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Peterson, A.

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Modular forms on the moduli space of polarised K3 surfaces

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Arie Peterson

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Faculteit der Natuurwetenschappen, Wiskunde en Informatica

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Contents

Introduction

The question we deal with in this thesis is the following:

Question. What is the Kodaira dimension of the moduli space of polarised K3 surfaces of degree $2d$, for a given positive integer d?

We shall introduce the objects involved in this question, give an overview of the partial answers that were obtained by others, and describe the strategy we have used to get some further answers and confirmation of existing results.

The geometric objects we study are smooth projective K3 surfaces over the complex numbers. K3 surfaces by definition have a trivial canonical bundle and trivial fundamental group. Standard examples of K3 surfaces are a double cover of $\mathbb{P}^2$ branched along a smooth sextic curve, a smooth quartic hypersurface in $\mathbb{P}^3$, a smooth complete intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$, a smooth complete intersection of three quadric hypersurfaces in $\mathbb{P}^5$, and the desingularisation of the quotient of an abelian surface by the group inversion $a \sim -a$.

Their trivial canonical bundle makes K3 surfaces an analogue of elliptic curves in higher dimension, like abelian varieties. They have been studied for a long time, for instance by Kummer as early as 1864, but also by the Italian school, in particular Enriques and Severi.

André Weil suggested the name "K3" in the 1950s after the geometers Kähler, Kodaira and Kummer, and also after the mountain K2 that was famously first summitted in those years. Weil proposed a programme to study these surfaces and their moduli space. A moduli space of a class of objects is a parameter space, in which all such objects are catalogued: points of the moduli space correspond to isomorphism classes of objects. The properties that Weil expected to hold of this moduli space together imply that this moduli space of K3 surfaces can be described in a concrete way, as the quotient of some 20-dimensional complex domain by a discrete group action. This would be the analogue of the description of the moduli space of elliptic curves as the quotient of the upper half-plane by the group $\mathrm{SL}(2, \mathbb{Z})$. Over the course of the next decades, Weil's expectations were shown to be correct by the work of many different mathematicians, among them Kodaira, Siu, Shah, Kulikov, Todorov, Looijenga, Piatetskii-Shapiro, Shafarevich and Friedman.

There exists a coarse moduli space of K3 surfaces, but it is not separated (Hausdorff, in complex-analytical language). Instead, one usually considers K3 surfaces together with a choice of (quasi-) polarisation (a primitive nef and big line bundle: see definition 1.1.3). A K3 surface S together with a choice of a polarisation H is called a polarised K3 surface. One may think of this as an object intermediate between an abstract K3 surface and one embedded in a projective space. Also note that on a K3 surface the map c_1 taking a line bundle to its first Chern class is injective, so we may identify the polarisation line bundle with a cohomology class.

Polarised K3 surfaces have a discrete invariant called the degree: the self-intersection H^2 of the polarisation H on the surface S, or equivalently, the cup product of the cohomology class $c_1(H)$ with itself. Because this degree is always even, it is commonly written as $2d$, where d is then a positive integer. Alternatively, we may parametrise the possible degrees by the number $g = d + 1$; this number $g \geq 2$ is called the genus of the polarised K3 surface: it equals the genus of any smooth divisor in the polarisation class H.

The moduli space of polarised K3 surfaces thus splits up as a disjoint union of components $\mathcal{F}_{2d}$, one for every positive integer d. These components turn out to be better behaved than the moduli space of K3 surfaces without polarisation: they are irreducible quasi-projective varieties by the work of Baily–Borel [2], each of dimension 19.

It is now a natural question to ask for the birational type of these moduli spaces $\mathcal{F}_{2d}$. More specifically, we would like to compute the Kodaira dimension of each of them.

Recall that the Kodaira dimension is a birational invariant of algebraic varieties which measures the dimension of the canonical model of the variety (i.e., the stable image of the variety under the maps to projective space given by sections of tensor powers of the canonical bundle). We write $\kappa(X)$ for the Kodaira dimension of a variety X. It takes value either $-\infty$ or a number between 0 and the dimension of X. For example, in the case of curves the Kodaira dimension discriminates between curves of genus 0 (having Kodaira dimension $-\infty$), genus 1 (of Kodaira dimension 0) and genus at least 2 (those all have Kodaira dimension 1). All rational varieties – even all unirational ones – have Kodaira dimension $-\infty$; they are in some sense the simplest possible. At the other end of the spectrum, the varieties having Kodaira dimension equal to their dimension are typically harder to classify, and are said to be of general type.

As a comparison, for the moduli space $\mathcal{M}_g$ of curves of genus g one can ask the same question about the Kodaira dimension. It was hailed as a big breakthrough in the 1980s when Harris and Mumford proved that for g large enough these moduli spaces are of general type. It was then natural to ask for the Kodaira dimension of other moduli spaces too, in particular for abelian varieties and for K3 surfaces.

In the next section, we give an overview of what is currently known about the Kodaira dimension of $\mathcal{F}_{2d}$.

Previous work

For $d \in \{1,2,3,4\}$ there are well-known explicit constructions of polarised K3 surfaces: as a branched cover of $\mathbb{P}^2$, respectively as a complete intersection of hypersurfaces in some projective space. These constructions each give rise to a birational map between the moduli space $\mathcal{F}_{2d}$ and a quotient of a projective space by a linear group. In particular, $\mathcal{F}_{2d}$ is unirational in those cases.

Mukai has extended that result to some slightly higher values of d, proving that $\mathcal{F}_{2d}$ is unirational (hence $\kappa(\mathcal{F}_{2d}) = -\infty$) for $d \in \{5,6,7,8,9\}$ (see [32]), for $d \in \{10\}$ (see [34]), for $d \in \{11,12\}$ (see [35]), for $d \in \{15\}$ (see [36]) and also for $d \in \{17,19\}$ (see [33]).

In the other direction, Kondō has shown in [25] that $\mathcal{F}_{2p^2}$ is of general type for large enough primes p, and later in [26] that the Kodaira dimension of $\mathcal{F}_{2d}$ is non-negative if

$$d \in \{42,43,51,53,55,57,59,61,66,67,69,74,83,85,105,119,133\} \ .$$

Gritsenko, Hulek and Sankaran more recently proved [18] that $\mathcal{F}_{2d}$ is of general type for all $d > 61$ and also for $d \in \{46,50,54,57,58,60\}$, and that its Kodaira dimension is non-negative for $d \in \{40,42,43,46,48,49,51,52,53,55,56,59,61\}$. It was noted by the author and Sankaran in [40] that their method also applies to the case $d = 52$, proving that $\mathcal{F}_{2\cdot 52}$ is of general type.

Arithmetic description

Mukai's unirationality results for low values of d depend on a construction of these polarised K3 surfaces as complete intersections in some homogeneous space. In all other results a crucial role is played by an alternative description of the moduli space $\mathcal{F}_{2d}$, as a so-called locally symmetric domain, or arithmetic quotient. See for instance [4] and [22]; more details and references can be found in section 4.1.

This description of the moduli space of polarised K3 surfaces as an arithmetic quotient is a natural analogue of the presentation of the moduli space of elliptic curves as the quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ of the upper half-plane by the special linear group. As in the case of elliptic curves, it connects the geometry of the moduli space to the theory of automorphic forms.

The starting point of this description is the observation that all K3 surfaces S have isomorphic second cohomology groups: $\mathrm{H}^2(S,\mathbb{Z})$ is torsion-free of rank 22. The cup product gives a bilinear form on this group, making it into a lattice, and the lattice isomorphism class of this cohomology group is also the same for all K3 surfaces: $\mathrm{H}^2(S,\mathbb{Z}) \cong L_{K3}$ as lattices, where L_{K3} is the so-called K3 lattice:

$$L_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \ ,$$

where U is the rank 2 hyperbolic lattice and $E_8(-1)$ the negative definite unimodular even lattice of rank 8.

This cohomology lattice thus does not distinguish among different K3 surfaces. However, we may retain some information about the complex algebraic structure of the K3 surface S by not only taking its middle cohomology group, but also including the Hodge decomposition of the second cohomology group:

$$\mathrm{H}^2(S,\mathbb{Z})\otimes\mathbb{C} = \mathrm{H}^{2,0}(S)\oplus\mathrm{H}^{1,1}(S)\oplus\mathrm{H}^{0,2}(S)\ .$$

This decomposition is completely determined by the position of the 1-dimensional subspace $\mathrm{H}^{2,0}(S)$ of holomorphic 2-forms on S within the 22-dimensional complex vector space $\mathrm{H}^2(S,\mathbb{Z})\otimes\mathbb{C}$.

These considerations lead us to the so-called period map, which associates to a K3 surface S and an identification $\mathrm{H}^2(S,\mathbb{Z})\cong L_{K3}$ the point of the projective space $\mathbb{P}(L_{K3})\otimes\mathbb{C}$ corresponding to the complex line $\mathrm{H}^{2,0}(S)\subset\mathrm{H}^2(S,\mathbb{Z})\otimes\mathbb{C}$. It is now a crucial fact due to Piatetskii-Shapiro and Shafarevich [41] that we can recover the complex-algebraic structure of the K3 surface from its period point (i.e., its image under the period map); this is a form of the Torelli theorem.

One would like to exploit this fact to get a concrete description of the moduli space of polarised K3 surfaces. To do so, we perform some modifications to the period map. First of all, because the isomorphism $\mathrm{H}^2(S,\mathbb{Z})\cong L_{K3}$ of lattices is not unique, we are forced to quotient out by the automorphism group $\mathrm{O}(L_{K3})$ of the K3 lattice. This gives us a well-defined map from the set of isomorphism classes of K3 surfaces (without choice of identification) to the quotient space $\mathrm{O}(L_{K3})\backslash(\mathbb{P}(L_{K3})\otimes\mathbb{C})$. Secondly, because we look at K3 surfaces with a choice of (quasi-)polarisation H (of degree $2d$, say), the subspace $\mathrm{H}^{1,1}(S)\cap\mathrm{H}^2(S,\mathbb{Z})$ always contains the class $c_1(H)$ of square $2d$, which is orthogonal to the subspace $\mathrm{H}^{2,0}(S)$. We may as well remove this superfluous piece of information by taking as codomain of the period map not $\mathbb{P}(L_{K3}\otimes\mathbb{C})$, but $\mathbb{P}(L_{2d}\otimes\mathbb{C})$, where

$$L_{2d} = \langle -2d\rangle\oplus U^{\oplus 2}\oplus E_8(-1)^{\oplus 2}\ .$$

In doing so, we have to change the group from $\mathrm{O}(L_{K3})$ to the stable orthogonal group $\tilde{\mathrm{O}}(L_{2d})$ (see section 2.3 for definitions). As a final modification, the Hodge–Riemann relations tell us that the image of the period map is contained in the subset

$$\mathcal{D}_{2d}\cup\overline{\mathcal{D}_{2d}} = \{\mathbb{C}z : (z,z)=0, (z,\bar{z})>0\}\subset\mathbb{P}(L_{2d}\otimes\mathbb{C})\ ,$$

and we may further restrict to a single component $\mathcal{D}_{2d}$ if we also replace the group $\tilde{\mathrm{O}}(L_{2d})$ by its index two subgroup $\tilde{\mathrm{O}}^+(L_{2d})$ (again see section 2.3). The space $\mathcal{D}_{2d}$ is called the period domain.

Once we have done all this, the injectivity given by the Torelli theorem and a surjectivity theorem due to Todorov [51] assert that the modified period map is an isomorphism from the coarse moduli space of polarised K3 surfaces of degree $2d$ to the arithmetic quotient space

$$\mathcal{F}_{2d} = \tilde{\mathrm{O}}^+(L)\backslash\mathcal{D}_{2d}\ .$$

This means that the general theory of arithmetic quotients can be applied to our moduli space.

Remark. Note that the surjectivity does not require removing hyperplanes orthogonal to -2-vectors from the period domain, as is commonly done: see for instance [22, theorem 2.9]. The reason is that we allow our quasi-polarisations to be non-ample.

Picard group of $\mathcal{F}_{2d}$

Our research has focused for a large part on the rational Picard group of the moduli space $\mathcal{F}_{2d}$.

The arithmetic description gives rise to a set of divisors on $\mathcal{F}_{2d}$ called Heegner divisors (see section 4.2). On the level of the period domain $\mathcal{D}_{2d}$ these are simply subsets of points orthogonal to a given vector $v\in L_{2d}\otimes\mathbb{Q}$ of negative norm. As any multiple of v gives the same divisor, we might as well normalise v to be a primitive vector in L_{2d}. To get a divisor on the quotient $\mathcal{F}_{2d}=\tilde{\mathrm{O}}^+(L_{2d})\backslash\mathcal{D}_{2d}$, just take the union of the orthogonal complements of an $\tilde{\mathrm{O}}^+(L_{2d})$-invariant set of such vectors v.

The $\tilde{\mathrm{O}}^+(L_{2d})$-orbits of primitive vectors $v\in L_{2d}$ are parametrised by two pieces of data (see lemma 2.3.8): in terms of the vector v^* (the rational positive multiple of v that is primitive in the dual lattice $L_{2d}^\vee$ – see section 2.2), these are the norm of v^* and the class of v^* in the discriminant group

$$D_{L_{2d}} = L_{2d}^\vee/L_{2d}\cong\mathbb{Z}/2d\mathbb{Z}\ ;$$

for a discussion of the discriminant group, see section 2.2.

Therefore, the Heegner divisors on $\mathcal{F}_{2d}$ are parametrised by a class $\gamma\in D_{L_{2d}}\cong\mathbb{Z}/2d\mathbb{Z}$ and a number $n\in\mathbb{Q}_{<0}$ such that $n\in\mathbb{Z}-\gamma^2/2$; we denote them by $H(\gamma,n)$. Through the period map, these divisors also correspond to interesting subsets of the moduli space of polarised K3 surfaces. The divisor $H(\gamma, n)$ corresponds to a so-called Noether–Lefschetz divisor $D_{h,a}$: that is the locus of polarised K3 surfaces (S, H) having a line bundle Γ independent of H with intersection numbers $\Gamma^2 = 2h - 2$ and $\Gamma \cdot H = a$. There is an explicit relation between (γ, n) and (h, a): see lemma 4.2.7.

A few questions immediately arise: can we compute linear relations between the Heegner divisors $H(\gamma, n)$? Do the Heegner divisors span the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$? What is the dimension of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ as a function of d?

To start with the last question: O'Grady showed [39] that as the polarisation degree $2d$ increases, the dimension of the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ is unbounded: this was a first indication that the situation is very different from the case of the moduli spaces of curves.

Borcherds [6] has devised a very general construction of modular forms on arithmetic quotient spaces, and the vanishing and singular locus of these modular forms is supported on Heegner divisors (see section 4.3). This gives us a way to compute relations between Heegner divisors. Additionally, Bruinier [10] has proved that in fact all relations between Heegner divisors arise in this way. That means that we can compute the image of all Heegner divisors in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$, including explicit linear relations, by exhaustively applying Borcherds' construction.

The input datum to Borcherds' construction is also a modular form, but of a much simpler type: it lives not on the big symmetric domain $\mathcal{D}_{2d}$, but on the upper half-plane. Specifically, it should be a vector-valued modular form (see 3.3) of half-integral weight with respect to the metaplectic group (see section 3.1). Using a well-known isomorphism, such modular forms can also be viewed as Jacobi forms of lattice index. The work of Raum [42] has made it possible to compute these spaces of Jacobi forms. We have written computer programs implementing his algorithms, and used these to compute the part of the rational Picard group generated by Heegner divisors for d up to around 50.

Finally, the very recent work of Bergeron, Li, Millson and Moeglin [5] shows that the rational Picard group of $\mathcal{F}_{2d}$ is spanned by the Heegner divisors, so our computations in fact give an explicit presentation of the full rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$.

Strategy

The strategy we follow to try to compute the Kodaira dimension of $\mathcal{F}_{2d}$ is a variant of the one of Gritsenko, Hulek and Sankaran [18]. Specifically, we apply [18, theorem 1.1]: if we can find a modular form on $\mathcal{F}_{2d}$ of weight less than 19 that vanishes (seen as a function on the period domain $\mathcal{D}_{2d}$) on the ramification divisor of the quotient map $\mathcal{D}_{2d} \to \mathcal{F}_{2d}$, and also vanishes at the cusps, then $\mathcal{F}_{2d}$ is of general type.

Let us translate their statement to a geometric language. Modular forms correspond to sections of powers of the Hodge bundle λ. We write $\overline{\mathcal{F}_{2d}}$ for a toroidal compactification (see section 5.2) of $\mathcal{F}_{2d}$, and B for the branch divisor of $\mathcal{D}_{2d} \to \mathcal{F}_{2d}$ and $\Delta = \overline{\mathcal{F}_{2d}} \backslash \mathcal{F}_{2d}$ for the boundary divisor. In those terms, [18, theorem 1.1] instructs us to find an $\varepsilon > 0$ such that the class $K - \varepsilon\lambda = (19 - \varepsilon)\lambda - 1/2 \cdot [B] - [\Delta]$ on $\mathcal{F}_{2d}$ is effective. The proof of [18, theorem 1.1] can be paraphrased as follows: if $K - \varepsilon\lambda$ is effective, then K is the sum of the ample divisor $\varepsilon\lambda$ and some effective divisor, so K is big, so $\mathcal{F}_{2d}$ is of general type. An essential problem of this simplified argument is that $\overline{\mathcal{F}_{2d}}$ and even $\mathcal{F}_{2d}$ itself are not smooth; it is therefore necessary to prove that the singularities of $\overline{\mathcal{F}_{2d}}$ are not too bad, and that pluricanonical forms on $\mathcal{F}_{2d}$ extend across them. In technical terms: $\overline{\mathcal{F}_{2d}}$ should have canonical singularities. A large part of [18] is devoted to proving that this is indeed the case; they do so for a large class of arithmetic quotient varieties. We gratefully use their work, and may thus reduce to finding modular forms on $\mathcal{F}_{2d}$ that have the required ratio of weight (coefficient of λ) versus vanishing order at the branch divisor (coefficient of B) and at the boundary (coefficient of Δ).

Our approach differs from [18] in the manner in which we obtain the modular forms on $\mathcal{F}_{2d}$ with the required properties. The method used in [18] – quasi-restriction of the Borcherds form on a bigger lattice – is very general, but it may or may not give a modular form with the right properties, depending on the value of d in a rather complex way. On the other hand, we compute the whole space of relevant modular forms for every d in the range we are interested in, and try to systematically search this space for the right ones.

We proceed as follows: at first, we restrict the class $K - \varepsilon\lambda$ to the open part $\mathcal{F}_{2d}$ of the moduli space; this gives us the class $K^\circ - \varepsilon\lambda$, where $K^\circ = 19\lambda - 1/2 \cdot [B]$. We compute an expression of this class in terms of some basis of Heegner divisors of the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$, using our computation of the latter. We compare the resulting expression to the large supply of effective divisors that we have on $\mathcal{F}_{2d}$, in the form of Heegner divisors and some slightly refined variants thereof (so-called irreducible Noether–Lefschetz divisors; see section 4.2.1).

If this gives a negative result, then K° may be non-effective. However, to be sure of that, we would need to know whether there are any effective divisors on $\mathcal{F}_{2d}$ that cannot be written as a positive linear combination of irreducible Noether–Lefschetz divisors. We have no answer to that question, so we are left with conditional results such as theorem 4.7.5.

On the other hand, if we get a positive result, and so $K^\circ - \varepsilon\lambda$ is indeed effective, then it remains to determine whether the unrestricted class $K - \varepsilon\lambda = (K^\circ - \varepsilon\lambda) - [\Delta]$ on the compactified moduli space $\overline{\mathcal{F}_{2d}}$ is still effective. Let us explore what this means in terms of modular forms. We are given some relation

$$K^\circ - \varepsilon\lambda = \sum_{\gamma,n} a_{\gamma,n}[H(\gamma,n)]$$

of divisor classes, or equivalently line bundles, on $\mathcal{F}_{2d}$. This relation is witnessed by a section of the line bundle corresponding to an integral multiple of $(19-\varepsilon)\lambda$ with given behaviour at Heegner divisors, i.e., a modular form Ψ on $\mathcal{F}_{2d}$. The completion of the above relation on $\mathcal{F}_{2d}$ to a valid relation on $\overline{\mathcal{F}_{2d}}$ is obtained by computing the vanishing order of the modular form Ψ at all the irreducible components of the boundary divisor Δ. If all these vanishing orders are at least 1, then the completed relation on $\overline{\mathcal{F}_{2d}}$ represented by Ψ shows that the class $K - \varepsilon\lambda = (K^\circ - \varepsilon\lambda) - [\Delta]$ is effective – or equivalently, that the modular form in the formulation of [18, theorem 1.1] is a cusp form.

We contribute a method to perform this completion of relations on $\mathcal{F}_{2d}$ to ones on $\overline{\mathcal{F}_{2d}}$, by computing the vanishing orders at the boundary components of the perfect cone toroidal compactification of any modular form on $\mathcal{F}_{2d}$ that is constructed using Borcherds' method; see section 5.3.

In practice, there are too many irreducible boundary components to deal with them one at a time. We have devised a method to compute bounds for the vanishing order of a fixed modular form on $\mathcal{F}_{2d}$ at all the different boundary components of the toroidal compactification associated to the perfect cone decomposition. This allows us to prove that many modular forms on $\mathcal{F}_{2d}$ are cusp forms.

Results

We summarise the most important results here.

First of all, we have computed an explicit basis of $\mathrm{Pic}(\mathcal{F}_{2d})$, for d up to 50, and coefficients that express Heegner divisors $H(\gamma,n)$ in terms in this basis. A few examples of the resulting relation between Heegner divisors and the Hodge class $\lambda = -H(\overline{0},0)$:

$$\begin{aligned}
d=1: &\quad 150\lambda \sim H(\overline{0},-1) + 56H(\overline{1},-1/4)\ ;\\
d=2: &\quad 108\lambda \sim H(\overline{0},-1) + 128H(\overline{1},-1/8) + 14H(\overline{2},-1/2)\ ;\\
d=3: &\quad 98\lambda \sim H(\overline{0},-1) + 108H(\overline{1},-1/12) + 54H(\overline{2},-1/3) + 2H(\overline{3},-3/4)\ ;\\
d=4: &\quad 80\lambda \sim H(\overline{0},-1) + 112H(\overline{1},-1/16) + 56H(\overline{2},-1/4) + 16H(\overline{3},-9/16)\ ;\\
d=10: &\quad 28\lambda \sim H(\overline{0},-1) + 96H(\overline{1},-1/40) + 94H(\overline{2},-1/10) + 16H(\overline{3},-9/40)\\
&\qquad + 16H(\overline{4},-2/5) + 16H(\overline{5},-5/8) - 2H(\overline{6},-9/10) + 16H(\overline{7},-9/40)\\
&\qquad + 18H(\overline{8},-3/5) + 96H(\overline{9},-1/40)\ .
\end{aligned}$$

More examples can be found in section 4.4.

We show how to complete any relation in $\mathrm{Pic}(\mathcal{F}_{2d})$ to the boundary of a specific toroidal compactification $\overline{\mathcal{F}_{2d}}$, getting valid relations in $\mathrm{Pic}(\overline{\mathcal{F}_{2d}})$:

Theorem (5.3.3). *Let $\sum_{\gamma,n} a_{\gamma,n}H(\gamma,n) \sim 0$ be a linear equivalence of Noether–Lefschetz divisors on $\mathcal{F}_{2d}$. Then the following linear equivalence holds on $\overline{\mathcal{F}_{2d}}$, the toroidal compactification of $\mathcal{F}_{2d}$ with the perfect cone decomposition:*

$$\sum_{\gamma,n} a_{\gamma,n}H(\gamma,n) + \sum_{F\in S_1}\sum_{\gamma,n} a_{\gamma,n}c(\gamma,n,F)\,\Delta_F \sim 0\ .$$

Here F ranges over the 1-cusps S_1 of $\mathcal{F}_{2d}$, Δ_F is the boundary divisor over the cusp F, and $c(\gamma,n,F)$ is some explicit function calculating the contribution of a given Heegner divisor $H(\gamma,n)$ at the cusp F.

See the full statement of theorem 5.3.3 in chapter 5 for details, including a formula for the function c.

For example, we may complete the earlier relation for $d = 1$ to include the boundary terms:

$$150\,\lambda \sim H(\overline{0},-1) + 56\,H(\overline{1},-1/4) + 30\,(\Delta_\beta + \Delta_\gamma) + 18\,(\Delta_\zeta + \Delta_\eta)\ ;$$

see section 5.3.3 for notation and details.

Using these results, we get

Theorem (4.7.3)**.** *The moduli spaces $\mathcal{F}_{13}$ and $\mathcal{F}_{14}$ have Kodaira dimension $-\infty$.*

Together with Mukai's results we get that $\kappa(\mathcal{F}_{2d}) = -\infty$ for $1 \le d \le 15$. In fact, our method easily reproves these results by Mukai in a uniform way, using no geometric construction of $\mathcal{F}_{2d}$, just the arithmetic description through the lattice L_{2d} and a computation of coefficients of Eisenstein series.

Moreover, our method also reproves some results by Gritsenko–Hulek–Sankaran [18] (and Peterson–Sankaran [40]): if $d \in \{46, 50, 52, 54\}$, then $\mathcal{F}_{2d}$ is of general type (i.e., the Kodaira dimension is $\kappa(\mathcal{F}_{2d}) = 19$); see theorem 5.5.1.

Our computation is very explicit: it gives an concrete linear relation for the canonical divisor on the open part of the moduli space in terms of Noether–Lefschetz divisors. As an example, for $d = 46$ we get:

$$\begin{aligned} K^\circ - \lambda \sim\ & 19H(\overline{1},-1/184) + 16H(\overline{2},-1/46) + 16H(\overline{3},-9/184) + 15H(\overline{4},-2/23) + 14H(\overline{5},-25/184) \\ & + 10H(\overline{6},-9/46) + 8H(\overline{7},-49/184) + 5H(\overline{8},-8/23) + 4H(\overline{9},-81/184) + 3H(\overline{10},-25/46) \\ & + 2H(\overline{11},-121/184) + 17H(\overline{14},-3/46) + 10H(\overline{15},-41/184) + 6H(\overline{16},-9/23) + 2H(\overline{17},-105/184) \\ & + 1H(\overline{18},-35/46) + 11H(\overline{20},-4/23) + 6H(\overline{21},-73/184) + 1H(\overline{22},-29/46) + 13H(\overline{24},-3/23) \\ & + 6H(\overline{25},-73/184) + 2H(\overline{26},-31/46) + 7H(\overline{28},-6/23) + 2H(\overline{29},-105/184) + 10H(\overline{31},-41/184) \\ & + 3H(\overline{32},-13/23) + 7H(\overline{34},-13/46) + 2H(\overline{35},-121/184) + 19H(\overline{36},-1/23) + 4H(\overline{37},-81/184) \\ & + 8H(\overline{39},-49/184) + 1H(\overline{40},-16/23) + 14H(\overline{41},-25/184) + 3H(\overline{42},-27/46) + 16H(\overline{43},-9/184) \\ & + 2H(\overline{44},-12/23) + 19H(\overline{45},-1/184) + 2H(\overline{46},-1/2)\ . \end{aligned}$$

For intermediate degrees, we have the following results.

Theorem (4.7.6)**.** *Let d be such that $16 \le d \le 39$ or $d \in \{41, 44, 45, 47\}$. Either $\kappa(\mathcal{F}_{2d}) = -\infty$, or there exists an irreducible codimension 1 subvariety of $\mathcal{F}_{2d}$ that is not a Noether–Lefschetz divisor.*

Theorem (4.7.9)**.** *Let $d \in \{40, 42, 43, 48, 49, 55, 56\}$. Either $\kappa(\mathcal{F}_{2d}) < 19$, or there exists an irreducible codimension 1 subvariety of $\mathcal{F}_{2d}$ that is not a Noether–Lefschetz divisor.*

We also reproved the result by Gritsenko–Hulek–Sankaran that for $d \in \{40, 42, 43, 48, 49, 55, 56\}$ the Kodaira dimension of $\mathcal{F}_{2d}$ is non-negative; see theorem 5.5.3.

Conditional results

We may rephrase the last results above as conditional calculations of the Kodaira dimension of $\mathcal{F}_{2d}$ for the relevant intermediate polarisation degrees $2d$, the hypothesis being that the effective cone of $\mathcal{F}_{2d}$ is generated by irreducible Noether–Lefschetz divisors. It is not clear at all if this hypothesis holds for all d. For low d, it seems plausible, for example from comparison to the moduli space of abelian surfaces. On the other hand, similar statements for some moduli spaces of curves turned out to be false in general.

Theorem (4.7.5)**.** *Let d be such that $16 \le d \le 39$ or $d \in \{41, 44, 45, 47\}$. If the effective cone of $\mathcal{F}_{2d}$ is generated by irreducible Noether–Lefschetz divisors and our list of generators is complete (see questions 4.5.2 and 4.5.3), then $\kappa(\mathcal{F}_{2d}) = -\infty$.*

Theorem (5.5.4)**.** *Let $d \in \{40, 42, 43, 48, 49, 55, 56\}$. If the effective cone of $\mathcal{F}_{2d}$ is generated by irreducible Noether–Lefschetz divisors and our list of generators is complete (see questions 4.5.2 and 4.5.3), then we have intermediate Kodaira dimension: $0 \le \kappa(\mathcal{F}_{2d}) < 19$.*

In table 1 we list the results – some conditional and some unconditional – of our computation of $\kappa(\mathcal{F}_{2d})$, and compare with what was known by earlier work.

Table 1: The Kodaira dimension of $\mathcal{F}_{2d}$, for $1 \leq d \leq 64$. The column marked κ_1 collects what was known about the value of $\kappa(\mathcal{F}_{2d})$ by earlier work. The results of our computation of the same number $\kappa(\mathcal{F}_{2d})$ stand in the column marked κ_2. Results printed in grey are conditional, relying on a positive answer to question 4.5.3, which at the moment is still wide open.

d	$\dim \mathrm{Pic}_{\mathbb{Q}}$	κ_1	κ_2
1	2	-∞	-∞
2	3	-∞	-∞
3	4	-∞	-∞
4	4	-∞	-∞
5	6	-∞	-∞
6	7	-∞	-∞
7	7	-∞	-∞
8	8	-∞	-∞
9	9	-∞	-∞
10	10	-∞	-∞
11	11	-∞	-∞
12	12	-∞	-∞
13	12		-∞
14	14		-∞
15	15	-∞	-∞
16	14	-∞	-∞
17	16	-∞	-∞
18	17		-∞
19	17	-∞	-∞
20	19		-∞
21	20		-∞
22	20		-∞
23	21		-∞
24	23		-∞
25	21		-∞
26	24		-∞
27	24		-∞
28	25		-∞
29	26		-∞
30	29		-∞
31	27		-∞
32	28		-∞

d	$\dim \mathrm{Pic}_{\mathbb{Q}}$	κ_1	κ_2
33	30		-∞
34	30		-∞
35	32		-∞
36	32		-∞
37	31		-∞
38	34		-∞
39	35		-∞
40	36	≥ 0	$0 \leq \kappa_2 < 19$
41	36		-∞
42	39	≥ 0	$0 \leq \kappa_2 < 19$
43	36	≥ 0	$0 \leq \kappa_2 < 19$
44	39		-∞
45	40		-∞
46	40	19	19
47	41		-∞
48	43	≥ 0	$0 \leq \kappa_2 < 19$
49	40	≥ 0	$0 \leq \kappa_2 < 19$
50	43	19	19
51	45	≥ 0	
52	45	19	19
53	45	≥ 0	
54	48	19	19
55	48	≥ 0	$0 \leq \kappa_2 < 19$
56	50	≥ 0	$0 \leq \kappa_2 < 19$
57	49	19	
58	49	19	
59	51	≥ 0	
60	55	19	
61	51	≥ 0	
62	54	19	
63	55	19	
64	53	19	

Overview of contents

Chapters 1, 2 and 3 are introductory, and give background information on K3 surfaces, lattices (including discriminant groups, genera of lattices, and orthogonal groups), and modular forms (especially vector-valued ones), respectively. Section 3.6 contains some non-standard material, explaining how to use the work of Raum [42] to compute some spaces of vector-valued modular forms. This provides the data necessary to compute relations among divisors on the moduli space of K3 surfaces.

In chapter 4 we discuss the central object of our research: the moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces of degree $2d$. We introduce a set of well-known divisors on the moduli space, called Noether–Lefschetz or Heegner divisors; we compute relations among these divisors using the modular forms from chapter 3; we study the effective cone of the moduli space, in particular defining and computing a large natural subcone of this effective cone. Finally, we compute the canonical divisor – for the moment disregarding its behaviour at the boundary – in terms of Noether–Lefschetz divisors, and draw some first conclusions about the Kodaira dimension of $\mathcal{F}_{2d}$ for low values of d (see theorem 4.7.3).

In chapter 5, we study the compactified moduli space. We describe the boundary of two different compactifications: the Satake compactification $\mathcal{F}_{2d}^*$, and toroidal compactifications $\overline{\mathcal{F}_{2d}}$, following old work by Scattone [44], and newer work by Gritsenko, Hulek and Sankaran [18]. We show how to extend the relations among Noether–Lefschetz divisors from chapter 4 to the boundary. Also, we study the set of vector-valued modular forms that occur as theta series of a lattice in a fixed lattice genus, and use this to compute bounds for variations in vanishing order of modular forms at the different boundary components of a toroidal compactification $\overline{\mathcal{F}_{2d}}$ of $\mathcal{F}_{2d}$. This allows us to compare the canonical divisor on $\overline{\mathcal{F}_{2d}}$ to our supply of effective divisors, and compute some more Kodaira dimensions (see theorem 5.5.1).

The final chapter 6 is not part of the main flow of arguments leading to the computation of the Kodaira dimension of $\mathcal{F}_{2d}$, but it sprouted from the same research, and seemed interesting enough to include here. In this chapter we return to the geometry of individual K3 surfaces. We compute the effective, nef and ample cones of K3 surfaces with a rank 2 Picard group, and show how to relate the Clifford index of a smooth section of the polarisation class to the lattice structure of the Picard group of the K3 surface. This way, we may take an interesting geometrically defined divisor on the moduli space $\mathcal{F}_{2d}$ – the subset where the Clifford index of the polarisation curve jumps – and compute it in terms of the well-known Noether–Lefschetz divisors on $\mathcal{F}_{2d}$.

Chapter 1

K3 surfaces

Definition 1.0.1. A K3 surface is a smooth surface S, such that

(i) $\omega_S \cong \mathcal{O}_S$, and

(ii) $\mathrm{H}^1(S, \mathcal{O}_S) = 0$.

We will only consider projective K3 surfaces over $\mathbb{C}$ in this thesis.

Example 1.0.2. Some examples of K3 surfaces are: a smooth quartic hypersurface in $\mathbb{P}^3$; a Kummer surface (the desingularised quotient of an abelian surface by the equivalence $x \sim -x$); the double cover of $\mathbb{P}^2$ branched along a smooth sextic curve.

Many invariants are completely determined by the K3 conditions (i) and (ii). The geometric genus is given by

$$p_g = \mathrm{h}^0(S, \omega_S) = \mathrm{h}^0(S, \mathcal{O}_S) = 1 \ . \tag{1.1}$$

The holomorphic Euler characteristic is

$$\chi(\mathcal{O}_S) = 2 \ . \tag{1.2}$$

The arithmetic genus then becomes

$$p_a = (-1)^2(\chi(\mathcal{O}_S) - 1) = 1 \cdot (2 - 1) = 1 \ . \tag{1.3}$$

Substituting this value of the arithmetic genus in Noether's formula, we get $c_2(\mathcal{T}_S) = 24$, where $\mathcal{T}_S$ is the tangent sheaf of S.

Also, since $c_1(\mathcal{T}_S) = c_1(\bigwedge^2 \mathcal{T}_S) = c_1(\omega_S^\vee)$, and ω_S is trivial, the first Chern class of a K3 surface vanishes: $c_1(\mathcal{T}_S) = 0$.

Now, let S be a projective surface. Its middle cohomology $\mathrm{H}^2(S, \mathbb{Z})$ carries an integer-valued bilinear form, given by the cup product. (This product is also called the intersection product, and the resulting form the intersection form.) This makes the second cohomology $\mathrm{H}^2(S, \mathbb{Z})$ into a lattice (see chapter 2 for definitions). If S is a K3 surface, the structure of this lattice is completely determined:

Proposition 1.0.3 ([4, proposition VIII.3.2])**.** *If S is a K3 surface, then $\mathrm{H}^2(S, \mathbb{Z})$ is isomorphic (as a lattice) to the K3 lattice $L_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.*

(Sketch of proof: use the geometry of S, in particular the Hodge decomposition of $\mathrm{H}^2(S, \mathbb{Z})$ and the Hodge index theorem, to deduce that $\mathrm{H}^2(S, \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$. Then apply a general lattice-theoretic result which says that these properties suffice to characterise $\mathrm{H}^2(S, \mathbb{Z})$ as a lattice.)

Because $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, the long sequence in cohomology associated to the exponential sequence in particular gives an injective map $\mathrm{H}^1(S, \mathcal{O}_S^\times) \to \mathrm{H}^2(S, \mathbb{Z})$, or equivalently, identifying $\mathrm{H}^1(S, \mathcal{O}_S^\times)$ with the Picard group $\mathrm{Pic}(S)$, an embedding $\mathrm{Pic}(S) \to \mathrm{H}^2(S, \mathbb{Z})$.

Definition 1.0.4. The Picard lattice of a K3 surface S is the Picard group $\mathrm{Pic}(S)$, with bilinear form given by restricting the one on $\mathrm{H}^2(S, \mathbb{Z})$ along the above embedding $\mathrm{Pic}(S) \to \mathrm{H}^2(S, \mathbb{Z})$.

1.1 Linear series on a K3 surface

We can analyse the behaviour of linear series on a K3 surface to a large extent. The basic results on which this analysis rests are due to Saint-Donat ([43]); the proposition below is a summary of the excellent overview given in [23].

Proposition 1.1.1. *Let D be a divisor on a K3 surface S. We write $\mathcal{O}(D)$ for the associated line bundle on S.*

(i) If D is effective, then $\mathrm{h}^2(S, \mathcal{O}(D)) = 0$.

(ii) If D is effective and nef, and $D^2 = D \cdot D > 0$, then $\mathrm{h}^1(S, \mathcal{O}(D)) = 0$. Also, either

(a) $|D|$ is base-point free, and a generic element of $|D|$ is smooth and irreducible; or

(b) $D \sim kE + \Delta$, where $k \geq 2$ is an integer, $|E|$ is a pencil, $E^2 = 0$, Δ is an irreducible curve with $\Delta^2 = -2$, and $E \cdot \Delta = 1$.

(iii) If D is effective and nef, and $D^2 = 0$, then $D \sim kE$, for some smooth genus 1 curve E and some $k > 0$, and $\mathrm{h}^1(S, \mathcal{O}(D)) = k - 1$.

(Reider's method provides a modern general way to derive these results.)

This proposition allows us to compute $\mathrm{h}^0(S, \mathcal{O}(D))$ for a sufficiently positive line bundle $\mathcal{O}(D)$ in terms of its self-intersection:

Corollary 1.1.2. *Let $\mathcal{O}(D)$ be an effective and nef line bundle on a K3 surface S such that $D^2 > 0$. Then $\mathrm{h}^0(S, \mathcal{O}(D)) = 2 + D^2/2$.*

Proof. Applying the Riemann–Roch theorem to $\mathcal{O}(D)$ gives $\chi(S, \mathcal{O}(D)) = 2 + D^2/2$. By proposition 1.1.1, we have $\mathrm{h}^1(S, \mathcal{O}(D)) = \mathrm{h}^2(S, \mathcal{O}(D)) = 0$. □

We will be interested in the moduli space of K3 surfaces. However, this moduli space is not even separated. Therefore, we instead consider K3 surfaces together with a choice of line bundle on it, satisfying some positivity condition. You may think of this as an object intermediate between an abstract surface and a surface embedded in a projective space, which is equivalent to a surface together with a choice of divisor, again with some positivity condition.

Definition 1.1.3. A line bundle H on a K3 surface is called a polarisation if H is primitive (i.e., H is not a proper tensor power of another line bundle) and nef, and the self-intersection $H^2 = H \cdot H$ is positive. A polarised K3 surface is a K3 surface S together with a choice of a polarisation H.

The positive integer H^2 is called the degree of a polarised K3 surface.

Because the degree is always even, we may write $H^2 = 2d$ for a positive integer d. Confusingly but understandably, this number d is sometimes also called the degree. Throughout this thesis, the symbol d will always refer to this number.

Instead of the number d, we can also parametrise the possible degrees by the number $g = d + 1$; this number g is called the genus of the polarised K3 surface. By the adjunction formula, a smooth section of the polarisation class H has a canonical divisor of degree $H^2 = 2g - 2$, so g is in fact the genus of such a curve.

Remark 1.1.4. Some prefer to strengthen the positivity condition in the above definition, and require a polarisation to be ample instead of nef; a polarisation as we call it, would be a quasi-polarisation in their terminology. Also, some do not include the requirement that a polarisation be primitive.

Definition 1.1.5. If the polarisation H on a K3 surface S is of the exceptional type of case iib in proposition 1.1.1, the polarised K3 surface is called monogonal.

We can see whether a polarised K3 surface (S, H) is monogonal from the Picard lattice alone:

Proposition 1.1.6. *A polarised K3 surface (S, H) is monogonal if and only if there exists a line bundle $E \in \mathrm{Pic}(S)$ such that $E^2 = 0$ and $H \cdot E = 1$.*

Note that this condition is equivalent to the requirement that the polarised surface is in the Noether–Lefschetz divisor $D_{1,1}$, or equivalently the Heegner divisor $H(\overline{1}, -1/4d)$ (these divisors on the moduli space are defined in section 4.2).

1.2 Existence of K3 surfaces with prescribed Picard lattice

Suppose given an even hyperbolic lattice L of rank 2, together with an element $H \in L$ such that H^2 is positive.

Nikulin ([37]) gives sufficient conditions for a lattice to have a primitive embedding into the K3 lattice. These conditions are satisfied for our rank 2 lattice L, so we get a primitive embedding $L \to L_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. From this, and the surjectivity of the period map (which will be defined in the chapter on the moduli space of K3 surfaces; see section 4.1), it is easy to show that there exists a K3 surface S with Picard lattice isomorphic to L.

So, pick a K3 surface S with Picard lattice isomorphic to L. Note that there are many lattice isomorphisms from L to $\mathrm{Pic}(S)$ (precomposing with automorphisms of L permutes them). We would like to choose an isomorphism that sends the distinguished element $H \in L$ of positive norm to a line bundle on S with strong positivity properties. Lemma 1.2.2 below shows that this is possible to some extent.

First, let us define some positivity notions that we need now and later; see [28] for a proper introduction. Let S be a K3 surface; we define some cones in the real hyperbolic vector space $V = (\mathrm{Pic}\, S) \otimes \mathbb{R}$.

Definition 1.2.1. The set $\{x \in V : x^2 > 0\}$ is a disjoint union of two cones $\mathcal{C} \cup -\mathcal{C}$; one of these, say $\mathcal{C}$, contains an ample class, and is called the positive cone. The cone of $\mathbb{R}^+$-linear combinations of ample classes is called the ample cone; similarly, we have the nef cone $\mathrm{Nef}(S)$, and the effective cone $\mathrm{Eff}(S)$.

Recall from [28] that the nef cone is the closure of the ample cone, and the ample cone is the interior of the nef cone.

Lemma 1.2.2. *We may choose the isomorphism between* $\mathrm{Pic}(S)$ *and* L *in such a way that* H *corresponds to a nef class on* S*.*

Proof. Pick some isomorphism $\alpha : L \longrightarrow \mathrm{Pic}(S)$, and let $h = \alpha(H)$ be the class corresponding to H. Because $h^2 > 0$, either h or $-h$ is in the positive cone $\mathcal{C}$. By [4, VIII.3.9], we may translate any element of the positive cone to an element of the nef cone by reflections in (-2)-curves. Composing α with these reflections, and with -1 if necessary, we get an isomorphism from L to $\mathrm{Pic}(S)$ sending H to a nef class. □

Remark 1.2.3. It may or may not be possible to let H correspond to an ample class on S. This is determined by the structure of the lattice L (and the choice $H \in L$):

(i) if there exists an $\delta \in L$ such that $\delta^2 = -2$ and $\delta \cdot H = 0$, then H cannot correspond to an ample class, because we know by [4, VIII.3.6.(i)] that either δ or $-\delta$ is effective;

(ii) if, on the other hand, $\delta \cdot H \neq 0$ for all $\delta \in L$ with $\delta^2 = -2$, then the class h corresponding to H cannot lie on the boundary of $\mathrm{Nef}(S)$ (because any codimension 1 face of the nef cone $\mathrm{Nef}(S)$ is given by the hypersurface orthogonal to some (-2)-curve), so h is in fact ample.

Chapter 2

Lattices

In this chapter, we review the classical theory of lattices (discrete subgroups of finite-dimensional vector spaces with a bilinear form). As general references for this theory, see [12],[37],[14],[24].

2.1 Definitions

Definition 2.1.1. A lattice is a free $\mathbb{Z}$-module L of finite rank, together with a symmetric bilinear map $(\cdot,\cdot) : L \times L \to \mathbb{Q}$. If the bilinear form takes values in $\mathbb{Z}$, then the lattice is called integral. If $v^2 = (v,v) \in 2\mathbb{Z}$ for all $v \in L$, then the lattice is called even; if a lattice is integral but not even, it is called odd.

Example 2.1.2. The free $\mathbb{Z}$-module of rank 1 has a natural bilinear form, given by $(m,n) = mn$. This makes it into an integral odd lattice of rank 1.

The hyperbolic lattice U is the rank 2 free $\mathbb{Z}$-module, generated by elements e,f, say, with the bilinear map specified by the 2×2-matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Explicitly, this means that $e^2 = f^2 = 0$, and $(e,f) = (f,e) = 1$. The hyperbolic lattice is integral and even (because $(me+nf)^2 = 2mn \in 2\mathbb{Z}$).

The lattice E_8 is a special even integral lattice of rank 8. We define it by giving its intersection matrix (the 8×8-matrix specifying the values of the bilinear form on the generators):

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} . \tag{2.1}$$

Alternatively, one may define this lattice as the root lattice of a Lie algebra of type E_8.

For any lattice L, and any rational number m, we define a new lattice $L(m)$ with the same underlying $\mathbb{Z}$-module as L, but scaling the bilinear form by m:

$$(v,w)_{L(m)} = m \cdot (v,w)_L \ . \tag{2.2}$$

As a special case, we define $\langle m \rangle = \mathbb{Z}(m)$. So $\langle m \rangle$ has rank 1, and its bilinear form is $(a,b) = mab$.

Given two lattices L_1, L_2, their direct sum $L_1 \oplus L_2$ as a $\mathbb{Z}$-module carries a natural bilinear form, where the summands are made orthogonal:

$$(a_1 \oplus a_2, b_1 \oplus b_2) = (a_1, b_1) + (a_2, b_2) \ . \tag{2.3}$$

Definition 2.1.3. A sublattice of a lattice is a submodule that carries the induced bilinear form. A subgroup $M \subseteq L$ of a free $\mathbb{Z}$-module (in particular, a sublattice) is called primitive if the quotient module L/M is torsion-free. (Equivalently, if no non-zero multiples of non-members of M are in M; equivalently, if $(M \otimes \mathbb{Q}) \cap L = M$.) A nonzero vector $v \in L$ is called primitive if the subgroup it generates is primitive; this just means that v is not a positive integral multiple of any element of L, except for the trivial $v = 1 \cdot v$.

