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Quadrature formulas on the unit circle based on rational functions

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Abstract

Quadrature formulas on the unit circle were introduced by Jones et al. in 1989. On the other hand, Bultheel et al. also considered such quadratures by giving results concerning error and convergence. In other recent papers, a more general situation was studied by the authors involving orthogonal rational functions on the unit circle which generalize the well-known Szegő polynomials. In this paper, these quadratures are again analyzed and results about convergence given. Furthermore, an application to the Poisson integral is also made.

Key words: Quadrature formulas; Positive measures; Orthogonal rational functions

1. Preliminaries

In this paper we shall use the notation $\mathbb{T} = \{z: |z| = 1\}$, $\mathbb{D} = \{z: |z| < 1\}$ and $\mathbb{E} = \{z: |z| > 1\}$ for the unit circle, the open unit disc and the exterior of the unit circle. Let μ be a finite Borel measure on $[-\pi, \pi]$. In order to estimate the integral

$$I_{\mu}\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta) = \int_{\mathbb{T}} f(z) d\mu(z) = \int f(z) d\mu(z)$$

(we have taken the freedom to write the previous integral in different forms, having in mind that integration will always be over the unit circle in one form or another), the so-called Szegő

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quadrature formulas were introduced in [11]. (See also [8] for a different approach.) Such quadratures are of the form

$$I_n\{f\} = \sum_{j=1}^n A_j^{(n)} f(x_j^{(n)}), \quad x_i^{(n)} \neq x_j^{(n)}, \quad x_j^{(n)} \in \mathbb{T} \text{ and } A_j^{(n)} > 0, \quad j = 1, \dots, n, \tag{1.1}$$

so that $I_\mu\{f\} = I_n\{f\}$ for all $f \in \Lambda_{-(n-1), (n-1)}$. For every pair (p, q) of integers, $p \leq q$, $\Lambda_{p,q}$ will denote the linear space of all Laurent polynomials of the form

$$\sum_{j=p}^q c_j z^j, \quad c_j \in \mathbb{C},$$

and Λ the space of all Laurent polynomials [10,13]. It is known that quadrature formulas (1.1) are of great interest to solve the trigonometric moment problem or equivalently the Schur coefficient problem (see [11]). On the other hand, Waadeland [14] recently studied such quadratures for the Poisson integral, that is, when the measure μ is given by

$$d\mu(\theta) = \frac{d\theta}{2\pi} \frac{1-r^2}{1-2r \cos \theta + r^2}, \quad r \in (0, 1),$$

or more generally,

$$d\mu(\theta) = \frac{1-|r|^2}{|z-r|^2} \frac{d\theta}{2\pi}, \quad r \in \mathbb{D}, \quad z = e^{i\theta}. \tag{1.2}$$

Observe that by taking $r=0$, we have the normalized Lebesgue measure $d\lambda(\theta) = d\theta/2\pi$. Szegő quadratures for such situations were also studied in [9]. Finally, in [4] aspects concerning error and convergence were analyzed.

In this paper, formulas (1.1) will be considered again, but instead of Laurent polynomials, more general rational functions with prescribed poles not on \mathbb{T} will be used, giving rise to the rational Szegő formulas which were earlier introduced in [2,5], where the so-called rational Szegő functions play a fundamental role.

For completeness, let $\{\alpha_i\}_1^\infty \subset \mathbb{D}$ be a given sequence and consider for $n = 0, 1, \dots$ the nested spaces \mathcal{L}_n of rational functions of degree at most n which are spanned by the basis of partial Blaschke products $\{B_k\}_0^n$, where $B_0 = 1$, $B_n = \xi_n B_{n-1}$ for $n = 1, 2, \dots$ and the Blaschke factors are defined as

$$\xi_n(z) = \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}.$$

By convention, we set $\bar{\alpha}_n/|\alpha_n| = -1$ for $\alpha_n = 0$. Sometimes, we shall also write

$$B_n(z) = \frac{\eta_n \omega_n(z)}{\pi_n(z)}, \quad \eta_n = (-1)^n \prod_{j=1}^n \frac{\bar{\alpha}_j}{|\alpha_j|}, \quad \omega_n = \prod_{j=1}^n (z - \alpha_j) \text{ and } \pi_n = \prod_{j=1}^n (1 - \bar{\alpha}_j z).$$

Note that if all the α_i are equal to zero, the spaces \mathcal{L}_n collapse to the space Π_n of polynomials of degree n .

