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SELF-DIFFUSION FOR A WEAKLY-COUPLED PLASMA IN A MAGNETIC FIELD

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The longitudinal self-diffusion coefficient for a magnetized plasma with a small plasma parameter is calculated from kinetic theory in the weak-coupling approximation. Asymptotic expressions for this coefficient are derived in the limits of weak and of strong magnetic field. For intermediate strength of the magnetic field numerical results are presented.

1. Introduction

Kinetic equations for a magnetized plasma with a small plasma parameter have been discussed by several authors[1]). However, up to now little is known about the quantitative dependence of transport phenomena on the magnetic field.

In this paper we study self-diffusion through a hot dilute plasma in a uniform magnetic field. The system considered is a one-component Coulomb plasma with a neutralizing background. We make use of a weak interaction approximation for the pair correlation function. The assumption that the single-particle distribution function satisfies a Markovian equation yields the Landau form for the collision term. For this model we derive the quantitative dependence on the magnetic field of the coefficient of self-diffusion along the field.

In the course of the calculation we have found that it is advantageous to avoid the usual unwieldy series involving Bessel functions and to retain an integration with respect to time. We have obtained asymptotic expansions in ω_B/ω_p, the ratio of the Larmor and the plasma frequencies, both for strong and for weak magnetic fields. For intermediate values of ω_B/ω_p the integrations have been performed numerically.

2. Derivation from kinetic theory of an explicit expression for the longitudinal self-diffusion coefficient

For a spatially homogeneous plasma in a uniform stationary magnetic field $\boldsymbol{B}$ the single-particle distribution function f_1 satisfies the equation

$$\left[\frac{\partial}{\partial t} + \omega_B(\boldsymbol{p}_1 \wedge \hat{\boldsymbol{B}})\cdot\frac{\partial}{\partial \boldsymbol{p}_1}\right] f_1(\boldsymbol{p}_1, t)$$
$$= \frac{\partial}{\partial \boldsymbol{p}_1}\cdot \int \mathrm{d}\boldsymbol{r}_2\, \mathrm{d}\boldsymbol{p}_2 [\nabla_{\boldsymbol{r}_1} v(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)] g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t)\,, \tag{2.1}$$

where $g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t) = f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t) - f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t)$ is the pair correlation function, $v(r)$ the interaction between the particles, $\omega_B = eB/mc$ the Larmor frequency, and $\hat{\boldsymbol{B}}$ a unit vector along the magnetic field. Neglecting triple correlations we have the following equation for the pair correlation function:

$$\frac{\partial}{\partial t} g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t) = -\mathrm{i} L_0^B(\boldsymbol{r}_1, \boldsymbol{p}_1) g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$$
$$+ [\nabla_{\boldsymbol{r}_1} v(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)]\cdot \frac{\partial}{\partial \boldsymbol{p}_1} [f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) + g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t)]$$
$$+ \frac{\partial}{\partial \boldsymbol{p}_1}\cdot \int \mathrm{d}\boldsymbol{r}_3\, \mathrm{d}\boldsymbol{p}_3 [\nabla_{\boldsymbol{r}_1} v(|\boldsymbol{r}_1 - \boldsymbol{r}_3|)] f_1(\boldsymbol{p}_1, t) g(\boldsymbol{r}_2 - \boldsymbol{r}_3, \boldsymbol{p}_2, \boldsymbol{p}_3, t)$$
$$+ (1 \leftrightarrow 2)\,, \tag{2.2}$$

where $L_0^B(\boldsymbol{r}, \boldsymbol{p}) = -\mathrm{i}(\boldsymbol{p}/m)\cdot\nabla_{\boldsymbol{r}} - \mathrm{i}\omega_B(\boldsymbol{p} \wedge \hat{\boldsymbol{B}})\cdot\partial/\partial\boldsymbol{p}$ is the Liouville operator for a single particle in a magnetic field and $(1 \leftrightarrow 2)$ stands for the preceding terms with indices 1 and 2 interchanged.

