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Analytical and numerical methods for massive two-loop self-energy diagrams [*]

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Abstract

Motivated by the precision results in the electroweak theory studies of two-loop Feynman diagrams are performed. Specifically this paper gives a contribution to the knowledge of massive two-loop self-energy diagrams in arbitrary and especially four dimensions. This is done in three respects: firstly results in terms of generalized, multivariable hypergeometric functions are presented giving explicit series for small and large momenta. Secondly the imaginary parts of these integrals are expressed as complete elliptic integrals. Finally one-dimensional integral representations with elementary functions are derived. They are very well suited for the numerical evaluations.

1. Introduction

The beautiful results of the LEP1 experiments have shown that the electroweak theory has a predictive power like that of QED several decades ago. It is to be expected that eventually the electroweak theory will provide high precision predictions for many experiments in the present and near future. One may in particular think in this respect of the measurement of the Z-mass and -width, a better determination of the W-mass and a first indication of the value of the top mass. This will in the future require two-loop calculations in the electroweak theory. One then has to face several problems: amongst others, a large number of diagrams and many different non-negligible masses. It is known that analytical results for arbitrary non-vanishing masses are in general not obtainable for two-loop diagrams in the form of generalized polylogarithms. The present

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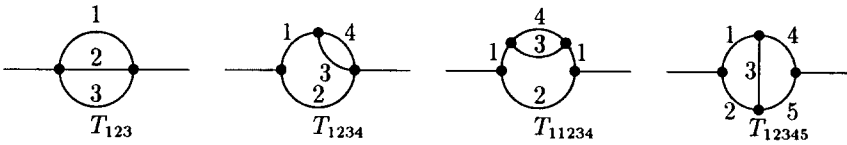[*] Supported by Stichting FOM and EU contract CHRX-CT-92-0004.

Fig. 1. Topologies.

paper contributes to the knowledge about this problem by introducing and applying new analytical and numerical approaches. A number of new results will be presented.

We focus on the simplest class of two-loop diagrams, i.e. the set of scalar two-loop self-energies. They are essential for physical predictions and moreover they show the typical problems one encounters in the evaluation of two-loop diagrams. There exist four non-trivial self-energy diagrams.

In order to classify our conventions we give the explicit form of the so-called master diagram:

$$T_{12345}\,(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = \left\langle\!\left\langle \frac{1}{k_1^2 - m_1^2} \cdots \frac{1}{k_5^2 - m_5^2} \right\rangle\!\right\rangle \tag{1}$$

where each bracket denotes a D-dimensional integration over one loop momentum q:

$$\langle \ldots \rangle = \int \frac{\mathrm{d}^D q}{i\pi^2 (2\pi\mu)^{D-4}} (\ldots)\,, \tag{2}$$

where μ is an arbitrary mass. The momenta of the propagators $k_1, \ldots, k_5$ are determined by momentum conservation in terms of the external momentum p and the integration momenta q_1 and q_2. It can be immediately seen that diagram $T_{11234}\,(p^2; m_1^2, m_1'^2, m_2^2, m_3^2, m_4^2)$ can be reduced to a difference of two T_{1234} diagrams by partial fractioning of the propagators with the same momentum:

$$T_{11234}\,(p^2; m_1^2, m_1'^2, m_2^2, m_3^2, m_4^2) = \frac{1}{m_1^2 - m_1'^2} \left(T_{1234}\,(p^2; m_1^2, m_2^2, m_3^2, m_4^2) \right.$$
$$\left. -T_{1234}\,(p^2; m_1'^2, m_2^2, m_3^2, m_4^2) \right)\,. \tag{3}$$

In the special case $m_1 = m_1'$ this has to be taken as a derivative with respect to m_1^2. So we are only left with three diagrams to be discussed. Of these the master diagram is convergent in four dimensions, the other two are divergent. After evaluating them in $D = 4 - 2\delta$ dimensions, one would like to derive an expansion in δ which starts in $1/\delta^2$ and $1/\delta$ terms.

In recent years several groups derived small- and large-p^2 expansions for these diagrams. For the general massive case they obtained algorithms for symbolic evaluation of these expansion coefficients. For the general coefficients in these expansions, no closed expression was derived. More recently multiple series were found for which the general coefficient is known and which lead to generalized hypergeometric functions in several variables. One aim of this paper is to extend these results, an other to express the imaginary parts by elliptic integrals. On the other hand an elegant numerical method has been worked out leading to a two-dimensional integral representation [1,2]. The

second aim of our paper is to improve on this by introducing a one-dimensional integral representation containing only elementary functions. For this a general procedure for diagrams containing one-loop self-energy expressions in a subloop is presented. The actual outline of the paper is as follows: in Sect. 2 the analytic approach leading to generalized hypergeometric functions and the corresponding explicit series representations are given, the elliptic integrals for the diagrams are calculated in Sect. 3. The next section describes the numerical approach by means of a one-dimensional representation which is derived by a self-energy insertion into one-loop diagrams. Finally Sect. 5 presents some numerical comparisons and draws conclusions.

2. Analytic approaches and hypergeometric functions

In this section we present analytic results in an arbitrary number of dimensions D for the two-loop scalar self-energy diagrams. They will be in the form of generalized hypergeometric functions, that is multiple series of ratios of p^2 and the masses. We also discuss the expansion of these results around $D = 4$.

The simplest case of a two-loop scalar self-energy diagram $T_{123}(p^2; m_1^2, m_2^2, m_3^2)$ was discussed in a previous paper [3]. In fact, the result turned out to be represented by Lauricella functions which could be derived by x-space techniques or Mellin–Barnes representation for a massive propagator. It turns out that there is a third method using dispersion relations. Since the latter method will be used several times in this paper we shall first rederive the result for $T_{123}(p^2; m_1^2, m_2^2, m_3^2)$ by means of dispersion relations. Then we focus our attention on the two-loop scalar self-energy diagram with four propagators $T_{1234}(p^2; m_1^2, m_2^2, m_3^2, m_4^2)$ where the small-p^2 behavior can again be derived by the dispersion method and the large-p^2 behavior by the Mellin–Barnes representation method.

Although we have three methods, the master diagram still poses many problems, the origin of which we will briefly discuss in the last section.

So we start with the rederivation of $T_{123}(p^2; m_1^2, m_2^2, m_3^2)$ by means of dispersion relations. The imaginary part is given by

$$
\begin{aligned}
\mathrm{Im}\left(T_{123}(p^2; m_1^2, m_2^2, m_3^2)\right) &= \frac{\Delta T_{123}(p^2; m_1^2, m_2^2, m_3^2)}{2i} \\
&= -\pi(4\pi\mu^2)^{2\delta}\frac{\Gamma^2(1-\delta)}{\Gamma^2(2-2\delta)}\Theta\left(p^2 - (m_1 + m_2 + m_3)^2\right) \\
&\quad \times \int_{(m_2+m_3)^2}^{(\sqrt{p^2}-m_1)^2} ds \, \frac{\lambda^{\frac{1}{2}-\delta}(s, m_2^2, m_3^2)}{s^{1-\delta}} \frac{\lambda^{\frac{1}{2}-\delta}(p^2, s, m_1^2)}{(p^2)^{1-\delta}} \\
&= \frac{1}{4\pi}\int_{(m_2+m_3)^2}^{(\sqrt{p^2}-m_1)^2} ds \, \Delta B_0(s; m_2^2, m_3^2)\Delta B_0(p^2; s, m_1^2) \qquad (4)
\end{aligned}
$$

where $\lambda(a, b, c) = (a - b - c)^2 - 4bc$ is the Källén function. The dispersion relation reads

$$
T_{123}\,(p^2; m_1^2, m_2^2, m_3^2) = \frac{1}{2\pi i} \int\limits_{(m_1+m_2+m_3)^2}^{\infty} dz \, \frac{1}{z - p^2} \Delta T_{123}\,(z; m_1^2, m_2^2, m_3^2)
$$

$$
= \frac{1}{4\pi^2} \int\limits_{(m_2+m_3)^2}^{\infty} ds \, \Delta B_0(s; m_2^2, m_3^2) \int\limits_{(\sqrt{s}+m_1)^2}^{\infty} \frac{dz}{z - p^2} \, \Delta B_0(z; s, m_1^2) \,.
$$

(5)

Using the expansion

$$
\frac{1}{z - p^2} = \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{p^2}{z} \right)^k
$$

(6)

we perform first the integration over z:

$$
A = \sum_{k=0}^{\infty} (p^2)^k \int\limits_{u}^{\infty} dz \, z^{\delta-2-k} (z - u)^{1/2-\delta} (z - v)^{1/2-\delta}
$$

$$
= \sum_{k=0}^{\infty} (p^2)^k u^{-\delta-k} B(k + \delta, 3/2 - \delta) \, _2F_1(\delta - 1/2, k + \delta; k + 3/2; v/u) \,,
$$

(7)

where $u = (m_1 + \sqrt{s})^2$ and $v = (m_1 - \sqrt{s})^2$. One transforms the Gauss hypergeometric function $_2F_1$ using relations which one can find in Ref. [4],

$$
_2F_1(\delta - 1/2, k + \delta; k + 3/2; v/u)
$$

$$
= \left(\frac{2\sqrt{s}}{m_1 + \sqrt{s}} \right)^{-2(k+\delta)} {}_2F_1(k + \delta, k + 1; 2k + 2; (1 - m_1^2/s))
$$

$$
= \left[\frac{\Gamma(1 - \delta)}{\Gamma(k + 2 - \delta)} \, _2F_1(k + \delta, k + 1; \delta; m_1^2/s) \right.
$$

$$
\left. + \left(\frac{m_1^2}{s} \right)^{1-\delta} \frac{\Gamma(\delta - 1)}{\Gamma(k + \delta)} \, _2F_1(k + 2 - \delta, k + 1; 2 - \delta; m_1^2/s) \right]
$$

$$
\times \left(\frac{2\sqrt{s}}{m_1 + \sqrt{s}} \right)^{-2(k+\delta)} \frac{\Gamma(2k + 2)}{k!} \,.
$$