Definition 2.1.4. A lattice L is called non-degenerate if the bilinear form is non-degenerate, i.e., if the map $L \to \mathrm{Hom}_{\mathbb{Q}}(L \otimes \mathbb{Q}, \mathbb{Q}) : v \mapsto (v, \cdot)$ is injective.

A lattice L of positive rank is called (totally) isotropic if the bilinear form on L vanishes.

Although isotropic lattices do not have an interesting lattice structure themselves, they occur naturally as sublattices (even of non-degenerate lattices).

Example 2.1.5. All the examples from 2.1.2 are non-degenerate (except $\langle n \rangle$ when $n = 0$). The lattice $\langle 0 \rangle$ is isotropic, as is the sublattice of U spanned by e (because $e^2 = 0$).

Definition 2.1.6. Let L be a lattice. We consider $L_{\mathbb{R}} = L \otimes \mathbb{R}$, with the induced bilinear form $(\cdot, \cdot) : L_{\mathbb{R}} \times L_{\mathbb{R}} \to \mathbb{R}$. By the theory of real quadratic forms (in particular, Sylvester's law of inertia), we can choose a basis of $L_{\mathbb{R}}$ such that the intersection matrix becomes diagonal, and the numbers of positive, negative and zero entries (say b^+, b^- and b^0) in this diagonal matrix are well defined (that is, independent of the basis). Now, the signature of L is defined to be the triple (b^0, b^+, b^-).

If the lattice L is non-degenerate, then the number b^0 is zero, and we will typically write the signature as (b^+, b^-).

A lattice L is called positive definite (negative definite) if $b^- = b^0 = 0$ (resp. $b^+ = b^0 = 0$). It is called definite if it is either positive definite or negative definite.

Note that if L has signature (b^+, b^-), the maximal rank of an isotropic sublattice is $\min\{b^+, b^-\}$.

2.2 Dual lattices and the discriminant group

Definition 2.2.1. Let L be a non-degenerate lattice. The dual lattice $L^\vee$ is given by

$$L^\vee = \{x \in L \otimes \mathbb{Q} : (x, v) \in \mathbb{Z} \text{ for all } v \in L\} \ . \tag{2.4}$$

The bilinear form on $L^\vee$ is induced by the one on L.

Example 2.2.2. The dual of $\langle n \rangle$ is a strict supergroup of $\langle n \rangle$, generated by $w/n \in L \otimes \mathbb{Q}$, where w is the generator of $\langle n \rangle$. We see that this generator has norm $(w/n, w/n) = (w, w)/n^2 = n/n^2 = 1/n$, so $\langle n \rangle^\vee$ is isomorphic as a lattice to $\langle 1/n \rangle$.

The dual of each of $\mathbb{Z}$, U, E_8 equals the lattice itself. Taking the dual distributes over direct sums.

The first of these examples shows that the dual of an integral lattice might not be integral.

There is another candidate for the dual of a lattice L: the abelian group $L^* = \mathrm{Hom}_{\mathbf{Ab}}(L, \mathbb{Z})$. There is a natural map from $L^\vee$ to L^*, sending x to the function $v \mapsto (x, v)$. Because L is non-degenerate, this map is a bijection, and we may use it to give L^* a lattice structure. We will identify these two incarnations of the dual lattice.

Definition 2.2.3. A morphism $f : K \to L$ of lattices is a linear map that preserves the bilinear maps: $(f(x), f(y))_L = (x, y)_K$. If we want to be more explicit, we may also call a morphism an isometric map.

Note that some natural linear maps between lattices may not be isometric: for example, the projection $K \oplus L \to K : x \oplus y \mapsto x$, or the dual map $f^\vee : L^\vee \to K^\vee$ of an isometric map $f : K \to L$.

Lemma 2.2.4. *Suppose A and L are lattices. If $j : A \to L$ is an injective linear map, and its image is primitive, then the dual map $j^\vee : L^\vee \to A^\vee$ is surjective.*

Proof. Because j is injective, we may view A as a subgroup of L. Note that $j^\vee$ sends $\varphi \in L^\vee$, say $\varphi : L \to \mathbb{Z}$, to $\varphi \circ j \in A^\vee$: in other words, $j^\vee$ restricts $\mathbb{Z}$-valued maps on L to A. Surjectivity of $j^\vee$ thus means, that for any map $\psi : A \to \mathbb{Z}$ of $\mathbb{Z}$-modules, we can extend the domain to L. Now, because A is primitive in L, the quotient L/A is torsion-free, and we may take a subgroup B of L such that $L = A \oplus B$ as $\mathbb{Z}$-modules. Extending ψ by zero on B, we get the desired map $\varphi : L \to \mathbb{Z}$. □

If L is integral and non-degenerate, it is naturally a sublattice of its dual $L^\vee$, by the map $v \mapsto v \otimes 1$ (or, viewing the dual as $\mathrm{Hom}(L, \mathbb{Z})$, by the map $v \mapsto (v, \cdot)$). However, this embedding might not be primitive. The failure of primitivity is measured by the quotient $L^\vee/L$. This is an important invariant of the lattice L.

Definition 2.2.5. Let L be an integral lattice. The discriminant group of L is the abelian group $D_L = L^\vee/L$. It carries a bilinear map $(\cdot,\cdot) : D_L \times D_L \to \mathbb{Q}/\mathbb{Z}$, induced by the bilinear form of L.

The lattice L is called unimodular if the discriminant group is trivial (equivalently, if $L^\vee = L$).

We may occasionally call D_L the discriminant module, if we want to emphasise its structure as an abelian group together with a quadratic form. (Such objects are sometimes called quadratic modules.)

In general, for a lattice L, the quadratic form on D_L takes values in $\mathbb{Q}/\mathbb{Z}$. If L is even, though, we may let the quadratic form take values in $\mathbb{Q}/2\mathbb{Z}$, and for any $\gamma \in D_L$, the number $\gamma^2/2$ is a well-defined element of $\mathbb{Q}/\mathbb{Z}$.

Example 2.2.6. The lattices $\mathbb{Z}$, U and E_8 are unimodular. The lattice $\langle n\rangle$ has discriminant group isomorphic to $\mathbb{Z}/\left|n\right|\mathbb{Z}$, with quadratic form given by $\overline{k}^2 = k^2/n \in \mathbb{Q}/\mathbb{Z}$. (The isomorphism sends the generator of $\langle n\rangle^\vee$ to the class $\overline{1} \in \mathbb{Z}/\left|n\right|\mathbb{Z}$; this is well defined, because $|n|$ times the generator of $\langle n\rangle^\vee$ is a vector in $\langle n\rangle \subseteq \langle n\rangle^\vee$, and the norm of this, $n^2/n = n \in \mathbb{Z}$ indeed vanishes in $\mathbb{Q}/\mathbb{Z}$.)

Definition 2.2.7. Let L be a non-degenerate integral lattice, and $v \in L$ a nonzero vector. The divisor of v is a number, written as $\operatorname{div}(v)$, that measures the imprimitivity of v in the lattice $L^\vee$. We define it to be the positive generator of the ideal $(v, L) \subseteq \mathbb{Z}$.

Also, we let $v^* = v/\operatorname{div}(v) \in L^\vee$.

Note that v^* is the unique positive rational multiple of v that is primitive in the dual lattice $L^\vee$.

2.2.1 Comparing discriminant groups

Suppose we are given a primitive embedding $j : A \to L$ of non-degenerate lattices. We would like to compare the discriminant groups of A and L.

As a first step, we may take the dual map $j^\vee : L^\vee \to A^\vee$, and compose this with the quotient map $A^\vee \to A^\vee/A$ to get a map $\pi : L^\vee \to A^\vee/A$. Note that, by lemma 2.2.4, the map π is surjective.

Next, we would like to let this map descend from $L^\vee$ to $L^\vee/L$ to get a map $L^\vee/L \to A^\vee/A$ of discriminant groups. However, this may not be possible.

Example 2.2.8. Let $L = U$, the hyperbolic plane. Let $A = \langle 2\rangle = \mathbb{Z}w$, a rank 1 lattice with generator w of norm 2. Let $j : A \to L$ be the map that sends w to $e + f \in U$. (Note that $(e + f)^2 = 2$, and $e + f$ is primitive in U, so this is a primitive embedding.)

Now, we have $U^\vee = U$, and $A^\vee = \frac{1}{2}\mathbb{Z}w$. Also, the map $j^\vee$ sends $e \in U$ to the map $w \mapsto (j(w), e) = (e + f, e) = 1$, which corresponds to $\frac{1}{2}w$; in the same way we get $j^\vee(f) = \frac{1}{2}w$. Therefore, the kernel of $\pi : L^\vee = U \to A^\vee/A \cong \mathbb{Z}/2\mathbb{Z}$ consists of all elements $me + nf$ such that $m + n$ is even. This is a strict sublattice of U (of index 2).

In particular, the kernel of π does not contain L, so π does not descend to a map $L^\vee/L \to A^\vee/A$ of discriminant groups.

We introduce a name for the case where this anomaly does not occur. It applies in a more general setting, not just for embeddings, but for any map of lattices.

Definition 2.2.9. If $f : A \to B$ is a map of non-degenerate lattices such that $B \subseteq \ker(B^\vee \to A^\vee/A)$, then we say that f is monomodular.

Lemma 2.2.10. *Suppose that $f : A \to B$ is a monomodular map.*

(i) In this case f extends to a linear map $f_\vee : A^\vee \to B^\vee$. (More explicitly: if $\alpha \in A^\vee$, say $\alpha = \frac{1}{n}a$ with $a \in A$, then $\frac{1}{n}f(a) \in B^\vee \subset B \otimes \mathbb{Q}$.)

(ii) If additionally f is isometric, then $f^\vee \circ f_\vee = \mathrm{id}_{A^\vee}$.

Proof. (i) Let $\alpha \in A^\vee$. Write $\alpha = \frac{1}{n}a$ with $a \in A$. We need to prove that $\frac{1}{n}f(a) \in B^\vee$, i.e., that for all $b \in B$, we have $(\frac{1}{n}f(a), b) \in \mathbb{Z}$.

Now, we know that $b : A \to B$ is monomodular, so $b \in B$ is in the kernel of the map $B^\vee \to A^\vee/A$. This means that $f^\vee(b) \in A^\vee$ is in fact an element of A, say a_b: this means that for all $x \in A$, we have $(a_b, x) = (f^\vee(b), x) = (b, f(x))$. Applying this to $x = a$, we see that $(a_b, a) = (b, f(a))$, so also $(a_b, \frac{1}{n}a) = (b, \frac{1}{n}f(a))$. The left-hand side of this last equation equals (a_b, α), which is in $\mathbb{Z}$, because $\alpha \in A^\vee$ and $a_b \in A$. Therefore the right-hand side, $(b, \frac{1}{n}f(a))$, is in $\mathbb{Z}$ as well, which was to be proved.

(ii) Let $\alpha \in A^\vee$, and take any $x \in A$. Then we have $(f^\vee(f_\vee(\alpha)), x) = (f_\vee(\alpha), f(x)) = (\alpha, x)$ (in the last step, we use that f is isometric). Because this holds for all $x \in A$, and A is non-degenerate, we conclude that $f^\vee(f_\vee(\alpha)) = \alpha$. □

Definition 2.2.11. If $f : A \to B$ is a linear map of non-degenerate lattices such that $B = \ker(B^\vee \to A^\vee/A)$, then we say that f is isomodular.

Proposition 2.2.12. *Let $f : A \to B$ be a linear map of non-degenerate lattices.*

(i) If f is injective with primitive image, and isomodular, then the induced map $B^\vee/B \to A^\vee/A$ is an isomorphism of $\mathbb{Z}$-modules.

(ii) If additionally f is isometric, then the induced map $B^\vee/B \to A^\vee/A$ is an isomorphism of discriminant modules.

Proof. (i) By lemma 2.2.4, the map $B^\vee \to A^\vee$ is surjective, so the composition $B^\vee \to A^\vee/A$ is surjective. Since $B = \ker(B^\vee \to A^\vee/A)$ (as f is isomodular), we get an isomorphism $B^\vee/B \to A^\vee/A$ of $\mathbb{Z}$-modules. (ii) We noted earlier that the dual map $f^\vee : B^\vee \to A^\vee$ might not be isometric. However, by lemma 2.2.10.ii, $f_\vee$ is a pre-inverse of $f^\vee$, so, descending to discriminant groups, we see that the map $A^\vee/A \to B^\vee/B$ induced by $f_\vee$ is a left inverse of $B^\vee/B \to A^\vee/A$; because the latter is an isomorphism, these two induced maps are in fact two-sided inverses. Finally, because $f_\vee$ is isometric, the induced map $A^\vee/A \to B^\vee/B$ preserves the quadratic structure, so its inverse $B^\vee/B \to A^\vee/A$ does as well. □

Note that the discriminant modules of A and B can be isomorphic, even though $f : A \to B$ is isometric but not isomodular. Example: $A = \langle 4\rangle$, $B = \mathbb{Z} \oplus \langle 4\rangle$, and $f : A \to B$ sends the generator of A to $2 \in \mathbb{Z}$.

We now formulate a result we will later need. It is a precise formulation of the following intuition: if $A \subset L$ is a sublattice, and all the "non-unimodularity" of L comes from A, then the complement $A^\perp$ should be unimodular.

Proposition 2.2.13. *Let L be a non-degenerate lattice, and A a non-degenerate sublattice. Write $B = A^\perp$ for the orthogonal complement of A in L; we assume that B is non-degenerate as well. If the inclusion $A \to L$ is isomodular, then B is unimodular.*

Proof. Let $\beta \in B^\vee$; we must prove that $\beta \in B$. Write $f : A \to L$ and $g : B \to L$ for the inclusions.

Because B is primitive in L (by construction as an orthogonal complement), the map $g^\vee : L^\vee \to B^\vee$ is surjective (lemma 2.2.4), so there is an $\lambda \in L^\vee$ such that $g^\vee(\lambda) = \beta$. Write $\alpha = f^\vee(\lambda) \in A^\vee$. Now, look at $b = \lambda - f_\vee(\alpha) \in L^\vee$. We have $f^\vee(b) = f^\vee(\lambda) - f^\vee(f_\vee(\alpha)) = \alpha - \alpha = 0$, where we use lemma 2.2.10.ii (which applies, since f is isometric, and isomodular, hence monomodular).

Because $f^\vee(b) = 0$, the vector b is in the kernel of the composed map $L^\vee \to A^\vee \to A^\vee/A$. Since f is isomodular, this kernel equals L, so $b \in L$.

Next, again because $f^\vee(b) = 0$, we see that b is in the kernel of the composed map $L \to L^\vee \to A^\vee$. This kernel is exactly $A^\perp = B$ (just unfold the definitions), so $b \in B$.

Finally, we prove that $b = \beta$. Because B is non-degenerate, it suffices to show that $(b, x) = (\beta, x)$ for all $x \in B$. Now, we have

$$(\beta, x) = (g^\vee(\lambda), x) = (\lambda, g(x)) = (g(b) + f_\vee(\alpha), g(x)) = (g(b), g(x)) + (f_\vee(\alpha), g(x)) = (b, x)\ , \quad (2.5)$$

where in the last step, we used that g is isometric, and that any elements $\alpha \in A^\vee$ and $x \in B$ are orthogonal (since $B = A^\perp$). This concludes the proof. □

2.3 Orthogonal groups

Definition 2.3.1. Let L be an integral lattice. We denote the group of automorphisms of L by $\mathrm{O}(L)$.

Recall that morphisms of lattices by definition preserve the bilinear form, so automorphisms are invertible $\mathbb{Z}$-linear maps that preserve the bilinear form. Such maps are sometimes called orthogonal maps, and the group $\mathrm{O}(L)$ is called the orthogonal group of L.

We get an important class of automorphisms by reflecting in hyperplanes orthogonal to a vector.

Definition 2.3.2. Let $v \in L$ be non-isotropic (i.e., $v^2 \neq 0$). Then we have an involution $\sigma_v \in \mathrm{O}(L_\mathbb{Q})$, given by

$$\sigma_v : x \mapsto x - 2\frac{(v, x)}{(v, v)}v\ , \quad (2.6)$$

called the reflection associated to v. (It is the reflection in the hyperplane orthogonal to v.) This map $\sigma_v : L_\mathbb{Q} \to L_\mathbb{Q}$ may not preserve the lattice L. If it does, and in addition v is primitive in L and $(v, v) < 0$, then we call v a generalised root.

Note that the reflection σ_v associated to a root v (i.e., a vector v with $v^2 = -2$) preserves L, so a root is indeed an example of a generalised root.

For even unimodular lattices, there are no others:

Lemma 2.3.3. *Let L be an even unimodular lattice. Then any generalised root $v \in L$ is a root.*

Proof. Let $v \in L$ be a generalised root, so v is primitive in L, and $v^2 < 0$, and σ_v preserves L. Looking at the definition of σ_v, and given the primitivity of v, we see that $\sigma_v(x) \in L$ if and only if v^2 divides $2(v, x)$. This happens for all $x \in L$ if and only if v^2 divides $2\,\mathrm{div}(v)$. Now, because L is unimodular, and v is primitive in L, we have in fact $\mathrm{div}(v) = 1$, so we conclude that v^2 divides 2. Since $v^2 < 0$ and L is even, we see that $v^2 = -2$, so v is a root. □

Lemma 2.3.4. *The group $\mathrm{O}(L_{\mathbb{Q}})$ is generated by the reflections $\{\sigma_v : v \in L\}$.*

See for instance [44, Section 3.5] for a proof.

The automorphism group of a lattice has a few natural subgroups that will be important to us.

An automorphism $\sigma \in \mathrm{O}(L)$ descends to a map $\overline{\sigma} : D_L \to D_L$, which then is an automorphism of abelian groups, preserving the bilinear form (i.e., an automorphism of quadratic modules). This gives a homomorphism of groups $\mathrm{O}(L) \to \mathrm{O}(D_L)$.

Definition 2.3.5. The subgroup $\tilde{\mathrm{O}}(L) \subseteq \mathrm{O}(L)$ is the kernel of the map $\mathrm{O}(L) \to \mathrm{O}(D_L)$. In other words, it is the set of automorphisms of L that act trivially on the discriminant group.

Another important subgroup of $\mathrm{O}(L)$ is the group of automorphisms of trivial spinor norm (to be defined below). This spinor norm distinguishes reflections associated to vectors of positive norm and negative norm.

Definition 2.3.6. Let $\sigma \in \mathrm{O}(L)$. Write σ as a product of reflections: $\sigma = \sigma_{v_1} \cdots \sigma_{v_n}$ (this is possible, by lemma 2.3.4). Then we define the spinor norm of σ by

$$(-1)^n \prod_{i=1}^{n} \operatorname{sign} v_i^2 \ . \tag{2.7}$$

The group $\mathrm{O}^+(L) \subseteq \mathrm{O}(L)$ is the subgroup of automorphisms of spinor norm 1.

We do not verify here that this does not depend on the chosen decomposition of σ as a product of reflections. We will only use the spinor norm for lattices L of signature $(2, n)$, and in that case $\mathrm{O}^+(L)$ is the subgroup of transformations that do not interchange the two components of the period domain associated to L (see section 4.1).

Definition 2.3.7. The group $\tilde{\mathrm{O}}^+(L) \subseteq \mathrm{O}(L)$ is the intersection of $\tilde{\mathrm{O}}(L)$ and $\mathrm{O}^+(L)$.

Lemma 2.3.8 ([22, lemma 7.5])**.** *Suppose that L is a lattice containing two orthogonal copies of the hyperbolic plane U. Then the $\tilde{\mathrm{O}}^+(L)$-orbit of a primitive vector $v \in L$ is determined by the norm v^2 and the discriminant class of $v^* \in L^\vee$.*

2.4 Lattice genera

Classifying lattices is hard in general. As an approximation to isomorphism, we can ask whether two lattices are isomorphic locally at each prime p.

Definition 2.4.1. Let K and L be lattices. We say that K and L are p-adically equivalent if $K \otimes \mathbb{Z}_p \cong L \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ are the p-adic integers. (We include the case of the infinite prime ∞, with the convention that $\mathbb{Z}_\infty = \mathbb{R}$.) We call K and L locally equivalent if they are p-adically equivalent at every prime p, including the infinite prime ∞.

If L is a lattice, we define the genus of L (denoted by $\mathcal{G}(L)$) to be the set of isomorphism classes of lattices that are locally equivalent to L.

This weaker equivalence notion turns out to be very useful. First of all, it is easy to decide whether two lattices are p-adically equivalent; the p-adic equivalence classes can be described by a relatively simple invariant, called the local symbol at p. Note that local equivalence at ∞ is just equivalence of real quadratic forms, so the local symbol at ∞ is the signature of the lattice.

Secondly, the genus of any given lattice turns out to be a finite set, so in this sense, local equivalence comes close to full isomorphism. In practice, the genus can be quite large, so the difference is still significant.

If we restrict to even lattices, the local equivalence class of a lattice at all finite primes is captured precisely by the discriminant module:

Theorem 2.4.2 ([37, 1.9.4])**.** *Even lattices are locally equivalent if and only if they have the same signature and isomorphic discriminant modules.*

2.5 Vector-valued theta series

Let L be a definite even lattice. Let us assume for convenience that L is positive definite; the negative definite case can be dealt with in the same way. There is a classical object derived from L, called the theta series. It counts the number of vectors in L of all possible lengths:

$$\Theta_L = \sum_{v \in L} q^{v^2/2} \ . \tag{2.8}$$

If L is non-unimodular, we may extract more information from it by considering vectors in the dual lattice $L^\vee$. We may keep track of the discriminant class (i.e., the coset of the vector in $D_L = L^\vee/L$) by letting the theta series take values in the group algebra $\mathbb{C}[D_L]$; we write $\mathbf{e}_\gamma \in \mathbb{C}[D_L]$ for the basis element associated to $\gamma \in D_L$:

$$\boldsymbol{\Theta}_L = \sum_{v \in L^\vee} q^{v^2/2} \mathbf{e}_{v+L} \ . \tag{2.9}$$

We call this the vector-valued theta series associated to the lattice L.

Proposition 2.5.1 ([6, section 4])**.** *If L is a definite even lattice of rank k, then $\boldsymbol{\Theta}_L$ is a vector-valued modular form (see section 3.3) of weight $k/2$, with values in the Weil representation ρ_L (see section 3.2).*

This is an improvement from the usual scalar-valued theta series, which is modular only with respect to a congruence subgroup. There is no fundamental difference though; components of a vector-valued modular form are themselves scalar-valued modular forms with level, and scalar-valued modular forms can be extended to get vector-valued ones.

Example 2.5.2. If we take $L = E_8$, a definite unimodular lattice of rank 8, we get as theta series a modular form $\boldsymbol{\Theta}_{E_8} = \Theta_{E_8}\mathbf{e}_0$ of weight $8/2 = 4$. Note that there is only one component, because E_8 is unimodular, so the discriminant group $L^\vee/L$ is trivial. We may as well identify $\boldsymbol{\Theta}_{E_8}$ with the scalar-valued form Θ_{E_8}.

Because the space of scalar-valued modular forms of weight 4 is 1-dimensional, Θ_{E_8} must be proportional to the classical Eisenstein series G_4. In fact, because the constant term of Θ_{E_8} equals 1 (being the number of vectors of length 0 in E_8), the theta series Θ_{E_8} equals the normalised Eisenstein series

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \ . \tag{2.10}$$

Incidentally, this gives a cheap method to compute the number of vectors in the lattice E_8 of given length.

In terms of the so-called Jacobi theta functions

$$\begin{aligned} \theta_2(q) &= \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2/2} \\ \theta_3(q) &= \sum_{n=-\infty}^{\infty} q^{n^2/2} \\ \theta_4(q) &= \sum_{n=-\infty}^{\infty} (-q)^{n^2/2} \ , \end{aligned} \tag{2.11}$$

we can write the theta series of the E_8 lattice as

$$\Theta_{E_8} = 1/2 \cdot \left(\theta_2(q)^8 + \theta_3(q)^8 + \theta_4(q)^8\right) \ ; \tag{2.12}$$

see [12, p. 122], but note that their q is $\exp(\pi i z)$, whereas we use $q = \exp(2\pi i z)$.

Example 2.5.3. As another example, let us take $L = D_n$, a definite lattice of rank n. This lattice is not unimodular; its discriminant group is the Klein four group if n is even, and the cyclic group C_4 of order four if n is odd. Let us write $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for the generators of the group algebra $\mathbb{C}[L^\vee/L]$ in either case (so if n is even, we have $\mathbf{e}_i^2 = \mathbf{e}_0$ for all i and $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3$; if n is odd, then $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_k$ with $k \equiv i + j \mod 4$).

Because the discriminant group has four elements, the vector-valued theta series, taking values in $\mathbb{C}[L^\vee/L]$, has four components. We may write it in terms of Jacobi theta functions as

$$\begin{aligned} \mathbf{\Theta}_{D_n} =& 1/2 \cdot \left(\theta_3(q)^n + \theta_4(q)^n\right) \mathbf{e}_0 \\ &+1/2 \cdot \theta_2(q)^n \, \mathbf{e}_1 \\ &+1/2 \cdot \left(\theta_3(q)^n - \theta_4(q)^n\right) \mathbf{e}_2 \\ &+1/2 \cdot \theta_2(q)^n \, \mathbf{e}_3 \ ; \end{aligned} \tag{2.13}$$

see for instance [12, p. 118].

Chapter 3

Modular forms

3.1 The metaplectic group

We will be dealing with modular forms of half-integral weight, as extensively studied by Shimura [48]. The transformation group governing the behaviour of these forms is not the special linear group, as in the integer weight case, but a double cover of it called the metaplectic group; the classic reference for this is [52].

Similarly to the integer weight case, we have a discrete subgroup $\mathrm{Mp}_2(\mathbb{Z})$ of a connected Lie group $\mathrm{Mp}_2(\mathbb{R})$. Unfortunately, we cannot realise this metaplectic group as a group of matrices. (More precisely: there is no faithful finite-dimensional representation of $\mathrm{Mp}_2(\mathbb{R})$.) We can get an explicit description from the covering map to $\mathrm{SL}_2(\mathbb{R})$, though.

Definition 3.1.1. The (real) metaplectic group is given as a set by

$$\mathrm{Mp}_2(\mathbb{R}) = \left\{ \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{cz+d} \right) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}) \right\} \ , \tag{3.1}$$

where $\sqrt{cz+d}$ denotes one of the two branches of the square root of the function $cz+d$ on the upper half-plane.

The group structure on $\mathrm{Mp}_2(\mathbb{R})$ is as follows:

$$(A_1, \varepsilon_1)(A_2, \varepsilon_2) = (A_1 A_2, \varepsilon) \ , \tag{3.2}$$

where

$$\varepsilon(z) = \varepsilon_1(A_2 \cdot z)\varepsilon_2(z) \ . \tag{3.3}$$

The covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ is the obvious map, forgetting about the choice of square root. The integral metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ is the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under this covering map.

We state some simple results on the structure of the integral metaplectic group.

Proposition 3.1.2.

(i) $\mathrm{Mp}_2(\mathbb{Z})$ *is generated by the elements* $\tilde{T} = (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$ *and* $\tilde{S} = (\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{z})$.

(ii) The centre of $\mathrm{Mp}_2(\mathbb{Z})$ *is generated by the element* $Z := \tilde{S}^2 = (\tilde{S}\tilde{T})^3 = (\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \mathrm{i})$.

Proof. (i) Let $g \in \mathrm{Mp}_2(\mathbb{Z})$. We know ([45]) that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T; so we can write

$$\pi(g) = \prod_i S^{m_i} T^{n_i} \tag{3.4}$$

for some integers m_i and n_i. Then, since $\pi(\tilde{S}) = S$ and $\pi(\tilde{T}) = T$, we have

$$\pi(g) = \pi\left(\prod_i \tilde{S}^{m_i} \tilde{T}^{n_i} \right) , \tag{3.5}$$

so g and $\prod_i \tilde{S}^{m_i} \tilde{T}^{n_i}$ are the same up to the choice of square root. Multiplying by $\tilde{S}^4 = (\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), -1)$ to change the square root if necessary, we see that $g \in \left\langle \tilde{S}, \tilde{T} \right\rangle$. (ii) Clearly, Z is in the centre of $\mathrm{Mp}_2(\mathbb{Z})$.

On the other hand, if c is an element of the centre, then $\pi(c)$ is in the centre of $\mathrm{SL}_2(\mathbb{Z})$ (because π is a surjective homomorphism), hence equal to $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$. Then c must be a pre-image of one of these elements, allowing only the four possibilities $1, Z, Z^2, Z^3$. □

Definition 3.1.3. The cyclic subgroup of $\mathrm{Mp}_2(\mathbb{Z})$ generated by $\tilde{T}$ is denoted by $\tilde{\Gamma}_\infty$.

3.2 The Weil representation associated to a lattice

Given an even lattice L, there is a natural representation ρ_L of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ on the vector space $\mathbb{C}[L^\vee/L]$. It is a special case of the representations of symplectic groups of [52], and thus it is called the Weil representation. It may be defined by the action of the generators $\tilde{T}, \tilde{S}$:

Definition 3.2.1. Let L be an even lattice of signature (b^+, b^-). Write $\mathrm{e}(t) = e^{2\pi i t}$ for convenience, and denote the natural generators of the group algebra $\mathbb{C}[L^\vee/L]$ by $\mathbf{e}_\gamma$ for $\gamma \in L^\vee/L$. We define the representation ρ_L by

$$\begin{aligned} \rho_L(\tilde{T})\,\mathbf{e}_\delta &= \mathrm{e}(\delta^2/2)\,\mathbf{e}_\delta \qquad\qquad \text{and} \\ \rho_L(\tilde{S})\,\mathbf{e}_\delta &= \frac{\mathrm{e}((b^- - b^+)/8)}{\sqrt{|L^\vee/L|}} \sum_{\gamma\in L^\vee/L} \mathrm{e}(-\gamma\cdot\delta)\,\mathbf{e}_\gamma \ . \end{aligned} \tag{3.6}$$

Note for reference that the central element Z acts by

$$\rho_L(Z)\,\mathbf{e}_\delta = \mathrm{e}((b^- - b^+)/4)\,\mathbf{e}_{-\delta} \ . \tag{3.7}$$

A formula giving the action of any element of the metaplectic group has been found by Shintani (see [49]), but we will not need it.

3.3 Vector-valued modular forms associated to a lattice

Throughout this section, let L be an even lattice and $k \in \frac{1}{2}\mathbb{Z}$. Following the conventions of [8], we introduce a class of modular forms associated to L of weight k with respect to the metaplectic group, taking values in the vector space $\mathbb{C}[L^\vee/L]$. First of all, we should specify how the metaplectic group acts on vector-valued functions on the upper half-plane.

Definition 3.3.1. Let $f : \mathbb{H} \to \mathbb{C}[L^\vee/L]$, and $(A, \varepsilon) \in \mathrm{Mp}_2(\mathbb{Z})$. The slash operator $|_k^*$ is defined by

$$f\,|_k^*(A,\varepsilon)(z) = \varepsilon(z)^{-2k}\rho_L^\vee(A,\varepsilon)^{-1} f(Az) \ . \tag{3.8}$$

Here, $\rho_L^\vee$ is the dual of the Weil representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L^\vee/L]$, defined in section 3.2.

Example 3.3.2. Suppose that $L = \mathbb{Z}$, the unimodular positive definite rank 1 lattice. Then $L^\vee/L$ is the trivial group, so the associated slash operator gives an action of $\mathrm{Mp}_2(\mathbb{Z})$ on the space of complex functions on the upper half-plane.

Suppose further that k is integer. Then the factor $\varepsilon(z)^{-2k}$ equals $(cz+d)^{-k}$ (if we write $A = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$), and the generators $\tilde{T}, \tilde{S}$ act by

$$\begin{aligned} f\,|_k^*\tilde{T} &= f(Tz) \ , \\ f\,|_k^*\tilde{S} &= \mathrm{e}(-1/8) z^{-k} f(Sz) \ ; \end{aligned} \tag{3.9}$$

this is the usual slash operator on (scalar) functions on the upper half-plane, except for the factor $\mathrm{e}(-1/8) = e^{-\pi i/4}$.

Definition 3.3.3. A holomorphic function $f : \mathbb{H} \to \mathbb{C}[L^\vee/L]$ is called a modular form of weight k on the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ if

(i) for every $\tilde{A} \in \mathrm{Mp}_2(\mathbb{Z})$, we have $f\,|_k^*\tilde{A} = f$; and

(ii) f is holomorphic at $\mathrm{i}\infty$.

The space of all modular forms of weight k is denoted by $M(k, L)$.

Remark 3.3.4. We will sometimes drop condition ii, and allow f to have a pole at the cusp. We will call this "a vector-valued modular form with possible pole at the cusp", although it is not a modular form in the above sense. Some authors refer to these objects as weak modular forms.

Suppose that $\varphi \in M(k, L)$ is a vector-valued modular form. We may write such a form as a Fourier expansion

$$\varphi(\tau) = \sum_{\gamma,n} a_{\gamma,n} q^n \mathbf{e}_\gamma \ , \tag{3.10}$$

where we write $q = \mathrm{e}(\tau) = e^{2\pi i\tau}$.

Definition 3.3.5. Given (γ, n) (such that $n \in -\gamma^2/2 + \mathbb{Z}$ and $n \geq 0$), let $c_{\gamma,n} : M(k, L) \to \mathbb{C}$ be the function taking a form φ to its (γ, n)-coefficient $a_{\gamma,n}$.

The modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ on which these vector-valued modular forms live has one cusp. However, we may want to distinguish the behaviour of the different components of a vector-valued form at the cusp. This leads to the following useful abuse of language:

Definition 3.3.6. Let L be an even lattice. The cusps of L are the isotropic elements of the discriminant group of L (i.e., the elements $\gamma \in D_L = L^\vee/L$ such that $\gamma^2/2 = 0 \in \mathbb{Q}/\mathbb{Z}$).

Using this terminology, we might say that a given vector-valued modular form φ vanishes at the cusp $\gamma \in D_L$ if the $\mathbf{e}_\gamma$-component of φ vanishes at the single cusp of the modular curve.

Remark 3.3.7. We will see later on (section 4.3) that if the lattice has signature $(2, n)$ there is a tight connection between the vector-valued modular forms associated to L and modular forms on the arithmetic quotient associated to L (see section 4.1). The cusps of L in the above sense correspond with the 0-dimensional cusps of that arithmetic quotient (see section 5.1.1), giving an alternative interpretation of definition 3.3.6.

Definition 3.3.8. We say that a modular form $\varphi \in M(k, L)$ is a cusp form if its coefficients satisfy $c_{\gamma,0}(\varphi) = 0$ for all cusps γ of L.

As a non-standard extension of this terminology, we say that φ is an almost cusp form if $c_{\gamma,0}(\varphi) = 0$ for all isotropic $\gamma \neq \bar{0}$. (In other words, an almost cusp form vanishes at all cusps except perhaps the standard cusp $\gamma = \bar{0} \in D_L$.)

Denote the space of cusp forms by $S(k, L)$, and the space of almost cusp forms by $AC(k, L)$.

Note that we have $S(k, L) \subsetneq AC(k, L) \subseteq M(k, L)$. If $k > 2$, the first inclusion is strict, and the last inclusion is an equality if and only if L has only one cusp: this follows from the existence of an Eisenstein series E_γ at every cusp γ of L (see [11] and section 3.4).

Note that L has only one cusp if and only if the quadratic module D_L has $\bar{0}$ as its only isotropic element (for details on this in the case of the lattice $L = L_{2d}$, which figures in the theory of K3 surfaces, see section 5.1.1).

3.3.1 Serre duality

Later on, we will need a result saying that the obstruction to constructing vector-valued modular forms with given singularities at the cusps of L is itself given by a space of holomorphic vector-valued modular forms (of different weight). This is fully explained in [7, section 3]; we give a short account here.

We introduce a few tools to describe the pole behaviour of vector-valued modular forms.

Definition 3.3.9. $\mathrm{Sing}(L)$ is the space of Laurent polynomials (in q) with values in $\mathbb{C}[D_L]$ having only non-positive powers of q.

Similarly, $\mathrm{Sing}(L)^-$ is the subspace of $\mathrm{Sing}(L)$ of elements having only negative powers of q, and $\mathrm{Sing}(L)^-_{\bar{0}}$ is the subspace of $\mathrm{Sing}(L)$ of elements having only negative powers of q except possibly a term $q^0\mathbf{e}_{\bar{0}}$.

We write $\mathrm{Obstruct}(k, L)$ for the obstruction space to the existence of vector-valued modular forms of weight k with given principal part. More formally, $\mathrm{Obstruct}(k, L)$ is the quotient of $\mathrm{Sing}(L)$ by the image of the map taking a vector-valued modular form with poles to its principal part.

By an application of Serre duality, the space $M(k,L)$ is dual to the obstruction space $\mathrm{Obstruct}(2-k,L)$ of obstructions to a given element of $\mathrm{Sing}(L)$ being the principal part of a meromorphic vector-valued modular form of weight $2-k$ and representation ρ_L. The duality is realised by the residue map

$$\begin{aligned} M(k,L) \times \mathrm{Sing}(L) &\to \mathbb{C} \\ (\varphi, f) &\mapsto \mathrm{Res}(\frac{\varphi f}{q^{1/N}}\, \mathrm{d}q^{1/N}) \ . \end{aligned} \tag{3.11}$$

Proposition 3.3.10 ([7]). *The above correspondence identifies the coefficient function* $c_{\gamma,n} \in M(k,L)^\vee$ *with* $[q^{-n}\mathbf{e}_\gamma] \in \mathrm{Obstruct}(2-k,L)$. *A linear combination*

$$\sum_{\gamma,n} a_{\gamma,n} c_{\gamma,n} \in M(k,L)^\vee \tag{3.12}$$

is zero on $M(k,L)$ *if and only if the corresponding obstruction*

$$[\sum_{\gamma,n>0} a_{\gamma,n} q^{-n}\mathbf{e}_\gamma] \in \mathrm{Obstruct}(2-k,L) \tag{3.13}$$

vanishes, i.e., if and only if this equals the principal part of some meromorphic vector-valued form of weight $2-k$ *and representation* ρ_L.

Remark 3.3.11. The analogous statement holds for the space $S(k,L)$ of cusp forms, if we restrict the principal parts to $\mathrm{Sing}^-(L)$, and likewise for the space $AC(k,L)$ of almost cusp forms, if we restrict the principal parts to $\mathrm{Sing}^-_0(L)$.

3.4 Eisenstein series

If the weight k is greater than 2, the space $M(k,L)$ of vector-valued modular forms contains some special forms called Eisenstein series. For a given weight k, there is one such a form for every cusp.

Definition 3.4.1. Let γ be a cusp of L, i.e., $\gamma \in L^\vee/L$ is such that $\gamma^2/2 = 0 \in \mathbb{Q}/\mathbb{Z}$; also, let $k > 2$ be a proper half-integer (i.e., $k \in \frac{1}{2}\mathbb{Z}$, and $k \notin \mathbb{Z}$). Then the Eisenstein series of weight k associated to γ is

$$E_\gamma = \sum_{\sigma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \mathbf{e}_\gamma |^*_k \sigma \ . \tag{3.14}$$

(Because $k > 2$, the sum is normally convergent.) We write $E = E_{\bar{0}}$ for the Eisenstein series associated to the standard cusp; if we refer to an Eisenstein series without mentioning a specific cusp, it shall be this one.

We will only be interested in the special case $L = L_{2d} = \langle -2d\rangle \oplus 2U \oplus 2E_8(-1)$, the lattice associated to the moduli space of polarised K3 surfaces of degree $2d$. From now on, let L denote that lattice (for some value of d).

The Eisenstein series $E_{\bar{0}}$ also has an interpretation involving the arithmetic quotient space $\mathcal{F}_{2d}$ associated to L (see section 4.1) and its so-called Heegner divisors (see section 4.2):

Proposition 3.4.2 ([27]). *Let* $H(\gamma,n)$ *be a Heegner divisor on* $\mathcal{F}_{2d}$. *The degree of* $H(\gamma,n)$, *i.e., the intersection number* $H(\gamma,n)\cdot\lambda^{18}$, *is equal to a negative constant (independent of* γ *and* n*) times* $c_{\gamma,n}(E_{\bar{0}})$.

Bruinier and Kuss [11] have found a formula for the Fourier coefficients of the Eisenstein series $E_{\bar{0}}$ which is explicit enough to allow a computer implementation.

Theorem 3.4.3 ([11]). *Let* $\gamma \in L^\vee$ *and* $n \in \mathbb{Z} - \gamma^2/2$, *with* $n > 0$. *The coefficient* $c_{\gamma,n}(E_{\bar{0}})$ *of the Eisenstein series* $E_{\bar{0}}$ *is given by*

$$\frac{-(2\pi)^{21/2} n^{19/2}}{\sqrt{|2d|}\Gamma(21/2)} \frac{L(10,\chi_{\mathcal{D}})}{\zeta(20)} \sum_{c|f} \mu(c)\chi_{\mathcal{D}}(c) c^{-10} \sigma_{-19}(f/c) \prod_{p|2d} \frac{p^{-20w_p} N_{\gamma,n}(p^{w_p})}{1-p^{-20}} \ , \tag{3.15}$$

where Γ *is the gamma function;* $L(\cdot,\chi_{\mathcal{D}})$ *is the Dirichlet L-series associated to the character* $\chi_{\mathcal{D}}$; $\sigma_i(x) = \sum_{b|x} b^i$ *is a divisor sum function; the product is taken over prime divisors* p *of* $2d$; *the* *number w_p is given by $w_p = 1 + 2v_p(2nd_\gamma)$ (where v_p denotes the additive valuation at p, and d_γ is the level of γ, i.e., the order of γ in the abelian group $L^\vee/L$); further, $N_{\gamma,n}(a)$ is a count of lattice congruence classes:*

$$N_{\gamma,n}(a) = \#\left\{r \in L/aL : (r-\gamma)^2/2 + n \equiv 0 \mod a\right\} \ ; \tag{3.16}$$

finally, the numbers $\mathcal{D}$ and f are given by the following procedure: write $n = n_0 f^2$, where $n_0 \in \mathbb{Q}, f \in \mathbb{N}$, and $(f, 2d) = 1$, and $v_l(n_0) \in \{0, 1\}$ for all prime l such that $(l, 2d) = 1$; then set $\mathcal{D} = 4dn_0 d_\gamma^2$.

Note that this is Theorem 4.8 from [11], specialised to our situation: $k = 21/2$ is the weight of the Eisenstein series we want to compute, $b^+ = 2$, $|L^\vee/L| = 2d$.

This formula can be implemented on a computer in a straightforward way. The special value of the L-series at the positive integer 10 can be calculated exactly. The only part that takes a significant amount of time to compute when done in a naive way is the congruence count $N_{\gamma,n}(a)$: counting this set of vectors would enumerate the set L/aL, which has $a^{\operatorname{rank} L} = a^{21}$ elements. So, if a is only slightly large, say $a = 32 = 2^5$, then this set already has $32^{21} \approx 4 * 10^{31}$ elements.

There are (at least) two ways around this problem.

In his thesis [3], Barnard rewrites these congruence counts in terms of Gauss sums. Unfortunately these cannot be computed directly, as this would involve a sum over a set of size similar to that of L/aL. Barnard presents an alternative way to compute these Gauss sums, using local analysis at each prime. However, he does not deal with the case of an odd 2-adic Jordan component in the lattice L. Our lattice $L = L_{2d}$ does have an odd 2-adic Jordan component, so Barnard's approach does not apply directly to our case, although it could possibly be extended to do so.

The approach that we will use rests on the observation that the Eisenstein series depends only on the discriminant group $L^\vee/L$ (together with its quadratic form $\gamma \mapsto \gamma^2/2 \in \mathbb{Q}/\mathbb{Z}$) and of course on the choice of weight k. Recall that our lattice $L = L_{2d} = \langle -2d\rangle \oplus 2U \oplus 2E_8(-1)$ has a big unimodular part $2U \oplus 2E_8(-1)$; its discriminant module is just the discriminant module of the lattice $\langle -2d\rangle$. So when computing the Eisenstein series associated to the rank 21 lattice L_{2d}, we might as well use the rank 1 lattice $\langle -2d\rangle$ instead. In doing so, we reduce the sum over L/aL from having size a^{21} to just a. The slight downside is that we can no longer use the easier formula of [11, Theorem 4.8], because that applies only to the special case where the weight k equals rank$(L)/2$. Instead, we may use the more general formula:

Theorem 3.4.4 ([11, Theorem 4.6])**.** *Let $\gamma \in D_{L_{2d}} \cong \mathbb{Z}/2d\mathbb{Z}$ and $n \in \mathbb{Z} - \gamma^2/2$, with $n > 0$. The coefficient $c_{\gamma,n}(E_{\bar{0}})$ of the Eisenstein series $E_{\bar{0}}$ is given by*

$$\frac{-2^{23/2}\pi^{21/2}n^{19/2}}{\sqrt{|2d|}\Gamma(21/2)} \frac{L(10,\chi_{\mathcal{D}})}{\zeta(20)} \prod_{p \mid 4dnd_\gamma^2} \frac{1-\chi_{\mathcal{D}}(p)p^{-10}}{1-p^{-20}} L_{\gamma,n}(p) \ , \tag{3.17}$$

where $\Gamma, L(\cdot,\chi_{\mathcal{D}}), \mathcal{D}, \zeta, d_\gamma$ all are as in theorem 3.4.3, and the local factor $L_{\gamma,n}(p)$ is given by

$$L_{\gamma,n}(p) = (1-p^{-10}) \sum_{\nu=0}^{w_p-1} N_{\gamma,n}(p^\nu)p^{-10\nu} + N_{\gamma,n}(p^{w_p})p^{-10w_p} \ , \tag{3.18}$$

where the congruence count $N_{\gamma,n}(a)$ is the same as before, but now for the lattice $L = \langle -2d\rangle$.

Note that the congruence count $N_{\gamma,n}(a)$ counts a subset of L/aL, which now has a elements (because $L = \langle -2d\rangle$ has rank 1). This means that a naive counting implementation will suffice for many values of γ and n: only if the number p^{w_p} is very large, for some prime divisors p of $4dnd_\gamma^2$, will this method take a significant amount of time.

Later, in section 3.6, we will see how to compute a basis of the space of modular forms M of which this Eisenstein series E is a member. Once we have such a basis, we need to compute only a finite number of coefficients of E; comparing with the coefficients of the basis forms, we may identify the modular form E. We can then compute all its other coefficients by linear algebra. Hence, for our purposes, we may avoid the computation of the coefficient $E(\gamma, n)$ for any particular value of γ and n, if we find it takes too much time.

We have implemented the formula given by theorem 3.4.4 using Sage [50]; see listing A.1.

Example 3.4.5. We give here some coefficients, for the case $d = 1$.

$$\begin{aligned}174611 \cdot E_{\bar{0}} = \mathbf{e}_{\bar{0}}&\left(349222 - 52377700q^1 - 37924141200q^2 - 1785624218400q^3 + O(q^4)\right)\\ + \mathbf{e}_{\bar{1}}&\left(-100q^{1/4} - 435865056q^{5/4} - 116224178500q^{9/4} - 3816045314400q^{13/4} + O(q^4)\right) .\end{aligned} \tag{3.19}$$

3.5 Jacobi forms

Modular forms are functions of one variable τ. Jacobi forms are a certain generalisation to functions of two variables τ and z.

Just as a modular form has a weight, which describes its behaviour under transformations of the domain, Jacobi forms have a weight and an index (an integer number), describing how it transforms under changes in the two variables.

Classically, the second variable z was a complex number. A natural further generalisation arises when we replace this number by a complex vector $z \in \mathbb{C}^N$. The index is then no longer an integer, but a lattice of rank N. We are interested in these lattice index Jacobi forms, because they are strongly related to vector-valued modular forms: see theorem 3.5.4 below.

A general reference on classical Jacobi forms is [15]. The generalisation to lattice index Jacobi forms is hard to attribute; we follow the notation of [42], although we use a rank N lattice L where Raum uses $\mathbb{Z}^N$ with a matrix L (representing the intersection form of our lattice L).