The Landau-approximation to (2.1) is obtained by retaining only the contribution that is quadratic in the interaction. Hence we require an approximate solution g_{L} of (2.2) that is correct to first order in v. If we assume that g_{L} vanishes in the infinite past we have

$$g_{\mathrm{L}}(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, t) = \int_{-\infty}^{t} \mathrm{d}s \Big\{ \mathrm{e}^{-\mathrm{i}[L_0^B(\boldsymbol{r}_1, \boldsymbol{p}_1) + L_0^B(\boldsymbol{r}_2, \boldsymbol{p}_2)](t-s)}$$
$$\times [\nabla_{\boldsymbol{r}_1} v(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)]\cdot \frac{\partial}{\partial \boldsymbol{p}_1} f_1(\boldsymbol{p}_1, s) f_1(\boldsymbol{p}_2, s) + (1 \leftrightarrow 2)\Big\}\,. \tag{2.3}$$

The time evolution which describes the free motion in a uniform stationary magnetic field is given by

$$\mathrm{e}^{\mathrm{i}L_0^B t} F(\boldsymbol{r}, \boldsymbol{p}) = F[\boldsymbol{r}(t), \boldsymbol{p}(t)]\,, \tag{2.4}$$

where $\boldsymbol{r}(t)$ and $\boldsymbol{p}(t)$ are the solutions at time t of the equations of motion with initial condition $(\boldsymbol{r}, \boldsymbol{p})$:

$$\boldsymbol{r}(t) = \boldsymbol{r} + \frac{\boldsymbol{p}_\parallel t}{m} + \frac{1}{m\omega_B}[\boldsymbol{p}_\perp \sin\omega_B t + \boldsymbol{p}_\perp \wedge \hat{\boldsymbol{B}}(1 - \cos\omega_B t)] \equiv \boldsymbol{r} - \frac{\boldsymbol{\alpha}_{-t}\cdot\boldsymbol{p}}{m}\,, \tag{2.5a}$$

$$\boldsymbol{p}(t) = \boldsymbol{p}_\parallel + \boldsymbol{p}_\perp \cos\omega_B t + \boldsymbol{p}_\perp \wedge \hat{\boldsymbol{B}} \sin\omega_B t \equiv \boldsymbol{\gamma}_{-t}\cdot\boldsymbol{p}\,. \tag{2.5b}$$

The labels $\|$ and $\perp$ indicate components parallel to and perpendicular to the magnetic field, respectively. Obviously we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}_t \cdot \boldsymbol{p}) = \boldsymbol{\gamma}_t \cdot \boldsymbol{p} \,. \tag{2.6}$$

Now the approximate pair correlation function g_{L} is inserted into (2.1) and one arrives at the kinetic equation[2])

$$\left[\frac{\partial}{\partial t} + \omega_B(\boldsymbol{p}_1 \wedge \hat{\boldsymbol{B}}) \cdot \frac{\partial}{\partial \boldsymbol{p}_1}\right] f_1(\boldsymbol{p}_1, t) = \frac{\partial}{\partial \boldsymbol{p}_1} \cdot \int \mathrm{d}\boldsymbol{r}_2\, \mathrm{d}\boldsymbol{p}_2 \Bigg\{ [\nabla_{\boldsymbol{r}_1} v(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)] \times \int_{-\infty}^{0} \mathrm{d}s [\nabla_{\boldsymbol{r}_1(s)} v[|\boldsymbol{r}_1(s) - \boldsymbol{r}_2(s)|]] \cdot \frac{\partial}{\partial \boldsymbol{p}_1(s)} f_1[\boldsymbol{p}_1(s), s+t] f_1[\boldsymbol{p}_2(s), s+t] + (1 \leftrightarrow 2) \Bigg\} . \tag{2.7}$$

Since in this paper we will be discussing self-diffusion along the magnetic field only, we may assume the single-particle distribution function to be gyrotropic, i.e. cylindrically symmetric around $\boldsymbol{B}$. Then the second term in the left-hand side of (2.7) vanishes and the arguments $\boldsymbol{p}_{1,2}(s)$ in the single-particle distribution functions can be replaced by $\boldsymbol{p}_{1,2}$. Furthermore, we assume the single-particle distribution function to relax much slower than the pair correlation function. Hence we may neglect the difference between the values of f_1 at times $t+s$ and t, respectively. In this way the kinetic equation becomes Markovian. Using the Fourier transform of the interaction $v(k) = \int \mathrm{d}\boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} v(r)$ and changing s to $-s$ we obtain from (2.7)

$$\frac{\partial f_1(\boldsymbol{p}_1, t)}{\partial t} = \frac{\partial}{\partial \boldsymbol{p}_1} \cdot \int \mathrm{d}\boldsymbol{p}_2 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} v(k)^2 \times \left[\boldsymbol{k} \int_0^{\infty} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\alpha}_s\cdot(\boldsymbol{p}_1 - \boldsymbol{p}_2)/m} \boldsymbol{k} \cdot \boldsymbol{\gamma}_s\right] \cdot \left(\frac{\partial}{\partial \boldsymbol{p}_1} - \frac{\partial}{\partial \boldsymbol{p}_2}\right) f_1(\boldsymbol{p}_1, t) f_1(\boldsymbol{p}_2, t) \,. \tag{2.8}$$

This kinetic equation has been put forward earlier[2–5]). The above discussion clearly shows how the assumptions of gyrotropy and Markovian behaviour enter into the derivation.

