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Allahverdyan, A.E.; Khrennikov, A.; Nieuwenhuizen, T.M.

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A. E. Allahverdyan,[1,2] A. Khrennikov,[3] and Th. M. Nieuwenhuizen[1]
[1]*Institute for Theoretical Physics, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*
[2]*Yerevan Physics Institute, Alikhanian Brothers Street 2, Yerevan 375036, Armenia*
[3]*International Centre for Mathematical Modeling in Physics and Cognitive Sciences, MSI, University of Växjö, S-35195, Sweden*
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For two classical Brownian particles an analog of continuous-variable quantum entanglement is presented: The common probability distribution of the two coordinates and the corresponding coarse-grained velocities cannot always be prepared via mixing of any factorized distributions referring to the two particles separately. This is possible for particles which have interacted in the past, but do not interact at present. Three factors are crucial for the effect: (1) separation of time scales of coordinate and momentum which motivates the definition of coarse-grained velocities; (2) the resulting uncertainty relations between the coordinate of the Brownian particle and the change of its coarse-grained velocity; (3) the fact that the coarse-grained velocity, though pertaining to a single Brownian particle, is defined on a common context of two particles. The Brownian entanglement is a consequence of a coarse-grained description and disappears for a finer resolution of the Brownian motion. Analogies with the quantum situation are discussed, as well as possibilities of experimental realization of the effect in examples of macroscopic Brownian motion.

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I. INTRODUCTION

There is a long tradition of seeking connections between quantum mechanics and classical statistical physics. A list of known examples includes (i) analogies between the notion of complementarity in quantum mechanics and statistical thermodynamics, in particular, between quantum-mechanical uncertainty relations and the energy-temperature uncertainty relation in statistical thermodynamics [1–4]; (ii) mathematical relations between Schrödinger and Fokker-Planck equations [5], which makes quantum intuition very useful, e.g., for polymers [6]; (iii) attempts to derive Schrödinger equations from a classical kinetic picture [7] or from classical stochastic electrodynamics [8]; (iv) a recently proposed general axiomatic framework to consider complementarity and entanglement beyond quantum mechanics[9].

More generally, both quantum mechanics and statistical physics are essentially probabilistic theories and already at this level one expects to find certain analogies between their concepts. The progress in this direction has been for a long time plagued by statements on incompatibility between quantum mechanics and classical probability theory, which lies in the basis of classical statistical physics. It was, however, noted that such opinions are not warranted [10], and recently it was shown explicitly that several basic relations of quantum mechanics can be derived from the classical probability theory provided the contexts of physical conditions (measurement setups) are properly taken into account [11].

Once the basic probabilistic ground of quantum mechanics and classical statistical physics is recognized to be the same, one wonders whether some unusual aspects of quantum mechanics, such as entanglement, can find analogies in classical statistical physics. We see at least two reasons for seeking such analogies. First, it is going to refresh our understanding of classical statistical physics, and may imply in the future that advantages offered by quantum mechanics in certain tasks of information processing and transfer are not unique to quantum mechanics, and can be looked for in classical areas of physics as well [11]. Second, it is useful for understanding quantum mechanics that certain concepts believed to be purely quantum—that is, incomprehensible in classical terms—can find natural classical analogies. In fact, the works and ideas mentioned in the above points (i)–(iii) were partially directed toward this goal. More recent results along these lines are classical analogies to quantum entanglement found in classical optics [12,13] (more generally, in classical wave physics), chaotic classical dynamics [14] and classical information theory [15] (secret classical correlations), as well as classical probabilistic models for certain aspects of (multitime) quantum measurements [16].

The purpose of the present paper is to show that quantum entanglement can have a natural analogy in the physics of Brownian particles (Brownian entanglement). The reason for the existence of this analogy can be qualitatively explained as follows. It is known that the dynamics of a Brownian particle can be observed at two levels [5]. Within the first, more fundamental, level the Brownian particle coupled to a thermal bath at temperature T is described via definite coordinate x and momentum p and moves under influence of external potential, friction force, and an external random force. The latter two forces are generated by the bath. The second, overdamped regime applies when the characteristic relaxation time of the coordinate τ_x is much larger than that of the momentum τ_p, $\tau_x \gg \tau_p$ (overdamped regime). On times much larger than τ_p one is interested in the change of the coordinate and defines the *coarse-grained* velocity as $v = \Delta x/\Delta t$ for $\tau_x \gg \Delta t \gg \tau_p$. This definition of v is the only operationally meaningful one for the (effective) velocity within the overdamped regime. It appears that the coarse-grained velocity, though pertaining to single particles, is defined in the context of the whole system of coupled Brownian particles. Together with uncertainty relations between the coordinate and the change of the coarse-grained velocity—the

role of Planck's constant is being played by the temperature of the bath—this contextuality feature will be shown to cause a phenomenon similar to quantum entanglement: The common probability distribution of the two coordinates and the corresponding coarse-grained velocities—for two Brownian particles which interacted in the past, but need not interact in the present—cannot be prepared via mixing of any factorized distributions referring to the two particles separately. This Brownian entanglement is a consequence of a coarse-grained description and disappears within the first (more fundamental) level of description, simply because entanglement is absent in classical mechanics.

The paper is organized as follows. In Sec. II we recall the phenomenon of (continuous-variable) quantum entanglement, focusing especially on the relations between the entanglement and the uncertainty relations. Section III discusses coarse-grained velocities and uncertainty relations for the classical Brownian motion. The next two sections define and study the phenomenon of Brownian entanglement. Sections V and VI offer a detailed comparison between the features of quantum entanglement and those of its Brownian counterpart. In Sec. VII we discuss possibilities of experimental realization of Brownian entanglement. Our conclusions are presented in the last section. Some technical questions are worked out in the Appendix.

II. QUANTUM ENTANGLEMENT

A. Statistical interpretation

This section recalls the phenomenon of entanglement in quantum mechanics, and especially underlines its connections with the uncertainty relations.

Before starting, it is useful to stress that in the present paper we adhere to the statistical (ensemble) interpretation of quantum mechanics, where a quantum "state" is described by a density matrix $\hat{\rho}$, and any state, including a pure state $|\psi\rangle\langle\psi|$, refers to an ensemble $\mathcal{E}(\hat{\rho})$ of identically prepared systems; see, e.g., Refs. [17–27].[1] To put it succinctly: quantum mechanics makes[2]

(i) statistical statements on

(ii) the results of measurements done

(iii) on ensembles of identically prepared systems.

As was stressed repeatedly [17–27], in particular by experimentalists [26], the experimentally relevant statements of quantum mechanics do not require more than the minimal statistical interpretation.[3] Moreover, this interpretation deals more successfully with the measurement problem, as instanced by a recent exactly solvable model [27], and allows us to reconcile quantum mechanics with classical probability theory [11]. The fact that discussions on quantum entanglement (and on Bell inequalities and related matters) do not employ the statistical interpretation is a mere prejudice; see Refs. [24,25,28] for examples of such discussions on the basis of statistical interpretation.

B. Definition of entanglement

Consider a quantum system S consisting of two subsystems S_1 and S_2. A state $\hat{\rho}$ of S is called entangled, see, e.g., Ref. [29], (or nonseparable) with respect to S_1 and S_2, if it *cannot* be represented as

$$\hat{\rho}=\sum_{k=1}^{n} p_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}, \quad \sum_{k=1}^{n} p_k = 1, \quad p_k \geqslant 0, \tag{1}$$

where $\hat{\rho}_i^{(1)}$ and $\hat{\rho}_i^{(2)}$ are arbitrary density matrices living in the Hilbert spaces of S_1 and S_2, respectively, n is an integer, and where $\{p_k\}_{k=1}^n$ is a probability distribution.

According to definition (1), a separable quantum state can always be prepared by means of mixing[4] noncorrelated states $\hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}$ of the two subsystems. For a pure state $\hat{\rho}=|\psi\rangle\langle\psi|$ we return to the more known definition of entanglement: $|\psi\rangle$ cannot be represented as $|\psi\rangle=|\psi\rangle_1 \otimes |\psi\rangle_2$. For this particular case the absence of entanglement implies the absence of any correlation. In contrast, for the more general situation given by Eq. (1), a nonentangled state can still possess certain (classical) correlations, since the totally uncorrelated situations will be given as $\hat{\rho}=\hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)}$.

Two important features of quantum entanglement should be noted. First, it can be tested only through measuring some correlations between the subsystems S_1 and S_2. Observables pertaining to S_1 or to S_2 alone are obviously insensitive to entanglement. Second, if the state (1) is prepared by two different observers 1 and 2, then this preparation is seen to involve correlated actions of them, and thus the corresponding observers have to communicate classically.

C. Classical systems

In classics the representation (1) is apparently always possible. Indeed, let us have a probability distribution $P(x_1,x_2)$ of two classical systems S_1 and S_2 represented by random variables x_1 and x_2. Assume for simplicity that x_1 can take

[1]The minimal statistical interpretation should of course be distinguished from various hidden-variable theories and assumptions, in particular, from the pre-assigned initial value assumptions, where quantum measurements are viewed as merely revealing pre-existing values of all observables. Unfortunately, some proponents of the statistical interpretation were unclear at this point, a fact that for a while discredited this interpretation. For well-balanced discussions on this and related points, see Refs. [21–24].

[2]This way of putting the message of statistical interpretation is adopted from Ref. [25].

[3]Still, as correctly pointed out in Refs. [28,30], the choice of interpretation can and does influence one's estimates of importance for various scientific problems.

