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Generalized minimum-phase relations for memory functions associated with wave phenomena

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SUMMARY

Memory functions occur as temporal convolution operators in governing equations of wave propagation and generally account for the instantaneous and non-instantaneous responses of a medium. The specific memory function that is causal and stable, and the inverse of which is causal and stable as well, is conventionally referred to as a minimum-phase (MP) function. Its amplitude and phase spectra are not independent, but related through MP-relations; that is, Kramers–Kronig relations between the amplitude and phase spectra. In this paper, we derive generalized MP-relations for a memory function that does not necessarily meet the stability requirements; such functions are often encountered in the geophysical context. We still address the function as MP because its phase spectrum exhibits minimum group delay, like that of a conventional MP function. We successfully tested the derived relations for the well-known Maxwell, Kelvin–Voigt and Zener compressibility models used in acoustics/elastodynamics, the dynamic permeability used in poroelasticity and the electrokinetic coefficient used in coupled acoustics and electromagnetics. In these fields, the derived relations can be applied for the determining the involved memory function using numerical or laboratory experiments; only the amplitude or the phase spectrum needs to be measured and the other can be calculated. The relations also have applications in effective-medium theory and for any other wave phenomenon that employs memory functions.

Key words: Persistence, memory, correlations, clustering; Elasticity and anelasticity; Seismic attenuation; Wave propagation; Acoustic properties.

1 INTRODUCTION

Memory functions show up as temporal convolution operators in the governing equations of wave propagation in lossy and homogenized scattering media. They describe the response of a medium due to certain excitations and can therefore also be addressed as medium-response functions. For instance, the material density $\rho = \rho(t)$ and compressibility $\kappa = \kappa(t)$ memory functions that occur in the equations for a homogeneous dissipative fluid (e.g. de Hoop 1995),

$$\rho * \partial_t^2 \mathbf{u} + \nabla p = \mathbf{0}, \quad (1)$$

$$\kappa * p + \nabla \cdot \mathbf{u} = 0, \quad (2)$$

generally account for both the instantaneous and non-instantaneous responses of the medium; κ relates pressure (excitation) and strain (response), and ρ relates acceleration (excitation) and pressure gradient (response). In eqs (1) and (2), $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the particle-displacement vector, $p = p(\mathbf{x}, t)$ the acoustic pressure, the asterisk represents temporal convolution, t denotes time, \mathbf{x} the Cartesian coordinate vector and $\partial_t^2 = \partial^2/\partial t^2$. The notion that memory functions

can be written as temporal convolution operators dates back to Boltzmann (1876).

Probably the most well-known models for memory functions in geophysics are the rheological models based on springs and dashpots that are particularly used to describe creep and relaxation phenomena (e.g. Bourbié *et al.* 1987; Dahlen & Tromp 1998; Carcione 2007). Memory functions are often employed in poroelasticity: for example, the frequency-dependent viscodynamic function (Norris 1986), the dynamic permeability (Johnson *et al.* 1987), the electrokinetic coefficient for coupled electromagnetic and acoustic wave propagation (Pride 1994) and effective-medium moduli that account for energy loss due to the presence of inhomogeneities (Norris 1993; Johnson 2001; Pride *et al.* 2004; Müller *et al.* 2010). In electromagnetics, memory functions are also widely used: the frequency-dependent permittivity, conductivity and magnetic permeability (e.g. de Hoop 1975, 1995; Landau & Lifshitz 1984; Carcione 2007). One can also think of memory functions that account for scattering losses in inhomogeneous media (e.g. Sheng 1995; Schubert & Koehler 2004).

A discussion of properties of memory functions is given by Fabrizio & Morro (1992, 2003) and Carcione (2007). One of the

most important properties of a memory function, which we denote as $f(t)$, is its causal character; that is, $f(t) = 0$ for $t < 0$, which is a consequence of the ‘primitive causality condition’, stating that the response (thus related to the excitation through the memory function) cannot precede the excitation (Nussenzveig 1972). The real and imaginary parts of its frequency-domain counterpart $\hat{f}(\omega)$, where ω denotes angular frequency, are related via the well-known Plemelj or Kramers–Kronig causality relations (Toll 1956; Nussenzveig 1972). These can be derived from the causality-imposed property that the Laplace-transformed function $\hat{f}(s)$ is free of singularities (poles, branch points) in the right half of the s -plane (s denotes the Laplace parameter).

It is obvious that the inverse of a memory function needs to be causal as well. For instance, the inverse of κ in eq. (2) should relate strain (excitation) and pressure (response) in a causal way. The specific memory function that is also stable (absolutely integrable) and has a stable inverse possesses the minimum-phase (MP) property (Bode 1959; Oppenheim & Schaffer 1999). This means that the phase spectrum has minimum group delay (Oppenheim & Schaffer 1999; Aki & Richards 2002), which implies that the phase is the smallest possible for a given amplitude spectrum so that $f(t)$ remains causal. It also implies that another pair of relations holds that arises from the fact that $\hat{f}(s)$ is free of both singularities and zeros in the right half-plane [and, in fact, on the imaginary axis/frequency axis (Oppenheim & Schaffer 1999)] so that its logarithm is regular as well. These ‘MP-relations’ express the interdependence of the amplitude and phase spectra, and provide a big advantage for the experimental determination of a memory function; either the amplitude or the phase spectrum needs to be determined, and the other can be calculated. In some cases, it is particularly hard to determine the phase reliably in an experiment. The relations can be useful in different kinds of laboratory or numerical experiments, both in the case of steady-state (harmonic) and transient (pulse) excitations (e.g. Blair 1996; Reppert & Morgan 2001; Carcione *et al.* 2003, 2011; Schoemaker *et al.* 2012). When both the amplitude and phase spectra are obtained from an experiment, the relations can be used to validate the measured data (Ehm *et al.* 2000). We address the relations as ‘conventional MP-relations’.

When a function and its inverse are causal and stable, both must consist of an instantaneous part (described by the Dirac delta function) and a delay part. It implies that both amplitude spectra tend to a constant value at infinite frequency. In many cases, however, the function and its inverse are not both stable. For instance, the function might include a Heaviside step function. For such a function, the conventional MP-relations do not hold as the logarithm of its amplitude spectrum becomes unbounded at infinite frequency. However, as both the function and its inverse are causal functions of time, its Laplace transform is still free of zeros and singularities in the right half-plane, and the phase spectrum should therefore also exhibit minimum group delay (Bode 1959; Aki & Richards 2002). Hence, it is challenging to see whether any relations can be derived between the amplitude and phase spectra.

To accomplish that, Bode (1959) suggested using some scaling of the involved function by frequency to overcome the unboundedness, depending on its specific behaviour at infinite frequency. Nussenzveig (1972) proposed the method of subtractions but only applied this to the Kramers–Kronig causality relations. Aki & Richards (2002) suggested more general scaling by frequency but used this for the MP-property of a propagating pulse (i.e. of a propagation factor with complex-valued wavenumber in the argument). This property results in Kramers–Kronig relations between real and imaginary parts of the wavenumber (e.g. Nussenzveig 1972; Fang & Müller

1991; Aki & Richards 2002), which combines different memory functions in one quantity [e.g. in acoustics both $\hat{\kappa}(s)$ and $\hat{\rho}(s)$; cf. eqs (1) and (2)].

