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Abstract

Poly-infix operators and operator families are introduced as an alternative for working modulo associativity and the corresponding bracket deletion convention. Poly-infix operators represent the basic intuition of repetitively connecting an ordered sequence of entities with the same connecting primitive.

Contents

1 Introduction

The source of inspiration for writing this paper we found in [13] where a seemingly compelling case is made for the use of sums of the form

$$n + n + n, \quad n + n + n + n, \quad n + n + n + n + n, \quad ...$$

and the way in which these expressions are created implies that brackets do not enter the picture in a meaningful way. In [13] only substitution instances like $7 + 7 + 7$ are used, and a reader of this source is supposed to know that $7 + 7 + 7$ is an expression that involves three occurrences of 7.

We will take an interest in the syntactic analysis of expressions like $7 + 7 + 7$ from the perspective that, e.g. in this particular case, we are looking at a single occurrence of a three-place operator rather than at two nested occurrences of a two-place infix operator with omitted bracketing (assuming associativity) or default bracketing (from the left or from the right).

We will call an operator that is made up from a repeated but unbracketed use of the same infix operator a *poly-infix operator*. All such operators together for a given infix operator constitute a poly-infix operator *family*. The two-place operator from which all operators from a poly-infix operator family are made up is called its *kernel*. The use of $+$ as a poly-infix operator is standard in Dutch primary education, see e.g. [9, 11]. The remarkable aspect of [13] which led us to the notion of a poly-infix operator is that it introduces expressions like $2 + 2 + 2 + 2$ in such a manner that an explanation of the structure of that expression as say $(2 + 2) + (2 + 2)$ is manifestly a detour, there simply is no role for brackets in the setting of [13].

1.1 Notation and axioms for poly-infix operator families

Given a sort name S and a function symbol Ψ, the poly-infix operator family with kernel Ψ for sort S contains for each positive number $n \geq 2$ an operator

$$\Psi_n : S^n \to S.$$

One axiom and two axiom schemes (AttL_{n+1}) and (AttR_{n+1}) (Association to the Left, and Association to the Right, respectively) with $n \geq 2$ are required for poly-infix operator families:[1]

$$\Psi_2(x_1, x_2) = x_1 \ \Psi \ x_2,$$
$$\Psi_{n+1}(x_1, x_2, ..., x_{n+1}) = \Psi_n(\Psi_2(x_1, x_2), ..., x_{n+1}), \qquad (\text{AttL}_{n+1})$$
$$\Psi_{n+1}(x_1, ..., x_n, x_{n+1}) = \Psi_n(x_1, ..., \Psi_2(x_n, x_{n+1})). \qquad (\text{AttR}_{n+1})$$

With induction to n it can be shown that for all positive numbers $n \geq 2$,

$$\Psi_{n+1}(x_1, ..., x_n, x_{n+1}) = \Psi_2(\Psi_n(x_1, .., x_n), x_{n+1}),$$
$$\Psi_{n+1}(x_1, x_2, ..., x_{n+1}) = \Psi_2(x_1, \Psi_n(x_2, .., x_{n+1})).$$

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[1] In some cases, it can be elegant, practical, or natural, to also introduce Ψ_1 defined by $\Psi_1(x) = x$, and, if a unit e_Ψ for Ψ is available, Ψ_0 defined by $\Psi_0 = e_\Psi$.

1.2 Poly-infix notation

As an alternative, and more appealing notation, it is plausible to use infix notation for $-\Psi-$ and to write

$$x_1 \Psi x_2 \ldots x_{n+1} \Psi x_{n+2} \quad \text{for} \quad \Psi_{n+2}(x_1, x_2, .., x_{n+1}, x_{n+2}).$$

This is called poly-infix notation for $-\Psi-$.

Associativity of Ψ as a binary operator follows immediately from the axioms of poly-infix families. It is obvious that both schemes are needed for associativity, by considering any example with Ψ a non-associative function.