(8)

With this result, $T_{123}(p^2; m_1^2, m_2^2, m_3^2)$ becomes

$$T_{123}(p^2; m_1^2, m_2^2, m_3^2)$$

$$= -(4\pi\mu^2)^\delta \frac{\Gamma(1-\delta)\Gamma(\delta-1)}{\Gamma(2-2\delta)} \int\limits_{(m_2+m_3)^2}^{\infty} ds\, s^{\delta-1} \lambda^{\frac{1}{2}-\delta}(s, m_2^2, m_3^2)$$

$$\times \left[\frac{m_1^2}{s} \left(\frac{m_1^2}{4\pi\mu^2} \right)^{-\delta} F_4(1, 2-\delta; 2-\delta, 2-\delta; p^2/s, m_1^2/s) \right.$$

$$\left. - \left(\frac{s}{4\pi\mu^2} \right)^{-\delta} F_4(1, \delta; 2-\delta, \delta; p^2/s, m_1^2/s) \right] . \tag{9}$$

Since in (5) the z-integration up to some factors represents the one-loop self-energy, we find as a by-product

$$B_0(p^2; m_1^2, m_2^2) = \Gamma(\delta-1) \left[\frac{m_1^2}{m_2^2} \left(\frac{m_1^2}{4\pi\mu^2} \right)^{-\delta} F_4(1, 2-\delta; 2-\delta, 2-\delta; p^2/m_2^2, m_1^2/m_2^2) \right.$$

$$\left. - \left(\frac{m_2^2}{4\pi\mu^2} \right)^{-\delta} F_4(1, \delta; 2-\delta, \delta; p^2/m_2^2, m_1^2/m_2^2) \right]$$

$$= \Gamma(\delta) \left(\frac{-p^2}{\mu^2} \right)^{-\delta} \left\{ x_2 \frac{(x_1 x_2)^{-\delta}}{1-\delta} {}_2F_1(1, \delta, 2-\delta; \frac{x_2}{x_1}) \right.$$

$$+ (1-x_1) \frac{(1-x_1)^{-\delta}(1-x_2)^{-\delta}}{1-\delta} {}_2F_1(1, \delta, 2-\delta; \frac{1-x_1}{1-x_2})$$

$$\left. + (x_1 - x_2)^{1-2\delta} B(1-\delta, 1-\delta)(-1)^{-\delta} \right\}, \tag{10}$$

where

$$x_{1,2} = \frac{1}{2p^2} \left(p^2 - m_1^2 + m_2^2 \pm \sqrt{\lambda(p^2, m_1^2, m_2^2)} \right) . \tag{11}$$

Using the definition of the F_4 functions we easily perform the integration over s. After some manipulations similar to the ones used in (8) we obtain the result for the London transport diagram which agrees with the one derived using x-space techniques or Mellin–Barnes representation

$$T_{123}(p^2; m_1^2, m_2^2, m_3^2) = -m_3^2 \left(\frac{m_3^2}{4\pi\mu^2} \right)^{2(\nu-1)}$$

$$\times \left\{ z_1^\nu z_2^\nu\, \Gamma^2(-\nu)\, F_C^{(3)}(1, 1+\nu; 1+\nu, 1+\nu, 1+\nu; z_1, z_2, z_3) \right.$$

$$\left. - z_1^\nu\, \Gamma^2(-\nu)\, F_C^{(3)}(1, 1-\nu; 1+\nu, 1-\nu, 1+\nu; z_1, z_2, z_3) \right.$$

$$-z_2^\nu \, \Gamma^2(-\nu) \, F_C^{(3)}(1, 1 - \nu; 1 - \nu, 1 + \nu, 1 + \nu; z_1, z_2, z_3)$$

$$-\Gamma(\nu)\Gamma(-\nu)\Gamma(1 - 2\nu) \, F_C^{(3)}(1 - 2\nu, 1 - \nu; 1 - \nu, 1 - \nu, 1 + \nu; z_1, z_2, z_3) \Big\} \, , (12)$$

where $z_i = m_i^2/m_3^2$, $i = 1, 2$, $z_3 = p^2/m_3^2$ and $\nu = 1 - \delta$. The above result is valid for small p^2 but it can be analytically continued to the large-p^2 region, using known analytic continuation formulae for the Lauricella function $F_C^{(3)}$.

With the dispersion method described above we derive the *small-p^2* result for T_{1234} in D dimensions. The discontinuity ΔT_{1234} is a sum of a two- and a three-particle cut, which we denote by $\Delta T_{1234}^{(2)}$ and $\Delta T_{1234}^{(3)}$, respectively. The two-particle cut is given by

$$\Delta T_{1234}^{(2)} (p^2; m_i^2) = \Delta B_0(p^2; m_1^2, m_2^2) B_0(m_1^2; m_3^2, m_4^2) \, , \tag{13}$$

where according to the Cutkosky rules m_1^2 in $B_0(m_1^2; m_3^2, m_4^2)$ is considered as $m_1^2 + i\epsilon$, when $m_1 > m_3 + m_4$. Inserting this result in the dispersion integral gives

$$T_{1234}^{(2)} (p^2; m_i^2) = \frac{1}{2\pi i} \int\limits_{(m_1+m_2)^2}^{\infty} dz \, \frac{1}{z - p^2} \Delta T_{1234}^{(2)}(z; m_i^2)$$

$$= B_0(p^2; m_1^2, m_2^2) B_0(m_1^2; m_3^2, m_4^2) \, . \tag{14}$$

The discontinuity $\Delta T_{1234}^{(3)}$ is related to the three-particle cut given in (4) but with an additional factor, which according to the Cutkosky rules is the complex conjugate of the propagator $(s - m_1^2 + i\epsilon)^{-1}$ and therefore we get the dispersion relation

$$T_{1234}^{(3)} (p^2; m_i^2) = -(4\pi\mu^2)^{2\delta} \frac{\Gamma^2(1-\delta)}{\Gamma^2(2-2\delta)} \int\limits_{(m_3+m_4)^2}^{\infty} ds \, \frac{\lambda^{\frac{1}{2}-\delta}(s, m_3^2, m_4^2)}{s^{1-\delta}(s - m_1^2 - i\epsilon)}$$

$$\times \int\limits_{(\sqrt{s}+m_2)^2}^{\infty} dz \, \frac{\lambda^{\frac{1}{2}-\delta}(s, z, m_2^2)}{z^{1-\delta}(z - p^2)} \, . \tag{15}$$

A further discussion of the analytic properties of $\Delta T_{1234}^{(2)}$, $\Delta T_{1234}^{(3)}$ and consequently $T_{1234}^{(2)}$, $T_{1234}^{(3)}$ is postponed to subsection 3.2. After performing the z-integration we obtain

$$T_{1234}^{(3)} (p^2; m_i^2) = -(4\pi\mu^2)^\delta \frac{\Gamma(1-\delta)\Gamma(\delta-1)}{\Gamma(2-2\delta)} \int\limits_{(m_3+m_4)^2}^{\infty} ds \, \frac{s^{\delta-1}}{s - m_1^2} \lambda^{\frac{1}{2}-\delta}(s, m_3^2, m_4^2)$$

$$\times \left[\frac{m_2^2}{s} \left(\frac{m_2^2}{4\pi\mu^2} \right)^{-\delta} F_4(1, 2-\delta; 2-\delta, 2-\delta; p^2/s, m_2^2/s) \right.$$

$$\left. - \left(\frac{s}{4\pi\mu^2} \right)^{-\delta} F_4(1, \delta; 2-\delta, \delta; p^2/s, m_2^2/s) \right] \, . \tag{16}$$

Expanding $(s - m_1^2)^{-1}$ in m_1^2/s and performing the integration over s one gets

$$T_{1234}^{(3)}(p^2; m_i^2) = \frac{\Gamma(1 - \delta)\Gamma(1 + \delta)}{\delta} \left(\frac{m_4^2}{4\pi\mu^2}\right)^{-2\delta}$$

$$\times \sum_{m,n,k,l=0}^{\infty} z_1^k z_2^n (1 - z_3)^l z_4^m \frac{\Gamma(1 + m + n)\Gamma(1 + \delta + m + n + k + l)}{\Gamma(2 - \delta + m)m!n!l!}$$

$$\times \left(z_2^{1-\delta} \frac{\Gamma(2 - \delta + m + n)\Gamma(2 + m + n + k + l)}{\Gamma(2 - \delta + n)\Gamma(2(m + n + k) + 4 + l)} \right.$$

$$\left. - \frac{\Gamma(\delta + m + n)\Gamma(2\delta + m + n + k + l)}{\Gamma(\delta + n)\Gamma(2(m + n + k) + 2 + 2\delta + l)} \right), \tag{17}$$

where $z_i = m_i^2/m_4^2$ with $i = 1, 2, 3$ and $z_4 = p^2/m_4^2$. Note that the contribution from the three-particle cut is written in terms of multiple series which are no longer Lauricella functions but rather belong to a special class of generalized hypergeometric functions. To our knowledge they have not been studied in the mathematical literature. One can however obtain information on the convergence region. The series in $(1 - z_3)$ can be written as $_2F_1$ functions which can be transformed into $_2F_1$ functions with variable z_3. One then obtains four quartic series in z_i. Applying the standard reasoning (see e.g. Ref. [5]) we get the following conditions for convergence:

$$m_2 + \sqrt{|p^2|} < m_1 \quad \text{and} \quad m_1 + m_3 < m_4. \tag{18}$$

With these results T_{1234} becomes

$$T_{1234}(p^2; m_i^2) = B_0(p^2; m_1^2, m_2^2) B_0(m_1^2; m_3^2, m_4^2) + T_{1234}^{(3)}(p^2; m_i^2). \tag{19}$$

Next the general small-p^2 result for T_{1234} will be expanded in δ.