If $f_1(\boldsymbol{p}_1, t) = n f_0(p)$, where n is the number density and $f_0(p) = (\beta/2\pi m)^{3/2} \exp(-\beta p^2/2m)$, the right-hand side of (2.8) vanishes. In fact, it becomes then

$$\mathrm{i}\, n^2 \beta \frac{\partial}{\partial \boldsymbol{p}_1} \cdot \int \mathrm{d}\boldsymbol{p}_2 f_0(p_1) f_0(p_2) \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} v(k)^2 \boldsymbol{k} \int_0^{\infty} \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s} [\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\alpha}_s\cdot(\boldsymbol{p}_1 - \boldsymbol{p}_2)/m}] \,. \tag{2.9}$$

Now the s-integration yields -1, so that the $\boldsymbol{k}$-integrand is anti-symmetric; thus the expression (2.9) vanishes.

In order to obtain a linearized collision operator we write

$$f_1(\boldsymbol{p}_1, t) = nf_0(p_1)[1 + h(\boldsymbol{p}_1, t)] . \tag{2.10}$$

For small deviations from equilibrium the kinetic equation (2.8) may be replaced by its linear form,

$$\frac{\partial f_1(\boldsymbol{p}_1, t)}{\partial t} = -n^2[Ih](\boldsymbol{p}_1, t) , \tag{2.11}$$

with a linearized collision operator I that follows directly from (2.8). Since we want to study self-diffusion we consider all particles but a small number of tagged ones to be in thermodynamic equilibrium. Correspondingly, the distribution function for the tagged particles satisfies a linearized kinetic equation, that contains I_s instead of I; explicitly,

$$[I_s h](\boldsymbol{p}_1, t) = -\frac{\partial}{\partial \boldsymbol{p}_1} \cdot \int \mathrm{d}\boldsymbol{p}_2 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} v(k)^2 f_0(p_1) f_0(p_2) \times \left[\boldsymbol{k} \int_0^\infty \mathrm{d}s \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\alpha}_s\cdot(\boldsymbol{p}_1-\boldsymbol{p}_2)/m} \boldsymbol{k}\cdot\boldsymbol{\gamma}_s \right] \cdot \frac{\partial}{\partial \boldsymbol{p}_1} h(\boldsymbol{p}_1, t) . \tag{2.12}$$

The longitudinal diffusion coefficient can now be expressed in terms of the so-called collision brackets corresponding to the linearized collision operator I_s. In lowest Chapman–Cowling approximation we have the simple form

$$\frac{1}{D_\parallel} = n\beta^2 [p_\parallel, p_\parallel]_s , \tag{2.13}$$

where $[p_\parallel, p_\parallel]_s = \int \mathrm{d}\boldsymbol{p}_1 \, p_{1,\parallel} I_s p_{1,\parallel}$. By performing a partial integration with respect to $\boldsymbol{p}_1$ we obtain the following expression:

$$\frac{1}{D_\parallel} = n\beta^2 \int \mathrm{d}\boldsymbol{p}_1 \, \mathrm{d}\boldsymbol{p}_2 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} v(k)^2 k_\parallel^2 f_0(p_1) f_0(p_2) \int_0^\infty \mathrm{d}s \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\alpha}_s\cdot(\boldsymbol{p}_1-\boldsymbol{p}_2)/m} . \tag{2.14}$$

The momentum integrations can be carried out

$$\int \mathrm{d}\boldsymbol{p} \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\alpha}_t\cdot\boldsymbol{p}/m} f_0(p) = \exp[-k_\parallel^2 t^2/(2m\beta) - k_\perp^2(1 - \cos\omega_B t)/(m\beta\omega_B^2)] . \tag{2.15}$$

