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Optimal Reinsurance with One Insurer and Multiple Reinsurers

Tim Boonen, Ken Seng Tan and Sheng Chao Zhuang

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Abstract

In this paper, we consider a one-period optimal reinsurance design model with n reinsurers and an insurer. For very general preferences of the insurer, we obtain that there exists a very intuitive pricing formula for all reinsurers that use a distortion premium principle. The insurer determines its optimal risk that it wants to reinsure via this pricing formula. This risk it wants to reinsure is then shared by the reinsurers via tranching. The optimal ceded loss functions among multiple reinsurers are derived explicitly under the additional assumptions that the insurer's preferences are given by an inverse- S shaped distortion risk measure and that the reinsurer's premium principles are some functions of the Conditional Value-at-Risk. We also demonstrate that under some prescribed conditions, it is never optimal for the insurer to cede its risk to more than two reinsurers.

Key Words: optimal reinsurance design, distortion risk measure, distortion premium principle, multiple reinsurers, representative reinsurer.

1 Introduction

The optimal risk sharing between an insurer and a reinsurer is one of the most challenging problems that has been heavily studied in the academic literature and actuarial practice. This problem is first formally analyzed by Borch (1960) who demonstrates that, under the assumption the reinsurance premium is calculated by the expected value principle, the stop-loss reinsurance treaty is the optimal strategy that minimizes the variance of the retained loss of the insurer. By maximizing the expected utility of the terminal wealth of a risk-averse insurer, Arrow (1963) similarly shows that the stop-loss reinsurance treaty is optimal. These pioneering results have subsequently been refined to incorporating more sophisticated optimality criteria and/or more realistic premium principles. See for example Kaluszka (2005) and Chi and Tan (2011) for a small sample of these generalizations. These results indicate that more exotic strategy such as that based on the limited stop-loss or truncated stop-loss could be optimal, as opposed to the classical stop-loss reinsurance.

While most of the existing literature on optimal reinsurance have predominantly confined to analyzing the optimal risk sharing between two parties; i.e. an insurer and a reinsurer, recently some progress have been made on addressing the optimal reinsurance in the presence of multiple reinsurers. See for example Asimit et al. (2013), Chi and Meng (2014), and Cong and Tan (2014). Such formulation is more reasonable since in a well established reinsurance market, an insurer could always use more than one reinsurer to reinsure its risk. In fact it may be desirable for the insurer to do so in view of the differences in reinsurers' risk attitude and the competitiveness of the reinsurance market. Some reinsures may have higher risk tolerance and maybe more aggressive in pricing certain layers of risk. As a result, the insurer that exploits such discrepancy among reinsurers might be able to achieve better risk sharing profile.

Motivated by these results, this paper studies the problem of optimal reinsurance in the presence of multiple reinsurers. The significant contributions of our proposed study can be described as follows. First, we allow for very general preferences of the insurer. In contrast, both Asimit et al. (2013) and Cong and Tan (2014) assume that the insurer's preference is to minimize its value at risk (VaR) while Chi and Meng (2014) assume conditional value at risk (CVaR), in addition to VaR. Second, we allow for more than two reinsurers while the optimal reinsurance models of Asimit et al. (2013) and Chi and Meng (2014) explicitly assume two reinsurers. Third, both Asimit et al. (2013) and Chi and Meng (2014) impose the condition that one of the reinsurers adopts the expected value premium principle. Our proposed model, on the other hand, does not have such constraint. In

fact, our premium principle is quite general in that we assume the reinsurers use distortion premium principles. Many authors, including Wang (1995, 2000), Wang et al. (1997), De Waegenaere et al. (2003), Chen and Kulperger (2006), Cheung (2010) and Assa (2015), use distortion functions to price risk. Special cases include the pricing principle induced by a Wang transform, Value-at-Risk, and expected value principle. Fourth, we allow for a risk exposure of the insurer after reinsurance that may be non-decreasing. This leads to a much larger set of possible reinsurance contracts. Fifth, we also analyze the uniqueness of proposed solution. Finally, we also demonstrate that under some additional assumptions, it is never optimal for an insurer to cede its loss to more than two reinsurers.

If there is only one reinsurer and the insurer maximizes dual utility (Yaari, 1987), the optimal reinsurance contract is given by tranching of the total insurance risk as shown by Assa (2015). We extend this result to the case of the presence of multiple reinsurers. We show that the structure remains the same, but there might be more tranches, and some tranches are allocated to the other reinsurers as well.

This paper is set out as follows. The model is defined in Section 2. In Section 3, we show our main results that characterizes the representative reinsurer. Section 4 describes the reinsurance contracts if the insurer uses dual utility. Section 5 provides an example where reinsurance prices are determined via the well-known CVaR. Finally, Section 6 concludes the paper and the Appendix contains the proof to one of the propositions.

2 Model Setup

The purpose of this section is to describe various important concepts including the distortion premium principles as well as our proposed formulation of the optimal reinsurance model in the presence of multiple reinsurers. These are described in Subsection 2.1 and 2.2, respectively.

2.1 Distortion premium principles

In this subsection, we introduce the pricing formula of the reinsurers. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the class of bounded random variables on it. For brevity, we use the notation L^∞ to denote $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ when there is no confusion. We interpret random variables as a loss. We now define the distortion risk measure, which is due to Wang (1995):

Definition 2.1 *The distortion risk measure is given by*

$$\rho^g(Z) = \int_0^\infty g(S_Z(z))dz, \text{ for all } Z \in L^\infty, \quad (1)$$

where S_Z is the survival function of the loss Z and the probability distortion function $g : [0, 1] \rightarrow [0, 1]$ is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$.

Corresponding to the distortion risk measure, we have the distortion premium principle, which is defined as follows:

Definition 2.2 *The distortion premium principle is given by*

$$\pi^{\theta, g}(Z) = (1 + \theta) \cdot \rho^g(Z), \text{ for all } Z \in L^\infty, \quad (2)$$

where $\theta > -1$ and g is a probability distortion function.

Note that when $g(x) = x$ and $\theta > 0$, the above distortion premium principle reduces to the (loaded) expected value premium principle. In this case θ can be interpreted as the loading factor. When the distortion function is concave, the distortion principle recovers Wang's premium principle. Wu and Wang (2003) and Wu and Zhou (2006) provide a characterization of the distortion premium principle based on additivity of comonotonic risks. The premium principle formulation (2) allows also for pricing formulas that include a risk component, i.e., $\pi(Z) = E[Z] + \alpha \cdot \rho^{\hat{g}}(Z)$ for $\alpha \geq 0$, where $\rho^{\hat{g}}$ can be a VaR or a CVaR (see, e.g., Acerbi and Tasche, 2002).

2.2 Reinsurance model set-up

We now turn to our proposed optimal reinsurance model. We assume that an insurer faces a non-negative and bounded random loss $X \in L^\infty$ and that $M = \text{esssup } X = \inf\{a \in R : \mathbb{P}(X > a) = 0\}$. We further assume that there are n reinsurers in this market and that each of these reinsurers is indexed by $\{1, \dots, n\}$. The probability distribution of the loss exposure X of the insurer is a common knowledge to all the participating reinsurers and let $f_i(X)$ represent the portion of the loss X that is ceded to reinsurer $i, i = 1, \dots, n$. The problem of optimal reinsurance is therefore concerned with the optimal partitioning of X into $f_i(X), i = 1, \dots, n$, and $X - \sum_{i=1}^n f_i(X)$. Note that $\sum_{i=1}^n f_i(X)$ captures the aggregate loss that is ceded to all n participating reinsurers so that $X - \sum_{i=1}^n f_i(X)$ denotes the loss that is retained by the insurer.

Before we proceed, let us introduce the definition of absolute continuous function.