In this paper, we derive ‘generalized MP-relations’ for individual memory functions without assuming stability of the function and its inverse. We employ the scaling method as suggested by Aki & Richards (2002) and, to verify the validity of the relations, we give particular attention to the presence of poles and zeros on the imaginary s -axis. These often occur in the origin due the presence of time integrals or derivatives in the memory functions. For instance, in view of eq. (2), depending on whether strain, strain rate or strain acceleration is measured for applied pressure, different memory functions $\kappa(t)$, $\partial_t \kappa(t)$ or $\partial_t^2 \kappa(t)$ can be determined (e.g. in an experiment); all of them possess the MP-property. Furthermore, we show that the obtained relations capture the conventional MP-relations and discuss the connection between the MP-property of individual memory functions and that of a propagating pulse to highlight their complementary character. Throughout the paper, causality is understood as primitive causality (Nussenzveig 1972), unless stated differently.

The paper is organized as follows. First, we review the derivation of the conventional MP-relations (Section 2). Then, we derive the generalized MP-relations and discuss their applicability (Section 3). In Section 4, we show how the MP-relations can be implemented numerically. The relations are tested for a few well-known models in Section 5. The discussion is given in Section 6, and we finish with conclusions in Section 7.

2 CONVENTIONAL MP-RELATIONS

In this section, we review the derivation of the conventional MP-relations. It is similar to the derivation of the Kramers–Kronig causality relations between the real and imaginary parts of the Laplace transform $\hat{f}(s)$ of a causal function as given by de Hoop (1995), and it clarifies the particular constraints of the involved functions. We consider a real-valued function $f(t)$. As $f(t)$ and its inverse are both causal, $\hat{f}(s)$ has no singularities in the right half of the s -plane (from here onwards abbreviated as ‘RHP’) and, in addition, no zeros in that part of the s -plane. As a result, $\ln(\hat{f}(s))$ is also regular in the RHP. Furthermore, as $f(t)$ is stable as well as its inverse, the behaviour of both $\hat{f}(s)$ and $\hat{f}^{-1}(s)$ is $o(s)$ as $|s| \rightarrow \infty$ in the RHP; that is, $\lim_{|s| \rightarrow \infty} |\hat{f}(s)/s| = 0$ and $\lim_{|s| \rightarrow \infty} |\hat{f}^{-1}(s)/s| = 0$ (note that o denotes the small Landau order symbol). This implies that $\lim_{|s| \rightarrow \infty} \hat{f}(s) = f_\infty$ is a real constant, which corresponds to the instantaneous (Dirac) part of $f(t)$. Poles and zeros on the imaginary axis are also excluded due to the stability constraint (Oppenheim & Schaffer 1999), but branch points are allowed (as long as they are not zeros).

To derive MP-relations between the amplitude and phase of $\hat{f}(s)$, we consider the ratio $(\ln(\hat{f}(s)) - \ln(f_\infty))/(s - i\Omega)$, where Ω is real and denotes (a fixed) frequency. In the numerator, we take $\ln(\hat{f}(s))$ instead of simply $\hat{f}(s)$, as in the derivation of the Kramers–Kronig causality relations; the subtraction is referred to below. We choose an integration contour C in the s -plane consisting of a vertical line, a semi-circle C_∞ of infinite radius, and a semi-circle C_Ω of infinitesimal radius around $s = i\Omega$ (see Fig. 1). We consider the limiting situation in which the vertical line approaches the imaginary axis from the RHP. Now, by virtue of Cauchy’s theorem (Titchmarsh 1939; Plemelj 1964), it holds that

$$\oint_C \frac{\ln(\hat{f}(s)) - \ln(f_\infty)}{s - i\Omega} ds = 0. \quad (3)$$

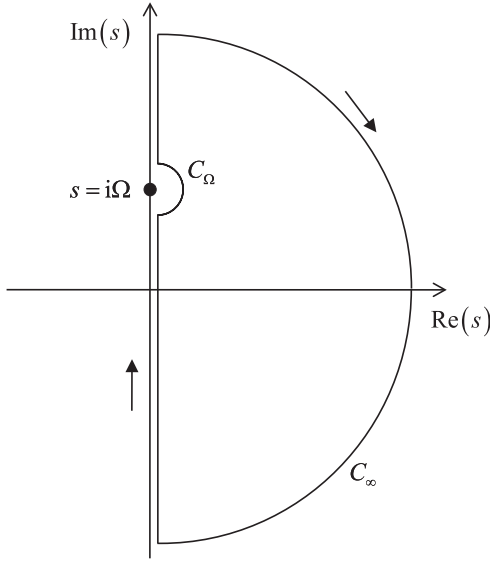


Figure 1. Closed contour of integration C in the complex s -plane used to establish the conventional MP-relations. It consists of a line parallel to imaginary axis with vanishing real part, an infinitesimal arc C_Ω around the pole (black dot) $s = i\Omega$ and an arc of infinite radius C_∞ . The direction of integration is indicated.

The contribution of C_∞ vanishes (Bode 1959; de Hoop 1975):

$$\int_{C_\infty} \frac{\ln(\hat{f}(s)) - \ln(f_\infty)}{s - i\Omega} ds \rightarrow 0, \quad (4)$$

which is due to the subtraction in the numerator:

$$\ln(\hat{f}(s)) - \ln(f_\infty) \rightarrow 0 \quad \text{as } |s| \rightarrow \infty \quad \text{in } \text{Re}(s) > 0. \quad (5)$$

The vanishing of the numerator is guaranteed by the above-described stability requirements of $\hat{f}(s)$ and its inverse [i.e. $\hat{f}(s) \rightarrow f_\infty$]. Now, we can establish a relation between the contributions along the imaginary axis and around $s = i\Omega$. Using a local Taylor expansion and $s = i\Omega + r \exp(i\theta)$ (where θ is the circumferential coordinate and r the radius), the integration along the infinitely small semi-circle C_Ω yields

$$\begin{aligned} & \int_{C_\Omega} \frac{\ln(\hat{f}(s)) - \ln(f_\infty)}{s - i\Omega} ds \\ &= i \lim_{r \rightarrow 0} \int_{-\pi/2}^{\pi/2} \left(\ln(\hat{f}(i\Omega) + \partial_s \hat{f}(s)|_{s=i\Omega} r e^{i\theta}) - \ln(f_\infty) \right) d\theta \\ &= i\pi \left(\ln(\hat{f}(i\Omega)) - \ln(f_\infty) \right), \end{aligned} \quad (6)$$

where $\hat{f}(i\Omega)$ is finite [i.e. $\ln(\hat{f}(s))$ is regular at $s = i\Omega$]. The contribution of the path along the imaginary s -axis yields, using $s \rightarrow i\omega$,

$$\begin{aligned} & \lim_{r \rightarrow 0} \left(\int_{-\infty}^{\Omega-r} + \int_{\Omega+r}^{\infty} \right) \frac{\ln(\hat{f}(\omega)) - \ln(f_\infty)}{\omega - \Omega} d\omega \\ &= \int_{-\infty}^{\infty} \frac{\ln(\hat{f}(\omega)) - \ln(f_\infty)}{\omega - \Omega} d\omega. \end{aligned} \quad (7)$$

Here, the \int denotes that the principal value of the integral should be taken. Taking the contributions of C_Ω and the imaginary axis together according to eq. (3), we get [replacing $\hat{f}(i\Omega)$ by $\hat{f}(\Omega)$]

$$\ln \left(\frac{\hat{f}(\Omega)}{f_\infty} \right) = -\frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\ln(\hat{f}(\omega)/f_\infty)}{\omega - \Omega} d\omega. \quad (8)$$

Now, using $\ln(\hat{f}(s)) = \ln|\hat{f}(s)| + i\varphi(s)$, where $-\pi < \varphi \leq \pi$, and separating between the real and imaginary parts, we obtain

$$\ln \left| \frac{\hat{f}(\Omega)}{f_\infty} \right| = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(\omega)}{\omega - \Omega} d\omega, \quad (9)$$

$$\varphi(\Omega) = +\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln|\hat{f}(\omega)/f_\infty|}{\omega - \Omega} d\omega. \quad (10)$$

Here, Ω is real and, without loss of generality, f_∞ is assumed positive so that $\varphi_\infty = 0$. Eqs (9) and (10) form a Hilbert transform pair and are referred to as the ‘conventional MP-relations’ between the amplitude and phase spectra. We address the functions for which eqs (9) and (10) hold as ‘conventional MP-functions’. The phase can be uniquely determined from the amplitude using eq. (10). The spectrum as obtained from eq. (9), however, can be a scaled version of the true one as the phase is insensitive to constant multiplication factors in $\hat{f}(\omega)$.