For v we insert the screened Coulomb potential $v(k) = e^2/(k^2 + k_\mathrm{D}^2)$, where $k_\mathrm{D} = e\sqrt{\beta n}$ is the inverse Debye-length. Hence it will be convenient to introduce the dimensionless variables: $x = k/k_D$ and $t = \omega_p s$ in (2.14). With the usual coupling constant $\Gamma = \beta e^2/4\pi a$, where $a = (3/4\pi n)^{1/3}$, one has $k_D = \sqrt{3\Gamma}/a$, and the self-diffusion coefficient is given by

$$\frac{1}{D_\parallel} = \frac{1}{\omega_p a^2} \frac{2\Gamma^{5/2}3^{3/2}}{\pi} \int_0^X dx \frac{x^4}{(1+x^2)^2} \int_0^1 dz z^2 \int_0^\infty dt\, e^{-x^2 u(b,t) - x^2 z^2 v(b,t)}, \tag{2.16}$$

with $z = \hat{x} \cdot \hat{B}$, $b = \omega_B/\omega_p$ and the abbreviations:

$$\begin{aligned} u(b,t) &= \frac{2}{b^2}(1-\cos bt), \\ v(b,t) &= t^2 - u(b,t). \end{aligned} \tag{2.17}$$

Because the x-integration is divergent for large x a cut-off at X has been imposed. Since the Landau approximation certainly ceases to be valid for wavevectors larger than the inverse Landau length $r_L^{-1} = 4\pi/\beta e^2$ one may choose $X = r_D/r_L = 1/\sqrt{3}\,\Gamma^{3/2}$ as an upper bound.

For the cases $b \to 0$ and $b \to \infty$, corresponding to vanishing and to infinite magnetic field strength, respectively, the integrals over z and t are easily evaluated. Namely, for $b = 0$,

$$\int_0^1 dz z^2 \int_0^\infty dt\, e^{-t^2 x^2} = \frac{\sqrt{\pi}}{6x} \tag{2.18a}$$

and for $b \to \infty$

$$\int_0^1 dz z^2 \int_0^\infty dt\, e^{-t^2 x^2 z^2} = \frac{\sqrt{\pi}}{4x}. \tag{2.18b}$$

As a consequence a simple proportionality relation holds

$$\frac{D_\parallel(b \to \infty, \Gamma)}{D_\parallel(b = 0, \Gamma)} = \frac{2}{3}. \tag{2.19}$$

3. Asymptotic expressions for the self-diffusion coefficient

For general values of b we write the z-integral in (2.16) as a confluent hypergeometric function. One has[6])

$$\int_0^1 dz\, e^{-wz^2} = \frac{1}{2}\sqrt{\frac{\pi}{w}}\, \mathrm{erf}(\sqrt{w}) = {}_1F_1(1/2, 3/2, -w), \tag{3.1}$$

and hence, by differentiation with respect to w and a Kummer transformation,

$$\int_0^1 \mathrm{d}z\, z^2\, \mathrm{e}^{-wz^2} = \tfrac{1}{3}\mathrm{e}^{-w}\,{}_1F_1(1, 5/2, w)\,. \tag{3.2}$$

The divergent contribution in (2.16) will appear as a separate term, if we use

$${}_1F_1(1, 5/2, w) = 1 + (2/5)w\,{}_1F_1(1, 7/2, w)\,. \tag{3.3}$$

Combining (2.16) and (3.1)–(3.3) now yields

$$\begin{aligned} 1/D_\parallel &= \frac{1}{\omega_\mathrm{p} a^2}\Gamma^{5/2}\sqrt{\frac{3}{\pi}}\left[\tfrac{1}{2}\log(1+X^2) - \tfrac{1}{2} + \frac{1}{2(1+X^2)} + J(b,\Gamma)\right] \\ &\simeq \frac{1}{\omega_\mathrm{p} a^2}\Gamma^{5/2}\sqrt{\frac{3}{\pi}}\,[-\log(\sqrt{3}\,\Gamma^{3/2}) - \tfrac{1}{2} + J(b,\Gamma)] \quad (\Gamma \ll 1)\,, \end{aligned} \tag{3.4}$$

where

$$J(b,\Gamma) = \int_0^{X/b} \mathrm{d}x\, \frac{x^4}{(b^{-2}+x^2)^2} K(x)\,, \tag{3.5a}$$

and

$$K(x) = \frac{4}{5\sqrt{\pi}} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t^2x^2} w\,{}_1F_1(1, 7/2, w)\,, \tag{3.5b}$$

with the abbreviation $w = [t^2 - 2(1-\cos t)]x^2$. The integral in (3.5a) remains convergent when the upper limit becomes infinitely large. Hence, $J(b, \Gamma)$ does not depend on X provided X/b is sufficiently large.