Definition 2.3 *A function f is absolutely continuous (or in short a.c.) on $[0, M]$ if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of $[0, M]$ satisfies $\sum_k (y_k - x_k) < \delta$, then $\sum_k |f(y_k) - f(x_k)| < \varepsilon$.*

It is a well-known result that, if f is a.c. on $[0, M]$, then there exists a Lebesgue integrable function h on $[0, M]$ such that $f(x) = f(0) + \int_0^x h(s)ds$ for all x on $[0, M]$. Moreover, h is unique almost everywhere (a.e.). For a given upper bound $N > 0$, we define the following set

$$\mathcal{F} = \left\{ f : [0, M] \rightarrow [0, N] \mid f(0) = 0, f \text{ is non-decreasing and a.c.} \right\}. \quad (3)$$

We will later require that $f_i \in \mathcal{F}$ for all $i = 1, \dots, n$ and that $\sum_{i=1}^n f_i \in \mathcal{F}$. The assumptions that reinsurance contracts are a.c. and bounded are mild technical conditions. It should be emphasized that we allow $x - \sum_{i=1}^n f_i(x)$ to be negative and decreasing. Insurance coverage increases if the risk X gets larger.

From the definition of the distortion risk measure (1) and for any $f \in \mathcal{F}$, we have

$$\rho^g(f(X)) = \int_0^M g(S_X(z))df(z), \quad (4)$$

which is due to Fubini's theorem, $f(0) = 0$, and the fact that f is a.c.

By partially transferring some of the losses to reinsurers, the insurer incurs an additional cost in the form of reinsurance premium that is payable to the reinsurers. The reinsurance premium depends on the ceded loss function $f_i(X)$, the loading factor θ_i , and the probability distortion function g_i of the reinsurer. We use $\pi^{\theta_i, g_i}(f_i(X))$ to denote the resulting reinsurance premium that is charged by the reinsurer i for assuming loss $f_i(X)$. Let W denote the future wealth of the insurer in the absence of insuring risk X . The future wealth W can be random or deterministic. By insuring and reinsuring X , the net worth of the insurer becomes $W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))$. Given that the reinsurance premium increases with the ceded losses, this implies that a conservative insurer could eliminate most of its risk by committing to a higher reinsurance premium and transferring most of the risk to reinsurers. On the other hand, a more aggressive insurer could reduce its reinsurance premium but at the expense of a higher exposure to loss. This illustrates the classical trade-off between risk retention and risk transfer.

From a risk management point of view, the existence of such a trade-off also implies that it is

important for the insurer to seek a best reinsurance strategy that optimally balances between risk retention and risk transfer. This can be accomplished by formulating the problem as an optimization problem. More specifically, let V captures the preference of an insurer's net worth. Here V is a function that maps any random variables to a real number. Furthermore, it satisfies $V(W + N) < \infty$ and that it is strictly monotonic; i.e., for all $X > Y$ almost surely, we have $V(X) > V(Y)$. Then the optimal strategy for the insurer to cede its risk to n reinsurers can be determined by solving the following optimization problem:

$$\begin{aligned} \max \quad & V\left(W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))\right) \\ \text{s.t.} \quad & f_i \in \mathcal{F}, \forall i = 1, \dots, n, \sum_{i=1}^n f_i \in \mathcal{F}. \end{aligned} \tag{5}$$

Recall that the term inside the function V gives the net worth of the insurer in the presence of n multiple reinsurers. The optimal ceded loss functions $f_i, i = 1, \dots, n$ therefore maximize the desirable preference of the insurer while subject to the condition as stipulated in \mathcal{F} . In the special case with only one reinsurer, we omit the subscript i so that the above optimal reinsurance problem simplifies to

$$\begin{aligned} \max \quad & V\left(W - X + f(X) - \pi^{\tilde{\theta}, \tilde{g}}(f(X))\right) \\ \text{s.t.} \quad & f \in \mathcal{F}. \end{aligned} \tag{6}$$

The above special case of the optimal reinsurance problem has been studied extensively in the literature. For instance, if the preference function V corresponds to the expected utility of the insurer and the reinsurance premium principle is the expected premium principle, the resulting problem is studied by Arrow (1963) who shows that the stop-loss function is optimal.

In this paper, our focus is the multiple reinsurance optimal problem (5). In particular, we will analyze the uniqueness and the construction of contracts $f_i(X), i = 1, \dots, n$.

3 Existence of a representative reinsurer - general case

We begin this section by first providing the following proposition which asserts the existence of the optimal reinsurance solution to (6) with one reinsurer.

Proposition 3.1 *For every V such that $V(W + N) < \infty$, there exists a solution to (6).*

PROOF: For any $f^1, f^2 \in \mathcal{F}$, define the norm $d(f^1, f^2) = \max_{t \in [0, M]} |f^1(t) - f^2(t)|$. The set \mathcal{F} is compact under this norm d . Moreover, we have $V(W + N) < \infty$. Hence, an optimal solution to (6) exists. \square

We now demonstrate that the above proposition facilitates us to obtain the solution to the optimal multiple reinsurance problem (5). This entails us to introduce the concept of *representative reinsurer*. We will show that there exists a representative reinsurer that uses distortion premium principle with safety loading $\tilde{\theta}$ and distortion function \tilde{g} such that $(f_i)_{i=1}^n$ solves (5) is equivalent to

- $\sum_{i=1}^n f_i$ solves (6) with $\theta = \tilde{\theta}$ and $g = \tilde{g}$;
- $\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) = \pi^{\tilde{\theta}, \tilde{g}}(\sum_{i=1}^n f_i(X))$.

This equivalence is useful in that the multiple reinsurance problem (5) can be solved by first studying problem (6).

Proposition 3.1 allows us to define the set

$$\mathcal{F}^* = \{f \in \mathcal{F} : f \text{ solves (6) with } \theta = \tilde{\theta} \text{ and } g = \tilde{g}\}.$$

For any given $f \in \mathcal{F}$, we define the following problem in which we minimize the total reinsurance premium given a total reinsurance contract $f(X)$:

$$\begin{aligned} \min \quad & \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) \\ \text{s.t.} \quad & f_i \in \mathcal{F}, \forall i = 1, \dots, n, \sum_{i=1}^n f_i = f. \end{aligned} \tag{7}$$

Given the total risk f that will be ceded to n reinsurers, the above optimization problem optimally determines the least expensive reinsurance strategy of allocating the total risk to these reinsurers. This is collected in the following set:

$$\mathcal{F}(f) = \{f_i \in \mathcal{F}, i = 1, \dots, n : (f_i)_{i=1}^n \text{ solves (7) for given } f\},$$

for all $f \in \mathcal{F}$. Since the functions f and $(f_i)_{i=1}^n$ are a.c., we can define h_i as the density of f_i for $i = 1, \dots, n$, and h as the density of f , satisfying $f_i(z) = \int_0^z h_i(x)dx$ and $f(z) = \int_0^z h(x)dx$ for any $z \in [0, M]$. In the following theorem, we characterize all optimal solutions to (7).

Theorem 3.2 Let $f \in \mathcal{F}$. Every $(f_i)_{i=1}^n \in \mathcal{F}(f)$ is such that

$$h_i(z) = \begin{cases} \lambda_i(z) & \text{if } i \in \operatorname{argmin}_{1 \leq j \leq n} (1 + \theta_j) g_j(S_X(z)), \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

for all $i = 1, \dots, n$, and all $z \in [0, M]$ almost surely, where $\lambda_i(z), z \in [0, M]$ is such that

$$\sum_{i=1}^n h_i(z) = h(z).$$

PROOF: We can rewrite the objective function of (7) as

$$\begin{aligned} \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) &= \sum_{i=1}^n (1 + \theta_i) \int_0^M g_i(S_X(z)) df_i(z) \\ &= \sum_{i=1}^n (1 + \theta_i) \int_0^M g_i(S_X(z)) h_i(z) dz. \end{aligned}$$

Hence, every h_i that minimizes the above expression such that $\sum_{i=1}^n h_i = h$ is optimal. It is easy to see that the set of optimal solutions is given by solutions satisfying (8) almost surely. \square

If $\theta_i = 0$ for all $i = 1, \dots, n$, and if the distortion functions $g_i, i = 1, \dots, n$ are all concave, the objective in (7) is the same as the objective to determine Pareto optimal risk redistributions as in Ludkovski and Young (2009). However, in the context of optimal reinsurance contracts, assuming $\theta_i = 0$ for all $i = 1, \dots, n$ is restrictive.

