Generalized algebra-valued models of set theory

Löwe, B.; Tarafder, S.

DOI
10.1017/S175502031400046X

Publication date
2015

Document Version
Final published version

Published in
Review of Symbolic Logic

THE REVIEW OF SYMBOLIC LOGIC
Volume 8, Number 1, March 2015

GENERALIZED ALGEBRA-VALUED MODELS OF SET THEORY

BENEDIKT LÖWE

Institute for Logic, Language and Computation, Universiteit van Amsterdam and Fachbereich Mathematik, Universität Hamburg

SOURAV TARAFDER

Department of Commerce (Morning), St. Xavier's College and Department of Pure Mathematics, Calcutta University

Abstract. We generalize the construction of lattice-valued models of set theory due to Takeuti, Titani, Kozawa and Ozawa to a wider class of algebras and show that this yields a model of a paraconsistent logic that validates all axioms of the negation-free fragment of Zermelo-Fraenkel set theory.

§1. Introduction. If $\mathbb{B}$ is any Boolean algebra and $\mathbf{V}$ a model of set theory, we can construct by transfinite recursion the *Boolean-valued model of set theory* $\mathbf{V}^{\mathbb{B}}$ consisting of names for sets, an extended language $\mathcal{L}_{\mathbb{B}}$, and an interpretation function $[\![\cdot]\!] : \mathcal{L}_{\mathbb{B}} \to \mathbb{B}$ assigning truth values in $\mathbb{B}$ to formulas of the extended language. Using the notion of validity derived from $[\![\cdot]\!]$, all of the axioms of ZFC are valid in $\mathbb{V}^{\mathbb{B}}$. Boolean-valued models were introduced in the 1960s by Scott, Solovay, and Vopěnka; an excellent exposition of the theory can be found in Bell (2005).

Replacing the Boolean algebra in the above construction by a Heyting algebra $\mathbb{H}$, one obtains a *Heyting-valued model of set theory* $\mathbf{V}^{\mathbb{H}}$. The proofs of the Boolean case transfer to the Heyting-valued case to yield that $\mathbf{V}^{\mathbb{H}}$ is a model of IZF, intuitionistic ZF, where the logic of the Heyting algebra $\mathbb{H}$ determines the logic of the Heyting-valued model of set theory (cf. Grayson, 1979; Bell, 2005, chap. 8). This idea was further generalized by Takeuti & Titani (1992), Titani (1999), Titani & Kozawa (2003), Ozawa (2007), and Ozawa (2009), replacing the Heyting algebra $\mathbb{H}$ by appropriate lattices that allow models of *quantum set theory* (where the algebra is an algebra of truth-values in quantum logic) or *fuzzy set theory*.

In this paper, we shall generalize this model construction further to work on algebras that we shall call *reasonable implication algebras* (§2). These algebras do not have a negation symbol, and hence we shall be focusing on the *negation-free fragment* of first-order logic: the closure under the propositional connectives $\wedge$, $\vee$, $\perp$, and $\rightarrow$. Classically, of course, every formula is equivalent to one in the negation-free fragment (since $\neg\varphi$ is equivalent to $\varphi \rightarrow \perp$). In §3, we define the model construction and prove that assuming a number of additional assumptions (among them a property we call the *bounded quantification property*), we have constructed a model of the negation-free fragment of ZF^- (which is classically equivalent to ZF^-).

In §4 and §5, we apply the results of §3 to a particular three-valued algebra where we prove the bounded quantification property (§4) and the axiom scheme of $\mathsf{Foundation}$ (§5).

Received: June 20, 2014.

doi:10.1017/S175502031400046X

Finally, in §6, we add a negation symbol to our language. With the appropriate negation, our example from §4 and §5 becomes a model of a paraconsistent set theory that validates all formulas from the negation-free fragment of ZF. We compare our paraconsistent set theory to other paraconsistent set theories from the literature and observe that it is fundamentally different from them.

We should like to mention that Joel Hamkins independently investigated the construction that is at the heart of this paper and proved a result equivalent to our Theorem 6.3 (presented at the *Workshop on Paraconsistent Set Theory* in Storrs, CT in October 2013).

§2. Reasonable implication algebras.

Implication algebras and implication-negation algebras. In this paper, all structures $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ will be complete distributive lattices with smallest element $\mathbf{0}$ and largest element $\mathbf{1}$. As usual, we abbreviate $x \wedge y = x$ as $x \leq y$. An expansion of this structure by an additional binary operation $\Rightarrow$ is called an *implication algebra* and an expansion with $\Rightarrow$ and another unary operation * is called an *implication-negation algebra*. We emphasize that no requirements are made for $\Rightarrow$ and * at this point.

Interpreting propositional logic in algebras. By $\mathcal{L}_{\mathrm{Prop}}$ we denote the language of propositional logic without negation (with connectives $\wedge$, $\vee$, $\rightarrow$, and $\perp$ and countably many variables Var); we write $\mathcal{L}_{\mathrm{Prop},\neg}$ for the expansion of this language to include the negation symbol $\neg$. Let $\mathcal{L}$ be either $\mathcal{L}_{\mathrm{Prop}}$ or $\mathcal{L}_{\mathrm{Prop},\neg}$, and let $\mathbb{A}$ be either an implication algebra or an implication-negation algebra, respectively. Any map ι from Var to A (called an *assignment*) allows us to interpret $\mathcal{L}$-formulas φ as elements $\iota(\varphi)$ of the algebra. *Par abus de langage*, for an $\mathcal{L}$-formula φ and some $X \subseteq A$, we write $\varphi \in X$ for "for all assignments $\iota : \mathrm{Var} \to A$, we have that $\iota(\varphi) \in X$". As usual, we call a set $D \subseteq A$ a *filter* if the following four conditions hold: (i) $\mathbf{1} \in D$, (ii) $\mathbf{0} \notin D$, (iii) if $x, y \in D$, then $x \wedge y \in D$, and (iv) if $x \in D$ and $x \leq y$, then $y \in D$; in this context, we call filters *designated sets of truth values*, since the algebra $\mathbb{A}$ and a filter D together determine a *logic* $\vdash_{\mathbb{A},D}$ by defining for every set Γ of $\mathcal{L}_{\mathrm{Prop}}$-formulas and every $\mathcal{L}_{\mathrm{Prop}}$-formula φ

$$\Gamma \vdash_{\mathbb{A},D} \varphi :\iff \text{if for all } \psi \in \Gamma, \text{ we have } \psi \in D, \text{ then } \varphi \in D.$$

We write $\mathrm{Pos}_{\mathbb{A}} := \{x \in A\,;\, x \neq \mathbf{0}\}$ for the set of positive elements in $\mathbb{A}$. In all of the examples considered in this paper, this set will be a filter.

