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Convergence of modified approximants associated with orthogonal rational functions

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Abstract

Let $\{\alpha_n\}$ be a sequence in the unit disk $\boldsymbol{D} = \{z \in \boldsymbol{C}: |z| < 1\}$ consisting of a finite number of points cyclically repeated, and let $\mathscr{L}$ be the linear space generated by the functions $B_n(z) = \prod_{k=0}^{n} - \alpha_k(z - \alpha_k)/|\alpha_k|(1 - \bar{\alpha}_k z)$. Let $\{\varphi_n(z)\}$ be orthogonal rational functions obtained from the sequence $\{B_n(z)\}$ (orthogonalization with respect to a given functional on $\mathscr{L}$), and let $\{\psi_n(z)\}$ be the corresponding functions of the second kind (with superstar transforms $\varphi_n^*(z)$ and $\psi_n^*(z)$ respectively). Interpolation and convergence properties of the modified approximants $R_n(z, u_n, v_n) = (u_n \psi_n(z) - v_n \psi_n^*(z))/(u_n \varphi_n(z) + v_n \varphi_n^*(z))$ that satisfy $|u_n| = |v_n|$ are discussed.

Keywords: Orthogonal rational functions; Rational interpolation

1. Preliminaries

We shall use the notation $\boldsymbol{T} = \{z \in \boldsymbol{C}: |z| = 1\}, \boldsymbol{D} = \{z \in \boldsymbol{C}: |z| < 1\}$ for the unit circle and the unit disk. The kernel $D(t, z)$ is defined by

$$D(t, z) = \frac{t + z}{t - z}. \tag{1.1}$$

Let μ be a finite Borel measure on $[-\pi, \pi]$. The integral transform Ω_μ is defined as the Carathéodory function

$$\Omega_\mu(z) = \int_{\boldsymbol{T}} D(t, z)\, \mathrm{d}\mu(t). \tag{1.2}$$

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(We use the simplified notation above for $\int_{-\pi}^{\pi} D(e^{i\theta}, z)\, d\mu(\theta)$, and analogously in similar cases.)

The real part of a Carathéodory function is a positive harmonic function in $\boldsymbol{D}$, and vice versa. (Recall the Riesz–Herglotz representation theorem. Note that the real part of the kernel $D(t, z)$ is the Poisson kernel.)

The substar conjugate f_* of a function f is defined as

$$f_*(z) = \overline{f(1/\bar{z})}. \tag{1.3}$$

When f is a rational function or a series expansion, this may also be written as

$$f_*(z) = \bar{f}(1/z), \tag{1.4}$$

where the bar denotes conjugation of the coefficients. The inner product $\langle\ ,\ \rangle_\mu$ is defined on $\boldsymbol{C}(\boldsymbol{T}) \times \boldsymbol{C}(\boldsymbol{T})$ by

$$\langle f, g \rangle_\mu = \int_{\boldsymbol{T}} f(t)\overline{g(t)}\, d\mu(t) = \int_{\boldsymbol{T}} f(t) g_*(t)\, d\mu(t). \tag{1.5}$$

Let $\{\alpha_n\colon n = 1, 2, \ldots\}$ be an arbitrary sequence of (not necessarily distinct) points (interpolation points) in $\boldsymbol{D}$. We define the Blaschke factor $\zeta_n(z)$ as the function

$$\zeta_n(z) = \frac{\overline{\alpha_n}}{|\alpha_n|} \frac{(\alpha_n - z)}{(1 - \bar{\alpha}_n z)}, \quad n = 1, 2, \ldots . \tag{1.6}$$

(Here $\overline{\alpha_n}/|\alpha_n| = -1$ if $\alpha_n = 0$.) We also define

$$\pi_0(z) = 1, \qquad \pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha_k} z), \quad n = 1, 2, \ldots , \tag{1.7}$$

$$\omega_0(z) = 1, \qquad \omega_n(z) = \prod_{k=1}^{n} (z - \alpha_k), \quad n = 1, 2, \ldots . \tag{1.8}$$

The Blaschke products $B_n(z)$ are defined by

$$B_0(z) = 1, \qquad B_n(z) = \prod_{k=1}^{n} \zeta_k(z) = \eta_n \frac{\omega_n(z)}{\pi_n(z)}, \quad n = 1, 2, \ldots , \tag{1.9}$$

where

$$\eta_n = (-1)^n \prod_{k=1}^{n} \frac{\overline{\alpha_k}}{|\alpha_k|}. \tag{1.10}$$

We shall also make use of the functions $B_{n\backslash k}(z)$ defined by

$$B_{n\backslash n}(z) = 1, \quad B_{n\backslash k}(z) = B_n(z)/B_k(z) = \prod_{j=k+1}^{n} \zeta_j(z) \quad \text{for } 0 \leqslant k < n, \qquad n = 1, 2, \ldots. \tag{1.11}$$

(The product means the constant 1 when $k = n$.)

