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The $\widehat{SU}(n)_1$ WZW models Spinon decomposition and yangian structure

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Abstract

We present a ‘spinon formulation’ of the $\widehat{SU}(n)_1$ Wess–Zumino–Witten models. Central to this approach are a set of massless quasi-particles, called ‘spinons’, which transform in the representation \bar{n} of $su(n)$ and carry fractional statistics of angle $\theta = \pi/n$. Multi-spinon states are grouped into irreducible representations of the yangian $Y(sl_n)$. We give explicit results for the $su(n)$ content of these yangian representations and present N -spinon cuts of the WZW character formulas. As a by-product, we obtain closed expressions for characters of the $su(n)$ Haldane–Shastry spin chains.

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1. Introduction

In this paper we analyze the quantum field theory description of a set of mass-less ‘quasi-particles’ in $1+1$ dimensions with the following two defining properties:

- (i) they transform in the representation \bar{n} of an $su(n)$ symmetry algebra;
- (ii) they carry fractional statistics of angle $\theta = \pi/n$.

For $n = 2$, such quasi-particles arise when one starts from electrons in one spatial dimension and decouples the charge degrees of freedom. The resulting quasi-particles, which have been called *spinons* [1], form a doublet under $su(2)$ -spin and carry half the statistics of the original electrons, i.e. they are *semions* with $\theta = \pi/2$. As a concrete example, one may think of the low-lying excitations of an XXX Heisenberg anti-ferromagnet or of a $d = 1$ Hubbard model at half filling. We shall be using the term

‘ $su(n)$ spinon’ for the quasi-particles transforming in the \bar{n} of $su(n)$ and with $\theta = \pi/n$ statistics.

The quantum field theories for massless $su(n)$ spinons are conformal field theories (CFT) with $su(n)$ symmetry. As such, they are invariant under an affine Lie algebra $\widehat{su(n)}_k$, where the positive integer k is called the level. Such theories are essentially unique; they are known as the $\widehat{SU(n)}_k$ Wess–Zumino–Witten (WZW) CFT. We are thus dealing with a class of field theories that are well known and well studied: detailed information on the spectrum and correlation functions of these theories is available.

The novel aspect of this work is that we shall work out a formulation of the $\widehat{SU(n)}_k$ WZW models that gives a central role to the fundamental quasi-particles, i.e. the $su(n)$ spinons. In particular, we shall interpret the (chiral) spectrum, which is usually viewed as a collection of irreducible highest weight modules (HWM) of the affine symmetry, as a collection of *multi-spinon states*. The motivation for such an approach comes from applications in condensed matter physics. Let us give two examples, which both involve CFT’s with $\widehat{SU(n)}_k$ symmetry.

- (i) The *multi-channel Kondo effect* has been formulated in terms of an $\widehat{SU(2)}_k$ CFT (this for the $s = \frac{1}{2}$, k -channel case). The essential dynamics is the boundary scattering of the fundamental spinons of this CFT. When $k > 1$ (the multi-channel case), these spinons carry non-trivial topological charges, which are at the origin of the non-Fermi-liquid behaviour at low temperatures.
- (ii) In the *fractional quantum Hall effect* edge excitations are described by chiral CFT’s. For filling fractions ν in the so-called Jain series, $\nu = n/(2np + 1)$, the edge theory is a $c = n$ CFT with extended symmetry $\widehat{U(1)} \times \widehat{SU(n)}_1$. Experimentally, the edge dynamics is probed *via* tunneling experiments. The tunneling events can be viewed as impurity scattering of the fundamental quasi-particles of the CFT. For $n = 1$ (and $p = 1$) such a picture was used for the exact computation of a universal tunneling conductance curve [2]. We expect that for $n > 1$ a similar analysis can be done as soon as one can properly deal with the quasi-particles of the $\widehat{su(n)}_1$ symmetry, which are precisely the spinons of this paper.