Definition 3.5.1. Let L be an even lattice. The Jacobi upper-half plane is the set

$$\mathbb{H}_L = \mathbb{H} \times L_{\mathbb{C}} \ , \tag{3.20}$$

where $\mathbb{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ is the usual upper-half plane, and $L_{\mathbb{C}} = L \otimes \mathbb{C}$, which is just $\mathbb{C}^N$ (if L has rank N), with inner product induced by that of L.

The Jacobi group is the semi-direct product

$$\Gamma^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \otimes L) \ , \tag{3.21}$$

with $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{Z}^2 \otimes L$ by the natural extension of the action on $\mathbb{Z}^2$. The Jacobi group acts on the Jacobi upper-half plane, as follows: if $\gamma^J = (\gamma, (\lambda, \mu)) \in \Gamma^J$ (so $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and $\lambda, \mu \in L$), and $(\tau, z) \in \mathbb{H}_L$, then

$$\gamma^J \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right) \ . \tag{3.22}$$

Given an integer number k, we define a slash operator, which is an action of the Jacobi group Γ^J on functions $\varphi : \mathbb{H}_L \to \mathbb{C}$:

$$(\varphi|_{k,L}\gamma^J)(\tau, z) = (c\tau + d)^{-k} \mathrm{e}\left(-c\frac{(z + \lambda\tau + \mu)^2}{2(c\tau + d)} + \tau\lambda^2/2 + (\lambda, z)\right) \varphi(\gamma^J \cdot (\tau, z)) \ , \tag{3.23}$$

where $\gamma^J = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)) \in \Gamma^J$. (Note that the inner product on $L_{\mathbb{C}}$ induced by that of L is used implicitly.)

Definition 3.5.2. Let L be an even lattice. A Jacobi form of weight k and index L is a holomorphic function $\varphi : \mathbb{H}_L \to \mathbb{C}$ such that

(i) $\varphi|_{k,L}\gamma^J = \varphi$ for all $\gamma^J \in \Gamma^J$, and

(ii) for every $\alpha, \beta \in L_{\mathbb{Q}}$, the function $\tau \mapsto \varphi(\tau, \alpha\tau + \beta)$ is bounded.

The space of Jacobi forms of weight k and index L is denoted by $J(k, L)$.

If L has rank 1, say generated by a vector of length $2m$, then we recover the definition of a Jacobi form in the classical sense (i.e., of scalar index) of weight k and index m.
Jacobi forms have a Fourier expansion, similar to usual modular forms: if φ is a Jacobi form of index L, then we may write

$$\varphi(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{\substack{r \in L^\vee \\ r^2/2 \leq n}} c(n, r)\, q^n \zeta^r \ , \tag{3.24}$$

where $c(n, r) \in \mathbb{C}$ are the Fourier coefficients of φ, and $q = \mathrm{e}(\tau)$, and $\zeta^r = \mathrm{e}((z, r))$ (using the natural pairing of $L_{\mathbb{C}}$ with $L^\vee$).

Some particularly useful relations among Fourier coefficients of Jacobi forms are the following:

Lemma 3.5.3. *Let φ be a Jacobi form of weight k and index L, with Fourier coefficients $c(n, r)$. If $\lambda \in L$, then*

$$c(n, r + \lambda) = c(n - \lambda^2/2 - (\lambda, r), r) \ . \tag{3.25}$$

Also, we have $c(n, -r) = (-1)^k c(n, r)$.

Proof. Use the invariance of φ under the slash operator applied to the Jacobi group element $((\begin{smallmatrix}1&0\\0&1\end{smallmatrix}),(\lambda,0))$, respectively $((\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}),(0,0))$. □

These relations imply that a Jacobi form is uniquely determined by the coefficients $c(n,r)$ for r in a set of representatives of $(L^\vee/L)/\pm$.

Theorem 3.5.4 ([53, p. 210])**.** *Let k be an integer and L a positive definite even lattice of rank N. There is an isomorphism $\Theta_L : J(k,L) \to M(k-N/2,L)$ between the space of Jacobi forms of weight k and index L, and the space of vector-valued modular forms of weight $k-N/2$ associated to the lattice L.*

Explicitly, if

$$f = \sum_{\gamma\in D_L} f_\gamma \mathbf{e}_\gamma \in M(k-N/2,L) \tag{3.26}$$

is such a vector-valued modular form, it corresponds under this isomorphism to

$$\Theta_L^{-1}(f)(\tau,z) = \sum_{r\in L^\vee} q^{r^2/2}\zeta^r f_{r+L}(\tau)\ . \tag{3.27}$$

This theorem will form the basis of our computations of spaces of vector-valued modular forms. We may compute spaces of Jacobi forms of lattice index by relating them to spaces of Jacobi forms of scalar index, which are well-known.

Definition 3.5.5 ([42])**.** Let φ be a Jacobi form of weight k and index L, and let $s \in L$. Then the restriction of φ along s is the function $\varphi[s]$ given by

$$\varphi[s](\tau,z) = \varphi(\tau,zs)\ . \tag{3.28}$$

Proposition 3.5.6 ([42, Lemma 3.6])**.** *If φ is a Jacobi form of weight k and index L, and $s \in L$, then the restriction $\varphi[s]$ is a Jacobi form of weight k and scalar index $s^2/2$.*

3.6 Computation of spaces of vector-valued modular forms

We want to compute spaces of vector-valued modular forms explicitly, in terms of Fourier expansions of a set that forms a basis.

In [42], Raum describes an algorithm to compute these expansions. We follow his strategy.

The first step is to relate the wanted vector-valued modular forms to Jacobi forms (of lattice index). This requires us to change the lattice to be positive definite, without changing the discriminant group. We explain this in section 3.6.1.

Next, the main tool to compute these Jacobi forms of lattice index is proposition 3.5.6: if φ is a Jacobi form of lattice index L, and $s \in L$ is an integral vector, then $\varphi[s]$ – the restriction (in its second variable) of the form φ to the span of s – is again a Jacobi form, but now of scalar index. Spaces of Jacobi forms of scalar index are well-known, and Fourier coefficients of such forms can be computed using existing algorithms. Therefore, if we can describe explicitly how the Fourier coefficients of $\varphi[s]$ are determined by those of φ, we get relations that the Fourier coefficients of φ must satisfy. We explain how to do this in section 3.6.2.

Now, the idea of the algorithm is to take several of these "restriction vectors" s, and for each of them compute the induced relations among coefficients of Jacobi forms. If we take enough restriction vectors, the induced relations may completely determine the space of Jacobi forms. We will know when we reach that point, because there is a formula for the dimension of the space of Jacobi forms. Moreover, Raum proves that there exists a finite set of restriction vectors that suffices, and gives an algorithm to compute such a set (although this algorithm is not very efficient, and in practice, it seems easier to generate restriction vectors at random).

3.6.1 Change to positive definite lattice

We would like to apply the isomorphism $\Theta_L : J(k,L) \to M(k-N/2,L)$ of theorem 3.5.4 to relate our space of vector-valued modular forms to a space of Jacobi forms, which we may compute in turn by Raum's restriction method. However, theorem 3.5.4 only works if the lattice L is positive definite, which our lattice $L_{2d} = \langle -2d\rangle \oplus 2E_8(-1) \oplus 2U$ is not.

Luckily, the space of vector-valued modular forms associated to a lattice L in fact only depends on the Weil representation ρ_L associated to L, and in turn this only sees the discriminant group D_L of the lattice. So we need to find a positive definite lattice N such that $D_N \cong D_{L_{2d}}$.

This is not so hard:

Proposition 3.6.1. *Let d be a positive integer. Pick any vector $v \in E_8$ such that $v^2 = 2d$, and define $N = v^\perp$, the orthogonal complement of v in E_8. Then N has the same discriminant module as L_{2d}.*

Proof. By the lemma below, since E_8 is unimodular, the discriminant module D_N of $N = v^\perp$ is minus that of $\langle v\rangle \cong \langle 2d\rangle$, so D_N is isomorphic to the discriminant module of $\langle -2d\rangle$. This in turn equals the discriminant module of $L_{2d} = \langle -2d\rangle \oplus 2U \oplus 2E_8(-1)$, because U and $E_8(-1)$ are unimodular. □

Lemma 3.6.2 ([37, proposition 1.6.1])**.** *Suppose that L is an even unimodular lattice, and that A, B are sublattices of L such that $A^\perp = B$ and $B^\perp = A$. Then the discriminant groups D_A and D_B satisfy $D_A \cong D_B(-1)$, where $D_B(-1)$ is just D_B with the sign of the quadratic form reversed. Explicitly: there is an isomorphism of groups $\alpha : D_A \to D_B$ such that $\alpha(t)^2/2 = -t^2/2 \in \mathbb{Q}/\mathbb{Z}$ for all $t \in D_A$.*

Proof. Let $t \in D_A$, say $t = a/m + A$, for some $a \in A$ and $m \in \mathbb{Z}_+$. We claim that there exists some $u \in mL$, such that $a - u \in B$.

As a general remark, note that the orthogonal complement of a lattice is by construction primitive; in particular A and B are primitive in L.

Proof of claim: since $A = B^\perp$, we know that A is primitive in L, so, by lemma 2.2.4, the map $L^\vee \to A^\vee$ is surjective. Because L is unimodular by assumption, $L^\vee = L$, so in fact $L \to A^\vee$ is surjective. Therefore, there is an $v \in L$ such that $(v, x) = (a/m, x)$ for all $x \in A$. Now, take $u = mv \in mL$; we then know that $(u, x) = (a, x)$ for all $x \in A$, so $(a - u, x) = 0$ for all $x \in A$. This means that $a - u \in A^\perp = B$, as required.

We now define the map α by declaring $\alpha(t) = (a - u)/m + B \in D_B$. We need to verify that this is well defined (independent of the choice of u), and that it gives an isomorphism of abelian groups, and finally that $\alpha(t)^2 = -t^2$.

First, we show that α is well defined. If $u' \in mL$ is another element such that $a - u' \in B$, then $(a - u') - (a - u) \in B$, but also $(a - u') - (a - u) = u - u' \in mL$. Because B is primitive in L, we have in fact that $(a - u') - (a - u) \in mB$, so $(a - u')/m + B = (a - u)/m + B \in D_B$.

Next, note that α is additive: if $t_i = a_i'/m_i \in D_A$, we can first take multiples to get a common denominator: $t_i = a_i/m$. Then, if $u_i \in mL$ is such that $a_i - u_i \in B$, then $(a_1 + a_2) - (u_1 + u_2) = a_1 - u_1 + a_2 - u_2 \in B$, and $u_1 + u_2 \in mL$, so $\alpha(t_1 + t_2) = \alpha((a_1 + a_2)/m) = ((a_1 + a_2) - (u_1 + u_2))/m + B = (a_1 - u_1)/m + (a_2 - u_2)/m + B = \alpha(t_1) + \alpha(t_2)$.

Furthermore, we may apply this construction also after switching A and B; this gives a map $\beta : D_B \to D_A$, such that if $s = b/m + B$, and $\overline{u} \in mL$ with $b - \overline{u} \in A$, then $\beta(s) = (b - \overline{u})/m$. Now, take $t \in D_A$, say $t = a/m$ with $a \in A$ and $m \in \mathbb{Z}_+$. Pick a $u \in mL$ such that $a - u \in B$, and write $s = \alpha(t) = (a - u)/m + B$. Note that $\overline{u} = -u$ has the property that $(a - u) - \overline{u} \in A$, so $\beta(s) = ((a - u) - \overline{u})/m + A = a/m + A = t$. This proves that $\beta \circ \alpha = \mathrm{id}_{D_A}$. Symmetrically, we have $\alpha \circ \beta = \mathrm{id}_{D_B}$, so α is an isomorphism of abelian groups.

Finally, let $t = a/m + A \in D_A$; we need to show that $\alpha(t)^2/2 = -t^2/2 \in \mathbb{Q}/\mathbb{Z}$. On the one hand, we have $t^2/2 = (a/m, a/m)/2 = (a, a)/2m^2$; on the other, we have

$$\begin{aligned} \alpha(t)^2/2 = ((a - u)/m)^2/2 &= (a - u, a - u)/2m^2 = (a, a)/2m^2 - 2(a, u)/2m^2 + (u, u)/2m^2 \\ &= -(a, a)/2m^2 \ , \end{aligned} \tag{3.29}$$

where in the final equation, we used the fact that $(a, u) = (a, a)$ (because $a - u \in B = A^\perp$), and that $(u, u)/2m^2 \in \mathbb{Z}$ (because $u \in mL$, and L is even). This proves that $t^2/2 = (a, a)/2m^2 = -\alpha(t)^2/2$. □

3.6.2 Restriction matrix

To compute Fourier coefficients of Jacobi forms using the restriction method, we need to describe explicitly how the restriction acts on the level of Fourier expansions.

Lemma 3.6.3. *Let φ be a Jacobi form of index L, and $s \in L$; write $c(n, r)$ for the Fourier coefficients of φ. Then the Fourier coefficients of $\varphi[s]$ are given by*

$$d(n, \tilde{r}) = \sum_{\substack{r \in L^\vee \\ (s, r) = \tilde{r} \\ r^2/2 \le n}} c(n, r) \ . \tag{3.30}$$

Proof. We simply insert the argument zs in the expansion of φ, and collect terms:

$$\begin{aligned} \varphi[s](\tau,z) &= \varphi(\tau,zs) \\ &= \sum_{n\in\mathbb{Z}} \sum_{\substack{r\in L^\vee \\ r^2/2\le n}} c(n,r)\, q^n\, \mathrm{e}((zs,r)) \\ &= \sum_{n\in\mathbb{Z}} \sum_{\substack{r\in L^\vee \\ r^2/2\le n}} c(n,r)\, q^n \zeta_1^{(s,r)} \\ &= \sum_{n\in\mathbb{Z}} \sum_{\tilde r\in\mathbb{Z}} \Big(\sum_{\substack{r\in L^\vee \\ (s,r)=\tilde r \\ r^2/2\le n}} c(n,r) \Big) q^n \zeta_1^{\tilde r} \ , \end{aligned} \tag{3.31}$$

where $\zeta_1 = \mathrm{e}(z)$. □

We want to compute the possible coefficients $c(n,r)$ of a lattice index Jacobi form, given the coefficients $d(n,\tilde r)$ of its restriction, using the above lemma. We can decrease the number of unknowns in equation (3.30), by employing the symmetries among the $c(n,r)$, as described by lemma 3.5.3.

Let us choose representatives r_i of the quotient $(L^\vee/L)/\pm$, and denote by ρ the map sending $r\in L^\vee$ to the representative of the class of r. Also, write

$$\nu(n,r) = n + \lambda^2/2 - (r,\lambda) \ , \tag{3.32}$$

where $\lambda\in L$ and $\varepsilon(r)\in\{1,-1\}$ are chosen in such a way that $\varepsilon(r)\rho(r) = r+\lambda$.

Then we have

$$d(n,\tilde r) = \sum_{\substack{r\in L^\vee \\ (s,r)=\tilde r \\ r^2/2\le n}} \varepsilon(r)\, c(\nu(n,r),\rho(r)) \ . \tag{3.33}$$

3.6.3 Examples

Our interest in vector-valued modular forms is motivated by the moduli space of polarised K3 surfaces (of degree $2d$, say), which is intimately related to the lattice $L_{2d} = \langle -2d\rangle \oplus 2U \oplus 2E_8(-1)$. In particular, we are interested in the space $M(d)$ of vector-valued modular forms associated to L_{2d}, of weight $21/2$. We may use the method outlined above to compute the Fourier expansion of a basis of this space $M(d)$, for any d.

Example 3.6.4. As a first example, take $d=1$. By theorem 4.2.10, we know $M(1)$ has dimension 2. Applying the restriction method, we get the following basis:

$$\begin{aligned} v_1 &= \frac{1}{169227}\Big[\mathbf{e}_{\bar 0}\cdot(169227 - 105457575250q - 25773956188200q^2 - 334332250520400q^3 + O(q^4)) \\ &\quad + \mathbf{e}_{\bar 1}\cdot\big(1882717700q^{1/4} + 677567176704q^{5/4} + 58249565705700q^{9/4} + O(q^3)\big)\Big] \ , \text{ and} \\ v_2 &= \frac{1}{56}\Big[\mathbf{e}_{\bar 0}\cdot(56q + 13680q^2 + 177120q^3 + O(q^4)) \\ &\quad + \mathbf{e}_{\bar 1}\cdot\big(-1q^{1/4} - 360q^{5/4} - 30969q^{9/4} + O(q^3)\big)\Big] \ . \end{aligned} \tag{3.34}$$

The subspace $S(1)$ of cusp forms is exactly the span of v_2. As there is only one cusp, the almost-cusp forms are just $AC(1) = M(1)$.

Example 3.6.5. As a second example, take $d=4$. Again by theorem 4.2.10, we know that

$\dim M(4) = 4$. Our implementation of the restriction method gives the following basis of $M(4)$:

$$
\begin{aligned}
v_1 = \mathbf{e}_{\bar 0} \cdot \Big(1 + &\frac{465139258835037405731374758478184425}{8674173235084289694627577456 8}q + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 1} + \mathbf{e}_{\bar 7}) \cdot \Big(&\frac{53516147782292641753445471897 5}{4337086617542144847313788728 4} q^{1/16} \\
&- \frac{370264222244751374080234548162000}{576433628062486024363874 1} q^{17/16} + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 2} + \mathbf{e}_{\bar 6}) \cdot \Big(&\frac{35103101398124208395824895824872 5}{2891391078361429898209192485 6} q^{1/4} \\
&+ \frac{1595098331580044666890691061071 36}{365074631106241148763786 93} q^{5/4} + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 3} + \mathbf{e}_{\bar 5}) \cdot \Big(&-\frac{606766741556901332116610451353675}{1606328376867461054560662492} q^{9/16} \\
&+ \frac{2296362523218756038043262949822170375}{43370866175421448473137887284} q^{25/16} + O(q^2)\Big) \\
+ \mathbf{e}_{\bar 4} \cdot \Big(&\frac{7597895932563135563414377343}{6425313507469844218242649968} - \frac{66262963242255774173341393628 41600}{985701503986851101662224711} q + O(q^2)\Big) \;; \\
v_2 = \mathbf{e}_{\bar 0} \cdot &(0 + q + O(q^2)) \\
+ (\mathbf{e}_{\bar 1} + \mathbf{e}_{\bar 7}) \cdot \Big(&\frac{112192763252842573 39}{124545224498724398002058 72} q^{1/16} \\
&- \frac{871258180695087441587760}{778407653117027487512867} q^{17/16} + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 2} + \mathbf{e}_{\bar 6}) \cdot \Big(&\frac{739772320056066866325}{778407653117027487512867} q^{1/4} + \frac{2662350506951504740208 64}{778407653117027487512867} q^{5/4} + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 3} + \mathbf{e}_{\bar 5}) \cdot \Big(&-\frac{819922747135906324152045}{1245452244987243980020587 2} q^{9/16} \\
&+ \frac{1148578501283318215988410 27}{1245452244987243980020587 2} q^{25/16} + O(q^2)\Big) \\
+ \mathbf{e}_{\bar 4} \cdot \Big(&\frac{265976041774352103}{1556815306234054975025734} - \frac{861282101166440052948992}{778407653117027487512867} q + O(q^2)\Big) \;; \\
v_3 = \mathbf{e}_{\bar 0} \cdot &(0 + 0 \cdot q + O(q^2)) \\
+ (\mathbf{e}_{\bar 1} + \mathbf{e}_{\bar 7}) \cdot \Big(&q^{1/16} - \frac{227879351918019379123 2}{4264745659198093} q^{17/16} + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 2} + \mathbf{e}_{\bar 6}) \cdot \Big(&\frac{1339786854908780246048}{149266098071933255} q^{1/4} + \frac{4823215796282086524510 72}{149266098071933255} q^{5/4} + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 3} + \mathbf{e}_{\bar 5}) \cdot \Big(&-\frac{670044384396765177399}{21323728295990465} q^{9/16} + \frac{939175261632704256004 73}{21323728295990465} q^{25/16} + O(q^2)\Big) \\
+ \mathbf{e}_{\bar 4} \cdot \Big(&\frac{386478697856256}{21323728295990465} - \frac{2143664765034516237516 8}{21323728295990465} q + O(q^2)\Big) \;; \\
v_4 = \mathbf{e}_{\bar 0} \cdot &(0 + 0 \cdot q + O(q^2)) \\
+ (\mathbf{e}_{\bar 1} + \mathbf{e}_{\bar 7}) \cdot \Big(&0 \cdot q^{1/16} - \frac{345397315221}{5804898746} q^{17/16} + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 2} + \mathbf{e}_{\bar 6}) \cdot \Big(&q^{1/4} + \frac{1044915751496}{2902449373} q^{5/4} + O(q^2)\Big) \\
+ (\mathbf{e}_{\bar 3} + \mathbf{e}_{\bar 5}) \cdot \Big(&\frac{-60950007503}{17414696238} q^{9/16} + \frac{1421881084638}{2902449373} q^{25/16} + O(q^2)\Big) \\
+ \mathbf{e}_{\bar 4} \cdot \Big(&-\frac{163352}{8707348119} - \frac{325066162176}{2902449373} q + O(q^2)\Big) \;.
\end{aligned}
\tag{3.35}
$$

The space of almost cusp forms $AC(4)$ has dimension 3: one less than $\dim M(4) = 4$, as $L_{2\cdot 4}$ has one non-standard cusp, represented by $\bar 4 \in \mathbb{Z}/8\mathbb{Z}$. The space of cusp forms thus has dimension 2.

Note that the dimension of the space of forms $M(d)$, the number of components of a single vector, and the size of a typical coefficient all increase as the parameter d grows. This is reflected in the time needed to compute a basis of the space $M(d)$.

3.6.4 Check with Maulik–Pandharipande

We may verify these numbers by comparing with some vector-valued modular forms that have been constructed in the past by different means. For example, Maulik and Pandharipande [30, section 6.3] used Rankin–Cohen brackets to construct forms for low values of d. In the case $d = 1$, they found the form

$$\begin{aligned}\vec{\Theta} &= \mathbf{e}_{\bar{0}} \cdot (-1 + 150q + 108600q^2 + 5113200q^3 + O(q^4)) \\ &+ \mathbf{e}_{\bar{1}} \cdot (1248q^{5/4} + 332800q^{9/4} + O(q^3)) \ .\end{aligned} \tag{3.36}$$

Comparing two coefficients of $\vec{\Theta}$, say of $q^0\mathbf{e}_{\bar{0}}$ and $q^{1/4}\mathbf{e}_{\bar{1}}$, with the corresponding coefficients of the basis vectors v_1 and v_2, given in the previous subsection, we see that we should have

$$\vec{\Theta} = -v_1 + \frac{105432191200}{169227} v_2 \ . \tag{3.37}$$

All other given coefficients of $\vec{\Theta}$ in fact agree with this equality.

3.6.5 Check with Eisenstein forms

As another verification, we may use the Eisenstein series E_γ; see section 3.4.

Example 3.6.6. Let us take the simplest case: $d = 1$. We know from example 3.4.5 that the Eisenstein series $E_{\bar{0}}$ in this case is given by

$$\begin{aligned}174611 \cdot E_{\bar{0}} = \mathbf{e}_{\bar{0}} &(349222 - 52377700q^1 - 37924141200q^2 - 1785624218400q^3 + O(q^4)) \\ + \mathbf{e}_{\bar{1}} &(-100q^{1/4} - 435865056q^{5/4} - 116224178500q^{9/4} - 3816045314400q^{13/4} + O(q^4)) \ .\end{aligned} \tag{3.19 revisited}$$

Now, this is a vector-valued modular form of weight 21/2 and representation associated to the lattice $\langle -2 \cdot 1 \rangle$, so we can write $E_{\bar{0}} = \alpha_1 v_1 + \alpha_2 v_2$ for some $\alpha_i \in \mathbb{Q}$, where v_1, v_2 are the two basis vectors of the space $M(1)$ we found in example 3.6.4. Comparing the coefficients of the term $q^0\mathbf{e}_{\bar{0}}$, we see that

$$2 = c_{\bar{0},0}(E_{\bar{0}}) = \alpha_1 c_{\bar{0},0}(v_1) + \alpha_2 c_{\bar{0},0}(v_2) = \alpha_1 \cdot 1 + \alpha_2 \cdot 0 \ , \tag{3.38}$$

so $\alpha_1 = 2$. Then comparing the $q^{1/4}\mathbf{e}_{\bar{1}}$-coefficients, we see that

$$\frac{-100}{174611} = c_{\bar{1},1/4}(E_{\bar{0}}) = \alpha_1 c_{\bar{1},1/4}(v_1) + \alpha_2 c_{\bar{1},1/4}(v_2) = 2 \cdot \frac{1882717700}{169227} + \alpha_2 \cdot \frac{-1}{56} \ , \tag{3.39}$$

and solving for α_2 gives $\alpha_2 = 36819241622917600/29548895697$. We can now verify whether this expression for $E_{\bar{0}}$ is correct by comparing some more coefficients. For instance, the coefficient of $E_{\bar{0}}$ of the term $q^1\mathbf{e}_{\bar{0}}$ is $-52377700/174611$; this indeed equals

$$\alpha_1 c_{\bar{0},1}(v_1) + \alpha_2 c_{\bar{0},1}(v_2) = 2 \cdot \frac{-105457575250}{169227} + \frac{36819241622917600}{29548895697} \cdot 1 = -\frac{52377700}{174611} \ . \tag{3.40}$$

Chapter 4

The moduli space of polarised K3 surfaces: open part

From this point on, we fix a positive integer d. We want to understand the moduli space of polarised K3 surfaces of degree $2d$ and its compactifications. For now, we will forget about the boundary, and focus on the moduli space itself; we will return to the compactified moduli space in chapter 5. In section 4.1, we explain the well-known explicit description of the moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces of degree $2d$ as a so-called arithmetic quotient.

A large part of our study of $\mathcal{F}_{2d}$ is focused on its (rational) Picard group. In section 4.2, we introduce an important class of divisors on the moduli space, called Noether–Lefschetz divisors or Heegner divisors.

An early result by O'Grady [39] showed that as the polarisation degree $2d$ increases, the rank of the Picard group $\mathrm{Pic}(\mathcal{F}_{2d})$ is unbounded. After the work of Borcherds [6], it became possible to use modular forms to study the Picard group. In particular, Bruinier [8] computed the dimension of the part of the rational Picard group generated by Noether–Lefschetz divisors, and in [5] it was shown that this part in fact equals the full rational Picard group. In sections 4.3 and 4.4, we use Borcherds' work, combined with work by Raum and others on modular forms, to give a complete and explicit description of the rational Picard group of $\mathcal{F}_{2d}$.

The strategy we follow to compute the Kodaira dimension of $\mathcal{F}_{2d}$ is the one of [18]: very roughly, we compute the canonical class K on a compactification $\overline{\mathcal{F}_{2d}}$ of $\mathcal{F}_{2d}$, and prove that it is effective, or even big. To that end, we study the effective cone of $\mathcal{F}_{2d}$ in section 4.5, and in section 4.6 we compute the restriction of the canonical class to the open part $\mathcal{F}_{2d}$ in terms of Noether–Lefschetz divisors; this amounts to the computation of the branch divisor as in [20]. Using these results, we can in fact draw some first conclusions on the Kodaira dimension of $\mathcal{F}_{2d}$ in section 4.7.

4.1 Description as a locally symmetric domain

In this section, we give a short explanation of the period map and the description of the moduli space $\mathcal{F}_{2d}$ as a locally symmetric domain. Good references for these old ideas are [4] and the recent survey [22].

A central role in this description of the moduli space is played by the lattice

$$L = L_{2d} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \langle w \rangle \ , \tag{4.1}$$

where U is the hyperbolic plane, w has square $w^2 = -2d$, and E_8 is the even unimodular positive lattice of rank 8. Throughout this chapter and the next, L will always refer to this lattice. The lattice L has signature $(2, 19)$, and discriminant module

$$D_L = L^\vee / L \cong \mathbb{Z}/2d\mathbb{Z} \tag{4.2}$$

generated by the element $w/2d$ of square $-1/2d$. (We will identify the class $\gamma = kw/2d + L \in D_L$ with the residue class $\overline{k} \in \mathbb{Z}/2d\mathbb{Z}$.)

Now, the moduli space $\mathcal{F}_{2d}$ can be described as a so-called locally symmetric space, or arithmetic quotient, associated to the lattice L. The basic idea is as follows: we try to identify a K3 surface by its second cohomology group $H^2(S, \mathbb{C})$, including the information contained in its Hodge decomposition.

The cohomology group is always the same, isomorphic to $L_{K3} \otimes \mathbb{C}$, so all information is in the Hodge decomposition. This decomposition is relatively simple (with dimensions $22 = 1 + 20 + 1$), and can be completely reconstructed from the position of the 1-dimensional subspace $H^{2,0}$; we can view that position as a point in the projective space $\mathbb{P}(L_{K3} \otimes \mathbb{C})$.

Let us make this more precise. One important point that we glossed over above is that the isomorphism $H^2(S, \mathbb{Z}) \cong L_{K3}$ is not unique; a choice of such an isomorphism is called a marking. Because we should in fact work with polarised K3 surfaces, we get the following definition.

Definition 4.1.1. Let S be a polarised K3 surface (with polarisation H of degree $2d$). A marking of S (with respect to H) is an isomorphism $\alpha : H^2(S, \mathbb{Z}) \to L_{K3}$ such that $\alpha(H) = e + df \in U \subset L_{K3}$, where e, f are the standard basis elements of U satisfying $e^2 = f^2 = 0$ and $(e, f) = 1$.

The choice of element $e + df$ is arbitrary; all that matters is that we take a fixed element of L_{K3} of norm $2d$. Note that the orthogonal complement of this element $e + df$ in L_{K3} is isomorphic to L_{2d}.

Definition 4.1.2. Let S be a polarised K3 surface with marking $\alpha : H^2(S, \mathbb{Z}) \to L_{K3}$. Then the period point of S is given by

$$\mathbb{C}\alpha_{\mathbb{C}}(\omega_S) \in \mathbb{P}(L_{K3} \otimes \mathbb{C}) \ , \tag{4.3}$$

where ω_S is any non-zero holomorphic 2-form on S.

The map from the moduli space of marked polarised K3 surfaces to $\mathbb{P}(L_{K3} \otimes \mathbb{C})$ that sends a K3 surface to its period point is called the period map.

As we indicated above, another way to write the period point is as $\alpha_{\mathbb{C}}(H^{2,0}(S))$, where $H^{2,0}(S) \subset H^2(S, \mathbb{C})$ is part of the Hodge decomposition.

Without the choice of marking, the period point is not well defined. However, different choices of marking differ only up to a lattice automorphism of L_{K3}, so the orbit of the period point under the action of $\mathrm{O}(L_{K3})$ is well defined. Because all different markings must send the polarisation class H to a fixed element of the K3 lattice, we can even reduce the action to the subgroup $\tilde{\mathrm{O}}(L_{2d}) = \tilde{\mathrm{O}}(L)$: the subgroup of $\mathrm{O}(L_{K3})$ fixing a given vector of norm $2d$ is isomorphic to the *stable* subgroup $\tilde{\mathrm{O}}(L_{2d})$ of the orthogonal complement L_{2d} of the vector; see for instance [22, example 7.6]. In this way, we get a well-defined map from the moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces to the quotient set $\tilde{\mathrm{O}}(L)\backslash\mathbb{P}(L_{K3} \otimes \mathbb{C})$.

Because the intersection form on $H^2(S, \mathbb{C})$ is the cup product, we know that $(\omega_S, \omega_S) = 0$ and $(\omega_S, \overline{\omega_S}) > 0$. We use this fact to restrict the codomain of the period map:

Definition 4.1.3. The period domain $\mathcal{D}_{2d}$ is defined by

$$\mathcal{D}_{2d} \cup \overline{\mathcal{D}_{2d}} = \{\mathbb{C}z : (z, z) = 0, (z, \bar{z}) > 0\} \subset \mathbb{P}(L \otimes \mathbb{C}) \ . \tag{4.4}$$

The set on the right-hand side of equation (4.4) consists of two connected components that are interchanged by complex conjugation; the left-hand side indicates that the period domain $\mathcal{D}_{2d}$ denotes one of these components.

Because the action of $\mathrm{O}(L)$ on the two-element set of components of the period domain is exactly given by the spinor norm, we may replace the earlier quotient $\tilde{\mathrm{O}}(L)\backslash\mathbb{P}(L_{K3} \otimes \mathbb{C})$ by $\tilde{\mathrm{O}}^+(L)\backslash\mathcal{D}_{2d}$: recall from 2.3 that $\tilde{\mathrm{O}}^+(L)$ is the subgroup of $\tilde{\mathrm{O}}(L)$ of elements of spinor norm 1, i.e., the stable lattice automorphisms that stabilise $\mathcal{D}_{2d}$.

Definition 4.1.4. The locally symmetric domain or arithmetic quotient $\mathcal{F}_{2d}$ is this quotient:

$$\mathcal{F}_{2d} = \tilde{\mathrm{O}}^+(L)\backslash\mathcal{D}_{2d} \ , \tag{4.5}$$

There are now two somewhat deeper facts that together show that the map from the moduli space of polarised K3 surfaces to $\mathcal{F}_{2d}$ is in fact an isomorphism.

Firstly, the image of the period map is the full period domain: this result is known as "surjectivity of the period map", and was first proved by Todorov. An updated proof can be found in [4, VIII.14].

Secondly, injectivity relies on a Torelli type theorem, which roughly says that the isomorphisms class of a polarised K3 surface is determined by its period point. This was first proved in [41], with later corrections by Rapoport and Shioda. The particular form that we need, including polarisations that need not be ample, is a result due to Morrison [31]. Again, see [22] for some more details.

4.2 Heegner and Noether–Lefschetz divisors

The two descriptions of $\mathcal{F}_{2d}$, as a locally symmetric variety on the one hand, and as a space parametrising geometric objects on the other, give rise to two sets of natural divisors, which turn out to coincide.

We will first introduce the divisors on the arithmetic side.

Definition 4.2.1. Given a vector $v \in L^\vee$, the Heegner divisor H_v is the subset of $\mathcal{D}_{2d} \subset \mathbb{P}(L \otimes \mathbb{C})$ orthogonal to v.

Note that this is a divisor on the symmetric domain $\mathcal{D}_{2d}$, not on the arithmetic quotient $\mathcal{F}_{2d}$. As any multiple of v will have the same orthogonal complement, we may as well assume v to be primitive in $L^\vee$.

The divisor H_v is not invariant under $\tilde{\mathrm{O}}^+(L)$. If we take the sum of a set of such divisors, associated to a set of vectors that is invariant under $\tilde{\mathrm{O}}^+(L)$, then we do get a divisor that is invariant and thus descends to a divisor on $\mathcal{F}_{2d}$.

Now, the orbits of primitive vectors of $L^\vee$ under the action of $\tilde{\mathrm{O}}^+(L)$ are classified exactly by the coset $v + L \in D_L$ and the square v^2 (see lemma 2.3.8). This motivates the following definition.

Definition 4.2.2. Let $\gamma \in D_L$ and $n \in \mathbb{Z} - \gamma^2/2$ such that $n < 0$. We denote the corresponding Heegner divisor by $H(\gamma, n)$:

$$H(\gamma, n) = \tilde{\mathrm{O}}^+(L) \Big\backslash \Big(\sum_{\substack{v \in L^\vee \\ v \equiv \gamma \mod L \\ v^2 = 2n}} H_v \Big) . \tag{4.6}$$

Remark 4.2.3. Note that $H(\gamma, n)$ has multiplicity 1 everywhere if $-\gamma \neq \gamma$, but it has multiplicity 2 everywhere if $-\gamma = \gamma$: in that latter case, the above sum runs over pairs $v, -v$ of vectors, and $H_v = H_{-v}$, so any orthogonal complement that occurs, occurs twice.

This consequence of the definition may seem strange or even undesirable from a geometric point of view. However, this choice makes the connection to modular forms very direct: see theorem 4.3.1, for example.

Apart from this, the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}(L \otimes \mathbb{C})$ descends to a line bundle λ on $\mathcal{F}_{2d}$, the class of which we will somewhat loosely refer to as the Hodge class. (Note that λ is isomorphic to $\pi_* \omega_\pi$, the pushforward of the relative dualising sheaf of the universal K3 surface $\pi : \mathcal{X}_{2d} \to \mathcal{F}_{2d}$.) The divisor (pick any) associated to the dual bundle $-\lambda$ is often considered together with the Heegner divisors, and denoted by $H(\overline{0}, 0)$. Note the minus sign: while the Heegner divisors are effective, this $H(\overline{0}, 0)$ is anti-ample. This may seem a strange choice, but it turns out to be natural.

The Heegner divisors we constructed are given in terms of the "arithmetic description" of the moduli space, i.e., its presentation as the arithmetic quotient associated to a lattice. We now define another class of divisors on the moduli space $\mathcal{F}_{2d}$, that are geometric in nature, that is, defined in terms of geometric properties of the K3 surfaces.

Recall that a very general (algebraic) K3 surface has a Picard group of rank 1. K3 surfaces with a Picard group of higher rank do exist, though, and such surfaces form special subsets of the moduli space. To get a useful, well-behaved locus, we should restrict the Picard group in some way. One possible way to do this, is by prescribing the isomorphism class of the Picard lattice (the Picard group with its intersection form): demanding that a K3 surface have a given rank k lattice in its Picard group, gives a codimension $k-1$ condition on the moduli space of K3 surfaces (if such surfaces exist at all). In particular, if we take a rank 2 lattice, then the corresponding locus, of K3 surfaces including this lattice in their Picard groups, is a divisor in $\mathcal{F}_{2d}$. We will look at these divisors in section 4.2.1.

Right now, we take a slightly different approach: we look at the locus of polarised K3 surfaces that have a class in their Picard group with given intersection numbers: specifically, self-intersection number and degree (i.e., intersection number with the polarisation class).

Definition 4.2.4. Given $h \in \mathbb{N}, a \in \mathbb{Z}$ such that $a^2 - 4d(h-1)$ is positive, the Noether–Lefschetz divisor $D_{h,a} \subset \mathcal{F}_{2d}$ is supported on the locus of polarised K3 surfaces (S, H) that have a divisor class $\beta \in \operatorname{Pic} S$ (with β not in the span of H) of square $\beta^2 = 2h - 2$ and degree $\beta \cdot H = a$.

The multiplicity of the irreducible component of $D_{h,a}$ consisting (generically) of K3 surfaces that have Picard lattice isomorphic to the rank 2 lattice L equals the number of elements $\beta \in L$ with $\beta^2 = 2h - 2$ and $\beta \cdot H = a$ (where we write H for the element of L corresponding to the polarisation class).

Remark 4.2.5. Note that the number $a^2 - 4d(h-1)$ equals minus the determinant of the lattice $\left(\begin{smallmatrix} 2d & a \\ a & 2h-2 \end{smallmatrix}\right)$, and this lattice is isomorphic to the the sublattice of $\mathrm{Pic}(S)$ that is generated by H and β. Therefore, the positivity of $a^2 - 4d(h-1)$ is necessary for such surfaces to exist, by the Hodge index theorem (which tells us that the lattice generated by H and β must have signature $(1,1)$).

Remark 4.2.6. The definition already refers to the irreducible components of $D_{h,a}$, that are given by K3 surfaces with rank 2 Picard lattice of fixed isomorphism class. We will look at those so-called irreducible Noether–Lefschetz divisors in section 4.2.1.

As we alluded to before, these geometrically defined Noether–Lefschetz divisors in fact coincide with the Heegner divisors:

Lemma 4.2.7 ([30, section 4.4, lemma 3])**.** *We have* $D_{h,a} = H(\gamma, n)$ *if* $\gamma = aw/2d \in D_L$ *and* $n = h - 1 - a^2/4d$.

Therefore, we will use the terms "Heegner divisor" and "Noether–Lefschetz divisor" interchangeably.

Having introduced a large set of divisors on the moduli space $\mathcal{F}_{2d}$, we would like to understand their role in the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$.

Definition 4.2.8. The Noether–Lefschetz Picard group $\mathrm{Pic}_{\mathbb{Q}}^{\mathrm{NL}}(\mathcal{F}_{2d})$ is the $\mathbb{Q}$-subspace of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ spanned by $H(\overline{0}, 0)$ and the (infinite) set of Heegner divisors $H(\gamma, n)$, for all $\gamma \in D_L$ and all $n \in \mathbb{Z} - \gamma^2/2$ with $n < 0$.

A first natural question is, whether the subspace $\mathrm{Pic}_{\mathbb{Q}}^{\mathrm{NL}}(\mathcal{F}_{2d})$ might in fact be equal to the full Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$.

Maulik and Pandharipande have conjectured [30] that this is indeed the case. It has been verified for $d \in \{1, 2\}$ in [46, 47], for $d \in \{3, 4\}$ in [29], and for $d \in \{5, 6, 7, 8, 9, 11\}$ in [17], by using a different description of the moduli space (as a GIT quotient). On the other hand, the corresponding conjecture does not hold for all arithmetic quotients: it fails for some Hilbert modular surfaces, for instance.

Very recently, this conjecture has been proved by Bergeron, Li, Millson and Moeglin:

Theorem 4.2.9 ([5])**.** *The rational Picard group is spanned by Noether–Lefschetz divisors, so indeed* $\mathrm{Pic}_{\mathbb{Q}}^{\mathrm{NL}}(\mathcal{F}_{2d}) = \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$.

The proof does not use the geometric interpretation of $\mathcal{F}_{2d}$, as a moduli space of polarised K3 surfaces. Instead, it uses the representation theory of the orthogonal groups occurring in the arithmetic description of $\mathcal{F}_{2d}$ as a locally symmetric domain.

A next question is to compute the dimension of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ as a function of d. A formula for $\dim \mathrm{Pic}_{\mathbb{Q}}^{\mathrm{NL}}(\mathcal{F}_{2d})$ has been found by Bruinier; by theorem 4.2.9 above, this is in fact the right answer.

Theorem 4.2.10 ([9])**.** *The dimension of* $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ *equals*

$$1 + \frac{15}{8}(d+1) - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 \ , \tag{4.7}$$

where

$$\begin{aligned} \alpha_1 &= \frac{d+1}{4} + \frac{1}{4\sqrt{2d}} \Re(G(2)) \ , \\ \alpha_2 &= \frac{d+1}{3} + \frac{1}{3\sqrt{6d}} \Re(\mathrm{e}(-19/24)(G(1) + G(-3)) \ , \\ \alpha_3 &= \sum_{k=0}^{d} \left\{ \frac{k^2}{4d} \right\} \ , \\ \alpha_4 &= \# \left\{ k \in (\mathbb{Z}/2d\mathbb{Z})/\pm : k^2 \in 4d\mathbb{Z} \right\} \ , \end{aligned} \tag{4.8}$$

and $\mathrm{e}(x) = e^{2\pi i x}$, $\{x\}$ *denotes the fractional part of* x, *and* $G(s)$ *is a Gauss sum associated to the lattice* L, *given by*

$$G(s) = \sum_{\gamma \in L^\vee/L} \mathrm{e}(s\gamma^2/2) = \sum_{k=0}^{2d-1} \mathrm{e}(-sk^2/(4d)) \ . \tag{4.9}$$

We list some of these values in table 4.1.

As these values indicate, the dimension grows roughly linearly in the polarisation degree $2d$, but with some variation. This is even clearer from a graph: see figure 4.1.

Table 4.1: The first few values of $\dim \operatorname{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$.

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\dim \operatorname{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$	2	3	4	4	6	7	7	8	9	10	11	12	12	14

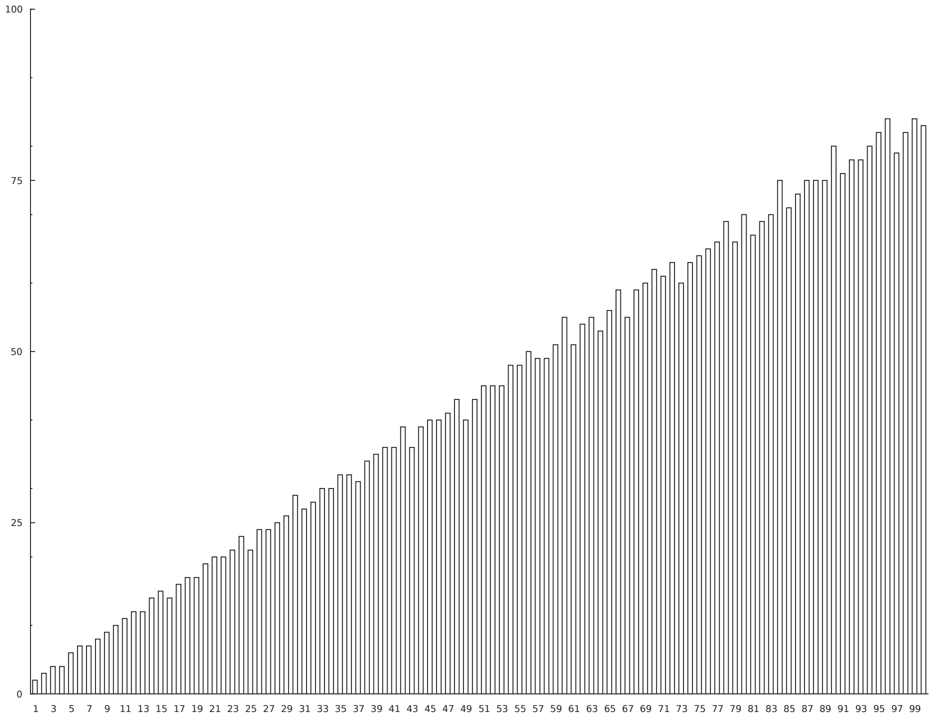


Figure 4.1: The dimension of $\operatorname{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ as a function of d.

4.2.1 Irreducible Noether–Lefschetz divisors

Let (S, H) be a polarised K3 surface. The moduli point corresponding to this surface lies on the Noether–Lefschetz divisor $D_{h,a}$ if and only if there exists a divisor class on S with intersection numbers h, a. This forces the Picard lattice of S to be at least rank 2. Even if we suppose that the Picard lattice of S has rank exactly 2, we cannot determine the lattice structure of $\operatorname{Pic}(S)$ from this condition alone. The reason is that, in general, there are several non-isomorphic rank 2 lattices having an element with the prescribed intersection numbers.

This is relevant from a geometric point of view: because of this phenomenon, the divisors $D_{h,a}$ are not irreducible.

So, instead of just imposing the existence of a divisor class with given intersection numbers, as in the definition of $D_{h,a}$, we will introduce slightly different, more refined divisors on the moduli space: the loci where the Picard lattice of the polarised K3 surface is of a given isomorphism class. To do so, we first present a convenient parametrisation of these isomorphism classes.

Definition 4.2.11. Let (L, H) be a $2d$-polarised even lattice of rank 2 and signature $(1, 1)$. Then the discriminant $\Delta \in \mathbb{Z}_+$ and coset $\delta \in (\mathbb{Z}/2d\mathbb{Z})/\pm$ of L are defined as follows: first choose an element $\Gamma \in L$ such that H and Γ form a basis of L. We write down the intersection matrix of L with respect to this basis:

$$\begin{pmatrix} 2d & y \\ y & 2x \end{pmatrix}$$

(so $y = H \cdot \Gamma$ and $2x = \Gamma^2$). Then $\Delta = y^2 - 4dx$, and $\delta = \overline{y} \in (\mathbb{Z}/2d\mathbb{Z})/\pm$.

(Note that the number Δ is in fact minus the usual discriminant of a lattice (i.e., the determinant of the intersection matrix); we prefer to work with positive numbers.) It is an easy exercise to check that the discriminant and coset are independent of the choice of element Γ, so they are invariants of this particular class of polarised lattices. In fact, they are complete invariants:

Lemma 4.2.12. *The $2d$-polarised even lattices of rank 2 and signature $(1,1)$ are classified by their discriminant and coset.*

The proof is not hard, and omitted here.