We also introduce the transformation $f_*(z) = \overline{f(1/\bar{z})}$, which allows to define for $f_n \in \mathcal{L}_n$ the superstar conjugate as

$$f_n^*(z) = B_n(z)f_n^*(z).$$

Let now the sequence $\{\phi_n: n = 0, 1, \dots\}$ be obtained by orthonormalization of the sequence $\{B_n: n = 0, 1, \dots\}$ with respect to the inner product induced by the measure μ , namely

$$\langle f, g \rangle_\mu = \int_{-\pi}^\pi f(e^{i\theta})\overline{g(e^{i\theta})} d\mu(\theta).$$

These functions are uniquely determined by the requirement that the leading coefficient k_n in

$$\phi_n(z) = \sum_{j=1}^n k_j B_j(z)$$

is positive. We then have $k_n = \phi_n^*(\alpha_n)$.

Finally, in order to summarize the main result given in [5] for the rational Szegő formulas, let us introduce the function spaces of the form

$$\mathcal{R}_{p,q} = \mathcal{L}_p^* + \mathcal{L}_q = \left\{ \frac{P}{\omega_p \pi_q}; P \in \Pi_{p+q} \right\}, \quad p \text{ and } q \text{ being nonnegative integers.}$$

Observe that $\mathcal{L}_n^* = \text{span}\{1, B_1^*, \dots, B_n^*\} = \text{span}\{1, 1/B_1, \dots, 1/B_n\}$. Therefore,

$$\mathcal{R}_{p,q} = \text{span} \left\{ \frac{1}{B_p}, \frac{1}{B_{p-1}}, \dots, \frac{1}{B_1}, 1, B_1, \dots, B_q \right\}, \quad \mathcal{R}_{0,n} = \mathcal{L}_n.$$

When all the α_i are equal to zero, then $B_k(z) = z^k$ and one has $\mathcal{R}_{p,q} = \text{span}\{z^k: k = -p, -p + 1, \dots, q\} = \mathcal{A}_{-p,q}$. Furthermore, for $w \in \mathbb{T}$, set $\chi_n(z, w) = \phi_n(z) + w\phi_n^*(z)$, so that the following holds (see [5]).

Theorem 1. (i) $\chi_n(z, w)$ has n simple zeros which lie on the unit circle.

(ii) Let x_1, \dots, x_n be the zeros of $\chi_n(z, w)$. Then there exist positive numbers A_1, \dots, A_n such that the formula $I_n\{f\} = \sum_{j=1}^n A_j f(x_j)$ is exact, that is, $I_n\{f\} = I_\mu\{f\}$ for all $f \in \mathcal{R}_{n-1, n-1}$.

In this case, $\mathcal{R}_{n-1, n-1}$ is said to be a maximal domain of validity. Moreover, it was also proved in [5] that the only quadrature formulas with such a maximal domain of validity are just those ones given in Theorem 1, where the weights A_j are given by

$$A_j = \int L_j(z) d\mu(z), \tag{1.3}$$

and $L_j(z) \in \mathcal{L}_{n-1} = \mathcal{R}_{0, n-1}$ is defined by the interpolation conditions $L_j(x_i) = \delta_{ij}$.

(Actually, a more general interpolating function space $\mathcal{R}_{p,q}$, p and q being nonnegative integers such that $p + q = n - 1$, can be considered, so that the resulting quadrature formula does not depend on p and q . See [5].)

2. An alternative approach

In this section we shall give an alternative approach to get the above quadrature formulas using Hermite interpolation in the space $\mathcal{R}_{n-1,n-1}$ (compare with the approach given by Markov for the classical Gauss formulas [12] and with the one given in [4] for the Szegő formulas on the unit circle in the polynomial case). Writing $A = \{\alpha_i\}_1^\infty$ and $\hat{A} = \{1/\bar{\alpha}_j: \alpha_j \in A\}$, it is easily seen that $\mathcal{R}_{p,q}$ represents a Chebyshev system on any set $X \subset \mathbb{C} - (A \cup \hat{A})$, so that, given the distinct nodes $\{x_j: j = 1, \dots, n\} \subset \mathbb{C} - (A \cup \hat{A})$, there exists a unique function $Q \in \mathcal{R}_{n-1,n-1}$, with

$$Q(x_i) = f(x_i), \quad i = 1, 2, \dots, n, \quad Q'(x_i) = f'(x_i), \quad i = 1, 2, \dots, n - 1.$$