[4]Mixing ensembles $\mathcal{E}(\hat{\rho}_1)$ and $\mathcal{E}(\hat{\rho}_2)$ with probabilities p_1 and p_2, respectively, means that one throws the dice with probabilities of outcomes equal to p_1 and p_2, and depending on the outcome one picks up a system from $\mathcal{E}(\hat{\rho}_1)$ or $\mathcal{E}(\hat{\rho}_2)$, keeping no information on where the system came from. Alternatively, one can join together Np_1 systems from $\mathcal{E}(\hat{\rho}_1)$ and Np_2 systems from $\mathcal{E}(\hat{\rho}_2)$ ($N \gg 1$), so that no information is kept on where a single system came from.

values $a_1,\ldots,a_n$, while x_2 can take values $b_1,\ldots,b_n$. Then $P(x_1,x_2)$ can be written as

$$P(x_1,x_2)=\sum_{\alpha=a_1,\ldots,a_n}\sum_{\beta=b_1,\ldots,b_n}P(\alpha,\beta)\delta_{\alpha x_1}\delta_{x_2\beta}, \tag{2}$$

where $\delta_{\alpha\beta}=1$ if $\alpha=\beta$, and $\delta_{\alpha\beta}=0$ otherwise. *Provided* that $\delta_{\alpha x_1}$ and $\delta_{x_2\beta}$ are two legitimate probability distributions belonging to S_1 and S_2, that is, provided there are no mechanisms prohibiting the realization of $\delta_{\alpha x_1}$ and $\delta_{x_2\beta}$ as physically acceptable distributions usable in the actual preparation, the representation (1) is realized for the classical situation: there is no entanglement.

In fact the quantum situation goes to the classical one right in Eq. (1) if we assume that the involved density matrices are always diagonal. So it is the presence of nondiagonal elements of the given density matrix that makes the situations different.

D. Semiclassical systems

We shall now localize the cause of quantum entanglement for systems which behave (semi)classically in several other respects. This will help us to understand the way of searching for entanglement in nonquantum situations.

Let S_1 and S_2 be two quantum (noninteracting) harmonic oscillators, and the overall density matrix (in the coordinate representation) $\rho(x_1,x_1';x_2,x_2')$ be a Gaussian function of its variables. The corresponding coordinate and momenta operators are denoted by $\hat{x}_1,\hat{p}_1$ and $\hat{x}_2,\hat{p}_2$, respectively. As is well known, a Gaussian state is conveniently dealt with help of the Wigner functions $W(x,p)$, which is a function of x and p and is equivalent to the density matrix.[5] In contrast to the latter it has several properties expected for the common probability distribution of the coordinate and momentum. In particular, the analysis via Wigner functions will provide us below with a richer intuition on the relations between entanglement and uncertainty relations, thus between entanglement and noncommutativity.

Since the overall Hamiltonian of S_1 and S_2 is assumed to be harmonic, the initially Gaussian state remains Gaussian for all times and the corresponding Wigner function will be positive and thus will *partially* admit a classical interpretation in terms of common probability distribution of the coordinate and momentum [31,32]. A related fact is that for two harmonic oscillators the Ehrenfest equations of motion can be recast into a classical form [32]. So many aspects of this system can be accounted for in classical terms.

In order to see why in spite of these classical features the system of two oscillators can be entangled, note that for the Wigner functions one can rewrite the condition (1) as

$$W(x_1,p_1;x_2,p_2)=\int d\lambda\ \mathcal{P}(\lambda)W_1(x_1,p_1|\lambda W_2(x_2,p_2|\lambda), \tag{3}$$

$$\mathcal{P}(\lambda)\geqslant 0,\quad \int d\lambda\ \mathcal{P}(\lambda)=1,$$

where W_1 and W_2 are separate Wigner functions for S_1 and S_2, respectively, and $\mathcal{P}(\lambda)$ is some probability distribution. If S_1 and S_2 were classical oscillators, then instead of Wigner functions we would have distribution functions, and one can always write down the analog of Eq. (2) for the common distribution function $P(x_1,p_1;x_2,p_2)$ and $\lambda=(\alpha_1\beta_1,\alpha_2,\beta_2)$:

$$\begin{aligned}P(x_1,p_1;x_2,p_2)=\int d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 P(\alpha_1,\beta_1;\alpha_2,\beta_2)\\ \times\delta(\alpha_1-x_1)\delta(\beta_1-p_1)\\ \times\delta(\alpha_2-x_2)\delta(\beta_2-p_2).\end{aligned} \tag{4}$$

In the classical situation this means that there is no entanglement. The same formula (4) can formally be written down also for the positive Wigner functions $W(x_1,p_1;x_2,p_2)$. However, in the quantum situation $\delta(\alpha_1-x_1)\delta(\beta_1-p_1)$ and $\delta(\alpha_2-x_2)\delta(\beta_2-p_2)$ are *not* legitimate Wigner functions, since they prescribe definite values to both coordinate and momentum and thus do not respect the uncertainty relations.

We conclude that the uncertainty relations are necessary—but not sufficient—for the existence of entanglement in semiclassical systems.

E. A simple sufficient condition for quantum entanglement

The definition of the entanglement as given by Eq. (1) is not practical (except for a pure density matrix $\hat{\rho}$, when no entanglement means no correlations). It is therefore useful to have certain sufficient conditions for the presence of entanglement which will be easy to handle in applications and which will have a transparent physical meaning.

We choose the units in such a way that the coordinate and the momentum have the same dimension as $\sqrt{\hbar}$. For a harmonic oscillator with mass m and frequency ω it will suffice to make the following canonical transformation: $\hat{x}\to\sqrt{m\omega}\hat{x}$, $\hat{p}\to\hat{p}/\sqrt{m\omega}$.

For a two-particle system S with coordinate and momenta operators $\hat{x}_1,\hat{x}_2$ and $\hat{p}_1,\hat{p}_2$, respectively, one can propose the following sufficient condition for the entanglement [33].

Let us first note that the standard uncertainty relation

$$\langle\Delta\hat{x}^2\rangle\langle\Delta\hat{p}^2\rangle\geqslant\frac{\hbar^2}{4}, \tag{5}$$

$$\Delta\hat{x}\equiv\hat{x}-\langle\hat{x}\rangle,\quad \Delta\hat{p}\equiv\hat{p}-\langle\hat{p}\rangle, \tag{6}$$

implies

$$\langle\Delta\hat{x}^2\rangle+\langle\Delta\hat{p}^2\rangle\geqslant\langle\Delta\hat{x}^2\rangle+\frac{\hbar^2}{4\langle\Delta\hat{x}^2\rangle}, \tag{7}$$

and then one gets via minimizing the right-hand side of Eq. (7) over $\langle\Delta\hat{x}^2\rangle$ [which produces $\langle\Delta\hat{x}^2\rangle\to\hbar/2$, to be put in the right-hand side of Eq. (7)]:

$$\langle\Delta\hat{x}^2\rangle+\langle\Delta\hat{p}^2\rangle\geqslant\hbar. \tag{8}$$

[5] For good reviews on the properties of Wigner function, see Ref. [32].

Now assume that the two-particle system is described by a factorized density matrix:

$$\hat{\rho} = \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)}. \tag{9}$$

Then due to Eq. (8) one can write

$$\langle (\Delta\hat{x}_1 - \Delta\hat{x}_2)^2 \rangle + \langle (\Delta\hat{p}_1 + \Delta\hat{p}_2)^2 \rangle = \langle \Delta\hat{x}_1^2 \rangle + \langle \Delta\hat{p}_1^2 \rangle + \langle \Delta\hat{x}_2^2 \rangle + \langle \Delta\hat{p}_2^2 \rangle \geqslant 2\hbar, \tag{10}$$

just because for noncorrelated systems the corresponding variances add up. If the overall system is in a separable state, then Eq. (10) is even strengthened, since the variance of *any* observable $\langle \Delta\hat{A}^2 \rangle$ increases under mixing, i.e., under a transformation:

$$\{p_k, \hat{\rho}_k\} \to \sum_k p_k \hat{\rho}_k, \quad p_k \geqslant 0, \quad \sum_k p_k = 1, \tag{11}$$

where $\hat{\rho}_k$ are normalized ($\mathrm{tr}\,\hat{\rho}_k = 1$) density matrices. Indeed,

$$\begin{aligned} \langle \Delta\hat{A}^2 \rangle &= \sum_k p_k \,\mathrm{tr}[\hat{\rho}_k (\hat{A} - \langle \hat{A} \rangle)^2] \\ &= \sum_k p_k \{\mathrm{tr}[\hat{\rho}_k \hat{A}^2] - \mathrm{tr}[\hat{\rho}_k \hat{A}]^2\} + \sum_k p_k \{\mathrm{tr}[\hat{\rho}_k \hat{A}] - \langle \hat{A} \rangle\}^2 \\ &= \sum_k p_k \langle \Delta\hat{A}^2 \rangle_k + \sum_k p_k \{\mathrm{tr}[\hat{\rho}_k \hat{A}] - \langle \hat{A} \rangle\}^2 \geqslant \sum_k p_k \langle \Delta\hat{A}^2 \rangle_k. \end{aligned} \tag{12}$$

Employing Eq. (12) for $\hat{A} = \Delta\hat{x}_1 - \Delta\hat{x}_2$ and for $\hat{A} = \Delta\hat{p}_1 - \Delta\hat{p}_2$, we get that though Eq. (10) was obtained for the factorized state (9), it remains valid for an arbitrary nonentangled state $\Sigma_k p_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}$. Thus the violation

$$\langle (\Delta\hat{x}_1 - \Delta\hat{x}_2)^2 \rangle + \langle (\Delta\hat{p}_1 + \Delta\hat{p}_2)^2 \rangle \leqslant 2\hbar \tag{13}$$

of Eq. (10) is a sufficient condition for entanglement.