Let us now study the asymptotic properties of $K(x)$ for either small or large values of its argument. If $x \ll 1$ we expand $w\,{}_1F_1(1, 7/2, w)$ about t^2x^2. Using $\mathrm{d}[w^a\,{}_1F_1(a, c, w)]/\mathrm{d}w = az^{a-1}\,{}_1F_1(a+1, c, w)$[6]) we arrive at

$$\begin{aligned} w\,{}_1F_1(1, 7/2, w) &\simeq t^2x^2\,{}_1F_1(1, 7/2, t^2x^2) \\ &\quad - 2x^2(1-\cos t)\,{}_1F_1(2, 7/2, t^2x^2) \quad (x \ll 1)\,. \end{aligned} \tag{3.6}$$

With this expansion we obtain from (3.5b)

$$\begin{aligned} K(x) \simeq \frac{4}{5\sqrt{\pi}} \int_0^\infty \mathrm{d}t[&t^2x^2\,{}_1F_1(5/2, 7/2, -t^2x^2) - 2x^2\,{}_1F_1(3/2, 7/2, -t^2x^2) \\ &+ 2x^2 \cos t\,{}_1F_1(3/2, 7/2, -t^2x^2)] \quad (x \ll 1)\,. \end{aligned} \tag{3.7}$$

The third term can be omitted, since it is $\mathcal{O}(x^5 \exp(-(1/4)x^{-2}))^7)$. The remaining terms yield[7])

$$K(x) \simeq \frac{1}{2x}(1-3x^2) \quad (x \ll 1). \tag{3.8}$$

For $x \gg 1$ we discuss the contribution of $[0, \pi]$ to K separately. To that end we write

$$K(x) = \sum_{n=0}^{\infty} K^{(n)}(x), \tag{3.9a}$$

where

$$K^{(0)}(x) = \frac{4}{5\sqrt{\pi}} \int_0^{\pi} \mathrm{d}t \, \mathrm{e}^{-t^2x^2} w \, {}_1F_1(1, 7/2, w), \tag{3.9b}$$

$$K^{(n)}(x) = \frac{4}{5\sqrt{\pi}} \int_{(2n-1)\pi}^{(2n+1)\pi} \mathrm{d}t \, \mathrm{e}^{-t^2x^2} w \, {}_1F_1(1, 7/2, w) \quad (n \geq 1). \tag{3.9c}$$

Let us now consider $K^{(0)}(x)$ for $x \gg 1$. Owing to the presence of the Gaussian weight factor we may use an expansion about $t = 0$ in the integrand

$$w \, {}_1F_1(1, 7/2, w) = x^2\left(\frac{t^4}{12} - \frac{t^6}{360} + \cdots\right) + \frac{2}{7}x^4\left(\frac{t^8}{144} + \cdots\right) + \cdots. \tag{3.10}$$

By extending the integration in (3.9b) over $[0, \infty)$ we get the following asymptotic expansion

$$K^{(0)}(x) \simeq \frac{1}{40x^3}\left(1 + \frac{1}{8x^2}\right) \quad (x \gg 1). \tag{3.11}$$

For $n \geq 1$ and $x \gg 1$ we may use the asymptotic expansion

$$\mathrm{e}^{-t^2x^2} {}_1F_1(1, 7/2, w) \simeq \Gamma(7/2) \frac{\mathrm{e}^{-2x^2(1-\cos t)}}{w^{5/2}} + \mathcal{O}\left(\frac{1}{w}\right). \tag{3.12}$$

The exponential in (3.12) is, for large x, sharply peaked, so that only values of t close to $2\pi n$ contribute significantly. Hence, the leading contribution is

$$K^{(n)}(x) \simeq \frac{3}{8} \frac{\mathrm{e}^{-2x^2}}{\pi^2 n^3 x^3} I_0(2x^2) \tag{3.13a}$$

$$\simeq \frac{3}{16\pi^{5/2} n^3 x^4}, \tag{3.13b}$$

where the third member of (3.13) follows from the asymptotic expansion for the modified Bessel function. Finally, combining (3.9), (3.11), (3.13) we have

$$K(x) \simeq \frac{1}{40x^3}\left[1 + \frac{15\zeta(3)}{2\pi^{5/2}x} + \frac{1}{8x^2}\right] \quad (x \gg 1)\,, \tag{3.14}$$

where $\zeta(3) = \Sigma_{n=1}^{\infty} 1/n^3$ is the Riemann ζ-function.