The following combination of the general massive case with a massless case is chosen in such a way that the infinite parts cancel [6]:

$$T_{1234N}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = T_{1234}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) - T_{1234}(p^2; m_1^2, m_2^2, 0, 0). \tag{20}$$

An analytic form of this combination is obtained by expanding the multiple series and their coefficients in δ, where the first and the second logarithmic derivatives of the Γ-function occur at integer arguments,

$$\psi(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \qquad \psi'(n + 1) = \zeta(2) - \sum_{k=1}^{n} \frac{1}{k^2}, \tag{21}$$

with the Euler constant γ and $\zeta(2) = \pi^2/6$.

The $1/\delta^2$, $1/\delta$ and γ terms indeed drop out from the result and a finite combination of various multiple series remains and some other terms corresponding to the finite parts of

$T_{1234}(p^2; m_1^2, m_2^2, 0, 0)$ and $B_0(p^2; m_1^2, m_2^2) B_0(m_1^2; m_3^2, m_4^2)$ which are given in Ref. [7]. The result is given in the appendix.

The *large-p^2 result* has been derived using the Mellin–Barnes representation for a massive propagator,

$$
\frac{1}{(k^2 - m^2)^\alpha} = \frac{1}{\Gamma(\alpha)} \frac{1}{2\pi i} \int\limits_{-i\infty}^{+i\infty} ds \frac{(-m^2)^s}{(k^2)^{\alpha+s}} \Gamma(-s) \Gamma(\alpha + s), \tag{22}
$$

where the integration contour in the s-plane must separate the series of poles of $\Gamma(-s)$ on the right from the series of poles of $\Gamma(s + \alpha)$ on the left. In the expression for T_{1234} we apply (22) with $\alpha = 1$ to all propagators, thereby relating the general massive case to the massless one, but with the propagators raised to arbitrary powers $1 + s_i$. The required expression is well known, see e.g. Ref. [8]

$$
T_{1234}(p^2; 0, 0, 0, 0; 1 + s_1, 1 + s_2, 1 + s_3, 1 + s_4) = \frac{e^{-i\pi \sum_{i=1}^4 s_i}}{(4\pi\mu^2)^{D-4}} (-p^2)^{D-4-\sum_{i=1}^4 s_i}
$$

$$
\times \frac{\Gamma(2 + s_3 + s_4 - D/2)\Gamma(D/2 - 1 - s_2)\Gamma(D/2 - 1 - s_3)\Gamma(D/2 - 1 - s_4)}{\Gamma(1 + s_2)\Gamma(1 + s_3)\Gamma(1 + s_4)\Gamma(D - 2 - s_3 - s_4)}
$$

$$
\times \frac{\Gamma(4 - D + \sum_{i=1}^4 s_i)\Gamma(D - 3 - s_1 - s_2 - s_3)}{\Gamma(3D/2 - 4 - \sum_{i=1}^4 s_i)\Gamma(3 + s_1 + s_3 + s_4 - D/2)}. \tag{23}
$$

Closing the integration contours in a way that the convergence is guaranteed we get

$$
T_{1234}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = \left(\frac{-p^2}{4\pi\mu^2}\right)^{2\nu-2} \left\{ \sum_{i=1}^6 c_i B_i + B \right\} \tag{24}
$$

where

$$
B = (-x_4)^{2\nu-1} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} F_4(1, 1-\nu; 1+\nu, 1-\nu; x_1, x_2)
$$

$$
\times \Big[\Gamma^2(\nu)\Gamma(1 - 2\nu) F_4(1 - \nu, 1 - 2\nu; 1 - \nu, 1 - \nu; r, t)
$$

$$
+ t^\nu \Gamma(\nu)\Gamma(-\nu) F_4(1, 1 - \nu; 1 - \nu, 1 + \nu; r, t)
$$

$$
+ r^\nu \Gamma(\nu)\Gamma(-\nu) F_4(1, 1 - \nu; 1 + \nu, 1 - \nu; r, t)
$$

$$
+ r^\nu t^\nu \frac{\Gamma^2(-\nu)\Gamma(1 + \nu)}{\Gamma(1 - \nu)} F_4(1, 1 + \nu; 1 + \nu, 1 + \nu; r, t) \Big] \tag{25}
$$

with $x_i = m_i^2/p^2$ with $i = 1, 2, 3, 4$, $r = m_1^2/m_4^2$ and $t = m_3^2/m_4^2$. Between the brackets we recognize up to some factors the result for the vacuum diagram with three massive lines [9]. Furthermore the coefficients c_i are

$$c_1 = \frac{\Gamma^2(\nu)\Gamma(\nu-1)\Gamma(1-2\nu)}{\Gamma(3\nu-1)},$$

$$c_2 = (-x_2)^\nu \frac{\Gamma(\nu)\Gamma(-\nu)\Gamma(\nu-1)\Gamma(2-\nu)}{\Gamma(2\nu-1)},$$

$$c_3 = (-x_3)^\nu \frac{\Gamma(\nu)\Gamma(-\nu)\Gamma(\nu-1)\Gamma(2-\nu)}{\Gamma(2\nu-1)},$$

$$c_4 = (-x_4)^\nu \frac{\Gamma^2(\nu)\Gamma(-\nu)\Gamma(1-\nu)}{\Gamma(2\nu)},$$

$$c_5 = (-x_2)^\nu (-x_4)^\nu \Gamma^2(-\nu),$$

$$c_6 = (-x_2)^\nu (-x_3)^\nu \Gamma^2(-\nu).$$

The other terms are quartic series,

$$B_1 = \sum_{j_1=0}^\infty \cdots \sum_{j_4=0}^\infty \frac{(1)_{j_1}(1-\nu)_{j_3+j_4}(1-2\nu)_{j_3+j_4}(2-2\nu)_{j_1+j_2+j_3+j_4}(2-3\nu)_{j_1+j_2+j_3+j_4}}{(1-\nu)_{j_2}(1-\nu)_{j_3}(1-\nu)_{j_4}(2-\nu)_{j_1+j_3+j_4}(2-2\nu)_{j_1+j_3+j_4}} q,$$

$$B_2 = \sum_{j_1=0}^\infty \cdots \sum_{j_4=0}^\infty \frac{(1)_{j_1}(1-\nu)_{j_3+j_4}(1-2\nu)_{j_3+j_4}(2-2\nu)_{j_1+j_2+j_3+j_4}(2-\nu)_{j_1+j_2+j_3+j_4}}{(1+\nu)_{j_2}(1-\nu)_{j_3}(1-\nu)_{j_4}(2-\nu)_{j_1+j_3+j_4}(2-2\nu)_{j_1+j_3+j_4}} q,$$

$$B_3 = \sum_{j_1=0}^\infty \cdots \sum_{j_4=0}^\infty \frac{(1)_{j_1}(1)_{j_3+j_4}(1-\nu)_{j_3+j_4}(2-2\nu)_{j_1+j_2+j_3+j_4}(2-\nu)_{j_1+j_2+j_3+j_4}}{(1-\nu)_{j_2}(1+\nu)_{j_3}(1-\nu)_{j_4}(2-\nu)_{j_1+j_3+j_4}(2)_{j_1+j_3+j_4}} q,$$

$$B_4 = \sum_{j_1=0}^\infty \cdots \sum_{j_4=0}^\infty \frac{(1)_{j_1}(1)_{j_3+j_4}(1-\nu)_{j_3+j_4}(2-2\nu)_{j_1+j_2+j_3+j_4}(2-\nu)_{j_1+j_2+j_3+j_4}}{(1-\nu)_{j_2}(1-\nu)_{j_3}(1+\nu)_{j_4}(2-\nu)_{j_1+j_3+j_4}(2)_{j_1+j_3+j_4}} q,$$

$$B_5 = \sum_{j_1=0}^\infty \cdots \sum_{j_4=0}^\infty \frac{(1)_{j_1}(1)_{j_3+j_4}(1-\nu)_{j_3+j_4}(2)_{j_1+j_2+j_3+j_4}(2-\nu)_{j_1+j_2+j_3+j_4}}{(1+\nu)_{j_2}(1-\nu)_{j_3}(1+\nu)_{j_4}(2-\nu)_{j_1+j_3+j_4}(2)_{j_1+j_3+j_4}} q,$$

$$B_6 = \sum_{j_1=0}^\infty \cdots \sum_{j_4=0}^\infty \frac{(1)_{j_1}(1)_{j_3+j_4}(1-\nu)_{j_3+j_4}(2)_{j_1+j_2+j_3+j_4}(2-\nu)_{j_1+j_2+j_3+j_4}}{(1+\nu)_{j_2}(1+\nu)_{j_3}(1-\nu)_{j_4}(2-\nu)_{j_1+j_3+j_4}(2)_{j_1+j_3+j_4}} q, \qquad (26)$$

where

$$q = \frac{x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4}}{j_1! j_2! j_3! j_4!}. \qquad (27)$$

The large-p^2 result given by (24) is valid for

$$|p^2| > (m_2 + m_3 + m_4)^2 \quad \text{and} \quad m_1 > m_3 + m_4. \qquad (28)$$

One may wonder what the relation is between the large-p^2 expansion of (24) and that given in Ref. [8]. In the latter approach the various terms in the p^2 expansion are obtained from the expansion of subgraphs. The subgraphs are obtained by distributing

the momentum p over the propagators in all possible ways. In the case of the diagram T_{1234} one has the following subgraphs: the diagram itself, the four diagrams where one internal line is removed, the two diagrams where two internal lines are removed and one diagram where three internal lines are removed. So one expects eight subgraphs, in fact seven because the contribution from the one where the line corresponding to propagator 1 in our convention is removed is zero.