In general, any causal function that has a causal inverse and obeys the constraint in eq. (5) should satisfy the conventional MP-relations. The opposite is, however, not generally true; once a certain function satisfies the MP-relations, that function and its inverse are not necessarily causal. This is because causality of the function and its inverse is only a necessary (i.e. not a necessary and sufficient) condition for eq. (3) to be true (Titchmarsh 1939).

It is important to note that, due to the limiting situation $s \rightarrow i\omega$ that we consider (as discussed earlier), $\hat{f}(\omega)$ is understood as the Laplace transform of $f(t)$ with $\text{Re}(s) \downarrow 0$, both in this section and throughout the entire paper.

3 GENERALIZED MP-RELATIONS

In many cases, the memory function does not obey the specific requirement of $\hat{f}(s) \rightarrow f_\infty$ as $|s| \rightarrow \infty$ in the RHP; cf. eq. (5). For instance, when an instantaneous part is absent, $\hat{f}(s)$ goes to zero, and when time derivatives are involved, it goes to infinity; in both cases $\ln(\hat{f}(s))$ becomes unbounded, which thus violates the basic requirements of conventional MP-functions. However, scaling by s , as suggested by Aki & Richards (2002) in a slightly different context, solves the problem of unboundedness (see further). For this reason, as a starting point we consider (cf. eq. 3)

$$\oint_C \frac{\ln(\hat{f}(s))}{s(s - i\Omega)} ds = 0, \quad (11)$$

where C now also includes an infinitesimal semi-circle C_0 around the pole at $s = 0$ (see Fig. 2). The contribution of C_∞ vanishes due to the scaling:

$$\int_{C_\infty} \frac{\ln(\hat{f}(s))}{s(s - i\Omega)} ds \rightarrow 0, \quad (12)$$

which is true provided that

$$\frac{\ln(\hat{f}(s))}{s} \rightarrow 0 \quad \text{as } |s| \rightarrow \infty \quad \text{in } \text{Re}(s) > 0. \quad (13)$$

This is, in turn, guaranteed by any function that can be expanded in a polynomial having finite powers of s as $|s| \rightarrow \infty$ in the RHP, because $\lim_{|s| \rightarrow \infty} \ln(s^n)/s = \lim_{|s| \rightarrow \infty} n \ln(s)/s = 0$ [here n represents the leading power of $\hat{f}(s)$]. Using local Taylor expansions of $\hat{f}(s)$ around $s = 0$ and $s = i\Omega$, and assuming $\ln(\hat{f}(s))$ regular at these points (cf. eq. 6), the integration along the two infinitely small

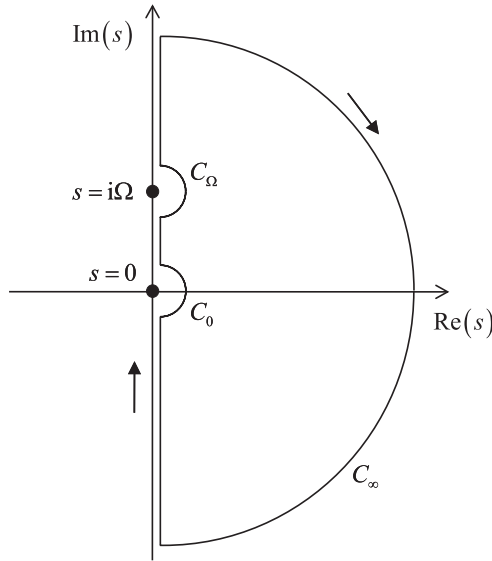


Figure 2. Closed contour of integration C in the complex s -plane used to establish the generalized MP-relations. It consists of a line parallel to imaginary axis with vanishing real part, two infinitesimal arcs C_Ω and C_0 around the poles (black dots) at $s = i\Omega$ and $s = 0$, respectively, and an arc of infinite radius C_∞ . The direction of integration is indicated.

semi-circles yields (for $\Omega \neq 0$)

$$\begin{aligned} \int_{C_\Omega} \frac{\ln(\hat{f}(s))}{s(s-i\Omega)} ds &= i \lim_{r \rightarrow 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\ln(\hat{f}(i\Omega) + \partial_s \hat{f}(s)|_{s=i\Omega} r e^{i\theta})}{i\Omega + r e^{i\theta}} d\theta \\ &= \pi \frac{\ln(\hat{f}(i\Omega))}{\Omega}, \end{aligned} \quad (14)$$

$$\begin{aligned} \int_{C_0} \frac{\ln(\hat{f}(s))}{s(s-i\Omega)} ds &= i \lim_{r \rightarrow 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\ln(\hat{f}(0) + \partial_s \hat{f}(s)|_{s=0} r e^{i\theta})}{-i\Omega + r e^{i\theta}} d\theta \\ &= -\pi \frac{\ln(\hat{f}(0))}{\Omega}, \end{aligned} \quad (15)$$

where $\hat{f}(0)$ denotes $\hat{f}(s=0)$. Taking now all contributions together (including that of the imaginary axis, which is similar to eq. 7) according to eq. (11), we get

$$\ln \left(\frac{\hat{f}(\Omega)}{\hat{f}(0)} \right) = -\frac{\Omega}{i\pi} \int_{-\infty}^{\infty} \frac{\ln(\hat{f}(\omega))}{\omega(\omega - \Omega)} d\omega. \quad (16)$$

Separating the real and imaginary parts again, we finally obtain

$$\ln \left| \frac{\hat{f}(\Omega)}{\hat{f}(0)} \right| = -\frac{\Omega}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(\omega)}{\omega(\omega - \Omega)} d\omega, \quad (17)$$

$$\varphi(\Omega) = +\frac{\Omega}{\pi} \int_{-\infty}^{\infty} \frac{\ln|\hat{f}(\omega)|}{\omega(\omega - \Omega)} d\omega, \quad (18)$$

where $\Omega \neq 0$ (see eqs 14 and 15); the principal-value integrals are taken at both $\omega = \Omega$ and $\omega = 0$, and $\varphi(0) = 0$ has been applied [which is generally true for real-valued $f(t)$, by virtue of $\varphi(s) = -\varphi(s^*)$]. We address eqs (17) and (18) as ‘generalized MP-relations’ as they hold for any memory function that obeys the constraint in eq. (13). This constraint is less strict than and captures that of the conventional MP-functions (eq. 5).