For $k > 1$ the $\widehat{SU(n)}_k$ spinons carry additional labels (‘topological charges’) related to a choice of ‘fusion channel’ in multi-spinon states. We shall in this paper focus on $k = 1$, where this complication does not occur. The *multi-spinon* formalism for the $\widehat{SU(n)}_1$ WZW theories is very different from the traditional approach, which was entirely based on the affine symmetry $\widehat{su(n)}_1$. In our new formulation, the affine symmetry is traded for an alternative symmetry, which is the yangian $Y(sl_n)$. In fact, the representation theory of the yangian quantum group can be interpreted as a Generalized Pauli Principle (see Section 4.3) that guides the construction of (a basis of) multi-spinon states.

For the case $n = 2$ (the original $su(2)$ spinons), a multi-spinon formulation has been worked out in great detail [3–5] and the connection with the representation theory of the yangian $Y(sl_2)$ has been made explicit.

For $n > 2$, the $su(n)$ representation content of the irreducible, finite-dimensional representations of $Y(sl_n)$ is not known in general. We shall therefore proceed in the following indirect manner. We first focus on a simpler (finite-dimensional) physical

system with exact yangian symmetry, which is the $su(n)$ Haldane–Shastry (HS) chain on L sites. In Section 3.1 we shall give an exact and closed form result for the $su(n)$ content of the yangian representations that appear in the spectrum of these models. The $\widehat{SU}(n)_1$ WZW model arises as the $L \rightarrow \infty$ limit of the HS model, and our results for the yangian representations simply carry over to the field theory (see Section 3.3).

One of our main goals in this paper is to find exact results for the N -spinon contribution to the CFT characters. Every yangian representation carries a well-defined spinon number N , and it is a matter of clever combinatorics to find the N -spinon cuts of the various characters. Our strategy will be to relate N -spinon characters of the $\widehat{SU}(n)_1$ WZW model to the characters of a set of *conjugate* HS chains on $N - nl$ sites, with $l = 0, 1, \dots$. Explicit results for the N -spinon cuts of various CFT characters follow from here. They are given in Section 4.4. It can be proved directly that the N -spinon characters sum up to the known CFT characters, which confirms the result of Section 3.1 for the $Y(sl_n)$ characters.

We would like to stress that even for $su(2)$ the results obtained in this paper are new. We shall find the following expression for the N -spinon contribution to the chiral spectrum

$$\text{ch}_{\text{WZW}}^N(q, z) = q^{-\frac{N^2}{4}} \sum_{j_1+j_2=N} \sum_{l \geq 0} (-1)^l \frac{q^{\frac{1}{2}l(l-1)}}{(q)_l} \frac{q^{\frac{1}{4}(j_1^2+j_2^2)}}{(q)_{j_1-l}(q)_{j_2-l}} z^{j_2-j_1} \quad (1.1)$$

(see Sections 4.4 and 4.5 for further discussion).

Comparing our results with quasi-particle character formulas proposed in Refs. [6,7], we would like to stress that, where we work with spinons labeled by the *weights* of a fundamental representation of $su(n)$ [6,7], use quasi-particles labeled by the *roots* of $su(n)$. To illustrate the difference, let us focus on the character of the ‘vacuum module’ $\widehat{L}(\Lambda_0)$ (see Section 2 for notations) of the $n = 2$ theory. The $su(2)$ spinons act back and forth between the modules $\widehat{L}(\Lambda_0)$ and $\widehat{L}(\Lambda_1)$ and the character of $\widehat{L}(\Lambda_0)$ is therefore gotten by restricting to even spinon numbers