The negation-free fragment. If $\mathcal{L}$ is any first-order language including the connectives $\wedge$, $\vee$, $\perp$ and $\rightarrow$ and Λ any class of $\mathcal{L}$-formulas, we denote closure of Λ under $\wedge$, $\vee$, $\perp$, $\exists$, $\forall$, and $\rightarrow$ by $\mathrm{Cl}(\Lambda)$ and call it the *negation-free closure of* Λ. A class Λ of formulas is *negation-free closed* if $\mathrm{Cl}(\Lambda) = \Lambda$. By NFF we denote the negation-free closure of the atomic formulas; its elements are called the *negation-free formulas*.[1]

Obviously, if $\mathcal{L}$ does not contain any connectives beyond $\wedge$, $\vee$, $\perp$, and $\rightarrow$, then NFF $= \mathcal{L}$. Similarly, if the logic we are working in allows to define negation in terms of the other connectives (as is the case, e.g., in classical logic), then every formula is equivalent to one in NFF.

[1] In some contexts, our negation-free fragment is called the *positive fragment*; in other contexts, the *positive closure* is the closure under $\wedge$, $\vee$, $\perp$, $\exists$, and $\forall$ (not including $\rightarrow$). In order to avoid confusion with the latter contexts, we use the phrase "negation-free" rather than "positive".

Reasonable implication algebras. We call an implication algebra $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1}, \Rightarrow)$ *reasonable* if the operation $\Rightarrow$ satisfies the following axioms:

P1 $(x \wedge y) \leq z$ implies $x \leq (y \Rightarrow z)$,
P2 $y \leq z$ implies $(x \Rightarrow y) \leq (x \Rightarrow z)$, and
P3 $y \leq z$ implies $(z \Rightarrow x) \leq (y \Rightarrow x)$.

We say that a reasonable implication algebra is *deductive* if

$$((x \wedge y) \Rightarrow z) = (x \Rightarrow (y \Rightarrow z)).$$

It is easy to see that any reasonable implication algebra satisfies that $x \leq y$ implies $x \Rightarrow y = \mathbf{1}$. Similarly, it is easy to see that in reasonable and deductive implication algebras, we have $(x \Rightarrow y) = (x \Rightarrow (x \wedge y))$. These facts are being used in the calculations later in the paper. It is easy to check that all Boolean algebras and Heyting algebras are reasonable and deductive implication algebras.

Recurring examples. The following two examples will be crucial during the rest of the paper: The three-valued Łukasiewicz algebra $Ł_3 = (\{0, 1/2, 1\}, \wedge, \vee, \Rightarrow, 0, 1)$ with operations defined as in Figure 1 is a reasonable, but non-deductive implication algebra. The three-valued algebra $\mathbb{PS}_3 = (\{0, 1/2, 1\}, \wedge, \vee, \Rightarrow, 0, 1)$ with operations defined as in Figure 2 is a reasonable and deductive implication algebra which is not a Heyting algebra. Let us emphasize that, contrary to usage in other papers, we consider $Ł_3$ and $\mathbb{PS}_3$ as implication algebras *without* negation (cf. §6 for adding negations to $\mathbb{PS}_3$).

§3. The model construction.

3.1. Definitions and basic properties. Our construction follows very closely the Boolean-valued construction as it can be found in Bell (2005). We fix a model of set theory $\mathbf{V}$ and an implication algebra $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1}, \Rightarrow)$ and construct a universe of *names* by transfinite recursion:

$$\mathbf{V}^{\mathbb{A}}_{\alpha} = \{x\,;\ \ x \text{ is a function and } \operatorname{ran}(x) \subseteq A$$
$$\text{and there is } \xi < \alpha \text{ with } \operatorname{dom}(x) \subseteq \mathbf{V}^{\mathbb{A}}_{\xi})\} \text{ and}$$
$$\mathbf{V}^{\mathbb{A}} = \{x\,;\ \exists\alpha(x \in \mathbf{V}^{\mathbb{A}}_{\alpha})\}.$$

We note that this definition does not depend on the algebraic operations in $\mathbb{A}$, but only on the set A, so any expansion of $\mathbb{A}$ to a richer language will give the same class of names $\mathbf{V}^{\mathbb{A}}$. By $\mathcal{L}_{\in}$, we denote the first-order language of set theory using only the propositional connectives $\wedge$, $\vee$, $\bot$, and $\to$. We can now expand this language by adding all of the

$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\Rightarrow$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

Fig. 1. Connectives for the algebra $Ł_3$.

$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\Rightarrow$	1	1/2	0
1	1	1	0
1/2	1	1	0
0	1	1	1

Fig. 2. Connectives for $\mathbb{PS}_3$.

elements of $\mathbf{V}^{\mathbb{A}}$ as constants; the expanded (class-sized) language will be called $\mathcal{L}_{\mathbb{A}}$. As in the Boolean case (Bell, 2005, Induction Principle 1.7), the *(meta-)induction principle* for $\mathbf{V}^{\mathbb{A}}$ can be proved by a simple induction on the rank function: for every property Φ of names, if for all $x \in \mathbf{V}^{\mathbb{A}}$, we have

$$\forall y \in \mathrm{dom}(x)(\Phi(y)) \text{ implies } \Phi(x),$$

then all names $x \in \mathbf{V}^{\mathbb{A}}$ have the property Φ.

As in the Boolean case, we can now define a map $[\![\cdot]\!]$ assigning to each negation-free formula in $\mathcal{L}_{\mathbb{A}}$ a truth value in A as follows. If u, v in $\mathbf{V}^{\mathbb{A}}$ and $\varphi, \psi \in \mathrm{NFF}$, we define

$$\begin{aligned}
[\![\bot]\!] &= \mathbf{0}, \\
[\![u \in v]\!] &= \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge [\![x = u]\!]), \\
[\![u = v]\!] &= \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]), \\
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \wedge [\![\psi]\!], \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \vee [\![\psi]\!], \\
[\![\varphi \rightarrow \psi]\!] &= [\![\varphi]\!] \Rightarrow [\![\psi]\!], \\
[\![\forall x \varphi(x)]\!] &= \bigwedge_{u \in \mathbf{V}^{\mathbb{A}}} [\![\varphi(u)]\!], \text{ and} \\
[\![\exists x \varphi(x)]\!] &= \bigvee_{u \in \mathbf{V}^{\mathbb{A}}} [\![\varphi(u)]\!].
\end{aligned}$$

As usual, we abbreviate $\exists x(x \in u \wedge \varphi(x))$ by $\exists x \in u\ \varphi(x)$ and $\forall x(x \in u \rightarrow \varphi(x))$ by $\forall x \in u\ \varphi(x)$ and call these *bounded quantifiers*. Bounded quantifiers will play a crucial role in this paper.

If D is a filter on A and σ is a sentence of $\mathcal{L}_{\mathbb{A}}$, we say that σ is D-valid in $\mathbf{V}^{\mathbb{A}}$ if $[\![\sigma]\!] \in D$ and write $\mathbf{V}^{\mathbb{A}} \models_D \sigma$.

In the Boolean-valued case, the names behave nicely with respect to their interpretations as names for sets. For instance, if two names denote the same object, then the properties of the object do not depend on the name you are using. In our generalized setting, we have to be very careful since many of these reasonable rules do not hold in general: cf. §4 for details.