We define the spaces $\mathscr{L}_n$ and $\mathscr{L}_{n*}$ by

$$\mathscr{L}_n = \operatorname{Span}\{B_k\colon k = 0, 1, \ldots, n\}, \tag{1.12}$$

$$\mathscr{L}_{n*} = \{f_*\colon f \in \mathscr{L}_n\}, \tag{1.13}$$

and set $\mathscr{L} = \bigcup_{n=0}^{\infty} \mathscr{L}_n$, $\mathscr{L}_* = \bigcup_{n=0}^{\infty} \mathscr{L}_{n*}$. We may then write

$$\mathscr{L}_n = \left\{\frac{p_n(z)}{\pi_n(z)}\colon p_n \in \Pi_n\right\}, \tag{1.14}$$

$$\mathscr{L}_{n*} = \left\{\frac{q_n(z)}{\omega_n(z)}\colon q_n \in \Pi_n\right\}, \tag{1.15}$$

where Π_n denotes the space of all polynomials of degree at most n.

For $f_n \in \mathscr{L}_n$ we define its superstar conjugate f_n^* by

$$f_n^*(z) = B_n(z) f_{n*}(z). \tag{1.16}$$

Note that this transformation depends on n. It must be clear from the context what n is. Also note that when $f_n \in \mathscr{L}_n$ then $f_n^* \in \mathscr{L}_n$.

The theory of the function spaces described above is connected with the Nevanlinna–Pick interpolation problem with interpolation points $\{\alpha_n\}$ (cf. [16, 17]). These function spaces were introduced by Djrbashian in 1969 (see [11]), and independently in [1, 2, 10]. The theory has recently been further developed in [3, 5, 6, 8] (cf. also [14]). For connections between Nevanlinna–Pick interpolation and system theory, see [9].

We shall in this paper mainly be concerned with a special case, which we shall call the cyclic case. In this case the sequence $\{\alpha_n\}$ consists of a finite number p of points cyclically repeated. Thus $\alpha_{qp+k} = \alpha_k$ for $k = 1, \ldots, p$, $q = 0, 1, 2, \ldots$. For more details on the cyclic case see [4, 7, 12].

When all the interpolation points coalesce at the origin, the space $\mathscr{L}$ reduces to the space of polynomials, and the orthogonal rational functions in $\mathscr{L}$ (see Section 2) are orthogonal polynomials, Szegő polynomials. For a survey of this special situation, see e.g. [13].

2. Orthogonal rational functions

Let the sequence $\{\varphi_n\colon n = 0, 1, 2, \ldots\}$ be obtained by orthonormalization of the sequence $\{B_n\colon n = 0, 1, 2, \ldots\}$ with respect to $\langle , \rangle_\mu$. These functions are uniquely determined by the requirement that the leading coefficient κ_n in

$$\varphi_n(z) = \sum_{k=0}^{n} \kappa_k B_k(z) \tag{2.1}$$

is positive. We then have $\kappa_n = \varphi_n^*(\alpha_n)$. The following orthogonality properties are valid:

$$\langle f, \varphi_n \rangle_\mu = 0 \quad \text{for } f \in \mathscr{L}_{n-1}, \tag{2.2}$$

$$\langle g, \varphi_n^* \rangle_\mu = 0 \quad \text{for } g \in \zeta_n \mathscr{L}_{n-1} \tag{2.3}$$

(see [3, 5]). We define the functions $\varphi_n(z, u, v)$ by

$$\varphi_n(z, u, v) = u\,\varphi_n(z) + v\,\varphi_n^*(z), \quad u,\ v \in \boldsymbol{C},\ (u, v) \neq (0, 0). \tag{2.4}$$

We note that $\varphi_n(z, u, v)$ belongs to $\mathscr{L}_n$ (as a function of z). We call these functions paraorthogonal when $|u| = |v|$.