Corollary 3.3 For any given $f \in \mathcal{F}$, the set $\mathcal{F}(f)$ is single-valued if and only if the Lebesgue measure of the set

$$\{z \in [0, M] : h(z) > 0, |\operatorname{argmin}_{1 \leq i \leq n} (1 + \theta_i) g_i(S_X(z))| > 1\},$$

is zero, where h is the density of f , and $|A|$ denotes the cardinality of the set A .

We now proceed to stating the main theorem of this paper.

Theorem 3.4 It holds that $f_i(X), i = 1, \dots, n$, solve (5) if and only if $\sum_{i=1}^n f_i \in \mathcal{F}^*$ and $(f_i)_{i=1}^n \in \mathcal{F}(\sum_{j=1}^n f_j)$.

Before proving the above theorem, let us first prove the following three lemmas.

Lemma 3.5 *There is a unique $\tilde{\theta} > -1$ and distortion function \tilde{g} on the range of S_X such that $(1 + \tilde{\theta})\tilde{g}(S_X(z)) = \min_{1 \leq i \leq n}(1 + \theta_i)g_i(S_X(z))$ for all $z \in [0, M]$, and it is such that $\tilde{\theta} = \min_{1 \leq i \leq n} \theta_i$ and $\tilde{g}(x) = \frac{\min_{1 \leq i \leq n}(1 + \theta_i)g_i(x)}{1 + \tilde{\theta}}$ for all x on the range of S_X .*

PROOF: It follows from $\min_{1 \leq i \leq n}(1 + \theta_i)g_i(1) = (1 + \tilde{\theta})\tilde{g}(1)$ and $g_i(1) = \tilde{g}(1) = 1$ that $\tilde{\theta} = \min_{1 \leq i \leq n} \theta_i$. Moreover, $\min_{1 \leq i \leq n}(1 + \theta_i)g_i(x) = (1 + \tilde{\theta}) \frac{\min_{1 \leq i \leq n}(1 + \theta_i)g_i(x)}{1 + \tilde{\theta}} = (1 + \tilde{\theta})\tilde{g}(x)$ for all x on the range of S_X . Hence the desired result follows directly. \square

Lemma 3.6 *For any $(f_i)_{i=1}^n$, with $\sum_{i=1}^n f_i \in \mathcal{F}$, we have*

$$\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) \geq \pi^{\tilde{\theta}, \tilde{g}}\left(\sum_{i=1}^n f_i(X)\right).$$

PROOF: This result follows from

$$\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) = \sum_{i=1}^n (1 + \theta_i) \int_0^M g_i(S_X(z)) df_i(z) \quad (9)$$

$$\geq \sum_{i=1}^n \int_0^M \min_{1 \leq i \leq n} (1 + \theta_i)g_i(S_X(z)) df_i(z) \quad (10)$$

$$= \sum_{i=1}^n \int_0^M (1 + \tilde{\theta})\tilde{g}(S_X(z)) df_i(z) \quad (11)$$

$$= \int_0^M (1 + \tilde{\theta})\tilde{g}(S_X(z)) d\sum_{i=1}^n f_i(z) \quad (12)$$

$$= \pi^{\tilde{\theta}, \tilde{g}}\left(\sum_{i=1}^n f_i(X)\right), \quad (13)$$

where (9) and (13) follow from (4), (11) follows from Lemma 3.5, and (12) follows from Fubini's theorem. This concludes the proof. \square

Lemma 3.7 *Let $f \in \mathcal{F}$. If $(f_i)_{i=1}^n \in \mathcal{F}(f)$, we have $\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) = \pi^{\tilde{\theta}, \tilde{g}}(\sum_{i=1}^n f_i(X)) = \pi^{\tilde{\theta}, \tilde{g}}(f(X))$.*

PROOF: Let $f \in \mathcal{F}$ and $(f_i)_{i=1}^n \in \mathcal{F}(f)$. Then, for $f_i(X), i = 1, \dots, n$, as in Theorem 3.2, it follows that

$$\begin{aligned} \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) &= \sum_{i=1}^n \int_0^M (1 + \theta_i) g_i(S_X(z)) df_i(z) \\ &= \int_0^M \min_{1 \leq i \leq n} (1 + \theta_i) g_i(S_X(z)) dz. \end{aligned}$$

Then, we obtain that the inequality (10) in the proof of Lemma 3.6 is an equality. Hence, $\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) = \pi^{\tilde{\theta}, \tilde{g}}(\sum_{i=1}^n f_i(X)) = \pi^{\tilde{\theta}, \tilde{g}}(f(X))$, as required. \square

Proof of Theorem 3.4: First, we show the “if” part of the proof. We prove this by contradiction. Let $f \in \mathcal{F}^*$ and $(f_i)_{i=1}^n \in \mathcal{F}(f)$, but assume that $f_i(X), i = 1, \dots, n$, do not solve the problem in (5). Then, there exist reinsurance contracts $\hat{f}_i(X), i = 1, \dots, n$, that solve (5). Then, we get

$$\begin{aligned} V\left(W - X + \sum_{i=1}^n \hat{f}_i(X) - \pi^{\tilde{\theta}, \tilde{g}}\left(\sum_{i=1}^n \hat{f}_i(X)\right)\right) &\geq V\left(W - X + \sum_{i=1}^n \hat{f}_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(\hat{f}_i(X))\right) \\ &> V\left(W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))\right) \\ &= V\left(W - X + \sum_{i=1}^n f_i(X) - \pi^{\tilde{\theta}, \tilde{g}}\left(\sum_{i=1}^n f_i(X)\right)\right), \end{aligned}$$

where the first inequality is due to Lemma 3.6 and the equality follows from Lemma 3.7. This is a contradiction with the assumption that $f \in \mathcal{F}^*$. Hence, $f_i(X), i = 1, \dots, n$, solve (5).

We now continue with the “only if” part of the proof. Let $f_i(X), i = 1, \dots, n$, solve (5). Assume that we do not have $f = \sum_{j=1}^n f_j \in \mathcal{F}^*$ and $(f_i)_{i=1}^n \in \mathcal{F}(\sum_{j=1}^n f_j)$. First, suppose that $f \notin \mathcal{F}^*$. Let $\hat{f} \in \mathcal{F}^*$ and $(\hat{f}_i)_{i=1}^n \in \mathcal{F}(\hat{f})$. It follows from the conclusion for the “if” part that $\hat{f}_i(X), i = 1, \dots, n$ solves (5). Hence, we have

$$\begin{aligned} V\left(W - X + \sum_{i=1}^n \hat{f}_i(X) - \pi^{\tilde{\theta}, \tilde{g}}\left(\sum_{i=1}^n \hat{f}_i(X)\right)\right) &= V\left(W - X + \sum_{i=1}^n \hat{f}_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(\hat{f}_i(X))\right) \\ &= V\left(W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))\right) \\ &\leq V\left(W - X + \sum_{i=1}^n f_i(X) - \pi^{\tilde{\theta}, \tilde{g}}\left(\sum_{i=1}^n f_i(X)\right)\right), \end{aligned}$$

where the first equality is due to Lemma 3.7, the second equality follows from the assumption that $f_i(X), i = 1, \dots, n$, solve (5), and the inequality follows from Lemma 3.6. However, this contradicts the assumption that $f \notin \mathcal{F}^*$.