Using poly-infix notation, it can be proven that the mentioned axiom schemes allow the introduction of arbitrary bracketing in an expression of the form $x_1 \Psi \ldots \Psi x_{n+1}$. For instance:

$$x_1 \Psi x_2 \Psi x_3 \Psi x_4 \Psi x_5 = x_1 \Psi(x_2 \Psi x_3) \Psi x_4 \Psi x_5.$$

2 Six examples of poly-infix operator kernels

We will call an operator *pre-arithmetical* if it can be reasonably introduced in an incremental hierarchy of datatype descriptions as well as in the explanation of that hierarchy in advance of an explanation of any arithmetical datatype. For the case of poly-infix operator kernels we found three cases where a pre-arithmetical introduction makes sense in our view. The property of being pre-arithmetical is an informal one and it depends on one's view of the relation between algebra and arithmetic. We assume the surrounding container view of algebra with respect to arithmetic (see [1]). That view reads as follows:

a) Arithmetic is about structures labelled as arithmetics,

b) each arithmetical structure is an algebra as well,

c) each arithmetical algebra is surrounded by a plurality of extended, restricted, and modified structures which are among the non-arithmetical algebras,

d) which algebras precisely must or may be classified as arithmetical may depend on the individual views which may differ from person to person, and

e) when taking that variation of judgement into account, and assuming that empirical work to determine the variation in a certain condition in a statistical manner, one finds that a degree of arithmetically may be assigned to an algebra rather than a sharp distinction.

For instance some may not count the meadow of rational complex numbers (see [8, 3]) as an arithmetic and others may do, which gives it a degree of arithmeticality between 0 and 1.

2.1 Three pre-arithmetical poly-infix operator kernels

Three most important examples of pre-arithmetical poly-infix operator families are these:

parallel composition: $p_1 \parallel .... \parallel p_n$ represents the parallel composition (also called merge) of n objects (processes, entities).[2]

sequential composition: $u_1; ...; u_n$ represents the sequential composition of n instructions.[3]

frame composition: $f_1 \oplus ... \oplus f_n$ represents the combination of n frames.[4]

The importance of these examples lies in the observation that expressions involving such operators can be found by looking at real life scenes or objects. The parallel presence of a number of static entities may be expressed as a merge of atoms for each entity. Straight-line computer programs may be modelled with an appropriate application of sequential composition for instruction sequences. A directed graph with labeled nodes and edges can be viewed as a sum of atomic frames.

2.2 Two intra-arithmetical poly-infix operator kernels

An operator kernel is called *intra-arithmetical* if it can be introduced on the basis of an initial fragment of an introduction of the arithmetic algebras, and if in addition it contributes to the further development of the development of a complete hierarchy of arithmetical algebras. Two operations stand out as examples; addition $(- + -)$ and multiplication $(- \cdot -)$.

Repeated addition, say $7+7+7+7+7$ can be used to explain multiplication, which is the approach taken in [13], and repeated multiplication, say $5 \cdot 5 \cdot 5 \cdot 5$ can be used to explain exponentiation.

2.3 A post-arithmetical poly-infix operator kernel

Matrix multiplication in the context of n dimensional square matrices of the rational numbers constitutes a prominent example of a post-arithmetical poly-infix operator kernel.

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[2]For an algebraic treatment of parallel composition we refer to [7].

[3]A more systematic name is text sequential composition, thus indicating that sequentiality in time of an effectuation process is not meant. For a theoretical account of these matters see e.g. [4].

[4]If edges and edge labels are deleted combinations of frames are mere sets of named objects, i.e. nodes. Frame composition extends naive set-theoretic union with deletable features for labeled directed graphs. See [5, 6].

3 Poly-infix operator specific equational logic

A non-Ψ expression is either a variable, or a constant, or an expression surrounded with brackets, or an expression with an operator different from Ψ as its leading function symbol. A Ψ-expression is an application of a poly-infix operator with kernel Ψ to a sequence of non-Ψ expressions. The *length* of a Ψ-expression is the number of arguments to its leading occurrence of a poly-infix operator with kernel Ψ. Non-Ψ expressions are considered Ψ expressions of length 1.

A Ψ-expression context $C_\Psi[-]$ is a Ψ-expression with a hole in it, the hole being denoted with $[-]$. The length of $C_\Psi[-]$ is the number of top-level occurrences of Ψ in it plus one. For instance,

$$x \, \Psi [-] \, \Psi \, y \, \Psi \, z$$

is a Ψ-expression context with length 4. Substitution of a Ψ-expression P of length n in a context $C_\Psi[-]$ of length m produces $C_\Psi[P]$, a Ψ-expression of length $m + n - 1$, for instance:

$$x \, \Psi [u \, \Psi \, v] \, \Psi \, y \, \Psi \, z = x \, \Psi \, u \, \Psi \, v \, \Psi \, y \, \Psi \, z.$$

An important rule of equational reasoning applies in this case:

Let $C_\Psi[-]$ be a Ψ-expression context. If $P = Q$, then $C_\Psi[P] = C_\Psi[Q]$.   (1)

The virtue of this derived rule is to allow derivations without any manipulation of bracket pairs.