We define the functions ψ_n of the second kind by

$$\psi_0(z) = 1, \qquad \psi_n(z) = \int_T D(t, z)[\varphi_n(t) - \varphi_n(z)]\,\mathrm{d}\mu(t),\ n = 1, 2, \ldots . \tag{2.5}$$

For the functions ψ_n and ψ_n^* various equivalent expressions can be given. Let us recall the following result (see [3, 5]).

Theorem 2.1. *For $n = 1, 2, \ldots$ the following formulas are valid:*

$$\psi_n(z) = \int_T D(t, z)\left[\frac{B_k(z)}{B_k(t)}\,\varphi_n(t) - \varphi_n(z)\right]\mathrm{d}\mu(t), \quad k = 0, 1, \ldots, n-1, \tag{2.6}$$

$$\psi_n^*(z) = -\int_T D(t, z)\left[\frac{B_{n\backslash k}(z)}{B_{n\backslash k}(t)}\,\varphi_n^*(t) - \varphi_n^*(z)\right], \quad k = 0, 1, \ldots, n-1. \tag{2.7}$$

We shall next prove a result valid in the cyclic situation.

Theorem 2.2. *In the cyclic case with p points the following formulas are valid for $n = p+1, p+2, \ldots$:*

$$\psi_n(z) = \int_T D(t, z)\left[\frac{B_{n\backslash qp}(z)}{B_{n\backslash qp}(t)}\,\varphi_n(t) - \varphi_n(z)\right]\mathrm{d}\mu(t) \quad \textit{where } qp < n, \tag{2.8}$$

$$\psi_n^*(z) = -\int_T D(t, z)\left[\frac{B_{qp}(z)}{B_{qp}(t)}\,\varphi_n^*(t) - \varphi_n^*(z)\right]\mathrm{d}\mu(t) \quad \textit{where } qp < n. \tag{2.9}$$

Proof. We may write

$$B_{n\backslash qp}(z) = \prod_{j=n-qp+1}^{n} \zeta_j(z) = \prod_{j=1}^{qp} \zeta_j(z) = B_{qp}(z).$$

The results now follow by using $k = qp$ in (2.6) and (2.7). □

We define the functions $\psi_n(z, u, v)$ of the second kind by

$$\psi_n(z, u, v) = u\psi_n(z) - v\psi_n^*(z), \quad u, v \in \boldsymbol{C}, \ (u, v) \neq (0, 0). \tag{2.10}$$

Theorem 2.3. *In the cyclic case with p points the following formulas are valid for $n = p+1, p+2, \ldots$:*

$$\psi_n(z, u, v) = \int_T D(t, z)\left[\frac{B_{qp}(z)}{B_{qp}(t)}\varphi_n(t, u, v) - \varphi_n(z, u, v)\right]\mathrm{d}\mu(t) \quad \textit{where } qp < n, \tag{2.11}$$

$$\psi_n(z, u, v) = \int_T D(t, z)\left[\frac{B_{n\backslash qp}(z)}{B_{n\backslash qp}(t)}\varphi_n(t, u, v) - \varphi_n(z, u, v)\right]\mathrm{d}\mu(t) \quad \textit{where } qp < n. \tag{2.12}$$

Proof. Follows by combining (2.7) and (2.8) (resp. (2.6) and (2.9)) for the situation $k = qp$. □

3. Interpolation by rational approximants

We shall in this section study interpolation properties of the rational functions

$$R_n(z, u, v) = \frac{\psi_n(z, u, v)}{\varphi_n(z, u, v)} \tag{3.1}$$

given by (2.4) and (2.10) to the function $-\Omega_\mu(z)$ defined in (1.2).
Let us recall the following result (see [8]).

Theorem 3.1. *The function $\Omega_\mu(z)$ has in $\boldsymbol{D}$ the following Newton series expansion:*

$$\Omega_\mu(z) = \left[\mu_0 + 2\sum_{m=1}^{\infty} \mu_m z\omega_{m-1}(z)\right], \tag{3.2}$$

where the general moments μ_m are given by

$$\mu_m = \int_T \frac{\mathrm{d}\mu(t)}{\omega_m(t)}, \quad m = 0, 1, 2, \ldots . \tag{3.3}$$

In the following we shall use the notation $q(n)$, $r(n)$ as defined below:

$$n = q(n)p + r(n), \qquad r(n) \in \{1, \ldots, p\}. \tag{3.4}$$

Theorem 3.2. *The rational function $R_n(z, u, v)$ interpolates the function $-\Omega_\mu(z)$ in the sense that for $n > p$:*

$$\psi_n(z, u, v) + \varphi_n(z, u, v)\Omega_\mu(z) = f_n(z)z\omega_{n-1}(z), \tag{3.5}$$

where $f_n(z)$ is analytic in $\boldsymbol{D}$.