These are $2n - 1$ constraints, which corresponds to the dimension of $\mathcal{R}_{n-1,n-1}$. In order to determine such Hermite rational interpolants, we can write

$$Q(z) = \sum_{j=1}^n H_{j,0}(z)f(x_j) + \sum_{j=1}^{n-1} H_{j,1}(z)f'(x_j),$$

where $H_{j,0}$ and $H_{j,1}$ belong to $\mathcal{R}_{n-1,n-1}$ and satisfy the interpolation conditions

$$H_{i,0}(x_j) = \delta_{ij}, \quad 1 \leq i, j \leq n, \quad H'_{i,0}(x_j) = 0, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n - 1, \tag{2.1}$$

$$H_{i,1}(x_j) = 0, \quad 1 \leq i \leq n - 1, \quad 1 \leq j \leq n, \quad H'_{i,1}(x_j) = \delta_{ij}, \quad 1 \leq i, j \leq n - 1. \tag{2.2}$$

Denoting $N_n(z) = \prod_1^n (z - x_j)$, we set for $i = 1, 2, \dots, n$,

$$L_i^{(2)}(z) = \left[\frac{N_n(z)}{z - x_i} \right]^2 \frac{\pi_{2n-2}(x_i)}{\pi_{2n-2}(z)[N'_n(x_i)]^2} \in \mathcal{L}_{2n-2},$$

which satisfies $L_i^{(2)}(x_j) = 0, 1 \leq i \neq j \leq n$. Define $L_{n,0}(z) = L_n^{(2)}(z)$ and

$$L_{i,0}(z) = L_i^{(2)}(z) + \lambda_i \frac{z - x_i}{z - x_n} L_i^{(2)}(z) \in \mathcal{L}_{2n-2}, \quad \text{for } i = 1, 2, \dots, n - 1,$$

with $\lambda_i \in \mathbb{C}$ chosen such that $L'_{i,0}(x_i) = 1$. Furthermore we set

$$L_{i,1}(z) = (x_i - x_n) \frac{z - x_i}{z - x_n} L_i^{(2)}(z) \in \mathcal{L}_{2n-2}, \quad i = 1, 2, \dots, n - 1.$$

It is simple to check that

$$L_{i,0}(x_j) = \delta_{i,j}, \quad 1 \leq i, j \leq n, \quad L'_{i,0}(x_j) = 0, \quad 1 \leq j \leq n - 1, \quad 1 \leq i \leq n, \tag{2.3}$$

and

$$L_{i,1}(x_j) = 0, \quad 1 \leq i \leq n - 1, \quad 1 \leq j \leq n, \quad L'_{i,1}(x_j) = \delta_{i,j}, \quad 1 \leq i, j \leq n - 1. \tag{2.4}$$

From (2.3) and (2.4), we can set for $i = 1, 2, \dots, n - 1$,

$$H_{i,0}(z) = \frac{\prod_n^{2n-2} (1 - \bar{\alpha}_j z)}{\prod_n (1 - \bar{\alpha}_j x_i)} \frac{\omega_{n-1}(x_i)}{\omega_{n-1}(z)} [L_{i,0}(z) + \mu_i L_{i,1}(z)] \in \mathcal{R}_{n-1, n-1},$$

where μ_i is uniquely determined by the condition $H'_{i,0}(x_i) = 0$ and

$$H_{n,0}(z) = \frac{\prod_n^{2n-2} (1 - \bar{\alpha}_j z)}{\prod_n (1 - \bar{\alpha}_j x_n)} \frac{\omega_{n-1}(x_n)}{\omega_{n-1}(z)} L_{n,0}(z) \in \mathcal{R}_{n-1, n-1}$$

satisfies the requirements (2.1). Similarly, the functions

$$H_{i,1}(z) = \frac{\prod_n^{2n-2} (1 - \bar{\alpha}_j z)}{\prod_n (1 - \bar{\alpha}_j x_i)} \frac{\omega_{n-1}(x_i)}{\omega_{n-1}(z)} L_{i,1}(z) \in \mathcal{R}_{n-1, n-1} \tag{2.5}$$

satisfy the conditions (2.2).