Following the analysis of Ref. [8] one can easily find the first term of each of the contributing series. For the subgraph representing the whole diagram the first term in the series should be the massless diagram. This series then corresponds to the first term in (24), i.e. B_1. The series which originates from the subgraph where two lines have been removed, e.g. 2 and 3, starts with the product of two massive tadpoles. They contribute a factor $\left(m_2^2 m_3^2\right)^\nu$ which can be identified with the sixth term in (24). The remaining subgraphs are obtained by removing one internal line, e.g. line 3. This yields a series starting with a massive tadpole proportional to $(m_3^2)^\nu$. This is the third term in (24). The last term in (24) corresponds to the diagram where three internal lines are removed whose reduced graph is the vacuum diagram. Thus the seven series in (24) can be related directly to the seven subgraphs which are required for the method of Ref. [8].

The large-p^2 expansion satisfies a system of four partial differential equations given below:

$$\left(T_1(1+D_1)x_1^{-1} - T_2(1+D_1)\right)f = 0, \tag{29}$$

$$\left((d+D_2)(1+D_2)x_2^{-1} - T_2\right)f = 0, \tag{30}$$

$$\left(T_1(d+D_3)(1+D_3)x_3^{-1} - T_2(d+D_3+D_4)(e+D_3+D_4)\right)f = 0, \tag{31}$$

$$\left(T_1(d+D_4)(1+D_4)x_4^{-1} - T_2(d+D_3+D_4)(e+D_3+D_4)\right)f = 0, \tag{32}$$

where

$$T_1 = (a + \sum_{i=1,3,4} D_i)(c + \sum_{i=1,3,4} D_i), \tag{33}$$

$$T_2 = (a + \sum_{i=1}^{4} D_i)(b + \sum_{i=1}^{4} D_i), \tag{34}$$

$a = (2-2\nu)$, $b = (2-3\nu)$, $c = (2-\nu)$, $d = (1-\nu)$, $e = (1-2\nu)$ and $D_i = x_i \partial/\partial x_i$.

The expressions for small and large p^2 are given in an arbitrary number of dimensions and for arbitrary masses. One may subsequently derive *special cases* setting masses equal to zero. Taking the small-p^2 expansion, m_3 and m_4 cannot be taken zero simultaneously. When enough masses are zero and others are equal it is not so difficult to recognize known hypergeometric functions and to perform an expansion in δ. We give the following examples.

We will use the following abbreviations:

$$L_m = \gamma + \ln\left(\frac{m^2}{4\pi\mu^2}\right), \qquad L_p = \gamma + \ln\left(\frac{-p^2}{4\pi\mu^2}\right). \tag{35}$$

In the case $m_1 = m_2 = m_3 = 0$ the three-particle cut contribution (17) reads

$$T^{(3)}_{1234}(p^2; 0, 0, 0, m^2) = \Gamma(1 - \delta)\Gamma(\delta - 1)\Gamma(2\delta)\, _3F_2(1, \delta, 2\delta; 2, 2 - \delta; x) \qquad (36)$$

and the two-particle cut contribution is given by

$$B_0(p^2; 0, 0)B_0(0; 0, m^2) = -\left(\frac{-p^2}{4\pi\mu^2}\right)^{-\delta}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\delta}\frac{\Gamma^2(1 - \delta)\Gamma(\delta)\Gamma(\delta - 1)}{\Gamma(2 - 2\delta)} \,. \quad (37)$$

Expanding the two results from above in δ we obtain

$$
\begin{aligned}
T_{1234}(p^2; 0, 0, 0, m^2) = &\frac{1}{2\delta^2} + \frac{1}{\delta}\left\{\frac{5}{2} - L_m - \ln(-x)\right\} \\
&+ \frac{19}{2} - \frac{3}{2}\zeta(2) + L_m^2 + (-5 + 2\ln(-x))L_m + \frac{1}{2}\ln^2(-x) \\
&- 3\ln(-x) - 2\frac{x - 1}{x}\ln(1 - x) - \frac{x + 1}{x}\mathrm{Li}_2(x)\,, \qquad (38)
\end{aligned}
$$

where $x = p^2/m^2$, in agreement with Ref. [7].

In the case $m_1 = m_2 = 0$ and $m_3 = m_4 = m$ the contribution from the three-particle cut (17) becomes

$$
\begin{aligned}
T^{(3)}_{1234}(p^2) = &2\frac{\Gamma(\delta - 1)\Gamma(2\delta)\Gamma(2 + \delta)}{\Gamma(3 + 2\delta)}\left(\frac{m^2}{4\pi\mu^2}\right)^{-2\delta}\,_3F_2(1, \delta, 2\delta; 2 - \delta, 3/2 + \delta; x/4) \\
= &-\frac{1}{2\delta^2} + \frac{1}{\delta}\left(\frac{1}{2} + L_m\right) - L_m^2 - L_m - \frac{3}{2} - \frac{1}{2}\zeta(2) + Y + O(\delta)\,, \qquad (39)
\end{aligned}
$$

where

$$
\begin{aligned}
Y = &-\sum_{n=1}^{\infty}\frac{\Gamma^2(n)}{(1 + n)\Gamma(2n + 2)}x^n \\
= &-\frac{x}{6}\left(\,_3F_2(1, 1, 1; 2, 5/2; x/4) - \frac{1}{2}\,_3F_2(1, 1, 2; 3, 5/2; x/4)\right)\,. \qquad (40)
\end{aligned}
$$

Knowing the analytic expression for the B_0 functions which occur in the two-particle cut contribution and expanding in δ we obtain

$$
\begin{aligned}
T_{1234}(p^2; 0, 0, m^2, m^2) = &\frac{1}{2\delta^2} + \frac{1}{\delta}\left(\frac{5}{2} - L_p\right) + \frac{19}{2} - \frac{1}{2}\zeta(2) \\
&+ L_p^2 - 5L_p + 3\ln\left(\frac{-p^2}{m^2}\right) - \frac{1}{2}\ln^2\left(\frac{-p^2}{m^2}\right) \\
&+ 3\frac{m^2}{p^2}(r_1 - r_2)\ln(r_1) + \frac{1}{2}\left(1 + \frac{2m^2}{p^2}\right)\ln^2(r_1)\,, \qquad (41)
\end{aligned}
$$

where r_1 and r_2 are the roots of the equation

$$m^2 r + \frac{m^2}{r} = 2m^2 - p^2\,, \qquad (42)$$

in agreement with Ref. [6].

When $m_2 = m_3 = 0$ and $m_1 = m_4 = m$, Eq. (17) gives

$$T_{1234}^{(3)}(p^2; m^2, 0, 0, m^2) = -\frac{\Gamma(\delta)\Gamma(2\delta)\Gamma(1-\delta)}{(1-\delta)(1-2\delta)}\left(\frac{m^2}{4\pi\mu^2}\right)^{-2\delta} {}_2F_1(2\delta, \delta; 2-\delta; x).$$

(43)

To this the contribution $B_0(p^2; m^2, 0) B_0(m^2; 0, m^2)$ should be added:

$$T_{1234}^{(2)}(p^2; m^2, 0, 0, m^2) = \frac{\Gamma^2(1+\delta)}{\delta^2(1-\delta)(1-2\delta)}\left(\frac{m^2}{4\pi\mu^2}\right)^{-2\delta} {}_2F_1(1, \delta; 2-\delta; x). \quad (44)$$

Expanding in δ we get

$$\begin{aligned}
T_{1234}(p^2; m^2, 0, 0, m^2) = {} & \frac{1}{2\delta^2} + \frac{1}{2\delta}\left\{5 - 2L_m + 2\left(\frac{1-x}{x}\right)\ln(1-x)\right\} \\
& + \frac{19}{2} - \frac{1}{2}\zeta(2) + L_m^2 - 5L_m - 2L_m\left(\frac{1-x}{x}\right)\ln(1-x) \\
& + 5\left(\frac{1-x}{x}\right)\ln(1-x) - \left(\frac{1-x}{x}\right)\ln^2(1-x) \\
& - \frac{1}{x}\text{Li}_2(x).
\end{aligned}$$

(45)

In principle there is one class of cases which has not yet been covered by the above multiple series. That is the case where $m_3 = m_4 = 0$. For completeness we give the small-p^2 case as a linear combination of an F_2 and an F_4 functions [5,10,11]. For the derivation we used the Mellin–Barnes representation method.

$$\begin{aligned}
T_{1234}(p^2; m_1^2, m_2^2, 0, 0) = {} & \left(\frac{m_2^2}{4\pi\mu^2}\right)^{-2\delta}\Gamma(1-\delta)\Gamma(\delta-1)\Gamma(2\delta-1) \\
& \times \left[(1 \pm \sqrt{y})^{-2(1+\delta)}F_2(1+\delta, 1, 3/2-\delta; 1+\delta, 3-2\delta; z_1, z_2)\right. \\
& \left. - x^{1-2\delta}F_4(1, 2-\delta; 2-\delta, 2-\delta; x, y)\right]
\end{aligned}$$

(46)

where $x = m_1^2/m_2^2$, $y = p^2/m_2^2$, $z_1 = x/(1 \pm \sqrt{y})^2$ and $z_2 = \pm 4\sqrt{y}/(1 \pm \sqrt{y})^2$.