$$\begin{aligned} \text{ch}_{\widehat{L}(\Lambda_0)}(q, z) &= \sum_{N \text{ even}} \text{ch}_{\text{WZW}}^N(q, z) \\ &= \sum_{\substack{j_1, j_2 \geq 0 \\ j_1+j_2 \text{ even}}} \sum_{l \geq 0} (-1)^l \frac{q^{\frac{1}{2}l(l-1)}}{(q)_l} \frac{q^{\frac{1}{4}(j_1-j_2)^2}}{(q)_{j_1-l}(q)_{j_2-l}} z^{j_2-j_1} \\ &= \sum_{\substack{j_1, j_2 \geq 0 \\ j_1+j_2 \text{ even}}} \frac{q^{\frac{1}{4}(j_1+j_2)^2}}{(q)_{j_1}(q)_{j_2}} z^{j_2-j_1}. \end{aligned} \quad (1.2)$$

The quasi-particles of Refs. [6,7] act within the module $\widehat{L}(\Lambda_0)$, giving a character formula with unrestricted summations over quasi-particle numbers r_1 and r_2

$$\widehat{\text{ch}}_{L(\Lambda_0)}(q, z) = \sum_{r_1, r_2 \geq 0} \frac{q^{r_1^2 + r_2^2 - r_1 r_2}}{(q)_{r_1} (q)_{r_2}} z^{2r_1 - 2r_2}. \quad (1.3)$$

Clearly, there are many ways to interpret affine characters in terms of quasi-particles. From the point of view of physics, the choice is usually clear; while there can be many options at the level of kinematics, the dynamics of a given system will dictate a particular choice of quasi-particle basis.

In Section 5 we briefly discuss other ways to set up a description of the $\widehat{SU(n)}_1$ WZW models, this time starting from a set of $n - 1$ different quasi-particles $\phi^{(i)}$, $i = 1, 2, \dots, n - 1$, transforming in the fundamental representations $L(\Lambda_i)$ of $su(n)$. For example, in the case $n = 3$ there will be two types of quasi-particles, one set transforming as the $\mathbf{3}$, the other as the $\bar{\mathbf{3}}$. In such a formulation, one may use a characterization of irreducible yangian representations in terms of parameters which are the zero's of a set of $n - 1$ Drinfel'd polynomials (Section 5.2). We shall also present examples of character formulas that are based on an approach that respects the symmetry between the conjugate representations $\mathbf{3}$ and $\bar{\mathbf{3}}$ (Section 5.3).

More details, including proofs of the various statements in this paper, will be given in a future publication [8]. In that paper, we shall also present character formulas for the higher-level ($k > 1$) $\widehat{SU(n)}_k$ WZW models, generalizing our results [9], see also Refs. [10,11], for the $\widehat{SU(2)}_k$ theories.

2. Basics

2.1. $\widehat{SU(n)}_k$ Wess–Zumino–Witten models

The $\widehat{SU(n)}_k$ WZW models are CFT's with abundant symmetry. In the traditional formulation, a central role is played by the (bosonic) symmetry algebra $\widehat{su(n)}_k$, which is a so-called affine Lie algebra. The defining commutation relations on the generators J_m^a , $m \in \mathbb{Z}$, $a = 1, \dots, n^2 - 1$, are

$$[J_m^a, J_n^b] = km d^{ab} \delta_{m+n,0} + f^{ab}_c J_{m+n}^c. \quad (2.1)$$

In this formula, d^{ab} is a Killing metric and the f^{ab}_c are the $su(n)$ structure constants, normalized as

$$f^a_{bc} f^{dbc} = -2n d^{aa}. \quad (2.2)$$

The Hilbert space of the $\widehat{SU(n)}_k$ WZW model consists of a collection of integrable highest weight modules (HWM) $\widehat{L}(\Lambda)$, where Λ is an integral dominant weight of $su(n)$. In Dynkin notation, such weights are written as $\Lambda = \sum_{i=1}^{n-1} m_i \Lambda_i$, where the Λ_i are the fundamental weights (we shall write $\Lambda_0 = 0$ for the weight of the singlet representation). For a given level k , only a finite number of HWM's are allowed. They are selected by the condition

$$\sum_{i=1}^{n-1} m_i \leq k. \quad (2.3)$$

When $k = 1$, the allowed highest weights are Λ_0 and the $n - 1$ fundamental weights of $su(n)$ (corresponding to the $\mathbf{3}, \bar{\mathbf{3}}$ for $su(3)$, the $\mathbf{4}, \mathbf{6}, \bar{\mathbf{4}}$ for $su(4)$, etc.).