Equation (13) has a transparent physical meaning: entanglement is present, if fluctuations do not sum up additively, i.e., if the changes $\Delta\hat{x}_1$ and $\Delta\hat{x}_2$ of the coordinates tend to correlate with each other, while those of the momenta tend to anticorrelate.

Alternative sufficient conditions for entanglement can be built based on $\langle (\Delta\hat{x}_1 + \Delta\hat{x}_2)^2 \rangle + \langle (\Delta\hat{p}_1 - \Delta\hat{p}_2)^2 \rangle$ or $\langle (\Delta\hat{x}_1 + \Delta\hat{x}_2)^2 \rangle + \langle (\Delta\hat{p}_1 + \Delta\hat{p}_2)^2 \rangle$, etc. Various conditions obtained in this way are obviously not equivalent to each other. An obvious way to strengthen it is to demand that

$$\langle (\Delta\hat{x}_1 + \epsilon\Delta\hat{x}_2)^2 \rangle + \langle (\Delta\hat{p}_1 + \zeta\Delta\hat{p}_2)^2 \rangle \leqslant 2\hbar, \tag{14}$$

at least for one of four independent choices $\epsilon = \pm 1$, $\zeta = \pm 1$.

For some special states (e.g., Gaussian states) there exist in literature necessary and sufficient conditions for entanglement [34]. For our purposes conditions (13), (14) are sufficient.

F. Operational issues

Let us finally recall how the condition (13) is checked operationally. To this end, we rewrite this inequality as

$$\langle \Delta\hat{x}_1^2 \rangle + \langle \Delta\hat{x}_2^2 \rangle + \langle \Delta\hat{p}_1^2 \rangle + \langle \Delta\hat{p}_2^2 \rangle - 2\langle \Delta\hat{x}_1 \Delta\hat{x}_2 \rangle + 2\langle \Delta\hat{p}_1 \Delta\hat{p}_2 \rangle \leqslant 2\hbar. \tag{16}$$

Within the traditional approach, one needs two ensembles of systems S, each one consisting of identically prepared correlated subsystems S_1 and S_2. Assuming that S_1 and S_2 are in possession of observers 1 and 2, respectively, the observer 1 measures on the first (second) ensemble $\hat{x}_1(\hat{p}_1)$, while the observer 2 measures on the first (second) ensemble $\hat{x}_2(\hat{p}_2)$. Two ensembles are involved, since $\hat{x}_1$ and $\hat{p}_1$ (respectively, $\hat{x}_2$ and $\hat{p}_2$) do not commute: $[\hat{x}_k, \hat{p}_l] = i\hbar\delta_{kl}$, with δ_{kl} being Kronecker's delta. Now the quantities in Eq. (15) can be estimated by each observer separately, while for the quantities in Eq. (16) the observers need to put the results of their measurements together (or to communicate in any other classical way) and to count the coinciding events.

Within an approach proposed recently [33], one needs only one single ensemble for measuring the averages in Eqs. (15) and (16). The corresponding measurement can be done by a *single* apparatus.[6]

III. COARSE-GRAINED VELOCITIES AND UNCERTAINTY RELATIONS FOR BROWNIAN PARTICLES

Consider N identical Brownian particles with coordinates $\mathbf{x} = (x_1, \ldots, x_N)$ and mass m interacting with N independent thermal baths at temperatures T_i and subjected to a potential $U(x_1, \ldots, x_N)$. The overdamped limit is defined by the following two conditions [5] (for a more detailed discussion, see the Appendix):

(i) The characteristic relaxation time[7] of the (real) momenta $m\dot{x}_i$ is much smaller than the one of the coordinates.

(ii) One is interested in times which are much larger than the relaxation time of the momenta, but which can be much smaller than or comparable to the relaxation time of the coordinates.

Under these conditions the dynamics of the system is described by the following Langevin equations [5]:

$$\begin{aligned} \dot{x}_i &= f_i(\mathbf{x}) + \eta_i(t), \quad f_i(\mathbf{x}) = -\partial_{x_i} U(\mathbf{x}), \\ \langle \eta_i(t)\, \eta_j(t') \rangle &= 2T_i \delta_{ij} \delta(t - t'), \end{aligned} \tag{17}$$

where for our convenience both the mass m and the damping constant (coupling constant of the particles to the bath) are in

[6] Strictly speaking, the argument in Ref. [35] was given for systems living in a finite-dimensional Hilbert space, but most likely it extends to the infinite-dimensional situations.

[7] The characteristic relaxation time τ_θ of a random time-dependent variable θ (e.g., coordinate or momentum) is an *ensemble notion* and is defined as the time necessary for the conditional distribution of θ to become memoryless: $P(\theta, t' + \tau_\theta | \theta', t') = P(\theta, t' + \tau_\theta)$. For the system of Brownian particles sufficiently strongly coupled to their thermal baths, the arbitrarinesses of the above definition (e.g., the precise choice of θ' and t') do not change the physical content of the definition [5].

the main text taken equal to 1 (they are re-inserted in the Appendix).[8]

The conditional probability $P(\mathbf{x},t|\mathbf{x}',t')$ is known to satisfy the following Fokker-Planck equation [5]:

$$\partial_t P(\mathbf{x},t|\mathbf{x}',t') = -\sum_i \partial_{x_i}[f_i(\mathbf{x})P(\mathbf{x},t|\mathbf{x}',t')] + \sum_i T_i \partial^2_{x_i x_i} P(\mathbf{x},t|\mathbf{x}',t'), \quad t \geqslant t'. \quad (18)$$

Eq. (18) is associated with the following initial condition:

$$P(\mathbf{x},t|\mathbf{x}',t) = \delta(\mathbf{x}-\mathbf{x}') \equiv \prod_{i=1}^{N} \delta(x_i - x_i'), \quad (19)$$

as follows from the very definition of the conditional probability.

Consider an ensemble $\Sigma(\mathbf{x},t)$ of all realizations of the whole N-particle system which at time t have a coordinate vector $\mathbf{x}$. Such an ensemble can be selected out of all possible realizations by measuring $\mathbf{x}=(x_1,\ldots,x_N)$. For this ensemble the average coarse-grained velocity for the Brownian particle with index j might naively be defined as

$$v_j(\mathbf{x},t) = \lim_{\varepsilon\to 0} \int d\mathbf{y} \frac{y_j - x_j}{\varepsilon} P(\mathbf{y},t+\varepsilon|\mathbf{x},t). \quad (20)$$

However, it was pointed out by Nelson [7] that the absence of regular trajectories enforces us to define different velocities for different directions of time:

$$v_{+,j}(\mathbf{x},t) = \lim_{\varepsilon\to +0} \int dy_j \frac{y_j - x_j}{\varepsilon} P(y_j,t+\varepsilon|\mathbf{x},t), \quad (21)$$

$$v_{-,j}(\mathbf{x},t) = \lim_{\varepsilon\to +0} \int dy_j \frac{x_j - y_j}{\varepsilon} P(y_j,t-\varepsilon|\mathbf{x},t). \quad (22)$$

The physical meaning of these expressions can be explained as follows.

(i) As seen from the definitions, $v_{+,j}(\mathbf{x},t)$ is the average velocity to move anywhere starting from $(\mathbf{x},t)$, whereas $v_{-,j}(\mathbf{x},t)$ is the average velocity to come from anywhere and to arrive at $\mathbf{x}$ at the moment t.

[8]Equation (17) can be obtained from the complete Langevin equations of motion [with the same definition of $\eta_i(t)$ as in Eq. (17)]: $m\ddot{x}_i + \dot{x}_i = f_i(\mathbf{x}) + \eta_i(t)$, by disregarding the first term $m\ddot{x}_i$ corresponding to acceleration. One might make this heuristically by requiring that the friction force $\dot{x}_i$ dominates, and that the second-derivative $\ddot{x}_i$ is small for long times. More rigorous derivation of this classical problem is presented in Ref. [5], and is recalled in the Appendix for a simple model. In their turn the above complete Langevin equations can be rigorously derived from the Newton equations of motion for the Brownian particles and the baths. Here are the basic conditions for this derivation: (i) the thermodynamical (macroscopic) limit for the baths; (ii) a reasonable model for the particle-bath interaction; (iii) the initially equilibrium state of the baths, described by the corresponding Gibbs distributions at temperature T_i [this is how the random noises $\eta_i(t)$ come into existence]. The details of the derivation can be looked up, e.g., in Refs. [36,37].

Since these velocities are defined already in the overdamped limit, ε is assumed to be much larger than the characteristic relaxation time of the (real) momentum which is small in the overdamped limit. Therefore we call Eqs. (21), (22) coarse-grained velocities. It is known that for the overdamped Brownian motion almost all trajectories are not smooth. This is connected to the chaotic influences of the bath(s) which randomize the real momenta on much smaller times, and this is also the reason for $v_{+,j}(\mathbf{x},t) \neq = v_{-,j}(\mathbf{x},t)$. The difference $v_{+,j}(\mathbf{x},t) - v_{-,j}(\mathbf{x},t)$ thus characterizes the degree of the above nonsmoothness.