Proof. One can easily establish the identity

$$1 + 2\sum_{m=1}^{n-1} \frac{z\omega_{m-1}(z)}{\omega_m(t)} = \frac{t+z}{t-z}\left[1 - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}\right] - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}. \tag{3.6}$$

Hence, after integrating (3.6) with measure μ, we get

$$\mu_0 + 2\sum_{m=1}^{n-1} \mu_m z\omega_{m-1}(z) = \int_T \left\{D(t,z)\left[1 - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}\right] - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}\right\} d\mu(t). \tag{3.7}$$

By combining (2.11) and (3.7) we then obtain (since $q(n)p < n$)

$$\begin{aligned}\psi_n(z,u,v) &+ \varphi_n(z,u,v)\left[\mu_0 + 2\sum_{m=1}^{n-1} \mu_m z\omega_{m-1}(z)\right] \\ &= \int_T D(t,z)\left[\frac{B_{q(n)p}(z)}{B_{q(n)p}(t)}\varphi_n(t,u,v) - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}\varphi_n(z,u,v)\right] d\mu(t) \\ &\quad - \varphi_n(z,u,v) z\omega_{n-1}(z)\int_T \frac{1}{t\omega_{n-1}(t)}\, d\mu(t)\end{aligned} \tag{3.8}$$

and hence

$$\begin{aligned}\psi_n(z,u,v) &+ \varphi_n(z,u,v)\left[\mu_0 + 2\sum_{m=1}^{n-1} \mu_m z\omega_{m-1}(z)\right] \\ &= -\mu_n' \varphi_n(z,u,v) z\omega_{n-1}(z) + \omega_{q(n)p}(z)\sigma_n(z),\end{aligned} \tag{3.9}$$

where

$$\mu_n' = \int_T \frac{1}{t\omega_{n-1}(t)}\, d\mu(t) \tag{3.10}$$

and

$$\sigma_n(z) = \int_T D(t,z)\left[\frac{\pi_{q(n)p}(t)}{\pi_{q(n)p}(z)\omega_{q(n)p}(t)}\varphi_n(t,u,v) - \frac{z\prod_{k=q(n)p+1}^{n-1}(z-\alpha_k)}{t\omega_{n-1}(t)}\varphi_n(z,u,v)\right] d\mu(t). \tag{3.11}$$

(If $q(n)p = n-1$, the product means the constant 1.)

We are going to prove that $\sigma_n(\alpha_k) = 0$ for $q(n)p + 1 \leqslant k \leqslant n-1$. Let $q(n)p + 1 \leqslant k \leqslant n-1$, if $n(q) < n-1$. Then

$$\sigma_n(\alpha_k) = \frac{1}{\pi_{q(n)p}(\alpha_k)}\int_T D(t,\alpha_k)\frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)}\varphi_n(t,u,v)\, d\mu(t). \tag{3.12}$$

We note that

$$D(t,\alpha_k)\left[\frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)}\right]_* = c\,\frac{1+\bar{\alpha}_k t}{1-\bar{\alpha}_k t}\,\frac{\omega_{q(n)p}(t)}{\pi_{q(n)p}(t)} = c\,\zeta_n(t)L(t),$$

where $L(t)\in\mathscr{L}_{n-1}$ and c is a constant, while also

$$D(t,\alpha_k)\frac{\omega_{q(n)p}(t)}{\pi_{q(n)p}(t)}\in\mathscr{L}_{n-1}.$$

Because we may note that

$$\frac{(1+\bar{\alpha}_k t)\omega_{q(n)p}(t)}{(1-\bar{\alpha}_k t)\pi_{q(n)p}(t)} = \frac{(t-\alpha_k)s_{q(n)p}(t)}{(1-\bar{\alpha}_k t)\pi_{q(n)p}(t)},$$

where $s_{q(n)p}(t)$ is a polynomial of degree $q(n)p$, that $(1-\bar{\alpha}_k t)\pi_{q(n)p}(t)$ is a factor in $\pi_n(t)$, and that $(t-\alpha_k)$ is a factor in $\omega_{q(n)p}(t)$, thus

$$\left[\frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)}\right]_* \in\mathscr{L}_{n-1}\cap\zeta_n\mathscr{L}_{n-1},$$

and hence

$$\sigma_n(\alpha_k) = \frac{1}{\pi_{q(n)p}(\alpha_k)}\left\langle \varphi_n(t,u,v), \left[\frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)}\right]_*\right\rangle_\mu = 0. \tag{3.13}$$

Analogously we find $\sigma_n(0)=0$.