As it was mentioned in the introduction the case T_{11234} with $m_1' = m_1$ can be obtained from T_{1234} by differentiating with respect to m_1^2. For instance, differentiating Eq. (19) leads to

$$\begin{aligned}
T_{11234}(p^2; m_1^2, m_1^2, m_2^2, m_3^2, m_4^2) = {} & \frac{\partial}{\partial m_1^2}T_{1234}^{(3)}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) \\
& + B_0(p^2; \nu_1 = 2, \nu_2 = 1; m_1^2, m_2^2)B_0(m_1^2; m_3^2, m_4^2) \\
& + B_0(p^2; m_1^2, m_2^2)\frac{\partial}{\partial m_1^2}B_0(m_1^2; m_3^2, m_4^2).
\end{aligned}$$

(47)

In this result the propagators in the B_0 functions have the usual powers $\nu_i = 1$ except when indicated $\nu_1 = 2$. One can now even take the limit $m_1 = 0$ and then expand the result in δ. In Sect. 4 more details on the products of B_0 functions and their expansion will be given.

From Eq. (47) one can obtain special cases. Taking $m_1 = 0$ and one other mass vanishing double series are obtained. In some cases one may obtain known hypergeometric functions, e.g. for $m_1 = m_2 = 0$, $m_3 = m_4 = m$, which leads to

$$T_{11234}\left(p^2; 0, 0, 0, m^2, m^2\right) = 2 \left(\frac{m^2}{4\pi\mu^2}\right)^{-2\delta} \frac{\Gamma(\delta - 1)\Gamma(1 + 2\delta)\Gamma(3 + \delta)}{\Gamma(5 + 2\delta)}$$

$$\times {}_3F_2(1, \delta, 2\delta + 1; 2 - \delta, \delta + 5/2; \frac{p^2}{4m^2})$$

$$+ \left(\frac{m^2}{4\pi\mu^2}\right)^{-\delta} \left(\frac{-p^2}{4\pi\mu^2}\right)^{-\delta} \frac{\Gamma(1 + \delta)\Gamma(\delta)\Gamma^2(1 - \delta)}{\Gamma(1 - 2\delta)} \left(\frac{1}{6m^2(1 - 2\delta)} - \frac{1}{p^2\delta}\right).$$

(48)

3. Analytic approaches and elliptic integrals

In this section we inspect the imaginary parts of the London transport diagram T_{123} and of T_{1234}. It turns out that they can be calculated in four dimensions in terms of complete elliptic integrals. These are well known functions and thus the results are of analytic interest. Furthermore fast and precise algorithms for the calculation of the elliptic integrals are available. Therefore the results provide also an efficient way to calculate the imaginary parts numerically.

3.1. Imaginary part of the London transport diagram

As can be seen from (4) the imaginary part of T_{123} is convergent in four dimensions and reads with a factorization of the Källén functions

$$\text{Im}(T_{123}(p^2)) = \frac{1}{2i}\Delta T_{123}(p^2) = -\frac{\pi}{p^2} \int_{x_2}^{x_3} \frac{dt}{t} \sqrt{(t - x_1)(t - x_2)(x_3 - t)(x_4 - t)}, \quad (49)$$

with

$$x_1 = (m_1 - m_2)^2; \quad x_2 = (m_1 + m_2)^2; \quad x_3 = (p - m_3)^2; \quad x_4 = (p + m_3)^2;$$

$$x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4; \quad p = \sqrt{|p^2|} \geqslant m_1 + m_2 + m_3.$$

The integration limits are zeros of the square roots, and thus (49) leads to complete elliptic integrals, defined by

$$K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}} = \frac{\pi}{2} \, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; x^2\right), \tag{50}$$

$$E(x) = \int_0^1 \frac{dt\,(1-x^2t^2)}{\sqrt{(1-t^2)(1-x^2t^2)}} = \frac{\pi}{2} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right), \tag{51}$$

$$\Pi(c,x) = \int_0^1 \frac{dt}{(1-ct^2)\sqrt{(1-t^2)(1-x^2t^2)}} = \frac{\pi}{2} F_1\left(\frac{1}{2}; 1, \frac{1}{2}; 1; c, x^2\right), \tag{52}$$

with the Gauss hypergeometric function $_2F_1$ and the Appell function F_1 [5,10,11]. Reduction of (49) to the Legendre normal form of the elliptic integrals [12,13] by decomposition into partial fractions and partial integration yields after some algebra

$$\text{Im}\left(T_{123}\left(p^2; m_1^2, m_2^2, m_3^2\right)\right) =$$

$$-\frac{\pi}{p^2}\Bigg\{ 4m_1m_2[(p+m_3)^2 - m_3p + m_1m_2]\sqrt{\frac{q_{--}}{q_{++}}}\, K\left(\sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right)$$

$$+ \frac{m_1^2 + m_2^2 + m_3^2 + p^2}{2}\sqrt{q_{++}q_{--}}\, E\left(\sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right)$$

$$+ \frac{8m_1m_2[(m_1^2+m_2^2)(p^2+m_3^2) - 2m_1^2m_2^2 - 2m_3^2p^2]}{\sqrt{q_{++}q_{--}}}\, \Pi\left(\frac{q_{-+}}{q_{--}}, \sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right)$$

$$- \frac{8m_1m_2(p^2-m_3^2)^2}{\sqrt{q_{++}q_{--}}}\, \Pi\left(\frac{(m_1-m_2)^2}{(m_1+m_2)^2}\frac{q_{-+}}{q_{--}}, \sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right)\Bigg\}$$

$$\times \Theta\left(p^2 - (m_1 + m_2 + m_3)^2\right), \tag{53}$$

with variables $q_{\pm\pm}$ corresponding to the physical and unphysical thresholds

$$q_{\pm\pm} := (p \pm m_3)^2 - (m_1 \pm m_2)^2. \tag{54}$$

This result is valid in all parameter regions. In special cases it leads to simpler formulae. For equal masses one gets

$$\text{Im}(T_{123}\left(p^2; m^2, m^2, m^2\right)) = -\frac{\pi}{p^2}\sqrt{(p-m)(p+3m)}$$

$$\times \left\{ -4m^2p\ K(\kappa) + \frac{(p-m)(p^2+3m^2)}{2}\ E(\kappa)\right\}\Theta(p^2 - 9m^2), \tag{55}$$

with
$$\kappa^2 := \frac{(p+m)^3(p-3m)}{(p-m)^3(p+3m)}, \tag{56}$$

involving only complete elliptic integrals of the first and second kind, i.e. $_2F_1$ Gauss hypergeometric functions. If at least one mass is zero, $\text{Im}(T_{123})$ reduces to logarithms. The most complicated case leads to

$$\text{Im}(T_{123}(p^2; 0, m_2^2, m_3^2)) = -\frac{\pi}{p^2}\left\{ \frac{p^2 + m_2^2 + m_3^2}{2}\sqrt{q_+ q_-} \right.$$

$$+ (2m_3^2 p^2 - m_2^2(m_3^2 + p^2))\log(\frac{p^2 + m_3^2 - m_2^2 - \sqrt{q_+ q_-}}{2m_3 p})$$

$$+ m_2^2(p^2 - m_3^2)\log(\frac{2m_2^2 m_3 p}{(p^2 - m_3)^2 - m_2^2(m_3^2 + p^2) + (p^2 - m_3^2)\sqrt{q_+ q_-}}) \left. \right\}$$

$$\times \Theta(p^2 - (m_2 + m_3)^2), \tag{57}$$

with $q_\pm := (p \pm m_3)^2 - m_2^2$.