Second, suppose that $(f_i)_{i=1}^n \notin \mathcal{F}(\sum_{j=1}^n f_j)$. Then, there exists an $(\hat{f}_i)_{i=1}^n \in \mathcal{F}(\sum_{j=1}^n f_j)$. Then, we have

$$V\left(W - X + \sum_{i=1}^n \hat{f}_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(\hat{f}_i(X))\right) > V\left(W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))\right),$$

which follows from strict monotonicity of V . This is a contradiction with the assumption that $f_i(X), i = 1, \dots, n$, solve (5). Consequently, we have $f \in \mathcal{F}^*$ and $(f_i)_{i=1}^n \in \mathcal{F}(\sum_{j=1}^n f_j)$. This concludes the proof. \square

The following proposition follows directly from Corollary 3.3 and Theorem 3.4. It provides a necessary and sufficient condition for verifying the uniqueness of the optimal solution to (5).

Proposition 3.8 *The optimal solution to (5) is unique if and only if the set \mathcal{F}^* is single-valued and the Lebesgue measure of the set*

$$\{z \in [0, M] : h(z) > 0, |\operatorname{argmin}_{1 \leq i \leq n} (1 + \theta_i)g_i(S_X(z))| > 1\}, \quad (14)$$

is zero, where h is the density of f and $f \in \mathcal{F}^*$.

The following proposition asserts that if it is optimal to cede f_i to reinsurer i with loading factor θ_i and distortion function g_i , then it is not possible to find another reinsurer that offers cheaper reinsurance premium for assuming the same ceded risk f_i . This also implies that in an optimal reinsurance arrangement, there is no incentive for the insurer to switch risks from one reinsurer to the other.

Proposition 3.9 *Assume that $f_i(X), i = 1, \dots, n$, solve (5), then for all $i, j = 1, \dots, n$, we have*

$$\pi^{\theta_i, g_i}(f_i(X)) \leq \pi^{\theta_j, g_j}(f_i(X)).$$

Moreover, if the Lebesgue measure of the set in (14) is zero, we have that there exist $i \neq j$ such that

$$\pi^{\theta_i, g_i}(f_i(X)) < \pi^{\theta_j, g_j}(f_i(X)).$$

PROOF: We first prove the first result by contradiction. Suppose that $f_i(X), i = 1, \dots, n$, solve (5) and that there exist i, j such that $\pi^{\theta_i, g_i}(f_i(X)) > \pi^{\theta_j, g_j}(f_i(X))$. Define the following contracts

$$\tilde{f}_k(z) = \begin{cases} f_k(X) & \text{if } k \neq i \text{ or } j, \\ 0 & \text{if } k = i, \\ f_i(X) + f_j(X) & \text{if } k = j, \end{cases}$$

for all $k = 1, \dots, n$. From the fact that $\pi^{\theta_i, g_i}(f_i(X)) + \pi^{\theta_j, g_j}(f_j(X)) > \pi^{\theta_j, g_j}(f_i(X)) + \pi^{\theta_j, g_j}(f_j(X)) = \pi^{\theta_j, g_j}(f_i(X) + f_j(X))$, we obtain $\sum_{i=1}^n \pi^{\theta_i, g_i}(\tilde{f}_i(X)) < \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))$. Moreover, we have $\sum_{i=1}^n \tilde{f}_i(X) = \sum_{i=1}^n f_i(X) = f(X)$. So, $f_i(X), i = 1, \dots, n$, is not minimizing (7). Hence, it follows from Theorem 3.4 that $f_i(X), i = 1, \dots, n$ does not solve (5). This is a contradiction.

The proof of the second result is analogue, where we note that from Corollary 3.3 there is a unique sequence $f_i(X), i = 1, \dots, n$, with $\sum_{i=1}^n f_i(X) = f(X)$ that minimizes $\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))$. Then, the result follows from Theorem 3.4. \square

Remark 1 *If there exists a reinsurer i such that $(1 + \theta_i)g_i(S_X(z)) \geq \min_{1 \leq j \leq n}(1 + \theta_j)g_j(S_X(z))$ for all $z \in [0, M]$, then there exists an optimal solution such that $f_i(X) = 0$. This follows from Theorem 3.2, Theorem 3.4 and Proposition 3.9. Moreover, if there exists a firm i such that $(1 + \theta_i)g_i(S_X(z)) > \min_{1 \leq j \leq n}(1 + \theta_j)g_j(S_X(z))$ for all $z \in (0, M]$, then all optimal solutions are such that $f_i(X) = 0$.*

Remark 2 *If there exists another reinsurer which cannot be represented by the distortion premium principle, then our results also hold in the sense that we can find a representative reinsurer to represent the n reinsurers that use a distortion premium principle. The proof is similar to Theorem 3.4 and hence is omitted.*

To conclude this section, let us now consider the following set of insurance contracts:

$$\mathcal{F}^L = \left\{ f : [0, M] \rightarrow [0, M] \mid f(0) = 0, 0 \leq f(x) - f(y) \leq x - y, \forall 0 \leq y < x \leq M \right\}. \quad (15)$$

All results in Section 3 still hold true if the feasible sets in (5) and (6) are changed to \mathcal{F}^L . This assumption is often imposed in optimal insurance contract design (see, e.g., Young, 1999; Chi and Tan, 2011; Asimit et al., 2013; Chi and Meng, 2014; Xu et al., 2015). The reason is that it prevents

moral hazard of the insurer (see, e.g., Bernard and Tian, 2009). For a given pair $(\tilde{\theta}, \tilde{g})$, the problem (6) with $\mathcal{F} = \mathcal{F}^L$ is well-studied for various functional forms of the utility function V . For instance, if V is a rank-dependent expected utility function and $\tilde{g}(s) = s$ for all $s \in [0, 1]$, this problem has a solution as shown by Xu et al. (2015). Moreover, if V is an expected utility function, the distortion function \tilde{g} is concave or piece-wise linear, and $\tilde{\theta} = 0$, then we get the optimal solution from Young (1999).

Remark 3 *As pointed earlier that the assumptions $f_i \in \mathcal{F}$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n f_i \in \mathcal{F}$ could lead to $x - \sum_{i=1}^n f_i(x)$ that is decreasing and even negative. This can be criticized for being unreasonable and exposes insurer to moral hazard. These issues could be rectified by imposing stronger assumptions of $f_i \in \mathcal{F}^L$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n f_i \in \mathcal{F}^L$. Under these assumptions, $X - \sum_{i=1}^n f_i(X)$ is non-decreasing and could be used to reduce moral hazard (see Asimit et al., 2013; Chi and Meng, 2014). The drawback of the latter model is that the resulting reinsurance model imposes severe limitation on the feasibility of the reinsurance designs. For example, suppose there are two reinsurers in the market and both reinsurers offer stop-loss reinsurance of the form $f_1(x) = (x-a)^+$ and $f_2(x) = (x-b)^+$ (with $a < b$), then $x - f_1(x) - f_2(x)$ is decreasing for $x > b$. In other words the latter model precludes the possibility of both reinsurers to offer the stop-loss reinsurance contract. In any case, the paper provides analytical solutions to both of these cases.*

4 Special case: preferences given by dual utility

In this section, we study the case where the preferences of the insurer are given by minimizing a distortion risk measure. This corresponds with dual utility, as introduced by Yaari (1987) via a modification of the independence axiom for expected utility. Preferences given by maximizing dual utility are equivalent with minimizing distortion risk measures. Hence we have $V(X) = -\rho^g(-X)$. These preferences have also been used as preferences of the insurer in bilateral reinsurance problems by Zheng and Cui (2014). Our preferences, however, are slightly less general than the law-invariant convex risk measures used by Cheung et al. (2014) in a bilateral reinsurance set-up. Nevertheless, both papers only show optimal reinsurance contracts for the expected value premium principle, whereas we use the more general distortion premium principle. For our problem, the optimal reinsurance

design problem is given by

$$\begin{aligned} \max \quad & -\rho^g\left(-W + X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))\right) \\ \text{s.t.} \quad & f_i \in \mathcal{F}, \forall i = 1, \dots, n, \sum_{i=1}^n f_i \in \mathcal{F}. \end{aligned} \tag{16}$$

where W is the deterministic wealth.