In a formulation based on the affine symmetry $\widehat{su(n)}_k$, the building blocks for the partition sum are the (chiral) characters of $\widehat{su(n)}_k$,

$$\text{ch}_{\widehat{L}(\Lambda)}(q, z) = \text{tr}_{\widehat{L}(\Lambda)}(q^{L_0} z^\Lambda). \quad (2.4)$$

The notation z^Λ is shorthand for $\prod z_i^{l_i}$ where $\Lambda = \sum l_i \Lambda_i$; this factor keeps track of the $su(n)$ quantum numbers of the states in the spectrum.

Related quantities are the string functions $c_\lambda^A(q)$ and the generating functionals $\Phi_\lambda^A(q)$. The former record the multiplicities of $su(n)$ weights λ throughout the spectrum

$$c_\lambda^A(q) = \sum_{L_0} (\text{mult. of weight } \lambda \text{ at energy level } L_0) q^{L_0}, \quad (2.5)$$

while the latter describe the branching of the $\widehat{su(n)}_k$ HWM $\widehat{L}(\Lambda)$ into $su(n)$ irreducible representations (irrep) $L(\lambda)$

$$\Phi_\lambda^A(q) = \sum_{L_0} (\text{mult. of irrep } L(\lambda) \text{ at energy } L_0) q^{L_0}. \quad (2.6)$$

For all these quantities, closed form expressions are known in the literature (see, e.g., Ref. [12] and references therein).

One of our goals in this paper is to recast the well-known character formulas for $\widehat{su(n)}_1$ in a way that keeps track of the *spinon number* of the various states. This then should be viewed as the first step in a program where the entire analysis of the WZW models (characters, form factors, correlation functions) is done in a novel fashion: one recognizes the spinons as the fundamental elementary excitations and builds the theory from there. One quickly discovers that the affine (Kac–Moody) symmetry is not very useful for such an approach, as the basic multi-spinon states do not transform nicely under the generators J_m^a with $m \neq 0$. A similar remark applies to the Virasoro symmetry. We shall therefore resort to an entirely different symmetry structure, which is the yangian $Y(sl_n)$, with generators $\{Q_p^a\}$, $p = 0, 1, \dots$, together with a set of conserved charges $\{H_q\}$, $q = 1, 2, \dots$. It turns out that this symmetry is precisely tailored for a *multi-spinon formulation* of the $\widehat{SU(n)}_1$ WZW models! Once the level-1 case has been understood in this manner, the higher-level models can be organized in an analogous way, see Refs. [9,8].

The algebraic relations between the $\{Q_p^a\}$ and $\{H_q\}$ are

$$[Q_p^a, H_q] = 0, \quad [H_q, H_{q'}] = 0, \quad (2.7)$$

together with the defining relations of the yangian $Y(sl_n)$ as given by Drinfel'd [13,14]

$$(Y1) \quad [Q_0^a, Q_0^b] = f^{ab}{}_c Q_0^c,$$

$$(Y2) \quad [Q_0^a, Q_1^b] = f^{ab}{}_c Q_1^c,$$

$$(Y3) \quad [Q_1^a, [Q_1^b, Q_0^c]] + (\text{cyclic in } a, b, c) = A^{abc}{}_{def} \{Q_0^d, Q_0^e, Q_0^f\},$$

$$(Y4) \quad [[Q_1^a, Q_1^b], [Q_0^c, Q_1^d]] + [[Q_1^c, Q_1^d], [Q_0^a, Q_1^b]] \\ = (A^{ab}{}_{p, qrs} f^{cdp} + A^{cd}{}_{p, qrs} f^{abp}) \{Q_0^q, Q_0^r, Q_1^s\},$$

where $A^{abc, def} = \frac{1}{4} f_{pqr} f^{adp} f^{beq} f^{cfr}$ and the curly brackets denote a completely symmetric (weighted) product.