3.2. The imaginary part of T_{1234}

The two-particle cut contribution to the discontinuity of T_{1234} was given in (13),

$$\Delta T_{1234}^{(2)}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = \Delta B_0(p^2; m_1^2, m_2^2) B_0(m_1^2 + i\epsilon; m_3^2, m_4^2). \tag{58}$$

As a product of a one-loop self-energy integral and a one-loop self-energy discontinuity it is composed of elementary functions and gets a real part for

$$(m_3 + m_4)^2 < m_1^2 \quad \text{and} \quad (m_1 + m_2)^2 < p^2. \tag{59}$$

The three particle cut contribution can be calculated in a fashion very similar to the case of the London transport diagram. Only one more (complex conjugated) propagator $1/(t - m_1^2 - i\epsilon)$ has to be added in (49). The calculation yields

$$\Delta T_{1234}^{(3)}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) =$$

$$\frac{2\pi i}{p^2}\left\{ 4m_3 m_4 \sqrt{\frac{q_{--}}{q_{++}}}\, K\left(\sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right) + \sqrt{q_{++}q_{--}}\, E\left(\sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right) \right.$$

$$+ \frac{8m_3 m_4(p^2 - m_1^2 + m_2^2 + m_3^2 + m_4^2)}{\sqrt{q_{++}q_{--}}}\, \Pi\left(\frac{q_{-+}}{q_{--}}, \sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right)$$

$$- \frac{8m_3 m_4(p^2 - m_2^2)^2}{m_1^2\sqrt{q_{++}q_{--}}}\, \Pi\left(\frac{(m_3 - m_4)^2}{(m_3 + m_4)^2}\frac{q_{-+}}{q_{--}}, \sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right)$$

$$+ \frac{8m_3 m_4 \lambda(p^2, m_1^2, m_2^2)}{m_1^2\sqrt{q_{++}q_{--}}}\, \Pi\left(\frac{m_1^2 - (m_3 - m_4)^2}{m_1^2 - (m_3 + m_4)^2}\frac{q_{-+}}{q_{--}} - i\epsilon, \sqrt{\frac{q_{+-}q_{-+}}{q_{++}q_{--}}}\right) \left. \right\}$$

$$\times \Theta(p^2 - (m_2 + m_3 + m_4)^2), \tag{60}$$

with

$$q_{\pm\pm} := (p \pm m_2)^2 - (m_3 \pm m_4)^2. \tag{61}$$

In the case (59) the characteristic c of the last Π-function in (60) is greater than 1,

$$c = \frac{m_1^2 - (m_3 - m_4)^2}{m_1^2 - (m_3 + m_4)^2} \frac{q_{-+}}{q_{--}} > 1 \, , \tag{62}$$

which requires an analytic continuation of that function. A comprehensive discussion of the analytic properties of the elliptic integrals can be found in Ref. [14]. The $m_1^2 + i\epsilon$ prescription in (60) ensures that $\Delta T_{1234}^{(3)}$ gets the correct real part, given through

$$\mathrm{Im}\,(\Pi(c - i\epsilon, \kappa)) = \frac{1}{2i} \, (\Pi(c - i\epsilon, \kappa) - \Pi(c + i\epsilon, \kappa))$$

$$= -\frac{\pi}{2} \sqrt{\frac{c}{(c - 1)(c - \kappa^2)}} \, . \tag{63}$$

This contribution cancels the real part of the two-particle cut $\Delta T_{1234}^{(2)}$. Consequently ΔT_{1234} is always purely imaginary.

Numerical checks show the agreement of the results of (53) for $\mathrm{Im}(T_{123})$ and of

$$\mathrm{Im}(T_{1234}) = \frac{1}{2i} \left(\Delta T_{1234}^{(2)} + \Delta T_{1234}^{(3)} \right) \tag{64}$$

with previously published tables [3,6].

4. One-dimensional integral representations

4.1. A general approach to two-loop integrals containing a self-energy subloop

An alternative method to the series expansion of the two-loop scalar diagrams consists in the derivation of one-dimensional integral representations. These are built up from one-loop self-energy functions B_0 coming from the self-energy subloop and the remaining one-loop integral. They can be derived by using a dispersion representation of the B_0 function.

A two-loop diagram with only three-vertices
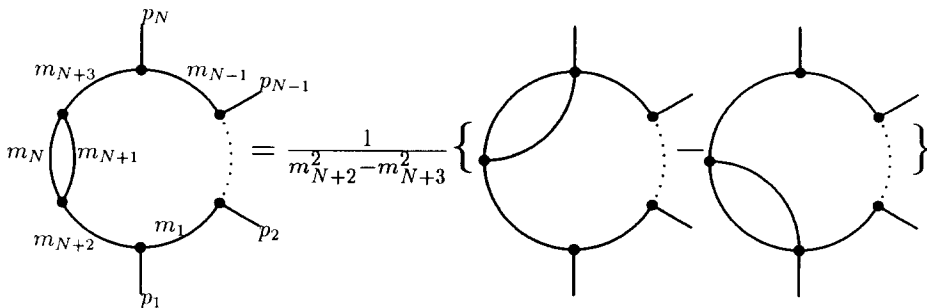
where k is the momentum flowing through the self-energy insertion, can in a first step be reduced to simpler diagrams by a decomposition into partial fractions

$$\frac{1}{k^2 - m_{N+2}^2} \frac{1}{k^2 - m_{N+3}^2} = \frac{1}{m_{N+2}^2 - m_{N+3}^2} \left(\frac{1}{k^2 - m_{N+2}^2} - \frac{1}{k^2 - m_{N+3}^2} \right) .$$

This yields for the diagram

$$T_{1\ldots N+3}(p_i; m_i^2) = \frac{1}{m_{N+2}^2 - m_{N+3}^2} \left(T_{1\ldots N+2}(p_i; m_1^2, \ldots, m_{N+1}^2, m_{N+2}^2) \right.$$

$$\left. - T_{1\ldots N+2}(p_i; m_1^2, \ldots, m_{N+1}^2, m_{N+3}^2) \right) . \qquad (65)$$

The difference has to be replaced by a derivative if $m_{N+2}^2 = m_{N+3}^2$.

Insertion of the dispersion representation for the self-energy subloop leads to

$$T_{1\ldots N+2}(p_i; m_1^2, \ldots, m_{N+1}^2, m_{N+2}^2)$$

$$= \left\langle B_0(k^2; m_N^2, m_{N+1}^2) \frac{1}{(k + p_1)^2 - m_1^2} \cdots \right.$$

$$\left. \times \frac{1}{(k + p_1 + \ldots + p_{N-1})^2 - m_{N-1}^2} \frac{1}{k^2 - m_{N+2}^2} \right\rangle$$

$$= \frac{1}{2\pi i} \int_{s_0}^{\infty} \mathrm{d}s \, \Delta B_0(s; m_N^2, m_{N+1}^2) \left\langle \frac{-1}{k^2 - s + i\epsilon} \right.$$

$$\left. \times \frac{1}{k^2 - m_{N+2}^2} \frac{1}{(k + p_1)^2 - m_1^2} \cdots \frac{1}{(k + p_1 + \ldots + p_{N-1})^2 - m_{N-1}^2} \right\rangle \qquad (66)$$

with $s_0 = (m_N + m_{N+1})^2$. After a further decomposition into partial fractions,

$$\frac{-1}{k^2 - s} \frac{1}{k^2 - m_{N+2}^2} = \frac{1}{s - m_{N+2}^2} \left(\frac{1}{k^2 - m_{N+2}^2} - \frac{1}{k^2 - s} \right) , \qquad (67)$$

the k-integrations and one of the s-integrations can be performed and yield

$$T_{1\ldots N+2}(p_i; m_1^2, \ldots, m_{N+1}^2, m_{N+2}^2)$$

$$= B_0(m_{N+2}^2; m_N^2, m_{N+1}^2) T^{(1)}(p_i; m_1^2, \ldots, m_{N-1}^2, m_{N+2}^2)$$

$$- \frac{1}{2\pi i} \int_{s_0}^{\infty} \mathrm{d}s \frac{\Delta B_0(s, m_N^2, m_{N+1}^2)}{s - m_{N+2}^2} T^{(1)}(p_i; m_1^2, \ldots, m_{N-1}^2, s) . \qquad (68)$$

$T^{(1)}$ denotes a one-loop N-point function in which s enters in the remaining one-dimensional integration as a mass variable.

A diagram with two four-vertices leads to a result which is similar to the remaining integration in (68),
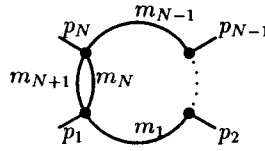
$$T_{1...N+1}(p_i; m_i^2) = \frac{1}{2\pi i} \int\limits_{s_0}^{\infty} ds\, \Delta B_0(s; m_N^2, m_{N+1}^2)$$

$$\times \left\langle \frac{-1}{k^2 - s} \frac{1}{(k+p_1)^2 - m_1^2} \cdots \frac{1}{(k+p_1+\ldots+p_{N-1})^2 - m_{N-1}^2} \right\rangle$$

$$= -\frac{1}{2\pi i} \int\limits_{s_0}^{\infty} ds\, \Delta B_0(s; m_N^2, m_{N+1}^2) T^{(1)}(p_i; m_1^2, \ldots, m_{N-1}^2, s) . \qquad (69)$$

4.2. Examples

An application of (69) to the London transport diagram leads to

$$T_{123}(p^2; m_1^2, m_2^2, m_3^2) = -\frac{1}{2\pi i} \int\limits_{(m_2+m_3)^2}^{\infty} ds\, \Delta B_0(s; m_2^2, m_3^2)\, B_0(p^2; s, m_1^2) , \qquad (70)$$

a result which would also follow from (5). In that case a suitable subtraction [6] is

$$T_{123N}(p^2; m_1^2, m_2^2, m_3^2) = T_{123}(p^2; m_1^2, m_2^2, m_3^2) - T_{123}(p^2; m_1^2, 0, m_3^2)$$

$$-T_{123}(p^2; 0, m_2^2, m_3^2) + T_{123}(p^2; 0, 0, m_3^2) . \qquad (71)$$

For T_{1234} one obtains from (68)

$$T_{1234}(p^2; m_1^2, m_2^2, m_3^2, m_4^2)$$

$$= \frac{1}{2\pi i} \int\limits_{(m_3+m_4)^2}^{\infty} ds\, \frac{\Delta B_0(s; m_3^2, m_4^2)}{s - m_1^2 - i\epsilon} \left(B_0(p^2; m_1^2, m_2^2) - B_0(p^2; s, m_2^2) \right) \qquad (72)$$

$$= B_0(m_1^2 + i\epsilon; m_3^2, m_4^2) B_0(p^2; m_1^2, m_2^2)$$

$$- \frac{1}{2\pi i} \int\limits_{(m_3+m_4)^2}^{\infty} ds\, \frac{\Delta B_0(s; m_3^2, m_4^2)}{s - m_1^2 - i\epsilon} B_0(p^2; s, m_2^2) . \qquad (73)$$

The representations (70) and (73) with the subtractions (71) and (20) provide efficient ways to calculate T_{123N} and T_{1234N} in all parameter regions. The results agree numerically with those published in Ref. [3].