In the case that $\Omega = 0$, the semi-circles C_Ω and C_0 merge to one semi-circle, the contribution of which can be shown to be propor-

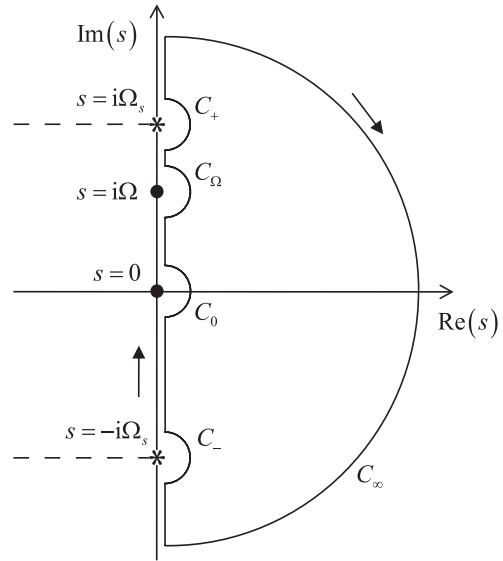


Figure 3. Closed contour of integration in the complex s -plane used to establish the generalized MP-relations for a function having poles or zeros at $s = \pm i\Omega_s$. These constitute logarithmic branch points (*) of $\ln(\hat{f}(s))$ from which branch cuts (dashed lines) depart. The contour consists of a line parallel to imaginary axis with vanishing real part, two infinitesimal arcs C_Ω and C_0 around the poles (black dots) at $s = i\Omega$ and $s = 0$, respectively, two infinitesimal arcs C_\pm around the branch points, and an arc of infinite radius C_∞ . The direction of integration is indicated.

tional to $\ln(\hat{f}(0))r^{-1}|_{r \rightarrow 0} = \infty$. Hence, the above-described scaling of $\hat{f}(s)$ does not give meaningful relations for $\Omega = 0$.

In the situation that $\hat{f}(s)$ has zeros or poles on the imaginary s -axis, logarithmic singularities/branch points are present at $s = \pm i\Omega_s$, where $\Omega_s \geq 0$ [poles or zeros always appear in pairs for real-valued $f(t)$, except when $\Omega_s = 0$]. The case that $\Omega_s \neq 0$ corresponds to some resonance in the medium response, and $\Omega_s = 0$ simply indicates the presence of time derivative(s) or integral(s). To account for the singularities, additional semi-circles C_\pm around $s = \pm i\Omega_s$ have to be incorporated in the contour integration (see eq. 11), and branch cuts need to be introduced (see Fig. 3). Distinguishing between four different cases, the integrations along the infinitely small semi-circles yield

$$\begin{aligned} \int_{C_\pm} \frac{\ln(\hat{f}(s))}{s(s-i\Omega)} ds &= i \lim_{r \rightarrow 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\ln(\hat{f}(\pm i\Omega_s + r e^{i\theta})) r e^{i\theta}}{(\pm i\Omega_s + r e^{i\theta})(i(\pm\Omega_s - \Omega) + r e^{i\theta})} d\theta \\ &= \begin{cases} 0, & \Omega \neq \pm\Omega_s, & \Omega_s \neq 0, & \text{(I, } C_\pm) \\ -\frac{\pi}{\Omega} \ln(\hat{f}(0)), & \Omega \neq 0, & \Omega_s = 0, & \text{(II, } C_\pm) \\ +\frac{\pi}{\Omega_s} \ln(\hat{f}(+i\Omega_s)), & \Omega = +\Omega_s, & \Omega_s \neq 0, & \text{(III, } C_+) \\ 0, & \Omega = +\Omega_s, & \Omega_s \neq 0, & \text{(III, } C_-) \\ 0, & \Omega = -\Omega_s, & \Omega_s \neq 0, & \text{(IV, } C_+) \\ -\frac{\pi}{\Omega_s} \ln(\hat{f}(-i\Omega_s)), & \Omega = -\Omega_s, & \Omega_s \neq 0, & \text{(IV, } C_-) \end{cases} \end{aligned} \quad (19)$$

where local Laurent [for poles in $\hat{f}(s)$] and Taylor [for zeros in $\hat{f}(s)$] series expansions of $\hat{f}(s)$ around $s = \pm i\Omega_s$ have been applied; obviously, $\ln(\hat{f}(0))$ and $\ln(\hat{f}(\pm i\Omega_s))$ are unbounded. The results are 0 for case I, provided that the leading power of r in the Laurent and

Taylor series expansions of $\hat{f}(s)$ is not infinite; thus $\hat{f}(s)$ should only have poles and zeros of finite order.

We can verify that in most cases the generalized MP-relations (eqs 17 and 18) are still obtained when all contributions along the contour C are taken together as in eq. (16) (i.e. now including the contribution of eq. 19). In case I of eq. (19), the integrations along C_{\pm} obviously do not add anything. Both eqs (17) and (18) thus remain valid. In case II, C_{\pm} merge with C_0 to one semi-circle, the contribution of which should be taken only once. The logarithm in the left-hand side of eq. (17) becomes unbounded, making this relation inapplicable for all Ω . In cases III and IV, either C_+ or C_- has a non-zero contribution (depending on whether $\Omega = +\Omega_s$ or $\Omega = -\Omega_s$), and Ω is replaced by $\pm\Omega_s$ in eqs (17) and (18). The left-hand side of the former equation thus becomes $\ln |\hat{f}(\pm\Omega_s)/\hat{f}(0)|$, which is unbounded. However, the relation is still applicable for $\Omega \neq \pm\Omega_s$ (which is case I). In the cases II–IV, eq. (18), which is probably the most important relation in view of geophysical applications, remains valid. In general, it is important to note that, when $\hat{f}(s)$ has zeros or poles at $s = \pm i\Omega_s$, the integrals in eqs (17) and (18) should be taken in the principal-value sense at these points.

In eq. (19), we left out the cases that $\Omega = \Omega_s = 0$ and $\Omega = 0$ and $\Omega_s \neq 0$, because we knew beforehand that no meaningful relations can be obtained for $\Omega = 0$, as explained above (cf. eqs 14 and 15).

The derivation in this section shows that MP-relations can thus also be obtained for any (not necessarily stable) causal memory function that has a causal inverse, and the Laplace transform of which can be written as a polynomial with finite powers of s when $|s| \rightarrow \infty$ in the RHP; the latter constraint covers most of the memory functions of physical interest. This confirms our expectation; a function without zeros and singularities in the RHP does possess the MP-property, and it should therefore be possible to find relations between its interdependent amplitude and phase spectra. The generalized MP-relations also capture memory functions that include distributions/generalized functions. We note that not all captured functions can be directly measured in a physical experiment, but given the excitation and the response, they can be extracted from measured data in band-limited sense [cf. Section 1, and eqs (1) and (2) in particular].

The phase spectrum can be uniquely calculated from the amplitude spectrum using eq. (18). To calculate the amplitude spectrum from the phase spectrum, one obviously also needs $\hat{f}(0)$, which is unknown beforehand. Hence, the amplitude spectrum can only be found relative to a constant (the dc-component). In addition, the integral in eq. (17) might give a scaled version of the left-hand side as the phase is insensitive to multiplication factors (see Section 2). Sometimes the low- or high-frequency limit of $\hat{f}(\omega)$ is known from theoretical considerations, or additional measurements are available. In any case, the value of \hat{f} at only one (fixed) ω is needed to determine the amplitude spectrum unambiguously. In the case that $\hat{f}(0)$ is zero or infinite, the amplitude spectrum cannot be calculated from the phase spectrum (as discussed earlier).

Like for the conventional MP-relations, we note that any causal function that has a causal inverse and obeys the constraint in eq. (13) should satisfy the generalized MP-relations. However, once a certain function satisfies the generalized MP-relations, that function and its inverse are not necessarily causal; causality of the function and its inverse is only a necessary condition for eq. (11) (Titchmarsh 1939).