Once the interpolating function $Q(z) \in \mathcal{R}_{n-1, n-1}$ has been characterized, one gets

$$\hat{I}_n\{f\} = \int Q(z) d\mu(z) = \sum_{j=1}^n \lambda_j^{(n)} f(x_j) + \sum_{j=1}^{n-1} \hat{\lambda}_j^{(n)} f'(x_j), \tag{2.6}$$

where $\lambda_j^{(n)} = \int H_{j,0}(z) d\mu$ and $\hat{\lambda}_j^{(n)} = \int H_{j,1}(z) d\mu$. Therefore, $\hat{I}_n\{f\}$ can be considered as a quadrature formula which makes use of values of the function f and its derivative. Clearly $\hat{I}_n\{f\}$ has a domain of validity $\mathcal{R}_{n-1, n-1}$. However, an adequate choice of the nodes $\{x_j\}$ can greatly simplify formulas (2.6). Indeed, when $\{x_j\}$ are the zeros of $\phi_n + w\phi_n^*$, $|w| = 1$, one has the following theorem.

Theorem 2. *The quadrature formula $\hat{I}_n\{f\}$ given by (2.6) reduces to an n -point Rational Szegő (or an R-Szegő, for short) formula when the nodes are the zeros of*

$$\phi_n + w\phi_n^*, \quad |w| = 1.$$

Proof. We write $\chi_n = \phi_n + w\phi_n^* = N_n(z)/\pi_n(z)$, $N_n \in \Pi_n$, and $N_n(x_j) = 0$, $j = 1, \dots, n$. Note that N_n is not necessarily monic. By the characterization theorem for R-Szegő formulas (Theorem 1), it suffices to show that $\hat{\lambda}_i^{(n)} = 0$, for $i = 1, \dots, n - 1$. But $\hat{\lambda}_i^{(i)} = \int H_{i,1}(z) d\mu$, where $H_{i,1}$ is given by (2.5). Hence, we have to prove

$$\int \frac{\prod_n^{2n-2} (1 - \bar{\alpha}_j z)}{\omega_{n-1}(z)} \frac{z - x_i}{z - x_n} L_i^{(2)}(z) d\mu = 0.$$

From the definition of $L_i^{(2)}(z)$, this integral can be written as (up to a constant factor)

$$\int \frac{N_n(z)}{\pi_{n-1}(z)} \frac{N_{n-1}(z)}{(z-x_i)\omega_{n-1}(z)} d\mu = \int \frac{N_n(z)}{\pi_n(z)} \frac{(1-\bar{\alpha}_n z)N_{n-1}(z)}{(z-x_i)\omega_{n-1}(z)} d\mu = \int \chi_n(z) h_*(z) d\mu = \langle \chi_n, h \rangle_\mu,$$

where

$$h(z) = \frac{(1-\bar{x}_1 z) \cdots (1-\bar{x}_{n-1} z)(z-\alpha_n)}{(1-\bar{x}_i z)\pi_{n-1}(z)} \in \mathcal{L}_{n-1} \quad \text{and} \quad h(\alpha_n) = 0.$$

Thus, h belongs to $\mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$, $\mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n : f(\alpha_n) = 0\}$. Now, by the orthogonality properties for χ_n , it follows that $\langle \chi_n, h \rangle_\mu = 0$. \square

Remark 3. Note that the same result can be obtained if the following interpolation problem is considered. Find $Q_i \in \mathcal{R}_{n-1, n-1}$, $i = 1, 2, \dots, n$, such that

$$Q_i(x_j) = f(x_j), \quad j = 1, 2, \dots, n, \quad Q_i'(x_j) = f'(x_j), \quad 1 \leq j \leq n, \quad i \neq j. \quad (2.7)$$

We can conclude that an n -point R-Szegő formula $I_n\{f\}$ is given by $I_n\{f\} = \int Q_i(z) d\mu$, where Q_i is the unique solution to the interpolation problem (2.7) and $\{x_j\}$ are the zeros of $\chi_n(z) = \phi_n + w\phi_n^*$, $|w| = 1$. Certainly, the given approach could be useful in order to give an expression for the error $E_n\{f\} = I_\mu\{f\} - I_n\{f\} = I_\mu\{f - Q_i\}$.