As in Assa (2015), we similarly assume that the premium is given by the general distortion premium principle. In contrast to Assa (2015), we solve the optimal reinsurance problem in the context of multiple reinsurers. Moreover, we show that under mild conditions, the optimal reinsurance contracts are Lipschitz as assumed by Assa (2015). The distortion risk measure used by the insurer might be generated by any non-decreasing distortion function, especially the inverse-S shaped distortion function which has recently gained popularity in behavioral finance (see Section 5).

By the translation invariance property, the objective function in (16) can be simplified as follows:

$$-\rho^g\left(-W + X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))\right) = W - \rho^g\left(X - \sum_{i=1}^n f_i(X)\right) - \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)), \tag{17}$$

since W and $-\sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X))$ are constants.

Proposition 4.1 *Let $\theta_i = 0$ for all $i = 1, \dots, n$. If ρ^g and all $g_i, i = 1, \dots, n$ are concave, there exists a solution $f_i(X), i = 1, \dots, n$, to (16) that is such that $f_i \in \mathcal{F}^L$ for all $i = 1, \dots, n$, and $\sum_{i=1}^n f_i \in \mathcal{F}^L$. Moreover, if ρ^g and $g_i, i = 1, \dots, n$, are all strictly concave, then every solution $f_i(X), i = 1, \dots, n$, to (16) that is such that $f_i \in \mathcal{F}^L$ for all $i = 1, \dots, n$, and $\sum_{i=1}^n f_i \in \mathcal{F}^L$.*

PROOF: First, we show the first result. From Theorem 3.4, it is sufficient to solve the problems:

$$\operatorname{argmin}_{f \in \mathcal{F}} \rho^g(-W + X - f(X) + \pi^{\tilde{\theta}, \tilde{g}}(f(X))). \tag{18}$$

From (17) it follows that we should solve

$$\operatorname{argmin}_{f \in \mathcal{F}} \rho^g(X - f(X)) + \rho^{\tilde{g}}(f(X)). \tag{19}$$

Existence of optimal $f(X)$ is shown in Proposition 3.1. Suppose we only require $f(0) = 0$, and not

$f \in \mathcal{F}$. Solving (19) is then a classical risk sharing problem, where we treat $\rho^{\tilde{g}}$ as preferences of a second representative party, and the unconstrained solutions of (19) as the set of all Pareto optima of cash-invariant preferences (see, e.g., Jouini et al., 2008). Ludkovski and Rüschendorf (2008) show that there exists a risk redistribution $(X - f(X), f(X))$ that is comonotonic with the aggregate risk X , and minimizes (19). Moreover, let $c = -f(0)$. Note that $(X - f(X) - c, f(X) + c)$ minimizes (19) as well. Moreover, it holds that $(f + c) \in \mathcal{F}^L$. Then, the first result follows from this, and that $f_i \in \mathcal{F}, i = 1, \dots, n$, and $\sum_{i=1}^n f_i \in \mathcal{F}^L$ implies $f_i \in \mathcal{F}^L$ for all $i = 1, \dots, n$.

Next, we show the second result. Strict concavity of a probability distortion function coincides with preferences that strictly preserve second order stochastic dominance (see Wirth and Hardy, 2001). We again solve the problem

$$\min \rho^g(X - \sum_{i=1}^n f_i(X)) + \sum_{i=1}^n \rho^{g_i}(f_i(X)), \quad (20)$$

such that $f_i(0) = 0$ for all $i = 1, \dots, n$. Carlier et al. (2012) show that since all preferences strictly preserve second order stochastic dominance, all (unconstrained) solutions to (20) are comonotonic with the aggregate risk X . Since the functions are cash-invariant, there exists a solution satisfying $f_i(0) = 0$ for all $i = 1, \dots, n$ as well (see, e.g., Jouini et al., 2008). \square

If the probability distortion functions $g_i, i = 1, \dots, n$, are all concave, then the function \tilde{g} is also concave. So, the representative distortion premium principle $\pi^{\tilde{\theta}, \tilde{g}}$ is concave for any $\tilde{\theta}$. Concavity of the distortion function resembles risk aversion of the corresponding reinsurer.

If $\theta_i \geq 0$ for all $i = 1, \dots, n$, we will later provide the same results as in Proposition 4.1 if the state space is finite.

As we want to allow for non-concave probability distortion functions, we explicitly consider only solutions to (16) with $f_i \in \mathcal{F}^L, \forall i = 1, \dots, n$, and $\sum_{i=1}^n f_i \in \mathcal{F}^L$. Then, we have

$$\rho^g\left(X - \sum_{i=1}^n f_i(X)\right) = \rho^g(X) - \sum_{i=1}^n \rho^g(f_i(X)), \quad (21)$$

which holds as $f_i(X), i = 1, \dots, n, \sum_{i=1}^n f_i(X)$ and $X - \sum_{i=1}^n f_i(X)$ are comonotonic random variables. We can ignore $\rho^g(X)$ in the objective function (21) since it is a constant and does not depend on $f_i(X), i = 1, \dots, n$.

Similarly, we have $\rho^g(f_i(X)) = \int_0^M g(S_X(z))df_i(z)$. To summarize, the objective function in (17)

can be replaced by

$$\begin{aligned} \sum_{i=1}^n \rho^g(f_i(X)) - \sum_{i=1}^n \pi^{\theta_i, g_i}(f_i(X)) &= \sum_{i=1}^n \int_0^M g(S_X(z)) df_i(z) - \sum_{i=1}^n (1 + \theta_i) \int_0^M g_i(S_X(z)) df_i(z) \\ &= \sum_{i=1}^n \int_0^M \left[g(S_X(z)) - (1 + \theta_i) g_i(S_X(z)) \right] df_i(z). \end{aligned} \quad (22)$$

So, we write (16) as follows

$$\begin{aligned} \min \quad & \sum_{i=1}^n \int_0^M \left[(1 + \theta_i) g_i(S_X(z)) - g(S_X(z)) \right] df_i(z) \\ \text{s.t.} \quad & f_i \in \mathcal{F}^L, \forall i = 1, \dots, n, \sum_{i=1}^n f_i \in \mathcal{F}^L, \end{aligned} \quad (23)$$

We denote $\mathcal{H} = \{h : [0, M] \rightarrow [0, 1] \mid 0 \leq h(z) \leq 1, \text{ a.s.}\}$. Since $f_i \in \mathcal{F}^L, \forall i = 1, \dots, n, \sum_{i=1}^n f_i \in \mathcal{F}^L$, we can write (23) as

$$\begin{aligned} \min \quad & \sum_{i=1}^n \int_0^M \left[(1 + \theta_i) g_i(S_X(z)) - g(S_X(z)) \right] h_i(z) dz \\ \text{s.t.} \quad & h_i \in \mathcal{H}, \forall i = 1, \dots, n, \sum_{i=1}^n h_i \in \mathcal{H}. \end{aligned} \quad (24)$$

By defining $A = \{z \in [0, M] : \min_{1 \leq i \leq n} \{(1 + \theta_i) g_i(S_X(z)) - g(S_X(z))\} < 0\}$, $B = \{z \in [0, M] : \min_{1 \leq i \leq n} \{(1 + \theta_i) g_i(S_X(z)) - g(S_X(z))\} = 0\}$ and $C = \{z \in [0, M] : \min_{1 \leq i \leq n} \{(1 + \theta_i) g_i(S_X(z)) - g(S_X(z))\} > 0\}$, it follows directly from Theorem 3.4 and Theorem 3.2 that we have the following proposition:

Proposition 4.2 *Every optimal solution to (24) satisfies*

$$h_i(z) = \begin{cases} \lambda_i(z) & \text{if } z \in A \text{ and } i \in \operatorname{argmin}_{1 \leq j \leq n} \{(1 + \theta_j) g_j(S_X(z)) - g(S_X(z))\}, \\ \beta_i(z) & \text{if } z \in B \text{ and } i \in \operatorname{argmin}_{1 \leq j \leq n} \{(1 + \theta_j) g_j(S_X(z)) - g(S_X(z))\}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $z \in [0, M]$, where $\lambda_i(z), \beta_i(z) \in [0, 1]$ are such that

$$\sum_{i=1}^n h_i(z) = \begin{cases} 1 & \text{if } z \in A, \\ \beta(z) & \text{if } z \in B, \\ 0 & \text{if } z \in C, \end{cases}$$

where $\beta(z) \in [0, 1]$ for any $z \in B$.