The asymptotic expansions for $K(x)$ will be used to derive asymptotic expressions for $J(b, \Gamma)$ for either small or large values of b. If $b \ll 1$ we split the integration interval in (3.5a) into $[0, x_0]$ and $[x_0, X/b]$. We choose x_0 such that $x_0 \gg 1$ and $bx_0 \ll 1$. Then for $x \leqslant x_0$ one has $x^4/(b^{-2}+x^2)^2 \simeq b^4x^4$, while for $x \geqslant x_0$ (3.14) is applicable. Hence we arrive at

$$J(b, \Gamma) \simeq \int_0^{x_0} \mathrm{d}x b^4 x^4 K(x) + \frac{1}{40}\int_{x_0}^{X/b} \mathrm{d}x \frac{x}{(b^{-2}+x^2)^2}\left[1 + \frac{15\zeta(3)}{2\pi^{5/2}x} + \frac{1}{8x^2}\right]. \tag{3.15}$$

The first term is $\mathcal{O}(b^4)$ and the remaining terms are easily evaluated

$$J(b, \Gamma) \simeq \frac{1}{80}b^2 + \frac{3\zeta(3)}{64\pi^{3/2}}b^3 + \mathcal{O}(b^{4-\delta}) \tag{3.16}$$

for arbitrarily small $\delta > 0$. In this asymptotic expression we omitted contributions that are $\mathcal{O}(b/X)$.

If $b \gg 1$ we first consider the case where $X/b \ll 1$. Equivalently, we have $r_B \ll r_L$, with r_B the thermal gyroradius $\omega_B^{-1}(m\beta)^{-1/2}$, so that we are dealing with a guiding centre plasma. We insert (3.8) into (3.5a) and we get the leading contribution

$$\begin{aligned} J(b, \Gamma) &\simeq \frac{1}{2}\int_0^{X/b} \mathrm{d}x \frac{x^3}{(b^{-2}+x^2)^2}(1 - 3x^2) \\ &\simeq \frac{1}{4}\left[\log(1+X^2) - 1 + \frac{1}{1+X^2}\right] \simeq \tfrac{1}{2}[-\log(\sqrt{3}\Gamma^{3/2}) - \tfrac{1}{2}]\,, \end{aligned} \tag{3.17}$$

for $b \to \infty$ and $\Gamma \ll 1$. The self-diffusion coefficient then follows from (3.4). Comparing it with $D_\parallel$ for $b \to 0$, which follows by inserting (3.16) into (3.4), one immediately recovers the proportionality relation (2.19).

If on the other hand $b \gg 1$, but not $X/b \ll 1$, the asymptotic expansion must be derived in a more subtle way. We now choose $bx_0 \gg 1$ such that $x_0 \ll 1$; using (3.8) for $x \leqslant x_0$ and $x^4/(b^{-2}+x^2)^2 \simeq 1 - 2/b^2x^2$ for $x \geqslant x_0$ we arrive at

$$J(b, \Gamma) \simeq \frac{1}{2}\int_0^{x_0} \mathrm{d}x \frac{x^3}{(b^{-2}+x^2)^2}(1 - 3x^2) + \int_{x_0}^{X/b} \mathrm{d}x\left(1 - \frac{2}{b^2x^2}\right)K(x)\,. \tag{3.18}$$

Consequently,

$$J(b,\Gamma) \simeq \tfrac{1}{2}\log b - \tfrac{1}{4} + C + \mathcal{O}(b^{-2+\delta}), \tag{3.19}$$

for arbitrarily small $\delta > 0$; here

$$C = \int_{x_0}^{X/b} \mathrm{d}x K(x) + \tfrac{1}{2}\log x_0 \tag{3.20}$$

is independent of $x_0 \ll 1$. If X/b is sufficiently large C is also independent of X/b. Numerically, we have found $C = -0.406$.