3. An application to the Poisson integral

We shall now characterize the n -point R-Szegő formulas for the measure μ induced by the Poisson integral kernel given by (1.2). In this sense, the first step is obtaining the orthonormal system $\{\phi_n\}$, $n = 0, 1, \dots$. (Djrbashian [7] already discussed how to find the corresponding orthogonal functions for some special cases.) We know $\phi_0 = 1$ and for $n = 1, 2, \dots$, ϕ_n has to verify the conditions (i) $\phi_n \in \mathcal{L}_n - \mathcal{L}_{n-1}$, $\langle \phi_n, \phi_n \rangle_\mu = 1$ and (ii) $\langle \phi_n, B_k \rangle_\mu = 0$, $k = 0, 1, \dots, n-1$.

From (i) one finds

$$\begin{aligned} \langle \phi_n, B_k \rangle_\mu &= \int \phi_n(z) \overline{B_k(z)} d\mu = \int \frac{\phi_n(z)(1-|r|^2)}{B_k(z)(z-r)(1-\bar{r}z)} d\lambda \\ &= \frac{1-|r|^2}{2\pi i} \int_{\mathbb{T}} \frac{\phi_n(z)}{B_k(z)(z-r)(1-\bar{r}z)} dz. \end{aligned}$$

The denominator $B_k(z)(z-r)(1-\bar{r}z)$ vanishes at $z = \alpha_i$, $i = 1, 2, \dots, k$, and $z = r$, which are all inside the unit disc \mathbb{D} and the other zero $1/\bar{r}$ is in \mathbb{E} . If this integral has to vanish, ϕ_n should be zero at $z = \alpha_i$, $i = 1, 2, \dots, k$, and $z = r$. This gives in combination with condition (i) that ϕ_n should have the following form:

$$\phi_n(z) = k_n \frac{(z-r)B_n(z)}{z-\alpha_n} \in \mathcal{L}_n \quad \text{and} \quad k_n \neq 0. \quad (3.1)$$

For $0 \leq k \leq n - 1$, it is easily seen that

$$\langle \phi_n, b_k \rangle_\mu = \frac{k_n(1 - |r|^2)}{2\pi i} \int_{\Gamma} \frac{B_{n/k}(z)}{z - \alpha_n} \frac{dz}{1 - \bar{r}z} = 0, \quad k_n \neq 0,$$

where $B_{n/k} = B_n/B_k$, and thus $B_{n/k}(\alpha_n) = 0$. The constant k_n is determined by

$$1 = \langle \phi_n, \phi_n \rangle_\mu = |k_n|^2 \frac{1 - |r|^2}{1 - |\alpha_n|^2}, \quad \text{so that } |k_n| = \left[\frac{1 - |\alpha_n|^2}{1 - |r|^2} \right]^{1/2}.$$

The leading coefficient is found as follows:

$$\phi_n^*(z) = \bar{k}_n \frac{1 - \bar{r}z}{1 - \bar{\alpha}_n z} \frac{1}{B_n(z)}, \tag{3.2}$$

so that

$$\phi_n^*(z) = \bar{k}_n \frac{1 - \bar{r}z}{1 - \bar{\alpha}_n z}, \quad \text{and thus } \overline{\phi_n^*(\alpha_n)} = k_n \frac{1 - \bar{\alpha}_n r}{1 - |\alpha_n|^2}.$$

Since the leading coefficient has to be positive,

$$k_n = \left[\frac{1 - |\alpha_n|^2}{1 - |r|^2} \right]^{1/2} \exp(i\gamma_n), \quad \gamma_n = -\arg(1 - \bar{\alpha}_n r).$$

We can check some particular cases.

(1) $r = 0$ delivers the Lebesgue measure. Then,

$$\gamma_n = -\arg(1 - \bar{\alpha}_n r) = -\arg(1) = 0 \quad \text{and} \quad \phi_n(z) = \sqrt{1 - |\alpha_n|^2} \frac{zB_n(z)}{z - \alpha_n}.$$

This corresponds with the result in [7]. When all the α_i are equal to zero, we recover the well-known result that $\phi_n(z) = z^n$.

(2) $r = \alpha_n$. One then has

$$\gamma_n = -\arg(1 - |\alpha_n|^2) = 0 \quad \text{and} \quad \phi_n(z) = B_n(z).$$

(3) $r \neq 0, \alpha_k = 0, k = 1, 2, \dots$. Then,

$$\phi_n(z) = \frac{(z - r)B_n(z)}{z} = (z - r)z^{n-1} = z^n - rz^{n-1}.$$

This was obtained in [14].