We have now seen that the second term on the right-hand side of (3.9) in addition to having the factor $\omega_{q(n)p}(z)$ also has the extra factor z and the extra factors $(z-\alpha_k)$ for $q(n)p+1\leqslant k\leqslant n-1$ (since $\sigma_n(0)$ and $\sigma_n(\alpha_k)=0$ for the values of k indicated).

It follows that the second term on the right of (3.9) is of the form $A_n(z)z\omega_{n-1}(z)$. Thus

$$\psi_n(z,u,v)+\varphi_n(z,u,v)\left[\mu_0+2\sum_{m=1}^{n-1}\mu_m z\omega_{m-1}(z)\right] = g_n(z)z\omega_{n-1}(z),\quad g_n(z)\text{ analytic.} \tag{3.14}$$

Since

$$\Omega_\mu(z)+\left[\mu_0+2\sum_{m=1}^{n-1}\mu_m z\omega_{m-1}(z)\right] = h_n(z)z\omega_{n-1}(z),\quad h_n(z)\text{ analytic,} \tag{3.15}$$

we conclude that (3.5) holds. □

4. Convergence of rational approximants

We recall that we call the function $\varphi_n(z,u,v)$ paraorthogonal when $|u|=|v|$. Paraorthogonal functions give rise to quadrature formulas. Let us recall the following result (see [3, 6]).

Theorem 4.1. *The zeros of $\varphi_n(z,u,v)$ for $|u|=|v|$ are all simple and lie on $\boldsymbol{T}$. Let the zeros be denoted by $\xi_k^{(n)}(u,v)$, $k=1,\dots,n$. Then there exist positive constants $\lambda_k^{(n)}(u,v)$ such that the quadrature* *formula*

$$\int_T L(t)\,\mathrm{d}\mu(t) = \sum_{k=1}^{n} \lambda_k^{(n)}(u,v)\,L(\xi_k^{(n)}(u,v)) \tag{4.1}$$

is valid for $L \in \mathscr{L}_{n-1} + \mathscr{L}_{(n-1)*}$.

We shall in the rest of this section again consider only the cyclic case with p points, and use the same notation as in Section 3 and Theorem 4.1.

Theorem 4.2. *Let* $|u| = |v|$, *and assume* $n > p$. *Then* $R_n(z,u,v)$ *has the partial fraction decomposition*

$$R_n(z,u,v) = -\sum_{m=1}^{n} \lambda_m^{(n)}(u,v)\,D(\xi_m^{(n)}(u,v), z). \tag{4.2}$$

Proof. Consider the function $f(t)$ defined by

$$f(t) = D(t,z)\left[\frac{B_p(z)}{B_p(t)}\,\varphi_n(t,u,v) - \varphi_n(z,u,v)\right]. \tag{4.3}$$

The function $\varphi_n(z,u,v)$ can be written as

$$\varphi_n(z,u,v) = \frac{p_n(z,u,v)}{\pi_n(z)}, \tag{4.4}$$

where $p_n(z,u,v) \in \Pi_n$. It follows that

$$f(t) = \frac{(t+z)[\omega_p(z)\pi_p(t)p_n(t,u,v)\pi_n(z) - \omega_p(t)\pi_p(z)\pi_n(t)p_n(z,u,v)]}{(t-z)\omega_p(t)\pi_p(z)\pi_n(t)}, \tag{4.5}$$

hence since $t - z$ is a factor in the numerator:

$$f(t) = \frac{P_{p+n-1}(z,t)(1-\overline{\alpha}_n t)}{\omega_p(t)\pi_n(t)}, \tag{4.6}$$

where P_{p+n-1} belongs to Π_{p+n-1} as a function of t. (Note that $(1-\overline{\alpha}_n t)$ is a factor both in $\pi_p(t)$ and in $\pi_n(t)$, and also in the numerator.)