PROPOSITION 3.1. *If $\mathbb{A}$ is a reasonable implication algebra and $u \in \mathbf{V}^{\mathbb{A}}$, we have that $[\![u = u]\!] = \mathbf{1}$ and $u(x) \leq [\![x \in u]\!]$ (for each $x \in \mathrm{dom}(u)$).*

Proof. This is an easy induction, using the fact that we have that in all reasonable implication algebras, $x \leq y$ implies $x \Rightarrow y = \mathbf{1}$. □

However, things break down rather quickly if you go beyond Proposition 3.1. The inequality $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$ representing transitivity of equality of names does not hold in general in the model constructed over Ł$_3$: consider the functions

$$\begin{aligned}
p_{\mathbf{0}} &= \{\langle \varnothing, \mathbf{0} \rangle\}, \\
p_{1/2} &= \{\langle \varnothing, 1/2 \rangle\}, \text{ and} \\
p_{\mathbf{1}} &= \{\langle \varnothing, \mathbf{1} \rangle\}.
\end{aligned}$$

Then it can be easily checked that $[\![p_{\mathbf{0}} = p_{1/2}]\!] = 1/2 = [\![p_{1/2} = p_{\mathbf{1}}]\!] > [\![p_{\mathbf{0}} = p_{\mathbf{1}}]\!] = \mathbf{0}$.

PROPOSITION 3.2. *If $\mathbb{A}$ is a reasonable implication algebra, $\varphi(x)$ an $\mathcal{L}_{\mathbb{A}}$-formula with one free variable x, and $u \in \mathbf{V}^{\mathbb{A}}$, then*

$$[\![\exists x \in u\, \varphi(x)]\!] \geq \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![\varphi(x)]\!]).$$

Proof. Easy calculation using Proposition 3.1. □

In the Boolean case, the inequality proved in Proposition 3.2 is an equality (Bell, 2005, p. 23):

$$[\![\exists x \in u\, \varphi(x)]\!] = \bigvee_{x \in \mathrm{dom}(u)} \big(u(x) \wedge [\![\varphi(x)]\!]\big) \text{ and}$$
$$[\![\forall x \in u\, \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]\big).$$

This once more breaks down for general reasonable implication algebras: in $\mathbf{V}^{Ł_3}$, we use the three names $p_{\mathbf{0}}$, $p_{1/2}$, and $p_{\mathbf{1}}$ defined above and consider the formula $\varphi(x) := (x = p_{\mathbf{0}})$ as well as the name $u = \{\langle p_{1/2}, 1/2\rangle\}$. We can calculate

$$1/2 = [\![\forall x \in u\, \varphi(x)]\!] < \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]\big) = \mathbf{1}.$$

This means that in the setting of reasonable implication algebras, the following equality

$$[\![\forall x \in u\, \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]\big). \tag{BQ$_\varphi$}$$

becomes a new axiom, one whose validity depends on the choice of the formula φ and on $\mathbb{A}$ (and conceivably on the model of set theory $\mathbf{V}$). If Λ is any class of formulas of the extended language, we say that the pair $(\mathbf{V}, \mathbb{A})$ *satisfies the Λ-bounded quantification property*, if BQ_φ holds for every $\varphi \in \Lambda$.

3.2. Set theory. The axiom system ZF^- consists of the axioms Extensionality, Pairing, Infinity, Union, and Power Set and the axiom schemes of Separation and Replacement. If add the axiom scheme of Foundation, we obtain ZF of Zermelo-Fraenkel set theory. For reference, we list the forms of the axioms and axiom schemes that we use in our proofs (in the schemes, φ is a formula with $n+2$ free variables); the concrete formulations follows Bell (2005) very closely:

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \to x = y] \tag{Extensionality}$$
$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y)) \tag{Pairing}$$
$$\exists x [\exists y (\forall z (z \in y \to \bot) \wedge y \in x) \wedge \forall w \in x \exists u \in x (w \in u)] \tag{Infinity}$$
$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w \in x (z \in x)) \tag{Union}$$
$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w \in z (w \in x)) \tag{Power Set}$$
$$\forall p_0 \cdots \forall p_n \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z, p_0, \dots, p_n)) \tag{Separation$_\varphi$}$$
$$\begin{aligned} &\forall p_0 \cdots \forall p_{n-1} \forall x [\forall y \in x \exists z \varphi(y, z, p_0, \dots, p_{n-1}) \\ &\quad \to \exists w \forall v \in x \exists u \in w\, \varphi(v, u, p_0, \dots, p_{n-1})] \end{aligned} \tag{Replacement$_\varphi$}$$
$$\begin{aligned} &\forall p_0 \cdots \forall p_n \forall x [\forall y \in x\, \varphi(y, p_0, \dots, p_n) \\ &\quad \to \varphi(x, p_0, \dots, p_n)] \to \forall z \varphi(z, p_0, \dots, p_n) \end{aligned} \tag{Foundation$_\varphi$}$$

We observe that all axioms and axiom schemes have natural forms that do not include any negation symbols,[2] so unless we instantiate one of the schemes with a formula containing a negation symbol, we will always have formulas in NFF. We write NFF-Separation and NFF-Replacement for the axiom schemes where we only allow the instantiation by negation-free formulas, and we write NFF-ZF^- and NFF-ZF for *negation-free set theory* using these schemes. We emphasize once more that in settings where negation can be defined in terms of negation-free formulas (such as classical logic), this coincides (up to provable equivalence) with standard Zermelo-Fraenkel set theory.

Theorems 3.3 and 3.4 are the core of this paper, establishing validity of NFF-ZF^- in our $\mathbb{A}$-valued model.

THEOREM 3.3. *Let $\mathbb{A}$ be a reasonable implication algebra such that* $(\mathbf{V}, \mathbb{A})$ *satisfies the* NFF-*bounded quantification property, and let D be any filter on A. Then* Extensionality*,* Pairing*,* Infinity*,* Union *and* NFF-Replacement *are D-valid in $\mathbf{V}^{\mathbb{A}}$; in fact, they all get the value* $\mathbf{1}$.

THEOREM 3.4. *Let $\mathbb{A}$ be a reasonable and deductive implication algebra such that* $(\mathbf{V}, \mathbb{A})$ *satisfies the* NFF-*bounded quantification property, and let D be any filter on A. Then* Power Set *and* NFF-Separation *are D-valid in $\mathbf{V}^{\mathbb{A}}$; in fact, they get the value* $\mathbf{1}$.

Proof of Theorem 3.3. The proofs follow closely the proofs of the Boolean cases and only use the axioms of complete distributive lattices and the additional axioms **P1**, **P2** and **P3** of reasonable and deductive implication algebras (and their simple consequences such as "if $x \leq y$, then $x \Rightarrow y = \mathbf{1}$", as mentioned above) and Proposition 3.1. Note that all of the calculations involve arguments with bounded quantifiers, relying on some equalities BQ_φ. Inspection of the proofs shows that the formulas in the scope of the bounded quantifiers are negation-free. All of the axioms get value $\mathbf{1}$ in $\mathbf{V}^{\mathbb{A}}$. □

Proof of Theorem 3.4. As in the proof of Theorem 3.3, we inspect the details of the proofs in the Boolean case and observe that they only use the axioms of reasonable implication algebras, their simple consequences and BQ_φ for $\varphi \in$ NFF. The proof of Power Set uses $x \Rightarrow y = x \Rightarrow (x \wedge y)$, as mentioned above. Again, all of the axioms get value $\mathbf{1}$ in $\mathbf{V}^{\mathbb{A}}$. □

§4. Application, Part 1: The bounded quantification property in $\mathbf{V}^{\mathbb{PS}_3}$.