The equation $\chi_n(z) = \phi_n(z) + \hat{w}\phi_n^*(z)$ which provides the nodes takes the form

$$\chi_n(z) = k_n \left[(z - r) \frac{B_n(z)}{z - \alpha_n} \right] + \hat{w} \bar{k}_n \frac{1 - \bar{r}z}{1 - \bar{\alpha}_n z} = 0,$$

or equivalently,

$$\left[(z-r) \frac{B_n(z)}{z-\alpha_n} \right] + w \frac{1-\bar{r}z}{1-\bar{\alpha}_n z} = 0, \quad \text{with } w = \hat{w} \frac{\bar{k}_n}{k_n} \in \mathbb{T}.$$

Using $B_n(z) = \eta_n \pi_n(z) / \omega_n(z)$, we get

$$\chi_n(z) = \frac{(z-r)\eta_n \omega_{n-1}(z) + w \pi_{n-1}(z)(1-\bar{r}z)}{\pi_n(z)} = \frac{N_n(z)}{\pi_n(z)}, \quad N_n \in \Pi_n, \quad (3.3)$$

The nodes x_j satisfy $N_n(x_j) = 0$. When $r = 0$, one obtains

$$zB_{n-1}(z) = \frac{|\alpha_n| w}{\bar{\alpha}_n}. \quad (3.4)$$

Note that when $\alpha_i = 0$, this reduces to $z^n = -w$ and the nodes x_j are uniformly distributed on \mathbb{T} , see [9]. Let us suppose that $n = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$; then (3.4) gives

$$2z^2 - (1+w)z + 2w = 0,$$

and the interpolation nodes x_1, x_2 will be:

for $w = -1$: $x_1 = 1, x_2 = -1$;

for $w = 1$: $x_1 = \frac{1}{2}(1 + \sqrt{3}i), x_2 = \frac{1}{2}(1 - \sqrt{3}i)$;

for $w = -i$: $x_1 = \frac{1}{4}[(1 + \sqrt{7}) - (1 - \sqrt{7})i], x_2 = \frac{1}{4}[(1 - \sqrt{7}) - (1 + \sqrt{7})i]$;

for $w = i$: $x_1 = \frac{1}{4}[(1 - \sqrt{7}) + (1 + \sqrt{7})i], x_2 = \frac{1}{4}[(1 + \sqrt{7}) + (1 - \sqrt{7})i]$.

For the weights, one has in the general case

$$\begin{aligned} A_j &= \int L_j(z) d\mu(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\chi_n(z)}{z-x_j} \frac{1-\alpha_n z}{1-\bar{\alpha}_n x_j} \frac{1-|r|^2}{\chi'_n(x_j)(z-r)(1-\bar{r}z)} dz \\ &= \frac{1}{\chi'_n(x_j)} \frac{\chi_n(r)}{r-x_j} \frac{1-\bar{\alpha}_n r}{1-\bar{\alpha}_n x_j}, \end{aligned}$$

where $\chi_n(z) = N_n(z) / \pi_n(z)$ is given by (3.3). Since $\chi_n(r) = w(1-|r|^2) / (1-\bar{\alpha}_n r)$, we find

$$A_j = \frac{w(1-|r|^2)}{\chi'_n(x_j)(r-x_j)(1-\bar{\alpha}_n x_j)}, \quad j = 1, 2, \dots, n.$$

On the other hand, since $\chi_n(x_j) = 0$, we get $\chi'_n(x_j) = N'_n(x_j) / \pi_n(x_j)$ where

$$N'_n(z) = (z-r)\eta_n \omega_{n-1}(z) + w \pi_{n-1}(z)(1-\bar{r}z). \quad (3.5)$$

Use $N_n(x_j) = 0$, to find from (3.5) that

$$\pi_n(x_j) = -(1-\bar{\alpha}_n x_j)(x_j-r) \frac{\eta_n \omega_{n-1}(x_j)}{w(1-\bar{r}x_j)},$$

which implies

$$A_j = \eta_n \frac{1 - |r|^2}{1 - \bar{r}x_j} \frac{\omega_{n-1}(x_j)}{N'_n(x_j)}. \tag{3.6}$$