Note that we do not claim that there exists such a lattice for any choice of Δ and δ. In fact, we have the following proposition.

Proposition 4.2.13. *There exists a rank 2 even hyperbolic $2d$-polarised lattice of discriminant Δ and coset δ if and only if*

$$\Delta \equiv \delta^2 \mod 4d \ . \tag{4.10}$$

Proof. Suppose that a lattice L with the given properties exists. Choose an element Γ that, together with the polarisation element H, generates L. Write the intersection form of L with respect to the basis H, Γ as

$$\begin{pmatrix} 2d & y \\ y & 2x \end{pmatrix} \ . \tag{4.11}$$

(Note that $2x = \Gamma^2$ is indeed even, because L is even by assumption.) We know that the discriminant Δ is minus the determinant of this matrix, so $\Delta = y^2 - 4dx$. Also, the coset of L is equal to $\overline{y} \in (\mathbb{Z}/2d\mathbb{Z})/\pm$, so $y^2 \equiv \delta^2 \mod 2d$, and certainly also $y^2 \equiv \delta^2 \mod 4d$. We conclude that $\Delta \equiv \delta^2 \mod 4d$.

On the other hand, suppose that $\Delta \equiv \delta^2 \mod 4d$. Let $k \in \mathbb{Z}$ be a representative of the class δ. By assumption, we can write $\Delta = k^2 + 4dl$, with $l \in \mathbb{Z}$. Let L be the rank 2 lattice with intersection matrix

$$\begin{pmatrix} 2d & k \\ k & -2l \end{pmatrix} \ . \tag{4.12}$$

Then this lattice has discriminant $k^2 - (2d)(-2l) = k^2 + 4dl = \Delta$, and coset $\overline{k} = \delta$. Also, L is clearly even, $2d$-polarised (by the first basis vector), and hyperbolic (because the intersection matrix has negative determinant, so it must have one positive and one negative eigenvalue). □

Now, using this description of rank 2 lattices in terms of their invariants Δ and δ, we arrive at the following definition of the refined Noether–Lefschetz divisors.

Definition 4.2.14. Given $\Delta \in \mathbb{Z}_+$ and $\delta \in (\mathbb{Z}/2d\mathbb{Z})/\pm$, the Noether–Lefschetz divisor $P_{\Delta,\delta}$ is the closure of the set of polarised K3 surfaces (S, H) such that $\operatorname{Pic} S$ has discriminant Δ and coset δ.

By taking the closure, we include surfaces with a Picard lattice that contains the given lattice, but has rank higher than 2.

Our work in refining the Noether–Lefschetz divisors is rewarded by the following result:

Proposition 4.2.15 ([38])**.** *For any given $\Delta \in \mathbb{Z}_+$ and $\delta \in (\mathbb{Z}/2d\mathbb{Z})/\pm$ such that there exists a lattice with discriminant Δ and coset δ, the divisor $P_{\Delta,\delta}$ on $\mathcal{F}_{2d}$ is irreducible.*

4.2.2 Triangular relations

It remains to relate the two sets of Noether–Lefschetz divisors $D_{h,a}$ and $P_{\Delta,\delta}$ in a precise way. To do so, we need to compute which rank 2 lattices (as specified by Δ, δ) contain a class with given intersection numbers h, a. The result is as follows.

Lemma 4.2.16. *We have the following expression for the $D_{h,a}$ in terms of the $P_{\Delta,\delta}$:*

$$D_{h,a} = \sum_{\Delta,\delta} \mu_{\Delta,\delta,h,a} P_{\Delta,\delta} \ , \tag{4.13}$$

where the coefficient $\mu_{\Delta,\delta,h,a} \in \{0,1,2\}$ is the number of integral solutions $(\xi,\eta) \in \mathbb{Z}^2$ to the following pair of quadratic and linear equations:

$$\begin{aligned} \Delta\eta^2 &= a^2 - 4d(h-1) \ , \\ 2d\xi &= a - y\eta \ , \end{aligned} \tag{4.14}$$

where $y \in \mathbb{Z}$ is any representative of $\delta \in (\mathbb{Z}/2d\mathbb{Z})/\pm$.

Proof. The number $\mu_{\Delta,\delta,h,a}$ is the number of vectors β in the lattice $L_{\Delta,\delta}$ that satisfy the equations $\beta \cdot H = a$ and $\beta^2 = 2h-2$. These are a linear and quadratic equation, respectively, in the coefficients (ξ, η) of β (with respect to any basis of $L_{\Delta,\delta}$).

We will rewrite these equations in more explicit terms. Choose a vector $\Gamma \in L_{\Delta,\delta}$ such that H and Γ form a basis of $L_{\Delta,\delta}$. Write $2d = H^2$ (as always), $y = H \cdot \Gamma$, and $2x = \Gamma^2$. Also, write ξ, η for the coefficients of β with respect to this basis. The equations for β become $2d\xi + y\eta = a$ and $2d\xi^2 + 2y\xi\eta + 2x\eta^2 = 2h-2$. We solve the first one for ξ, and insert this in the second one. (We will check a posteriori that ξ is an integer.) This gives the equation

$$2d\left(\frac{a^2}{4d^2} - \frac{2ay\eta}{4d^2} + \frac{y^2\eta^2}{4d^2}\right) + 2y\eta\frac{a-y\eta}{2d} + 2x\eta^2 = 2h-2\ . \tag{4.15}$$

Rewriting, and completing the square for η, we get

$$(y^2 - 4dx)\eta^2 = a^2 - 4d(h-1)\ . \tag{4.16}$$

Note that the factor $y^2 - 4dx$ is just the (positive) discriminant of the lattice $L_{\Delta,\delta}$, hence equals Δ. Also, the right-hand side is just the (positive) discriminant of the lattice described by h and a (with intersection matrix $\left(\begin{smallmatrix} 2d & a \\ a & 2h-2 \end{smallmatrix}\right)$).

In the form (4.16), it is easy to solve for η. Finally, a solution of (4.16) gives a solution to the original equations if and only if $\xi = (a - y\eta)/2d$ is an integer. □

In view of proposition 4.2.15 and lemma 4.2.16, we will refer to the $D_{h,a}$ as the reducible Noether–Lefschetz divisors, and to the $P_{\Delta,\delta}$ as the irreducible ones. (Even though in particular cases, $D_{h,a}$ could happen to be irreducible.) The irreducible divisors are perhaps more natural from a geometric point of view, but the reducible combinations $D_{h,a}$ are better behaved in the arithmetic description of the moduli space, as witnessed by their direct correspondence to the Heegner divisors (lemma 4.2.7).

Example 4.2.17. We work out (4.13) for the case $d = 1, h = 0, a = 0$. This gives the decomposition of the Heegner divisor $D_{0,0} = H(\bar{0}, -1)$ in $\mathscr{F}_{2\cdot 1}$, as a sum of irreducible divisors.

Note that $a^2 - 4d(h-1) = 4$. From (4.16), we see that Δ must be a divisor of 4, such that the quotient $4/\Delta$ is a square. So, there are two possibilities: $\Delta = 1$ and $\Delta = 4$.

First, we find the contributions with $\Delta = 1$. For every $\delta \in (\mathbb{Z}/2d\mathbb{Z})/\pm = \{\bar{0}, \bar{1}\}$, we need to find a rank 2 lattice $L_{1,\delta}$ with discriminant $\Delta = 1$ and coset δ (if it exists), and compute the coefficient $\mu_{1,\delta,0,0}$ (i.e., the number of vectors β in this lattice $L_{1,\delta}$ with $\beta \cdot H = 0$ and $\beta^2 = 0$). We can write $\left(\begin{smallmatrix} 2 & y \\ y & 2x \end{smallmatrix}\right)$ for the intersection matrix of the lattice, where x and y are to be determined. Now, observe that $1 = \Delta = y^2 - 4x$, and $\delta = \overline{y} \in (\mathbb{Z}/2\mathbb{Z})/\pm$; combining both equations modulo 2, we get that $\delta = \bar{1}$. We conclude that there is no rank 2 lattice with $\Delta = 1$ and $\delta = \bar{0}$, so $\mu_{1,\bar{0},0,0} = 0$. For $\delta = \bar{1}$, we see that the choice $x = 0, y = 1$ gives the right discriminant and coset. To get the multiplicity, we take all solutions of (4.16), in this case $\eta = \pm 2$, and verify for each of them whether ξ is integer. For $\eta = 2$, we get $\xi = (a - y\eta)/2d = -1 \in \mathbb{Z}$, so this gives a solution. For $\eta = -2$, we get $\xi = (a - y\eta)/2d = 1 \in \mathbb{Z}$, giving another solution. We conclude that there are two vectors $\beta \in L_{1,\bar{1}}$ with the properties $\beta \cdot H = a = 0$ and $\beta^2 = 2h - 2 = -2$ (to wit, $\beta = (\xi, \eta) \in \{(-1, 2), (1, -2)\}$), so the multiplicity in this case is $\mu_{1,\bar{1},0,0} = 2$.

Next, we find the contributions with $\Delta = 4$; we leave out some details for brevity. In this case, only $\delta = \bar{0}$ gives a contribution. One possible lattice is $L_{4,\bar{0}}$ with intersection matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$. This has again two vectors with the required intersection numbers: $\beta \in \{(0, 1), (0, -1)\}$. So also in this case the multiplicity is $\mu_{4,\bar{0},0,0} = 2$.

The above shows that, for $d = 1$,

$$D_{0,0} = 2 \cdot P_{1,\bar{1}} + 2 \cdot P_{4,\bar{0}}\ . \tag{4.17}$$

Recalling the geometric description of these Noether–Lefschetz divisors, we can interpret this as follows. The locus $D_{0,0}$ consists of the K3 surfaces whose Picard lattice has a divisor class β with $\beta \cdot H = 0$, $\beta^2 = -2$. This can happen in two ways: in (a dense subset of) the sublocus $P_{4,\bar{0}}$, the Picard lattice of the K3 surface is in fact generated by the classes H and β. In (a dense subset of) $P_{1,\bar{1}}$, the lattice generated by H and $\beta = \pm(H - 2\Gamma)$ has index 2 in the full Picard lattice of the surface (which is generated by H and Γ). Because in both cases there are two elements β with the required properties, both components contribute to $D_{0,0}$ with multiplicity two.

We now derive some properties of equation (4.13).

Proposition 4.2.18.

(i) The coefficient $\mu_{\Delta,\delta,h,a}$ *is nonzero only if* Δ *divides* $a^2 - 4d(h-1)$.

(ii) If Δ *equals* $a^2 - 4d(h-1)$*, the coefficient* $\mu_{\Delta,\delta,h,a}$ *is nonzero if and only if* $\delta = \overline{a} \in (\mathbb{Z}/2d\mathbb{Z})/\pm$.

Proof. (i) This follows directly from equation (4.16). (ii) Suppose first that $\delta = \overline{a} \in (\mathbb{Z}/2d\mathbb{Z})/\pm$. Look at the lattice generated by H and Γ, with $H^2 = 2d, H \cdot \Gamma = a, \Gamma^2 = 2h-2$. This has discriminant $a^2 - 4d(h-1)$ and coset δ. The vector Γ satisfies the equations for β by construction, so there is at least one vector with the right intersection numbers, hence $\mu_{a^2-4d(h-1),\overline{a},h,a} > 0$.

Now, suppose that $\mu_{a^2-4d(h-1),\delta,h,a}$ is nonzero. Since $a^2 - 4d(h-1) = \Delta$, we see from equation (4.16) that $\eta = \pm 1$. Therefore, from the final remark of the proof of lemma 4.2.16, we conclude that $a \pm y$ is divisible by $2d$; recall that the class of y in $(\mathbb{Z}/2d\mathbb{Z})/\pm$ is exactly δ, so $\overline{a} = \overline{y} = \delta \in (\mathbb{Z}/2d\mathbb{Z})/\pm$. □

By part (i) of the above proposition, the sum in (4.13) is finite. We can say something even stronger: the set of equations (4.13) is triangular with respect to the divisibility ordering on Δ, i.e., in the equation for $D_{h,a}$, only terms occur with Δ a divisor of $\Delta_0 = a^2 - 4d(h-1)$. This implies that we can invert the (infinite!) set of equations (4.13), solving for $P_{\Delta,\delta}$ in terms of $D_{h,a}$.

4.3 Borcherds' construction of modular forms on $\mathcal{F}_{2d}$

In [6], Borcherds gives a construction of modular forms on the arithmetic quotient associated to a lattice of signature $(2,n)$. We apply this to our situation, where the lattice is $L = L_{2d}$ (of signature $(2,19)$), and the associated arithmetic quotient is exactly our moduli space $\mathcal{F}_{2d}$. This gives us a large supply of modular forms on our space $\mathcal{F}_{2d}$.

In the following, recall from section 3.3.1 the space $\operatorname{Sing}(L)$ (Laurent polynomials with values in $\mathbb{C}[D_L]$, which we use to describe the pole behaviour of meromorphic modular forms), and the space $\operatorname{Obstruct}(k,L)$ of obstructions to the existence of a meromorphic modular form of weight k with given principal part.

Theorem 4.3.1 ([6, Theorem 13.3]). *Let* $f = \sum_{\gamma,n\geq 0} a_{\gamma,n} q^{-n} \mathbf{e}_\gamma \in \operatorname{Sing}(L)$ *be the principal part of a meromorphic modular form of weight* $1 - b^-/2 = 1 - 19/2 = -17/2$ *(i.e.,* $[f] = 0 \in \operatorname{Obstruct}(-17/2, L)$*). Assume that all coefficients* $a_{\gamma,n}$ *are integral. Then there is a meromorphic modular form* Ψ *(scalar valued) on* $\mathcal{F}_{2d}$*, of weight* $a_{\bar{0},0}/2$*, with divisor* $1/2 \cdot \sum_{\gamma,n>0} a_{\gamma,n} H(\gamma,n)$*. Moreover,* Ψ *has the following product expansion around the cusp associated to* z*, in the tube domain parametrisation (see section 5.2.1):*

$$\Psi_z(Z_M) = C\,\mathrm{e}((Z_M,\rho_M)) \prod_{\substack{\lambda\in M^\vee \\ (\lambda,W_M)>0}} \prod_{\substack{\delta\in D_L \\ \delta|_M=\lambda}} (1 - \mathrm{e}((\lambda, Z_M) + (\delta, z')))^{a_{\lambda,\lambda^2/2}}\ . \tag{4.18}$$

Here, the number C *is some nonzero constant;* W_M *is a Weyl chamber (with respect to* f*) that has* z *in its closure (this is the subset of the period domain on which the expansion will be valid);* $\rho_M = \rho(M, W_M, f) \in M \otimes \mathbb{Q}$ *is the corresponding Weyl vector; the notation* $(\lambda, W_M) > 0$ *means that* $(\lambda, w) > 0$ *for all* $w \in W_M$.

About the condition $(\lambda, W_M) > 0$: if λ is such that $a_{\lambda,\lambda^2/2} \neq 0$ (and note that other λ do not contribute!), then it suffices to check this for any single $w_0 \in W_M$.

Note that even though $f \in \operatorname{Sing}(L)$, only the terms of f with $n < 0$ or $(\gamma, n) = (\bar{0}, 0)$ are used, so we might as well take $f \in \operatorname{Sing}_0^-(L)$. The theorem shows that if the combination

$$a_{\bar{0},0}\mathbf{e}_{\bar{0}} + \sum_{\gamma,n>0} a_{\gamma,n} q^{-n} \mathbf{e}_\gamma \in \operatorname{Sing}_0^-(L) \tag{4.19}$$

vanishes in the obstruction space $\operatorname{Obstruct}(-17/2, L)$, then

$$a_{\bar{0},0} H(\bar{0},0) + \sum_{\gamma,n>0} a_{\gamma,n} H(\gamma,n) \tag{4.20}$$

is linearly equivalent to the zero divisor on $\mathcal{F}_{2d}$. (Recall that a modular form of weight k is a section of the line bundle $k\lambda = -kH(\bar{0}, 0)$.) On the other hand, recall from section 3.3.1 (in particular proposition 3.3.10 and remark 3.3.11) that

$$\sum_{\gamma, n \geq 0} a_{\gamma,n} q^{-n} \mathbf{e}_\gamma \in \mathrm{Sing}_0^-(L) \tag{4.21}$$

vanishes in Obstruct$(-17/2, L)$ if and only if the corresponding functional of coefficients of almost cusp forms of weight $2 - (-17/2) = 21/2$ vanishes, i.e., if

$$\sum_{\gamma, n \geq 0} a_{\gamma,n} c_{\gamma,n} = 0 \in AC(d)^\vee \; . \tag{4.22}$$

Therefore, we may produce relations in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ among Heegner divisors and the Hodge class λ by computing equalities among coefficients of vector-valued almost cusp forms.

Moreover, Bruinier [10, Theorem 1.2] shows that any meromorphic modular form on $\mathcal{F}_{2d}$ with divisor supported on Heegner divisors occurs as a result of Borcherds' construction. Translating to geometric terms, this means that all relations among Noether–Lefschetz divisors come from linear combinations of coefficients that vanish on all almost cusp forms.

Finally, by the recent work of [5], the rational Picard group of $\mathcal{F}_{2d}$ is generated by Noether–Lefschetz divisors, so we may summarise all the above as follows.

Theorem 4.3.2. *The rational Picard group of $\mathcal{F}_{2d}$ is isomorphic to the dual of the space of rational vector-valued almost cusp forms of weight $21/2$. This isomorphism $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d}) \to AC(d)^\vee_{\mathbb{Q}}$ sends $[H(\gamma, n)]$ to the coefficient function $c_{\gamma,n} : AC(d)^\vee_{\mathbb{Q}} \to \mathbb{Q}$; as a special case, $\lambda = -[H(\bar{0}, 0)]$ is sent to $-c_{\bar{0},0}$.*

Proof. As described above, this is a direct combination of Borcherds' construction of forms on arithmetic quotients [6], Bruinier's converse theorem [10], and the result [5] by Bergeron, Li, Millson and Moeglin that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ is generated by Noether–Lefschetz divisors. □

4.4 Computing relations in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$

As we saw in the previous section, the relations among divisors on $\mathcal{F}_{2d}$ are exactly given by linear relations between coefficients of vector-valued modular forms.

By the work of [42], explained in section 3.6, we may compute a basis of that space of vector-valued modular forms up to any wanted number of Fourier coefficients. These data then give all relations among Noether–Lefschetz divisors in concrete form, and by theorem 4.2.9, that gives us a complete description of the rational Picard group.

Let us do a couple of examples by hand.

4.4.1 Example: $d = 1$

We may use the data on vector-valued modular forms we computed in section 3.6.3 to get a basis of the rational Picard group. Recall that the space $M(1)$ of vector-valued modular forms has dimension 2, and that there is only a single cusp, so that the space of almost cusp forms is the whole space $AC(1) = M(1)$.

As a basis for $AC(1)^\vee$, we pick $\{\varphi_1, \varphi_2\}$, where

$$\begin{aligned} \varphi_1 &= c_{\bar{0},0} \; , \\ \varphi_2 &= \frac{105457575250}{169227} c_{\bar{0},0} + c_{\bar{0},-1} \; . \end{aligned} \tag{4.23}$$

Note that this is the basis dual to the basis $\{v_1, v_2\}$ of M that we computed in 3.6.3. Employing the isomorphism between $AC(d)^\vee$ and $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$, we conclude that a basis of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2\cdot 1})$ is formed by

$$\begin{aligned} &H(\bar{0}, 0) = -\lambda \qquad \text{and} \\ &-\frac{105457575250}{169227}\lambda + H(\bar{0}, -1) \; . \end{aligned} \tag{4.24}$$

Alternatively, we might take as a basis the class λ and the class of the Noether–Lefschetz divisor $H(\bar{0},-1) = D_{0,0}$.

Given any other Noether–Lefschetz divisor $D_{h,a} = H(\gamma, n)$, we may read off its coefficients with respect to the basis $\{\varphi_1, \varphi_2\}$ directly from the corresponding coefficients of v_1 and v_2. For instance, take $D_{0,1} = H(\bar{1}, -1/4)$:

$$\begin{aligned}
D_{0,1} &= H(\bar{1}, -\tfrac{1}{4}) \\
&\sim c_{\bar{1},-\frac{1}{4}}(v_1) \cdot H(\bar{0},0) + c_{\bar{1},-\frac{1}{4}}(v_2) \cdot \left(\frac{105457575250}{169227} H(\bar{0},0) + H(\bar{0},-1)\right) \\
&= \frac{1882717700}{169227} H(\bar{0},0) - \frac{1}{56} \cdot \left(\frac{105457575250}{169227} H(\bar{0},0) + H(\bar{0},-1)\right) \\
&= -\frac{75}{28} H(\bar{0},0) - \frac{1}{56} H(\bar{0},-1) \\
&= \frac{75}{28}\lambda - \frac{1}{56} D_{0,0} \ .
\end{aligned} \tag{4.25}$$

4.4.2 Example: $d = 3$

Take $d = 3$. By theorem 4.2.10, $M(3)$ has dimension 4. Because there is again only one cusp, the space of almost cusp forms $AC(3)$ coincides with $M(3)$, so $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2\cdot 3})$ has dimension 4. We use Jacobi forms and the restriction method to determine that $M(3)$ is spanned by the following vectors:

$$\begin{aligned}
v_1 &= \mathbf{e}_{\bar{0}} \cdot (1 + 0\,q + -63756q^2 + ...) \\
&\quad + (\mathbf{e}_{\bar{1}} + \mathbf{e}_{\bar{5}}) \cdot (0\,q^{1/12} - 315q^{13/12} - 90531q^{25/12} + O(q^3)) \\
&\quad + (\mathbf{e}_{\bar{2}} + \mathbf{e}_{\bar{4}}) \cdot (0\,q^{1/3} - 1017q^{4/3} - 275580q^{7/3} + O(q^3)) \\
&\quad + \mathbf{e}_{\bar{3}} \cdot (-49q^{3/4} - 17190q^{7/4} - 1293075q^{11/4} + O(q^3)) \ , \\
v_2 &= \mathbf{e}_{\bar{0}} \cdot (0 + 1\,q - 12q^2 + ...) \\
&\quad + (\mathbf{e}_{\bar{1}} + \mathbf{e}_{\bar{5}}) \cdot \left(0\,q^{1/12} - \frac{3}{2}q^{13/12} + \frac{45}{2}q^{25/12} + O(q^3)\right) \\
&\quad + (\mathbf{e}_{\bar{2}} + \mathbf{e}_{\bar{4}}) \cdot (0\,q^{1/3} + 3q^{4/3} - 51q^{7/3} + O(q^3)) \\
&\quad + \mathbf{e}_{\bar{3}} \cdot \left(-\frac{1}{2}q^{3/4} + 5q^{7/4} - \frac{3}{2}q^{11/4} + O(q^3)\right) \ , \\
v_3 &= \mathbf{e}_{\bar{0}} \cdot (0 + 0 \cdot q - 67392q^2 + ...) \\
&\quad + (\mathbf{e}_{\bar{1}} + \mathbf{e}_{\bar{5}}) \cdot (1\,q^{1/12} - 320q^{13/12} - 44335q^{25/12} + O(q^3)) \\
&\quad + (\mathbf{e}_{\bar{2}} + \mathbf{e}_{\bar{4}}) \cdot (0\,q^{1/3} + 1248q^{4/3} + 120640q^{7/3} + O(q^3)) \\
&\quad + \mathbf{e}_{\bar{3}} \cdot (-54q^{3/4} + 20736q^{7/4} + 1053000q^{11/4} + O(q^3)) \ , \\
v_4 &= \mathbf{e}_{\bar{0}} \cdot (0 + 0\,q - 7452q^2 + O(q^3)) \\
&\quad + (\mathbf{e}_{\bar{1}} + \mathbf{e}_{\bar{5}}) \cdot (0\,q^{1/12} - 25q^{13/12} + 5375q^{25/12} + O(q^3)) \\
&\quad + (\mathbf{e}_{\bar{2}} + \mathbf{e}_{\bar{4}}) \cdot (1\,q^{1/3} + 352q^{4/3} + 5780q^{7/3} + O(q^3)) \\
&\quad + \mathbf{e}_{\bar{3}} \cdot (-27q^{3/4} - 2754q^{7/4} - 31185q^{11/4} + O(q^3)) \ .
\end{aligned} \tag{4.26}$$

Classical numbers of special fibres in families

We have a method to compute relations among Noether–Lefschetz divisors, by using coefficients of vector-valued modular forms. As Noether–Lefschetz divisors have a direct geometrical interpretation (in terms of the existence of divisors on the K3 surface with prescribed intersection numbers), these relations have a geometrical meaning as well.

To extract some of this information, we can take a 1-dimensional family of K3 surfaces, i.e., a curve C in our moduli space $\mathcal{F}_{2d}$. Intersecting a Noether–Lefschetz divisor $H(\gamma, n)$ with this curve gives us a number, which we will denote by $C(\gamma, n)$ (this equals the degree of $H(\gamma, n)$ on C). Then a relation among Noether–Lefschetz divisors gives an equation among these degrees.

Following [30, section 6.4], we will apply this to a case where we have a way to compute some of these degrees; then our relation will compute another one.

We express $H(\overline{3}, -3/4)$ in terms of the basis

$$\lambda = -v_1^\vee, \quad H(\overline{0}, 1) = v_2^\vee, \quad H(\overline{1}, -1/12) = v_3^\vee, \quad H(\overline{2}, -1/3) = v_4^\vee \ , \tag{4.27}$$

by reading off the coefficients of $\mathbf{e}_{\bar{3}} q^{3/4}$ for the vectors in the table above. These are -49, $-1/2$, -54, -27 respectively, so we get that

$$\begin{aligned} H(\overline{3}, -\frac{3}{4}) &\sim -49 v_1^\vee - \frac{1}{2} v_2^\vee - 54 v_3^\vee - 27 v_4^\vee \\ &= 49\lambda - \frac{1}{2} H(\overline{0}, 1) - 54 H(\overline{1}, -\frac{1}{12}) - 27 H(\overline{2}, -\frac{1}{3}) \ , \end{aligned} \tag{4.28}$$

or equivalently, multiplying by the common denominator,

$$2H(\overline{3}, -\frac{3}{4}) = 98\lambda - H(\overline{0}, -1) - 108 H(\overline{1}, -\frac{1}{12}) - 54 H(\overline{2}, -\frac{1}{3}) \ . \tag{4.29}$$

Now, recall that there is an open subset of $\mathcal{F}_{2\cdot 3}$ of K3 surfaces that can be constructed as smooth intersections of a quadric and a cubic in $\mathbb{P}^4$. We take a fixed quadric, and a Lefschetz pencil of cubics. This gives a 1-dimensional family of K3 surfaces, but not all of them are smooth. It is possible to resolve the singularities (see [30, section 5.1]); this results in a curve C_1 in $\mathcal{F}_{2\cdot 3}$. We intersect this curve with the above relation (4.29).

By classical computations the degree of the Hodge bundle λ on C_1 is 1 (note that what [30] calls the Hodge bundle is the dual of λ, so they get -1 instead). Also, the degree of $H(\overline{0}, -1)$ on C_1 equals the number of singular fibres in the family, which is 98 in this case.

By a generic choice of the Lefschetz pencil of cubics, we can make sure that the points on C_1 that are on $H(\overline{1}, -1/12)$ or $H(\overline{2}, -1/3)$ have a very ample polarisation. Then Castelnuovo's bound, in the form of [30, Lemma 7], shows that such points in fact cannot exist. This implies that the degrees $C_1(\overline{1}, -1/12)$ and $C_1(\overline{2}, -1/3)$ vanish.

All this gives the following result for the intersection of (4.29) with C_1:

$$\begin{aligned} 2C_1(\bar{3}, -\frac{3}{4}) &= 98 \cdot 1 - C_1(\bar{0}, 1) - 108 C_1(\bar{1}, -\frac{1}{12}) - 54 C_1(\bar{2}, -\frac{1}{3}) \\ &= 98 - 98 - 108 \cdot 0 - 54 \cdot 0 \\ &= 0 \ , \end{aligned} \tag{4.30}$$

so we conclude that

$$C_1(\bar{3}, -\frac{3}{4}) = 0 \ . \tag{4.31}$$

To extract the geometric information contained in this last equality, it is convenient to translate it to a statement about prime Noether–Lefschetz divisors $P_{\Delta,\delta}$, using the triangular relations of 4.2.2.

A simple computation gives

$$H(\overline{3}, -\frac{3}{4}) = D_{1,3} = 2P_{1,\bar{1}} + 2P_{9,\bar{3}} \ . \tag{4.32}$$

Here $P_{9,\bar{3}}$ corresponds to the lattices with $\Delta = 9, \delta = \bar{3}$; these are isomorphic to $\left(\begin{smallmatrix} 6 & 3 \\ 3 & 0 \end{smallmatrix}\right)$. The other occurring primitive divisor, $P_{1,\bar{1}}$, corresponds to the lattices with $\Delta = 1, \delta = \bar{1}$; these are isomorphic to $\left(\begin{smallmatrix} 6 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

As $P_{1,\bar{1}} = D_{1,1} = H(\overline{1}, -1/12)$ has degree 0 on our family C_1 (as we saw above, by Castelnuovo's bound), this does not contribute. (By appropriately choosing the pencil of cubics, we may ensure that C_1 cannot be contained in $P_{1,\bar{1}}$.) So, from (4.31), we conclude that in fact the number of K3 surfaces in our family C_1 that have Picard lattice $\left(\begin{smallmatrix} 6 & 3 \\ 3 & 0 \end{smallmatrix}\right)$ vanishes.

Now, this Noether–Lefschetz divisor consists exactly of the K3 surfaces with a primitive class E such that $E^2 = 0$ and $E \cdot H = 3$, so a generic section will be a smooth elliptic plane curve. (Indeed a plane curve, because $\mathcal{O}(H)$ restricted to E is degree 3, hence very ample, so gives an embedding of E in $\mathbb{P}^2$.)

The conclusion of this exercise is thus as follows:

Proposition 4.4.1. *There are no K3 surfaces in the family C_1 that contain a smooth elliptic plane curve.*

4.4.3 Writing the Hodge class in terms of Noether–Lefschetz divisors

We get interesting relations in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$, by writing the Hodge class λ in terms of Noether–Lefschetz divisors. There is more than one way to do this, as there are many (even infinitely many) distinct Noether–Lefschetz divisors.

We get a particularly interesting relation if we take a simple set of Noether–Lefschetz divisors:

Definition 4.4.2. For every $\gamma \in \{\overline{0}, \dots, \overline{d}\}$, let us write B_γ for the Heegner divisor $H(\gamma, n)$ with the highest possible value of n (recall that $n = -\Delta/4d$ is always negative; we exclude the Hodge bundle $H(\overline{0}, 0) = -\lambda$ in this context). Let us write $D + 1 = \dim \mathrm{Pic}_{\mathbb{Q}} \mathcal{F}_{2d}$. Note that $D \leq d$ (for all values of d we are interested in, we can see this by computing D using theorem 4.2.10). Now, we define the minimal basis to be the set of divisor classes $[B_{\overline{0}}], \dots, [B_{\overline{D}}]$.

The Hodge relation is the expression of the Hodge bundle λ in terms of the minimal basis, normalised in such a way that the coefficient of $H(\overline{0}, -1)$ equals one.

Remark 4.4.3. The definition assumes that the given divisor classes $[B_{\overline{0}}], \dots, [B_{\overline{D}}]$ are indeed independent; we need to check this by explicit computation, using the basis we have of the space of almost cusp forms.

It turns out that for some higher values of d, this assumption in fact fails: for example for $d \in \{37, 41, 43, 47, 49\}$.

One might wonder how natural this basis is: the dimension D has a seemingly random variation as a function of the parameter d, and it is not clear why the B_γ with $0 \leq \gamma \leq D$ should be more important than the ones with $\gamma > D$. We will see later, when we compute the boundary coefficients that complete this relation to a valid relation on $\overline{\mathcal{F}_{2d}}$, that this particular choice has some nice numerical consequences.

Example 4.4.4. For $d = 1$, we have $D + 1 = \dim \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2\cdot 1}) = 2$. Therefore, the minimal basis consists of $B_{\overline{0}} = H(\overline{0}, -1)$ and $B_{\overline{1}} = H(\overline{1}, -1/4)$.

We have seen before that

$$H(\overline{1}, -1/4) \sim -75/28 \cdot H(\overline{0}, 0) - 1/56 \cdot H(\overline{0}, -1) \ . \qquad \text{(4.25 revisited)}$$

Rearranging to isolate $\lambda = -H(\overline{0}, 0)$, and multiplying by 56, we get

$$150\lambda \sim H(\overline{0}, -1) + 56H(\overline{1}, -1/4) \qquad (4.33)$$

as the Hodge relation for polarisation degree 1.

We may compute the Hodge relation for other genera in the same way, from a basis of the space of almost cusp forms. Some results:

Table 4.2: The Hodge relation for low values of d.

d	Hodge relation
1	$150\lambda \sim H(\overline{0}, -1) + 56H(\overline{1}, -1/4)$
2	$108\lambda \sim H(\overline{0}, -1) + 128H(\overline{1}, -1/8) + 14H(\overline{2}, -1/2)$
3	$98\lambda \sim H(\overline{0}, -1) + 108H(\overline{1}, -1/12) + 54H(\overline{2}, -1/3) + 2H(\overline{3}, -3/4)$
4	$80\lambda \sim H(\overline{0}, -1) + 112H(\overline{1}, -1/16) + 56H(\overline{2}, -1/4) + 16H(\overline{3}, -9/16)$
5	$84\lambda \sim H(\overline{0}, -1) + 88H(\overline{1}, -1/20) + 66H(\overline{2}, -1/5) + 24H(\overline{3}, -9/20)$ $+ 2H(\overline{4}, -4/5) + 32H(\overline{5}, -1/4)$
6	$70\lambda \sim H(\overline{0}, -1) + 96H(\overline{1}, -1/24) + 60H(\overline{2}, -1/6) + 32H(\overline{3}, -3/8)$ $+ 6H(\overline{4}, -2/3) + 96H(\overline{5}, -1/24) + 10H(\overline{6}, -1/2)$
7	$96\lambda \sim H(\overline{0}, -1) + 56H(\overline{1}, -1/28) + 54H(\overline{2}, -1/7) + 54H(\overline{3}, -9/28)$ $+ 2H(\overline{4}, -4/7) + 2H(\overline{5}, -25/28) + 54H(\overline{6}, -2/7)$

Table 4.2: The Hodge relation for low values of d (continued).

d	Hodge relation
8	$66\lambda \sim H(\overline{0},-1) + 84H(\overline{1},-1/32) + 56H(\overline{2},-1/8) + 42H(\overline{3},-9/32)$ $+ 14H(\overline{4},-1/2) + 2H(\overline{5},-25/32) + 72H(\overline{6},-1/8) + 14H(\overline{7},-17/32)$
9	$58/3 \cdot \lambda \sim H(\overline{0},-1) + 112H(\overline{1},-1/36) + 308/3 \cdot H(\overline{2},-1/9) - 56/9 \cdot H(\overline{3},-1/4)$ $+ 280/9 \cdot H(\overline{4},-4/9) + 56/9 \cdot H(\overline{5},-25/36) - 10/9 \cdot H(\overline{6},-1)$ $+ 448/9 \cdot H(\overline{7},-13/36) + 28/9 \cdot H(\overline{8},-7/9)$
10	$28\lambda \sim H(\overline{0},-1) + 96H(\overline{1},-1/40) + 94H(\overline{2},-1/10) + 16H(\overline{3},-9/40)$ $+ 16H(\overline{4},-2/5) + 16H(\overline{5},-5/8) - 2H(\overline{6},-9/10) + 16H(\overline{7},-9/40)$ $+ 18H(\overline{8},-3/5) + 96H(\overline{9},-1/40)$
11	$66\lambda \sim H(\overline{0},-1) + 56H(\overline{1},-1/44) + 70H(\overline{2},-1/11) + 42H(\overline{3},-9/44)$ $+ 16H(\overline{4},-4/11) + 14H(\overline{5},-25/44) + 0H(\overline{6},-9/11) + 70H(\overline{7},-5/44)$ $+ 14H(\overline{8},-5/11) + 2H(\overline{9},-37/44) + 42H(\overline{10},-3/11)$
12	$56\lambda \sim H(\overline{0},-1) + 72H(\overline{1},-1/48) + 54H(\overline{2},-1/12) + 44H(\overline{3},-3/16)$ $+ 24H(\overline{4},-1/3) + 12H(\overline{5},-25/48) + 2H(\overline{6},-3/4) + 72H(\overline{7},-1/48)$ $+ 30H(\overline{8},-1/3) + 4H(\overline{9},-11/16) + 54H(\overline{10},-1/12) + 12H(\overline{11},-25/48)$

Many of the coefficients in the Hodge relations have a geometric interpretation, counting curves with special properties; see section 4.4.2 for a worked example.

4.5 The effective cone of $\mathcal{F}_{2d}$

Now that we have a good description of the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$, we may next try to understand the effective cone inside the Picard group.

There is a natural subcone of the effective cone, generated by the irreducible Noether–Lefschetz divisors $P_{\Delta,\delta}$:

Definition 4.5.1. The Noether–Lefschetz cone $\mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_{2d}) \subseteq \mathrm{Eff}(\mathcal{F}_{2d})$ is the cone generated by the set of all irreducible Noether–Lefschetz divisors $P_{\Delta,\delta}$.

Remark that the reducible Noether–Lefschetz divisors $H(\gamma, n)$ are positive linear combinations of the irreducible ones (by lemma 4.2.16), so they will all lie in this subcone.

First of all, it would be nice to understand the structure of the Noether–Lefschetz cone. In particular:

Question 4.5.2. Is the Noether–Lefschetz cone finitely generated? If so, can we give a list of generators and/or a list of bounding hyperplanes?

Apart from this, it would be nice to know how much we lose by restricting to this special subcone:

Question 4.5.3. Is the Noether–Lefschetz cone $\mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_{2d})$ equal to the effective cone $\mathrm{Eff}(\mathcal{F}_{2d})$?

Note that the equality $\mathrm{Pic}^{\mathrm{NL}}_{\mathbb{Q}}(\mathcal{F}_{2d}) = \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$, recently proved by [5], does not imply a positive answer to this last question: there could be effective divisors on $\mathcal{F}_{2d}$ that are linearly equivalent to some combination of irreducible Noether–Lefschetz divisors, but with some of the coefficients necessarily negative.

Also note that if $\mathrm{Eff}^{\mathrm{NL}}(\mathcal{F}_{2d}) \neq \mathrm{Eff}(\mathcal{F}_{2d})$, then there are modular forms on $\mathcal{F}_{2d}$ with a vanishing locus containing prime divisors that are not Noether–Lefschetz divisors.

4.5.1 Results

We have done computer calculations to determine the structure of the Noether–Lefschetz cone for many values of d. Let us outline the steps that we took.

Algorithm 4.5.4.

(i) Choose some large but finite set of irreducible Noether–Lefschetz divisors. (Possibly look at earlier results when determining this set: on the one hand, we want to take enough elements to have some confidence that we do not miss any generators; on the other hand, we would like to avoid an unnecessarily high number of redundant elements that only lengthen the computation. In practice, we took for every $\delta \in (\mathbb{Z}/2d\mathbb{Z})/\pm$ the 4 divisors $P_{\Delta,\delta}$ with lowest possible Δ; this seems to be on the safe side.)

(ii) Write each of these irreducible divisors $P_{\Delta,\delta}$ as a linear combination of reducible Noether–Lefschetz divisors $H(\gamma, n)$, using the triangular relations of section 4.2.2.

(iii) Rewrite the resulting expressions in terms of a chosen basis of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$, using the relations among divisors we found in section 4.4. This gives a finite set of points in the finite-dimensional $\mathbb{Q}$-vector space $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$.

(iv) Compute the cone generated by these points, as a rational convex cone inside $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$.

Our experiments indicate that this cone is generated by a relatively small number of irreducible Noether–Lefschetz divisors, all of small discriminant Δ. See table 4.3.

As the polarisation degree $2d$ increases, the dimension of the (rational) Picard group grows, and the Noether–Lefschetz cone inside this space becomes more and more complicated. The number of bounding hyperplanes, which gives an indication of the complexity of the structure of the cone, grows to unmanageable numbers: ten of thousands for d around 30, and at least hundreds of thousands for $d = 40$ – we did not finish this last computation, because the amount of computer memory needed became outrageous.

Table 4.3: Noether–Lefschetz cone of $\mathcal{F}_{2d}$

d	dimension of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$	number of generators of cone	number of bounding hyperplanes of cone	generators
1	2	2	2	$P_{4,\overline{0}}, P_{1,\overline{1}}$
2	3	3	3	$P_{8,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}$
3	4	4	4	$P_{12,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}$
4	4	5	6	$P_{16,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}$
5	6	6	6	$P_{20,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{5,\overline{5}}$
6	7	7	7	$P_{24,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{1,\overline{5}}, P_{12,\overline{6}}$
7	7	8	12	$P_{28,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{8,\overline{6}}, P_{21,\overline{7}}$
8	8	10	26	$P_{32,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{4,\overline{6}}, P_{17,\overline{7}}, P_{32,\overline{8}}, P_{33,\overline{1}}$
9	9	13	56	$P_{36,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{13,\overline{7}}, P_{28,\overline{8}}, P_{9,\overline{9}}, P_{37,\overline{1}}, P_{40,\overline{2}}, P_{45,\overline{9}}$
10	10	11	28	$P_{40,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{9,\overline{7}}, P_{24,\overline{8}}, P_{1,\overline{9}}, P_{20,\overline{10}}$
11	11	16	158	$P_{44,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{5,\overline{7}}, P_{20,\overline{8}}, P_{37,\overline{9}}, P_{12,\overline{10}}, P_{33,\overline{11}}, P_{45,\overline{1}}, P_{48,\overline{2}}, P_{53,\overline{3}}, P_{49,\overline{7}}$
12	12	15	106	$P_{48,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{1,\overline{7}}, P_{16,\overline{8}}, P_{33,\overline{9}}, P_{4,\overline{10}}, P_{25,\overline{11}}, P_{48,\overline{12}}, P_{49,\overline{1}}, P_{49,\overline{7}}$
13	12	15	94	$P_{52,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{49,\overline{7}}, P_{12,\overline{8}}, P_{29,\overline{9}}, P_{48,\overline{10}}, P_{17,\overline{11}}, P_{40,\overline{12}}, P_{13,\overline{13}}, P_{53,\overline{1}}$
14	14	18	208	$P_{56,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{49,\overline{7}}, P_{8,\overline{8}}, P_{25,\overline{9}}, P_{44,\overline{10}}, P_{9,\overline{11}}, P_{32,\overline{12}}, P_{1,\overline{13}}, P_{28,\overline{14}}, P_{57,\overline{1}}, P_{57,\overline{13}}, P_{84,\overline{14}}$
15	15	20	324	$P_{60,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{49,\overline{7}}, P_{4,\overline{8}}, P_{21,\overline{9}}, P_{40,\overline{10}}, P_{1,\overline{11}}, P_{24,\overline{12}}, P_{49,\overline{13}}, P_{16,\overline{14}}, P_{45,\overline{15}}, P_{61,\overline{1}}, P_{64,\overline{2}}, P_{64,\overline{8}}, P_{61,\overline{11}}$
16	14	20	514	$P_{64,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{49,\overline{7}}, P_{64,\overline{8}}, P_{17,\overline{9}}, P_{36,\overline{10}}, P_{57,\overline{11}}, P_{16,\overline{12}}, P_{41,\overline{13}}, P_{4,\overline{14}}, P_{33,\overline{15}}, P_{64,\overline{16}}, P_{65,\overline{1}}, P_{68,\overline{2}}, P_{68,\overline{14}}$
17	16	23	1104	$P_{68,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{49,\overline{7}}, P_{64,\overline{8}}, P_{13,\overline{9}}, P_{32,\overline{10}}, P_{53,\overline{11}}, P_{8,\overline{12}}, P_{33,\overline{13}}, P_{60,\overline{14}}, P_{21,\overline{15}}, P_{52,\overline{16}}, P_{17,\overline{17}}, P_{69,\overline{1}}, P_{72,\overline{2}}, P_{77,\overline{3}}, P_{76,\overline{12}}, P_{85,\overline{17}}$
18	17	25	1276	$P_{72,\overline{0}}, P_{1,\overline{1}}, P_{4,\overline{2}}, P_{9,\overline{3}}, P_{16,\overline{4}}, P_{25,\overline{5}}, P_{36,\overline{6}}, P_{49,\overline{7}}, P_{64,\overline{8}}, P_{9,\overline{9}}, P_{28,\overline{10}}, P_{49,\overline{11}}, P_{72,\overline{12}}, P_{25,\overline{13}}, P_{52,\overline{14}}, P_{9,\overline{15}}, P_{40,\overline{16}}, P_{1,\overline{17}}, P_{36,\overline{18}}, P_{73,\overline{1}}, P_{76,\overline{2}}, P_{88,\overline{4}}, P_{81,\overline{9}}, P_{73,\overline{17}}, P_{108,\overline{18}}$

4.6 Computing the canonical class of $\mathcal{F}_{2d}$

There is an explicit formula for the canonical class K of $\overline{\mathcal{F}_{2d}}$. Because $\overline{\mathcal{F}_{2d}}$ is singular, we should be careful and restrict ourselves to the regular part. The formula is

$$K = 19\lambda - 1/2 \cdot [B] - [\Delta] \ , \tag{4.34}$$

where B is the branch divisor of the quotient map $\mathcal{D}_{2d} \to \mathcal{F}_{2d}$, and $\Delta = \overline{\mathcal{F}_{2d}} \setminus \mathcal{F}_{2d} = \sum_i \Delta_i$ is the boundary divisor.

We want to know whether $\mathcal{F}_{2d}$ is of general type. More precisely, as $\mathcal{F}_{2d}$ is neither smooth nor projective, we take a smooth projective birational model of $\mathcal{F}_{2d}$, and ask the same question of that model.

If $\mathcal{F}_{2d}$ were smooth and projective, being of general type would correspond to K being a big divisor; since the class λ is ample, that is the case if and only if $K - \varepsilon\lambda$ is $\mathbb{Q}$-effective for some positive ε.

The results of [18] show that the singularities of $\mathcal{F}_{2d}$, and even of a toroidal compactification $\overline{\mathcal{F}_{2d}}$ (if we choose it in the right way), are canonical. This allows our naive formulation, using ampleness and effectiveness of divisor classes, to go through.

Our strategy will be to rewrite $K - \varepsilon\lambda$ in terms of only Noether–Lefschetz divisors $H(\gamma, n)$ and boundary divisors Δ_i, and then compare the resulting expression to our supply of Noether–Lefschetz divisors on $\mathcal{F}_{2d}$ (which are all effective by construction).

The hardest part of this is the computation of what happens at the boundary (concretely, computing boundary coefficients of relations among Noether–Lefschetz divisors). As a first approximation, we can try to ignore that hard part, and restrict to the open part $\mathcal{F}_{2d}$. This might seem like a silly thing to do: for the moduli space of curves, the Picard group of the open part has rank 1, so in that case all essential information (the ratio between the λ-coefficient and the boundary coefficients) is lost by restricting to the open part. In our case, though, the Picard group of the open part is (much) bigger (how big exactly depends on the polarisation degree), so more information may be preserved. In fact, when looking over the results (see section 4.7) of this approximation – ignoring the boundary completely – it seems that we retain enough information to see whether the Kodaira dimension is $-\infty$ or not (although we need boundary calculations to make some of the cases rigorous).

But let us not get ahead of ourselves. We first need to compute the branch divisor B.

