon^2) \quad (n = 1, 3, 5, \ldots), \end{aligned} \tag{34}$$

$$\begin{aligned} \gamma_{+,+}(n) &= \left(\frac{7}{720} + O(\varepsilon)\right) + \left(\frac{24}{720} + O(\varepsilon)\right)\left(\frac{1}{1+2\varepsilon}\right)^{n-2} \\ &\quad + \left(\frac{4}{45} + O(\varepsilon)\right)\left(\frac{4-8\varepsilon}{(3-2\varepsilon)^2}\right)^{(n-2)/2} + O(\varepsilon^2) \quad (n = 2, 4, 6, \ldots), \end{aligned} \tag{35}$$

and

$$\gamma_{+,+}(0) = \tfrac{35}{144} + O(\varepsilon). \tag{36}$$

(see Fig. 12). The third term in the formulae (34) and (35) is of order $(\frac{2}{3})^n$ and therefore decays rapidly and independently of the value of ε. In contradistinction, the second term is of order $\exp(-2\varepsilon n)$ and will have decreased considerably only for values of n greater than the average length of the laminar intervals $\langle n_{\text{lam}} \rangle = 1/2\varepsilon$. In the limit $n \to \infty$ we obtain:

$$\lim_{n\to\infty} \gamma_{+,+}(n) = \tfrac{7}{720} + O(\varepsilon). \tag{37}$$

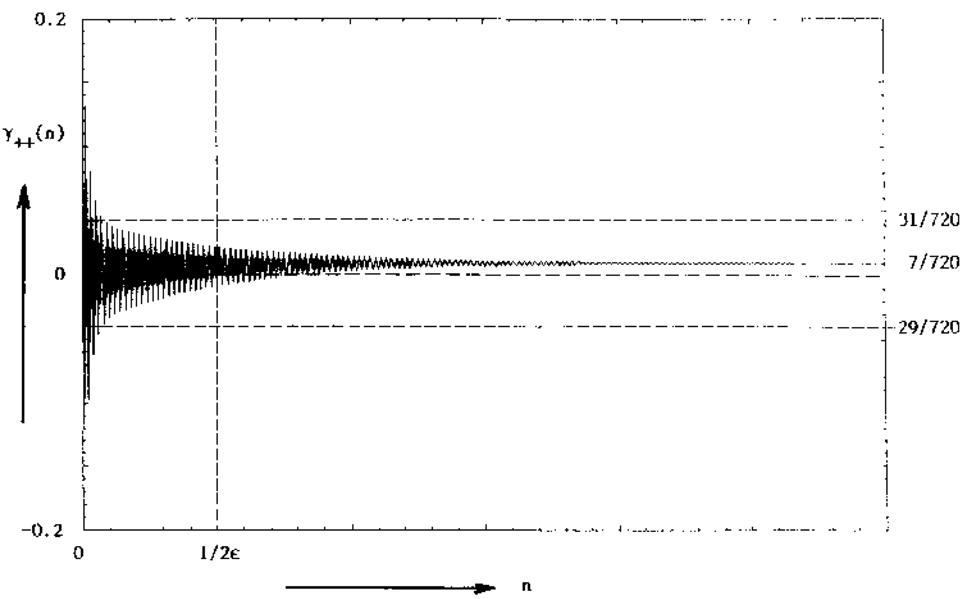

Fig. 12. The correlation function $\gamma_{+,+}(n)$ just before the critical value, i.e. for $a = \frac{1}{3} - \varepsilon$ with $\varepsilon = 0.001$.

Thus the correlations do not die out, indicating the existence of long-range ordering just before the critical value, see Fig. 12. Also notice that the slowly decaying term in equations (34) and (35) is of order

$$\gamma_{+,+}(n) \sim e^{-n/\tau} \tag{38}$$

where

$$\tau = \frac{1}{\log(1 + 2\varepsilon)} \tag{39}$$

goes to infinity if $\varepsilon \to 0$, i.e. if the critical value is approached from below. This phenomenon is called *critical slowing down*.

4.4. *The case* $\lim_{\varepsilon \downarrow 0} \frac{1}{2} - \varepsilon$ *(the critical value from below)*

Taking the limit $\varepsilon \to 0$ in equations (34)–(36), one finds the following expressions for the correlation function $\gamma_{+,+}(n)$:
For $n = 0$:

$$\gamma_{+,+}(0) = \tfrac{35}{144}. \tag{40}$$

For n is odd:

$$\gamma_{+,+}(n) = -\tfrac{29}{720} - \tfrac{2}{15}(\tfrac{2}{3})^{n-1}, \tag{41}$$

and for $n = 2, 4, 6, \ldots$:

$$\gamma_{+,+}(n) = \tfrac{31}{720} + \tfrac{4}{45}(\tfrac{2}{3})^{n-2}, \tag{42}$$

see Fig. 13. One sees that the correlation function, after a short 'set-in' time, remains oscillating for ever between the values 31/720 and −29/720. This indicates a very strong communication over large distances since the difference between odd and even is always felt, in contradistinction with the case of finite ε, see Fig. 12.