Now use

$$\pi'_k(x_j) = -\pi_k(x_j) \sum_{i=1}^k \frac{\bar{\alpha}_i}{1 - \bar{\alpha}_i x_j} \quad \text{and} \quad \omega'_k(x_j) = \omega_k(x_j) \sum_{i=1}^k \frac{1}{x_j - \alpha_i}$$

to obtain from (3.5) and (3.6)

$$A_j = (1 - |r|^2)/(1 - \bar{r}x_j) \left[1 + (x_j - r) \sum_{k=1}^{n-1} \left(\frac{1}{x_j - \alpha_k} - \frac{\bar{\alpha}_k}{1 - \bar{\alpha}_k x_j} \right) - \frac{\bar{r}(r - x_j)}{1 - \bar{r}x_j} \right],$$

so that for $j = 1, 2, \dots, n$,

$$A_j = (1 - |r|^2) / \left[1 - |r|^2 + |x_j - r|^2 \sum_{k=1}^{n-1} \frac{1 - |\alpha_k|^2}{|x_j - \alpha_k|^2} \right], \tag{3.7}$$

where the positivity of the weights is clearly exposed. Again, when all the α_i are equal to zero, one gets

$$A_j = \frac{1 - |r|^2}{1 - |r|^2 + (n - 1)|x_j - r|^2}, \quad j = 1, 2, \dots, n.$$

If we set $x_j = \exp(i\theta_j)$ are $r \in (0, 1)$, then

$$|x_j - r|^2 = |\exp(i\theta_j) - r|^2 = 1 + r^2 - 2r \cos \theta_j.$$

Therefore,

$$A_j = \frac{1 - |r|^2}{1 - |r|^2 + (n - 1)(1 + r^2 - 2r \cos \theta_j)}, \quad j = 1, 2, \dots, n. \tag{3.8}$$

The same expression was obtained in [14] for the polynomial case.

If $r = 0$ (Lebesgue measure), it follows from (3.8) that $A_j = n^{-1}$ for $j = 1, \dots, n$. That means that the corresponding Szegő formula for the polynomial case has all its nodes equally spaced on the unit circle and all its weights are equal to n^{-1} . (Compare with the results given in [9].)

In the special case $r = \alpha_n$, (3.7) yields

$$A_j = (1 - |\alpha_n|^2) / \left[|x_j - \alpha_n|^2 \sum_{k=1}^n \frac{1 - |\alpha_k|^2}{|x_j - \alpha_k|^2} \right]. \tag{3.9}$$

Concluding example: assume that $\alpha_k = \alpha$ for $k = 1, 2, \dots, n$ and take also $r = \alpha$. Then,

$$\phi_n(z) = B_n(z), \quad \phi_n^*(z) = 1, \quad \chi_n = \phi_n + w\phi_n^* = B_n(z) + w.$$

Because now $B_n = \zeta^n$ with $\zeta(z) = (\bar{\alpha}/|\alpha|)(\alpha - z)/(1 - \bar{\alpha}z)$, the nodes x_j are solutions of

$$\left(\frac{\alpha - z}{1 - \bar{\alpha}z} \right)^n = -w \frac{|\alpha|^n}{\bar{\alpha}^n} = \hat{w} \in \mathbb{T}.$$

Setting $r_j = (\hat{w})^{1/n}$, $j = 1, 2, \dots, n$, we get $x_j = (\alpha - r_j)/(1 - \bar{\alpha}r_j)$. As for the weights, one gets from (3.9)

$$A_j = \frac{1}{n}, \quad j = 1, 2, \dots, n.$$

4. Convergence

Let $I_n\{f\}$, $n = 1, 2, \dots$, be a sequence of R-Szegő formulas (take into account that for each n , $I_n\{f\}$ represents a one-parameter family of quadrature formulas), that is,

$$I_n\{f\} = \sum_{j=1}^n A_j^{(n)} f(x_j^{(n)}), \quad x_i^{(n)} \neq x_j^{(n)}, \quad i \neq j, \quad x_j^{(n)} \in \mathbb{T} \text{ and } A_j^{(n)} > 0, \quad j = 1, 2, \dots, n,$$

where the weights $A_j^{(n)}$ are given by (1.3). In this section, we shall study the convergence of such quadratures for any function f in the class $R_\mu(\mathbb{T})$ of the integrable functions on \mathbb{T} with respect to the measure μ . For this purpose a first result we shall need is the following.