4.6.1 Computation of the branch divisor

In [20, section 2], the branch divisor B is computed: a formula is given for the number of components, and the proofs in fact indicate a procedure to determine exactly which components occur. We will formulate these results, and do a few example computations for low polarisation degree.

Proposition 4.6.1 ([20])**.** *The ramification divisor of the map $\mathcal{D}_{2d} \to \mathcal{F}_{2d}$ is given by*

$$\bigcup_{\pm r \in S/\pm} \mathcal{D}_r \ , \tag{4.35}$$

where $\mathcal{D}_r = r^\perp \subset \mathcal{D}_{2d}$ is the subvariety associated to the vector r, and $S = S_1 \cup S_2$ is a set of vectors given by

$$\begin{aligned} S_1 &= \{r \in L : r^2 = -2\} \ , \\ S_2 &= \{r \in L : r^2 = -2d, r \text{ is primitive, and } \operatorname{div}(r) \in \{d, 2d\}\} \ . \end{aligned} \tag{4.36}$$

Note that, in order to get the branch divisor B, we need to take the quotient of the ramification divisor under the action of the group $\widetilde{\mathrm{O}}^+(L)$. So, we need to know how S_1 and S_2 decompose in orbits of this action. This can also be found in [20]:

Proposition 4.6.2 ([20])**.**

(i) The number of $\widetilde{\mathrm{O}}^+(L)$-orbits in S_1 is

$$\begin{cases} 2, & \text{if } d \equiv 1 \mod 4 \\ 1, & \text{if } d \not\equiv 1 \mod 4 \end{cases} . \tag{4.37}$$

(ii) *For $d > 1$, the number of $\widetilde{\mathrm{O}}^+(L)$-orbits in S_2 with $\mathrm{div}(r) = 2d$ is $2^{\rho(d)}$, where $\rho(d)$ is the number of distinct prime divisors of d. The number of $\widetilde{\mathrm{O}}^+(L)$-orbits in S_2 with $\mathrm{div}(r) = d$ is equal to*

$$\begin{cases} 2^{\rho(d)}, & \text{if } d \text{ is odd or } d \equiv 4 \mod 8 \\ 2^{\rho(d)+1}, & \text{if } d \equiv 0 \mod 8 \\ 2^{\rho(d)-1}, & \text{if } d \equiv 2 \mod 8 \end{cases} . \tag{4.38}$$

Let us calculate the branch divisor B in a few cases.

Example 4.6.3. First take $d = 1$. In this case all terms of the ramification divisor come from -2-vectors. By proposition 4.6.2, since $d \equiv 1 \mod 4$, there are two orbits of such vectors: one is represented by $e - f \in U$ (in either of the copies of the hyperbolic plane), the other by w (the generator of the non-unimodular part $\langle -2 \rangle$). So, we get two terms in the branch divisor.

The hyperspace orthogonal to $e - f$ is just the Heegner divisor $H(\bar{0}, -1)$, except that the latter has multiplicity two (since $\bar{0} = -\bar{0}$). So the contribution to the branch divisor is $1/2H(\bar{0}, -1)$.

Similarly, the hyperspace orthogonal to w, or equivalently to $w^* = w/2$ (the primitive positive multiple of w in $L^\vee$), is the Heegner divisor $H(\bar{1}, -1/4)$, except that the latter is multiplicity 2 (because $-\bar{1} = \bar{1}$). So here, the contribution is $1/2H(\bar{1}, -1/4)$.

We conclude that the branch divisor in this case is $B = 1/2H(\bar{0}, -1) + 1/2H(\bar{1}, -1/4)$.

Example 4.6.4. Next take $d = 2$. Now we get terms both from -2-vectors and from -4-vectors.

As $d \not\equiv 1 \mod 4$, there is only one orbit of -2-vectors, and we may again take $e - f$ as a representative. This gives the contribution $1/2H(\bar{0}, -1)$, again because the Heegner divisor $H(\bar{0}, -1)$ has multiplicity 2.

There are $2^{\rho(2)} = 2$ orbits of -4-vectors of divisibility $2d = 4$. To find these, first solve $x^2 \equiv 1 \mod 8$, for x modulo 4; this gives $x = \pm 1 \mod 4$. For each solution x, we need to find a vector $u \in 2U \oplus 2E_8(-1)$ such that $u^2 = (x^2 - 1)/2d$; we may take $u = e \in U$ for both. The two orbits are now represented by $2du + xw$, so $4e + w$ and $4e - w$. The corresponding primitive elements of $L^\vee$ are $e + w/4$ and $e - w/4$, giving Heegner divisors $H(\bar{1}, -1/8)$ and $H(\bar{3}, -1/8) = H(\bar{1}, -1/8)$, respectively. However, according to proposition 4.6.1, we should only take the Heegner divisor corresponding to one of these vectors, because the orbit of one is the orbit of minus the other. So, the total contribution from -4-vectors of divisibility 4 is $H(\bar{1}, -1/8)$.

Finally, as $d \equiv 2 \mod 8$, there is $2^{\rho(2)-1} = 1$ orbit of -4-vectors of divisibility $d = 2$. To find it, we solve $x^2 \equiv 1 \mod 2$ for x modulo 2, giving $x = 1 \mod 2$; next we take a u such that $u^2 = 2(x^2 - 1)/d = 0$, say $u = e$; then we get as our vector $du + xw = 2u + w$. The corresponding primitive vector of $L^\vee$ is $u + w/2$, giving rise to half the Heegner divisor $H(\bar{2}, -1/2)$ (since the latter has multiplicity two!).

All in all, the branch divisor is $B = 1/2H(\bar{0}, -1) + H(\bar{1}, -1/8) + 1/2H(\bar{2}, -1/2)$.

Example 4.6.5. Next take $d = 3$. In this case, we get terms from -2-vectors, and from -6-vectors.

As $d \not\equiv 1 \mod 4$, there is only one orbit of -2-vectors, say of $e - f$, giving the contribution $1/2H(\bar{0}, -1)$, as before.

There are $2^{\rho(3)} = 2$ orbits of -6-vectors of divisibility $2d = 6$. To find these, first solve $x^2 \equiv 1 \mod 12$, for x modulo 6; this gives $x = \pm 1 \mod 6$. As before, we may take $u = e \in U$ for both solutions. The two orbits are now represented by $2du + xw$, so $6e + w$ and $6e - w$. The corresponding primitive elements of $L^\vee$ are $e + w/6$ and $e - w/6$, giving Heegner divisors $H(\bar{1}, -1/12)$ and $H(\bar{5}, -1/12) = H(\bar{1}, -1/12)$, respectively. Again, according to proposition 4.6.1, we should only take the Heegner divisor corresponding to one of these vectors. So, the total contribution from -6-vectors of divisibility 6 is $H(\bar{1}, -1/12)$.

Finally, as d is odd, there are $2^{\rho(3)} = 2$ orbits of -6-vectors of divisibility $d = 3$. To find them, we solve $x^2 \equiv 1 \mod 3$ for x modulo 3, giving $x = \pm 1 \mod 3$; next we take a u such that $u^2 = 2(x^2 - 1)/d = 0$, say, once more, $u = e$; then we get as our vectors $du + xw = 3u \pm w$. The corresponding primitive vectors of $L^\vee$ are $u + w/3$ and $u - w/3$; the first gives a contribution $H(\bar{2}, -1/3)$, and the second $H(\bar{4}, -1/3)$. The opposite of one of these vectors is in the orbit of the other, so we should take only one of these.

All in all, the branch divisor is $B = 1/2H(\bar{0}, -1) + H(\bar{1}, -1/12) + H(\bar{2}, -1/3)$.

We summarise the results of these and some more cases in table 4.4.

d	B
1	$1/2\cdot H(\overline{0},-1)+1/2\cdot H(\overline{1},-1/4)$
2	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/8)+1/2\cdot H(\overline{2},-1/2)$
3	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/12)+H(\overline{2},-1/3)$
4	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/16)+H(\overline{2},-1/4)$
5	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/20)+H(\overline{2},-1/5)+1/2\cdot H(\overline{5},-1/4)$
6	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/24)+H(\overline{2},-1/6)+H(\overline{5},-1/24)$
7	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/28)+H(\overline{2},-1/7)$
8	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/32)+H(\overline{2},-1/8)+H(\overline{6},-1/8)$
9	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/36)+H(\overline{2},-1/9)+1/2\cdot H(\overline{9},-1/4)$
10	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/40)+H(\overline{2},-1/10)+H(\overline{9},-1/40)$
11	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/44)+H(\overline{2},-1/11)$
12	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/48)+H(\overline{2},-1/12)+H(\overline{7},-1/48)+H(\overline{10},-1/12)$
13	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/52)+H(\overline{2},-1/13)+1/2\cdot H(\overline{13},-1/4)$
14	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/56)+H(\overline{2},-1/14)+H(\overline{13},-1/56)$
15	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/60)+H(\overline{2},-1/15)+H(\overline{8},-1/15)+H(\overline{11},-1/60)$
16	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/64)+H(\overline{2},-1/16)+H(\overline{14},-1/16)$
17	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/68)+H(\overline{2},-1/17)+1/2\cdot H(\overline{17},-1/4)$
18	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/72)+H(\overline{2},-1/18)+H(\overline{17},-1/72)$
19	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/76)+H(\overline{2},-1/19)$
20	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/80)+H(\overline{2},-1/20)+H(\overline{18},-1/20)+H(\overline{9},-1/80)$
21	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/84)+H(\overline{2},-1/21)+H(\overline{16},-1/21)+H(\overline{13},-1/84)$ $+1/2\cdot H(\overline{21},-1/4)$
22	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/88)+H(\overline{2},-1/22)+H(\overline{21},-1/88)$
23	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/92)+H(\overline{2},-1/23)$
24	$1/2\cdot H(\overline{0},-1)+H(\overline{1},-1/96)+H(\overline{2},-1/24)+H(\overline{10},-1/24)+H(\overline{14},-1/24)$ $+H(\overline{22},-1/24)+H(\overline{17},-1/96)$

Table 4.4: The branch divisor of $\mathcal{F}_{2d}$ for some values of d.

4.6.2 Example: $d = 1$

We are now in a position to compute the canonical class K, and the important class $K - \varepsilon\lambda$, in terms of Noether–Lefschetz divisors, at least on the open part $\mathcal{F}_{2d}$. Let us take the simplest case: $d = 1$. We write K° for the restriction of K to the open part $\mathcal{F}_{2d}$.

We have seen before that $150\lambda \sim H(\overline{0},-1)+56H(\overline{1},-1/4)$, and we just saw that $B = 1/2H(\overline{0},-1)+1/2H(\overline{1},-1/4)$, so we get

$$\begin{aligned}
K^\circ &= 19\lambda - B \\
&\sim \frac{19}{150}\big(H(\overline{0},-1)+56H(\overline{1},-1/4)\big) - \big(1/2H(\overline{0},-1)+1/2H(\overline{1},-1/4)\big) \\
&= -\frac{28}{75}H(\overline{0},-1)+\frac{989}{150}H(\overline{1},-1/4) \\
&= -\frac{56}{75}P_{4,\overline{0}}+\frac{933}{75}P_{1,\overline{1}}\ ,
\end{aligned} \tag{4.39}$$

where we have used the triangular equations $H(\overline{0},-1) = 2P_{1,\overline{1}} + 2P_{4,\overline{0}}$ and $H(\overline{1},-1/4) = 2P_{1,\overline{1}}$.

Suppose we knew the answer to question 4.5.3 to be positive. Then the effective cone of $\mathcal{F}_{2\cdot 1}$ would be generated by $P_{4,\bar{0}}$ and $P_{1,\bar{1}}$, and we could conclude from the above calculation that K° is not in the effective cone of $\mathcal{F}_{2\cdot 1}$. Then a fortiori, the canonical divisor $K = K^\circ - \Delta$ would also not be effective on $\overline{\mathcal{F}_{2\cdot 1}}$, so we could conclude that the Kodaira dimension of $\mathcal{F}_{2\cdot 1}$ is $-\infty$. We know of course that this conclusion holds, by earlier results.

In this particular case, though, we arrive at this same conclusion without knowing the answer to question 4.5.3. (This is essentially a worked example of theorem 4.7.3.) The reason is that we know (by proposition 3.4.2) that the effective cone of $\mathcal{F}_{2d}$ is inside the positive half-space of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ given by the Eisenstein form $-E_{\bar{0}}$. Because

$$K^\circ \sim -\frac{28}{75} H(\bar{0}, -1) + \frac{989}{150} H(\bar{1}, -1/4) \;, \tag{4.40}$$

we may compute whether K° is in this positive half-space by computing the linear combination of coefficients

$$-\frac{28}{75} c_{\bar{0},-1}(E_{\bar{0}}) + \frac{989}{150} c_{\bar{1},-1/4}(E_{\bar{0}}) \tag{4.41}$$

of the Eisenstein series. Substituting the coefficients of the Eisenstein series that we know from example 3.4.5, we get

$$-\frac{28}{75} c_{\bar{0},-1}(E_{\bar{0}}) + \frac{989}{150} c_{\bar{1},-1/4}(E_{\bar{0}}) = -\frac{28}{75} \cdot \frac{-52377700}{174611} + \frac{989}{150} \cdot \frac{-100}{174611} = \frac{19553682}{174611} \;. \tag{4.42}$$

Now the degree $K^\circ \cdot \lambda^{18}$ of K° is a negative constant times this last number (by proposition 3.4.2), so it is negative. This implies that K° cannot be an effective divisor on $\mathcal{F}_{2d}$, so certainly $K = K^\circ - \Delta$ cannot be effective on $\overline{\mathcal{F}_{2d}}$.

4.7 Deciding effectivity of the canonical class

In the cases for which we were able to compute the Noether–Lefschetz cone completely (up to $d = 32$), we may use those results to compute whether the canonical class K° is inside or outside the cone.

However, for the cases in the interesting region – say d around 40, where we expect the Kodaira dimension of $\mathcal{F}_{2d}$ to change – it seemed not feasible to compute the Noether–Lefschetz cone in full detail. Fortunately, we do not need the full structure of the cone per se: we just need to know the relative position of the canonical class with respect to the cone, and if it is inside, we need to express the canonical class explicitly as a positive combination of (irreducible) Noether–Lefschetz divisors (as input for the calculation of the boundary coefficients).

It turns out that we may formulate this as a so-called linear programming problem. We want to write

$$K^\circ = \sum_{\Delta,\delta} t_{\Delta,\delta}[P_{\Delta,\delta}] \;, \tag{4.43}$$

where $K^\circ = 19\lambda - 1/2 \cdot [B]$ is the canonical class restricted to the open part of the moduli space, the $P_{\Delta,\delta}$ are irreducible Noether–Lefschetz divisors (see section 4.2.1), and the $t_{\Delta,\delta}$ are non-negative rational numbers.

Remark 4.7.1. We must restrict to a finite subset of the (infinite) set of irreducible Noether–Lefschetz divisors in order to get a finite problem. We may use the information gathered in our calculations of section 4.5 to guess which ones suffice to generate the full Noether–Lefschetz cone. This is of course not rigorous, and as a result we cannot conclude with certainty that a point is outside the Noether–Lefschetz cone. However, we do not even know that this cone equals the full effective cone of $\mathcal{F}_{2d}$ (see question 4.5.3), so this only adds to the uncertainty of an argument that was already incomplete.

Moreover, the results of this procedure and their perfect agreement with the results of [18] suggest that in practice no information is lost at all: see remark 4.7.2.

So, pick a finite set of irreducible Noether–Lefschetz divisors $P_{\Delta,\delta}$, and introduce corresponding variables $t_{\Delta,\delta}$. Equation (4.43) is then a set of (linear) constraints for these variables (one constraint for every coordinate on the $\mathbb{Q}$-vector space $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$). Existing programs for linear programming can solve these constraints, for non-negative values of the variables $t_{\Delta,\delta}$ – or assert that this is impossible.

We used the software package QSopt-Exact (for exact linear programming over the rationals). The results are listed in table 4.5. The entry “negative” means that the canonical class has negative degree, that is, it is not even inside the positive half-space defined by the Eisenstein series $E_{\bar{0}}$; this implies that the canonical class is outside the Noether–Lefschetz cone (as all effective divisors certainly have positive degree), but the stronger negativity statement facilitates a direct argument that the Kodaira dimension must be $-\infty$ in such a case (see theorem 4.7.3).

d	position of K°	d	position of K°
1	negative	33	outside
2	negative	34	outside
3	negative	35	outside
4	negative	36	outside
5	negative	37	outside
6	negative	38	outside
7	negative	39	outside
8	negative	40	inside
9	negative	41	outside
10	negative	42	inside
11	negative	43	inside
12	negative	44	outside
13	negative	45	outside
14	negative	46	inside
15	negative	47	outside
16	outside	48	inside
17	outside	49	inside
18	outside	50	inside
19	outside	51	
20	outside	52	inside
21	outside	53	
22	outside	54	inside
23	outside	55	inside
24	outside	56	inside
25	outside	57	
26	outside	58	
27	outside	59	
28	outside	60	
29	outside	61	
30	outside	62	
31	outside	63	
32	outside	64	

Table 4.5: The position of K° with respect to the Noether–Lefschetz cone of $\mathcal{F}_{2d}$.

Remark 4.7.2. We note one feature of these results right away: we have found that K° is inside the Noether–Lefschetz cone exactly in the cases where Gritsenko, Hulek and Sankaran have found (see [18]) that the Kodaira dimension is non-negative ($\kappa(\mathcal{F}_{2d}) \geq 0$ in some cases, $\kappa(\mathcal{F}_{2d}) = 19$ in others).

This compatibility between our results and those of [18] – obtained by different methods – can be viewed as a confirmation of both.

Interestingly, it also shows that the method of [18] is more powerful than one might expect. Recall that their method relies on the construction (as a quasi-pullback of the Borcherds form Φ_{12}) of a special modular form on $\mathcal{F}_{2d}$; it is not clear at all that the failure of this particular construction to result in a form with the wanted properties signifies non-existence of such a form. However, their special construction apparently works and manages to find a good modular form in all the cases where our results indicate that one exists.

We now discuss the implications of these results for the Kodaira dimension of $\mathcal{F}_{2d}$. We start with the lowest values of d, where K° is most negative.

Theorem 4.7.3. *If $1 \leq d \leq 15$, the moduli space $\mathcal{F}_{2d}$ has Kodaira dimension $-\infty$.*

Proof. Let us assume that the Kodaira dimension is not $-\infty$. Then there is some positive integer m such that the multiple mK is effective on $\overline{\mathcal{F}_{2d}}$. Restricting to the open part of the moduli space, we conclude that mK° is effective on $\mathcal{F}_{2d}$. It follows that the intersection product $mK^\circ \cdot \lambda^{18}$ is positive, because λ is ample. Now, by proposition 3.4.2, the number $K^\circ \cdot \lambda^{18} = \deg(K^\circ)$ is, up to a positive constant, given by the modular form $-E_{\bar{0}}$ (seen as a function on $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$) applied to K°. However, we saw above that if d is as in the hypothesis of the theorem, then K° is not in the positive half-space determined by $E_{\bar{0}}$. This contradiction proves that $\kappa(\mathcal{F}_{2d}) = -\infty$. □

Remark 4.7.4. The use of intersection numbers on the quasi-projective variety $\mathcal{F}_{2d}$ may appear to be dubious. However, because the boundary components in the Satake compactification have very low dimension (only 0 and 1), and because λ is ample even on this compactification, we may represent the class λ^{18} – or in fact even λ^2 – on the compactification by a subvariety that is supported away from the boundary.

Most of these cases have been known for a long time by the work of Mukai ([32] [33] [34] [35] [36]), which uses the more explicit structure of the moduli space that is known in these cases. Our proof is a lot simpler, using only coefficients of Eisenstein series and computation of the branch divisor. Moreover, the cases $d \in \{13, 14\}$ are new.

We now turn to the intermediate values of d where K° is inside the positive half-space, but outside the Noether–Lefschetz cone.

Theorem 4.7.5. *Let d be such that $16 \leq d \leq 39$ or $d \in \{41, 44, 45, 47\}$. If the effective cone of $\mathcal{F}_{2d}$ is generated by irreducible Noether–Lefschetz divisors and our list of generators is complete (see questions 4.5.2 and 4.5.3), then $\kappa(\mathcal{F}_{2d}) = -\infty$.*

Proof. We have computed (see table 4.5) that for these values of d, the open part K° of the canonical class cannot be written as a non-negative combination of the supposedly generating Noether–Lefschetz divisors. By assumption these divisors indeed generate the effective cone, so we conclude that K° is not effective. Then $K = K^\circ - \Delta$ is definitely not effective, so $\kappa(\mathcal{F}_{2d}) = -\infty$. □

An unconditional proof that $\kappa(\mathcal{F}_{2d}) = -\infty$ in these cases thus needs a positive answer to question 4.5.3. It is suggestive that these values of d (together with the cases $1 \leq d \leq 15$) are exactly the ones for which the alternative approach of [18], which aims to prove that $\kappa(\mathcal{F}_{2d}) \geq 0$, fails.

We may formulate this result positively as follows:

Theorem 4.7.6. *Let d be such that $16 \leq d \leq 39$ or $d \in \{41, 44, 45, 47\}$. Either $\kappa(\mathcal{F}_{2d}) = -\infty$, or there exists an irreducible codimension 1 subvariety of $\mathcal{F}_{2d}$ that is not a Noether–Lefschetz divisor.*

Finally, we turn to the cases where K° is inside the Noether–Lefschetz cone.

We can make a further distinction, by looking at the expression $K^\circ - \varepsilon\lambda$, where ε is some non-negative rational number. Recall that the canonical divisor K is big if and only if $K - \varepsilon\lambda$ is effective for some positive number ε. So, if that is the case, then its restriction to the open part $\mathcal{F}_{2d}$, which is $K^\circ - \varepsilon\lambda$, must also be effective. Under the assumption that the answers to questions 4.5.3 and 4.5.2 are positive, we may compute whether this is possible. We simply extend the linear programming problem described by equation (4.43) by adding another variable ε, and changing the equations to

$$K^\circ - \varepsilon\lambda = \sum_{\Delta,\delta} t_{\Delta,\delta}[P_{\Delta,\delta}] \,. \tag{4.44}$$

We instruct the linear solver to minimise the solution with respect to the value of the variable ε.

The full results of this procedure, in the form of equations for $K^\circ - \varepsilon\lambda$ as a sum of irreducible Noether–Lefschetz divisors $P_{\Delta,\delta}$, are listed in appendix B. Moreover, we also converted the resulting equations by rewriting the irreducible divisors $P_{\Delta,\delta}$ in terms of their reducible counterparts $H(\gamma, n)$.

Remark 4.7.7. It is interesting to note that the expression of $K^\circ - \varepsilon\lambda$ in terms of reducible Noether–Lefschetz divisors $H(\gamma, n)$ does not include any negative coefficients. This is somewhat surprising, because we have seen before that – at least for some small values of d – the cone generated by the positive combinations of the irreducible divisors $P_{\Delta,\delta}$ is strictly bigger than the one generated by the reducible ones $H(\gamma, n)$. Apparently, either this difference does not occur for larger values of d, or at least the position of our specific element $K^\circ - \varepsilon\lambda$ is insensitive to the difference.

We list in table 4.6 the most important part of the results: the minimal value of ε among solutions of equation (4.44).

Now, let us consider what these data mean for the Kodaira dimension of $\mathcal{F}_{2d}$.

Table 4.6: The minimal value of ε among solutions of equation (4.44).

d	40	42	43	46	48	49	50	51	52	53	54	55	56	57	58	59	60	61
ε	0	0	0	1	0	0	1		1		1	0	0					

Theorem 4.7.8. *Let d be one of $\{40, 42, 43, 48, 49, 55, 56\}$. If the effective cone of $\mathcal{F}_{2d}$ is generated by irreducible Noether–Lefschetz divisors (and our list of generators is complete), then the Kodaira dimension of $\mathcal{F}_{2d}$ satisfies $\kappa(\mathcal{F}_{2d}) < 19$.*

Proof. We see from table 4.6 that for these values of d, equation (4.44) can only be solved for $\varepsilon = 0$; therefore, by the assumption on the effective cone, $K^\circ - \varepsilon\lambda$ cannot be effective for positive ε. In such a case K cannot be big, so the Kodaira dimension of $\mathcal{F}_{2d}$ is less than 19. □

Again, we may formulate this result positively as follows:

Theorem 4.7.9. *Let $d \in \{40, 42, 43, 48, 49, 55, 56\}$. Either $\kappa(\mathcal{F}_{2d}) < 19$, or there exists an irreducible codimension 1 subvariety of $\mathcal{F}_{2d}$ that is not a Noether–Lefschetz divisor.*

Remark 4.7.10. In fact, for these values of d we can also prove unconditionally that $\kappa(\mathcal{F}_{2d}) \geq 0$, but for that we need to consider the boundary of the moduli space; see theorem 5.5.3.

The agreement with the results of [18] is again striking: these d are exactly the values for which their method only proves that $\kappa(\mathcal{F}_{2d}) \geq 0$.

For the other values of d, the ones giving a solution to (4.44) with positive ε, the divisor $K - \varepsilon\lambda$ on $\overline{\mathcal{F}_{2d}}$ stands a chance of being effective, which would mean that $\kappa(\mathcal{F}_{2d}) = 19$. However, in order to prove this we need to take the boundary of the moduli space into account. Specifically, we would like to take the expression of $K^\circ - \varepsilon\lambda$ as a combination of Heegner divisors, and extend it to a valid relation in the Picard group of $\overline{\mathcal{F}_{2d}}$. In the next chapter, we show how to do this, by computing boundary coefficients of relations among Noether–Lefschetz divisors (see theorem 5.3.3); in theorem 5.5.1 we apply this to prove that $\mathcal{F}_{2d}$ is of general type for the relevant values of d.

Chapter 5

The moduli space of polarised K3 surfaces: boundary

The moduli space of polarised K3 surfaces, like most moduli spaces, is not complete ('compact' in the complex-analytical language). Therefore, we will need to consider compactifications of our moduli space, even to answer some questions about the moduli space itself.

We will use two compactifications of $\mathcal{F}_{2d}$: the Satake compactification, and toroidal compactifications.

The Satake compactification is easy to define in terms of the lattice L_{2d}. However, it is not so well-behaved from a geometric point of view; in particular, it is very singular.

Toroidal compactifications resolve most of the singularities of the Satake compactification, but they are less natural, and somewhat harder to work with.

5.1 Satake compactification

The Satake compactification $\mathcal{F}_{2d}^*$, also called Baily–Borel compactification, was introduced by Baily and Borel [2]; see also [22, section 5.2]. It adds only 0- and 1-dimensional components, called cusps.

Definition 5.1.1. We may define the Satake compactification abstractly as the projective variety associated to the graded ring of modular forms associated to the lattice L

$$\mathcal{F}_{2d}^* = \operatorname{Proj}\left(\bigoplus_k M(k, \tilde{\mathrm{O}}^+(L))\right), \tag{5.1}$$

where $M(k, \tilde{\mathrm{O}}^+(L))$ is the space of modular forms on the period domain $\mathcal{D}_{2d}$ with respect to the group $\tilde{\mathrm{O}}^+(L)$.

A more explicit description can be derived from the description of the moduli space as an arithmetic quotient. Among other things, this gives a convenient description of the cusps in terms of the lattice L:

Proposition 5.1.2. *The k-dimensional cusps correspond bijectively to the orbits of the set of $(k+1)$-dimensional isotropic subspaces of $L \otimes \mathbb{Q}$ under the action of the group $\tilde{\mathrm{O}}^+(L)$.*

Given this proposition, it is clear why the boundary components all have dimension 0 or 1: since the signature of L is $(2, 19)$, its non-trivial isotropic subspaces have dimension 1 or 2.

Additionally, the correspondence of proposition 5.1.2 respects inclusions, in the following sense:

Lemma 5.1.3. *A given 0-cusp corresponding to the isotropic subspace $E \subset L$ of rank 1 is a limit point of the 1-cusp corresponding to the isotropic subspace $F \subset L_{2d}$ of rank 2 if and only if some $\tilde{\mathrm{O}}^+(L)$-translate of E is contained in F.*

Definition 5.1.4. We denote the set of 0-cusps by S_0 and the set of 1-cusps by S_1. Note that they depend implicitly on the parameter d.

We will now analyse the structure of the sets of 0-cusps and 1-cusps for varying d.

5.1.1 0-cusps

As proposition 5.1.2 tells us, the 0-cusps of $\mathcal{F}_{2d}^*$ correspond to orbits of 1-dimensional isotropic subspaces of $L \otimes \mathbb{Q}$ under the action of $\widetilde{\mathrm{O}}^+(L)$. Now, a 1-dimensional isotropic subspace is just an isotropic vector up to a scalar; we may as well normalise the vector, and take it to be a primitive vector of L.

Furthermore, we know (lemma 2.3.8) that the orbits of primitive vectors $v \in L$ under $\widetilde{\mathrm{O}}^+(L)$ are classified by their norm v^2 and the discriminant class $v^* + L \in D_L$. Therefore, the orbits of isotropic vectors are classified by their discriminant class alone. The only thing left to determine is: which discriminant classes arise as the class of v^* for some isotropic vector v?

Proposition 5.1.5 ([44, Lemma 4.1.1])**.** *Let $\gamma \in D_L$. There exists an isotropic vector $v \in L$ with $\gamma = v^* + L$ if and only if $\gamma^2/2 = 0 \in \mathbb{Q}/\mathbb{Z}$.*

Proof. If $\gamma = v^* + L$, then $\gamma^2/2 = (v^*)^2/2 + \mathbb{Z} = (v/\operatorname{div}(v))^2/2 + \mathbb{Z} = v^2/2\operatorname{div}(v)^2 + \mathbb{Z} = 0 + \mathbb{Z}$.

On the other hand, suppose that $\gamma^2/2 = 0 + \mathbb{Z}$. Represent γ by any $x \in L^\vee$. We want to construct a different representative v^* of γ, for some vector $v \in L$ with $v^2 = 0$.

As a first step, we strip off the part of x that lies in a hyperbolic plane: recalling that $L = 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle = U \oplus M$, write $x = u \oplus r$, with $u \in U^\vee$ and $r \in M^\vee$. Because U is unimodular, $U^\vee = U$, so $\gamma = x + L = r + L$.

Next, we put back a vector in the first copy of U, in order to get the right norm, and to ensure primitivity. We know that $0 + \mathbb{Z} = \gamma^2/2 = r^2/2 + \mathbb{Z}$, so $r^2 \in 2\mathbb{Z}$, say $r^2 = 2N$, with $N \in \mathbb{Z}$. Take $u' = e - Nf \in U$; then $(u')^2 = -2N$, so if we let $y = u' + r$, then $y^2 = -2N + 2N = 0$. Moreover, because u' is already primitive in $U^\vee = U$, definitely y will be primitive in $L^\vee$. Therefore, if we let v be the unique multiple of y that is primitive in L, then $v^* = y$, and of course v^2 is a multiple of $y^2 = 0$, so $v^2 = 0$.

Finally, note that $\gamma = x + L = r + L = y + L$, so we have found a primitive isotropic $v \in L$ such that $\gamma = v^* + L$. □

This proposition gives us an easy formula to count the number $\#S_0$ of 0-cusps for a given polarisation degree $2d$: this number equals

$$\#\left\{\gamma \in D_L : \gamma^2/2 = 0 \in \mathbb{Q}/\mathbb{Z}\right\} = \#\left\{k \in \{0, \dots, 2d-1\} : 4d \mid k^2\right\} . \tag{5.2}$$

Note that this is exactly the term α_4 from theorem 4.2.10. In the formula stated in that theorem, it occurs as the codimension of the space $S(d)$ of cusp forms in the space $M(d)$ of all modular forms.

Let us list this number for some low values of d.

Table 5.1: The number of 0-cusps $\#S_0$ in $\mathcal{F}_{2d}^*$.

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\#S_0$	1	1	1	2	1	1	1	2	2	1	1	2	1	1

In the range that interests us, say $1 \leq d \leq 61$, the maximum number that occurs is 4.

5.1.2 1-cusps

By proposition 5.1.2, the 1-cusps of $\mathcal{F}_{2d}^*$ correspond to orbits of 2-dimensional isotropic subspaces of $L \otimes \mathbb{Q}$ under the action of $\widetilde{\mathrm{O}}^+(L)$.

Example 5.1.6. There is an obvious choice of isotropic plane, which we can make for every degree $2d$: take the plane spanned by the two vectors e (one in each of the two copies of U). We will call the associated cusp the standard 1-cusp.

Unlike the number of 0-cusps, which grows sublinearly in d, the number of 1-cusps grows very fast, and the set of 1-cusps is hard to compute exactly for larger d. Scattone [44] has computed it for $d = 1$ (giving 4 1-cusps) and for $d = 2$ (giving 9 1-cusps); he also found some bounds for the size of this set, showing the asymptotics:

Theorem 5.1.7 ([44])**.** *The number of* 1*-cusps in $\mathcal{F}_{2d}^*$ is at least the number of lattices in the genus of L ([44, Theorem 5.0.2]). This latter number grows as d^8 ([44, Example 3.4.2]).*

Imprimitivity

Recall that orbits of primitive isotropic vectors in L are classified by their discriminant class. This class, an element of the discriminant group D_L, measures how far the vector is from being primitive in $L^\vee$ as well. Something similar happens for isotropic planes in L.

Definition 5.1.8 ([44])**.** Let I be an isotropic plane in L. Define $H_I = (I^\perp_{L^\vee})^\perp_{L^\vee}/I$; if $H_I \cong \mathbb{Z}/N\mathbb{Z}$, then the positive integer N is by definition the imprimitivity of I.

Note that the group H_I is naturally a subgroup of the discriminant group $L^\vee/L = D_L$, so, because $D_L \cong \mathbb{Z}/2d\mathbb{Z}$ is cyclic, the subgroup H_I is cyclic as well; thus the definition makes sense. Moreover, because the elements of H_I are isotropic elements of the discriminant module D_L, we can conclude that N^2 divides d. (Proof: viewing H_I as a subgroup of $D_L \cong \mathbb{Z}/2d\mathbb{Z}$ of cardinality N, we write $\gamma = k + 2d\mathbb{Z}$ for the generator of H_I, where $k = 2d/N \in \mathbb{N}_+$. Then γ is isotropic if and only if $\gamma^2/2 = k^2/4d = 0 \in \mathbb{Q}/\mathbb{Z}$ if and only if $(2d/N)^2/4d = 0 \in \mathbb{Q}/\mathbb{Z}$ if and only if $d/N^2 = 0 \in \mathbb{Q}/\mathbb{Z}$ if and only if N^2 divides d.)

Also note that if σ is a stable orthogonal transformation of L, then by definition σ acts as the identity on D_L, so $H_{\sigma I} = H_I$, and in particular σI has the same imprimitivity as I. In view of proposition 5.1.2, this means that the subgroup H_I and the imprimitivity index N are in fact properties of the cusp, not just its representative I:

Definition 5.1.9. If F is a 1-cusp represented by an isotropic plane I, then we set $H_F = H_I$, and the imprimitivity N_F of F is by definition the imprimitivity N of I.

(The invariant N is called e in [44], but that letter is already overloaded in our context.)

From a lattice point of view, the imprimitivity measures the failure of the embedding $I \subset L$ to be isomodular (roughly, isomodularity means preserving discriminant classes; see section 2.2.1).

Example 5.1.10. Let N be such that N^2 divides d. We construct an isotropic plane I_N in L of imprimitivity N.

Because N divides $2d$, we can take $k = 2d/N \in \mathbb{N}_+$, so that $k + 2d\mathbb{Z}$ is an element of $\mathbb{Z}/2d\mathbb{Z}$ of order N. Also write $m = d/N^2$, a positive integer. Now define

$$z = w + Ne + Nmf \in L \ . \tag{5.3}$$

Note that $z^2 = -2d + 2N^2m = -2d + 2d = 0$, so z is isotropic; also z is clearly primitive in L. Further, remark that the divisor of z is exactly N, so

$$z^* = \frac{1}{N}w + e + mf \in L^\vee \ , \tag{5.4}$$

from which we see that the discriminant class of z^* is $2d/N + 2d\mathbb{Z} = k + 2d\mathbb{Z}$.

Now, the isotropic plane I_N we take is the one generated by this $z \in L$ and the vector e_2 (this is just e in the other copy of the hyperbolic plane U).

We claim that I_N has imprimitivity N. To see this, note that the double orthogonal complement $I_N^{\perp\perp}$ is just $\mathbb{Z}w + \mathbb{Z}e_2 = I_N$ if taken in L; if we take these complements in $L^\vee$ instead, we get $((I_N)^\perp_{L^\vee})^\perp_{L^\vee} = \mathbb{Z}w/N + \mathbb{Z}e_2$, so $H_{I_N} = ((I_N)^\perp_{L^\vee})^\perp_{L^\vee}/I_N = (\mathbb{Z}w/N + \mathbb{Z}e_2)/(\mathbb{Z}w + \mathbb{Z}e_2) \cong \mathbb{Z}/N\mathbb{Z}$, as claimed.

The set of 1-cusps thus decomposes into sets of cusps of the same imprimitivity N, where N ranges over all positive integers such that N^2 divides d. The above example shows that each of these sets is non-empty.

Associated definite lattices

Given a 1-cusp F, represented by a rank 2 isotropic sublattice I of L, we can take the orthogonal complement $I^\perp$ of I (inside L). Because I is isotropic, it is itself contained in its orthogonal complement.

Definition 5.1.11. Let a 1-cusp F be given, represented by the isotropic plane $I \subset L$. The associated definite lattice is the quotient $K = K(F) = K(I) = I^\perp/I$.

Note that the quotient is a non-degenerate lattice of signature $(2-2, 19-2) = (0, 17)$, so it is indeed (negative) definite.

This associated definite lattice $K(F)$ turns out to capture some essential properties of the 1-cusp F. In particular, our computations of coefficients of relations in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{F}_{2d}})$ at a given 1-cusp F will only depend on the lattice $K(F)$.

Example 5.1.12. If F is the standard 1-cusp, the associated definite lattice is $K_0 = 2E_8(-1) \oplus \langle -2d \rangle$.

The genus of the definite lattice $K(F)$ is not the same for all 1-cusps F. However, the genus only depends on the imprimitivity invariant N of the cusp. This is not so strange from a lattice point of view: the genus of an even lattice is captured by the signature and the discriminant module; since the signature of K is fixed, only the discriminant module matters, and the discriminant module of the quotient $I^\perp/I$ is directly related to the failure of isomodularity (see definition 2.2.11) of the sublattice $I \subset L$, in the spirit of proposition 2.2.13.

The precise statement is as follows:

Proposition 5.1.13 ([44, Lemma 5.1.3])**.** *If the isotropic plane $I \subset L$ has imprimitivity N, then the discriminant module of $K(F) = I^\perp/I$ is isomorphic to $\mathbb{Z}/2m\mathbb{Z}$, where $m = d/N^2$ is as in example 5.1.10.*

We will later need the following explicit version of the above statement, giving the relation between the groups $\mathbb{Z}/2d\mathbb{Z}$ and $\mathbb{Z}/2m\mathbb{Z}$. The discriminant module $D_K \cong \mathbb{Z}/2m\mathbb{Z}$ of the subquotient lattice $K(F) = I^\perp/I$ of L is naturally a subquotient of the discriminant module D_L: it is $H_F^\perp/H_F$ (see below).

Definition 5.1.14. The subgroup $H_F^\perp \subseteq D_L$ is the subgroup orthogonal to H_F (recall from definition 5.1.9 that $H_F \subset D_L$ is the isotropic subgroup associated to the cusp F):

$$H_F^\perp = \{\gamma \in D_L : \text{for all } \delta \in H_F : (\gamma, \delta) = 0\} \ . \tag{5.5}$$

We define $p : H_F^\perp \to D_K$ to be the surjective map derived from the isomorphism $D_K \cong H_F^\perp/H_F$.

For example, if the cusp F has imprimitivity $N = 1$ (this is the case for the standard cusp, for instance), then $H_F = \{\bar{0}\}$, so $H_F^\perp = D_L$, and the map $p : D_L \to D_K$ is an isomorphism of discriminant modules.

Now, proposition 5.1.13 implies that if F has imprimitivity N, then $K(F) \in \mathcal{G}(K(I_N)) = \mathcal{G}(\langle -2m \rangle \oplus 2E_8(-1))$.

It is not clear if every definite lattice in the genus of $K(I_N)$ is obtained in this way. It is true under the assumption that $\gcd(N, 2m) = 1$: see [44, Section 5.4].

The relevance of this for our applications is the following: when considering the vanishing behaviour of modular forms at the 1-cusps, instead of enumerating all 1-cusps, it suffices to enumerate the lattices in the genera $\mathcal{G}(K(I_N))$ instead, for all N such that N^2 divides d.

Scattone gives an algorithm to compute this list of lattices $\mathcal{G}(K(I_N)) = \mathcal{G}(\langle -2m \rangle \oplus 2E_8(-1))$.

Algorithm 5.1.15 ([44, Proposition 6.1.2])**.** Choose a primitive vector $v \in E_8(-1)$ of length $-2m$. Compute $C = v^\perp$, the orthogonal complement of v in $E_8(-1)$; this is a rank 7 negative definite lattice.

Compute the set of equivalence classes of all possible embeddings of the lattice C in a unimodular negative definite lattice T of rank 24. (Note that such lattices are classified up to isomorphism; this is Niemeier's list.) For every such embedding, compute $K = C_T^\perp$, the orthogonal complement of C in T.

The set of lattices K obtained in this way is exactly the genus $\mathcal{G}(\langle -2m \rangle \oplus 2E_8(-1))$.

This algorithm is not very practical for large values of m. In fact, the size of the genus grows quite rapidly as m increases, so any procedure that needs to deal with each of the lattices in the genus separately will be prohibitively expensive for large values of m.

However, let us apply the algorithm to the simplest case, $m = 1$, to get a feeling for the type of lattices that occur.

Example 5.1.16. Take $m = 1$. We want to compute the genus $\mathcal{G}(\langle -2 \rangle \oplus 2E_8(-1))$. We apply the above algorithm. It is a standard fact that the complement N in E_8 of a vector of length 2 is always isomorphic to E_7. Going through Niemeier's list of unimodular definite lattices of rank 24, we see that four of them allow an embedding of E_7: in the notation of [12], these are the lattices called β, γ, ζ, and η. In each of these four cases, the embedding of E_7 is unique up to isomorphism. Computing the orthogonal complement in all cases, we get four lattices M that make up the genus in this case: β gives $K_\beta = D_{16}^+ \oplus \langle -2 \rangle$; γ gives $K_\gamma = 2E_8 \oplus \langle -2 \rangle = K_0$; ζ gives $K_\zeta = A_{17}[6]$; η gives $K_\eta = (D_{10} \oplus E_7)[1, 1]$.

Let us see what this means for the set of 1-cusps of $\mathcal{F}_{2\cdot 1}$. First of all, note that the only possible value of N is $N=1$, so there is only one genus to consider, the one above, with $m=1$. However, we still need to analyse the fibres of the map that sends a 1-cusp represented by I to the element $I^{\perp}/I \in \mathcal{G}(\langle -2\rangle \oplus 2E_8(-1))$.

Clearly, if two isotropic planes I_1, I_2 are related by an automorphism $\sigma \in \tilde{\mathrm{O}}^+(L)$, then $K(I_1) \cong K(I_2)$. However, the reverse implication does not hold in general: if $K(I_1) \cong K(I_2)$, then there exists an automorphism $\tau \in \mathrm{O}^+(L)$ such that $\tau(I_1) = I_2$, but this automorphism may not lie in $\tilde{\mathrm{O}}^+(L)$.

Under some quite strong conditions on the divisibility of d, this problem disappears, and the set of 1-cusps of imprimitivity N is in fact equal to the genus $\mathcal{G}(\langle -2d/N^2\rangle \oplus 2E_8(-1))$: see [44, Corollary 5.6.10]. These conditions are satisfied for $d=1$, so the above example shows that $\mathcal{F}_{2\cdot 1}$, the moduli space of polarised K3 surfaces with $2d=2$, has exactly four 1-cusps.

Theta series

As a further simplification, note that in the calculation of boundary coefficients (see theorem 5.3.3), we use the definite lattice $K(F)$ associated to the cusp F only through its vector-valued theta series $\mathbf{\Theta}_{K(F)}$. In the above example case of $m=1$, we get four distinct definite lattices K, but only two different theta series: D_{16}^+ has the same theta series as $2E_8$, so the lattices K_β and K_γ have identical vector-valued theta series; the same holds for K_ζ and K_η.

For future use, let us compute a few terms of the vector-valued theta series of these lattices.

Example 5.1.17. Again, take $m=1$. We have

$$\mathbf{\Theta}_{K_\beta} = \mathbf{\Theta}_{K_\gamma} = \mathbf{e}_{\bar{0}}\,(1 + 482\,q + \ldots) + \mathbf{e}_{\bar{1}}(2\,q^{1/4} + \ldots)\ . \tag{5.6}$$

and

$$\mathbf{\Theta}_{K_\zeta} = \mathbf{\Theta}_{K_\eta} = \mathbf{e}_{\bar{0}}\,(1 + 306\,q + \ldots) + \mathbf{e}_{\bar{1}}(0\,q^{1/4} + \ldots)\ . \tag{5.7}$$

It would be interesting to know how many distinct vector-valued theta series the lattices in the genus $\mathcal{G}(\langle -2m\rangle \oplus 2E_8(-1))$ have, as m increases. This determines how many different classes of 1-cusps need to be dealt with separately, when doing boundary calculations for Noether–Lefschetz divisors.

In section 5.4, we try to determine what vector-valued modular forms can occur as the theta series of a definite lattice of a given genus.

5.1.3 Centrality of the standard 0-cusp

The standard 0-cusp plays a special role in the Satake compactification, because of the following fact:

Lemma 5.1.18 ([18, Lemma 4.1])**.** *The standard 0-cusp is a limit point of every 1-cusp of $\mathcal{F}_{2d}$.*

5.2 Toroidal compactification

The main disadvantage of the Satake compactification is that it is very singular. In particular, it is not suited to investigating the birational geometry of the moduli space via the ampleness of the pluricanonical bundle.

Toroidal compactifications are extensively studied in the classic book [1]. They replace the cusps by divisors, essentially blowing up the singularities of the Satake compactification. This process is not completely canonical: it needs some combinatorial data as input.

It is possible to choose these combinatorial data in such a way that the resulting toroidal compactification has only finite quotient singularities; this is proved in [1]. (This is best possible, in the sense that the open part $\mathcal{F}_{2d}$ itself may have finite quotient singularities.) Moreover, with the right choice of combinatorical data these quotient singularities are canonical singularities; this result is a core part of [18].

In this section, we will describe toroidal compactifications $\overline{\mathcal{F}_{2d}}$ of our locally symmetric domain $\mathcal{F}_{2d}$.

5.2.1 Tube domain realisation of the period space

Before we begin our description of the toroidal compactifications, we recall the tube domain model, a well-known alternative parametrisation of the period space $\mathcal{D}_{2d} \subseteq \mathbb{P}(L \otimes \mathbb{C})$ (see for instance [8, p. 79]). This parametrisation depends on a choice of 0-cusp, i.e., a 1-dimensional isotropic subspace of $L\otimes\mathbb{Q}$, together with some extra data. So fix a primitive $z \in L$ with $z^2 = 0$ (this spans a 1-dimensional isotropic subspace, corresponding to a 0-cusp), together with a $z' \in L^\vee$ such that $(z, z') = 1$.

Now, define the sublattice

$$M = L \cap z^\perp \cap z'^\perp \ , \tag{5.8}$$

of signature $(1, 18)$. Then we have the decompositions $L \otimes \mathbb{Q} = (M \otimes \mathbb{Q}) \oplus \mathbb{Q}z \oplus \mathbb{Q}z'$ and $L \otimes \mathbb{R} = (M \otimes \mathbb{R}) \oplus \mathbb{R}z \oplus \mathbb{R}z'$. Typically, we will write Z_M for an element of $M \otimes \mathbb{C}$, and X_M, Y_M for its real and imaginary parts (and similarly for L).