It follows that we may write

$$f(t) = \frac{P_{p+n-1}(z,t)}{\omega_p(t)\pi_{n-1}(t)}, \tag{4.7}$$

hence $f(t) \in \mathscr{L}_{n-1} + \mathscr{L}_{p*} \subset \mathscr{L}_{n-1} + \mathscr{L}_{(n-1)*}$, by partial fraction decomposition. (Note that $\omega_p(t)$ and $\pi_{n-1}(t)$ have no common factors.) Since $f(\xi_m^{(n)}(u,v)) = -D(\xi_m^{(n)}(u,v), z)\,\varphi_n(z,u,v)$, as

$\varphi_n(\xi_m^{(n)}(u, v), u, v)$ equals zero, application of Theorem 4.1 and formula (2.11) yields

$$\psi_n(z, u, v) = -\varphi_n(z, u, v) \sum_{m=1}^{n} \lambda_m^{(n)}(u, v) D(\xi_m^{(n)}(u, v), z), \tag{4.8}$$

which is equivalent to (4.2). □

Since (4.1) is valid for $L = 1$, the following equality holds:

$$\sum_{m=1}^{n} \lambda_m^{(n)}(u, v) = \mu_0. \tag{4.9}$$

Theorem 4.3. *Let $|u_n| = |v_n|$ for $n = 1, 2, \ldots$. Then the sequence $\{R_n(z, u_n, v_n)\}$ converges locally uniformly on $\boldsymbol{D}$ to $-\Omega_\mu(z)$.*

Proof. It easily follows by (4.2) and (4.9) that the functions $R_n(z, u, v)$, $|u| = |v|$, are uniformly bounded on every compact subset of $\boldsymbol{D}$, and thus form a normal family. So there exist subsequences of $\{R_n(z, u_n, v_n)\}$ converging locally uniformly on $\boldsymbol{D}$. Let $\nu_n(t, u_n, v_n)$ be the measure on $\boldsymbol{T}$ having masses $\lambda_m^{(n)}(u_n, v_n)$ at the points $\xi_m^{(n)}(u_n, v_n)$. By Theorem 4.2 we may then write

$$R_n(z, u_n, v_n) = -\int_T D(t, z)\, \mathrm{d}\nu_n(t, u_n, v_n). \tag{4.10}$$

A standard argument shows that a subsequence of $\{R_n(z, u_n, v_n)\}$ converges locally uniformly on $\boldsymbol{D}$ to a function $F(z)$ if and only if the corresponding subsequence of $\{\nu_n(t, u_n, v_n)\}$ converges to a measure ν such that $F(z) = -\Omega_\nu(z)$.

Furthermore $\int_T \mathrm{d}\nu_n(u_n, v_n, t)/\omega_m(t)$ converges to $\int_T \mathrm{d}\nu(t)/\omega_m(t)$ for $m = 0, 1, 2, \ldots$. On the other hand Theorem 3.2 shows that $R_n(z, u_n, v_n) + \Omega_\mu(z) = g_n(z) z \omega_{n-1}(z)$, where $g_n(z)$ is analytic in $\boldsymbol{D}$. It follows from this and (4.10) that $\int_T \mathrm{d}\nu_n(t, u_n, v_n)/\omega_m(t) = \int_T \mathrm{d}\mu(t)/\omega_m(t)$ for $m = 0, 1, \ldots, n-1$.

Consequently $\int_T \mathrm{d}\nu(t)/\omega_m(t) = \int_T \mathrm{d}\mu(t)/\omega_m(t)$ for $m = 0, 1, 2, \ldots$ (cf. [7, 8] where related problems are treated). It is known that the measure giving rise to the moments $\mu_m = \int_T \mathrm{d}\mu(t)/\omega_m(t)$ is unique when $\sum_{m=1}^{\infty}(1 - |\alpha_n|) = \infty$ (this follows e.g. from the convergence result in [3, Section 21]). This is the case in the cyclic situation. Thus $\nu = \mu$ and the whole sequence $\{R_n(z, u_n, v_n)\}$ converges to $-\Omega_\mu(z)$. □

For convergence properties of the rational approximants $R_n(z, 0, 1)$ and $R_n(z, 1, 0)$ see [3]. For a more detailed study of convergence of multipoint Padé approximants, see especially [15].

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