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# Is QED inconsistent in the presence of Dirac monopoles?

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**Abstract.** We examine a recently made claim that an unphysical coupling associated with the longitudinal component of the gauge field appears in the Dirac quantization condition thus rendering QED inconsistent in the presence of Dirac monopoles. We point out that a conventional quantization condition can still be obtained.

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Recently, He, Qiu and Tze [1] have proposed a generalized formulation of QED where they allow for two different coupling constants $e$ and $\tilde{e}$ associated, respectively, with the transverse (physical) and longitudinal (unphysical) components of the gauge field. By considering Dirac monopoles with strings along the $\pm\hat{z}$ axes they then argue that the two potentials are related via a gauge transformation such that the transverse parts are left unchanged; this has a nontrivial consequence, namely, that the conventional Dirac quantization condition is replaced by one where the unphysical coupling $\tilde{e}$ enters, *i.e.*, $\tilde{e}g = n/2, n \in Z$. Thus, they conclude, the only viable scenario is that the monopole charge $g$ has to be zero. Given the importance that such a statement bears for studies of confinement mechanisms driven by abelian monopoles [2] we wish to examine this argument carefully. We show that a conventional type of Dirac quantization condition (that is to say, one where the physical charge $e$ enters and not $\tilde{e}$) can still be obtained in this scheme. Let us start by writing the "generalized" covariant derivative used in Ref. [1]

$$D_\mu = \partial_\mu - ie\mathcal{A}_\mu - i\tilde{e}\tilde{\mathcal{A}}_\mu \ . \tag{1}$$

Here the gauge field $A_\mu$ is decomposed into transverse, $\mathcal{A}_\mu = T_{\mu\nu}A^\nu$, and longitudinal, $\tilde{\mathcal{A}}_\mu = L_{\mu\nu}A^\nu$, components, coupled to charges $e$ and $\tilde{e}$, respectively; we employ the projectors $L_{\mu\nu} = \partial_\mu\partial_\nu/\partial^2$ and $T_{\mu\nu} = g_{\mu\nu} - L_{\mu\nu}$. The longitudinal components do not enter the field strength tensor $F_{\mu\nu}$ and are unphysical. This theory is invariant under local $U(1)$ transformations

$$e\mathcal{A}_\mu(x) + \tilde{e}\tilde{\mathcal{A}}_\mu(x) \to e\mathcal{A}_\mu(x) + \tilde{e}\tilde{\mathcal{A}}_\mu(x) - \partial_\mu\Omega(x) \ . \tag{2}$$

By applying the projectors $L_{\mu\nu}$, $T_{\mu\nu}$ on both sides of (2) one obtains

$$\begin{aligned} &\mathcal{A}_\mu(x) \to \mathcal{A}_\mu(x) - \frac{1}{e}\partial_\mu\Omega(x) + \frac{1}{e}\partial_\mu\frac{1}{\partial^2}\partial^2\Omega(x) \\ &\tilde{\mathcal{A}}_\mu(x) \to \tilde{\mathcal{A}}_\mu(x) - \frac{1}{\tilde{e}}\partial_\mu\frac{1}{\partial^2}\partial^2\Omega(x) \ . \end{aligned} \tag{3}$$

Thus, as noticed in [1], the transverse components are left invariant and only the longitudinal ones change. However, this statement is ambiguous when $\partial^2\Omega = 0$.

Let us now discuss Dirac monopoles with string singularities in the context of the above theory. The simplest examples are static Dirac strings in the $\mp\hat{z}$ axes,

$$\mathbf{A}_{\mp\hat{z}} = \frac{g}{r}\frac{\pm 1 - \cos\theta}{\sin\theta}\hat{\phi} \ . \tag{4}$$

The authors offer the following decomposition into transverse and longitudinal components:

$$\begin{aligned} &\mathcal{A}_{\mp\hat{z}} = -\frac{g}{r}\frac{\cos\theta}{\sin\theta}\hat{\phi} \\ &\tilde{\mathcal{A}}_{\mp\hat{z}} = \pm\frac{g}{r\sin\theta}\hat{\phi} \ . \end{aligned} \tag{5}$$

Thus, the transverse piece is the same for the two strings and the gauge transformation that maps one string solution to the other is of the type

$$\begin{aligned} &\mathcal{A}(\mathbf{r}) \to \mathcal{A}(\mathbf{r}) \\ &\tilde{\mathcal{A}}(\mathbf{r}) \to \tilde{\mathcal{A}}(\mathbf{r}) - \frac{1}{\tilde{e}}\nabla\Omega(\mathbf{r}) \ , \end{aligned} \tag{6}$$

with $\Omega(\mathbf{r}) = 2\tilde{e}g\phi$. The usual argument [3] pertaining to the single–valueness of the wavefunction leads then to the pathological quantization condition