The original intuition of Boolean-valued models was that the names represent objects and that the equivalence classes of names under the equivalence relation defined by $u \sim v$ if and only if $[\![u = v]\!] \in D$ can serve as the ontology of the new model. In particular, this means that if two names represent the same object, they should instantiate the same properties. This is known as "indiscernibility of identicals", one of the directions of Leibniz's Law. In our setting, we can represent this by a statement of the type

$$[\![u = v]\!] \wedge [\![\varphi(u)]\!] \leq [\![\varphi(v)]\!]. \tag{†}$$

Unfortunately, it will turn out that these statements are not in general true in reasonable implication algebras and thus we have to be considerably more careful.

[2] Note that this is only the case because we formulated the occurrence of the empty set in Infinity appropriately and because we used the axiom scheme of $\in$-induction instead of the usual formulation of Foundation; the latter is not negation-free.

 https://doi.org/10.1017/S175502031400046X

In this section (Theorem 4.5), we are going to prove the bounded quantification property for $(\mathbf{V}, \mathbb{PS}_3)$. We start by making some algebraic observations about $\mathbb{PS}_3$: Since the truth table for the connective $\Rightarrow$ does not contain the value $1/2$, we immediately know that for any $u, v \in \mathbf{V}^{\mathbb{PS}_3}$, the value of $[\![u = v]\!]$ will be either $\mathbf{0}$ or $\mathbf{1}$. Similarly, any formula with $\rightarrow$ as the outermost connective will be assigned value either $\mathbf{0}$ or $\mathbf{1}$. Furthermore, since all of the axioms of set theory except for Infinity are of the logical form $\mathsf{Q}\Psi$ where Q is a block of quantifiers and Ψ is a conjunction of implications, axioms of set theory can only get the values $\mathbf{0}$ and $\mathbf{1}$ as well. Also, we use that by the truth table for $\wedge$, we have that any conjunction that gets the value $\mathbf{0}$ must have one conjunct that gets value $\mathbf{0}$; similarly, every disjunction that gets value $\mathbf{1}$ must have a disjunct that gets value $\mathbf{1}$.

PROPOSITION 4.1. *For any three elements $u, v, w \in \mathbf{V}^{(\mathbb{PS}_3)}$, we have*

1. $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$ *and*
2. $[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$.

Proof. (1) We will prove $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$ by induction on w: assume that for all $z \in \mathrm{dom}(w)$, we have

$$[\![u = v]\!] \wedge [\![v = x]\!] \leq [\![u = z]\!].$$

By the above remark, we know that all of the values are $\mathbf{0}$ or $\mathbf{1}$. If $[\![u = w]\!] = \mathbf{1}$, then we have nothing to prove. Therefore, suppose

$$[\![u = w]\!] = \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![x \in w]\!]\big) \wedge \bigwedge_{z \in \mathrm{dom}(w)} \big(w(z) \Rightarrow [\![z \in u]\!]\big) = \mathbf{0}.$$

Case 1. Suppose $\bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![x \in w]\!]\big) = \mathbf{0}$. So, there exists $x_0 \in \mathrm{dom}(u)$ such that

$$\begin{aligned} \mathbf{0} &= \big[u(x_0) \Rightarrow [\![x_0 \in w]\!]\big] \\ &= \left[u(x_0) \Rightarrow \bigvee_{z \in \mathrm{dom}(w)} \big(w(z) \wedge [\![x_0 = z]\!]\big)\right]. \end{aligned}$$

This can only be the case if

$$u(x_0) \neq \mathbf{0} \text{ and } \bigvee_{z \in \mathrm{dom}(w)} \big(w(z) \wedge [\![x_0 = z]\!]\big) = \mathbf{0}. \tag{‡}$$

CLAIM 4.2. *For any $y_0 \in \mathrm{dom}(v)$ with $v(y_0) \neq \mathbf{0}$, we have either $[\![y_0 \in w]\!] = \mathbf{0}$ or $[\![x_0 = y_0]\!] = \mathbf{0}$.*

Proof of Claim 4.2. If $[\![y_0 \in w]\!] \neq \mathbf{0}$, i.e., $\bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![y_0 = z]\!]) \neq \mathbf{0}$, then there exists $z_0 \in \mathrm{dom}(w)$, such that $w(z_0) \neq \mathbf{0}$ and $[\![y_0 = z_0]\!] \neq \mathbf{0}$. Since $w(z_0) \neq \mathbf{0}$, equation (‡) yields $[\![x_0 = z_0]\!] = \mathbf{0}$. Now by induction hypothesis, $[\![x_0 = y_0]\!] \wedge [\![y_0 = z_0]\!] \leq [\![x_0 = z_0]\!]$. Hence we get $[\![x_0 = y_0]\!] = \mathbf{0}$. □

Using Claim 4.2, we either have that there is some $y_0 \in \mathrm{dom}(v)$ with $v(y_0) \neq \mathbf{0}$ and $[\![y_0 \in w]\!] = \mathbf{0}$ or for all such y_0, we have $[\![x_0 = y_0]\!] = \mathbf{0}$. In the first case, we immediately calculate that $[\![v = w]\!] = \mathbf{0}$. In the second case

$$[\![x_0 \in v]\!] = \bigvee_{y \in \mathrm{dom}(v)} \big(v(y) \wedge [\![x_0 = y]\!]\big) = \mathbf{0},$$

and therefore $[\![u = v]\!] = \mathbf{0}$.

Case 2. Suppose $\bigwedge_{z\in\mathrm{dom}(w)} \big(w(z) \Rightarrow [\![z \in u]\!]\big) = \mathbf{0}$. This case is proved analogously.
Claim (2) in the statement of the proposition follows easily from (1):

$$\begin{aligned}
[\![u = v]\!] \wedge [\![u \in w]\!] = [\![u = v]\!] \wedge & \bigvee_{z\in\mathrm{dom}(w)} \big(w(z) \wedge [\![u = z]\!]\big) \\
&= \bigvee_{z\in\mathrm{dom}(w)} \big[w(z) \wedge \big([\![u = z]\!] \wedge [\![u = v]\!]\big)\big] \\
&\le \bigvee_{z\in\mathrm{dom}(w)} \big(w(z) \wedge [\![v = z]\!]\big) \\
&= [\![v \in w]\!].
\end{aligned}$$

□

Proposition 4.1 proves the instances of (†) where $\varphi(x)$ is $x = w$ or $x \in w$ for some fixed w, respectively. However, the case where $\varphi(x)$ is $w \in x$ is not valid in $\mathbf{V}^{\mathbb{PS}_3}$ in general: let $w \in \mathbf{V}^{\mathbb{PS}_3}$ be arbitrary and u and v with $\mathrm{dom}(u) = \mathrm{dom}(v) = \{w\}$ defined by $u(w) = \mathbf{1}$ and $v(w) = 1/2$. Then $[\![u = v]\!] = \mathbf{1} = [\![w \in u]\!]$, but $[\![w \in v]\!] = 1/2$.