Similar relations to eqs (17) and (18) were found by Papoulis (1962). These relations were derived separately, using different starting points, while our derivation shows that the relations can be derived from a joint basis (eq. 11). Furthermore, our derivation unambiguously reveals the broad class of functions for which the re-

lations hold true (cf. eq. 13). It also clarifies that the relations remain valid when there are singularities on the imaginary s -axis, depending on their specific locations and the order of the singularities (cf. eq. 19).

4 IMPLEMENTATION

For application of the derived MP-relations, it is convenient to rewrite the principal-value integrals to proper integrals. Using symmetry properties and a standard integral, eqs (9) and (10) can be written as (see Appendix A)

$$\ln \left| \frac{\hat{f}(\Omega)}{\hat{f}_{\infty}} \right| = -\frac{2}{\pi} \int_0^{\infty} \frac{\varphi(\omega)\omega - \varphi(\Omega)\Omega}{\omega^2 - \Omega^2} d\omega, \quad (20)$$

$$\varphi(\Omega) = \frac{2\Omega}{\pi} \int_0^{\infty} \frac{\ln |\hat{f}(\omega)/\hat{f}(\Omega)|}{\omega^2 - \Omega^2} d\omega, \quad (21)$$

with $0 < \Omega < \infty$, and where the integrals around $\omega = \Omega$ are no longer principal-value integrals. The integrands are integrable, which allows straightforward numerical evaluation. The relations are known as the ‘Bode relations’ (Bode 1959; de Hoop 1995).

The generalized MP-relations (eqs 17 and 18) can be rewritten in a similar way (Appendix A). The result is

$$\ln \left| \frac{\hat{f}(\Omega)}{\hat{f}(0)} \right| = -\frac{2\Omega}{\pi} \int_0^{\infty} \frac{\varphi(\omega)\Omega - \varphi(\Omega)\omega}{\omega(\omega^2 - \Omega^2)} d\omega, \quad (22)$$

$$\varphi(\Omega) = \frac{2\Omega}{\pi} \int_0^{\infty} \frac{\ln |\hat{f}(\omega)/\hat{f}(\Omega)|}{\omega^2 - \Omega^2} d\omega, \quad (23)$$

with $0 < \Omega < \infty$. The integrals are now proper ones at both $\omega = 0$ and at $\omega = \Omega$, but still their principal values should be taken when a singularity is present at $\omega = \Omega_s$. There are various methods available to accomplish this (e.g. Criscuolo & Scuderi 1998). It should be noted that the integral in eq. (22) is only finite when $\varphi(\omega) \rightarrow 0$ when $\omega \downarrow 0$. If this is not the case, the integral is divergent. This is to be expected because, in that case, the left-hand side is also infinite due to the presence of a zero or a pole in $\hat{f}(s)$ at $s = 0$ (cf. case II of eq. 19); for example, see the Maxwell model in Section 5. Furthermore, eqs (21) and (23) are very similar, but the assumed properties of the integrands are different [in particular, the behaviour of $\ln |\hat{f}(\omega)|$ as $\omega \rightarrow \infty$, and the possible singularities].

Limited bandwidth data can be dealt with using a method proposed by Ehm *et al.* (2000).

5 NUMERICAL EXAMPLES

In this section, we test the derived generalized MP-relations for a few well-known memory functions from acoustics/elastodynamics, poroelasticity, and coupled acoustics and electromagnetics. In reality, one applies the relations to compute either the phase or the amplitude spectrum when one of them is unknown. For the models that we consider, we know both of them, but in all examples, we used either the amplitude or the phase spectrum to simulate a measurement. We computed the other spectrum using eq. (23) or (22), respectively, using an adaptive eight-point Legendre-Gauss algorithm (Abramowitz & Stegun 1972; Davis & Rabinowitz 1975) and compare it with the known result (other standard numerical-integration algorithms can be used as well).

We first consider the Maxwell, Kelvin–Voigt and Zener models that are associated with viscoacoustic or viscoelastic behaviour

of materials (e.g. Bourbié *et al.* 1987; Dahlen & Tromp 1998; Carcione 2007). These rheological models are particularly related to the compressibility $\kappa(t)$, as introduced in eq. (2). The specific expressions for the three models are known (see Appendix B); here, we only show their Laplace transforms:

$$\hat{\kappa}(s) = \begin{cases} \kappa_\infty + \frac{1}{\eta s}, & \text{(Maxwell)} \\ \frac{1}{K_\infty + \eta s}, & \text{(Kelvin-Voigt)} \\ \frac{1}{K_\infty} \left(1 + \frac{\tau_1 - \tau_2}{\tau_1 \tau_2} \frac{1}{s + \tau_1^{-1}} \right). & \text{(Zener)} \end{cases} \quad (24)$$

Here, η denotes the dynamic fluid viscosity, κ_∞ and K_∞ are the instantaneous parts of $\kappa(t)$ and its inverse $K(t)$, respectively, where the latter is the compression-modulus memory function; τ_1 and τ_2 are timescales related to strain and stress, respectively, with $\tau_1 > \tau_2$ (Carcione 2007). Note that $K_\infty \neq \lim_{|s| \rightarrow \infty} \hat{K}(s)$ (in the RHP) for the Kelvin-Voigt model; this limit is unbounded. Yet, the constant K_∞ is meaningful as $K(t)$ does have an instantaneous part.

We can observe that the functions in eq. (24) do not have singularities and zeros in the RHP. Hence, in all models $\kappa(t)$ and its inverse are causal functions of time. However, not all models provide conventional MP-functions (*cf.* eq. 5); only the Zener model does. We applied the generalized MP-relations and the results for the phase (using eq. 23) are displayed in Fig. 4 for all three models. The results for the amplitude (using eq. 22) are displayed in Fig. 5, for the Zener and Kelvin-Voigt models only; here, the Maxwell model is absent as there is a pole at $s = 0$ and hence eq. (22) (or eq. 17) cannot be used (see Section 3). The involved parameter values are taken from Carcione (2007, p. 75). The computed spectra coincide with the analytical ones, which confirms the validity of the generalized MP-relations. We emphasize that the Zener model could have been handled with the conventional MP-relations, but the current result verifies that the generalized MP-relations also capture conventional MP-functions, as argued in Section 3.

Now, we consider two memory functions that occur in poroelasticity. Incorporating the coupling of acoustic and electromagnetic fields (Pride 1994; Schoemaker *et al.* 2012), Biot's equation of

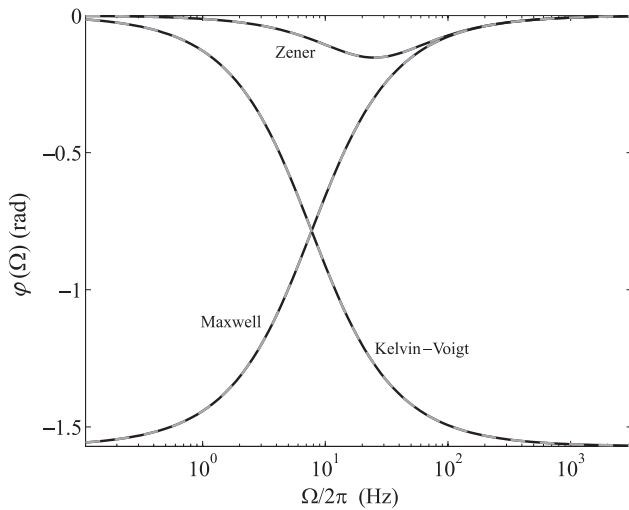


Figure 4. Phase spectra of the Maxwell, Kelvin-Voigt and Zener models: analytical (solid black line) and computed using eq. (23) (dashed grey lines); for each of the models, the lines coincide. Ω denotes angular frequency. For K_∞ , $\kappa_\infty = 1/K_\infty$, η , τ_1 and τ_2 , the same parameter values have been used as in Carcione (2007, p. 75).