Recall that if one would take ε much smaller than the characteristic relaxation time of the momentum—which would amount to applying definitions (21) and (22) to a smoother trajectory—then $v_{+,j}(\mathbf{x},t)$ and $v_{-,j}(\mathbf{x},t)$ would be equal to each other and equal to the average momentum. These points are discussed in detail in the Appendix.

(ii) Here is an operational procedure for measuring $v_{+,j}(\mathbf{x},t)$ and $v_{-,j}(\mathbf{x},t)$. Consider the overall ensemble of the Brownian particles. The single members of this ensemble are N coupled Brownian particles. For each such single member one measures

(i) the coordinate of the Brownian particle with index j at the moment $t-\varepsilon$;

(ii) the coordinates of *all* Brownian particles at the moment t;

(iii) the coordinate of the same Brownian particle with index j at the moment $t+\varepsilon$.

Repeating these points many times on various members of the above overall ensemble, ignoring all the results from (i) and (iii), and selecting events from the second step, we reconstruct the ensemble $\Sigma(\mathbf{x},t)$. Employing the results from (i) and (iii) and conditioning repeatedly upon a single member from $\Sigma(\mathbf{x},t)$, we estimate the conditional probabilities $P(y_j,t\pm\varepsilon|\mathbf{x},t)$ and then finally calculate $v_{+,j}(\mathbf{x},t)$ and $v_{-,j}(\mathbf{x},t)$ via Eqs. (21), (22).

(iii) The above ensemble $\Sigma(\mathbf{x},t)$ is different from an ensemble $\Sigma_j(x_j,t)$ obtained by measuring x_j (at time t) *irrespective* of other coordinates. Indeed, $\Sigma(\mathbf{x},t)$ is defined globally, and if there are different observers in possession of each Brownian particle, they have to communicate to each other in order to be able to construct $\Sigma(\mathbf{x},t)$. In contrast, $\Sigma_j(x_j,t)$ is defined exclusively with respect to the Brownian particle with index j. This point will be discussed in more detail later on in Sec. V.

The calculation of $v_{+,i}(x,t)$ and $v_{-,i}(x,t)$ is straightforward upon using the following three things: first, the relation

$$P(\mathbf{x},t+\varepsilon|\mathbf{y},t) = \delta(\mathbf{x}-\mathbf{y}) + \varepsilon\sum_i [-f_i(\mathbf{y})\partial_{x_i}\delta(\mathbf{x}-\mathbf{y}) + T_i\partial^2_{x_i x_i}\delta(\mathbf{x}-\mathbf{y})], \quad (23)$$

which follows from Eqs. (18), (19); second, the Bayes formula, and, third, partial integration assuming natural boundary conditions at infinity: $P(x,t)\to 0$ if $x_j \to \pm\infty$. Starting from definitions (21), (22), we obtain

$$v_{+,j}(\mathbf{x},t) = \lim_{\varepsilon\to+0}\int d\mathbf{y}\frac{y_j - x_j}{\varepsilon}P(\mathbf{y},t+\varepsilon|\mathbf{x},t)$$
$$= \int d\mathbf{y}(y_j - x_j)\sum_i [-f_i(\mathbf{x})\partial_{y_i}\delta(\mathbf{x}-\mathbf{y}) + T_i\partial^2_{y_iy_i}\delta(\mathbf{x}-\mathbf{y})] = f_j(\mathbf{x}), \quad (24)$$

$$v_{-,j}(\mathbf{x},t) = \lim_{\varepsilon\to+0}\int d\mathbf{y}\frac{x_j - y_j}{\varepsilon}P(\mathbf{y},t-\varepsilon|\mathbf{x},t)$$
$$= \lim_{\varepsilon\to+0}\int d\mathbf{y}\frac{x_j - y_j}{\varepsilon}P(\mathbf{x},t|\mathbf{y},t-\varepsilon)\frac{P(\mathbf{y},t-\varepsilon)}{P(\mathbf{x},t)}$$
$$= \int d\mathbf{y}(x_j - y_j)\frac{P(\mathbf{y},t)}{P(\mathbf{x},t)}\sum_i [-f(\mathbf{y})\partial_{x_i}\delta(\mathbf{x}-\mathbf{y}) + T_i\partial_{x_ix_i}\delta(\mathbf{x}-\mathbf{y})]$$
$$= f_j(\mathbf{x}) - 2T_j\partial_{x_j}\ln P(\mathbf{x},t). \quad (25)$$

The difference between the coarse-grained velocities $v_{+,j}(\mathbf{x},t)$ and $v_{-,j}(\mathbf{x},t)$ is

$$u_j(\mathbf{x},t) = \frac{v_{-,j}(\mathbf{x},t) - v_{+,j}(\mathbf{x},t)}{2} = -T_j\partial_{x_j}\ln P(\mathbf{x},t). \quad (26)$$

Recall that this quantity is nonzero due to the action of the thermal bath [the factor T_j in Eq. (26)], and due to the fact that at the coarse-grained level of description almost all trajectories of the Brownian particles are not smooth. Sometimes $u_j(\mathbf{x},t)$ is referred to as "osmotic velocity" of the Brownian particle with index j [7]. We shall use this word as a useful shorthand for the more precise term "change of the coarse-grained velocity."

Note that once the interactions between the Brownian particles are absent: $U(\mathbf{x})=\Sigma_k U(x_k)$—and then correlations are absent, $P(\mathbf{x},t)=\Pi_k P(x_k,t)$, if they were absent initially—the coarse-grained velocities $v_{\pm,j}$ depend only the corresponding coordinate: $v_{\pm,j}(\mathbf{x},t)=v_{\pm,j}(x_j,t)$.

The regular (or mechanical) counterpart of the coarse-grained velocity for the Brownian particle with unit mass and index j can be naturally associated with the Newtonian force $f_j(\mathbf{x})=-\partial_{xj}U(\mathbf{x})$, as defined by Eq. (17). When the bath(s) are absent, this is the only contribution to the coarse-grained velocity. The fact that velocity appears to be proportional to force reminds us that we are in the overdamped regime of description, where, in particular, friction $\dot{\mathbf{x}}$ dominates over acceleration $\ddot{\mathbf{x}}$.

Uncertainty relations and their interpretation

Uncertainty relations exist not only in quantum mechanics but also in physics of underdamped [8] and overdamped [38] Brownian motion. As the very subject of statistical physics, they arise out of ignorance reasons, or more precisely due to separation of time scales: though the real classical particles involved in the classical Brownian motion certainly do have sharply defined coordinates and momenta at the microscopical level, at the coarse-grained (overdamped) level of description each Brownian particle does not have a well-defined trajectory, and cannot possess sharply defined coordinate and coarse-grained velocity, as verified below.

We saw above that the coarse-grained velocities $v_{+,j}(\mathbf{x},t)$ and $v_{-,j}(\mathbf{x},t)$ are defined with respect to the ensemble $\Sigma(\mathbf{x},t)$ of all Brownian particles which at the moment t pass via coordinate vector $\mathbf{x}$. On the other hand, $\Sigma(\mathbf{x},t)$ is by itself a subensemble embedded with probability (weight) $P(\mathbf{x},t)$ into the ensemble of all realizations of the random coordinate vector $\mathbf{x}$ at the time t. Thus both $v_{+,j}(\mathbf{x},t)$ and $v_{-,j}(\mathbf{x},t)$, as well as the osmotic velocity $u_j(\mathbf{x},t)$ are random quantities as functions of the random configuration of N Brownian particles $\mathbf{x}=(x_1,\ldots,x_N)$. In other words, this randomness enters via the context $(\mathbf{x},t)$ which was chosen to define the ensemble $\Sigma(\mathbf{x},t)$.

More specifically, let us focus on the common distribution function $P(\mathbf{x},\mathbf{u};t)$ of

$$\mathbf{x}=(x_1,\ldots,x_N) \quad \text{and} \quad \mathbf{u}=(u_1,\ldots,u_N), \quad (27)$$

$$P(\mathbf{x},\mathbf{u};t) = P(\mathbf{x},t)\prod_{j=1}^{N}\delta(u_j - u_j(\mathbf{x},t)), \quad (28)$$

where $u_j(\mathbf{x},t)$ is defined by Eq. (26).

One has after averaging over $P(\mathbf{x},\mathbf{u};t)$

$$\langle u_j\rangle = \int du_j d\mathbf{x}\, u_j P(\mathbf{x},t)\delta(u_j - u_j(\mathbf{x},t))$$
$$= -T_j\int d\mathbf{x}\,\partial_{x_j}P(\mathbf{x},t) = 0, \quad (29)$$

$$\langle (x_k - \langle x_k\rangle)(u_j - \langle u_j\rangle)\rangle = \int d\mathbf{x}\, u_j(\mathbf{x},t)P(\mathbf{x},t)(x_k - \langle x_k\rangle)$$
$$= -T_j\int d\mathbf{x}(x_k - \langle x_k\rangle)\partial_{x_j}P(\mathbf{x},t)$$
$$= T_j\delta_{kj}. \quad (30)$$

The fact of $\langle u_j\rangle=0$ is natural, since u_j is the (coarse-grained) velocity difference generated due to the interaction to the bath. It is also seen from Eq. (30) that u_j of the corresponding Brownian particle correlates only with its own coordinate. This is related to the assumed independence of the thermal baths which act on different Brownian particles.