PROPOSITION 4.3. *For any three elements $u, v, w \in \mathbf{V}^{(\mathbb{PS}_3)}$, we have the following:*

1. $[\![u = v]\!] \Rightarrow [\![u = w]\!] \;=\; [\![u = v]\!] \Rightarrow [\![v = w]\!]$.
2. $[\![u = v]\!] \Rightarrow [\![u \in w]\!] \;=\; [\![u = v]\!] \Rightarrow [\![v \in w]\!]$.
3. $[\![u = v]\!] \Rightarrow [\![w \in u]\!] \;=\; [\![u = v]\!] \Rightarrow [\![w \in v]\!]$.

Proof. Claims (1) and (2) are easy calculations using Proposition 4.1 and the axioms for reasonable implication algebras. Claim (3) is different, since we do not have the analogue of Proposition 4.1 for the formula $w \in x$ (as seen above). As observed above, $[\![x = y]\!]$ will always take either the value $\mathbf{0}$ or the value $\mathbf{1}$. If $[\![u = v]\!] = \mathbf{0}$, then both sides of the equation are $\mathbf{1}$, so we have nothing to prove. Thus, we can assume that $[\![u = v]\!] = \mathbf{1}$. Checking the truth table for $\Rightarrow$, we realize that (without loss of generality) we only need to exclude the case that $[\![w \in u]\!] = \mathbf{0}$ and $[\![w \in v]\!] \neq \mathbf{0}$.

So, let us assume that

$$[\![w \in u]\!] = \bigvee_{x\in\mathrm{dom}(u)} \big(u(x) \wedge [\![w = x]\!]\big) = \mathbf{0}. \tag{\#}$$

We also assumed

$$[\![u = v]\!] = \bigwedge_{x\in\mathrm{dom}(u)} \big(u(x) \Rightarrow [\![x \in v]\!]\big) \wedge \bigwedge_{y\in\mathrm{dom}(v)} \big(v(y) \Rightarrow [\![y \in u]\!]\big) = \mathbf{1}. \tag{\S}$$

If for all $y \in \mathrm{dom}(v)$, we have $v(y) = \mathbf{0}$, then $[\![w \in v]\!] = \mathbf{0}$ and we are done, so we can assume that there is some y_0 such that $v(y_0) \neq \mathbf{0}$. Therefore, (§) implies that

$$[\![y_0 \in u]\!] = \bigvee_{x\in\mathrm{dom}(u)} \big(u(x) \wedge [\![y_0 = x]\!]\big) \neq \mathbf{0},$$

so there exists $x_0 \in \mathrm{dom}(u)$ such that $u(x_0) \neq \mathbf{0} \neq [\![y_0 = x_0]\!]$, from which we get $[\![w = x_0]\!] = \mathbf{0}$ via (#). Proposition 4.1 gives $[\![w = y_0]\!] \wedge [\![y_0 = x_0]\!] \le [\![w = x_0]\!]$, thus $[\![w = y_0]\!] = \mathbf{0}$. This, together with $v(y_0) \neq \mathbf{0}$, gives $[\![w \in v]\!] = \mathbf{0}$, and we are done. □

THEOREM 4.4. *If $\varphi \in$ NFF, then for all $u, v \in \mathbf{V}^{\mathbb{PS}_3}$, we have*

$$[\![u = v]\!] \Rightarrow [\![\varphi(u)]\!] \;=\; [\![u = v]\!] \Rightarrow [\![\varphi(v)]\!],$$

Proof. This is proved by induction on the formula complexity. Proposition 4.3 provides the atomic cases. As before, we know that $[\![u = v]\!]$ is either $\mathbf{0}$ or $\mathbf{1}$. If it is $\mathbf{0}$, then the claim is obvious, so we can assume that $[\![u = v]\!] = \mathbf{1}$. All cases are simple calculations using this assumption and the truth tables of the algebra $\mathbb{PS}_3$. □

Theorem 4.4 is enough to establish the appropriate amount of the bounded quantification property that we need:

THEOREM 4.5. *The pair* $(\mathbf{V}, \mathbb{PS}_3)$ *has the* NFF-*bounded quantification property.*

Proof. We have to prove BQ_φ for any negation-free formula φ, i.e., for any $u \in \mathbf{V}^{\mathbb{PS}_3}$, we need to show

$$[\![\forall x(x \in u \to \varphi(x))]\!] = \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]\big).$$

First of all, an easy calculation using the properties of reasonable implication algebras and Theorem 4.4 shows that

$$[\![\forall x(x \in u \to \varphi(x))]\!] = \bigwedge_{y \in \mathbf{V}^{\mathbb{PS}_3}} \bigwedge_{x \in \mathrm{dom}(u)} \big[\big(u(x) \wedge [\![y = x]\!]\big) \Rightarrow [\![\varphi(x)]\!]\big].$$

Furthermore,

$$\begin{aligned}\bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]\big) &= \bigwedge_{y \in \mathbf{V}^{\mathbb{PS}_3}} \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]\big) \\ &\leq \bigwedge_{y \in \mathbf{V}^{\mathbb{PS}_3}} \bigwedge_{x \in \mathrm{dom}(u)} \big[\big(u(x) \wedge [\![y = x]\!]\big) \Rightarrow [\![\varphi(x)]\!]\big].\end{aligned}$$

For the other direction, take any $x \in \mathrm{dom}(u)$ and obtain

$$\begin{aligned}\bigwedge_{y \in \mathbf{V}^{\mathbb{PS}_3}} \big[\big(u(x) \wedge [\![y = x]\!]\big) \Rightarrow [\![\varphi(x)]\!]\big] &\leq \big(u(x) \wedge [\![x = x]\!]\big) \Rightarrow [\![\varphi(x)]\!] \\ &= u(x) \Rightarrow [\![\varphi(x)]\!] \text{ (by Proposition 4.1)},\end{aligned}$$

and hence,

$$\bigwedge_{x \in \mathrm{dom}(u)} \bigwedge_{y \in \mathbf{V}^{\mathbb{PS}_3}} \big[\big(u(x) \wedge [\![y = x]\!]\big) \Rightarrow [\![\varphi(x)]\!]\big] \leq \bigwedge_{x \in \mathrm{dom}(u)} \big(u(x) \Rightarrow [\![\varphi(x)]\!]\big).$$ □

§5. Application, Part 2: Foundation in $\mathbf{V}^{\mathbb{PS}_3}$. In this section, we discuss the axiom scheme of Foundation (for which we do not have a general theorem along the lines of Theorems 3.3 and 3.4) and some related formulas such as $\exists x(x \in x)$.

THEOREM 5.1. *For any filter D, the axiom scheme of* NFF-Foundation *is D-valid in* $\mathbf{V}^{\mathbb{PS}_3}$.