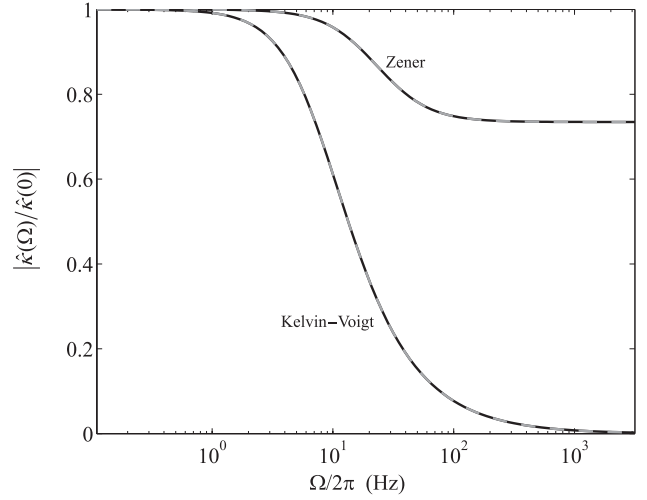


Figure 5. Normalized amplitude spectra of the Kelvin-Voigt and Zener models: analytical (solid black line) and computed using eq. (22) (dashed grey lines); for both models, the lines coincide. Ω denotes angular frequency. For K_∞ , $\kappa_\infty = 1/K_\infty$, η , τ_1 and τ_2 , the same parameter values have been used as in Carcione (2007, p. 75).

motion of the fluid phase can be written in the Laplace domain as

$$s \hat{\mathbf{w}} = -\frac{\hat{k}(s)}{\eta} (\nabla \hat{p} + s^2 \rho_f \hat{\mathbf{u}}) + \hat{L}(s) \hat{\mathbf{E}}, \quad (25)$$

where $\hat{\mathbf{w}} = \phi(\hat{\mathbf{U}} - \hat{\mathbf{u}})$ denotes the displacement of the fluid $\hat{\mathbf{U}}$ relative to that of the porous frame $\hat{\mathbf{u}}$ [multiplied by the porosity ϕ], \hat{p} the fluid pressure, $\hat{\mathbf{E}}$ the electric field, ρ_f the fluid density and η the dynamic fluid viscosity [the fields $\hat{\mathbf{u}}$, $\hat{\mathbf{U}}$, $\hat{\mathbf{w}}$, \hat{p} and $\hat{\mathbf{E}}$ are dependent on \mathbf{x} and s]; $\hat{k}(s)$ and $\hat{L}(s)$ are memory functions that denote the dynamic permeability and the electrokinetic coupling factor, respectively. Johnson *et al.* (1987) postulated the following function for the dynamic permeability [based on rigid-porous frame ($\hat{\mathbf{u}} = \mathbf{0}$) considerations]:

$$\hat{k}(s) = \frac{k_0}{\sqrt{1 + s\tau_c + \frac{2}{M}s\tau_c}}. \quad (26)$$

Here, $\text{Re}(\sqrt{1 + s\tau_c}) \geq 0$ in the entire s -plane, k_0 denotes the Darcy permeability, $\tau_c = M/(2\omega_c)$ and M denotes a pore shape factor; $\omega_c = \eta\phi/(\alpha\rho_f k_0)$ is Biot's transition frequency in which α denotes the tortuosity of the porous frame. An expression for the electrokinetic coupling factor was postulated by Pride (1994):

$$\hat{L}(s) = \frac{L_0}{\sqrt{1 + s\tau_d \left(1 - 2\frac{d}{\Lambda}\right)^2 \left(1 + d\sqrt{s\frac{\rho_f}{\eta}}\right)^2}}, \quad (27)$$

where $\text{Re}(\sqrt{s}) \geq 0$ in the entire s -plane, L_0 is the static electrokinetic coupling coefficient, $\tau_d = 2/(M\omega_c)$, d is the Debye length and Λ a characteristic pore size parameter.

We can verify that the functions in eqs (26) and (27) are not conventional MP-functions (eq. 5). The calculated phase and amplitude spectra are displayed in Figs 6 and 7, respectively, and coincide with the analytical results (see Table 1 for the involved parameter values). In fact, it is not a surprise that the generalized MP-relations hold true for the dynamic permeability and the electrokinetic coupling factor. These functions were postulated, exactly based on causality arguments of the functions themselves and their inverses (Johnson *et al.* 1987; Pride 1994); hence, the generalized MP-relations must be satisfied.

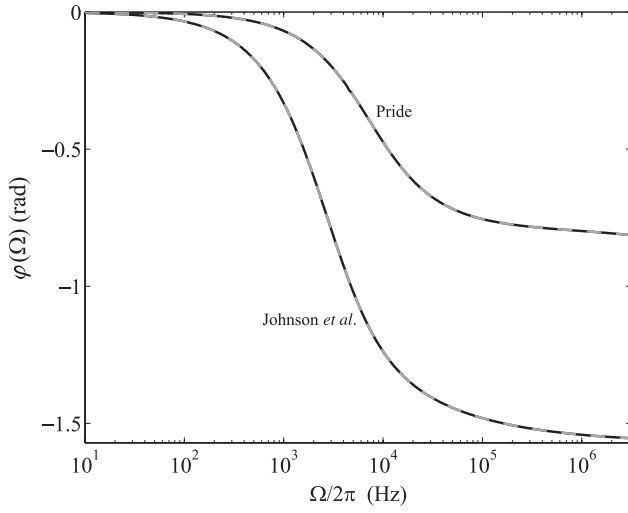


Figure 6. Phase spectra of the dynamic permeability (Johnson *et al.* 1987) [$\hat{\psi}(\Omega) = \hat{k}(\Omega)$] and the electrokinetic coupling factor (Pride 1994): analytical (solid black line) and computed using MP-relation (eq. 23) (dashed grey lines); for both models, the lines coincide. Ω denotes angular frequency. The used parameter values are given in Table 1; results are independent of L_0 .

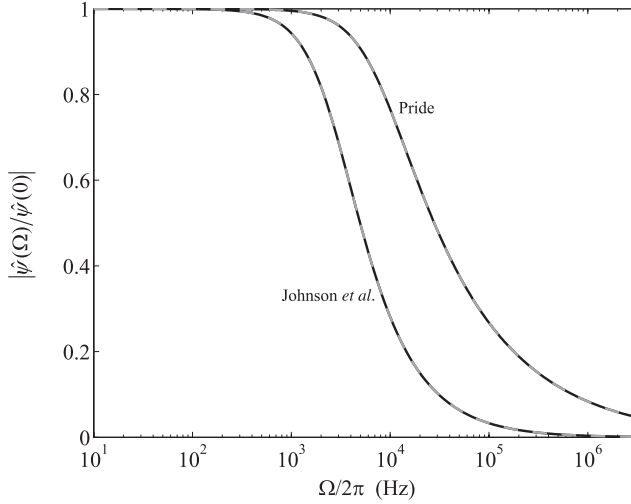


Figure 7. Normalized amplitude spectra of the dynamic permeability (Johnson *et al.* 1987) [$\hat{\psi}(\Omega) = \hat{k}(\Omega)$] and the electrokinetic coupling factor (Pride 1994) [$\hat{\psi}(\Omega) = \hat{L}(\Omega)$]: analytical (solid black line) and computed using MP-relation (eq. 22) (dashed grey lines); for both models, the lines coincide. Ω denotes angular frequency. For the used parameter values, see Table 1.