It follows immediately from Proposition 4.2 that there is a unique optimal solution to (24) if and only if the Lebesgue measure of B is zero and $\operatorname{argmin}_{1 \leq j \leq n} \{(1 + \theta_j)g_j(S_X(z)) - g(S_X(z))\}$ is unique almost surely for $z \in A$.

When we solve

$$\begin{aligned} \min \quad & \rho^g\left(X - f(X) + \pi^{\tilde{\theta}, \tilde{g}}(f(X))\right) \\ \text{s.t.} \quad & f \in \mathcal{F}, \end{aligned} \tag{25}$$

we get in a similar fashion as for (21) and (22) that every solution to (25) is such that

$$h(z) = \begin{cases} 1 & \text{if } z \in A, \\ \beta(z) & \text{if } z \in B, \\ 0 & \text{if } z \in C, \end{cases}$$

where $\beta(z) \in [0, 1]$ for all $z \in B$ almost surely. This result with one reinsurer is alternatively established in Assa (2015).

To conclude this section, we demonstrate that under the assumption of finite state space such that every event on the state space has a positive probability, the result of Proposition 4.1 can be extended to the case where $\theta_i \geq 0$ for all $i = 1, \dots, n$. We relegate its proof to the appendix.

Proposition 4.3 *Let the state space be finite and $\theta_i \geq 0$ for all $i = 1, \dots, n$. If ρ^g and all $g_i, i = 1, \dots, n$ are concave, there exists a solution to (16) that is in \mathcal{F}^L . Moreover, if ρ^g and $g_i, i = 1, \dots, n$ are all strictly concave, then every solution to (16) is in \mathcal{F}^L .*

Remark 4 *We hypothesize that Propositions 4.1 and 4.3 hold true also when $\theta_i \geq 0$ for all $i = 1, \dots, n$ and when there is an infinite state space. This research question is related to the papers of Landsberger and Meilijson (1994), Ludkovski and Rüschemdorf (2008), Ludkovski and Young (2009), and Carlier et al. (2012). We leave this question open for future research.*

5 Numerical illustration with Conditional Value-at-Risk (CVaR)

In this section, we illustrate the concept of representative reinsurer by means of an example involving CVaR. First, let us recall the definitions of VaR and CVaR risk measures. The VaR

of a non-negative random variable Z at a confidence level $1 - \alpha$ where $0 < \alpha < 1$ is given by $VaR_\alpha(Z) = \inf\{z \geq 0 : \mathbb{P}(Z > z) \leq \alpha\}$. The CVaR of Z at a confidence level $1 - \alpha$ is given by $CVaR_\alpha(Z) = \frac{1}{\alpha} \int_0^\alpha VaR_s(Z) ds$. Here, $CVaR_\alpha$ is the distortion premium principle with $\theta = 0$ and $g(s) = \min\{\frac{s}{\alpha}, 1\}$ for all $s \in [0, 1]$ (see, e.g., Dhaene et al., 2003).

In this example, reinsurer i is assumed to adopt the following premium principle:

$$\pi_i(X) = E[X] + \alpha_i(CVaR_{\beta_i}(X) - E[X]) \text{ for all } i = 1, \dots, n, \quad (26)$$

where $\alpha_i, \beta_i \in (0, 1)$. This is an example of a distortion premium principle, as can be seen by setting $\theta_i = 0$ and g_i to

$$g_i(s) = \begin{cases} (1 - \alpha_i + \frac{\alpha_i}{\beta_i})s & \text{if } 0 \leq s \leq \beta_i, \\ (1 - \alpha_i)s + \alpha_i & \text{if } \beta_i < s \leq 1. \end{cases}$$

By defining $i^* \in \operatorname{argmax}_{1 \leq i \leq n} \alpha_i \left(1 - \frac{1}{\beta_i}\right)$, $j^* \in \operatorname{argmin}_{1 \leq j \leq n} \alpha_j$, and setting

$$\hat{s} = \frac{\alpha_{j^*}}{\alpha_{j^*} + \alpha_{i^*} \left(\frac{1}{\beta_{i^*}} - 1\right)} \in (0, 1), \quad (27)$$

it is easy to show that $g_{i^*}(s) \leq g_{j^*}(s)$ for $s \in [0, \hat{s}]$ and $g_{i^*}(s) \geq g_{j^*}(s)$ for $s \in [\hat{s}, 1]$. We derive that the pricing principle of the representative reinsurer is such that $\tilde{\theta} = 0$, and the distortion function \tilde{g} is given by

$$\tilde{g}(s) = \begin{cases} (1 - \alpha_{i^*} + \frac{\alpha_{i^*}}{\beta_{i^*}})s & \text{if } 0 \leq s \leq \hat{s}, \\ (1 - \alpha_{j^*})s + \alpha_{j^*} & \text{if } \hat{s} < s \leq 1. \end{cases}$$

So, if $i^* = j^*$, we have $\tilde{g} = g_{i^*}$.

In this example, we wish to study an insurer that is endowed with an inverse- S shaped distortion function. First, we provide the definition.

Definition 5.1 *A distortion function g is inverse- S shaped if:*

- *it is continuously differentiable;*
- *there exists $b \in (0, 1)$ such that g is strictly concave on the domain $(0, b)$ and strictly convex on the domain $(b, 1)$;*

- it holds that $g'(0) = \lim_{s \downarrow 0} g'(s) > 1$ and $g'(1) = \lim_{s \uparrow 1} g'(s) > 1$.

Tversky and Kahneman (1992) propose an inverse- S distortion function, which is given by $g^\gamma(s) = s^\gamma / (s^\gamma + (1-s)^\gamma)^{\frac{1}{\gamma}}$, for all $s \in [0, 1]$, where $\gamma > 0$. As noted by Rieger and Wang (2006) and Ingersoll (2008), this probability distortion function is increasing and exhibit inverse- S shaped for any $\gamma \in (0.279, 1)$.

For inverse- S shaped distortion functions, the next property of the function

$$p(s) = \frac{1 - g(s)}{1 - s}, \text{ for all } s \in [0, 1),$$

follows from Xu et al. (2015).

Lemma 5.1 *The function p is continuous. Moreover, there exists $a \in (0, b)$ such that $p(s)$ is strictly decreasing on $[0, a]$ and strictly increasing on $[a, 1]$.*

According to the results established in the previous section, we need to study the sign of the difference between the distortion functions g and \tilde{g} . It turns out that in our example, there are six cases to consider. Each case corresponds to a specific structure of the reinsurance contracts. We first split these cases depending on the relative magnitude of $\tilde{g}'(0)$ and $g'(0)$. In particular, the condition $\tilde{g}'(0) = 1 - \alpha_{i^*} + \frac{\alpha_{i^*}}{\beta_{i^*}} < g'(0)$ leads to four cases for locations of $(\hat{s}, \tilde{g}(\hat{s}))$, as depicted graphically in Figure 1. In what follows, we determine $f_{i^*}(X)$ and $f_{j^*}(X)$ separately. If $i^* = j^*$, then this reinsurer reinsures the risk $f_{i^*}(X) + f_{j^*}(X)$.