Applying to Eq. (30) the standard Cauchy-Schwartz inequality one deduces

$$\langle (x_k - \langle x_k\rangle)^2\rangle\langle (u_j - \langle u_j\rangle)^2\rangle \geq |\langle (x_k - \langle x_k\rangle)(u_j - \langle u_j\rangle)\rangle|^2 = T_j^2\delta_{kj}, \quad (31)$$

which implies an uncertainty relation between the uncertainty of the coordinate and that of the osmotic velocity [38].

Obviously, the regular counterpart $f_j(x,t)$ of the coarse-grained velocity does not (and should not) enter the uncertainty relation.

IV. BROWNIAN ENTANGLEMENT

Consider now two Brownian particles. An analog of entanglement can be introduced in the following way. Call the state of two particles nonentangled (separable) if the common distribution function (28) of the coordinates and the osmotic velocities can be prepared by mixing noncorrelated distributions,[9]

$$P(x_1,u_1,x_2,u_2) = \int d\lambda\, \mathcal{P}(\lambda) P_1(x_1,u_1|\lambda P_2(x_2,u_2|\lambda,$$
$$\mathcal{P}(\lambda) \geqslant 0, \quad \int d\lambda\, \mathcal{P}(\lambda) = 1. \tag{32}$$

We shall naturally require that the separate distributions $P_1(x_1,u_1|\lambda)$ and $P_2(x_2,u_2|\lambda)$ ensure the basic properties (29), (30) of the osmotic velocity u_j. For the rest they can be completely arbitrary. The requirements (29), (30) make the basic difference when comparing to the (naive) classical discussion in Sec. II C, where any distribution was admissible. More detailed discussion on the physical meaning of Eq. (32), and, in particular, on its similarities and differences with the definition (1) of quantum entanglement, is postponed until Sec. VI.

For simplicity we choose units such that x and u have the same dimension as $\sqrt{T}$ (e.g., we measure time in units of square root of the damping constant; see the Appendix). We take also $T_1=T_2=T$ again for simplicity. In the same way as we derived Eqs. (5), (13), one can write the Brownian uncertainty relation (31) as

$$\langle \Delta u_1^2\rangle + \langle \Delta x_1^2\rangle \geqslant 2T, \quad \langle \Delta u_2^2\rangle + \langle \Delta x_2^2\rangle \geqslant 2T, \tag{33}$$

and proceed to derive

$$\langle (\Delta u_1 + \Delta u_2)^2\rangle + \langle (\Delta x_1 - \Delta x_2)^2\rangle < 4T, \tag{34}$$

as a sufficient condition for the entanglement. More general relations (14) can also be obviously transferred to the Brownian situation:

$$\langle (\Delta u_1 + \zeta \Delta u_2)^2\rangle + \langle (\Delta x_1 + \epsilon \Delta x_2)^2\rangle < 4T, \tag{35}$$

where ζ and ϵ can independently assume values ± 1.

A. Gaussian state of two interacting Brownian particles

It will be useful to work out a simple model for two coupled Brownian particles, where the above general concepts can be visualized and studied in detail.

Consider now two harmonically interacting Brownian particles with an overall potential energy,

$$U(x_1,x_2) = \frac{ax_1^2}{2} + \frac{ax_2^2}{2} + gx_1x_2, \tag{36}$$

where $a>0$ and g characterizes the interaction between the particles.[10] The particles interact with independent bath at the same temperature T.

The Fokker-Planck equation (18) for the dynamics of these Brownian particles can be solved directly and the outcome is known to be given by the following two-dimensional Gaussian (provided the initial distribution was Gaussian):

$$P(x_1,x_2;t) = \frac{\sqrt{\det C}}{2\pi} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{2} C_{i,j}x_ix_j \right],$$
$$C^{-1} = \begin{pmatrix} \langle x_1^2(t)\rangle & \langle x_1(t)x_2(t)\rangle \\ \langle x_1(t)x_2(t)\rangle & \langle x_2^2(t)\rangle \end{pmatrix} \equiv \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{12}(t) & \sigma_{22}(t) \end{pmatrix}, \tag{37}$$

where for simplicity we assumed that

$$\langle x_1(t)\rangle = \langle x_2(t)\rangle = 0, \tag{38}$$

for all times. It is also useful to note the marginal distributions of $P(x_1,x_2;t)$:

$$P_j(x_j;t) = \exp\left[-\frac{x_j^2}{2\sigma_{jj}(t)} \right], \quad j=1,2. \tag{39}$$

The easiest way to obtain $\sigma_{11}(t)$, $\sigma_{22}(t)$, and $\sigma_{12}(t)$ is to look directly at the Langevin equations (17) with the potential (36):

$$\dot{x}_1(t) + ax_1(t) + gx_2(t) = \eta_1(t),$$
$$\dot{x}_2(t) + ax_2(t) + gx_1(t) = \eta_2(t), \tag{40}$$

which can be more conveniently rewritten in terms of the relative coordinate $r_-=(x_1-x_2)/2$ and the center-of-mass coordinate $r_+=(x_1+x_2)/2$:

$$\dot{r}_\pm(t) = -(a\pm g)r_\pm(t) + \frac{1}{2}[\eta_1(t) \pm \eta_2(t)], \tag{41}$$

and then solved directly:

$$r_\pm(t) = e^{-(a\pm g)t}r_\pm(0) + \frac{1}{2}\int_0^t ds\, e^{-(a\pm g)s}[\eta_1(t-s) \pm \eta_2(t-s)]. \tag{42}$$

Equations (26), (37) produce for the coarse-grained velocity difference (osmotic velocity) of the first and the second particle, respectively,

[9] We consider only the common distribution function of the coordinates and the osmotic velocities, and not, e.g., the common distribution of the coordinates and forward velocities $v_{+,j}(\mathbf{x},t)$, simply because due to the Brownian uncertainty relation (30) only the former can lead to a nontrivial definition of entanglement. Recall in this context that the quantum uncertainty relations are necessary for the existence of the quantum entanglement.

[10] Note that the positivity of $U(x_1,x_2)$ is implied by $g^2<a^2$.

$$u_1(\mathbf{x},t)=\frac{T(\sigma_{22}x_1-\sigma_{12}x_2)}{\sigma_{11}\sigma_{22}-\sigma_{12}^2},\quad u_2(\mathbf{x},t)=\frac{T(\sigma_{11}x_2-\sigma_{12}x_1)}{\sigma_{11}\sigma_{22}-\sigma_{12}^2}. \tag{43}$$

These results display the fact that the osmotic velocity of the first particle depends explicitly on the coordinate of the second one (and vice versa).

The sufficient condition (35) reduces to

$$\langle(\Delta u_1+\zeta\Delta u_2)^2\rangle+\langle(\Delta x_1+\epsilon\Delta x_2)^2\rangle=\frac{T^2(\sigma_{22}+\sigma_{11}-2\zeta\sigma_{12})}{\sigma_{11}\sigma_{22}-\sigma_{12}^2}+\sigma_{22}+\sigma_{11}+2\epsilon\sigma_{12}<4T. \tag{44}$$

A simpler expression is obtained for $\sigma_{11}=\sigma_{22}$:[11]

$$\frac{T^2}{\sigma_{11}+\zeta\sigma_{12}}+\sigma_{11}+\epsilon\sigma_{12}<2T. \tag{45}$$

This condition is of course not satisfied for $\sigma_{12}=0$, as the minimal value of $\sigma_{11}+T^2/\sigma_{11}$ over σ_{11} is just equal to $2T$. For the correlated situation $\sigma_{12}>0$ and we apply Eq. (45) for $\zeta=1$, $\epsilon=-1$, while for the anticorrelated situation $\sigma_{12}<0$ we use in the same way Eq. (45) for $\zeta=-1$, $\epsilon=1$. With these we get from Eq. (45) the following condition:

$$(\sigma_{11}-T)^2<\sigma_{12}^2+2T|\sigma_{12}|. \tag{46}$$

It is seen that this sufficient condition for the Brownian entanglement is symmetric with respect to correlation and anticorrelation, and it can be satisfied if σ_{11} is sufficiently close to T and $|\sigma_{12}|$ is finite. Equation (46) can be also satisfied when the particles are either correlated or anticorrelated sufficiently strongly, e.g., when $|\sigma_{12}|$ is sufficiently large and sufficiently close to its upper bound σ_{11}. Note that Eq. (46) is not satisfied for $T\to 0$. This is natural, since there are no Brownian uncertainty relations in this limit.

Thus the Brownian entanglement can exist for sufficiently strongly fluctuating and/or sufficiently strongly interacting Brownian particles.

Let us check Eq. (45) more specifically with the stationary (equilibrium) state of the Brownian particles which is established for long times provided the potential energy $U(x_1,x_2)$ is (strictly) positive, and where

$$\sigma_{11}=\sigma_{22}=\frac{Ta}{a^2-g^2},\quad \sigma_{12}=-\frac{Tg}{a^2-g^2}. \tag{47}$$

These relations can be obtained either by directly solving Eq. (42) and then taking the limit $t\to\infty$, or directly via the stationary distribution for the particles which is known to be Gibbsian: $P(x_1,x_2)\propto\exp[-U(x_1,x_2)/T]$.

Equation (46) reduces to[12]

$$|g|>-1+\sqrt{1+(a-1)^2}. \tag{48}$$

Thus if $|g|$ is large enough, there can be entanglement in the equilibrium state.

B. Entanglement for two noninteracting Brownian particles

In the quantum case two subsystems can be entangled even if they interacted in the past, but they do not interact at the moment when the entanglement is tested. This is also the case with the Brownian particles, as we show now.