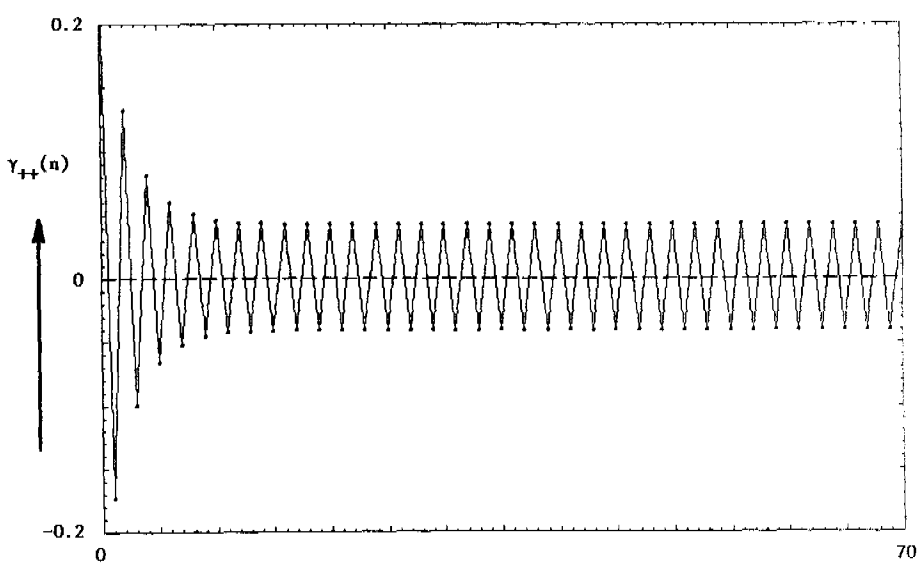

Fig. 13. The correlation function $\gamma_{+,+}(n)$ in the limit $a \uparrow \frac{1}{2}$.

In summary, the following behaviour of the correlation functions occurs if the control parameter a is raised from $\frac{1}{2}-\varepsilon$ to $\frac{1}{2}$: For $a=\frac{1}{2}-\varepsilon$, the correlation function $\gamma_{+,+}(n)$ shows a critical slowing down. The characteristic critical-slowing-down time τ is equal to the mean length of the laminar intervals $\langle n_{\text{lam}}\rangle \sim 1/2\varepsilon$. Notice that the correlation function does not go to zero, but to the value 7/720. So, even for finite ε there is a (weak) long-range order. In the limit $\varepsilon \downarrow 0$ or, equivalently, $a \uparrow \frac{1}{2}$, the characteristic critical-slowing-down time has become infinite, and the correlation function keeps oscillating between 31/720 and −29/720. But if ε is exactly equal to zero ($a=\frac{1}{2}$), long-range order is absent: the correlation function falls off exponentially.

Thus the iterated bungalow-tent map shows a discontinuity in the correlation function, with long-range order on the critical value from below being absent on the critical value from above.

5. RELATION WITH ISING CHAINS AND CONCLUDING REMARKS

The symbolic (+−)-description of the jump phenomenon in the iterated bungalow-tent map can be regarded as a phase transition [14] of a one-dimensional Ising model in which the probabilities of the various configurations are specified rather than the interactions between groups of spins. The transition between the one description to the other can be achieved by applying a generalized Legendre transformation. It would be of interest to investigate this feature in more detail and to find out what kind of interactions would produce the phase transition. Here we have no definitive answer, but we would anticipate that the transition would yield an Ising model with rather complicated many-body interactions. The non-occurrence of successive minuses in the symbolic description (cf. the remark just after equation (25)) suggests a relation to antiferromagnetic systems close to paramagnetic behaviour.

Formally, we can link the (+−)-description of the bungalow-tent map (but also, more generally, of any one-hump FDC map) to an Ising model using the relation:

$$\beta E(\vec{w}^k) = -\log P_{\bar{s}}(\vec{w}^k) - \log Z_k \tag{43}$$

where β is the inverse temperature and $E(\vec{w}^k)$ is the energy of a configuration $\vec{w}^k$. Z_k is the canonical partition function of an Ising chain with k spins. Spins are associated with the symbols w_i in the usual manner: spin up corresponds to a plus and spin down to a minus. The pair correlation function $\Gamma(n)$ between two spins on a distance $n<k$ is then linked to the symbolic correlation functions as follows:

$$\Gamma(n) \equiv Z_k^{-1}\sum_{\{\vec{w}^k\}}\left(\frac{1}{k-n}\sum_{i=1}^{k-n} w_i w_{i+n}\right)\exp(-\beta E(\vec{w}^k)) - \left[Z_k^{-1}\sum_{\{\vec{w}^k\}}\left(\frac{1}{k}\sum_{i=1}^{k} w_i\right)\exp(-\beta E(\vec{w}^k))\right]^2 \tag{44}$$

$$= 4\gamma_{+,+}(n), \tag{45}$$

as follows after a straightforward calculation [15]. So the long-range order as well as the discontinuity in the symbolic correlation function of the bungalow-tent map at the transition $(a\uparrow\frac{1}{2})\rightarrow(a=\frac{1}{2})$ is translated to the spin-pair correlation function $(k\rightarrow\infty)$, indicating a real phase transition in the one-dimensional Ising chain. This phase transition appears to be different from the phase transitions of the Dyson type [16] in the case of long-range interactions.

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