3.1 A formal example

As an application of rule (1), which may be understood as a formal underpinning of the work in [13] we notice that it allows to derive

$$2 + 2 + 3 = 4 + 3$$

from $2 + 2 = 4$. Furthermore, we notice that by considering repeated addition in this way, no talk about associativity or about conventions of not writing bracket pairs which at closer inspection must be assumed to be present is needed.

3.2 Motivation of the example

The relevance of the example above is the following: if a teacher is aware of the theory of poly-infix operators and operator families as discussed above, then:

i) (s)he can speak with confidence and correctly about repeated addition as a primitive operator which allows its own dedicated equational logic,

ii) this exposition strategy is meaningful too in the direction of an audience of students who have not been exposed to that theory and who will not be exposed to that theory,

iii) further when introducing expressions like $2+2+2+2+2$ (s)he may assume with full confidence that in terms of talk of syntax and expressions one is dealing with a five-place operator, which is a member of the poly-infix operator family with $-+-$ as its kernel,

iv) and as a consequence (s)he may be in no doubt that $2+2+2+2+2$ is an expression, in particular a $+$-expression of length five, and finally

v) an equation like $2+2+2+2+2 = 2+2+4+2$ can be understood as stating of two expressions that these have the same value, while a useful and dedicated proof rule for deriving that equation is readily available.

4 Concluding remarks

The use of a dedicated equational logic given the reading of an occurrence of unbracketed repetition of an infix operator as an occurrence of a poly-infix operator falls within the degrees of freedom allowed by the relativism of [14]. Pragmatically speaking, logic does not come for free given an algebra or an arithmetic, it calls for dedicated design, even if in a more fundamental sense there are few degrees of freedom.

Reasoning about poly-infix expressions, while thinking of these as values, introduces complications that are comparable with the resolution of inconsistencies that arise if an expression-oriented view on fractions and a value-oriented view on fractions are mixed naively. In [2] the latter topic is dealt with by means of an application of paraconsistent reasoning in the style of the chunk and permeate paradigm of [10].

The motivation for this work has come from three sides and can be characterized as follows:

a) to provide a theoretic foundation of the educational proposals made in [13] which to the best of our knowledge necessitates considering poly-infix operators as first class citizens,

b) to proceed with efforts on contrasting an expression-oriented view and a value-oriented view on topics in elementary mathematics such as reported in [2] by taking an expression-oriented view on repeated occurrences of the same infix operator seriously, and

c) to provide foundations for the efforts described in [12] on defining new and "low" reference levels of arithmetical competence which are expected to be helpful for improving the organization of special education and remedial teaching in the Netherlands.

It seems that the proposals in [12] are entirely consistent with the approach taken in [13] regarding the objective to avoid the use of brackets in connection with what we have proposed to call "poly-infix operators".

We end with a philosophical remark. Albert Visser from Utrecht University has explained to one of us (JAB) that the use of poly-infix operators introduces strong typing in a way which on the long run may prove counter-productive in view of its complexity. In particular when modelling natural language strong typing may lead to impractical complexity. Flexible arity allows an operator to have variable number and structure of arguments. Personally, I prefer strong typing, where each operator has a fixed number of arguments, over flexible typing because of its clarity, but admittedly that is a point of view which may need to be compromised once the proximity to natural language increases.[5] Albert Visser has also pointed out that Leibniz has contemplated the representation of natural numbers with poly-infix addition applied to units, including the remarkably simple definition of addition in that context. Frege amongst others has criticized that view noticing that more precision with brackets is needed.

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[5]Indeed, poly-infix operators are implicitly used in the process of learning the written form of a natural language based on a Roman alphabet, in the construction of words from letters, of sentences from words (and punctuation symbols with special spacing directives), and the construction of texts from sentences. According to some conventions, Arabic numerals can also be used as words in sentences (and as letters in words in sms-messaging (text messaging)).

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