Return to the above system of two harmonically interacting Brownian particles and assume that they did interact for $t<0$ but are not interacting for positive times.[13] One has from Eq. (40) [or from Eq. (42)]

$$\sigma_{jj}(t)=e^{-2at}\sigma_{jj}(0)+\frac{T}{a}(1-e^{-2at}),\qquad \sigma_{12}(t)=e^{-2at}\sigma_{12}(0),\quad j=1,2. \tag{49}$$

We assume that the condition (45) (with $\sigma_{11}=\sigma_{22}$ at all times) was satisfied at the initial time, so that the Brownian entanglement was present. For positive times and $a>0$, $\sigma_{12}(t)$ gradually disappears. This means that for long times the sufficient condition (45) for the Brownian entanglement will not be valid. Clearly, no Brownian entanglement can persist for long times, since the particles become in this limit completely noncorrelated, i.e., the common probaility distribution for $t\to\infty$ factorizes $P(x_1,x_2,t)=P(x_1,t)P(x_2,t)$ [compare with our discussion after Eq. (26)].

It is, however, to be stressed that for certain not very long times we shall surely have the condition (45) satisfied, at least once it was satisfied initially. Moreover, if even this condition was not satisfied at $t=0$, it can curiously get valid for some (not very long) times. Here is an example. For free Brownian particles with $a=0$, the correlation $\sigma_{12}(t)$ does not change in time at all, but instead the dispersion of the coordinates increases linearly with time: $\sigma_{jj}(t)=\sigma_{jj}(0)+2Tt$. Condition (46) for the presence of entanglement now reads

$$|\sigma_{11}(0)-T+2Tt|<\sqrt{\sigma_{12}^2(0)+2T|\sigma_{12}(0)|}. \tag{50}$$

Even if this condition was not valid at $t=0$, $|\sigma_{11}(0)-T|>\sqrt{\sigma_{12}^2(0)+2T|\sigma_{12}(0)|}$, and in the case that one has additionally $\sigma_{11}(0)<T$, Eq. (50) can still be satisfied in the time window

$$t_-<t<t_+, \tag{51}$$

where

$$t_\pm=\frac{T-\sigma_{11}(0)\pm\sqrt{\sigma_{12}^2+2T|\sigma_{12}|}}{2T}. \tag{52}$$

Our general conclusion is that the Brownian entanglement is possible if the two Brownian particles do interact or had

[11] This equality will be satisfied for two identical Brownian particles if their initial conditions are the same.

[12] Conditions (48) are compatible with the stability requirement $a>|g|$, which comes from demanding positivity of $U(x_1,x_2)$. Indeed, once $|g|$ satisfies Eq. (48), $a>|g|$ becomes $a(2+a)>(a-1)^2$ and is satisfied provided $a>1/4$.

[13] To avoid possible misunderstanding, recall that the thermal baths acting on two particles were taken to be completely independent of each other, so that there is no influence via them.

interacted strongly enough. The absence of interactions at present need not always destroy the Brownian entanglement, for some finite times it may even facilitate the sufficient conditions for its existence, as we just saw.

V. LOCAL OSMOTIC VELOCITIES

In Sec. III we have seen that the definition of the coarse-grained velocities (21), (22), (26) is given via the ensemble $\Sigma(x_1,x_2,t)$ in the common context of the two Brownian particles.

Let us now turn to the local velocities determined with respect to the ensemble $\Sigma_1(x_1,t)$, which is obtained by measuring at time t only the coordinate x_1 of the first particle.

There are two completely equivalent ways for determining the coarse-grained velocities over this ensemble. The first way amounts to repeating definitions (21), (22) for the ensemble $\Sigma_1(x_1,t)$,

$$\nu_{\pm,1}(x_1,t) = \pm \lim_{\varepsilon\to+0} \int dy_1 \frac{y_1 - x_1}{\varepsilon} P(y_1, t \pm \varepsilon | x_1, t), \quad (53)$$

where all the involved probability distributions contain no references on the second Brownian particle. The corresponding osmotic velocity reads, in complete analogy to Eq. (26),

$$\mu_1(x_1,t) = \nu_{-,1}(x_1,t) - \nu_{+,1}(x_1,t). \quad (54)$$

The second way is to note that if observer 1 has no information at all from observer 2, then he effectively sums the ensemble $\Sigma(x_1,x_2,t)$ over all possible results of the second coordinate x_2 at time t with his result x_1 being fixed. Each value of x_2 during this summation is then met with probability $P(x_2,t|x_1,t)$, and this results in

$$\nu_{\pm,1}(x_1,t) = \int dx_2 v_{\pm,1}(x_1,x_2,t) P(x_2,t|x_1,t). \quad (55)$$

The equivalence between definitions (53), (55) can be established via Eqs. (24), (25) and the Bayes formula.[14]

Completely similar definitions can be given for the second particle, employing the ensemble $\Sigma(x_2)$ obtained by measuring the second particle's coordinate only. For the corresponding osmotic component one now has from Eqs. (26), (54), (55):

$$\mu_j(x_j,t) = -T_j \partial_{x_j} P(x_j,t), \quad j=1,2. \quad (56)$$

For the Brownian particle in the harmonic potential, as described, e.g., by Eq. (39), one has

$$\mu_j(x_j,t) = T_j \frac{x_j}{\sigma_{jj}(t)}. \quad (57)$$

If now the definition (32) of nonentanglement is applied to the common distribution function,

$$P(x_1,\mu_1,x_2,\mu_2) = P_1(x_1)\delta[\mu_1 - \mu_1(x_1)]P_2(x_2)\delta[\mu_2 - \mu_2(x_2)], \quad (58)$$

then it is seen to be satisfied trivially: no entanglement occurs with locally defined osmotic velocities

VI. DISCUSSION

Let us compare in more detail the physical meaning of the quantum-mechanical entanglement versus its Brownian analog.

(i) In analogy to continuous-variable quantum entanglement, Brownian entanglement is defined as a type of correlation between the coordinates and the changes of the coarse-grained velocities (osmotic velocities) of two Brownian particles, which is impossible to reproduce by mixing noncorrelated—that is, referring to each particle separately—distributions.

(ii) The Brownian uncertainty relation (30) between the coordinate and the change of the coarse-grained velocity is necessary for the very existence of the Brownian entanglement. This is again similar to the quantum situation, where the analogous role of a necessary condition is being played by the quantum uncertainty relations.

(iii) The momentum operator $\hat{p}$ in quantum mechanics—though being equal in Heisenberg representation to the time derivative of the coordinate operator $\hat{p}=m(d/dt)\hat{x}$—does not in general fully characterize the intuitive notion of "change of the coordinate for an infinitesimal time."[15]

In contrast, the Brownian entanglement is about the coordinates and the coarse-grained velocities which do character-

[14]Note that for obtaining the local coarse-grained velocities $\nu_{\pm,1}(x_1,t)$ we had to average $v_{\pm,1}(x_1,x_2,t)$ over the conditional distribution $P(x_2,t|x_1,t)$, and *not* over the unconstrained probability of the second coordinate $P(x_2,t)$. This inequivalence can be illustrated with the help of the following fact of probability theory concerning three random variables A,B,C: $\Sigma_C P(C)P(A|BC) \neq \Sigma_C P(C|B)P(A|BC) = \Sigma_C P(AC|B) = P(A|B)$. Would we adopt the second possibility, e.g., $\tilde{u}(x_1,t) = \int dx_2 u(x_1,x_2,t)P(x_2,t)$, this would lead us to an explicitly common-context (nonlocal) quantity, e.g., for the above example of harmonic oscillators we would get $\tilde{u}(x_1,t) = T\sigma_{22}(t)x_1/[\sigma_{11}(t)\sigma_{22}(t) - \sigma_{12}^2(t)]$, which explicitly depends on the dispersion $\sigma_{22}(t)$ of the second particle. If the two particles do not interact for positive times, but did interact in the past (for $t<0$), $\tilde{u}(x_1,t)$ can still be influenced, e.g., by external fields which apply on the second particle for positive times (apparent or false nonlocality).

[15]This is a general point. A difference $\hat{A}(t) - \hat{A}(0) = \int_0^t dt (d/dt)\hat{A}$ of Heisenberg operators does not fully characterize the change in time of the observable $\hat{A}$, because there are ensembles described by (time independent in the Heisenberg representation) states $|\psi\rangle\langle\psi|$ for which $[\hat{A}(t) - \hat{A}(0)]|\psi\rangle = 0$. This seems to imply that the value of $\hat{A}$ did not change at all, but this is not correct, since the above eigenvalue relation may be still compatible—due to noncommutativity $[\hat{A}(t), \hat{A}(0)] \neq 0$—with different statistics of $\hat{A}(t)$ and $\hat{A}(0)$, e.g., $\langle\psi|\hat{A}^3(t)|\psi\rangle \neq \langle\psi|\hat{A}^3(0)|\psi\rangle$. For more elaborated discussion and concrete examples, see Ref. [39]. For unbound operators, such as $\hat{x}$ and $\hat{p}$, the above reasonings may need to be technically modified, since eigenstates of unbound operators are not normalizable. In this context it may suffice to require $[\hat{A}(t) - \hat{A}(0)]|\psi\rangle \approx 0$.

ize the change of the coordinates on the coarse-grained scale of time.

(iv) Both quantum entanglement and its Brownian analog can exist for subsystems which interacted in the past, but do not interact at the present.