One may also consider vertex functions, for example
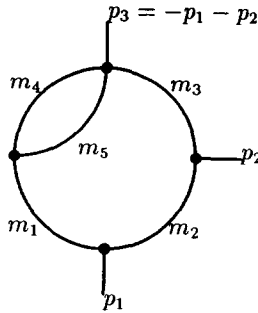
$$T(p_1^2, p_2^2, p_3^2; m_1^2, \ldots, m_5^2) = B_0(m_1^2; m_4^2, m_5^2)\, C_0(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2)$$

$$-\frac{1}{2\pi i} \int\limits_{s_0}^{\infty} ds\, \frac{\Delta B_0(s; m_4^2, m_5^2)}{s - m_1^2}\, C_0(p_1^2, p_2^2, p_3^2; s, m_2^2, m_3^2) \,. \quad (74)$$

In that case the counterterm of the self-energy subloop can be subtracted for a numerical evaluation.

In the cases of T_{123N} and T_{1234N} equivalent one-dimensional integral representations can be obtained with a direct integration in the momentum space or using the dispersion representation.

The case T_{11234} for $m_1 = 0$ gets the representation

$$T_{11234}(p^2; 0, 0, m_2^2, m_3^2, m_4^2) = B_0(p^2; \nu_1 = 2, \nu_2 = 1; 0, m_2^2)\, B_0(0; m_3^2, m_4^2)$$

$$-\frac{1}{2\pi i} \int\limits_{(m_3+m_4)^2}^{\infty} ds\, \frac{\Delta B_0(s; m_3^2, m_4^2)}{s^2} \left[B_0(p^2; s, m_2^2) - B_0(p^2; 0, m_2^2) \right] \,. \quad (75)$$

The integral is convergent and the result in an arbitrary number of dimensions for the product $B_0 B_0$ which we denote by Z is given by

$$Z = \frac{1}{m_2^2} \left(\frac{m_2^2}{4\pi\mu^2} \right)^{-\delta} \left(\frac{m_4^2}{4\pi\mu^2} \right)^{-\delta} \left(1 - \frac{m_3^2}{m_4^2} \right)^{-1} \left[1 - \left(\frac{m_3^2}{m_4^2} \right)^{1-\delta} \right]$$

$$\times \frac{\Gamma^2(1+\delta)}{\delta^2(1-\delta)^2}\, {}_2F_1(1+\delta, 2; 2-\delta; p^2/m_2^2). \quad (76)$$

One can perform the expansion of Z in δ and obtains

$$Z = \frac{(1-x)^{-1}}{m_2^2} \left(1 - \frac{m_3^2}{m_4^2} \right)^{-1} \left\{ \frac{1}{\delta^2} - \frac{1}{\delta} \left(L_{m_4} + a + b - 2 \right) + 3 + \zeta(2) + X \right.$$

$$\left. + a \left(L_{m_4} + b - 2 \right) + \frac{1}{2} \left(L_{m_4} + b \right)^2 - 2 \left(L_{m_4} + b \right) - \frac{m_3^2}{m_4^2} \left[L_{m_4} \to L_{m_3} \right] \right\}, \quad (77)$$

where $x = p^2/m_2^2, a = 1 + (1/x - 1) \ln(1 - x), b = L_{m_2} + 2\ln(1 - x)$ and

$$X = \left(\frac{1-x}{x}\right)\ln(1-x) - \left(\frac{1-x}{x}\right)\ln^2(1-x) + \left(\frac{1+x}{x}\right)\text{Li}_2(x). \tag{78}$$

In the above representation masses can be set to zero, except the case $m_3 = m_4 = 0$. However, for this case an explicit result can be derived,

$$T_{11234}(p^2; 0, 0, m^2, 0, 0) = \frac{1}{2m^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-2\delta}\frac{\Gamma(1+\delta)\Gamma^2(-\delta)\Gamma(1+2\delta)}{(1-2\delta)\Gamma(2-\delta)}$$
$$\times {}_2F_1(1+2\delta, 2+\delta; 2-\delta; x) \tag{79}$$

with $x = p^2/m^2$. Expanding the result around $D = 4$ we get

$$T_{11234}(p^2; 0, 0, m^2, 0, 0) = \frac{1}{2m^2(1-x)}\left\{\frac{1}{\delta^2} - \frac{2}{\delta}\left(c + d - \frac{3}{2}\right) + 9 + 3\zeta(2)\right.$$
$$+2d^2 - 6d + 4c\left(d - \frac{3}{2}\right) + 4\left(\frac{x-1}{x}\right)\ln^2(1-x)$$
$$\left.-4\left(\frac{x-1}{x}\right)\ln(1-x) + 2\left(\frac{2x+1}{x}\right)\text{Li}_2(x)\right\}, \tag{80}$$

where $c = 1 + (1/x - 1)\ln(1-x)$ and $d = L_m + 2\ln(1-x)$.

For the other cases where some masses are zero the integral can also be calculated analytically. A simple case is

$$p^2 T_{11234}(p^2; 0, 0, 0, m^2, 0) = -\frac{1}{\delta^2} + \frac{1}{\delta}\{-1 + L_p + L_m\} - \frac{3}{2} - \frac{1}{2}(L_p + L_m)^2$$
$$+L_p + L_m - \frac{1}{2}x\ln(-x) + \text{Li}_2(x)$$
$$+\frac{1}{2}\left(x - \frac{1}{x}\right)\ln(1-x) \tag{81}$$

where $x = p^2/m^2$, in agreement with Ref. [7].

5. Numerical comparisons and conclusions

In this section numerical comparisons are made between the various results of this paper and the literature. Moreover some concluding remarks and an outlook are presented. Various numerical evaluations for small p^2 are presented in Table 1. In the first column are values from the two-dimensional numerical integral representation [6], in the second results from the series obtained from the hypergeometric functions (appendix), whereas column three uses three terms of the Taylor expansion of [15] and finally the last column is calculated from the one-dimensional integral of Sect. 4. The comparison gives confidence in the various calculational methods.

In conclusion the analytic results in terms of generalized hypergeometric functions offer the possibility to use well-known mathematical techniques for analytic continuation, partial differential equations, contour integral representations, etc. Moreover the result is

Table 1
Real part of T_{1234N} $(p^2; m_1^2, m_2^2, m_3^2, m_4^2)$ for small values of p^2. The masses are $m_1 = 1$, $m_2 = 3$, $m_3 = 5$ and $m_4 = 7$. A: Result from VEGAS integration. B: Result of Eq. (A.5). C: Approximation using 3 terms of Taylor expansion. D: One-dimensional integral representation (73)

p^2	A	B	C	D
2.66667	−8.454752	−8.45044	−8.445577	−8.45038
1.77778	−8.286016	−8.28753	−8.286116	−8.28747
1.18519	−8.186514	−8.18486	−8.184417	−8.18481
0.790124	−8.116017	−8.11883	−8.118665	−8.11878
0.526749	−8.077571	−8.07583	−8.075740	−8.07577
0.351166	−8.053491	−8.04759	−8.047528	−8.04754
0.234111	−8.033367	−8.02896	−8.028900	−8.02890
0.156074	−8.013727	−8.01662	−8.016561	−8.01656
0.104049	−8.018667	−8.00843	−8.008371	−8.00837
0.069366	−7.999787	−8.00298	−8.002926	−8.00292
0.046244	−7.997535	−7.99936	−7.999303	−7.99930
0.030829	−7.985865	−7.99695	−7.996891	−7.99689
0.020553	−8.006871	−7.99534	−7.995285	−7.99528
0.013702	−7.900682	−7.99427	−7.994214	−7.99421
0.009135	−7.957410	−7.99356	−7.993501	−7.99350
0.006090	−7.793835	−7.99308	−7.993026	−7.99302

derived for arbitrary masses from which one can get many special cases. In other words the several formulas for the cases with vanishing masses [7] are unified in one result in D dimensions. When one is only interested in the imaginary parts an alternative analytic result in four dimensions is obtained in terms of complete elliptic integrals. Since their properties are well known they are easily accessible for numerical evaluations. Finally we have derived a one-dimensional integral representation for all two-loop diagrams containing a self-energy insertion. For the two-loop self-energy diagrams treated in this paper the integrand is composed of elementary functions only and the representation is valid for all values of p^2. The main application of these integrals in this paper is for numerical evaluations, giving a good alternative to the existing two-dimensional integrals. In order to complete the treatment of massive two-loop self-energy diagrams one has to make the similar study of the master diagram. As to the analytic approaches the prospects are not so good since it is not of the self-energy insertion type. In practice this means for instance that the massless case with arbitrary powers of the propagators is not available in the literature. This prevents an application of the Mellin–Barnes representation. On the other hand symmetry properties, special cases and imaginary parts have been studied in [16]. The two-dimensional integral representation was derived in particular for this diagram [1]. Nevertheless further studies will be needed. Once adequate techniques for the self-energy diagrams are available one could envisage practical applications for physics predictions. For the electroweak theory the obvious application is to the gauge boson self-energies which play a role in the $M_W - M_Z$ mass relation and details of the Z line shape. For QED the two-loop vacuum polarization has been known for a long time [17], but the electron two-loop self-energy was never fully calculated.