Proof. We show Foundation in the form of $\in$-induction: for every negation-free φ, we have that

$$[\![\forall x[\forall y \in x\, \varphi(y) \to \varphi(x)] \to \forall x \varphi(x)]\!] = \mathbf{1}.$$

Case 1. Suppose $[\![\varphi(x)]\!] \neq \mathbf{0}$ for every $x \in \mathbf{V}^{\mathbb{PS}_3}$. Hence in this case $[\![\forall x \varphi(x)]\!] \in \{1/2, \mathbf{1}\}$ and therefore by definition of $\Rightarrow$,

$$[\![\forall x[\forall y \in x\, \varphi(y) \to \varphi(x)] \to \forall x \varphi(x)]\!] = \mathbf{1}.$$

Case 2. Now let $x \in \mathbf{V}^{\mathbb{PS}_3}$ with $[\![\varphi(x)]\!] = \mathbf{0}$. Take a minimal $u \in \mathbf{V}^{\mathbb{PS}_3}$ satisfying this, i.e., $[\![\varphi(u)]\!] = \mathbf{0}$ but for any $y \in \mathrm{dom}(u)$; $[\![\varphi(y)]\!] \neq \mathbf{0}$. Since there exist $x \in \mathbf{V}^{\mathbb{PS}_3}$ for which $[\![\varphi(x)]\!] = \mathbf{0}$, clearly $[\![\forall x \varphi(x)]\!] = \mathbf{0}$. Once more, the definition of $\Rightarrow$ gives us:

$$\begin{aligned}
[\![\forall x[(\forall y \in x\ \varphi(y)) \to \varphi(x)]]\!] &\leq [\![(\forall y \in u\ \varphi(y)) \to \varphi(u)]\!] \\
&= \bigwedge_{y \in \mathrm{dom}(u)} (u(y) \Rightarrow [\![\varphi(y)]\!]) \Rightarrow [\![\varphi(u)]\!] \\
&= \mathbf{0}
\end{aligned}$$

Hence we get

$$[\![\forall x[\forall y \in x\ \varphi(y) \to \varphi(x)] \to \forall x \varphi(x)]\!] = \mathbf{1} \qquad \square$$

COROLLARY 5.2. *For any filter D, all axioms of* NFF-ZF *are D-valid in* $\mathbf{V}^{\mathbb{PS}_3}$.

Proof. The claim follows from Theorems 3.3, 3.4, 4.5, and 5.1. $\square$

THEOREM 5.3. *For all $u \in \mathbf{V}^{\mathbb{PS}_3}$, $[\![u \in u]\!] = \mathbf{0}$. So, in particular, $[\![\exists x(x \in x)]\!] = \mathbf{0}$.*

Proof. By meta-induction, if there is a counterexample to the claim, there is a minimal counterexample, i.e., a name u with $[\![u \in u]\!] \neq \mathbf{0}$, but for every $x \in \mathrm{dom}(u)$, we have that $[\![x \in x]\!] = \mathbf{0}$. The first claim means that there is some $x_0 \in \mathrm{dom}(u)$ with $u(x_0) \neq \mathbf{0}$ and $[\![u = x_0]\!] \neq \mathbf{0}$. Since $[\![u = x_0]\!]$ is defined in terms of a conjunction in which all expressions of the form $u(x) \Rightarrow [\![x \in x_0]\!]$ for $x \in \mathrm{dom}(u)$ occur, each of these must be non-zero. Take one of these and let $x = x_0$ in this expression; we obtain $u(x_0) \Rightarrow [\![x_0 \in x_0]\!]$. But we assumed that $u(x_0) \neq \mathbf{0}$ and $[\![x_0 \in x_0]\!] = \mathbf{0}$. Contradiction! $\square$

§6. Adding negation: A model of paraconsistent set theory.

The model construction. As mentioned in §3, the construction of the $\mathbb{A}$-names does not depend on the algebraic structure at all, so if $\mathbb{A}$ is an implication algebra and $\mathbb{A}'$ is an implication-negation algebra expanding it, they define the same class of names $\mathbf{V}^{\mathbb{A}} = \mathbf{V}^{\mathbb{A}'}$. The language $\mathcal{L}_{\mathbb{A}'}$ is then the closure of $\mathcal{L}_{\mathbb{A}}$ under negation, and we can now easily extend the map $[\![\cdot]\!]$ to include all formulas in $\mathcal{L}_{\mathbb{A}'}$ by adding the condition $[\![\neg\varphi]\!] := [\![\varphi]\!]^*$.

Negation and paraconsistency. Let $\mathbb{A}' = (A, \wedge, \vee, \mathbf{0}, \mathbf{1}, \Rightarrow, ^*)$ be an implication-negation algebra and D a filter on A. We call the pair $(\mathbb{A}', D)$ *paraconsistent* if there are formulas φ and ψ such that

$$\{\varphi, \neg\varphi\} \not\models_{\mathbb{A}',D} \psi.$$

In the Boolean and Heyting cases, as well as in the algebras considered by Takeuti & Titani (1992), Titani (1999), Titani & Kozawa (2003), and Ozawa (2007, 2009), negation is defined in terms of implication via $a^* := a \Rightarrow \mathbf{0}$. This definition, together with minimal requirements, makes it impossible to have paraconsistency. E.g., Titani (1999) requires that negation is defined in terms of negation by $a^* := a \Rightarrow \mathbf{0}$ and, furthermore, $(x \Rightarrow y) = \mathbf{1}$ iff $x \leq y$ and that $x \wedge (x \Rightarrow y) \leq y$. These three conditions together immediately imply that any such lattice with any filter D of designated truth values will not be paraconsistent in the above sense.

Adding a negation to $\mathbb{PS}_3$. If we expand $\mathbb{PS}_3$ with a negation * defined by $\mathbf{1}^\star = 1/2^\star = \mathbf{0}$, and $\mathbf{0}^\star = \mathbf{1}$, then the results from §4 extend to give the bounded quantification property for all formulas (including negations) and Theorems 3.3, 5.1 and 3.4 extend to give full ZF in the resulting model. However, for none of the two possible filters D on $\mathbb{PS}'_3$ is the pair $(\mathbb{PS}'_3, D)$ paraconsistent, and the resulting logic $\vdash_{(\mathbb{PS}_3,\star),\mathrm{Pos}_{\mathbb{PS}_3}}$ will just be classical logic.

If, however, we supplement $\mathbb{PS}_3$ with the negation * defined by $\mathbf{1}^* = \mathbf{0}$, $1/2^* = 1/2$, and $\mathbf{0}^* = \mathbf{1}$, then $(\mathbb{PS}_3, {}^*, \mathrm{Pos}_{\mathbb{PS}_3})$ is paraconsistent, since $1/2^* = 1/2 \in D$.[3]

The positive results of §4 cannot be extended to $(\mathbb{PS}_3, {}^*)$: consider the analogue of Theorem 4.4 for the formula $\varphi(x) := \neg(w \in x)$. Again, we let $w \in \mathbf{V}^{\mathbb{PS}_3}$ be an arbitrary name and u and v with $\mathrm{dom}(u) = \mathrm{dom}(v) = \{w\}$ defined by $u(w) = \mathbf{1}$ and $v(w) = 1/2$. We calculate $[\![u = v]\!] = \mathbf{1}$, $[\![\varphi(u)]\!] = \mathbf{0}$, and $[\![\varphi(v)]\!] = 1/2$. Therefore,

$$[\![u = v]\!] \Rightarrow [\![\varphi(u)]\!] = \mathbf{0} \neq \mathbf{1} = [\![u = v]\!] \Rightarrow [\![\varphi(v)]\!],$$

so the φ-instance of Theorem 4.4 is not valid in $\mathbf{V}^{\mathbb{PS}_3}$. This gives us the following result immediately:

THEOREM 6.1. *There is a formula $\varphi \in \mathcal{L}_{(\mathbb{PS}_3,{}^*)}$ such that $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$ does not have the property* BQ_φ.