(v) In quantum mechanics the sufficient condition (13) for entanglement involves correlations between the coordinate operators $\hat{x}_1$ and $\hat{x}_2$, and the momentum operators $\hat{p}_1, \hat{p}_2$ of the subsystems S_1 and S_2. Quantum entanglement can only be found after the results obtained via measurements on the corresponding subensembles for the subsystems S_1 and S_2 are put together. The situation with the Brownian entanglement is similar, since it also requires correlation experiments.

(vi) The traditional point of view is that for checking quantum entanglement [e.g., via sufficient condition (13)] one needs to make measurements of the noncommuting coordinate $\hat{x}_k$ and momentum $\hat{p}_k$ for each quantum particle ($k=1,2$). To this end one usually employs two different ensembles of particles S_1 and S_2 and correspondingly two different measuring apparatuses. In contrast, the Brownian entanglement involves only consecutive coordinate measurements done on a single ensemble of the Brownian pairs. It may seem that this constitutes an operational difference between the quantum entanglement and its Brownian analog, possibly preventing the complete unification of the matters. However, this standard difference in ensembles to be employed can be reconsidered taking into the fact that a quantum state may be determined using a single apparatus [35]. This scheme employs an auxiliary system in a known state such that it first couples with the target system, and only later the commuting variables of the full system are measured. This setup may be invoked here to ascertain that both clasically and quantum-mechanically commutative measurements may completely suffice for the state determination of a single ensemble, and thus for detecting the entanglement. In this respect the two concepts of entanglement (quantum and Brownian) are similar.

(vii) Here is finally the main conceptual difference between the quantum entanglement and its Brownian analog. In quantum mechanics the above operators of coordinate and momentum pertain to the corresponding subsystems S_1 and S_2 *independently* of the full system context. By this we mean that all the statistics of, e.g., $\hat{p}_1$ can be collected via local measurements on the corresponding quantum subensemble, whether or not this subensemble forms a part of any larger ensemble.

In contrast, the very definition of the coarse-grained velocities (21), (22), (26) involves a global (that is, depending on the two subsystems) ensemble $\Sigma(\mathbf{x},t)$. As seen in Sec. V, the purely local definition of coarse-grained velocities can also be given, but there will not be any entanglement on that level, for the same reason as there is no entanglement in other classical systems (see Sec. II C).

This conclusion on the main difference is close to the analogous conclusion of Ref. [12], which discusses similarities between quantum entanglement and certain correlations in classical optics.

VII. POSSIBILITIES OF EXPERIMENTAL REALIZATION

One hopes that an experimental verification of the Brownian entanglement is going to be easier than that of its quantum analog: since Brown's discovery in 1828, various examples of Brownian motion are routinely observed in many systems. On the other hand, the basic conditions needed for observation of the Brownian entanglement amount to two coupled Brownian particles and a resolution of the Brownian motion sufficient for observing the osmotic velocities. Recall that on the coarse-grained time scale the osmotic velocities can be visualized as average kicks obtained by the Brownian particle due to its interaction with bath particles.

Here we discuss two experimentally studied examples of Brownian motion. The pecularity of these examples is that the Brownian motion can be detected by the human eye (without microscopes). These are thus macroscopic examples of Brownian motion. Recall that the typical examples of Brownian motion involve much smaller scales: pollen molecules in water observed originally by Brown had a size $\approx 10^{-3}$ cm, and even smaller lengths are typical for many other realizations of the Brownian motion.

The first experiment [40] considers a two-dimensional circular container with elastic walls. Inside of the container there are motorized balls with a mass of 120 g and a diameter of 8 cm. Each ball is driven by a battery-powered motor inside, and moves chaotically due to elastic collisions with the walls of the container and with other motorized balls. The set of balls (with concentration 50 balls/m^2) models the bath particles. The Brownian particles are modeled with a set of ping-pong balls (with a mass 2 g and diameter 4 cm) which undergo random collisions with the motorized balls. The primary purpose of Ref. [40] was to study polymer statistics, so that the ping-pong balls were jointed into a long chain by means of springs. The resulting coupled Brownian motion produced a probability distribution for the end-to-end distance of the ping-pong chain in close agreement with the existing theories of two-dimensional (self-avoiding) random walks. In this experiment the Langevin equation as such was not tested directly, but there were several indirect reasons supporting its validity [40]. This system would fit for observing the Brownian entanglement, because almost all needed ingredients of this phenomenon are present.

A curious pedagogical analog of the above experiment is the motion of a child's toy called Bumble Ball [41]: a plastic sphere about 11 cm in diameter with a symmetrical pattern of rubber knobs extending about 3 cm from its surface. An internal motor rotates an off-axis mass such that when activated and placed on a hard surface, a Bumble Ball simulates a random walk [41]. Joining two such balls with a spring and placing them into a container creates a toy model which might provide at least the most rough indications of the Brownian entanglement.

The second experiment [42] studied a single sphere (a ping-pong ball of a mass 2.5 g and diameter 4 cm) rolling stochastically in an upflow of gas under flow speed 280 cm/c. The sphere rolls stochastically due to the turbulence it generates in the gas stream (the corresponding Reynolds number is ≈ 40). The study focussed on the full time-dependent dynamics of the sphere, as captured by video imaging. It appears that the dynamics is that of the underdamped Brownian particle in a harmonic potential. This correspondence was thoroughly checked from various perspectives. When looking for the Brownian entanglement in such a

situation, it would be necessary to go to the overdamped motion regime (e.g., by decreasing the mass of the sphere and by increasing the flow speed of the stream) and to add there a second sphere.

VIII. CONCLUSION

We have uncovered the phenomenon of Brownian entanglement: a correlation effect between the coordinates and the coarse-grained velocities of two classical Brownian particles, which resembles the quantum entanglement. In contrast to the latter, which is presently given a fundamental status, the Brownian entanglement—as the very subject of statistical physics—arises out of coarse-graining (incomplete description) reasons. In that respect it is similar to other basic relations of the statistical physics, such as the second law [39]. In the present situation the coarse-graining comes due to the time-scale separation: the evolution of the momenta of the Brownian particles is very fast and cannot be resolved on the time scales available to the experiment. The idea of time-scale separations is one of the most pertinent ones in nonequilibrium statistical physics. In a qualitative form it appears already in good textbooks on this subject [4,43], and has been since then formalized in various contexts and on various levels of generality [44–48].

Once there is time-scale separation between the (stochastic) motion of the coordinate and the momentum of the Brownian particles, the operational definition of velocity via the rate of the coordinate leads to the coarse-grained velocity which is not equal to the real momentum. The change of the coarse-grained velocity is controlled by the Brownian uncertainty relation. Moreover, the coarse-grained velocities appear to be contextual random quantities, i.e., they depend on the concrete setting of the coordinate measurement used to define them. These two aspects (uncertainty relations and contextuality) suffice to define Brownian entanglement, similarly to quantum entanglement, as the impossibility to prepare the common distribution of the coordinates and the coarse-grained velocities of two Brownian particles via mixing locally independent (noncorrelated) distributions referring to the two particles separately. Again in analogy to quantum entanglement, Brownian entanglement can be witnessed via the uncertainty relations. Alternatively, it is also possible to understand entanglement from the perspective of noncommuting observables if the coarse-graining provides nongenerating partitions[14].

In this paper we demonstrated the Brownian entanglement on the simplest, exactly solvable, one-dimensional models of two interacting Brownian particles. It should be, however, kept in mind that interacting Brownian particles (random walks) are basic for several fields of modern statistical physics, such as colloids [49] or polymers [6]. One of the most transparent examples from the polymer science is a DNA macromolecule which consists of two interacting random walks (strands) [6]. As we argued, the experimental observation of the Brownian entanglement might be easier than those of the quantum entanglement, since there are realizations of macroscopic Brownian motion that are visible with the human eye.

One of our motivations was to understand the precise relations between the quantum entanglement and its Brownian analog. In Sec. VI we scrutinized this question in detail. The basic difference between the two concepts can be characterized (loosely speaking) as overcontextuality of the overdamped Brownian model as compared to quantum mechanics. Qualitatively similar differences were noted in classical-optics analogs of the quantum entanglement [12]. It is, however, our current opinion that it might be possible to find more complex examples of Brownian motion where the analogy between the two concepts is more complete. In particular, one may consider underdamped Brownian motion such as stochastic electrodynamics [8].

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APPENDIX

This appendix has two closely related purposes. First, we shall apply the definition (21), (22) of the coarse-grained velocities in a nonoverdamped situation, that is, we shall take ε in those definitions much smaller than the characteristic relaxation time of the (real) momenta. We shall convince ourselves that the expected answer is obtained, relating the coarse-grained velocities to the average (real) momentum. In addition, we shall see that the the coarse-grained velocity difference (osmotic velocity) (26) disappears in this situation, i.e., the definitions (21), (22) become equivalent. Second, based on an exactly solvable situation, we shall follow in detail to the behavior of the coarse-grained velocity as a function of ε. For simplicity we operate here only with one Brownian particle.

The evolution of the common probability distribution of the coordinate x and the momentum p of the Brownian particle is described by the following Fokker-Planck-Kramers-Klein equation [5]:

$$\partial_t \mathcal{P} = -\frac{p}{m}\partial_x \mathcal{P} + \partial_p\left(\frac{\gamma}{m}p + V'(x) + \gamma T\partial_p\right)\mathcal{P}, \tag{A1}$$

where m, γ, and $V(x)$ are mass, damping constant, and potential, respectively, and where

$$\mathcal{P}(x,p,t|x',p',t'), \quad t \geqslant t',$$
$$\mathcal{P}(x,p,t|x',p',t) = \delta(x-x')\,\delta(p-p') \tag{A2}$$

is the conditional probability to move from (x',p') at time t' to (x,p) at time t. This equation corresponds to the Langevin equation

$$m\ddot{x} = -ax - \gamma\dot{x} + \eta(t), \quad \langle \eta(t)\rangle = 0, \tag{A3}$$

$$\langle \eta(t)\,\eta(t')\rangle = 2\gamma T\delta(t-t'),$$

where $-\gamma\dot{x}$ is the friction force, and where $\eta(t)$ is the random Gaussian white noise.