For a model in which the involved memory function has not been postulated based on explicit causality considerations, the generalized MP-relations can offer a sanity check (e.g. for the alternative expression for the dynamic permeability derived by Müller & Sahay 2011). Violation of the relations indicates that the function and/or its inverse are not causal. However, we emphasize that causality is not necessarily guaranteed when the MP-relations are satisfied (see Sections 2 and 3).

6 DISCUSSION

In many studies, the MP-property of a propagating pulse is considered (i.e. of a propagation factor with complex-valued wavenumber in the argument). This results in Kramers–Kronig relations

Table 1. Material parameters as used for the dynamic permeability and the electrokinetic coupling factor (see Figs 6 and 7).

Fluid density ρ_f (kg m ⁻³)	1000
Porosity ϕ	0.35
Darcy permeability k_0 (m ²)	8×10^{-12}
Tortuosity α	1.93
Dynamic fluid viscosity η (Pa·s)	0.001
Pore shape factor M	1
Debye length d (m)	9.6×10^{-9}
Length scale pores Λ (m)	1.88×10^{-5}

between the real and imaginary parts of the wavenumber, or equivalently, between the phase velocity and the attenuation factor (e.g. Nussenzweig 1972; Fang & Müller 1991; Blair 1996; Aki & Richards 2002). Here, we discuss the connection between the MP-property of an individual memory function, as considered in the main part of this paper, and the MP-property of a propagation factor, which combines different memory functions in the wavenumber.

The basic assumption from which the MP-property of a propagation factor originates is the consideration that a pulse needs a finite time to travel over a finite distance. For the infinite-space Green's function $G(\mathbf{x}, t)$, this comes down to the fact that $G(\mathbf{x}, t) = 0$ for $t < |\mathbf{x}|/c_\infty$, where c_∞ denotes a finite maximum propagation velocity. Using this condition, and taking the Laplace transform $\hat{G}(\mathbf{x}, s) = \exp(-\hat{\gamma}(s)|\mathbf{x}|)/(4\pi|\mathbf{x}|)$, with $\text{Re}(\hat{\gamma}(s)) \geq 0$ in the RHP (de Hoop 1995), one can find that the function

$$\hat{w}(s) = \exp(-\hat{\gamma}(s)|\mathbf{x}|) \exp(+s|\mathbf{x}|/c_\infty), \quad (28)$$

is free of singularities and zeros in the RHP (Nussenzweig 1972; Aki & Richards 2002); $\hat{\gamma}(s)$ is in general complex-valued and related to the wavenumber: $\hat{q}(s) = -i\hat{\gamma}(s)$. The function $\hat{w}(s)$ is sometimes referred to as the attenuation operator (Fang & Müller 1991). It is found by disregarding the factor $4\pi|\mathbf{x}|$ of $\hat{G}(\mathbf{x}, s)$ and removing the linear phase that is related to the finite traveltime $|\mathbf{x}|/c_\infty$ of the pulse. Starting with $\ln(\hat{w}(s))$, which is thus regular in the RHP, the contour integration can be performed in a similar way as in Section 2. Taking $s \rightarrow i\omega$ along the imaginary s -axis, one finds that

$$\Omega \left(\frac{1}{c(\Omega)} - \frac{1}{c_\infty} \right) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\omega)}{\omega - \Omega} d\omega, \quad (29)$$

$$\alpha(\Omega) = +\frac{1}{\pi} \int_{-\infty}^{\infty} \omega \left(\frac{1}{c(\omega)} - \frac{1}{c_\infty} \right) d\omega, \quad (30)$$

where $c(\omega)$ denotes the phase velocity and $\alpha(\omega)$ is the attenuation factor (imaginary part of the wavenumber). Eqs (29) and (30) are obtained, provided that $\hat{\gamma}(s) \rightarrow s/c_\infty$ as $|s| \rightarrow \infty$ in the RHP (to let the contribution of the infinite semi-circle vanish; cf. Fig. 1); this corresponds to the finite-propagation-velocity requirement. For example, in acoustic wave propagation (cf. eqs 1 and 2), this requirement implies that $s\sqrt{\hat{k}(s)\hat{\rho}(s)} \rightarrow s/c_\infty$, which comes down to the fact that $c(\omega) = 1/\text{Re}\sqrt{\hat{k}(\omega)\hat{\rho}(\omega)} \rightarrow c_\infty$ and $\alpha(\omega) = \omega \text{Im}\sqrt{\hat{k}(\omega)\hat{\rho}(\omega)} \rightarrow 0$ as $|\omega| \rightarrow \infty$. Hence, the phase velocity should be bounded at infinite frequency to guarantee the finite propagation velocity of the pulse. Note that for situations in which $\alpha(\omega)$ behaves differently at infinite frequency, modified versions of eqs (29) and (30) exist (e.g. Aki & Richards 2002).

In Section 3, we derived MP-relations for individual memory functions that are not necessarily finite as $|s| \rightarrow \infty$ in the RHP, but to get a realistic model, the combination of the memory functions apparently needs to yield a finite phase velocity in this limit. The

MP-property of a propagation factor is thus more strict in the sense that it relates to the so-called ‘macroscopic causality condition’ or ‘relativistic causality condition’, stating that there is a limiting velocity for the propagation of signals (Nussenzweig 1972).

On the other hand, the MP-property of a propagation factor relates to the constraint that the combination of memory functions is only free of singularities in the RHP. Zeros in $\hat{\gamma}(s)$ are permitted as $\ln(\hat{w}(s))$ remains regular when $\hat{\gamma}(s)$ is zero. In many cases, this comes down to the fact that all involved memory functions should only be free of singularities in the RHP, as $\hat{\gamma}(s)$ simply contains products of memory functions, like in acoustics and electromagnetics; in poroelasticity, this constraint also applies to an individual memory function (the viscodynamic operator). The MP-property of an individual memory function, however, relates to the constraint that its Laplace transform is free of both singularities and zeros in the RHP. Hence, we argue that the MP-property of an individual memory function is complementary to (and in a way more strict than) that of a propagation factor because it also incorporates primitive causality of the inverse of the considered memory function.

As an example, consider the application of the Kelvin–Voigt model for the Lamé parameters $\lambda(t)$ and $\mu(t)$ to model viscoelastic wave propagation (e.g. Carcione *et al.* 2004). The memory function associated with this model possesses the MP-property of an individual memory function (see Section 5). The fact that $\hat{\lambda}(s)$ and $\hat{\mu}(s)$ become unbounded as $|s| \rightarrow \infty$ (*cf.* eq. 24), however, requires the material density to be s -dependent as well to ensure that $\text{Re}\sqrt{\hat{\rho}(s)/(\hat{\lambda}(s) + 2\hat{\mu}(s))}$ is finite in this limit (i.e. to satisfy the macroscopic causality condition). Otherwise, errors are introduced especially at the onset of the arriving pulse. One can avoid these errors by band-limiting the pulse (i.e. by suppressing the infinitely fast propagating high-frequency harmonics, as it was done by Carcione *et al.* 2004), but strictly speaking, the model is physically inadmissible.

7 CONCLUSIONS

Memory functions occur as temporal convolution operators in the governing equations of wave propagation in lossy and homogenized scattering media, and capture the medium response due to certain excitations. Based on the requirement of primitive causality of such a memory function and its inverse, and assuming that the memory function and its inverse are both stable (absolutely integrable in time), relations can be derived between its amplitude and phase spectra. These relations express that both spectra are interdependent and are addressed as ‘conventional MP-relations’.