Proof. We use $u, v, w \in \mathbf{V}^{\mathbb{PS}_3}$ and $\varphi(x) := \neg(w \in x)$ as in the above example. Define a name $z := \{(v, \mathbf{1})\}$. We readily calculate $[\![\forall x(x \in z \to \varphi(x))]\!] = \mathbf{0}$. But, on the other hand,

$$\bigwedge_{x \in \mathrm{dom}(z)} (z(x) \Rightarrow [\![\varphi(x)]\!]) = z(v) \Rightarrow [\![\varphi(v)]\!] = \mathbf{1} \Rightarrow 1/2 = \mathbf{1}. \qquad \square$$

Paraconsistency in $\mathbf{V}^{(\mathbb{PS}_3,*)}$ and ontology of $\mathbf{V}^{(\mathbb{PS}_3,*)}$. Exactly this phenomenon can now be used to show that the resulting set theory is paraconsistent:

THEOREM 6.2. *There is a sentence $\sigma \in \mathcal{L}_\in$ such that both σ and $\neg\sigma$ are* $\mathrm{Pos}_{\mathbb{PS}_3}$*-valid in* $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$.

Proof. We use the three names u, v, and w from above: $w \in \mathbf{V}^{\mathbb{PS}_3}$ is arbitrary and u and v with $\mathrm{dom}(u) = \mathrm{dom}(v) = \{w\}$ defined by $u(w) = \mathbf{1}$ and $v(w) = 1/2$. These three names witness that the sentence

$$\sigma := \exists u, v, w(u = v \wedge w \in u \wedge w \notin v)$$

has value $1/2$, and thus both σ and $\neg\sigma$ are $\mathrm{Pos}_{\mathbb{PS}_3}$-valid. $\square$

Corollary 5.2 and Theorem 6.2 together show that $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$ is a model of set theory with paraconsistent phenomena, in short, a *model of paraconsistent set theory*. As in the Boolean-valued case, the algebra-valued construction does not produce a model of a set theory in the standard sense of ordinary model theory. As discussed in §4, the natural approach here would be consider the $\sim$-equivalence classes of names as objects where $u \sim v$ if and only if $[\![u = v]\!] \in D$.[4] Due to the proof of Theorem 6.1, we cannot expect that (the scheme of) Leibniz's Law

$$\forall x \forall y(x = y \wedge \varphi(x) \to \varphi(y))$$

[3] This implication-negation algebra was introduced by Marcos (2000) as one of the 8,192 maximal paraconsistent three-valued logics mentioned in the title of the paper; it was further studied in Carnielli & Marcos (2002, § 3.11), Marcos (2005), and Coniglio & da Cruz Silvestrini (2014). It was recently independently rediscovered by Chakraborty and the second author.

[4] Note that by Proposition 4.1, the relation $\sim$ is an equivalence relation on $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$.

holds for arbitrary formulas (even though we proved the negation-free fragment of Leibniz's Law in Theorem 4.4).

Not all formulas defining a unique object in ordinary set theory do so in our model: e.g., the formula $N(x) := \forall z(z \notin x)$ usually uniquely defines the empty set, but in $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$, the formula $N(x)$ is valid if and only if x is a name such that $\mathrm{ran}(x) \subseteq \{\mathbf{0}, 1/2\}$. Now let u be such a name with $\mathrm{ran}(u) \subseteq \{\mathbf{0}\}$ and v be such a name with $1/2 \in \mathrm{ran}(u)$. Then

$$[\![N(u) \wedge N(v) \wedge u \neq v]\!] = 1/2.$$

In particular, the class of names x such that $N(x)$ does not form a $\sim$-equivalence class.[5] We can modify the formula N to $\widetilde{N}(x) := \forall y \forall z(x = y \rightarrow z \notin y)$ which is classically equivalent to $N(x)$. Then it is easy to see that for a name x, the formula $\widetilde{N}(x)$ is valid if and only if $\mathrm{ran}(x) \subseteq \{\mathbf{0}\}$, and this class forms a $\sim$-equivalence class: the class is thus is a good candidate for the ontology of the empty set in $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$.

And yet, the failure of Leibniz's Law affects these concrete mathematical objects as well, as can be seen by applying the proof of Theorem 6.2: Define $E(x) := \exists e(\widetilde{N}(e) \wedge \forall z(z \in x \leftrightarrow z = e))$; this is the canonical formula defining the von Neumann ordinal one. We observe that the class of names x such that $E(x)$ is valid forms a $\sim$-equivalence class, and thus is a good candidate for the ontology of the von Neumann ordinal one. However, this equivalence class contains names of different nature: let w be any name such that $\widetilde{N}(w)$ is valid, and let $u = \{(w, \mathbf{1})\}$ and $v = \{(w, 1/2)\}$. Then $[\![E(u)]\!] = \mathbf{1}$ and $[\![E(v)]\!] = 1/2$, so both u and v are names for the von Neumann ordinal one. However,

$$[\![\exists x(\widetilde{N}(x) \wedge E(u) \wedge x \in u \wedge x \notin u)]\!] = \mathbf{0} \text{ and}$$

$$[\![\exists x(\widetilde{N}(x) \wedge E(v) \wedge x \in v \wedge x \notin v)]\!] = 1/2,$$

so the truth value of the statement "zero is both an element of one and not an element of one" depends on which name for one is chosen. A first discussion of the behaviour of von Neumann ordinals in $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$ can be found in Tarafder (2015).

Comparison to other paraconsistent set theories. Paraconsistent set theories have been studied by many authors (Brady, 1971; Brady & Routley, 1989; Restall, 1992; Libert, 2005; Weber, 2010a,b, 2013); all of these accounts start from the observation that ZF was created to avoid the contradiction that can be obtained from the axiom scheme of Comprehension

$$\exists x \forall y(y \in x \leftrightarrow \varphi(y))$$

via Russell's paradox. Arguing that contradictions are not necessarily devastating in a paraconsistent setting, these authors reinstate the axiom scheme of Comprehension as acceptable, allow the formation of the Russell set R, and conclude that both $R \in R$ and $R \notin R$ are true.

Our paraconsistent set theory behaves very differently from the considerations of paraconsistent set theory in the mentioned papers, as we can show that the axiom scheme of Comprehension is not valid in our model:

THEOREM 6.3. *If* $(\mathbb{PS}_3, \mathrm{Pos}_{\mathbb{PS}_3})$, *we have* $[\![\exists x \forall y(y \in x)]\!] = \mathbf{0}$. *Since this formula is an instance of* Comprehension, *the axiom scheme of* Comprehension *is not* $\mathrm{Pos}_{\mathbb{PS}_3}$*-valid in* $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$.