Next, we look at the map

$$M \otimes \mathbb{C} \to \mathbb{P}(L \otimes \mathbb{C}) \tag{5.9}$$

$$Z_M \mapsto [Z_M \oplus (-Z_M^2/2 - (z')^2/2)z \oplus z'] \ . \tag{5.10}$$

We may restrict this to a bijection $M \otimes \mathbb{R} \oplus i\mathcal{C} \to \mathcal{D}_{2d}$, where $\mathcal{C}$ is some real cone (specifically, one of the two components of $\{Y \in M \otimes \mathbb{R} : Y^2 > 0\}$).

This gives a realisation of the symmetric domain $\mathcal{D}_{2d}$ as a subset of the complex space $M_\mathbb{C}$: it is the subset of vectors such that the real part is unrestricted, and the imaginary part is restricted to lie in the cone $\mathcal{C}$. This parametrisation of $\mathcal{D}_{2d}$ is therefore called the tube domain realisation.

5.2.2 Abstract definition

Throughout this section, we follow the notation and description of [22, section 5.3]; see also [1] for more details.

We will describe the toroidal compactifications at first on the level of the period domain $\mathcal{D}_{2d}$ (i.e., before the gluing procedure, and before locally taking the quotient by the orthogonal group). As said before, we need to add divisors over each of the cusps; we will do this one cusp at a time.

So, let F be a cusp. We introduce some associated subgroups of the (real) orthogonal group of the lattice.

Definition 5.2.1. We define $N(F) \subset \tilde{\mathrm{O}}^+(L)_\mathbb{R}$ to be the stabiliser of F; also, let $W(F) \subset N(F)$ be the unipotent radical of $N(F)$, and $U(F) \subseteq W(F)$ the centre of $W(F)$.

($N(F)$ is also called the parabolic subgroup associated to the cusp.)

Now, the partial compactification of $\mathcal{D}_{2d}$ at F is taken inside the larger space $\mathcal{D}_L(F)$. This space $\mathcal{D}_L(F)$ can be abstractly defined as

$$\mathcal{D}_L(F) = U(F)_\mathbb{C}\mathcal{D}_{2d} \ , \tag{5.11}$$

where $U(F)_\mathbb{C}$ acts on the period domain $\mathcal{D}_{2d}$ within the larger space (the so-called compact dual)

$$\check{\mathcal{D}}_{2d} = \{\mathbb{C}z : (z, z) = 0\} \subset \mathbb{P}(L \otimes \mathbb{C}) \ . \tag{5.12}$$

This space $\mathcal{D}_L(F)$ has a product decomposition

$$\mathcal{D}_L(F) \cong F \times V(F) \times U(F)_\mathbb{C} \ , \tag{5.13}$$

where $V(F) = W(F)/U(F)$.

We now have an alternative description of the symmetric domain $\mathcal{D}_{2d}$, at least locally around this cusp F, as the subset $\mathcal{D}$ of $\mathcal{D}_L(F)$ defined by the condition that the imaginary part of the projection to $U(F)_\mathbb{C}$ is contained in a certain cone $C(F) \subset U(F)$.

Let us see what the group action looks like in this alternative description. We will want to look at the action of the stabiliser $N(F) \cap \tilde{\mathrm{O}}^+(L)$ on $\mathcal{D}$. However, let us first restrict to the action of the unipotent subgroup $U(F)_\mathbb{Z} = U(F) \cap \tilde{\mathrm{O}}^+(L)$. This action induces a fibre bundle $\mathcal{D}/U(F)_\mathbb{Z} \to \mathcal{D}/U(F)_\mathbb{C}$ with fibre $U(F)_\mathbb{C}/U(F)_\mathbb{Z}$, which is a torus: a product of $\dim U(F)$ copies of $\mathbb{C}^\times$. This is where the toroidal compactifications get their name from.

The compactification procedure now proceeds by replacing this torus by a toric variety $X_{\Sigma(F)}$ (see below), and accordingly modifying the map $\mathcal{D}/U(F)_{\mathbb{Z}} \to \mathcal{D}/U(F)_{\mathbb{C}}$ by replacing its fibres by this $X_{\Sigma(F)}$, and then extending $\mathcal{D}/U(F)_{\mathbb{Z}}$ to its closure in the resulting fibre map.

The toric variety $X_{\Sigma(F)}$ in this procedure is not unique; this is where the compactification process requires us to make a choice. The datum $\Sigma(F)$ describing the toric variety is a fan: a decomposition of the rational closure of the cone $C(F)$ as a union of subcones (with some required properties).

There are two further steps to take in constructing the toroidal compactifications. Firstly, we must take a further quotient of each of these unfinished partial compactifications, dividing out by the full integral stabiliser group $N(F)_{\mathbb{Z}}$, instead of just $U(F)_{\mathbb{Z}}$. This gives an actual (partial) compactification of the arithmetic quotient space (at the chosen cusp F). Secondly, we must glue together the resulting partial compactifications by taking a Hausdorff open neighbourhood of each and identifying the copies of the arithmetic quotient $\mathcal{D}_{2d}/\widetilde{\mathrm{O}}^+(L)$ in each of them. As a result, we get two conditions on the fans $\Sigma(F)$: the decomposition at every cusp F must be respected by the action of $N(F)$, and if we have two cusps, one of which is in the closure of the other, then the corresponding decompositions must be compatible in some way.

We now describe how this abstract definition unfolds in our case of the moduli space $\mathcal{F}_{2d}$ associated to the lattice $L = \langle -2d \rangle \oplus 2U \oplus 2E_8(-1)$. We start with the 1-cusps: although there are many of them, their treatment in any toroidal compactification is easy.

5.2.3 1-cusps

Suppose that the cusp F is a 1-cusp corresponding to an isotropic plane in L. The stabiliser $N(F)$, unipotent radical $W(F)$ and its centre $U(F)$ can be described explicitly with respect to a particular basis of L: see [18, section 2.3]. The subgroup $U(F)$ is 1-dimensional in this case, and with respect to that basis it looks like

$$U(F) = \left\{ \begin{pmatrix} I & 0 & \left(\begin{smallmatrix} 0 & Nx \\ -x & 0 \end{smallmatrix}\right) \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \mid x \in \mathbb{R} \right\} , \tag{5.14}$$

where N is the imprimitivity of the 1-cusp F (see section 5.1.2).

The decomposition (5.13) in this case looks like

$$\mathcal{D}_L(F) \cong \mathbb{C} \times \mathbb{C}^{17} \times \mathbb{H} ; \tag{5.15}$$

let us write s for the coordinate on the first factor $\mathbb{C}$.

The group $U(F)$ is 1-dimensional, so the torus $U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}$ is just $\mathbb{C}^\times$. A calculation shows that the element of $U(F)$ parametrised by $x \in \mathbb{R}$ acts on $\mathcal{D}_L(F)$ by increasing s by Nx (and fixing the other coordinates) – see [18, proposition 2.26]. Therefore, we choose as a coordinate on the torus $U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}} \cong \mathbb{C}^\times$ the function $u = \exp(2\pi i s/N)$; the compactification then adds the point $u = 0$.

1-cusp as a limit in the tube domain

We may describe a point on the boundary divisor in the tube domain model (see 5.2.1) as the limit of the path

$$t \mapsto Z_M = ity + \tau y' + \kappa , \tag{5.16}$$

where $\tau \in \mathbb{H}$ and $\kappa \in K \otimes \mathbb{C}$ together parametrise the point within the boundary divisor (up to the action of the arithmetic group).

The order of vanishing of a function at this particular boundary divisor can be computed by reading off the power of the local parameter u in the local description of the function:

Proposition 5.2.2. *Let Ψ be a modular form on $\mathcal{F}_{2d}$, seen as a function on the period domain $\mathcal{D}_{2d}$; write Ψ_z for Ψ as a function on the tube domain. Suppose that the value of Ψ_z on the above path is asymptotically equal to $c \cdot \exp(-2\pi\alpha t)$ for some constants $c \in \mathbb{C}$ and $\alpha \in \mathbb{Z}$. Then the vanishing order of Ψ at the boundary divisor under consideration equals $N\alpha$.*

(Recall that N is the imprimitivity of the 1-cusp F under consideration.)

Proof. We compute that the function u (the local parameter of the boundary divisor; see section 5.2.3) along this path is given by $u = \exp(2\pi i(it)/N) = \exp(-2\pi t/N)$.

Therefore the the given asymptotical value of Ψ is $c \cdot \exp(-2\pi\alpha t) = c \cdot u^{N\alpha}$, so the order of vanishing of Ψ is $N\alpha$. □

Because the group $U(F)$ is 1-dimensional, the real cone $C(F)$ is just $\mathbb{R}_+$, and for such a trivial cone there is only a single choice of fan. Therefore, as far as the toroidal boundary over the 1-cusps is concerned, we do not need to make any choices. This is different for the 0-cusps, though, as we will see below.

5.2.4 0-cusps

Suppose that the cusp F is a 0-cusp, corresponding to a primitive isotropic vector $z \in L$. Writing $M = z^\perp/\mathbb{Z}z$ as before, the stabiliser is given by the semi-direct product $N(F) = \tilde{\mathrm{O}}^+(M) \rtimes E_z(L)$, where $E_z(L)$ is the group of Eichler transvections associated to z:

Definition 5.2.3. Let L, z and M be as above. For any $m \in M$, the Eichler transvection associated to m is the map $L \to L$ given by

$$x \mapsto x - (x,z)m + (x,m)z - m^2/2 \cdot (x,z)z\ . \tag{5.17}$$

The group of all such transformations is denoted by $E_z(L)$.

In fact, this group of Eichler transvections is isomorphic to the additive group of the lattice M.

Furthermore, the unipotent radical $W(F)$ is the subgroup $E_z(L) \cong M$; as this is an abelian group, its centre $U(F)$ coincides with $W(F)$.

Now, looking at equation (5.13), we see that in this case $F \cong \{\mathrm{pt}\}$ (as F is a 0-cusp), and $V(F) = W(F)/U(F) = \{0\}$, so the space $\mathcal{D}_L(F)$ is actually isomorphic to $U(F)_\mathbb{C} \cong M \otimes \mathbb{C}$. In fact, the inclusion $\mathcal{D}_{2d} \subseteq \mathcal{D}_L(F)$ is just the tube domain realisation associated to z (see section 5.2.1), so under the isomorphism $\mathcal{D}_L(F) \cong M \otimes \mathbb{C}$, the period domain $\mathcal{D}_{2d}$ is given by the set $M \otimes \mathbb{R} \oplus iC(F)$, where $C(F)$ is a real cone in $M \otimes \mathbb{R}$. In fact, $C(F)$ is just the positive cone in $M \otimes \mathbb{R}$, given by $C(F) = \{v \in M \otimes \mathbb{R} : v^2 > 0, (z', v) > 0\}$. (Recall that z' is an element of $L \otimes \mathbb{Q}$ such that $(z, z') = 1$; here, it just serves to pick out one of the two connected components of the set of vectors of positive norm.)

The torus $T(F)$ is $U(F)_\mathbb{C}/U(F)_\mathbb{Z} \cong (M \otimes \mathbb{C})/M \cong (\mathbb{C}^\times)^{19}$. The toroidal compactifications complete this torus to a toric variety, but this now essentially depends on a choice of fan $\Sigma(F)$ (i.e., cone decomposition) of the rational closure of the cone $C(F)$, a real cone in the 19-dimensional vector space $M \otimes \mathbb{R}$.

The components of the boundary divisor over a cusp in general correspond to the rays in the cone decomposition of the rational closure of $C(F)$, up to action by the orthogonal group $\tilde{\mathrm{O}}^+(M)$ and identification in the final gluing procedure. In the case of a 0-cusp, this set of rays depends on the choice of fan $\Sigma(F)$: at the very least, it will include the boundary rays of the rational closure of $C(F)$, but there may be more (internal) rays.

Note that the boundary rays of the rational closure of the positive cone $C(F) \subset M \otimes \mathbb{R}$ are exactly the rays through the isotropic vectors of M. Such a vector, taken together with the isotropic vector $z \in L$ representing the 0-cusp under consideration, gives an isotropic plane in L, which in turns represents a 1-cusp of $\mathcal{F}_{2d}$. Moreover, the component of the boundary divisor over the 0-cusp that corresponds to this ray is identified by the gluing procedure with the boundary component over this 1-cusp (as described in section 5.2.4).

There is one somewhat natural choice of fan in this case: the perfect cone decomposition (see [1] for details). It is minimal, in the sense that the set of rays in this decomposition consists of only the boundary rays. This is useful: by the above paragraph, we see that if we choose this fan for every 0-cusp, the only boundary components that we get are the ones over 1-cusps. The main disadvantage of the perfect cone decomposition is that it gives a compactification that may have bad (i.e., non-canonical) singularities. A further subdivision of the decomposition is necessary to get rid of these singularities ([18, section 2] proves that that is possible), but this reintroduces additional boundary components that are harder to control.

One feature of our case simplifies the situation significantly: because the cone decompositions over the 1-cusps are unique, the compatibility conditions between the fans associated to a 1-cusp and 0-cusps in its closure are trivially satisfied.

Sufficient conditions for cuspidality

For our applications we want modular forms on $\mathcal{F}_{2d}$ that are cusp forms in a very strong sense: for any toroidal compactification, we want the modular form to vanish on every component of the boundary divisor. In fact, because the vanishing order need not be an integer number, we should strengthen this by demanding that the vanishing order is at least 1.

Lemma 5.2.4 ([19])**.** *If the modular form Ψ on $\mathscr{F}_{2d}$ vanishes at all cusps, then it vanishes to order at least* 1 *on every component of the boundary divisor in any toroidal compactification.*

This fact is used implicitly in [18], without proof. The essential point is that the vanishing order of a modular form on $\mathscr{F}_{2d}$ at a component of the boundary divisor is an integer number; this is proved in [21, proposition 2.1] (also see [19]). The proof relies on the fact that the arithmetic group $\widetilde{\mathrm{O}}^+(L_{2d})$ has only one non-trivial character (the determinant); see [21, corollary 1.8].

Moreover, it is enough to have cuspidality at the 1-cusps:

Proposition 5.2.5. *If the modular form Ψ on $\mathscr{F}_{2d}$ vanishes at all* 1*-cusps, then it vanishes to order at least* 1 *on every component of the boundary divisor in any toroidal compactification.*

Proof. Every 0-cusp occurs as a limit point of at least one 1-cusp, so Ψ vanishes at every 0-cusp as well by continuity. Lemma 5.2.4 gives the desired result. ☐

5.3 Extending divisor relations to the compactification

We would like to understand the rational Picard group of the toroidal compactifications $\overline{\mathscr{F}_{2d}}$ of the moduli space.

We should be careful though: depending on the choice of cone decomposition, the toroidal compactification might not be locally ($\mathbb{Q}$-)factorial, so the Picard group might be smaller than the divisor class group: some boundary components might not be ($\mathbb{Q}$-)Cartier divisors. We could deal with this by looking at the full divisor class group instead of the Picard group. However, because in the end we are only interested in (combinations of) divisors that arise as vanishing locus of modular forms, and those are Cartier, we may as well restrict to the Picard group.

We already know the rational Picard group of the open part $\mathscr{F}_{2d}$: see the previous chapter. The Picard group of the compactification is generated by the Picard group of the open part, together with (some combinations of) the boundary divisors. So the task that remains is to understand the relations in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathscr{F}_{2d}})$.

Now, a relation in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathscr{F}_{2d}})$ is by definition given by the vanishing of the divisor of a meromorphic function on $\overline{\mathscr{F}_{2d}}$. This function restricts to a meromorphic function on $\mathscr{F}_{2d}$, giving a relation in $\mathrm{Pic}_{\mathbb{Q}}(\mathscr{F}_{2d})$. So, the relations in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathscr{F}_{2d}})$ are just the relations in $\mathrm{Pic}_{\mathbb{Q}}(\mathscr{F}_{2d})$, together with the extra information of the vanishing behaviour of the corresponding functions at all the boundary components.

Remark 5.3.1. Note that in a precise sense we do not lose information by restricting relations to the open part $\mathscr{F}_{2d}$: a relation on $\overline{\mathscr{F}_{2d}}$ that becomes trivial after restriction corresponds to a meromorphic modular form on $\overline{\mathscr{F}_{2d}}$ that is holomorphic (and non-zero) on $\mathscr{F}_{2d}$. Koecher's principle implies that this modular form is in fact holomorphic on the whole of $\overline{\mathscr{F}_{2d}}$, but then it must be constant (thus in particular have weight 0), so it represents the trivial relation on $\overline{\mathscr{F}_{2d}}$ as well.

5.3.1 Weyl chambers

Given $f \in \mathrm{Sing}(M)$ (the space of Laurent polynomials), we decompose the cone $\mathcal{C} \subseteq M \otimes \mathbb{R}$ into so-called Weyl chambers W_M associated to f (see for instance [8, page 88]): these are the connected components of the set

$$\mathcal{C} \setminus \bigcup_{\substack{\beta, n \\ c_{\beta,n}(f) \neq 0}} H(\beta, n) \ . \tag{5.18}$$

Here $H(\beta, n)$ is a Heegner divisor in $M \otimes \mathbb{R}$. This also gives a decomposition of the tube domain model of the period space $\mathcal{D}_{2d} \cong M \otimes \mathbb{R} \oplus i\mathcal{C}$ into Weyl chambers W_L, as the inverse images under the imaginary part map $\mathfrak{I}$ of the Weyl chambers W_M.

5.3.2 Completing relations

To complete a given relation in $\mathrm{Pic}_{\mathbb{Q}}(\mathscr{F}_{2d})$ to a relation in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathscr{F}_{2d}})$, we will proceed as follows. Recall that the map $c_{\gamma,n} \mapsto H(\gamma, n)$ sends identities among coefficients of modular forms to linear equivalences of divisors. The function – in fact, a meromorphic modular form – on $\mathcal{F}_{2d}$ exhibiting this linear equivalence can be described quite explicitly, by the work of Borcherds. We will determine the vanishing order of this function at the cusps, and consequently compute the boundary terms of its divisor. This will give a relation in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{F}_{2d}})$.

Remark 5.3.2. In view of the discussion of section 5.2.4, for this to make sense we need to pick a specific toroidal compactification, because that choice determines the structure of the boundary divisors over the 0-cusps.

We pick the toroidal compactification determined by the perfect cone decomposition. We have seen in section 5.2.4 that in that case the irreducible components of the boundary divisor are easy to describe: there is one component for every 1-cusp, and there are no others. Let us write Δ_F for the component corresponding to the 1-cusp F.

Now, we can formulate the boundary behaviour of the modular form Ψ.

Theorem 5.3.3. *Let $\sum_{\gamma,n} a_{\gamma,n} H(\gamma, n) \sim 0$ be a linear equivalence of Noether–Lefschetz divisors on $\mathcal{F}_{2d}$. Then the following linear equivalence holds on $\overline{\mathcal{F}_{2d}}$ (the toroidal compactification of $\mathcal{F}_{2d}$ with the perfect cone decomposition):*

$$\sum_{\gamma,n} a_{\gamma,n} H(\gamma, n) + \sum_{F \in S_1} \sum_{\gamma,n} a_{\gamma,n} c(\gamma, n, N_F, K(F))\, \Delta_F \sim 0\ . \tag{5.19}$$

Here F ranges over the 1-cusps S_1 of $\mathcal{F}_{2d}$, N_F is the imprimitivity of the cusp F (see section 5.1.2), and $K(F)$ is the negative definite lattice of rank 17 associated to the cusp F (also explained in section 5.1.2). The function $c(\gamma, n, N_F, K(F))$ calculating the contribution of a given Heegner divisor $H(\gamma, n)$ at the cusp of imprimitivity N_F having definite lattice $K(F)$ is given by

$$c(\gamma, n, N, K) = \begin{cases} N/24 \cdot (E_2 \mathbf{\Theta}_K)(p(\gamma), n) & \text{if } \gamma \in H_F^{\perp} \\ 0 & \text{otherwise} \end{cases}\ , \tag{5.20}$$

where E_2 is the usual Eisenstein series, $\mathbf{\Theta}_K$ is the vector-valued theta series of the lattice K, and the subgroup $H_F^{\perp} \subseteq D_L$ and the map $p : H_F^{\perp} \to D_K$ are defined in 5.1.14.

The proof of this theorem will be given in section 5.3.4; we first look at an example application in the next section.

5.3.3 Example: completing the Hodge relation for $d = 1$

As an example, we take the Hodge relation in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2\cdot 1})$:

$$150\,\lambda \sim H(\overline{0}, -1) + 56\, H(\overline{1}, -1/4)\ . \tag{5.21}$$

For $d = 1$, there is one 0-cusp, and there are four 1-cusps, $F_\beta, F_\gamma, F_\zeta, F_\eta$ (see the example at the end of section 5.1.2). Let us write $\Delta = \Delta_\beta, \Delta_\gamma, \Delta_\zeta, \Delta_\eta$ for the corresponding decomposition of the boundary divisor.

These four 1-cusps give only two distinct vector-valued theta series; see example 5.1.17. Applying theorem 5.3.3 to this relation, we get

$$\begin{aligned}
0 &\sim 150\,H(\bar{0},0) + H(\bar{0},-1) + 56\,H(\bar{1},-1/4) \\
&\quad + (150\,c(\bar{0},0,1,K_\beta) + c(\bar{0},1,1,K_\beta) + 56\,c(\bar{1},-1/4,1,K_\beta))(\Delta_\beta + \Delta_\gamma) \\
&\quad + (150\,c(\bar{0},0,1,K_\zeta) + c(\bar{0},1,1,K_\zeta) + 56\,c(\bar{1},-1/4,1,K_\zeta))(\Delta_\zeta + \Delta_\eta) \\
&\sim 150\,H(\bar{0},0) + H(\bar{0},-1) + 56\,H(\bar{1},-1/4) \\
&\quad + \frac{1}{24}(150\,\mathbf{\Theta}_{K_\beta}(\bar{0},0) + 1\cdot(\mathbf{\Theta}_{K_\beta}(\bar{0},1) + E_2(1)\mathbf{\Theta}_{K_\beta}(\bar{0},0)) + 56\,\mathbf{\Theta}_{K_\beta}(\bar{1},-1/4))\,(\Delta_\beta + \Delta_\gamma) \\
&\quad + \frac{1}{24}(150\,\mathbf{\Theta}_{K_\zeta}(\bar{0},0) + 1\cdot(\mathbf{\Theta}_{K_\zeta}(\bar{0},1) + E_2(1)\mathbf{\Theta}_{K_\zeta}(\bar{0},0)) + 56\,\mathbf{\Theta}_{K_\zeta}(\bar{1},-1/4))\,(\Delta_\zeta + \Delta_\eta) \\
&\sim 150\,H(\bar{0},0) + H(\bar{0},-1) + 56\,H(\bar{1},-1/4) \\
&\quad + \frac{1}{24}(150\cdot 1 + 1\cdot(482 + (-24)\cdot 1) + 56\cdot 2)\,(\Delta_\beta + \Delta_\gamma) \\
&\quad + \frac{1}{24}(150\cdot 1 + 1\cdot(306 + (-24)\cdot 1) + 56\cdot 0)\,(\Delta_\zeta + \Delta_\eta) \\
&\sim 150\,H(\bar{0},0) + H(\bar{0},-1) + 56\,H(\bar{1},-1/4) + 30\,(\Delta_\beta + \Delta_\gamma) + 18\,(\Delta_\zeta + \Delta_\eta)\ .
\end{aligned} \tag{5.22}$$

We thus see that

$$150\,\lambda \sim H(\bar{0},-1) + 56\,H(\bar{1},-1/4) + 30\,(\Delta_\beta + \Delta_\gamma) + 18\,(\Delta_\zeta + \Delta_\eta)\ . \tag{5.23}$$

Remark 5.3.4. Computations such as this one are possible, in principle, for every d. However, the number of 1-cusps increases very rapidly with d, and so does the potential number of different boundary coefficients that must be computed. This quickly becomes unfeasible.

There is one particular pattern that we observed, though, which seems to hold for all d. (At least, for all d for which the minimal basis exists.)

Observation 5.3.5. For every d such that the minimal basis exists, if we complete the Hodge relation in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$, then the coefficient at the standard 1-cusp Δ_γ (the one associated to the isotropic subspace $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset U \oplus U \subset L_{2d}$, with definite lattice $K = \langle -2d\rangle \oplus 2E_8(-1))$ is always 30. At other 1-cusps, the coefficient can be smaller (as seen above, for Δ_ζ and Δ_η) or greater, though.

This observation might be explained by the existence of a series of modular forms, uniformly defined for all relevant d, whose vanishing divisor is supported on the minimal basis, and which always has vanishing order 30 at Δ_γ.

5.3.4 Analysis of the product expansion at the cusps

In this section we prove theorem 5.3.3.

Let

$$\sum_{\gamma,n} a_{\gamma,n} H(\gamma,n) \sim 0 \tag{5.24}$$

be a linear equivalence of Noether–Lefschetz divisors on $\mathcal{F}_{2d}$. By theorem 4.3.2, this relation corresponds to a principal part

$$f = \sum_{\gamma,n} a_{\gamma,n} q^n \mathbf{e}_\gamma \in \mathrm{Sing}_0^-(L) \tag{5.25}$$

that vanishes in the obstruction space Obstruct; the modular form Ψ on $\mathcal{F}_{2d}$ that gives rise to the given relation is exactly the result of applying Borcherds construction 4.3.1 to this principal part. See section 4.3 for details.

We will determine the vanishing order (negative in the case of a pole) of the function Ψ at the boundary divisor over a given 1-cusp F; by remark 5.3.2, if we can prove that this vanishing order equals

$$a_{\gamma,n} c(\gamma,n,N,K) = N \cdot a_{\gamma,n} (E_2 \mathbf{\Theta}_K)(\gamma,n)\,/\,24\ , \tag{5.26}$$

then this will also be the vanishing order at any component of Δ_F, and we are done.

Lemma 5.3.6. *We may assume, without loss of generality, that the isotropic plane $I \subset L$ representing F is generated by $z = e \in U \subset L$ and another element $y \in L$ (so in particular, $y^2 = 0$ and $(z,y) = 0$) and moreover $(z,L) = \mathbb{Z}$ (i.e., $\mathrm{div}(z) = 1$).*

For a proof, see [18, lemmas 2.23 and 4.1].

Notation: we write $M = z^\perp/\mathbb{Z}z \cong U \oplus 2E_8(-1) \oplus \langle -2d\rangle$; this is a lattice of signature $(1, 18)$. We employ a slight abuse of notation and also write $y \in M$ for the image of $y \in L$ in M; also, we write $K = y_M^\perp/\mathbb{Z}y$; this lattice K thus also equals $I^\perp/I = K(F)$, and has signature $(0, 17)$.

Now, we analyse the product expansion (4.18) of Ψ at the standard 0-cusp, corresponding to the isotropic vector $z = e \in U$. Let us recall the formula for the product expansion:

$$\Psi_z(Z_M) = C\,\mathrm{e}((Z_M, \rho_M)) \prod_{\substack{\lambda\in M^\vee \\ (\lambda,W_M)>0}} \prod_{\substack{\delta\in D_L \\ \delta|_M=\lambda}} (1 - \mathrm{e}((\lambda, Z_M) + (\delta, z')))^{a_{\lambda,\lambda^2/2}} \qquad \text{(4.18 revisited)}$$

This expansion is valid only on a single Weyl chamber $W_M \subset M \otimes \mathbb{R}$. (When the coordinate Z_M crosses a Heegner divisor $H(\gamma, n)$ that occurs in the relation $\sum_{\gamma,n} a_{\gamma,n} H(\gamma, n) \sim 0$, the product formula changes. See [6] for details on this wall-crossing phenomenon.) The choice of Weyl chamber W_M is not completely free, but determined – perhaps not completely – by the choice of 1-cusp F: we must choose W_M in such a way that the isotropic vector $y \in M \subset M_\mathbb{R}$ lies in its closure.

Recall from section 5.2.3 that we may describe a point on the boundary divisor in the tube domain model as the limit of the path

$$t \mapsto Z_M = ity + \tau y' + \kappa\ , \qquad (5.27)$$

where $\tau \in \mathbb{H}$ and $\kappa \in K \otimes \mathbb{C}$ together parametrise the limit point in the boundary divisor (up to the action of the arithmetic group).

Lemma 5.3.7. *The value of Ψ_z along the above path is asymptotically (as $t \to \infty$) equal to $\exp(-2\pi(y, \rho_M)t)$.*

Proof. We analyse the value of the product expansion on this path. The non-zero constant C is irrelevant. We will deal with the factor $\mathrm{e}((Z_M, \rho_M))$ later, and now first look at a single factor in the big product. Let us introduce a notation for such a factor: say $F_\lambda := 1 - \mathrm{e}((\lambda, Z_M))$ for fixed $\lambda \in M^\vee$ with $(\lambda, W_M) > 0$. Note that, because $(z, L) = \mathbb{Z}$, we may choose z' in such a way that $z' \in L$, so $(\delta, z') \in \mathbb{Z}$, and thus we may leave that term out.

We distinguish two cases. If $(\lambda, y) = 0$, then

$$(\lambda, Z_M) = (\lambda, ity + \tau y' + \kappa) = \tau(\lambda, y') + (\lambda, \kappa)\ . \qquad (5.28)$$

This shows that F_λ is constant along the path to the cusp, and F_λ vanishes only if $(\lambda, Z_M) \in \mathbb{Z}$, or equivalently if $\tau(\lambda, y') + (\lambda, \kappa) \in \mathbb{Z}$, which cannot happen for generic (τ, κ).

On the other hand, suppose that $(\lambda, y) \neq 0$. Because $(\lambda, W_M) > 0$, and y is in the closure of W_M, this inner product cannot be negative, so $(\lambda, y) > 0$. Then

$$\lim_{t\to\infty} \mathrm{e}((\lambda, Z_M)) = \lim_{t\to\infty} \mathrm{e}((\lambda, ity + \tau y' + \kappa)) = \lim_{t\to\infty} \mathrm{e}(it(\lambda, y)) = \lim_{t\to\infty} \exp(-2\pi(\lambda, y)) = 0 \qquad (5.29)$$

and hence the limit of F_λ is $1 - 0 = 1$.

Finally, we look at the factor $\mathrm{e}((Z_M, \rho_M)) = \mathrm{e}((ity + \tau y' + \kappa, \rho_M))$. In the limit, the term with ity dominates (unless $(y, \rho_M) = 0$; in that case, this factor is constant on the path to the boundary point, and non-zero for a generic choice of point on the boundary divisor), so asymptotically $\mathrm{e}((Z_M, \rho_M)) = \exp(-2\pi(y, \rho_M)t)$.

We conclude that the only factor that contributes asymptotically is $\mathrm{e}((Z_M, \rho_M))$, and it has the asymptotics of $\exp(-2\pi(y, \rho_M)t)$, as claimed. □

We see that as we approach the boundary divisor, the function Ψ_z can get a zero or a pole, depending on the sign of the constant (y, ρ_M). By proposition 5.2.2, since our function is asymptotically given by $\mathrm{e}(it(y, \rho_M))$, the vanishing order that we want to compute equals $N \cdot (y, \rho_M)$ (where N is the imprimitivity of the 1-cusp F, as always).

Finally, we may evaluate the constant (y, ρ_M) using the theorems in Chapter 10 of Borcherds' paper [6].

Lemma 5.3.8. *The inner product with the Weyl vector ρ_M is given by*

$$(y, \rho_M) = \sum_{\substack{\gamma,n \\ \gamma\in H_F^\perp}} a_{\gamma,n}(E_2\mathbf{\Theta}_K)(p(\gamma), n)/24\ , \qquad (5.30)$$

where $H_F^\perp \subseteq D_L$ and the map $p : H_F^\perp \to D_K$ are defined in 5.1.14.

Proof. By [6, Theorem 10.4], we know that ρ_M equals $\rho_K + r_{y'}y' + r_y y$, where

$$\rho_K = -\frac{1}{2} \sum_{\lambda \in K^\vee} \sum_{\substack{\delta \in M^\vee/M \\ \delta|_K = \lambda \\ (\delta, W_M) > 0}} a_\delta(\lambda^2/2) \; ; \tag{5.31}$$

$$r_{y'} \text{ is the constant term of } \overline{\mathbf{\Theta}_K}(\tau) F_K(\tau) E_2(\tau)/24 \; ; \tag{5.32}$$

$$r_y = -r_{y'}(y')^2/2 + \frac{1}{2} \sum_{\lambda \in K^\vee} \sum_{\substack{\delta \in M^\vee/M \\ \delta|_K = \lambda \\ (\delta, W_M) > 0}} a_\delta(\lambda^2/2) B_2((\delta, z')) \; ; \tag{5.33}$$

further, $B_2(x)$ is a modified Bernoulli function, given by

$$B_2(x+1) = B_2(x) \text{ for all } x \in \mathbb{R} \text{ , and} \tag{5.34}$$

$$B_2(x) = x^2 - x + \frac{1}{6} \text{ for } 0 \le x < 1 \; ; \tag{5.35}$$

also, $\mathbf{\Theta}_K$ is the vector-valued theta series associated to the lattice K, and $\overline{\mathbf{\Theta}_K}$ its complex conjugate (or equally $\mathbf{\Theta}_{K(-1)}$, the theta series of the positive definite lattice $K(-1)$), so

$$\overline{\mathbf{\Theta}_K}(\tau) = \sum_{\lambda \in K^\vee} q^{-\lambda^2/2} \mathbf{e}_{\lambda + K} \; ; \tag{5.36}$$

F_K is the adaptation of the principal part f to the subquotient lattice K:

$$F_K = \sum_{\substack{\gamma, n \\ \gamma \in H_F^\perp}} a_{\gamma,n} q^n \mathbf{e}_{p(\gamma)} \in \operatorname{Sing}_0^-(K) \; , \tag{5.37}$$

and $E_2(\tau)$ is the usual Eisenstein series of weight 2, given by

$$E_2(\tau) = 1 - 24 \sum_{n > 0} \sigma_1(n) q^n \; , \tag{5.38}$$

where σ_1 is the divisor function (i.e., the sum of the prime divisors of its argument).

This description of ρ_M is explicit enough to compute the constant (y, ρ_M). First note that $(y, y) = 0$ (recall that y is isotropic) and also $(y, \rho_K) = 0$ (by definition of K), leaving only $(y, \rho_M) = r_{y'}$, the constant term of $\overline{\mathbf{\Theta}_K}(\tau) F_K(\tau) E_2(\tau)/24$. Substituting the expression for F_K in terms of its coefficients $a_{\gamma,n}$, we get the claimed equality (5.30). ☐

We conclude that the vanishing order of Ψ at the boundary divisor Δ_F over F is equal to

$$N \cdot \sum_{\substack{\gamma, n \\ \gamma \in H_F^\perp}} a_{\gamma,n} (E_2 \mathbf{\Theta}_K)(p(\gamma), n) / 24 \; . \tag{5.39}$$

This finishes the proof.

5.4 Theta ghosts

We would like to use theorem 5.3.3 to complete relations in $\operatorname{Pic}(\mathcal{F}_{2d})$ to the boundary, in particular to prove that given modular forms on $\mathcal{F}_{2d}$ are cusp forms.

Looking at equations 5.19 and 5.20, this means that we have to enumerate the 1-cusps of $\mathcal{F}_{2d}$ and for each of them compute the associated definite lattice K and its vector-valued theta series $\mathbf{\Theta}_K$. We know from proposition 5.1.13 that the lattice genus of K is known (although dependent on the imprimitivity N of F): it is $\mathcal{G}(2E_8(-1) \oplus \langle -2m \rangle)$ where $m = d/N^2$.

Let us rephrase our task in a slightly more general setting. Let K_0 be a non-degenerate definite lattice. We would like to determine what vector-valued modular forms occur as theta series of a definite lattice of genus $\mathcal{G}(K_0)$. (For an introduction to vector-valued modular forms, see section 3.3; for vector-valued theta series (these count the number of vectors in the dual lattice of given length and discriminant class), see 2.5; for an explanation of lattice genera, see 2.4.) In other words, we want to identify the image of the map

$$\begin{aligned} \mathbf{\Theta} &: \mathcal{G}(K_0) \to M_{\mathrm{rank}(K_0)/2}(K_0) \\ &: K \mapsto \mathbf{\Theta}_K \ . \end{aligned} \tag{5.40}$$

In our application we have $K_0 = 2E_8 \oplus \langle 2m \rangle$, where $m = d/N^2$ is directly related to the polarisation degree of the K3 surface (in particular there is always the case $N = 1, m = d$). As m increases, computing the image of $\mathbf{\Theta}$ is computationally too hard to approach directly, i.e., by enumerating the lattices in the genus $\mathcal{G}(K_0)$ and computing the theta series of each lattice. This is already clear from the size of the set $\mathcal{G}(K_0)$, which grows rapidly (as m^8: see theorem 5.1.7).

We will take another approach: we use some properties shared by all theta series to define a finite superset of the image of $\mathbf{\Theta}$. For some reasonably low values of d, we can compute this superset explicitly. More importantly, for all d in the range that interests us we can compute a lower bound for the boundary coefficients of any given relation in $\mathrm{Pic}(\mathcal{F}_{2d})$ by solving a linear programming problem, minimising a linear expression for these boundary coefficients over the superset.

Because the superset might be a strict superset, this does not answer the original question (of computing the exact image of the image of the map $\mathbf{\Theta}$). It turns out that this loss of information is acceptable for our needs, and we may use this method to prove that many modular forms on $\mathcal{F}_{2d}$ are cusp forms.

5.4.1 Definition and first examples

Definition 5.4.1. Let K be a definite lattice (let us say positive definite). A vector-valued modular form $\Psi = \sum_{\gamma,n} \Psi(\gamma, n) q^n \mathbf{e}_\gamma \in M_{\mathrm{rank}\, K/2}(K)$ is a theta ghost if

(i) it is an almost-cusp form (see definition 3.3.8);

(ii) $\Psi(\gamma, n) \in \mathbb{N}$ for all $\gamma \in D_K$ and $n \geq 0$;

(iii) $\Psi(\gamma, n) = \Psi(-\gamma, n)$ for all $\gamma \in D_K$ and $n \geq 0$;

(iv) $\Psi(\gamma, n) \in 2\mathbb{N}$ for all $\gamma \in D_K$ such that $\gamma = -\gamma$ and for all $n > 0$;

(v) $\Psi(0, 0) = 1$.

Proposition 5.4.2. *If K is a definite lattice, then $\mathbf{\Theta}_K$ is a theta ghost.*

Proof. (i) Let $\gamma \in D_K$ be a non-standard cusp (i.e., $\gamma^2 = 0$ but $\gamma \neq 0$). We look at $\mathbf{\Theta}_K(\gamma, 0)$, the number of elements $v \in K^\vee$ of discriminant class γ and norm $v^2 = 0$. Now, since K is definite, $v^2 = 0$ implies that $v = 0$, but the zero vector has discriminant class 0, not γ. This shows that $\mathbf{\Theta}_K(\gamma, 0) = 0$, so $\mathbf{\Theta}_K$ is indeed an almost-cusp form. (ii) As a count of a finite set, $\Psi(\gamma, n)$ is of course a natural number. (iii) The map $v \mapsto -v$ gives a bijection from the set of vectors of class γ to the set of vectors of class $-\gamma$, preserving the norm $2n$. (iv) The map $v \mapsto -v$ gives a bijective map from the set of vectors $v \in K^\vee$ of class γ and norm $2n$ to itself. Because $v^2 = 2n > 0$ for these vectors, the zero vector is not among them, so this involution has no fixed points, and the size of the set must be even. (v) The number $\Psi(0, 0)$ counts the elements $v \in K^\vee$ of discriminant class 0 (that is, elements of K) with norm $v^2 = 0$; because K is definite, the only such vector is the zero vector. □

Next, we would like to compute the set of theta ghosts. Using the methods of chapter 3, we may compute a Heegner basis $c^\vee_{\gamma_i, n_i}$ of the space $M_{17/2}(K_0)$, and write any other coefficient function $c_{\gamma,n}$ as an explicit (rational) linear combination of these basis vectors. Write $\Psi = \sum_i m_i c^\vee_{\gamma_i, n_i}$ for a general theta ghost. For every choice of γ and n, we rewrite condition (ii) on the coefficient $\Psi(\gamma, n)$ as a condition on the numbers m_i. This gives a restriction of the form

$$\sum_i a_i m_i \in \mathbb{N} \ , \tag{5.41}$$

where the a_i are rational numbers. Clearing denominators, we see that this is a problem of linear integer programming, and our task is to enumerate all solutions.

Remark 5.4.3. If we are given a modular form in a numerical way, where the calculation of every coefficient requires some amount of work, then it is impossible to verify the infinite set of conditions (ii) directly.

However, examples show that a finite number of these conditions (ii) seem to suffice: if a modular form satisfies condition (ii) for this finite set of coefficients (γ, n), then apparently it is fulfilled for all other coefficients as well (as far as we could check). We will call such modular forms *apparent* theta ghosts. Note that this situation is analogous to the question of finite generation of the Noether–Lefschetz cone (see in particular question 4.5.2).

Although the sets of theta ghosts and apparent theta ghosts apparently coincide, we do not know for sure that they do, and we will not use that fact. We do know that the set of theta ghosts is contained in the set of apparent theta ghosts, and thus the set of theta series is contained in the set of apparent theta ghosts; that last fact is enough for our purposes.

Example 5.4.4. Let us compute the set of apparent theta ghosts in the case $d = 1$.

The theta series of definite lattices in the genus of $K_0 = \langle 2 \rangle \oplus 2E_8$ are vector-valued modular forms of weight 17/2 in the Weil representation associated to the quadratic module $\mathbb{Z}/2d\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$. We can use the methods of section 3.6 to compute the space of all such forms. It turns out that this space is 2-dimensional; one possible choice of basis is the set $c^\vee_{\bar{0},0}, c^\vee_{\bar{0},1}$.

Let $\mathbf{\Theta}$ be a theta ghost. Thus, we may write

$$\mathbf{\Theta} = m_1 c^\vee_{\bar{0},0} + m_2 c^\vee_{\bar{0},1} \ , \tag{5.42}$$

for some $m_i \in \mathbb{Q}$. Moreover, because $\mathbf{\Theta}(\bar{0}, 0) = 1$, we see that $m_1 = 1$; also, because $\mathbf{\Theta}(\bar{0}, 1) \in \mathbb{N}$, we see that $m_2 \in \mathbb{N}$. So, the theta ghosts are in this case completely determined by a single natural number m_2.

We can constrain this further, by using the fact that other coefficients must also be natural numbers. For instance, we have

$$c_{\bar{1},1/4} = -\frac{153}{44} c_{\bar{0},0} + \frac{1}{88} c_{\bar{0},1} \tag{5.43}$$

on this space of modular forms, so we get the extra condition on $\mathbf{\Theta}$ that

$$-\frac{153}{44}\mathbf{\Theta}(\bar{0}, 0) + \frac{1}{88}\mathbf{\Theta}(\bar{0}, 1) \in \mathbb{N} \ . \tag{5.44}$$

This just means that $-153m_1/44 + m_2/88 \in \mathbb{N}$, or equivalently – since $m_1 = 1$ – that $m_2 \in 306 + 88\mathbb{N}$, giving a lower bound and divisibility condition on m_2.

Taking another coefficient, we get another condition: since

$$c_{\bar{1},5/4} = \frac{30804}{11} c_{\bar{0},0} - \frac{42}{11} c_{\bar{0},1} \ , \tag{5.45}$$

we get that

$$\frac{30804}{11}\mathbf{\Theta}(\bar{0}, 0) - \frac{42}{11}\mathbf{\Theta}(\bar{0}, 1) \in \mathbb{N} \ , \tag{5.46}$$

or equivalently, that $42m_2 \in 30804 - 11\mathbb{N}$.

Combining the two conditions $m_2 \in 306 + 88\mathbb{N}$ and $42m_2 \in 30804 - 11\mathbb{N}$, we see that $m_2 \in \{306, 394, 482, 570, 658\}$.

Finally, we can use the evenness condition (iv) of theta ghosts to cut down this list even further. Since $\gamma = \bar{1} \in \mathbb{Z}/2\mathbb{Z}$ satisfies $-\gamma = \gamma$, condition (iv) applies for the coefficient $(\gamma, n) = (\bar{1}, 1/4)$, so we know that $\mathbf{\Theta}(\bar{1}, 1/4)$ must be even. This holds in the cases $m_2 \in \{306, 482, 658\}$ but not in the cases $m_2 \in \{394, 570\}$.

We conclude that there are 3 apparent theta ghosts in the case $d = 1$. (Calculation of some tens of coefficients did not reveal any further relevant conditions of type (ii).)

This example indicates that the set of theta ghosts might be bigger than the set of actual theta series of definite lattices: we know from example 5.1.17 that there are only two distinct actual theta series in this case, one with $\mathbf{\Theta}(\bar{0}, 1) = 482$ and one with $\mathbf{\Theta}(\bar{0}, 1) = 306$; the third apparent theta ghost, with $\mathbf{\Theta}(\bar{0}, 1) = 658$, does not come from a definite lattice.

We will later (section 5.4.2) describe further conditions on vector valued modular forms – adding to the list of conditions of definition 5.4.1 – that eliminate some of these theta ghosts.

There is one further example where we can compare the set of theta series and the set of theta ghosts: for $d = 2$, we know that the size of the genus of the lattice $K_0 = \langle 4\rangle \oplus 2E_8$ is 9, so the number of distinct theta series is at most 9. Computing a basis of vector-valued modular forms in this case, and solving the linear programming problem, we count that there are 35 apparent theta ghosts. This shows that the number of apparent theta ghosts can be substantially larger than the number of theta series.

We have counted the number of apparent theta ghosts for d at most 10: see table 5.2.

Table 5.2: The number G of apparent theta ghosts for given d.

d	1	2	3	4	5	6	7	8	9	10
G	3	35	11	107	58	164	483	1344	196	4887

Interestingly, the number increases with d, but not nearly as rapidly as the size of the genus (which grows as d^8). In particular, this proves that the fibres of the map $K \mapsto \mathbf{\Theta}_K$ are large: there must be many distinct definite lattices with the same theta series. (Note that if there are relevant conditions on theta ghosts that we disregarded, or the map $K \mapsto \mathbf{\Theta}_K$ is not surjective, this further decreases the possible number of theta series, so this only strengthens the conclusion that there must be many lattices with the same theta series.)

This is useful, because it allows us to prove cuspidality of modular forms on the moduli space $\mathcal{F}_{2d}$ by enumerating only the much smaller set of apparent theta ghosts, instead of the huge set of 1-cusps. The validity of this simplification stems from the fact that when completing relations on $\mathcal{F}_{2d}$ to $\overline{\mathcal{F}_{2d}}$, the coefficient at the 1-cusp corresponding to the definite lattice K only depends on the theta series $\mathbf{\Theta}_K$ (see theorem 5.3.3). The uncertainty in determining the set of theta ghosts, and the non-surjectivity of the map $K \mapsto \mathbf{\Theta}_K$, result in spurious conditions in our cuspidality criterion. This means that we may well be able to prove that a given modular form is a cusp form, but the failure of this method does not imply that it is not a cusp form.

5.4.2 Further conditions

The above definition of a theta ghost applies to any genus of definite lattices. In the case that interests us, where K has rank 17, we can analyse the behaviour of the theta series of a lattice in $\mathcal{G}(K_0)$ in more detail, and formulate properties that such a theta series must have. Any theta ghost that does not have these extra properties can then be deleted from the list of potential theta series.

As a first step, we show that there cannot be many distinct dual vectors that contribute to $\mathbf{\Theta}_K(\bar{1}, 1/4d)$. For brevity, let us call such a dual vector – having discriminant class corresponding to $1 + 2d\mathbb{Z}$ and norm $2 \cdot 1/4d$ – a $(\bar{1}, 1/4d)$-vector.