In many cases, the stability requirements associated with the conventional MP-relations are not met, and the relations therefore do not hold. However, for any memory function encountered in wave phenomena, the causality requirements are satisfied, and its phase spectrum therefore still exhibits minimum group delay, like in the conventional case. In this paper, we derived ‘generalized MP-relations’ between the amplitude and phase spectra of a memory function that does not necessarily meet the stability requirements. The relations capture any memory function for which the Laplace Transform can be expanded in a polynomial of the Laplace parameter as long as the involved powers are finite; this covers most functions of physical interest. The memory function can have finite-order poles and zeros on the imaginary axis in the Laplace domain (i.e. the frequency axis). In specific cases, their presence makes one of the relations inapplicable, but the relation to compute the phase from the amplitude spectrum, being the most important one in

geophysical applications, remains valid. The derived relations (both conventional and generalized) are complementary to the widely used relations between the phase velocity and attenuation factor that express the MP-property of a propagating pulse.

We successfully tested the generalized MP-relations for the well-known Maxwell, Kelvin–Voigt and Zener compressibility models used in acoustics/elastodynamics, the dynamic permeability used in poroelasticity and the electrokinetic coefficient used in coupled acoustics and electromagnetics. The relations can also be used in effective-medium theory and for any other wave phenomenon that involves memory functions; either the amplitude or the phase spectrum needs to be measured, and the other can be calculated.

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APPENDIX A: REWRITING OF INTEGRALS

In this appendix, we rewrite the derived minimum phase (MP)-relations to the more convenient forms given in eqs (20)–(23). First, we address the conventional MP-relations. Using $\bar{\omega} = -\omega$ and $\varphi(-\omega) = -\varphi(\omega)$, eq. (9) can be rewritten as

$$\begin{aligned} \ln \left| \frac{\hat{f}(\Omega)}{f_\infty} \right| &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(\omega)}{\omega - \Omega} d\omega \\ &= -\frac{1}{\pi} \left(-\int_0^{\infty} \frac{\varphi(-\bar{\omega})}{\bar{\omega} + \Omega} d\bar{\omega} + \int_0^{\infty} \frac{\varphi(\omega)}{\omega - \Omega} d\omega \right) \\ &= -\frac{2}{\pi} \left(\int_0^{\infty} \frac{\varphi(\omega)\omega}{\omega^2 - \Omega^2} d\omega \right). \end{aligned} \quad (\text{A1})$$

Now, using the following standard integral (de Hoop 1975)

$$\int_0^{\infty} \frac{1}{\omega^2 - \Omega^2} d\omega = 0, \quad (\text{A2})$$

where $0 < \Omega < \infty$, we get

$$\begin{aligned} \ln \left| \frac{\hat{f}(\Omega)}{f_\infty} \right| &= -\frac{2}{\pi} \left(\int_0^{\infty} \frac{\varphi(\omega)\omega}{\omega^2 - \Omega^2} d\omega - \int_0^{\infty} \frac{\varphi(\Omega)\Omega}{\omega^2 - \Omega^2} d\omega \right) \\ &= -\frac{2}{\pi} \int_0^{\infty} \frac{\varphi(\omega)\omega - \varphi(\Omega)\Omega}{\omega^2 - \Omega^2} d\omega. \end{aligned} \quad (\text{A3})$$

Note that the integrand in the final expression is not singular at $\omega = \Omega$; the integral is therefore a proper one. In a similar way, using $|\hat{f}(\omega)| = |\hat{f}(-\omega)|$ and eq. (A2), eq. (10) can be rewritten as

$$\begin{aligned} \varphi(\Omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\hat{f}(\omega)/f_\infty|}{\omega - \Omega} d\omega \\ &= \frac{2\Omega}{\pi} \int_0^{\infty} \frac{\ln |\hat{f}(\omega)/f_\infty|}{\omega^2 - \Omega^2} d\omega \\ &= \frac{2\Omega}{\pi} \left(\int_0^{\infty} \frac{\ln |\hat{f}(\omega)/f_\infty|}{\omega^2 - \Omega^2} d\omega - \int_0^{\infty} \frac{\ln |\hat{f}(\Omega)/f_\infty|}{\omega^2 - \Omega^2} d\omega \right) \\ &= \frac{2\Omega}{\pi} \int_0^{\infty} \frac{\ln |\hat{f}(\omega)/\hat{f}(\Omega)|}{\omega^2 - \Omega^2} d\omega. \end{aligned} \quad (\text{A4})$$

Now, we rewrite the generalized MP-relations. Eq. (17) becomes

$$\begin{aligned} \ln \left| \frac{\hat{f}(\Omega)}{\hat{f}(0)} \right| &= -\frac{\Omega}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(\omega)}{\omega(\omega - \Omega)} d\omega \\ &= -\frac{2\Omega}{\pi} \int_0^{\infty} \frac{\varphi(\omega)\Omega}{\omega(\omega^2 - \Omega^2)} d\omega \\ &= -\frac{2\Omega}{\pi} \left(\int_0^{\infty} \frac{\varphi(\omega)\Omega}{\omega(\omega^2 - \Omega^2)} d\omega - \int_0^{\infty} \frac{\varphi(\Omega)\omega}{\omega(\omega^2 - \Omega^2)} d\omega \right) \\ &= -\frac{2\Omega}{\pi} \int_0^{\infty} \frac{\varphi(\omega)\Omega - \varphi(\Omega)\omega}{\omega(\omega^2 - \Omega^2)} d\omega, \end{aligned} \quad (\text{A5})$$

and eq. (18) can be rewritten as

$$\begin{aligned} \varphi(\Omega) &= \frac{\Omega}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\hat{f}(\omega)|}{\omega(\omega - \Omega)} d\omega \\ &= \frac{2\Omega}{\pi} \int_0^{\infty} \frac{\ln |\hat{f}(\omega)|}{\omega^2 - \Omega^2} d\omega \\ &= \frac{2\Omega}{\pi} \left(\int_0^{\infty} \frac{\ln |\hat{f}(\omega)|}{\omega^2 - \Omega^2} d\omega - \int_0^{\infty} \frac{\ln |\hat{f}(\Omega)|}{\omega^2 - \Omega^2} d\omega \right) \\ &= \frac{2\Omega}{\pi} \int_0^{\infty} \frac{\ln |\hat{f}(\omega)/\hat{f}(\Omega)|}{\omega^2 - \Omega^2} d\omega. \end{aligned} \quad (\text{A6})$$

Here, the remaining principal-value integrals relate to a possible singularity in $\hat{f}(\omega)$ at $\omega = \Omega_s$. Eqs (A3)–(A6) only hold for $0 < \Omega < \infty$.

APPENDIX B: TIME-DOMAIN EXPRESSIONS OF RHEOLOGICAL MODELS

Here, we give the time-domain expressions of the Maxwell, Kelvin–Voigt and Zener models, whose Laplace transforms are used in Section 5. The expressions are known in the literature, but we include them for clarity; in particular, the Zener model is sometimes addressed differently. The expressions read (e.g. Bourbié *et al.* 1987; Dahlen & Tromp 1998; Carcione 2007)

$$\kappa(t) = \begin{cases} \kappa_\infty \delta(t) + \frac{1}{\eta} H(t), & \text{(Maxwell)} \\ \frac{1}{k_\infty} e^{-\frac{\eta}{k_\infty} t} H(t), & \text{(Kelvin–Voigt)} \\ \frac{1}{k_\infty} \left(\delta(t) + \frac{\tau_1 - \tau_2}{\tau_1 \tau_2} e^{-\frac{t}{\tau_1}} H(t) \right), & \text{(Zener)} \end{cases} \quad (\text{B1})$$

where $\delta(t)$ and $H(t)$ denote the Dirac and Heaviside step functions, respectively. The involved parameters are explained in Section 5.