Acknowledgements

Appendix A

Here we give the result for the small-p^2 expansion in δ. We introduce the dimensionless variables x, y, u and v defined by

$$x = \frac{p^2}{m_2^2}, \quad y = \frac{m_1^2}{m_2^2}, \tag{A.1}$$

and

$$u = \frac{m_1^2}{m_4^2}, \quad v = \frac{m_3^2}{m_4^2}. \tag{A.2}$$

We also need the roots r_1 and r_2 of the equation

$$m_2^2 r + \frac{m_1^2}{r} = m_1^2 + m_2^2 - p^2 \tag{A.3}$$

and the roots r_3 and r_4 of the equation

$$m_4^2 r + \frac{m_3^2}{r} = m_3^2 + m_4^2 - m_1^2. \tag{A.4}$$

$$T_{1234N}\left(p^2; m_1^2, m_2^2, m_3^2, m_4^2\right) = -A1 + A2 + C1 + C2 + C3$$
$$+C4 + C5 + C6 + C7 \tag{A.5}$$

where

$$A1 = 8 - 2\zeta(2) + \frac{1}{2}\left(1 + \frac{y-1}{x}\right)\ln^2(y)$$

$$+ \left(-2\left(1 + \frac{y-1}{x}\right) - \frac{3}{4}\frac{r_1 - r_2}{x}\left(\ln(r_1) - \ln(r_2)\right)\right)\ln(y)$$

$$+ \frac{1-x}{x}\ln(1-x) + \frac{1}{2}\left(\frac{1-y+x}{x}\right)\text{Li}_2(x)$$

$$+ \frac{1}{2}\frac{r_1 - r_2}{x}\left\{4\ln(r_1) - 4\ln(r_2)\right.$$

$$+ \ln\left(\frac{1-r_1}{r_2 - r_1}\right)\ln\left(\frac{r_1(1-r_2)}{r_1 - r_2}\right) - \ln\left(\frac{1-r_2}{r_1 - r_2}\right)\ln\left(\frac{r_2(1-r_1)}{r_2 - r_1}\right)$$

$$+ \text{Li}_2\left(\frac{r_1(1-r_2)}{r_1 - r_2}\right) - \text{Li}_2\left(\frac{r_2(1-r_1)}{r_2 - r_1}\right)$$

$$-\mathrm{Li}_2\left(\frac{1-r_2}{r_1-r_2}\right)+\mathrm{Li}_2\left(\frac{1-r_1}{r_2-r_1}\right)-\mathrm{Li}_2(1-r_1)+\mathrm{Li}_2(1-r_2)$$

$$-\mathrm{Li}_2\left(\frac{r_2(1-r_1)}{-r_1}\right)-\eta\left(1-x,\frac{1}{r_1}\right)\ln\left(\frac{r_2(1-r_1)}{-r_1}\right)$$

$$+\mathrm{Li}_2\left(\frac{r_1(1-r_2)}{-r_2}\right)+\eta\left(1-x,\frac{1}{r_2}\right)\ln\left(\frac{r_1(1-r_2)}{-r_2}\right)\Bigg\}. \tag{A.6}$$

$$A2=\left\{-2+\frac{1}{2}\frac{y-1}{x}\ln(y)-\frac{1}{2}\frac{r_1-r_2}{x}\left(\ln(r_1)-\ln(r_2)\right)\right\}$$

$$\times\left\{-2+\frac{1}{2}\frac{v-1}{u}\ln(v)-\frac{1}{2}\frac{r_3-r_4}{u}\left(\ln(r_3)-\ln(r_4)\right)\right\}$$

$$+\frac{1}{2}\left\{-2+\frac{1}{2}\frac{y-1}{x}\ln(y)-\frac{1}{2}\frac{r_1-r_2}{x}\left(\ln(r_1)-\ln(r_2)\right)\right\}\left\{\log\frac{m_1^2}{m_2^2}+\log\frac{m_3^2}{m_2^2}+\log\frac{m_4^2}{m_2^2}\right\}$$

$$+\frac{1}{2}\left\{-2+\frac{1}{2}\frac{v-1}{u}\ln(v)-\frac{1}{2}\frac{r_3-r_4}{u}\left(\ln(r_3)-\ln(r_4)\right)\right\}\left\{\log\frac{m_1^2}{m_4^2}+\log\frac{m_2^2}{m_4^2}+\log\frac{m_3^2}{m_4^2}\right\}$$

$$+\frac{1}{8}\log^2\frac{m_3^2}{m_4^2}+\frac{1}{8}\log^2\frac{m_1^2}{m_2^2}+\frac{1}{4}\log\frac{m_1^2}{m_3^2}\log\frac{m_4^2}{m_2^2}+\frac{1}{4}\log\frac{m_1^2}{m_4^2}\log\frac{m_3^2}{m_2^2}+\log\frac{m_2^2}{m_4^2}$$

$$+\frac{1}{2}\Bigg\{\zeta(2)+8+\frac{1}{4}\ln^2(y)$$

$$-2\frac{y-1}{x}\ln(y)+\frac{r_1-r_2}{x}\left(2\ln(r_1)-2\ln(r_2)\right.$$

$$+\ln\left(\frac{1-r_1}{r_2-r_1}\right)\ln\left(\frac{r_1(1-r_2)}{r_1-r_2}\right)-\ln\left(\frac{1-r_2}{r_1-r_2}\right)\ln\left(\frac{r_2(1-r_1)}{r_2-r_1}\right)$$

$$+\mathrm{Li}_2\left(\frac{r_1(1-r_2)}{r_1-r_2}\right)-\mathrm{Li}_2\left(\frac{r_2(1-r_1)}{r_2-r_1}\right)-\mathrm{Li}_2\left(\frac{1-r_2}{r_1-r_2}\right)+\mathrm{Li}_2\left(\frac{1-r_1}{r_2-r_1}\right)\Bigg)\Bigg\}$$

$$+\frac{1}{2}\Bigg\{\zeta(2)+8+\frac{1}{4}\ln^2(v)$$

$$-2\frac{v-1}{u}\ln(v)+\frac{r_3-r_4}{u}\left(2\ln(r_3)-2\ln(r_4)\right.$$

$$+\ln\left(\frac{1-r_3}{r_4-r_3}\right)\ln\left(\frac{r_3(1-r_4)}{r_3-r_4}\right)-\ln\left(\frac{1-r_4}{r_3-r_4}\right)\ln\left(\frac{r_4(1-r_3)}{r_4-r_3}\right)$$

$$+\mathrm{Li}_2\left(\frac{r_3(1-r_4)}{r_3-r_4}\right)-\mathrm{Li}_2\left(\frac{r_4(1-r_3)}{r_4-r_3}\right)-\mathrm{Li}_2\left(\frac{1-r_4}{r_3-r_4}\right)+\mathrm{Li}_2\left(\frac{1-r_3}{r_4-r_3}\right)\Bigg)\Bigg\}. \tag{A.7}$$

$$C1 = \sum_{\substack{m,n=1 \\ k,l=0}}^{\infty} \frac{(m+n)!(m+n-1)!(m+n+k+l-1)!(m+n+k+l)!}{m!(m+1)!n!(n-1)!l!(2(m+n+k)+l+1)!}$$

$$\times \left[-\psi(m+n) - 2\psi(m+n+k+l) - \psi(m+n+k+l+1) \right.$$

$$\left. -\psi(m+2) + \psi(n) + 2\psi(2(m+n+k)+2+l) \right] z_1^k z_2^n (1-z_3)^l z_4^m. \quad (A.8)$$

$$C2 = \sum_{m,n,k,l=0}^{\infty} \frac{(m+n)!(m+n+1)!(m+n+k+l+1)!(m+n+k+l)!}{m!(m+1)!n!(n+1)!l!(2(m+n+k)+l+3)!}$$

$$\times \left[\log(m_4^2/m_2^2) + \psi(m+n+k+l+1) + \psi(m+2) \right.$$

$$\left. +\psi(n+2) - \psi(m+n+2) \right] z_1^k z_2^{n+1} (1-z_3)^l z_4^m. \quad (A.9)$$

$$C3 = -\sum_{\substack{k,l=0 \\ m=1}}^{\infty} \frac{(m-1)!(m+k+l-1)!(m+k+l)!}{(m+1)!l!(2(m+k)+l+1)} z_1^k (1-z_3)^l z_4^m \quad (A.10)$$

$$C4 = \sum_{\substack{k,l=0 \\ n=1}}^{\infty} \frac{(n+k+l-1)!(n+k+l)!}{l!(2(n+k)+l+1)!} z_1^k z_2^n (1-z_3)^l$$

$$\times \left[-1 + 2\psi(2(n+k)+l+2) - 2\psi(n+k+l) - \psi(n+k+l+1) \right]. \quad (A.11)$$

$$C5 = \sum_{k,l=1}^{\infty} \frac{(k+l-1)!(k+l)!}{l!(2k+l+1)!} z_1^k (1-z_3)^l \left[-1 - 2\psi(k+l) \right.$$

$$\left. -\psi(k+l+1) + 2\psi(2k+l+2) \right]. \quad (A.12)$$

$$C6 = \sum_{k=1}^{\infty} \frac{k!(k-1)!}{(2k+1)!} z_1^k \left[-1 - 2\psi(k) - \psi(k+1) + 2\psi(2k+2) \right] \quad (A.13)$$

$$C7 = \sum_{l=1}^{\infty} \frac{(l-1)!}{(l+1)!} (1-z_3)^l \left[-1 - 2\psi(l) - \psi(l+1) + 2\psi(l+2) \right] \quad (A.14)$$

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