Lemma 5.4.5. *Let $K \in \mathcal{G}(K_0)$, so K is a positive definite even lattice of rank* 17 *and discriminant module $\mathbb{Z}/2d\mathbb{Z}$.*

(i) If $d = 1$, then $\mathbf{\Theta}_K(\bar{1}, 1/4d) \leq 2$.

(ii) If $d > 1$, then $\mathbf{\Theta}_K(\bar{1}, 1/4d) \leq 1$.

Proof. Let us first show that there cannot be two linearly independent $(\bar{1}, 1/4d)$-vectors. Suppose otherwise; let v and w be such vectors. Then v and w span a 2-dimensional subspace of $K_{\mathbb{Q}}$. Because K is positive definite, this subspace is positive definite as well, so we have the following inequality:

$$0 < \begin{vmatrix} v^2 & (v,w) \\ (v,w) & w^2 \end{vmatrix} = \frac{1}{2d} \cdot \frac{1}{2d} - (v,w)^2 , \tag{5.47}$$

so $|(v,w)| < \frac{1}{2d}$. However, from the structure of the quadratic module D_K, we know that $(v,w) \in \frac{1}{2d} + \mathbb{Z}$. This is a contradiction.

Now, we may prove the lemma. (i) Suppose $\mathbf{\Theta}_K(\bar{1}, 1/4d) > 2$. Note that the only proper rational multiple of a $(\bar{1}, 1/4d)$-vector v that is again a $(\bar{1}, 1/4d)$-vector is $-v$ (other multiples have the wrong norm). Thus, among the $(\bar{1}, 1/4d)$-vectors (at least 3) there must be two that are linearly independent. By the above, this is impossible. (ii) Suppose $\mathbf{\Theta}_K(\bar{1}, 1/4d) > 1$. This time there is no proper rational multiple of a $(\bar{1}, 1/4d)$-vector v that is again a $(\bar{1}, 1/4d)$-vector: as before, because of the norm condition, it could only be $-v$, but that has discriminant class $-1 + 2d\mathbb{Z} \neq 1 + 2d\mathbb{Z}$ (the inequality needs $d > 1$). Therefore any two of the $(\bar{1}, 1/4d)$-vectors are linearly independent, and we get a contradiction as before. □

Let us see if this gives us any new information on the possible theta series.

For $d = 1$, we saw in example 5.1.17 that there are three theta ghosts in that case (with $\Psi(\bar{0}, 1) \in \{306, 482, 658\}$), only two of which are actual theta series. Using our knowledge of the relations on vector-valued modular forms, we may compute that these three theta ghosts have $\Psi(\bar{1}, 1/4) \in \{0, 2, 4\}$. The third one, with $\Psi(\bar{1}, 1/4) = 4$, is prohibited by the above lemma. In this case, the extra condition in fact allows us to separate the theta series from the unwanted theta ghost.

Unfortunately, for $d > 1$ this particular condition seems to be satisfied by all theta ghosts – as far as we have tried – and thus gives no new information.

As a second step, we look at dual vectors of the form $v + w$, where w is a $(\bar{1}, 1/4d)$-vector, and v is a root of K orthogonal to w. As we vary v, this gives a set of $(\bar{1}, 1 + 1/4d)$-vectors, thus giving a lower bound on $\mathbf{\Theta}_K(\bar{1}, 1 + 1/4d)$. We now make this precise.

Lemma 5.4.6. *Let $K \in \mathcal{G}(K_0)$. If $\mathbf{\Theta}_K(\bar{1}, 1/4d) \neq 0$, then $\mathbf{\Theta}_K(\bar{1}, 1 + 1/4d) \geq 480$.*

Proof. Pick an element of $K^\vee$ of discriminant class corresponding to $1 + 2d\mathbb{Z}$ and of norm $2 \cdot 1/4d$. By the structure of the discriminant group, we may write this element as $w/2d$, where $w \in K$.

We claim that the embedding $\mathbb{Z}w \subset K$ is primitive and isomodular; let us prove this first. If w were not primitive in K, then we could write $w = mv$, with $v \in K$ and $m > 1$ an integer. Then $2d = w^2 = m^2 v^2 \in m^2\mathbb{Z}$, so we may write $2d = km$ with $1 < k < 2d$. But then $v = w/m = kw/2d$ has discriminant class corresponding to $k \mod 2d$, which is impossible because $v \in K$. This contradiction proves that w is primitive.

For the isomodularity, look at the map $\pi : K^\vee \to (\mathbb{Z}w)^\vee \to (\mathbb{Z}w)^\vee/\mathbb{Z}w$; we must prove that the kernel of π is exactly K. The first part of π sends an element $\kappa \in K^\vee$ to the function that maps w to $(\kappa, w) \in \mathbb{Z}$; the second part of π sends this function to the discriminant class that corresponds to $(\kappa, w) + 2d\mathbb{Z}$.

Now, if $v \in K$, then $(v, w/2d) \in \mathbb{Z}$, so $(v, w) \in 2d\mathbb{Z}$, so $\pi(v) = (v, w) + 2d\mathbb{Z} = 0 + 2d\mathbb{Z}$, and we see that indeed $v \in \ker(\pi)$. Conversely, suppose that $\kappa \in K^\vee$ is in the kernel of π. By the structure of the discriminant group, we may write $\kappa = kw/2d + v$, with $k \in \mathbb{Z}$ and $v \in K$. Then $0 + 2d\mathbb{Z} = \pi(\kappa) = (\kappa, w) + 2d\mathbb{Z} = k + (v, w) + 2d\mathbb{Z} = k + 2d\mathbb{Z}$, so k is a multiple of $2d$, but then $kw/2d \in K$, so $\kappa \in K$. This proves the claim.

Since the embedding $\mathbb{Z}w \subset K$ is isomodular, we can apply lemma 2.2.13, and conclude that the orthogonal complement $J = w^\perp$ is unimodular. It is therefore a unimodular even positive definite lattice of rank 16. Such lattices are classified, and there are only two of them up to isomorphism: $E_8 \oplus E_8$ and D_{16}^+. Moreover, these two lattices have the same theta series. In particular, J has exactly 480 roots (elements $v \in J$ with $v^2 = 2$).

Now, look at the elements $v + w/2d$, where v is a root of J. These are 480 different elements of $K^\vee$. They have discriminant class corresponding to $1 + 2d\mathbb{Z}$ by construction, and because $(v, w) = 0$, we have $(v + w/2d)^2/2 = (2 + \frac{1}{2d})/2 = 1 + 1/4d$. This shows that $\mathbf{\Theta}_K(\bar{1}, 1 + 1/4d) \geq 480$. □

Let us apply this lemma to $d = 2$, and see if it excludes any theta ghosts. We noted earlier that for $d = 2$ there are 35 theta ghosts. Of these 35, 17 have non-zero $(\bar{1}, 1/8)$-coefficient. Of these 17, only 2 have $(\bar{1}, 9/8)$-coefficient at least 480. The other 15 cannot be theta series, because they would contradict lemma 5.4.6. So, for $d = 2$ this new condition reduces the number of possible theta series from 35 to 20.

One might continue along these lines: try to find even more conditions on theta series (from properties of definite lattices, as the lemmas above), and perform more enumerations of lattice genera to see if there are any theta ghosts left that can be excluded. For our purposes though, we have done enough.

5.4.3 Application to cuspidality of modular forms on $\mathcal{F}_{2d}$

We show how to use theta ghosts to prove that a given modular form on $\mathcal{F}_{2d}$ is a cusp form.

Given any modular form on $\mathcal{F}_{2d}$ that has vanishing locus supported on Noether–Lefschetz divisors, we know by Bruinier's result that it arises from Borcherds' construction (see section 4.3). Our theorem 5.3.3 then describes the vanishing order of the modular form at every 1-cusp (or more precisely: at the components over all the 1-cusps of any toroidal compactification). Because the number of 1-cusps increases rapidly with the polarisation degree $2d$, we use our idea of theta ghosts to bound these vanishing orders.

We recall here the contents of theorem 5.3.3, reinterpreted in terms of modular forms instead of relations in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$. Let Ψ be any modular form on $\mathcal{F}_{2d}$ with vanishing locus supported on Noether–Lefschetz divisors; we may assume without loss of generality that Ψ is associated to the linear relation $\sum_{\gamma,n} a_{\gamma,n} c_{\gamma,n} = 0$ of coefficients functions $c_{\gamma,n}$ on $AC(d)$, the space of almost cusp forms associated to the lattice L_{2d}. Then the vanishing order of Ψ at the boundary components associated to the 1-cusp F is given by

$$\sum_{\substack{\gamma,n \\ \gamma\in H_F^\perp}} a_{\gamma,n} \cdot N/24 \cdot (E_2\mathbf{\Theta}_K)(p(\gamma),n) \ , \tag{5.48}$$

where N is the imprimitivity of the cusp F (see definition 5.1.9), K is the definite lattice of rank 17 associated to the cusp F (see definition 5.1.11), $\mathbf{\Theta}_K$ is the vector-valued theta series of the lattice K (see section 2.5), and E_2 is the usual Eisenstein series.

For a fixed modular form Ψ, this vanishing order is thus a linear expression in the coefficients $\mathbf{\Theta}_K(\gamma,n)$.

We have complete knowledge of the space of which $\mathbf{\Theta}_K$ is a member, and this is the same space for all 1-cusps F of the same imprimitivity N: it is $AC_{17/2}(m)$, the space of almost cusp forms of weight 17/2 associated to the lattice $\mathbb{Z}/2m\mathbb{Z}$, where $m = d/N^2$. In particular, using the methods from section 3.6 we may compute a basis of that space and express any coefficient $c_{\gamma,n}$ in terms of that basis. Let us write λ_i for the unknown coefficients in the expression for $\mathbf{\Theta}_K$ in terms of the basis. Now, the fact that the coefficients $\mathbf{\Theta}_K(\gamma,n)$ are natural numbers gives us an infinite set of inequalities and integrality constraints for the numbers λ_i.

All in all, this means that the vanishing order of Ψ, viewed as a function of the cusp F, has a linear expression in terms of variables λ_i, and we have linear inequalities and integrality constraints for these λ_i. Therefore, we may apply integer linear programming techniques to minimise the expression (5.48) with respect to the constraints. This gives us a lower bound for the vanishing order of Ψ among all the 1-cusps with imprimitivity N.

Repeating this for all possible imprimitivities N (i.e., the positive integers with N^2 dividing d), and taking the lowest outcome, we get a lower bound for the vanishing order of Ψ among all the 1-cusps. If this is a positive number, then we conclude that Ψ is a cusp form.

Remark 5.4.7. Because there might be theta ghosts that do not come from definite lattices, we cannot use this to prove that a form is not a cusp form.

Results

Applying the procedure outlined above to the modular forms represented by the equations in appendix B, we get the lower bounds for the vanishing orders listed in table 5.3.

Table 5.3: A lower bound l for the vanishing order at the cusps – of imprimitivity N – of the modular forms on $\mathcal{F}_{2d}$ represented by the equations in appendix B.

d	N	l
40	1	3/2
	2	4
42	1	2
43	1	3/2
46	1	3/2
48	1	3/2
	2	53/12
	4	19

d	N	l
49	1	2
	7	140/3
50	1	3/2
	5	45/2
52	1	3/2
	2	47/12
54	1	3/2
	3	37/4

d	N	l
55	1	3/2
56	1	2
	2	15/4

Remark 5.4.8. To get these results we have only used the inequalities $\Psi(\gamma,n) \geq 0$, not the integrality constraints $\Psi(\gamma,n) \in \mathbb{N}$. Using those last constraints, and perhaps further conditions as in section 5.4.2, the bounds might be improved.

From the fact that all the lower bounds l in the table are positive we may conclude that all the corresponding modular forms are cusp forms.

5.5 Application to the Kodaira dimension of $\mathcal{F}_{2d}$

Theorem 5.5.1. *If $d \in \{46, 50, 52, 54\}$, then $\mathcal{F}_{2d}$ is of general type (i.e., the Kodaira dimension is $\kappa(\mathcal{F}_{2d}) = 19$).*

Proof. By the results of section 4.7, for these values of d we can write $K^\circ - \varepsilon\lambda = (19-\varepsilon)\lambda - 1/2 \cdot [B]$ as a positive linear combination of (irreducible) Noether–Lefschetz divisors for some positive value of ε:

$$(19-\varepsilon)\lambda - 1/2 \cdot [B] = \sum_{\Delta,\delta} t_{\Delta,\delta}[P_{\Delta,\delta}] \ . \tag{5.49}$$

Concrete equations for various d are provided in appendix B.

Translating, this means that there is a modular form Ψ on $\mathcal{F}_{2d}$ of weight $19-\varepsilon < 19$ that vanishes on the ramification divisor (which is exactly the pullback of $1/2 \cdot B$). By [18, theorem 1.1], if we can prove that Ψ is a cusp form, then we know that $\mathcal{F}_{2d}$ is of general type.

First of all, we rewrite the irreducible Noether–Lefschetz divisors $P_{\Delta,\delta}$ in terms of the reducible divisors $H(\gamma, n)$ using the triangular relations of section 4.2.2.

Next, we apply theorem 5.3.3 to relation (5.49), completing it to a relation on some toroidal compactification. In theory, this gives us the boundary coefficient of (5.49) at every 1-cusp. The set of 1-cusps is very large, though, so we cannot compute all of these boundary coefficients explicitly.

We apply the method of theta ghosts (see section 5.4) to compute a lower bound for the boundary coefficient of Ψ at all the 1-cusps: we take the expression produced by theorem 5.3.3 for the vanishing order in terms of theta coefficients, and use linear programming to get a lower bound for this expression. From the results of this procedure (see section 5.4.3), we see that for the values of d under consideration the lower bound is at least 1. This means that the modular form Ψ vanishes at every 1-cusp.

By proposition 5.2.5, this is enough to conclude that Ψ is indeed a cusp form, and we are done. □

Remark 5.5.2. Note that in the proof we use two distinct toroidal compactifications of $\mathcal{F}_{2d}$: theorem 5.3.3 (which completes relation on $\mathcal{F}_{2d}$ to $\overline{\mathcal{F}_{2d}}$) applies to the toroidal compactification associated to the perfect cone decomposition; in the proof of [18, theorem 1.1] on the other hand it is essential to use a toroidal compactification with canonical singularities. This is not a problem: the only thing we need from theorem 5.3.3 is the vanishing order of Ψ at the 1-cusps, and this is independent of the choice of toroidal compactification.

Theorem 5.5.3. *If $d \in \{40, 42, 43, 48, 49, 55, 56\}$, then $\kappa(\mathcal{F}_{2d}) \geq 0$.*

Proof. The only difference with the theorem above is that we now have a modular form of weight exactly 19 (because $\varepsilon = 0$ in these cases). This means we can apply the second case of [18, theorem 1.1], and conclude that the Kodaira dimension of $\mathcal{F}_{2d}$ is non-negative. (Again, the vanishing order at the standard 1-cusp of the form we find equals 15 in all of these cases.) □

Combining this result with theorem 4.7.8, we get the following.

Theorem 5.5.4. *Let $d \in \{40, 42, 43, 48, 49, 55, 56\}$. If the effective cone of $\mathcal{F}_{2d}$ is generated by irreducible Noether–Lefschetz divisors and our list of generators is complete (see questions 4.5.2 and 4.5.3), then we have intermediate Kodaira dimension: $0 \leq \kappa(\mathcal{F}_{2d}) < 19$.*

Chapter 6

The Koszul divisor

This chapter is not part of the main thread of this thesis: it plays no role in the computation of the Kodaira dimension of the moduli space $\mathcal{F}_{2d}$ of polarised K3 surfaces.

In our investigation of the moduli space, we had a particular need to construct effective divisors. A natural way to get an effective divisor on a moduli space is to impose a geometric condition on the objects that it parametrises. Often, such conditions are given in terms of an invariant of the object that takes a constant value on the generic object but jumps at a divisorial subset. The Noether–Lefschetz divisors we met in chapter 4 are an example of this, where the invariant is the Picard lattice of the K3 surface.

In the case of polarised K3 surfaces, another such invariant is the Clifford index of a smooth curve in the polarisation class. In this chapter, we will compute the divisor where this invariant jumps. For some cases (values of d), there are a few terms of which we could not determine the multiplicity with certainty.

6.1 Clifford index

Recall that a line bundle A on a curve C is said to contribute to the Clifford index of the curve C if $\mathrm{h}^0(C,A) \geq 2$ and $\mathrm{h}^1(C,A) \geq 2$; for such a line bundle, we define its Clifford index by $\mathrm{Cliff}(A) = \deg_C(A) - 2r(A)$, where $r(A) = \mathrm{h}^0(C,A) - 1$ is the projective dimension of the linear system $|A|$. The Clifford index of C is the minimum of the values $\mathrm{Cliff}(A)$, where A ranges over all line bundles that contribute. Any bundle A with this minimal Clifford index is said to compute the Clifford index of C.

Thus, the Clifford index of a curve measures the presence of line bundles with many sections relative to their degree.

A few facts we will use later:

(i) The Clifford index of a generic curve of genus g is $\lfloor (g-1) / 2 \rfloor$.

(ii) For a line bundle B on a curve C, we have $\mathrm{Cliff}(B) = \mathrm{Cliff}(\omega_C \otimes B^{-1})$; therefore, for a line bundle A on a K3 surface S, by adjunction, $\mathrm{Cliff}(A|_C) = \mathrm{Cliff}((\mathcal{O}_S(C) \otimes A^{-1})|_C)$.

Remark 6.1.1. Note that we try to use multiplicative notation for the tensor product of line bundles. However, because line bundles on K3 surfaces form a discrete group – even a lattice: see definition 1.0.4 – we will at some places switch to additive notation to make the connection to the corresponding lattice more explicit.

6.2 Clifford index of a K3 section

Suppose given a K3 surface S with a polarisation H of degree $H^2 = 2d \geq 2$ and with Picard lattice L of rank 2. We want to compute the Clifford index of a smooth curve C in the class H (so C has genus $g = d + 1 \geq 2$). The main tool that makes this possible is the following result by Green and Lazarsfeld.

Theorem 6.2.1 ([16])**.** *In the situation described above, if* $\mathrm{Cliff}(C)$ *is less than the generic value* $\lfloor (g-1) / 2 \rfloor$*, then there is a line bundle* D *on* S *such that* $D|_C$ *computes the Clifford index of* C*.*

Moreover, we may choose D in such a way that

(i) $\mathrm{h}^0(S, D) \geq 2$,

(ii) $\mathrm{h}^0(S, \mathcal{O}(C) \otimes D^{-1}) \geq 2$, and

(iii) $C \cdot D \leq g - 1$.

Let us call a line bundle D on S with these last three properties a *candidate bundle*. The fact that the bundle D from the theorem may be chosen to have these properties is not explicitly mentioned in [16]; variants of the theorem, such as the older version in [13], do mention this. An inspection of the proof by Green and Lazarsfeld shows that it holds in their general case as well.

This theorem gives us a strategy to compute $\mathrm{Cliff}(C)$: we enumerate all candidate line bundles D on S (there will be only finitely many). For each of these, we compute the Clifford index of $D|_C$; the smallest of these values will equal $\mathrm{Cliff}(C)$. If there are no candidate line bundles, then the Clifford index must be the generic value $\lfloor (g-1) / 2 \rfloor$.

Now, to compute the Clifford index of the restriction $D|_C$, we may use an exact sequence relating it to the bundles $\mathcal{O}_S(C)$ and D on S. Standard results describe how to compute cohomology groups of line bundles on S, in terms of their lattice-theoretic properties in $\operatorname{Pic} S \cong L$; see section 1.1.

6.3 Procedure

We describe the general procedure to compute, given a rank 2 even hyperbolic lattice L, the Clifford index of irreducible curves in the polarisation class of the corresponding K3 surfaces.

Procedure 6.3.1.

(i) Compute the set $\{E \in L : E^2 = 0\}$, and check if any of these E has $E \cdot H = 1$. If so, then H is monogonal (by proposition 1.1.6), so there are no smooth irreducible curves in the class H, so there is no Clifford index to calculate.

(ii) Compute the set of (-2)-vectors $\Delta = \{\delta \in L : \delta^2 = -2\}$. Compute the intersection of each of these with H; if any has $\delta \cdot H = 0$, then we know that we can choose H to be nef, but not ample; if all intersections are nonzero, then we may even choose H to be ample.

(iii) Compute the positive cone $\mathcal{C}$. Use Vinberg's algorithm to compute a fundamental domain of the action of the group $\{s_\delta : \delta \in \Delta\}$ on $\mathcal{C}$; by [4, VIII.3.9], this fundamental domain is exactly the nef cone. This also tells us what the *irreducible* (-2)-curves are: these are exactly the ones that define a face of the nef cone.

(iv) Determine the set of effective classes: by [4, VIII.3.7], this is the semigroup generated by the irreducible (-2)-curves together with the classes that lie in the closure of the positive cone. Compute the set of classes D such that D is effective, and $H \otimes D^{-1}$ is effective, and $D \cdot H \leq g-1$. This contains all candidates. It will be a finite set, even without the last condition.

(v) For each of these D, calculate the Betti numbers $\mathrm{h}^0(S, D)$, $\mathrm{h}^1(S, D)$, $\mathrm{h}^0(S, \mathcal{O}(C) \otimes D^{-1})$ and $\mathrm{h}^1(S, \mathcal{O}(C) \otimes D^{-1})$.

(vi) From the Betti numbers of D and $\mathcal{O}(C) \otimes D^{-1}$, compute the Betti numbers of $D|_C$. This gives us the Clifford index of $D|_C$.

(vii) Take the minimum of these Clifford indices to get the Clifford index of C.

Remark 6.3.2. The only part that might not succeed, is step (vi): in some cases, we do not have enough information to isolate the Betti numbers of $D|_C$ from the long exact sequence that relates them to D and $C \otimes D^{-1}$.

6.4 Examples

6.4.1 $H^2 = 4$, $H \cdot \Gamma = 3$, $\Gamma^2 = 0$

Take the lattice $L = \mathbb{Z}H \oplus \mathbb{Z}\Gamma$, where $H^2 = 4$, $H \cdot \Gamma = 3$, $\Gamma^2 = 0$. Let S be a K3 surface with Picard lattice L, and choose the isomorphism in such a way that H corresponds to a nef class, using lemma 1.2.2. In the following, we will identify $\operatorname{Pic} S$ and L using this isomorphism.

The equation $x^2 > 0$, for $x = aH + b\Gamma$, gives $4a^2 + 6ab > 0$. Since H is nef, hence positive, the positive cone must be $\mathcal{C} = \{aH + b\Gamma \in V : a > 0, 2a + 3b > 0\}$.

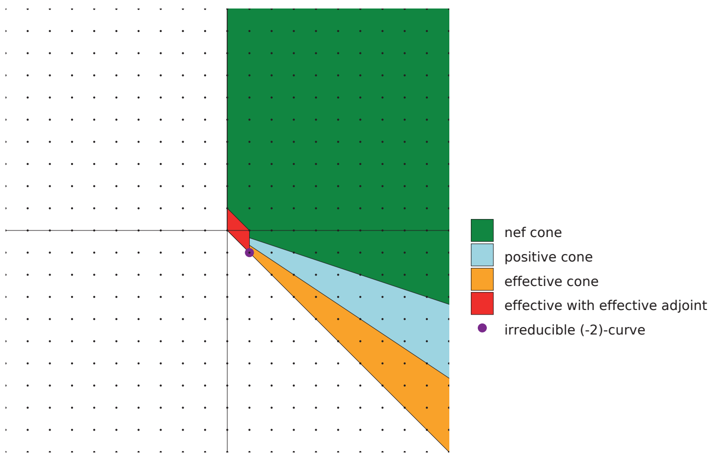
First of all, we compute the set of $E \in L$ with $E^2 = 0$, to see if H could be monogonal. Now, $4a^2 + 6ab = 0$ gives $a = 0$ or $3b = -2a$, so this has solutions $E \in \mathbb{Z} \cdot \Gamma$ and $E \in \mathbb{Z} \cdot (3H - 2\Gamma)$. We need to check whether there is such an E with the property $E \cdot H = 1$ (in that case, E is necessarily primitive). This does not hold in our case, since $\Gamma \cdot H = 3$, and $(3H - 2\Gamma) \cdot H = 6$. We conclude that H is not monogonal.

Next, we compute the set of $\delta \in L$ such that $\delta^2 = -2$. Writing $\delta = aH + b\Gamma$, we get $4a^2 + 6ab = -2$, with solutions $\delta = H - \Gamma$ and $-\delta$. Now, the nef cone is a fundamental domain of the action of s_δ (the reflection in $\pm\delta$) on $\mathcal{C}$. Since $\delta \cdot H = 1$, we know that H is ample, and that δ is effective (not $-\delta$), and the nef cone is given by

$$\operatorname{Nef}(S) = \left\{x \in \overline{\mathcal{C}} : \delta \cdot x \geq 0\right\} = \{aH + b\Gamma \in V : a \geq 0, a + 3b \geq 0\} \ . \tag{6.1}$$

See figure 6.1.

Figure 6.1: The various cones in $\operatorname{Pic}(S) \otimes \mathbb{R} \cong \mathbb{R}^2$ of a K3 surface with rank 2 Picard lattice with intersection numbers $H^2 = 4$, $H \cdot \Gamma = 3$, $\Gamma^2 = 0$. (Note that the colours are superimposed: the effective cone includes the positive cone, which includes the ample cone.)



The effective classes are generated by δ together with the lattice points in $\overline{\mathcal{C}}$. The ones of these that satisfy $D \cdot H \leq g - 1$ are $H - \Gamma (= \delta)$ and $2H - 2\Gamma (= 2\delta)$. Neither of these are candidates: δ has $h^0(S, \delta) = 1$, since δ corresponds to a (-2)-curve, while 2δ has a non-effective adjoint bundle ($H - 2\delta = -H + 2\Gamma$, and $h^0(S, H - 2\delta) = 0$).

We conclude that the Clifford index of C is $\lfloor (g-1)/2 \rfloor = 1$.

6.4.2 $H^2 = 4$, $H \cdot \Gamma = 2$, $\Gamma^2 = 0$

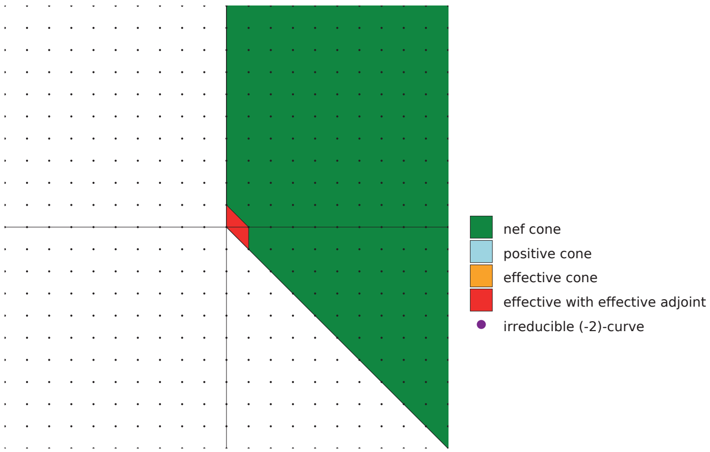
Take the lattice $L = \mathbb{Z}H \oplus \mathbb{Z}\Gamma$, where $H^2 = 4$, $H \cdot \Gamma = 2$, $\Gamma^2 = 0$. Let S be a K3 surface with Picard lattice L, and choose the isomorphism in such a way that H corresponds to a nef class, using lemma 1.2.2.

The equation $x^2 > 0$, for $x = aH + b\Gamma$, gives $4a^2 + 4ab > 0$. Since H is nef, hence positive, the positive cone must be $\mathcal{C} = \{aH + b\Gamma \in V : a > 0, a + b > 0\}$.

First, we compute the set of $E \in L$ with $E^2 = 0$. Now $4a^2 + 4ab = 0$ gives $a = 0$ or $b = -a$, so this has solutions $E \in \mathbb{Z} \cdot \Gamma$ and $E \in \mathbb{Z} \cdot (H - \Gamma)$; these have an intersection with H of $\Gamma \cdot H = 2 \neq 1$, and $(H - \Gamma) \cdot H = 2 \neq 1$, so we see that H is not monogonal.

Next, we compute the set of $\delta \in L$ such that $\delta^2 = -2$. Writing $\delta = aH + b\Gamma$, we get $4a^2 + 4ab = -2$; since the right-hand side is $2 \mod 4$, this has no solutions. Because there are no negative curves, the nef cone equals the closure of the positive cone. See figure 6.2.

Figure 6.2: The various cones in $\mathrm{Pic}(S) \otimes \mathbb{R} \cong \mathbb{R}^2$ of a K3 surface with rank 2 Picard lattice with intersection numbers $H^2 = 4$, $H \cdot \Gamma = 2$, $\Gamma^2 = 0$. All three cones coincide (up to differences at the boundary).



As there are no irreducible (-2)-curves, the effective classes are exactly the lattice points in $\overline{\mathcal{C}}$. The ones of these that satisfy $D \cdot H \leq g - 1$ are Γ and $H - \Gamma$. Since $\mathrm{Cliff}(D) = \mathrm{Cliff}(H \otimes D^{-1})$, we only need to compute the Clifford index for $D = \Gamma$.

So, look at the long exact sequence associated to

$$0 \longrightarrow \mathcal{O}_S(\Gamma - H) \longrightarrow \mathcal{O}_S(\Gamma) \longrightarrow \mathcal{O}_C(A) \longrightarrow 0 \,, \tag{6.2}$$

where A is the restriction of $\mathcal{O}_S(\Gamma)$ to the smooth curve $C \in |H|$ of genus $g = 3$:

$$0 \longrightarrow \mathrm{H}^0(S, \Gamma - H) \longrightarrow \mathrm{H}^0(S, \Gamma) \longrightarrow \mathrm{H}^0(C, A) \tag{6.3}$$
$$\longrightarrow \mathrm{H}^1(S, \Gamma - H) \longrightarrow \mathrm{H}^1(S, \Gamma) \longrightarrow \mathrm{H}^1(C, A) \tag{6.4}$$
$$\longrightarrow \mathrm{H}^2(S, \Gamma - H) \longrightarrow \mathrm{H}^2(S, \Gamma) \longrightarrow 0 \,. \tag{6.5}$$

We use our knowledge of linear series on a K3 surface to compute the left and middle terms.

(i) Since $H - \Gamma$ is effective, $\mathrm{H}^2(S, H - \Gamma) = 0$; because it is also nef and primitive, $\mathrm{H}^1(S, H - \Gamma) = 0$ and $\mathrm{h}^0(S, H - \Gamma) = 2 + (H - \Gamma)^2/2 = 2$. Using $\mathrm{h}^i(S, D) = \mathrm{h}^{2-i}(S, -D)$, we get all left terms.

(ii) Because Γ is effective, $\mathrm{H}^2(S, \Gamma) = 0$; since it is nef and primitive, $\mathrm{H}^1(S, \Gamma) = 0$, so $\mathrm{h}^0(S, \Gamma) = 2 + \Gamma^2/2 = 2$. This gives all middle terms.

From this, we get that $\mathrm{h}^0(C, A) = 2$ and $\mathrm{h}^1(C, A) = 2$, so A contributes, and $\mathrm{Cliff}(A) = \deg_C(A) - 2(\mathrm{h}^0(C, A) - 1) = H \cdot \Gamma - 2 \cdot (2 - 1) = 0$. Therefore the Clifford index of C is 0, which is lower than the expected value $\lfloor (g - 1) \,/\, 2 \rfloor = 1$.

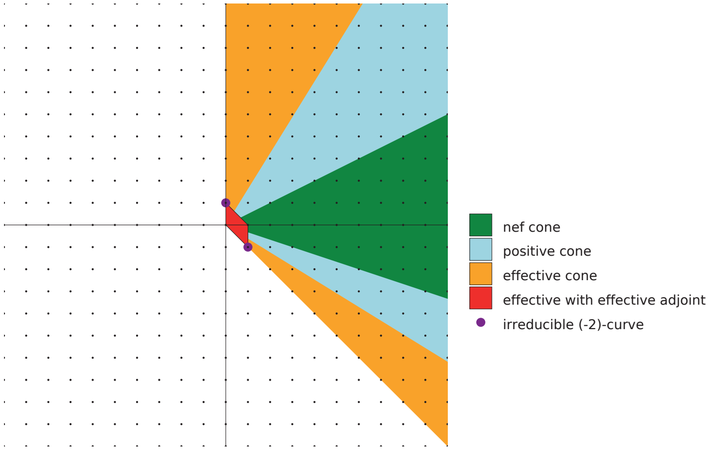
6.4.3 $H^2 = 2$, $H \cdot \Gamma = 1$, $\Gamma^2 = -2$

For the next example, we take the lattice $L = \mathbb{Z}H \oplus \mathbb{Z}\Gamma$, where $H^2 = 2$, $H \cdot \Gamma = 1$, $\Gamma^2 = -2$. We just give the results.

There are two -2-curves. All the relevant cones are drawn in figure 6.3.

There are four candidate bundles $H, \Gamma, H - \Gamma, 0$, but none of them contribute. The polarisation curve therefore has as Clifford index the generic value $\lfloor (g - 1) \,/\, 2 \rfloor = 0$ (which we could have known without computation, because this is the only possible Clifford index of a curve of genus 2).

Figure 6.3: The various cones in $\mathrm{Pic}(S) \otimes \mathbb{R} \cong \mathbb{R}^2$ of a K3 surface with rank 2 Picard lattice with intersection numbers $H^2 = 2$, $H \cdot \Gamma = 1$, $\Gamma^2 = -2$.

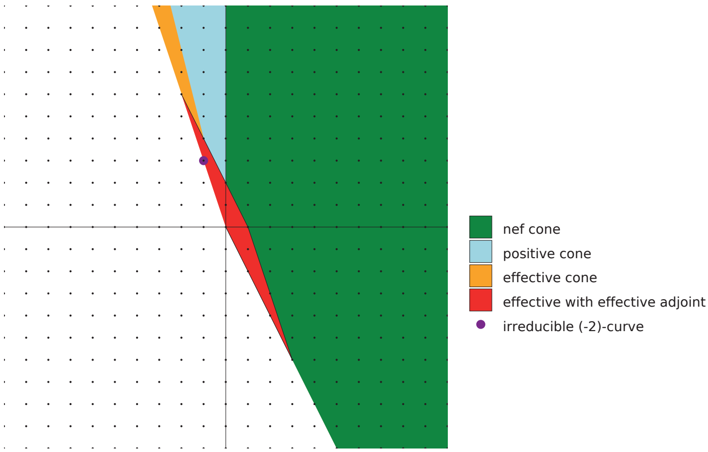


6.4.4 $H^2 = 16$, $H \cdot \Gamma = 6$, $\Gamma^2 = 2$

For the next example, we take the lattice $L = \mathbb{Z}H \oplus \mathbb{Z}\Gamma$, where $H^2 = 16$, $H \cdot \Gamma = 6$, $\Gamma^2 = 2$. We just give the results.

There are two -2-curves. All the relevant cones are drawn in figure 6.4.

Figure 6.4: The various cones in $\mathrm{Pic}(S) \otimes \mathbb{R} \cong \mathbb{R}^2$ of a K3 surface with rank 2 Picard lattice with intersection numbers $H^2 = 16$, $H \cdot \Gamma = 6$, $\Gamma^2 = 2$.



We list the candidate bundles together with their possible Clifford indices in table 6.1.

Because the Clifford index of the bundle $3H - 6\Gamma$ and its adjoint $-2H + 6\Gamma$ (located in figure 6.4 at the top and bottom vertex of the red area) could be either 0 or 2 (we cannot tell this from the Betti numbers in the long exact sequence: see remark 6.3.2), the Clifford index of the polarisation curve is also either 0 or 2.

Table 6.1: Candidate bundles on a K3 surface with rank 2 Picard lattice with intersection numbers $H^2 = 16$, $H \cdot \Gamma = 6$, $\Gamma^2 = 2$.

Candidate bundle	possible Clifford indices
$3H - 6\Gamma$	0,2
$2H - 4\Gamma$	2,4
$2H - 3\Gamma$	(does not contribute)
$1H - 2\Gamma$	2
$1H - 1\Gamma$	2
$0H + 0\Gamma$	(does not contribute)
$1H + 0\Gamma$	(does not contribute)
$0H + 1\Gamma$	2
$0H + 2\Gamma$	2
$-1H + 3\Gamma$	(does not contribute)
$-1H + 4\Gamma$	2,4
$-2H + 6\Gamma$	0,2

6.5 Results

We present some results of this procedure, for $d = 3$ and $d = 6$, in table 6.2.

There is a clear pattern in the distribution of blanks vs. non-blanks in the table: the non-blanks form an infinite set of parabolas, shifted in the Δ-direction. This is explained by proposition 4.2.13.

To get a more compact representation of these results, we choose another parametrisation of the lattices.

Definition 6.5.1. The rank 2 even hyperbolic $2d$-polarised lattice of coset $k \in \mathbb{Z}$ (note that the coset of the lattice is in fact $[k] \in (\mathbb{Z}/2d\mathbb{Z})/\pm$) and series $l \in \mathbb{Z}$ is the lattice $\left(\begin{smallmatrix} 2d & k \\ k & -2l \end{smallmatrix}\right)$. The Clifford index of a K3 surface with this Picard lattice is denoted by $C(k, l)$, if this is well defined; it is not clear whether the Picard lattice of a K3 surface always determines the Clifford index of a smooth section of the polarisation class, due to the problematic step (vi) in the procedure.

Note that this lattice is only hyperbolic under the condition that $\Delta = k^2 + 4dl$ is positive, so for fixed k, the parameter l must not be too negative.

In terms of these new parameters, we give some more results of our Clifford index computations in table 6.3.

Table 6.2: The Clifford index of a polarised K3 surface with rank 2 Picard lattice with discriminant $-\Delta$ and coset δ. The rows are labelled by Δ, the columns by δ. A blank indicates that there exists no rank 2 lattice with those particular invariants. The symbol **M** denotes that the corresponding polarised K3 surface is monogonal (definition 1.1.5).

(a) $d = 3$

	0	1	2	3
1		M		
2				
3				
4			0	
5				
6				
7				
8				
9				1
10				
11				
12	1			
13		1		
14				
15				
16			1	
17				
18				
19				
20				
21				1
22				
23				
24	1			
25		1		
26				
27				
28			1	
29				
30				
31				
32				
33				1
34				
35				
36	1			
37		1		
38				
39				
40			1	
41				
42				
43				
44				
45				1
46				
47				
48	1			

(b) $d = 6$

	0	1	2	3	4	5	6
1		M				0	
2							
3							
4			0				
5							
6							
7							
8							
9				1			
10							
11							
12							2
13							
14							
15							
16					2		
17							
18							
19							
20							
21							
22							
23							
24	3						
25		3				3	
26							
27							
28			3				
29							
30							
31							
32							
33				3			
34							
35							
36							3
37							
38							
39							
40					3		
41							
42							
43							
44							
45							
46							
47							
48	3						

Table 6.3: The Clifford index of a polarised K3 surface with rank 2 Picard lattice of given coset k (column) and series l (row). A blank indicates that there exists no rank 2 lattice with those particular invariants. The symbol **M** denotes that the corresponding polarised K3 surface is monogonal (definition 1.1.5). A question mark denotes that our procedure failed to give a definite result (see remark 6.3.2).

(a) $d = 1$

	0	1
0		M
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0

(b) $d = 2$

	0	1	2
0		M	0
1	1	1	1
2	1	1	1
3	1	1	1
4	1	1	1
5	1	1	1

(c) $d = 3$

	0	1	2	3
0		M	0	1
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1
5	1	1	1	1

(d) $d = 4$

	0	1	2	3	4
0		M	0	1	2
1	2	2	2	2	2
2	2	2	2	2	2
3	2	2	2	2	2
4	2	2	2	2	2
5	2	2	2	2	2

(e) $d = 5$

	0	1	2	3	4	5
-1						1
0		M	0	1	2	2
1	2	2	2	2	2	2
2	2	2	2	2	2	2
3	2	2	2	2	2	2
4	2	2	2	2	2	2

(f) $d = 6$

	0	1	2	3	4	5	6
-1						**0**	2
0		M	0	1	2	3	3
1	3	3	3	3	3	3	3
2	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3
4	3	3	3	3	3	3	3

(g) $d = 7$

	0	1	2	3	4	5	6	7
-1							2	3
0		M	0	1	2	3	3	3
1	3	3	3	3	3	3	3	3
2	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3
4	3	3	3	3	3	3	3	3

(h) $d = 8$

	0	1	2	3	4	5	6	7	8
-1							**?**	3	4
0		M	0	1	2	3	4	4	4
1	4	4	4	4	4	4	4	4	4
2	4	4	4	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4

(i) $d = 9$

	0	1	2	3	4	5	6	7	8	9
-2										3
-1								3	4	4
0		M	0	1	2	3	4	4	4	4
1	4	4	4	4	4	4	4	4	4	4
2	4	4	4	4	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4	4	4

(j) $d = 10$

	0	1	2	3	4	5	6	7	8	9	10
-2										**0**	4
-1								3	4	5	5
0		M	0	1	2	3	4	5	5	5	5
1	5	5	5	5	5	5	5	5	5	5	5
2	5	5	5	5	5	5	5	5	5	5	5
3	5	5	5	5	5	5	5	5	5	5	5

(k) $d = 11$

	0	1	2	3	4	5	6	7	8	9	10	11
-2											4	5
-1								3	4	5	5	5
0		M	0	1	2	3	4	5	5	5	5	5
1	5	5	5	5	5	5	5	5	5	5	5	5
2	5	5	5	5	5	5	5	5	5	5	5	5
3	5	5	5	5	5	5	5	5	5	5	5	5

(l) $d = 12$

	0	1	2	3	4	5	6	7	8	9	10	11	12
-2											**2**	5	6
-1								**?**	4	5	6	6	6
0		M	0	1	2	3	4	5	6	6	6	6	6
1	6	6	6	6	6	6	6	6	6	6	6	6	6
2	6	6	6	6	6	6	6	6	6	6	6	6	6
3	6	6	6	6	6	6	6	6	6	6	6	6	6

(m) $d = 13$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
-3														5
-2												5	6	6
-1									4	5	6	6	6	6
0		M	0	1	2	3	4	5	6	6	6	6	6	6
1	6	6	6	6	6	6	6	6	6	6	6	6	6	6
2	6	6	6	6	6	6	6	6	6	6	6	6	6	6

(n) $d = 14$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
-3														**0**	6
-2												**4**	6	7	7
-1									4	5	6	7	7	7	7
0		M	0	1	2	3	4	5	6	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
2	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7

(o) $d = 15$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
-3															**4**	7
-2												?	6	7	7	7
-1									?	5	6	7	7	7	7	7
0		M	0	1	2	3	4	5	6	7	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
2	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7

(p) $d = 16$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-3															**2**	7	8
-2													6	7	8	8	8
-1										5	6	7	8	8	8	8	8
0		M	0	1	2	3	4	5	6	7	8	8	8	8	8	8	8
1	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
2	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8

(q) $d = 17$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
-4																		7
-3																7	8	8
-2													6	7	8	8	8	8
-1										5	6	7	8	8	8	8	8	8
0		M	0	1	2	3	4	5	6	7	8	8	8	8	8	8	8	8
1	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8

(r) $d = 18$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
-4																		**0**	8
-3																**4**	8	9	9
-2														7	8	9	9	9	9
-1										5	6	7	8	9	9	9	9	9	9
0		M	0	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9
1	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9

(s) $d = 19$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
-4																			8	9
-3																	8	9	9	9
-2														7	8	9	9	9	9	9
-1										5	6	7	8	9	9	9	9	9	9	9
0		M	0	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9	9	9
1	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9

(t) $d = 20$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
-4																			**2**	9	10
-3																	**6**	9	10	10	10
-2														?	8	9	10	10	10	10	10
-1										?	6	7	8	9	10	10	10	10	10	10	10
0		M	0	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	10	10
1	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10

(u) $d = 21$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
-5																						9
-4																				9	10	10
-3																	**4**	9	10	10	10	10
-2														?	8	9	10	10	10	10	10	10
-1											6	7	8	9	10	10	10	10	10	10	10	10
0		M	0	1	2	3	4	5	6	7	8	9	10	10	10	10	10	10	10	10	10	10

(v) $d = 22$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
-5																						**0**	10
-4																				**4**	10	11	11
-3																		**8**	10	11	11	11	11
-2															8	9	10	11	11	11	11	11	11
-1											6	7	8	9	10	11	11	11	11	11	11	11	11
0		M	0	1	2	3	4	5	6	7	8	9	10	11	11	11	11	11	11	11	11	11	11

6.6 Analysis of results

Most of the results can be described by a relatively simple formula, but a few values differ from it.

The easiest cases are the positive series, and the series $l = 0$.

Conjecture 6.6.1. *If $l \geq 0$, we have the following formula for $C(\delta, l)$:*

(i) $C(\delta, l) = \lfloor (g-1) / 2 \rfloor$ if $l > 0$;

(ii) $C([k], 0) = \min\{k-2, \lfloor (g-1) / 2 \rfloor\}$ if $2 \leq k \leq d$.

All values that we have computed agree with this conjecture.

Note that case (ii) essentially covers the whole series $l = 0$, because $C([0], 0)$ and $C([1], 0)$ are not defined: the first because the corresponding lattice $\left(\begin{smallmatrix} 2d & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is degenerate, and the second because the lattice $\left(\begin{smallmatrix} 2d & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is always monogonal.

For the negative series, we have to work a little harder.

For convenience, let us parametrise the rank 2 lattices in yet another way: by d (as before) and integers b and c, where the lattice is $\left(\begin{smallmatrix} 2d & b \\ b & 2c \end{smallmatrix}\right)$. We also introduce some derived quantities

$$x = b - 2c - 2 \tag{6.6}$$

$$y = b - d - c - 1 \tag{6.7}$$

$$e = \left\lfloor \frac{d}{2} \right\rfloor = \lfloor (g-1) / 2 \rfloor \ . \tag{6.8}$$

Let us write C_1 for the Clifford index conjectured by 6.6.1. In terms of the above parameters, the formula for C_1 becomes

$$C_1 = \begin{cases} \min\{x, e\} & \text{if } x \geq 0 \text{ and } c \geq 0 \\ e & \text{otherwise.} \end{cases} \tag{6.9}$$

It is easiest to see how to adapt this formula to the negative series by keeping c fixed and varying d and b. There is a triangle of points in the (d, b)-plane where the Clifford index differs from C_1 in a regular way: see table 6.4.

This leads us to submit the following refined formula for the Clifford index:

$$C_2 = \begin{cases} \max\{x, 0\} + t & \text{if } t \geq 1 \text{ and } y \geq 0 \\ \min\{x, e\} & \text{if } x \geq 0 \text{ and } c \geq 0 \\ e & \text{otherwise;} \end{cases} \tag{6.10}$$

where

$$t = \left\lfloor \frac{c+1}{2} - \frac{|x|}{2} - \frac{|y|}{2} \right\rfloor \ . \tag{6.11}$$

Unfortunately, there are still some lattices for which this formula gives the wrong result. (These are typeset in bold in the tables of results.)

Testing our formula for all hyperbolic and non-monogonal lattices with $2 \leq d \leq 100, 0 \leq b \leq 100, 1 \leq c \leq 50$ gives the following statistics: of these 137000 lattices,

- 3451 have a non-generic Clifford index;
- 14130 do not comply with the simple formula C_1;
- 1105 do not comply with the refined formula C_2;
- of these last ones, 1023 are only narrowly hyperbolic, in the sense that decreasing b by one makes it non-hyperbolic.