Applying the definition (21) of the coarse-grained velocity and using Eqs. (A1), (A2) one has

$$\begin{aligned}\nu_+(x,t) &= \lim_{\varepsilon\to+0}\int dy\frac{y-x}{\varepsilon}\mathcal{P}(y,t+\varepsilon|x,t)\\ &= \lim_{\varepsilon\to+0}\int dydpdp'\frac{y-x}{\varepsilon}\\ &\quad\times\mathcal{P}(y,p,t+\varepsilon|x,p',t)\mathcal{P}(p',t|x,t)\\ &= -\int dydp(y-x)\mathcal{P}(p,t|x,t)\frac{p}{m}\partial_y\delta(x-y)\\ &= \int dp\,\mathcal{P}(p,t|x,t)\frac{p}{m},\end{aligned} \tag{A4}$$

where $\mathcal{P}(p,t|x,t)$ is the conditional probability for having the momentum equal to p at time t, provided the coordinate was equal to x at the same time.

Likewise,

$$\begin{aligned}\nu_-(x,t) &= \lim_{\varepsilon\to+0}\int dy\frac{x-y}{\varepsilon}P(y,t-\varepsilon|x,t)\\ &= \lim_{\varepsilon\to+0}\int dydpdp'\frac{x-y}{\varepsilon}\\ &\quad\times\mathcal{P}(x,p,t|y,p',t-\varepsilon)\frac{\mathcal{P}(y,p',t-\varepsilon)}{\mathcal{P}(x,t)}\\ &= \int dydp(y-x)\frac{\mathcal{P}(y,p,t)}{\mathcal{P}(x,t)}\frac{p}{m}\partial_x\delta(x-y)\\ &= \int dp\,\mathcal{P}(p,t|x,t)\frac{p}{m} = \nu_+(x,t).\end{aligned} \tag{A5}$$

The fact is that $\nu_-(x,t)=\nu_+(x,t)$ can be easily generalized to any number of interacting Brownian particles.

Now if there are no correlations between the momentum and coordinate $\mathcal{P}(p,t|x,t)=\mathcal{P}(p,t)$, then $\nu_-(x,t)=\nu_+(x,t)$ reduce to the usual average momentum. This is the case, in particular, for the deterministic situation, where $\mathcal{P}(x,p,t)=\delta[x-x(t)]\delta[p-p(t)]$.

Thus provided ε has been taken much smaller than any relevant time scale related to the coordinate and/or the momentum, the coarse-grained velocity reduces to the average momentum as it should. In this situation the osmotic velocity is zero.

Now let us show the exactly solvable situation with a harmonic potential

$$V(x) = \frac{ax^2}{2}, \tag{A6}$$

that if the characteristic relaxation times of the momentum

$$\tau_p = \frac{m}{\gamma}, \tag{A7}$$

and the coordinate

$$\tau_x = \frac{\gamma}{a}, \tag{A8}$$

are well separated,

$$\tau_p \ll \tau_x, \tag{A9}$$

and if additionally

$$\tau_x \gg \varepsilon \gg \tau_p, \tag{A10}$$

one gets back the values for osmotic velocity which were obtained in the main text by means of the overdamped Fokker-Planck equation (18).

The case with $V(x)=ax^2/2$ can be solved either directly from Eq. (A1), or using the equivalent Langevin equation (A3). The solution of the latter—obtained, e.g., via Laplace transformation—reads

$$x(t) = x(0)g(t) + \frac{1}{m}p(0)f(t) + \frac{1}{m}\int_0^t dt' f(t-t')\,\eta(t'), \tag{A11}$$

where

$$f(t) = \frac{e^{-\omega_2 t}-e^{-\omega_1 t}}{\omega_1-\omega_2}, \quad g(t) = \frac{\omega_1 e^{-\omega_2 t}-\omega_2 e^{-\omega_1 t}}{\omega_1-\omega_2}, \tag{A12}$$

$$\omega_{1,2} = \frac{\gamma}{2m}\left(1 \pm \sqrt{1-\frac{4am}{\gamma^2}}\right). \tag{A13}$$

For simplicity reasons we shall put $x(0)=p(0)=0$. This can be done, since these quantities are assumed to be independent of the noise by the very definition of the considered stochastic process.

As the process (A11) is Gaussian, the two-time probability distribution of the coordinate can be written down as

$$\mathcal{P}(y,s;x,t) = \frac{\sqrt{d}}{2\pi}\exp\left(-\frac{1}{2}[A_{11}y^2 + A_{22}x^2 + 2A_{12}xy]\right), \tag{A14}$$

where

$$d \equiv \sigma(t,t)\sigma(s,s) - \sigma^2(s,t), \tag{A15}$$

$$\begin{pmatrix} A_{11} & A_{12}\\ A_{12} & A_{22}\end{pmatrix}^{-1} = \begin{pmatrix}\sigma(s,s) & \sigma(s,t)\\ \sigma(s,t) & \sigma(t,t)\end{pmatrix} = \frac{1}{d}\begin{pmatrix}\sigma(t,t) & -\sigma(s,t)\\ -\sigma(s,t) & \sigma(s,s)\end{pmatrix}, \tag{A16}$$

and where $\sigma(s,t)$ is the correlation function of the coordinate,

$$\sigma(s,t) = \langle x(s)x(t)\rangle = \frac{2\gamma T}{m^2}\int_0^s\int_0^t dt_1 dt_2 f(s-t_2)f(t-t_2)\delta(t_1-t_2) = \frac{2\gamma T}{m^2}\int_0^{\min(s,t)} dt' f(t')f(t'+|s-t|). \tag{A17}$$

Using

$$\int dy\, y\, \mathcal{P}(y,s|x,t) = \int dy\, y \frac{\mathcal{P}(y,s;x,t)}{\mathcal{P}(x,t)} = -x\frac{A_{12}}{A_{11}} = x\frac{\sigma(s,t)}{\sigma(t,t)}, \tag{A18}$$

one gets, keeping ε finite and fixed,

$$\nu_{+(x,t)} = \int dy \frac{y-x}{\varepsilon}\mathcal{P}(y,t+\varepsilon|x,t) = \frac{x}{\varepsilon}\left(\frac{\sigma(t+\varepsilon,t)}{\sigma(t,t)} - 1\right), \tag{A19}$$

$$\nu_-(x,t) = \int dy \frac{x-y}{\varepsilon}P(y,t-\varepsilon|x,t) = \frac{x}{\varepsilon}\left(1 - \frac{\sigma(t-\varepsilon,t)}{\sigma(t,t)}\right), \tag{A20}$$

for any $\varepsilon \geqslant 0$ smaller than t: $t-\varepsilon > 0$.

If now ε is the smallest time scale,

$$\nu_-(x,t) - \nu_+(x,t) = \frac{2\gamma T}{m^2}\frac{x\varepsilon}{\sigma(t,t)}\left[\int_0^t dt' \dot{f}^2(t') - f(t)\dot{f}(t)\right] + O(\varepsilon^2) \tag{A21}$$

goes to zero with $\varepsilon \to 0$ as was predicted above.

As seen from Eq. (A13), the overdamped limit is realized with the dimensionless parameter $4am/\gamma^2$ being small:

$$\frac{4am}{\gamma^2} \ll 1, \tag{A22}$$

and then the characteristic relaxation times of the momentum and the coordinate [obtained from Eqs. (A11)–(A13)] are well separated,

$$\tau_p = \frac{m}{\gamma} \approx \frac{1}{\omega_1} \ll \tau_x = \frac{\gamma}{a} \approx \frac{1}{\omega_2}. \tag{A23}$$

Now if

$$\omega_1 t \gg 1, \quad \omega_1 s \gg 1, \quad \omega_1|s-t| \gg 1, \tag{A24}$$

one gets from Eq. (A17)

$$\sigma(s,t) = \frac{T}{a}(e^{-\omega_2|s-t|} - e^{-\omega_2(t+s)}), \tag{A25}$$

which coincides with the corresponding correlator obtained from the overdamped Fokker-Planck equation directly. Taking now $s=t\pm\varepsilon$ in Eqs. (A19), (A20) and employing Eq. (A25) with the limit $\varepsilon \to 0$—which now should be understood in the context of conditions (A23), (A24)—one can get the expressions for $\nu_\pm(x,t)$,

$$\nu_+(x,t) = -\frac{ax}{\gamma}, \quad \nu_-(x,t) = -\frac{ax}{\gamma} - \frac{2T}{\gamma}\frac{x}{\sigma(t,t)}, \tag{A26}$$

which can alternatively be derived from Eqs. (24), (25), if we put there $N=1$ (only one Brownian particle), $f(x) = -ax/\gamma$ (the linear force divided over the friction constant), and $P(x,t) \propto \exp[-x^2/2\sigma(t,t)]$ (Gaussian distribution function).

Equation (A26) exhibits the fundamental difference for the coarse-grained velocity of a Brownian particle between arriving at a position $x[\nu_-(x)]$ and leaving this position $[\nu_+(x)]$.

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