$$\tilde{e}g = \frac{n}{2}, \ n = \{0, \pm 1, \pm 2 \ldots\} \ , \tag{7}$$

where a *physical* coupling $g$ is supposed to be constrained by an *unphysical* coupling $\tilde{e}$. Notice, however, that both string potentials (4) have zero divergence (only $\partial_\phi$ appears and there is no $\phi$ dependence) and $\nabla^2\phi = 0$. Thus, $\partial^2\Omega = 0$ and this is a case where (6) does not necessarily follow from (3). In fact, given that $\nabla \cdot \mathbf{A}_{\mp\hat{z}} = 0$, another possible (or even natural, if one sets $\partial^2\Omega = 0$ in (3)) decomposition is one with no longitudinal piece altogether, $\tilde{\mathcal{A}}_{\mp\hat{z}} = 0$. Then, the gauge transformation that connects the two strings is of a type *dual* to (6)

$$\mathscr{A}(\mathbf{r}) \to \mathscr{A}(\mathbf{r}) - \frac{1}{e}\nabla\Omega(\mathbf{r})$$
$$\tilde{\mathscr{A}}(\mathbf{r}) \to \tilde{\mathscr{A}}(\mathbf{r}) \ , \qquad (8)$$

with $\Omega(\mathbf{r}) = 2eg\phi$. Now it is the longitudinal component that remains the same while the transverse changes and the corresponding quantization condition is the conventional one with $e$, the *physical* charge, appearing. We wish to emphasize here that the choice of longitudinal piece made in [1] can not be considered false, as $\tilde{\mathscr{A}}_{\mp\hat{z}}$ in (5) has both zero curl and zero divergence and thus can be considered to be *either* transverse *or* longitudinal. The authors are aware of this ambiguity but imply that this is not the case for "other[1] singular monopole potentials". A less trivial example to check this assertion would be one with a nontrivial $\phi$ dependence. Consider therefore Dirac strings along the $\pm\hat{x}$ axes, with corresponding potentials

$$\mathbf{A}_{\mp\hat{x}} = \mp\frac{g}{(r \pm r\sin\theta\cos\phi)}\left[\sin\phi\hat{\theta} + \cos\theta\cos\phi\hat{\phi}\right] \ . \qquad (9)$$

It is easy to check that these solutions are divergenceless as well. In fact one can show that this holds for an *arbitrary* Dirac string that traces a path $\Gamma$ with potential

$$\mathbf{A}(\mathbf{r}) = -\int_\Gamma d\mathbf{u} \times \mathbf{B}(\mathbf{r}-\mathbf{u}) \qquad (10)$$

with $\mathbf{B}(\mathbf{x}) = -g\nabla(\frac{1}{4\pi x})$. Then we get for the divergence

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = -\int_\Gamma \mathbf{B}(\mathbf{r}-\mathbf{u})\cdot\nabla_\mathbf{r} \times d\mathbf{u} + \int_\Gamma d\mathbf{u}\cdot\nabla_\mathbf{r} \times \mathbf{B}(\mathbf{r}-\mathbf{u})$$
$$= 0 \ . \qquad (11)$$

Thus, it appears that this generalization of QED does not affect Dirac monopoles since their fields have always a zero divergence and can therefore always be taken to be purely transverse. The gauge transformation connecting two strings along paths $l$ and $l'$ will then be of the type (8), with angle $\Omega = e\Omega_{l,l'}$ that satisfies $\nabla^2\Omega_{l,l'} = 0$. Then the Dirac quantization condition will be oblivious to the unphysical coupling $\tilde{e}$ and there is no reason to deduce from such a construction that QED is inconsistent with Dirac monopoles.

It is perhaps instructive to address an alternative argument based on the quantization of total angular momentum that the authors present to support the same idea: consider the total angular momentum in the presence of a Dirac string on the $\mp\hat{z}$ axis [4]. Using (1) and the transverse/longitudinal decomposition (5) one gets

$$\mathbf{J} = m\mathbf{r} \times \frac{d\mathbf{r}}{dt} - eg\hat{r}$$
$$= i\mathbf{r} \times \mathbf{D} - eg\hat{r}$$
$$= \mathbf{r} \times \mathbf{p} + g\left[\frac{-e\cos\theta \mp \tilde{e}}{\sin\theta}\hat{\theta} - e\hat{r}\right]$$
$$\Rightarrow J_z = -i\partial_\phi \pm g\tilde{e} \ . \qquad (12)$$

Requiring half integer eigenvalues for $J_z$ results to the pathological quantization condition (7) since $\tilde{e}$ (and not $e$) appears in (12). This argument however depends totally on the choice of longitudinal component one makes. The choice $\tilde{\mathscr{A}}_{\mp\hat{z}} = 0$, dictated by the fact that $\nabla\cdot\mathbf{A}_{\mp\hat{z}} = 0$, leads instead to

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} + ge\left[\frac{-\cos\theta \mp 1}{r\sin\theta}\hat{\theta} + \hat{r}\right]$$
$$\Rightarrow J_z = -i\partial_\phi \pm ge \ . \qquad (13)$$

Thus it is now the *physical* coupling $e$ that will appear in the quantization condition and we reach the same conclusion as above, namely, that this version of QED is indeed consistent with Dirac monopoles.

[1] we take this to mean "string-like" potentials

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