[5] This is not in conflict with the fact that Extensionality is valid in $\mathbf{V}^{(\mathbb{PS}_3,{}^*)}$: in order to apply Extensionality, we need $\forall z(z \in u \leftrightarrow z \in v)$, but $N(u) \wedge N(v)$ is not strong enough in our logic to conclude this.

Proof. This follows immediately from Theorem 5.3: if $[\![\exists x \forall y (y \in x)]\!] \neq \mathbf{0}$ and u is a name witnessing this (i.e., $[\![\forall y (y \in u)]\!] \neq \mathbf{0}$), then $[\![u \in u]\!] \neq \mathbf{0}$ in contradiction to Theorem 5.3. □

THEOREM 6.4. *In* $(\mathbb{PS}_3, {}^*, \mathrm{Pos}_{\mathbb{PS}_3})$, *we have* $[\![\exists x \forall y (y \in x \leftrightarrow y \notin y)]\!] = \mathbf{0}$. *This means that there is no Russell set.*

Proof. Again, assume towards a contradiction that u satisfies $[\![\forall y (y \in u \leftrightarrow y \notin y)]\!] \neq \mathbf{0}$. By Theorem 5.3, $[\![y \notin y]\!] = [\![y \in y]\!]^* = \mathbf{0}^* = \mathbf{1}$ for all y and $[\![u \in u]\!] = \mathbf{0}$. But then $[\![u \notin u \to u \in u]\!] = \mathbf{0}$. Contradiction! □

§7. Acknowledgments. Part of this research was done while both authors were Visiting Fellows of the *Isaac Newton Institute for Mathematical Sciences* in Cambridge (in the programme *Semantics & Syntax*). The second author was funded by a grant from the *Council of Scientific and Industrial Research*, India (CSIR, sanction number 09/028(0754)/2009–EMR1) during his doctoral studies; a grant from the *Erasmus Mundus External Co-operation Window India Lot 13* (EMECW13) for a stay at the *Universiteit van Amsterdam* in 2010/11; a travel grant from the *University of Calcutta* for the stay in Cambridge in 2012; a travel grant from the *National Board for Higher Mathematics*, India (NBHM) for a research trip to Hamburg and Amsterdam in 2013; and a travel grant from the *Indo-European Research Training Network in Logic* (IERTNiL) for a trip to Amsterdam in 2014.

BIBLIOGRAPHY

Bell, J. L. (2005). *Set Theory, Boolean-Valued Models and Independence Proofs* (third edition). Oxford Logic Guides, Vol. 47. Oxford: Oxford University Press.

Brady, R. (1971). The consistency of the axioms of abstraction and extensionality in a three valued logic. *Notre Dame Journal of Formal Logic*, **12**, 447–453.

Brady, R., & Routley, R. (1989). The non-triviality of extensional dialectical set theory. In Priest, G., Routley, R., and Norman, J., editors. *Paraconsistent Logic: Essays on the Inconsistent*. Analytica. Munich: Philosophia Verlag, pp. 415–436.

Carnielli, W. A., & Marcos, J. (2002). A taxonomy of **C**-systems. In Carnielli, W. A., Coniglio, M. E., and D'Ottaviano, I. M. L., editors. *Paraconsistency: The Logical Way to the Inconsistent. Proceedings of the 2nd World Congress on Paraconsistency (WCP 2000)*. Lecture Notes in Pure and Applied Mathematics, Vol. 228. New York: Marcel Dekker, pp. 1–94.

Coniglio, M. E., & da Cruz Silvestrini, L. H. (2014). An alternative approach for quasi-truth. *Logic Journal of the IGPL*, **22**(2), 387–410.

Grayson, R. J. (1979). Heyting-valued models for intuitionistic set theory. In Fourman, M. P., Mulvey, C. J., & Scott, D. S., editors. *Applications of Sheaves, Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra and Analysis held at the University of Durham, Durham, July 9–21, 1977*, Lecture Notes in Mathematics, Vol. 753. Berlin: Springer, pp. 402–414.

Libert, T. (2005). Models for paraconsistent set theory. *Journal of Applied Logic*, **3**(1), 15–41.

Marcos, J. (2000). 8k solutions and semi-solutions to a problem of da Costa. Unpublished.

Marcos, J. (2005). On a problem of da costa. In Sica, G., editor. *Essays on the Foundations of Mathematics and Logic*, Advanced Studies in Mathematics and Logic, Vol. 2. Monza: Polimetrica, pp. 39–55.

Ozawa, M. (2007). Transfer principle in quantum set theory. *Journal of Symbolic Logic*, **72**(2), 625–648.

Ozawa, M. (2009). Orthomodular-valued models for quantum set theory. Preprint, arXiv 0908.0367.

Restall, G. (1992). A note on naïve set theory in LP. *Notre Dame Journal of Formal Logic*, **33**(3), 422–432.

Takeuti, G., & Titani, S. (1992). Fuzzy logic and fuzzy set theory. *Archive for Mathematical Logic*, **32**(1), 1–32.

Tarafder, S. (2015). Ordinals in an algebra-valued model of a paraconsistent set theory. In Banerjee, M. and Krishna, S., editors. *Logic and Its Applications, 6th International Conference, ICLA 2015, Mumbai, India, January 8–10, 2015, Proceedings*, Lecture Notes in Computer Science, Vol. 8923. Berlin: Springer-Verlag, pp. 15–206.

Titani, S. (1999). A lattice-valued set theory. *Archive for Mathematical Logic*, **38**(6), 395–421.

Titani, S., & Kozawa, H. (2003). Quantum set theory. *International Journal of Theoretical Physics*, **42**(11), 2575–2602.

Weber, Z. (2010a). Extensionality and restriction in naive set theory. *Studia Logica*, **94**(1), 87–104.

Weber, Z. (2010b). Transfinite numbers in paraconsistent set theory. *Review of Symbolic Logic*, **3**(1), 71–92.

Weber, Z. (2013). Notes on inconsistent set theory. In Tanaka, K., Berto, F., Mares, E., and Paoli, F., editors. *Paraconsistency: Logic and Applications*, Logic, Epistemology, and the Unity of Science, Vol. 26. Dordrecht: Springer-Verlag, pp. 315–328.

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION
UNIVERSITEIT VAN AMSTERDAM
POSTBUS 94242, 1090 GE AMSTERDAM
THE NETHERLANDS
E-mail: b.loewe@uva.nl

FACHBEREICH MATHEMATIK
UNIVERSITÄT HAMBURG
BUNDESSTRASSE 55
20146 HAMBURG
GERMANY

and

DEPARTMENT OF COMMERCE (MORNING)
ST. XAVIER'S COLLEGE
30 MOTHER TERESA SARANI
KOLKATA, 700016
INDIA
E-mail: souravt09@gmail.com

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA
35 BALLYGUNGE CIRCULAR ROAD
KOLKATA, 700019
INDIA