Lemma 4. *Let us define $\mathcal{R}_n = \mathcal{R}_{n,n} = \mathcal{L}_n + \mathcal{L}_n^*$ and $\mathcal{R} = \mathcal{R}_\infty$; then \mathcal{R} is dense in the class $C(\mathbb{T})$ of continuous functions on \mathbb{T} , iff $\sum(1 - |\alpha_n|) = \infty$.*

Proof. This is a direct consequence of the “closure criterion” discussed in [1, Addendum A.2, p.244]. \square

We are now ready to prove a first result asserting the convergence in the class $C(\mathbb{T})$. Indeed, one has the following theorem.

Theorem 5. *Let f be a continuous function on \mathbb{T} ; then*

$$\lim_{n \rightarrow \infty} I_n\{f\} = I_\mu\{f\} = \int f(z) \, d\mu(z)$$

if $\sum(1 - |\alpha_n|) = \infty$.

Proof. Let ϵ be a given real positive number. Take

$$\epsilon' = \frac{\epsilon}{2\mu_0}, \quad \text{where } \mu_0 = \int d\mu(z).$$

By Lemma 4, there exists $R_N \in \mathcal{R}_N$ such that

$$|f(z) - R_N(z)| < \epsilon', \quad \forall z \in \mathbb{T}.$$

Assume $n > N$ and write $I_n\{f\} = \sum_{j=1}^n A_j^{(n)} f(x_j^{(n)})$; then,

$$I_\mu\{f\} - I_n\{f\} = I_\mu\{f - R_N\} + I_n\{R_N - f\}.$$

Hence,

$$|I_\mu\{f\} - I_n\{f\}| \leq \int |f(z) - R_N(z)| d\mu(z) + \sum_{j=1}^n A_j^{(n)} |f(x_j^{(n)}) - R_N(x_j^{(n)})| \leq 2\mu_0\epsilon' = \epsilon.$$

(Recall that $A_j^{(n)} > 0$, $j = 1, 2, \dots, n$, and $\sum_{j=1}^n A_j^{(n)} = \mu_0$.) \square

Assume now f a complex function defined on the unit circle \mathbb{T} which is integrable with respect to the measure μ . We can write

$$f(z) = f_1(\theta) + i f_2(\theta), \quad z = \exp(i\theta), \quad (4.1)$$

where $f_j(\theta)$, $j = 1, 2$, are real-valued functions defined on the interval $[-\pi, \pi]$. Let us first suppose f is a continuous function, or equivalently f_j , $j = 1, 2$, are continuous functions. From Theorem 5, we can write

$$\lim_{n \rightarrow \infty} I_n\{f_j\} = I_\mu\{f_j\}, \quad j = 1, 2. \quad (4.2)$$

Now paralleling rather closely the arguments given in [6, pp. 127–129], it can be seen that (4.2) is also valid for integrable functions, because any sequence of integration rules with positive weights which converges for all continuous functions converges for all integrable functions with respect to a finite Borel measure μ on $[-\pi, \pi]$.

Let now $f \in R_\mu(\mathbb{T})$. From (4.1), one can write

$$I_\mu\{f\} = I_\mu\{f_1\} + i I_\mu\{f_2\} \quad \text{and} \quad I_n\{f\} = I_n\{f_1\} + i I_n\{f_2\}. \quad (4.3)$$

Thus, by (4.2) and (4.3) the next corollary immediately follows.

Corollary 6. *Under the same hypothesis as in Theorem 5, one has*

$$\lim_{n \rightarrow \infty} I_n\{f\} = I_\mu\{f\}, \quad \text{for any } f \in R_\mu(\mathbb{T}).$$

Remark 7. When all the α_i are equal to zero, then the Blaschke condition $\sum(1 - |\alpha_n|) = \infty$ holds trivially, and the convergence of the Szegő quadrature formulas introduced in [11] is guaranteed in the class $R_\mu(\mathbb{T})$ (see also [4] for a direct proof). On the other hand, in the special case when the sequence $\{\alpha_n\}$ consists of a finite number p of points cyclically repeated (see, e.g., [3]), the Blaschke condition also holds and therefore the convergence of the corresponding quadrature process is assured.

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