Case 5.1 If $\tilde{g}(\hat{s}) \leq g(\hat{s})$, then there exists a $c \in [\hat{s}, 1)$ such that $\tilde{g}(s) < g(s)$ for $s \in (0, c)$ and $\tilde{g}(s) > g(s)$ for $z \in (c, 1)$. Therefore, it follows from Proposition 4.2 that the optimal solution to (25) is given by

$$h(z) = \begin{cases} 1 & \text{if } 0 \leq S_X(z) \leq c, \\ 0 & \text{if } c < S_X(z) \leq 1, \end{cases}$$

for all $z \in [0, M]$ almost surely, or

$$h(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_c(X), \\ 1 & \text{if } VaR_c(X) < z \leq M. \end{cases}$$

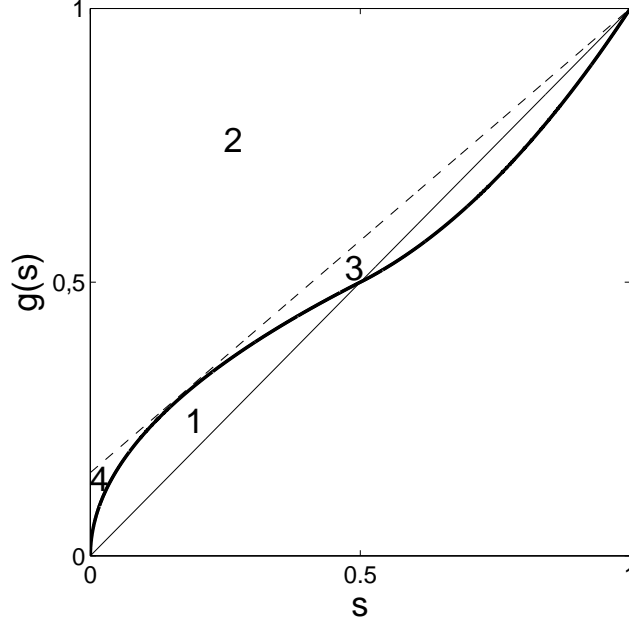


Figure 1: This figure displays an inverse- S shaped probability distortion function. Moreover, it shows the first four cases for any location of $(\hat{s}, \tilde{g}(\hat{s}))$, where $1 \leq \tilde{g}'(0) < g'(0)$.

This leads to

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_c(X), \\ z - VaR_c(X) & \text{if } VaR_c(X) < z \leq M, \end{cases}$$

or, equivalently, $f(X) = (X - VaR_c(X))^+$, where $(Y)^+ = \max\{Y, 0\}$. Moreover, Proposition 4.2 implies that an optimal solution for the individual reinsurance contracts is $f_{i^*}(X) = (X - VaR_{\hat{s}}(X))^+$, $f_{j^*}(X) = \min\{(X - VaR_c(X))^+, VaR_{\hat{s}}(X) - VaR_c(X)\}$, and $f_i(X) = 0$ for all $i \neq i^*, j^*$.

Case 5.2 If $\tilde{g}(\hat{s}) > g(\hat{s})$ and $p(a) \geq 1 - \alpha_{j^*}$, then there exists a point $c \in (0, \hat{s})$ such that $\tilde{g}(s) < g(s)$ for $s \in (0, c)$ and $\tilde{g}(s) > g(s)$ for $s \in (c, 1)$. The optimal solution to (25) is given by $f(X) = (X - VaR_c(X))^+$. Moreover, an optimal solution for the individual reinsurance contracts is $f_{i^*}(X) = f(X)$, and $f_i(X) = 0$ for all $i \neq i^*$.

Case 5.3 If $\tilde{g}(\hat{s}) > g(\hat{s})$, $\hat{s} \geq a$ and $p(a) < 1 - \alpha_{j^*}$, the solution coincides with the solution of Case 5.2.

Case 5.4 If $\tilde{g}(\hat{s}) > g(\hat{s})$, $\hat{s} < a$ and $p(a) < 1 - \alpha_{j^*}$, then there exists three points $c \in (0, \hat{s})$, $d \in (\hat{s}, a)$ and $e \in (a, 1)$ such that $\tilde{g}(s) < g(s)$ for $s \in (0, c)$, $\tilde{g}(s) > g(s)$ for $s \in (c, d)$, $\tilde{g}(s) < g(s)$ for $s \in (d, e)$

and $\tilde{g}(s) > g(s)$ for $s \in (e, 1)$. The optimal solution to (25) is given by

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_e(X), \\ z - VaR_e(X) & \text{if } VaR_e(X) < z \leq VaR_d(X), \\ VaR_d(X) - VaR_e(X) & \text{if } VaR_d(X) < z \leq VaR_c(X), \\ z - VaR_c(X) + VaR_d(X) - VaR_e(X) & \text{if } VaR_c(X) < z \leq M, \end{cases}$$

or, equivalently, $f(X) = \min\{(X - VaR_e(X))^+, VaR_d(X) - VaR_e(X)\} + (X - VaR_c(X))^+$. Moreover, an optimal solution for the individual reinsurance contracts is $f_{i^*}(X) = (X - VaR_c(X))^+$, $f_{j^*}(X) = \min\{(X - VaR_e(X))^+, VaR_d(X) - VaR_e(X)\}$, and $f_i(X) = 0$ for all $i \neq i^*, j^*$.

We now consider the remaining two cases, as stipulated by the condition $\tilde{g}'(0) = 1 - \alpha_{i^*} + \frac{\alpha_{j^*}}{\beta_{j^*}} \geq g'(0)$. These two cases are graphically displayed in Figure 2.

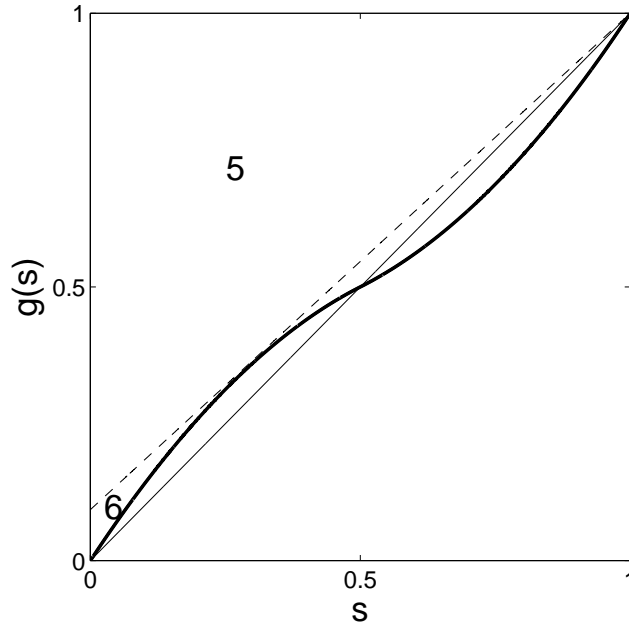


Figure 2: This figure displays the two different cases where $\tilde{g}'(0) > g'(0)$ holds.

Case 5.5 If $p(a) \leq 1 - \alpha_{j^*}$, then $\tilde{g}(s) > g(s)$ for $s \in (0, 1)$. The optimal solution to (25) is $f(z) = 0$ for all $z \in [0, M]$, i.e., no reinsurance is optimal.