So, formula C_2 fails in less than a percent of all tested lattices. Of the ones for which it fails, more than 90 percent is only narrowly hyperbolic. This is significant, as of all 137000 tested lattices only 2361 are narrowly hyperbolic. It appears that the problematic lattices – the ones for which our refined formula fails – are near the boundary of the set of hyperbolic lattices.

As a further analysis, we may look at the discriminants $\Delta = b^2 - 4dc$ of the problematic lattices. This is motivated by the observation that hyperbolic lattices are just the ones having $\Delta > 0$, so ones that are narrowly hyperbolic have a small positive Δ.

Table 6.4: The Clifford index of polarised K3 surface with rank 2 Picard lattice of degree $2d$ (row), parametrised by b (column) and $c = 5$. The colours indicate the difference of the computed value with simple formula C_1: black means 0, red means 1, blue means 2, green means 3.

$2d$ \ b	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4			1	1	1	1	1	1	1	1	1	1	1	1
6				**0**	1	1	1	1	1	1	1	1	1	1
8					M	2	2	2	2	2	2	2	2	2
10							2	2	2	2	2	2	2	2
12							M	3	3	3	3	3	3	3
14								0	3	3	3	3	3	3
16									1	4	4	4	4	4
18										2	4	4	4	4
20											3	5	5	5
22											3	4	5	5
24												4	5	6
26													5	6
28													**4**	6
30														6
32														**2**

Looking at the list of values of Δ for the problematic lattices, together with their number of occurrences, one immediately notices the predominance of perfect squares. In fact, the 12 values occurring most often are exactly the first 12 squares. Among the rest, squares and sums of two squares are in majority, although other numbers occur as well. Again, we have found an interesting, relatively simple pattern, together with a small minority of cases that do not conform, in some way that is not readily recognisable.

Appendix A

Computer implementation

A.1 Sage implementation of Eisenstein coefficients

We implemented the formula of theorem 3.4.4, to compute the coefficients of the Eisenstein series $E_{\bar{0}}$, in Sage [50].

Listing A.1: Sage code computing coefficients of Eisenstein series

```
from sage.quadratic_forms.special_values import *

k = 21/2
bplus = 0; bmin = 1
m = bplus + bmin

def eisenstein(d,gamma,n):
  (n_0,f) = decomp(d,n)
  d_g = d_gamma(d,n)
  n_bar = n * d_g^2
  n_0_bar = n_0 * d_g^2
  detS = - 2 * d
  D_curl = ZZ(2 * (-1)^((m + 1) / 2) * n_0_bar * detS)
  c = kronecker_character(D_curl)
  x1 = 2^(k + 1) * pi^k * n^(k - 1) * (-1)^((2 * k - bmin + bplus) / 4) \
    / (sqrt(2 * d) * gamma__exact(k))
  x2 = quadratic_L_function__exact(ZZ(k - 1/2),D_curl) / zeta__exact(2 * k - 1)
  x3 = 1
  for p in (2 * n_bar * detS).support():
    x3 *= (1 - c(p) * p^(ZZ(1/2 - k))) / (1 - p^(1 - 2 * k)) \
      * L_local(d,gamma,n,p,w_p(d,d_g,n,p))
  result = (x1 * x2 * x3).simplify_radical()
  return result

def d_gamma(d,gamma):
  result = 2 * d / gcd(2 * d,gamma)
  return result

def w_p(d,d_gamma,n,p):
  return 1 + 2 * (2 * n * d_gamma).valuation(p)

def decomp(d,n):
  nOutsideS = n.prime_to_S_part(QQ(2 * d).support())
  nOnS = n/nOutsideS
  nOutsideS_0 = nOutsideS.squarefree_part()
  n_0 = nOutsideS_0 * nOnS
  f = (nOutsideS/nOutsideS_0).sqrt(None,False)
  return (n_0,f)

def L_local(d,gamma,n,p,w_p):
  result = (1 - p^(m / 2 - k)) * \
```

```
    sum(N(d,gamma,n,p^nu) * p^(nu * (1 - m / 2 - k)) \
      for nu in [0 .. (w_p - 1)]) \
    + N(d,gamma,n,p^w_p) * p^(w_p * (1 - m / 2 - k))
  return result

def N(d,gamma,n,a):
  np = ZZ(n - gamma ^ 2 / (4 * d))
  count = 0
  for x in [0 .. (a - 1)]:
    q = x * gamma - d * x^2 + np
    if q.mod(a) == 0:
      count += 1
  return count
```

Appendix B

Equations for $K^\circ - \varepsilon\lambda$ as an effective class

We list here the expressions we found of the class $K^\circ - \varepsilon\lambda$ as a positive combination of Noether–Lefschetz divisors, with the smallest possible (non-negative) value of $\varepsilon \in \mathbb{Q}$. The procedure is described in section 4.7.

B.1 In terms of irreducible Noether–Lefschetz divisors

$d = 40$:

$$\begin{aligned}
K^\circ - 0\lambda = {}& 111\,[P_{1,\overline{1}}] + 48\,[P_{4,\overline{2}}] + 29\,[P_{9,\overline{3}}] + 21\,[P_{16,\overline{4}}] + 13\,[P_{25,\overline{5}}] + 9\,[P_{36,\overline{6}}] + 9\,[P_{49,\overline{7}}] + 6\,[P_{64,\overline{8}}] \\
& + 3\,[P_{81,\overline{9}}] + 1\,[P_{100,\overline{10}}] + 1\,[P_{121,\overline{11}}] + 29\,[P_{9,\overline{13}}] + 11\,[P_{36,\overline{14}}] + 6\,[P_{65,\overline{15}}] + 3\,[P_{96,\overline{16}}] + 48\,[P_{4,\overline{18}}] \\
& + 9\,[P_{41,\overline{19}}] + 3\,[P_{80,\overline{20}}] + 1\,[P_{121,\overline{21}}] + 56\,[P_{4,\overline{22}}] + 9\,[P_{49,\overline{23}}] + 2\,[P_{96,\overline{24}}] + 9\,[P_{36,\overline{26}}] + 3\,[P_{89,\overline{27}}] \\
& + 9\,[P_{41,\overline{29}}] + 3\,[P_{100,\overline{30}}] + 111\,[P_{1,\overline{31}}] + 4\,[P_{64,\overline{32}}] + 11\,[P_{36,\overline{34}}] + 2\,[P_{105,\overline{35}}] + 21\,[P_{16,\overline{36}}] + 3\,[P_{89,\overline{37}}] \\
& + 56\,[P_{4,\overline{38}}] + 3\,[P_{81,\overline{39}}]\,.
\end{aligned} \tag{B.1}$$

$d = 42$:

$$\begin{aligned}
K^\circ - 0\lambda = {}& 111\,[P_{1,\overline{1}}] + 48\,[P_{4,\overline{2}}] + 32\,[P_{9,\overline{3}}] + 19\,[P_{16,\overline{4}}] + 15\,[P_{25,\overline{5}}] + 10\,[P_{36,\overline{6}}] + 8\,[P_{49,\overline{7}}] + 5\,[P_{64,\overline{8}}] \\
& + 4\,[P_{81,\overline{9}}] + 2\,[P_{100,\overline{10}}] + 1\,[P_{121,\overline{11}}] + 111\,[P_{1,\overline{13}}] + 15\,[P_{28,\overline{14}}] + 6\,[P_{57,\overline{15}}] + 4\,[P_{88,\overline{16}}] + 1\,[P_{121,\overline{17}}] \\
& + 15\,[P_{25,\overline{19}}] + 5\,[P_{64,\overline{20}}] + 2\,[P_{105,\overline{21}}] + 15\,[P_{25,\overline{23}}] + 6\,[P_{72,\overline{24}}] + 1\,[P_{121,\overline{25}}] + 48\,[P_{4,\overline{26}}] + 6\,[P_{57,\overline{27}}] \\
& + 1\,[P_{112,\overline{28}}] + 111\,[P_{1,\overline{29}}] + 8\,[P_{60,\overline{30}}] + 1\,[P_{121,\overline{31}}] + 19\,[P_{16,\overline{32}}] + 4\,[P_{81,\overline{33}}] + 8\,[P_{49,\overline{35}}] + 1\,[P_{120,\overline{36}}] \\
& + 15\,[P_{25,\overline{37}}] + 2\,[P_{100,\overline{38}}] + 32\,[P_{9,\overline{39}}] + 4\,[P_{88,\overline{40}}] + 111\,[P_{1,\overline{41}}] + 2\,[P_{84,\overline{42}}]\,.
\end{aligned} \tag{B.2}$$

$d = 43$:

$$\begin{aligned}
K^\circ - 0\lambda = {}& 111\,[P_{1,\overline{1}}] + 56\,[P_{4,\overline{2}}] + 29\,[P_{9,\overline{3}}] + 22\,[P_{16,\overline{4}}] + 15\,[P_{25,\overline{5}}] + 11\,[P_{36,\overline{6}}] + 7\,[P_{49,\overline{7}}] + 6\,[P_{64,\overline{8}}] \\
& + 3\,[P_{81,\overline{9}}] + 3\,[P_{100,\overline{10}}] + 1\,[P_{121,\overline{11}}] + 14\,[P_{24,\overline{14}}] + 9\,[P_{53,\overline{15}}] + 3\,[P_{84,\overline{16}}] + 2\,[P_{117,\overline{17}}] + 21\,[P_{17,\overline{19}}] \\
& + 6\,[P_{56,\overline{20}}] + 3\,[P_{97,\overline{21}}] + 30\,[P_{13,\overline{23}}] + 5\,[P_{60,\overline{24}}] + 3\,[P_{109,\overline{25}}] + 9\,[P_{41,\overline{27}}] + 2\,[P_{96,\overline{28}}] + 10\,[P_{40,\overline{30}}] \\
& + 3\,[P_{101,\overline{31}}] + 6\,[P_{57,\overline{33}}] + 1\,[P_{124,\overline{34}}] + 17\,[P_{21,\overline{35}}] + 3\,[P_{92,\overline{36}}] + 6\,[P_{68,\overline{38}}] + 9\,[P_{52,\overline{40}}] + 1\,[P_{133,\overline{41}}] \\
& + 9\,[P_{44,\overline{42}}]\,.
\end{aligned} \tag{B.3}$$

$d = 46$:

$$\begin{aligned}
K^\circ - 1\lambda = {}& 112\,[P_{1,\overline{1}}] + 49\,[P_{4,\overline{2}}] + 30\,[P_{9,\overline{3}}] + 20\,[P_{16,\overline{4}}] + 17\,[P_{25,\overline{5}}] + 10\,[P_{36,\overline{6}}] + 8\,[P_{49,\overline{7}}] + 5\,[P_{64,\overline{8}}] \\
& + 4\,[P_{81,\overline{9}}] + 3\,[P_{100,\overline{10}}] + 2\,[P_{121,\overline{11}}] + 27\,[P_{12,\overline{14}}] + 10\,[P_{41,\overline{15}}] + 6\,[P_{72,\overline{16}}] + 2\,[P_{105,\overline{17}}] + 1\,[P_{140,\overline{18}}] \\
& + 12\,[P_{32,\overline{20}}] + 6\,[P_{73,\overline{21}}] + 1\,[P_{116,\overline{22}}] + 15\,[P_{24,\overline{24}}] + 6\,[P_{73,\overline{25}}] + 2\,[P_{124,\overline{26}}] + 7\,[P_{48,\overline{28}}] + 2\,[P_{105,\overline{29}}] \\
& + 10\,[P_{41,\overline{31}}] + 3\,[P_{104,\overline{32}}] + 7\,[P_{52,\overline{34}}] + 2\,[P_{121,\overline{35}}] + 37\,[P_{8,\overline{36}}] + 4\,[P_{81,\overline{37}}] + 8\,[P_{49,\overline{39}}] + 1\,[P_{128,\overline{40}}] \\
& + 17\,[P_{25,\overline{41}}] + 3\,[P_{108,\overline{42}}] + 30\,[P_{9,\overline{43}}] + 2\,[P_{96,\overline{44}}] + 112\,[P_{1,\overline{45}}] + 4\,[P_{92,\overline{46}}]\,.
\end{aligned} \tag{B.4}$$

$d = 48$:

$$\begin{aligned}
K^\circ - 0\lambda = {}& 111\,[P_{1,\overline{1}}] + 48\,[P_{4,\overline{2}}] + 32\,[P_{9,\overline{3}}] + 21\,[P_{16,\overline{4}}] + 15\,[P_{25,\overline{5}}] + 10\,[P_{36,\overline{6}}] + 9\,[P_{49,\overline{7}}] + 6\,[P_{64,\overline{8}}] \\
& + 3\,[P_{81,\overline{9}}] + 3\,[P_{100,\overline{10}}] + 3\,[P_{121,\overline{11}}] + 56\,[P_{4,\overline{14}}] + 11\,[P_{33,\overline{15}}] + 6\,[P_{64,\overline{16}}] + 3\,[P_{97,\overline{17}}] + 2\,[P_{132,\overline{18}}] \\
& + 19\,[P_{16,\overline{20}}] + 9\,[P_{57,\overline{21}}] + 3\,[P_{100,\overline{22}}] + 9\,[P_{49,\overline{25}}] + 3\,[P_{100,\overline{26}}] + 1\,[P_{153,\overline{27}}] + 21\,[P_{16,\overline{28}}] + 6\,[P_{73,\overline{29}}] \\
& + 111\,[P_{1,\overline{31}}] + 9\,[P_{64,\overline{32}}] + 1\,[P_{129,\overline{33}}] + 48\,[P_{4,\overline{34}}] + 6\,[P_{73,\overline{35}}] + 1\,[P_{144,\overline{36}}] + 15\,[P_{25,\overline{37}}] + 3\,[P_{100,\overline{38}}] \\
& + 6\,[P_{64,\overline{40}}] + 12\,[P_{36,\overline{42}}] + 3\,[P_{121,\overline{43}}] + 19\,[P_{16,\overline{44}}] + 2\,[P_{105,\overline{45}}] + 56\,[P_{4,\overline{46}}] + 3\,[P_{97,\overline{47}}]\,.
\end{aligned} \tag{B.5}$$

$d = 49$:

$$\begin{aligned}
K^\circ - 0\lambda = {} & 111\,[P_{1,\overline{1}}] + 56\,[P_{4,\overline{2}}] + 29\,[P_{9,\overline{3}}] + 24\,[P_{16,\overline{4}}] + 16\,[P_{25,\overline{5}}] + 11\,[P_{36,\overline{6}}] + 8\,[P_{49,\overline{7}}] + 8\,[P_{64,\overline{8}}] \\
& + 3\,[P_{81,\overline{9}}] + 4\,[P_{100,\overline{10}}] + 1\,[P_{121,\overline{11}}] + 1\,[P_{144,\overline{12}}] + 15\,[P_{29,\overline{15}}] + 7\,[P_{60,\overline{16}}] + 4\,[P_{93,\overline{17}}] + 2\,[P_{128,\overline{18}}] \\
& + 36\,[P_{8,\overline{20}}] + 7\,[P_{49,\overline{21}}] + 4\,[P_{92,\overline{22}}] + 1\,[P_{137,\overline{23}}] + 14\,[P_{37,\overline{25}}] + 4\,[P_{88,\overline{26}}] + 1\,[P_{141,\overline{27}}] + 6\,[P_{57,\overline{29}}] \\
& + 2\,[P_{116,\overline{30}}] + 9\,[P_{44,\overline{32}}] + 4\,[P_{109,\overline{33}}] + 9\,[P_{49,\overline{35}}] + 1\,[P_{120,\overline{36}}] + 5\,[P_{72,\overline{38}}] + 1\,[P_{149,\overline{39}}] + 14\,[P_{32,\overline{40}}] \\
& + 2\,[P_{113,\overline{41}}] + 6\,[P_{85,\overline{43}}] + 6\,[P_{65,\overline{45}}] + 9\,[P_{53,\overline{47}}] + 1\,[P_{148,\overline{48}}] + 7\,[P_{49,\overline{49}}]\ .
\end{aligned} \tag{B.6}$$

$d = 50$:

$$\begin{aligned}
K^\circ - 1\lambda = {} & 112\,[P_{1,\overline{1}}] + 49\,[P_{4,\overline{2}}] + 36\,[P_{9,\overline{3}}] + 20\,[P_{16,\overline{4}}] + 14\,[P_{25,\overline{5}}] + 12\,[P_{36,\overline{6}}] + 8\,[P_{49,\overline{7}}] + 6\,[P_{64,\overline{8}}] \\
& + 6\,[P_{81,\overline{9}}] + 2\,[P_{100,\overline{10}}] + 2\,[P_{121,\overline{11}}] + 1\,[P_{144,\overline{12}}] + 17\,[P_{25,\overline{15}}] + 7\,[P_{56,\overline{16}}] + 4\,[P_{89,\overline{17}}] + 3\,[P_{124,\overline{18}}] \\
& + 10\,[P_{41,\overline{21}}] + 3\,[P_{84,\overline{22}}] + 2\,[P_{129,\overline{23}}] + 16\,[P_{25,\overline{25}}] + 6\,[P_{76,\overline{26}}] + 2\,[P_{129,\overline{27}}] + 10\,[P_{41,\overline{29}}] + 3\,[P_{100,\overline{30}}] \\
& + 18\,[P_{24,\overline{32}}] + 4\,[P_{89,\overline{33}}] + 1\,[P_{156,\overline{34}}] + 17\,[P_{25,\overline{35}}] + 3\,[P_{96,\overline{36}}] + 10\,[P_{44,\overline{38}}] + 2\,[P_{121,\overline{39}}] + 6\,[P_{81,\overline{41}}] \\
& + 8\,[P_{49,\overline{43}}] + 2\,[P_{136,\overline{44}}] + 14\,[P_{25,\overline{45}}] + 1\,[P_{116,\overline{46}}] + 36\,[P_{9,\overline{47}}] + 3\,[P_{104,\overline{48}}] + 112\,[P_{1,\overline{49}}] + 4\,[P_{100,\overline{50}}]\ .
\end{aligned} \tag{B.7}$$

$d = 52$:

$$\begin{aligned}
K^\circ - 1\lambda = {} & 112\,[P_{1,\overline{1}}] + 49\,[P_{4,\overline{2}}] + 30\,[P_{9,\overline{3}}] + 22\,[P_{16,\overline{4}}] + 16\,[P_{25,\overline{5}}] + 10\,[P_{36,\overline{6}}] + 10\,[P_{49,\overline{7}}] + 6\,[P_{64,\overline{8}}] \\
& + 4\,[P_{81,\overline{9}}] + 3\,[P_{100,\overline{10}}] + 3\,[P_{121,\overline{11}}] + 1\,[P_{144,\overline{12}}] + 22\,[P_{17,\overline{15}}] + 10\,[P_{48,\overline{16}}] + 4\,[P_{81,\overline{17}}] + 3\,[P_{116,\overline{18}}] \\
& + 1\,[P_{153,\overline{19}}] + 16\,[P_{25,\overline{21}}] + 6\,[P_{68,\overline{22}}] + 3\,[P_{113,\overline{23}}] + 112\,[P_{1,\overline{25}}] + 10\,[P_{52,\overline{26}} + 3\,[P_{105,\overline{27}}] + 1\,[P_{160,\overline{28}}]] \\
& + 30\,[P_{9,\overline{29}}] + 6\,[P_{68,\overline{30}}] + 1\,[P_{129,\overline{31}}] + 10\,[P_{49,\overline{33}}] + 3\,[P_{116,\overline{34}}] + 7\,[P_{48,\overline{36}}] + 3\,[P_{121,\overline{37}}] + 6\,[P_{65,\overline{39}}] \\
& + 1\,[P_{144,\overline{40}}] + 22\,[P_{17,\overline{41}}] + 3\,[P_{100,\overline{42}}] + 9\,[P_{64,\overline{44}}] + 1\,[P_{153,\overline{45}}] + 10\,[P_{36,\overline{46}}] + 1\,[P_{129,\overline{47}}] + 23\,[P_{16,\overline{48}}] \\
& + 3\,[P_{113,\overline{49}}] + 49\,[P_{4,\overline{50}}] + 3\,[P_{105,\overline{51}}]\ .
\end{aligned} \tag{B.8}$$

$d = 54$:

$$\begin{aligned}
K^\circ - 1\lambda = {} & 112\,[P_{1,\overline{1}}] + 49\,[P_{4,\overline{2}}] + 33\,[P_{9,\overline{3}}] + 20\,[P_{16,\overline{4}}] + 17\,[P_{25,\overline{5}}] + 11\,[P_{36,\overline{6}}] + 10\,[P_{49,\overline{7}}] + 5\,[P_{64,\overline{8}}] \\
& + 6\,[P_{81,\overline{9}}] + 3\,[P_{100,\overline{10}}] + 2\,[P_{121,\overline{11}}] + 2\,[P_{144,\overline{12}}] + 30\,[P_{9,\overline{15}}] + 11\,[P_{40,\overline{16}}] + 6\,[P_{73,\overline{17}}] + 3\,[P_{108,\overline{18}}] \\
& + 2\,[P_{145,\overline{19}}] + 36\,[P_{9,\overline{21}}] + 7\,[P_{52,\overline{22}}] + 4\,[P_{97,\overline{23}}] + 1\,[P_{144,\overline{24}}] + 17\,[P_{28,\overline{26}}] + 4\,[P_{81,\overline{27}}] + 3\,[P_{136,\overline{28}}] \\
& + 10\,[P_{36,\overline{30}}] + 4\,[P_{97,\overline{31}}] + 36\,[P_{9,\overline{33}}] + 5\,[P_{76,\overline{34}}] + 2\,[P_{145,\overline{35}}] + 6\,[P_{73,\overline{37}}] + 1\,[P_{148,\overline{38}}] + 30\,[P_{9,\overline{39}}] \\
& + 6\,[P_{88,\overline{40}}] + 12\,[P_{36,\overline{42}}] + 2\,[P_{121,\overline{43}}] + 6\,[P_{81,\overline{45}}] + 1\,[P_{172,\overline{46}}] + 10\,[P_{49,\overline{47}}] + 17\,[P_{25,\overline{49}}] + 2\,[P_{124,\overline{50}}] \\
& + 33\,[P_{9,\overline{51}}] + 3\,[P_{112,\overline{52}}] + 112\,[P_{1,\overline{53}}] + 4\,[P_{108,\overline{54}}]\ .
\end{aligned} \tag{B.9}$$

$d = 55$:

$$\begin{aligned}
K^\circ - 0\lambda = {} & 111\,[P_{1,\bar{}}1] + 56\,[P_{4,\bar{}}2] + 29\,[P_{9,\bar{}}3] + 22\,[P_{16,\bar{}}4] + 13\,[P_{25,\bar{}}5] + 11\,[P_{36,\bar{}}6] + 9\,[P_{49,\bar{}}7] + 7\,[P_{64,\bar{}}8] \\
& + 6\,[P_{81,\bar{}}9] + 3\,[P_{100,\bar{}}10] + 3\,[P_{121,\bar{}}11] + 1\,[P_{169,\bar{}}13] + 56\,[P_{5,\bar{}}15] + 11\,[P_{36,\bar{}}16] + 9\,[P_{69,\bar{}}17] + 3\,[P_{104,\bar{}}18] \\
& + 2\,[P_{141,\bar{}}19] + 111\,[P_{1,\bar{}}21] + 9\,[P_{44,\bar{}}22] + 3\,[P_{89,\bar{}}23] + 3\,[P_{136,\bar{}}24] + 30\,[P_{16,\bar{}}26] + 5\,[P_{69,\bar{}}27] + 3\,[P_{124,\bar{}}28] \\
& + 21\,[P_{20,\bar{}}30] + 3\,[P_{81,\bar{}}31] + 2\,[P_{144,\bar{}}32] + 9\,[P_{56,\bar{}}34] + 3\,[P_{125,\bar{}}35] + 9\,[P_{49,\bar{}}37] + 1\,[P_{124,\bar{}}38] + 5\,[P_{60,\bar{}}40] \\
& + 3\,[P_{141,\bar{}}41] + 56\,[P_{4,\bar{}}42] + 6\,[P_{89,\bar{}}43] + 11\,[P_{45,\bar{}}45] + 1\,[P_{136,\bar{}}46] + 29\,[P_{9,\bar{}}47] + 3\,[P_{104,\bar{}}48] + 6\,[P_{80,\bar{}}50] \\
& + 1\,[P_{181,\bar{}}51] + 9\,[P_{64,\bar{}}52] + 9\,[P_{56,\bar{}}54]\ .
\end{aligned} \tag{B.10}$$

B.2 In terms of reducible Noether–Lefschetz divisors

In the following variants of the above equations, we have rewritten the irreducible divisors $P_{\Delta,\delta}$ in terms of reducible Noether–Lefschetz divisors $H(\gamma, n)$.

$d = 40$:

$$\begin{aligned}
K^\circ - 0\lambda = {} & 21[H(\overline{1}, -1/160)] + 17[H(\overline{2}, -1/40)] + 17[H(\overline{3}, -9/160)] + 15[H(\overline{4}, -1/10)] + 12[H(\overline{5}, -5/32)] \\
& + 9[H(\overline{6}, -9/40)] + 9[H(\overline{7}, -49/160)] + 6[H(\overline{8}, -2/5)] + 3[H(\overline{9}, -81/160)] + 1[H(\overline{10}, -5/8)] \\
& + 1[H(\overline{11}, -121/160)] + 17[H(\overline{13}, -9/160)] + 11[H(\overline{14}, -9/40)] + 6[H(\overline{15}, -13/32)] + 3[H(\overline{16}, -3/5)] \\
& + 17[H(\overline{18}, -1/40)] + 9[H(\overline{19}, -41/160)] + 3[H(\overline{20}, -1/2)] + 1[H(\overline{21}, -121/160)] + 21[H(\overline{22}, -1/40)] \\
& + 9[H(\overline{23}, -49/160)] + 2[H(\overline{24}, -3/5)] + 9[H(\overline{26}, -9/40)] + 3[H(\overline{27}, -89/160)] + 9[H(\overline{29}, -41/160)] \\
& + 3[H(\overline{30}, -5/8)] + 21[H(\overline{31}, -1/160)] + 4[H(\overline{32}, -2/5)] + 11[H(\overline{34}, -9/40)] + 2[H(\overline{35}, -21/32)] \\
& + 15[H(\overline{36}, -1/10)] + 3[H(\overline{37}, -89/160)] + 21[H(\overline{38}, -1/40)] + 3[H(\overline{39}, -81/160)]\ .
\end{aligned} \tag{B.11}$$

$d = 42$:

$$
\begin{aligned}
K^\circ - 0\lambda = {} & 19\,[H(\overline{1},-1/168)] + 17\,[H(\overline{2},-1/42)] + 18\,[H(\overline{3},-3/56)] + 14\,[H(\overline{4},-2/21)] + 13\,[H(\overline{5},-25/168)] \\
& + 10\,[H(\overline{6},-3/14)] + 8\,[H(\overline{7},-7/24)] + 5\,[H(\overline{8},-8/21)] + 4\,[H(\overline{9},-27/56)] + 2\,[H(\overline{10},-25/42)] \\
& + 1\,[H(\overline{11},-121/168)] + 19\,[H(\overline{13},-1/168)] + 14\,[H(\overline{14},-1/6)] + 6\,[H(\overline{15},-19/56)] + 4\,[H(\overline{16},-11/21)] \\
& + 1\,[H(\overline{17},-121/168)] + 13\,[H(\overline{19},-25/168)] + 5\,[H(\overline{20},-8/21)] + 2\,[H(\overline{21},-5/8)] + 13\,[H(\overline{23},-25/168)] \\
& + 6\,[H(\overline{24},-3/7)] + 1\,[H(\overline{25},-121/168)] + 17\,[H(\overline{26},-1/42)] + 6\,[H(\overline{27},-19/56)] + 1\,[H(\overline{28},-2/3)] \\
& + 19\,[H(\overline{29},-1/168)] + 8\,[H(\overline{30},-5/14)] + 1\,[H(\overline{31},-121/168)] + 14\,[H(\overline{32},-2/21)] + 4\,[H(\overline{33},-27/56)] \\
& + 8\,[H(\overline{35},-7/24)] + 1\,[H(\overline{36},-5/7)] + 13\,[H(\overline{37},-25/168)] + 2\,[H(\overline{38},-25/42)] + 18\,[H(\overline{39},-3/56)] \\
& + 4\,[H(\overline{40},-11/21)] + 19\,[H(\overline{41},-1/168)] + 1\,[H(\overline{42},-1/2)]\ .
\end{aligned}
\tag{B.12}
$$

$d = 43$:

$$
\begin{aligned}
K^\circ - 0\lambda = {} & 17\,[H(\overline{1},-1/172)] + 20\,[H(\overline{2},-1/43)] + 15\,[H(\overline{3},-9/172)] + 16\,[H(\overline{4},-4/43)] + 12\,[H(\overline{5},-25/172)] \\
& + 11\,[H(\overline{6},-9/43)] + 7\,[H(\overline{7},-49/172)] + 6\,[H(\overline{8},-16/43)] + 3\,[H(\overline{9},-81/172)] + 3\,[H(\overline{10},-25/43)] \\
& + 1\,[H(\overline{11},-121/172)] + 12\,[H(\overline{14},-6/43)] + 9\,[H(\overline{15},-53/172)] + 3\,[H(\overline{16},-21/43)] + 2\,[H(\overline{17},-117/172)] \\
& + 15\,[H(\overline{19},-17/172)] + 6\,[H(\overline{20},-14/43)] + 3\,[H(\overline{21},-97/172)] + 19\,[H(\overline{23},-13/172)] + 5\,[H(\overline{24},-15/43)] \\
& + 3\,[H(\overline{25},-109/172)] + 9\,[H(\overline{27},-41/172)] + 2\,[H(\overline{28},-24/43)] + 10\,[H(\overline{30},-10/43)] \\
& + 3\,[H(\overline{31},-101/172)] + 6\,[H(\overline{33},-57/172)] + 1\,[H(\overline{34},-31/43)] + 14\,[H(\overline{35},-21/172)] + 3\,[H(\overline{36},-23/43)] \\
& + 6\,[H(\overline{38},-17/43)] + 9\,[H(\overline{40},-13/43)] + 1\,[H(\overline{41},-133/172)] + 9\,[H(\overline{42},-11/43)]\ .
\end{aligned}
\tag{B.13}
$$

$d = 46$:

$$
\begin{aligned}
K^\circ - 1\lambda = {} & 19\,[H(\overline{1},-1/184)] + 16\,[H(\overline{2},-1/46)] + 16\,[H(\overline{3},-9/184)] + 15\,[H(\overline{4},-2/23)] + 14\,[H(\overline{5},-25/184)] \\
& + 10\,[H(\overline{6},-9/46)] + 8\,[H(\overline{7},-49/184)] + 5\,[H(\overline{8},-8/23)] + 4\,[H(\overline{9},-81/184)] + 3\,[H(\overline{10},-25/46)] \\
& + 2\,[H(\overline{11},-121/184)] + 17\,[H(\overline{14},-3/46)] + 10\,[H(\overline{15},-41/184)] + 6\,[H(\overline{16},-9/23)] + 2\,[H(\overline{17},-105/184)] \\
& + 1\,[H(\overline{18},-35/46)] + 11\,[H(\overline{20},-4/23)] + 6\,[H(\overline{21},-73/184)] + 1\,[H(\overline{22},-29/46)] + 13\,[H(\overline{24},-3/23)] \\
& + 6\,[H(\overline{25},-73/184)] + 2\,[H(\overline{26},-31/46)] + 7\,[H(\overline{28},-6/23)] + 2\,[H(\overline{29},-105/184)] + 10\,[H(\overline{31},-41/184)] \\
& + 3\,[H(\overline{32},-13/23)] + 7\,[H(\overline{34},-13/46)] + 2\,[H(\overline{35},-121/184)] + 19\,[H(\overline{36},-1/23)] + 4\,[H(\overline{37},-81/184)] \\
& + 8\,[H(\overline{39},-49/184)] + 1\,[H(\overline{40},-16/23)] + 14\,[H(\overline{41},-25/184)] + 3\,[H(\overline{42},-27/46)] + 16\,[H(\overline{43},-9/184)] \\
& + 2\,[H(\overline{44},-12/23)] + 19\,[H(\overline{45},-1/184)] + 2\,[H(\overline{46},-1/2)]\ .
\end{aligned}
\tag{B.14}
$$

$d = 48$:

$$
\begin{aligned}
K^\circ - 0\lambda = {} & 17\,[H(\overline{1},-1/192)] + 14\,[H(\overline{2},-1/48)] + 19\,[H(\overline{3},-3/64)] + 15\,[H(\overline{4},-1/12)] + 12\,[H(\overline{5},-25/192)] \\
& + 10\,[H(\overline{6},-3/16)] + 9\,[H(\overline{7},-49/192)] + 6\,[H(\overline{8},-1/3)] + 3\,[H(\overline{9},-27/64)] + 3\,[H(\overline{10},-25/48)] \\
& + 3\,[H(\overline{11},-121/192)] + 20\,[H(\overline{14},-1/48)] + 11\,[H(\overline{15},-11/64)] + 6\,[H(\overline{16},-1/3)] + 3\,[H(\overline{17},-97/192)] \\
& + 2\,[H(\overline{18},-11/16)] + 12\,[H(\overline{20},-1/12)] + 9\,[H(\overline{21},-19/64)] + 3\,[H(\overline{22},-25/48)] + 9\,[H(\overline{25},-49/192)] \\
& + 3\,[H(\overline{26},-25/48)] + 1\,[H(\overline{27},-51/64)] + 15\,[H(\overline{28},-1/12)] + 6\,[H(\overline{29},-73/192)] + 17\,[H(\overline{31},-1/192)] \\
& + 9\,[H(\overline{32},-1/3)] + 1\,[H(\overline{33},-43/64)] + 14\,[H(\overline{34},-1/48)] + 6\,[H(\overline{35},-73/192)] + 1\,[H(\overline{36},-3/4)] \\
& + 12\,[H(\overline{37},-25/192)] + 3\,[H(\overline{38},-25/48)] + 6\,[H(\overline{40},-1/3)] + 12\,[H(\overline{42},-3/16)] + 3\,[H(\overline{43},-121/192)] \\
& + 12\,[H(\overline{44},-1/12)] + 2\,[H(\overline{45},-35/64)] + 20\,[H(\overline{46},-1/48)] + 3\,[H(\overline{47},-97/192)]\ .
\end{aligned}
\tag{B.15}
$$

$d = 49$:

$$
\begin{aligned}
K^\circ - 0\lambda = {} & 16\,[H(\overline{1},-1/196)] + 18\,[H(\overline{2},-1/49)] + 15\,[H(\overline{3},-9/196)] + 15\,[H(\overline{4},-4/49)] \\
& + 12\,[H(\overline{5},-25/196)] + 10\,[H(\overline{6},-9/49)] + 8\,[H(\overline{7},-1/4)] + 8\,[H(\overline{8},-16/49)] + 3\,[H(\overline{9},-81/196)] \\
& + 4\,[H(\overline{10},-25/49)] + 1\,[H(\overline{11},-121/196)] + 1\,[H(\overline{12},-36/49)] + 13\,[H(\overline{15},-29/196)] + 7\,[H(\overline{16},-15/49)] \\
& + 4\,[H(\overline{17},-93/196)] + 2\,[H(\overline{18},-32/49)] + 17\,[H(\overline{20},-2/49)] + 7\,[H(\overline{21},-1/4)] + 4\,[H(\overline{22},-23/49)] \\
& + 1\,[H(\overline{23},-137/196)] + 13\,[H(\overline{25},-37/196)] + 4\,[H(\overline{26},-22/49)] + 1\,[H(\overline{27},-141/196)] \\
& + 6\,[H(\overline{29},-57/196)] + 2\,[H(\overline{30},-29/49)] + 9\,[H(\overline{32},-11/49)] + 4\,[H(\overline{33},-109/196)] + 9\,[H(\overline{35},-1/4)] \\
& + 1\,[H(\overline{36},-30/49)] + 5\,[H(\overline{38},-18/49)] + 1\,[H(\overline{39},-149/196)] + 12\,[H(\overline{40},-8/49)] + 2\,[H(\overline{41},-113/196)] \\
& + 6\,[H(\overline{43},-85/196)] + 6\,[H(\overline{45},-65/196)] + 9\,[H(\overline{47},-53/196)] + 1\,[H(\overline{48},-37/49)] + 7/2\cdot[H(\overline{49},-1/4)]\ .
\end{aligned}
\tag{B.16}
$$

$d = 50$:

$$\begin{aligned}
K^\circ - 1\lambda = {} & 17\,[H(\overline{1},-1/200)] + 16\,[H(\overline{2},-1/50)] + 18\,[H(\overline{3},-9/200)] + 13\,[H(\overline{4},-2/25)] + 12\,[H(\overline{5},-1/8)] \\
& + 11\,[H(\overline{6},-9/50)] + 8\,[H(\overline{7},-49/200)] + 6\,[H(\overline{8},-8/25)] + 6\,[H(\overline{9},-81/200)] + 2\,[H(\overline{10},-1/2)] \\
& + 2\,[H(\overline{11},-121/200)] + 1\,[H(\overline{12},-18/25)] + 14\,[H(\overline{15},-1/8)] + 7\,[H(\overline{16},-7/25)] + 4\,[H(\overline{17},-89/200)] \\
& + 3\,[H(\overline{18},-31/50)] + 10\,[H(\overline{21},-41/200)] + 3\,[H(\overline{22},-21/50)] + 2\,[H(\overline{23},-129/200)] + 12\,[H(\overline{25},-1/8)] \\
& + 6\,[H(\overline{26},-19/50)] + 2\,[H(\overline{27},-129/200)] + 10\,[H(\overline{29},-41/200)] + 3\,[H(\overline{30},-1/2)] + 15\,[H(\overline{32},-3/25)] \\
& + 4\,[H(\overline{33},-89/200)] + 1\,[H(\overline{34},-39/50)] + 14\,[H(\overline{35},-1/8)] + 3\,[H(\overline{36},-12/25)] + 10\,[H(\overline{38},-11/50)] \\
& + 2\,[H(\overline{39},-121/200)] + 6\,[H(\overline{41},-81/200)] + 8\,[H(\overline{43},-49/200)] + 2\,[H(\overline{44},-17/25)] + 12\,[H(\overline{45},-1/8)] \\
& + 1\,[H(\overline{46},-29/50)] + 18\,[H(\overline{47},-9/200)] + 3\,[H(\overline{48},-13/25)] + 17\,[H(\overline{49},-1/200)] + 2\,[H(\overline{50},-1/2)]\ .
\end{aligned} \tag{B.17}$$

$d = 52$:

$$\begin{aligned}
K^\circ - 1\lambda = {} & 17\,[H(\overline{1},-1/208)] + 15\,[H(\overline{2},-1/52)] + 16\,[H(\overline{3},-9/208)] + 15\,[H(\overline{4},-1/13)] + 13\,[H(\overline{5},-25/208)] \\
& + 9\,[H(\overline{6},-9/52)] + 10\,[H(\overline{7},-49/208)] + 6\,[H(\overline{8},-4/13)] + 4\,[H(\overline{9},-81/208)] + 3\,[H(\overline{10},-25/52)] \\
& + 3\,[H(\overline{11},-121/208)] + 1\,[H(\overline{12},-9/13)] + 15\,[H(\overline{15},-17/208)] + 10\,[H(\overline{16},-3/13)] + 4\,[H(\overline{17},-81/208)] \\
& + 3\,[H(\overline{18},-29/52)] + 1\,[H(\overline{19},-153/208)] + 13\,[H(\overline{21},-25/208)] + 6\,[H(\overline{22},-17/52)] \\
& + 3\,[H(\overline{23},-113/208)] + 17\,[H(\overline{25},-1/208)] + 10\,[H(\overline{26},-1/4)] + 3\,[H(\overline{27},-105/208)] + 1\,[H(\overline{28},-10/13)] \\
& + 16\,[H(\overline{29},-9/208)] + 6\,[H(\overline{30},-17/52)] + 1\,[H(\overline{31},-129/208)] + 10\,[H(\overline{33},-49/208)] \\
& + 3\,[H(\overline{34},-29/52)] + 7\,[H(\overline{36},-3/13)] + 3\,[H(\overline{37},-121/208)] + 6\,[H(\overline{39},-5/16)] + 1\,[H(\overline{40},-9/13)] \\
& + 15\,[H(\overline{41},-17/208)] + 3\,[H(\overline{42},-25/52)] + 9\,[H(\overline{44},-4/13)] + 1\,[H(\overline{45},-153/208)] + 9\,[H(\overline{46},-9/52)] \\
& + 1\,[H(\overline{47},-129/208)] + 16\,[H(\overline{48},-1/13)] + 3\,[H(\overline{49},-113/208)] + 15\,[H(\overline{50},-1/52)] \\
& + 3\,[H(\overline{51},-105/208)]\ .
\end{aligned} \tag{B.18}$$

$d = 54$:

$$\begin{aligned}
K^\circ - 1\lambda = {} & 15\,[H(\overline{1},-1/216)] + 17\,[H(\overline{2},-1/54)] + 16\,[H(\overline{3},-1/24)] + 13\,[H(\overline{4},-2/27)] + 14\,[H(\overline{5},-25/216)] \\
& + 9\,[H(\overline{6},-1/6)] + 10\,[H(\overline{7},-49/216)] + 5\,[H(\overline{8},-8/27)] + 6\,[H(\overline{9},-3/8)] + 3\,[H(\overline{10},-25/54)] \\
& + 2\,[H(\overline{11},-121/216)] + 2\,[H(\overline{12},-2/3)] + 14\,[H(\overline{15},-1/24)] + 11\,[H(\overline{16},-5/27)] + 6\,[H(\overline{17},-73/216)] \\
& + 3\,[H(\overline{18},-1/2)] + 2\,[H(\overline{19},-145/216)] + 18\,[H(\overline{21},-1/24)] + 7\,[H(\overline{22},-13/54)] + 4\,[H(\overline{23},-97/216)] \\
& + 1\,[H(\overline{24},-2/3)] + 14\,[H(\overline{26},-7/54)] + 4\,[H(\overline{27},-3/8)] + 3\,[H(\overline{28},-17/27)] + 10\,[H(\overline{30},-1/6)] \\
& + 4\,[H(\overline{31},-97/216)] + 18\,[H(\overline{33},-1/24)] + 5\,[H(\overline{34},-19/54)] + 2\,[H(\overline{35},-145/216)] + 6\,[H(\overline{37},-73/216)] \\
& + 1\,[H(\overline{38},-37/54)] + 14\,[H(\overline{39},-1/24)] + 6\,[H(\overline{40},-11/27)] + 11\,[H(\overline{42},-1/6)] + 2\,[H(\overline{43},-121/216)] \\
& + 6\,[H(\overline{45},-3/8)] + 1\,[H(\overline{46},-43/54)] + 10\,[H(\overline{47},-49/216)] + 14\,[H(\overline{49},-25/216)] + 2\,[H(\overline{50},-31/54)] \\
& + 16\,[H(\overline{51},-1/24)] + 3\,[H(\overline{52},-14/27)] + 15\,[H(\overline{53},-1/216)] + 2\,[H(\overline{54},-1/2)]\ .
\end{aligned} \tag{B.19}$$

$d = 55$:

$$\begin{aligned}
K^\circ - 0\lambda = {} & 14\,[H(\overline{1},-1/220)] + 20\,[H(\overline{2},-1/55)] + 12\,[H(\overline{3},-9/220)] + 15\,[H(\overline{4},-4/55)] + 10\,[H(\overline{5},-5/44)] \\
& + 11\,[H(\overline{6},-9/55)] + 9\,[H(\overline{7},-49/220)] + 7\,[H(\overline{8},-16/55)] + 6\,[H(\overline{9},-81/220)] + 3\,[H(\overline{10},-5/11)] \\
& + 3\,[H(\overline{11},-11/20)] + 1\,[H(\overline{13},-169/220)] + 21\,[H(\overline{15},-1/44)] + 9\,[H(\overline{16},-9/55)] + 9\,[H(\overline{17},-69/220)] \\
& + 3\,[H(\overline{18},-26/55)] + 2\,[H(\overline{19},-141/220)] + 15\,[H(\overline{21},-1/220)] + 9\,[H(\overline{22},-1/5)] + 3\,[H(\overline{23},-89/220)] \\
& + 3\,[H(\overline{24},-34/55)] + 19\,[H(\overline{26},-4/55)] + 5\,[H(\overline{27},-69/220)] + 3\,[H(\overline{28},-31/55)] + 15\,[H(\overline{30},-1/11)] \\
& + 3\,[H(\overline{31},-81/220)] + 2\,[H(\overline{32},-36/55)] + 9\,[H(\overline{34},-14/55)] + 3\,[H(\overline{35},-25/44)] + 9\,[H(\overline{37},-49/220)] \\
& + 1\,[H(\overline{38},-31/55)] + 5\,[H(\overline{40},-3/11)] + 3\,[H(\overline{41},-141/220)] + 14\,[H(\overline{42},-1/55)] + 6\,[H(\overline{43},-89/220)] \\
& + 11\,[H(\overline{45},-9/44)] + 1\,[H(\overline{46},-34/55)] + 15\,[H(\overline{47},-9/220)] + 3\,[H(\overline{48},-26/55)] + 6\,[H(\overline{50},-4/11)] \\
& + 1\,[H(\overline{51},-181/220)] + 9\,[H(\overline{52},-16/55)] + 9\,[H(\overline{54},-14/55)]\ .
\end{aligned} \tag{B.20}$$

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Summary

This thesis concerns a subject from algebraic geometry, a branch of mathematics. Geometry is the study of spatial structures; algebraic geometry looks at spatial objects that can be described using polynomial formulas and uses abstract algebraic methods to study properties of those objects. The possibility to use the power and precision of algebraic methods in combination with geometric intuition makes this a beautiful subject.

K3 surfaces are a class of 2-dimensional geometric objects. There are infinitely many distinct K3 surfaces; it is not possible to enumerate them all. However, it is possible to create a "catalogue", in which every possible K3 surface occurs exactly once. This catalogue itself can be seen to be a geometric object; it is called the moduli space of K3 surfaces. A point of this moduli space corresponds to a particular K3 surface; a small displacement within the moduli space gives a small deformation of the surface.

In this thesis we study the structure of the moduli space of K3 surfaces. It turns out that so-called modular forms are relevant to this. These are functions that behave in a very special way under the action of a discrete group of transformations. These modular forms contain a surprising amount of number-theoretic information.

Samenvatting

Dit proefschrift behandelt een onderwerp uit de algebraische meetkunde, een vakgebied binnen de wiskunde. Meetkunde is de studie van ruimtelijke structuren; algebraische meetkunde bestudeert ruimtelijke objecten die kunnen worden beschreven met polynomiale formules, en gebruikt abstracte rekenkundige methoden om eigenschappen van die objecten te onderzoeken. De mogelijkheid om zowel de precisie en kracht van algebraische methoden alsook ruimtelijke intuïtie te gebruiken maakt dit tot een boeiend vakgebied.

K3-oppervlakken vormen een klasse van 2-dimensionale meetkundige objecten. Er zijn oneindig veel verschillende K3-oppervlakken; het is niet goed mogelijk om ze allemaal op te sommen. Toch is het mogelijk om een "catalogus" te maken, waarin elk mogelijk K3-oppervlak precies een keer voorkomt. Deze catalogus heeft zelf ook een ruimtelijke structuur en wordt de moduliruimte van K3-oppervlakken genoemd: een punt van deze moduliruimte correspondeert met een specifiek K3-oppervlak; een kleine verplaatsing binnen de moduliruimte geeft een kleine vervorming van het oppervlak.

In dit proefschrift wordt de structuur van de moduliruimte van K3-oppervlakken onderzocht. Hierbij blijken zogeheten modulaire vormen een grote rol te spelen. Dit zijn functies die zich op een bijzondere manier gedragen onder de werking van een discrete groep van transformaties. Deze modulaire vormen bevatten een verrassende hoeveelheid getaltheoretische informatie.