From (3.4) and (3.19)–(3.20) it is seen that in leading order $1/D_\parallel$ is proportional to $\log(X\sqrt{b}) = \log(r_\mathrm{D}^{3/2}/r_\mathrm{L} r_B^{1/2})$ for $r_\mathrm{L} \ll r_B \ll r_\mathrm{D}$. We have found here that the leading order contribution for the self-diffusion coefficient is not obtained by simply replacing the Debye length in the Coulomb logarithm by the gyroradius, as is sometimes suggested[1,4].

4. Numerical results

For intermediate values of b the double integral $J(b,\Gamma)$, which determines the dependence of the longitudinal diffusion coefficient on the magnetic field, must be evaluated numerically. The integrand of $J(b,\Gamma)$ contains the function $K(x)$ defined in (3.5b). If either $x \ll 1$ or $x \gg 1$ we may use for $K(x)$ the asymptotic expansions (3.8) and (3.14), respectively. For arbitrary values of x the infinite domain of the t-integration is divided into finite subintervals, as in (3.9). For n sufficiently large one may use (3.12) to evaluate $K^{(n)}$, as in (3.13a). Consequently, we get

$$K(x) = \sum_{n=0}^{N} K^{(n)}(x) + \frac{3}{8\pi^2 x^3}\mathrm{e}^{-2x^2} I_0(2x^2)\left[\zeta(3) - \sum_{n=1}^{N}\frac{1}{n^3}\right]. \tag{4.1}$$

Only the first $N+1$ terms at the right-hand side must be evaluated numerically. The actual value of N depends on x and on the required precision. Once numerical values of $K(x)$ have been obtained the evaluation of $J(b,\Gamma)$ is straightforward.

The results of the calculation of $J(b,\Gamma)$ are presented in table I. The column marked $\Gamma = 0$ contains the values for $J(b,\Gamma)$ that are obtained by putting the upper limit in the x-integral equal to ∞. For $b \gg 1$ the values of $J(b,0)$ agree with those found from the asymptotic expansion (3.19), while for $\Gamma \neq 0$ the numerical results approach the asymptotic values obtained from (3.17). For all Γ the tabulated results are consistent with (3.16) as b approaches 0.

Using (3.4) and table I one finally obtains the curves for the reduced

TABLE I
The contribution $J(b, \Gamma)$ to the inverse longitudinal self-diffusion coefficient

b \ Γ	0	0.05	0.1	0.2
0.1	1.36×10^{-4}	1.36×10^{-4}	1.36×10^{-4}	1.33×10^{-4}
1	2.04×10^{-2}	2.04×10^{-2}	2.04×10^{-2}	2.01×10^{-2}
2	7.88×10^{-2}	7.88×10^{-2}	7.87×10^{-2}	7.75×10^{-2}
5	2.88×10^{-1}	2.88×10^{-1}	2.87×10^{-1}	2.79×10^{-1}
10	5.48×10^{-1}	5.48×10^{-1}	5.44×10^{-1}	4.94×10^{-1}
20	8.60×10^{-1}	8.58×10^{-1}	8.37×10^{-1}	6.32×10^{-1}
50	1.30	1.29	1.11	6.85×10^{-1}
100	1.65	1.55	1.18	6.93×10^{-1}
200	1.99	1.68	1.20	6.94×10^{-1}
∞	—	1.72	1.20	6.94×10^{-1}

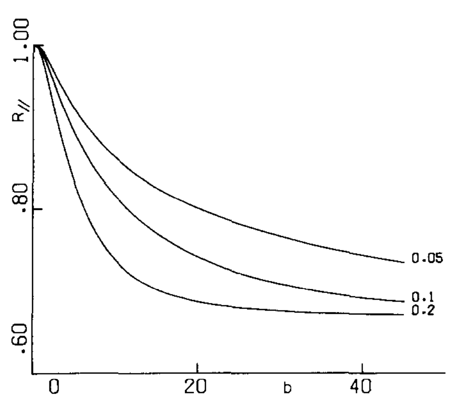

Fig. 1. The reduced longitudinal self-diffusion coefficient $R_\parallel$ as a function of the dimensionless magnetic field b, for the values 0.05, 0.1 and 0.2 of the coupling constant Γ.

longitudinal self-diffusion coefficient

$$R_\parallel(b, \Gamma) = \frac{D_\parallel(b, \Gamma)}{D_\parallel(0, \Gamma)}, \tag{4.2}$$

as drawn in fig. 1. As this figure shows the longitudinal diffusion process is gradually impeded as the magnetic field becomes stronger. For large field strengths the reduction factor for the diffusion coefficient is 2/3, as mentioned already in (2.19).

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