Case 5.6 If $p(a) > 1 - \alpha_{j^*}$, then there exist two points $c \in (0, a)$ and $d \in (a, 1)$ such that $\tilde{g}(s) > g(s)$ for $s \in (0, c)$, $\tilde{g}(s) < g(s)$ for $s \in (c, d)$, and $\tilde{g}(s) > g(s)$ for $s \in (d, 1)$. The optimal solution to (25)

is given by

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq VaR_d(X), \\ z - VaR_d(X) & \text{if } VaR_d(X) < z \leq VaR_c(X), \\ VaR_c(X) - VaR_d(X) & \text{if } VaR_c(X) < z \leq M, \end{cases}$$

or, equivalently, $f(X) = \min\{(X - VaR_d(X))^+, VaR_c(X) - VaR_d(X)\}$. Moreover, an optimal solution for the individual reinsurance contracts is $f_{j^*}(X) = f(X)$, and $f_i(X) = 0$ for all $i \neq j^*$.

We now provide another example for which the n reinsurers adopt the following premium principles:

$$\pi_i(X) = E[X] + \alpha_i CVaR_{\beta_i}(X), \text{ for all } i = 1, \dots, n, \quad (28)$$

where $\alpha_i \geq 0$ and $\beta_i \in (0, 1)$. Note that this is another example of a distortion premium principle with $\theta_i = \alpha_i$ and g_i given by

$$g_i(s) = \begin{cases} \frac{\beta_i + \alpha_i}{\beta_i(1 + \alpha_i)} s & \text{if } 0 \leq s \leq \beta_i, \\ \frac{s + \alpha_i}{1 + \alpha_i} & \text{if } \beta_i < s \leq 1. \end{cases}$$

Let $i^* \in \operatorname{argmin}_{1 \leq i \leq n} \frac{\alpha_i}{\beta_i}$ and $j^* \in \operatorname{argmin}_{1 \leq j \leq n} \alpha_j$. For $\hat{s} = \frac{\alpha_{j^*} \beta_{i^*}}{\alpha_{i^*}} \in (0, 1)$, we have $(1 + \alpha_{i^*})g_{i^*}(s) \leq (1 + \alpha_{j^*})g_{j^*}(s)$ for $s \in [0, \hat{s}]$, and $(1 + \alpha_{i^*})g_{i^*}(s) \geq (1 + \alpha_{j^*})g_{j^*}(s)$ for $s \in [\hat{s}, 1]$. We derive that the pricing principle of the representative reinsurer is such that $\tilde{\theta} = \alpha_{j^*}$, and the distortion function \tilde{g} is given by

$$\tilde{g}(s) = \begin{cases} \frac{\beta_{i^*} + \alpha_{i^*}}{\beta_{i^*}(1 + \alpha_{j^*})} s & \text{if } 0 \leq s \leq \hat{s}, \\ \frac{s + \alpha_{j^*}}{1 + \alpha_{j^*}} & \text{if } \hat{s} < s \leq 1. \end{cases}$$

So, if $i^* = j^*$, we have $\tilde{g} = g_{i^*}$. The remaining analysis of deriving the optimal ceded loss function is similar to the previous example and hence is omitted for brevity.

From the representations of \tilde{g} in this section, we obtain the following proposition directly from Theorem 3.4.

Proposition 5.2 *If $\pi_i, i = 1, \dots, n$, are all of the form of either (26) or (28), then there exist*

reinsurance contracts $f_i, i = 1, \dots, n$, solving (16) such that $f_i(X) \neq 0$ for at most two reinsurers.

The above result is interesting and to some extent surprising. It implies that in a well functioning reinsurance market in which an insurer could cede its risk to multiple reinsurers, it is never optimal to cede its risk to more than two reinsurers.

6 Conclusion

In this paper, we study the problem of optimal reinsurance in the presence of multiple reinsurers. When all reinsurers use a generalized distortion premium principle, we derive that there exists a representative pricing principle in the market. This pricing principle is a distortion pricing principle as well, and is used to determine the optimal aggregate reinsurance contract and its price.

If the insurer minimizes a distortion risk measure of its own risk, the optimal reinsurance contract is such that there exists “tranching” of the insurance risk. The insurance risk will be partitioned in layers, and any layer will be either retained, or reinsured by a particular reinsurer. The optimal ceded loss functions among multiple reinsurers are derived explicitly under the additional assumptions that the insurer’s preferences are given by an inverse- S shaped distortion risk measure and that the reinsurer’s premium principles are some functions of the Conditional Value-at-Risk. An interesting result of our analysis is that for our prescribed example, it is never optimal for the insurer to cede its risk to more than two reinsurers.

Appendix: Proof of Proposition 4.3

Let the state space be finite. Without loss of generality, we let the state space be given by $\{\omega_1, \dots, \omega_p\}$ with $p < \infty$ and $X(\omega_1) \leq \dots \leq X(\omega_p)$. Existence of an $f \in \mathcal{F}^*$ is shown in Proposition 3.1. We find all solutions by minimizing $\rho^g(X - f(X)) + \pi^{\tilde{\theta}, \tilde{g}}(f(X))$ over all $f \in \mathcal{F}$. Let $g^*(s) = \min\{g(s), (1 + \tilde{\theta})\tilde{g}(s)\}$ for all $s \in [0, 1]$, which is a distortion function due to the fact that $\tilde{\theta} \geq 0$. Since $g_i, i = 1, \dots, n$, are all concave, the function g^* is concave as well. From Proposition 4.2, we derive that there exists an $f \in \mathcal{F}^L$ such that

$$\rho^g(X - f(X)) + \pi^{\tilde{\theta}, \tilde{g}}(f(X)) = \rho^{g^*}(X). \quad (29)$$

Let $f \in \mathcal{F}^*$, and suppose $X - f(X)$ is not non-decreasing in X , i.e., there exist states ω_k, ω_{k+1}

such that $X(\omega_k) - f(X(\omega_k)) > X(\omega_{k+1}) - f(X(\omega_{k+1}))$. Recall that $\rho^{g^*}(X) = E_{\mathbb{Q}}[X]$, where \mathbb{Q} is the additive probability measure such that $\mathbb{Q}(\{\omega_\ell\}) = g^*(\mathbb{P}(\{\omega_1, \dots, \omega_\ell\})) - g^*(\mathbb{P}(\{\omega_1, \dots, \omega_{\ell-1}\}))$ for all $\ell = 1, \dots, p$. Due to concavity of the function g^* and due to $g^*(s) \leq g(s)$ for all $s \in [0, 1]$, it holds that (see Boonen, 2015):

$$\rho^g(Y) \geq \rho^{g^*}(Y) \geq E_{\mathbb{Q}}[Y], \text{ for all } Y. \quad (30)$$

Moreover, since $f \in \mathcal{F}$ and $g^*(s) \leq (1 + \tilde{\theta}_i)\tilde{g}(s)$ for all $s \in [0, 1]$, we obtain

$$\pi^{\tilde{\theta}, \tilde{g}}(f(X)) \geq \tilde{E}_{\mathbb{Q}}[f(X)]. \quad (31)$$

From this, it follows that the any solution satisfying (29) is optimal. Hence, any solution as in Proposition 4.2 is optimal. This concludes the first part of the proof.

We now consider the second part of the proof. Let $g_i, i = 1, \dots, n$, be strictly concave; then so is the function g^* . It follows from Boonen (2015; see the proof of Proposition 3.7)¹ is that

$$\rho^{g^*}(X - f(X)) > E_{\mathbb{Q}}[X - f(X)]. \quad (32)$$

Then, it follows that

$$\rho^g(X - f(X)) + \pi^{\tilde{\theta}, \tilde{g}}(f(X)) > E_{\mathbb{Q}}[X - f(X)] + \pi^{\tilde{\theta}, \tilde{g}}(f(X)) \quad (33)$$

$$\geq E_{\mathbb{Q}}[X], \quad (34)$$

where (33) follows from (30) and (32), and (34) follows from (31). Hence, the contract f does not solve (18). This concludes the proof.

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¹For review only: Proposition 3.7 in Boonen (2015) can also be find as Lemma 4.3.13 in https://pure.uvt.nl/portal/files/1569038/dissertation_Tim_Boonen.pdf (a Ph.D.-thesis).

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