The yangian $Y(sl_n)$ is a non-trivial example of a quantum group, and as such it is equipped with a non-co-commutative co-multiplication

$$\Delta_{\pm}(Q_0^a) = Q_0^a \otimes 1 + 1 \otimes Q_0^a, \\ \Delta_{\pm}(Q_1^a) = Q_1^a \otimes 1 + 1 \otimes Q_1^a \pm \frac{1}{2} f^a{}_{bc} Q_0^b \otimes Q_0^c. \quad (2.8)$$

The right-hand sides of the relations (Y3) and (Y4) can be derived from the homomorphism property of this co-multiplication. For the case of sl_2 , the cubic relation (Y3) is superfluous while for sl_n , with $n \geq 3$, (Y4) follows from (Y2) and (Y3).

In Ref. [15] it was established that the $\widehat{SU(n)}_1$ WZW models carry a representation of $Y(sl_n)$, with the following explicit expressions for the lowest yangian generators Q_0^a and Q_1^a and charges H_1 and H_2 :

$$Q_0^a = J_0^a, \quad Q_1^a = \frac{1}{2} f^a{}_{bc} \sum_{m>0} (J_{-m}^b J_m^c) \mp \frac{n}{2(n+2)} W_0^a, \\ H_1 = L_0, \quad H_2 = d_{ab} \sum_{m>0} (m J_{-m}^a J_m^b) \pm \frac{n}{(n+1)(n+2)} W_0. \quad (2.9)$$

These expressions contain the zero-modes of the following conformal fields (the brackets denote standard normal ordering, see, e.g., Ref. [16] for conventions)

$$W^a(z) = \frac{1}{2} d^a{}_{bc} (J^b J^c)(z), \quad W(z) = \frac{1}{6} d_{abc} (J^a (J^b J^c))(z). \quad (2.10)$$

The 3-index d -symbol that occurs in these expressions is completely symmetric, traceless and has been normalized according to

$$d^a{}_{bc} d^{dbc} = \frac{2(n^2 - 4)}{n} d^{ad}. \quad (2.11)$$

It was shown in Ref. [15] that the explicit expressions (2.9) satisfy the defining relations (2.7) and (2.8) when acting on integrable HWM's.

With this result, one immediately concludes that the HWM's $\widehat{L}(\Lambda)$ of $\widehat{su(n)}_1$ decompose into collections of irreducible finite-dimensional representations of $Y(sl_n)$. Such yangian representations have a natural interpretation in terms of 'multi-spinon' states

and each yangian representation carries a well-defined ‘spinon number’ N . In Section 4 below we analyze this structure in full detail and present ‘ N -spinon cuts’ of the CFT characters.

2.2. Haldane–Shastry $su(n)$ spin chains

In 1988 Haldane [17] and Shastry [18] proposed a class of integrable quantum spin chains that are different from those that can be solved by means of the Bethe ansatz. A characteristic feature is that the spin–spin exchange is not restricted to nearest neighbors. Instead, it has a non-trivial dependence on distance, which, in the simplest case, is of the form $1/r^2$. Here we shall recall a few aspects of $su(n)$ HS chains. In later sections we shall use these results in our analysis of the $\widehat{SU}(n)_1$ WZW CFT’s.

The hamiltonian H_2 of the $su(n)$ Haldane–Shastry (HS) chain with $1/r^2$ exchange acts on a Hilbert space that has n states (transforming as the fundamental \bar{n} of $su(n)$) for each site i , $i = 1, 2, \dots, L$. It has the form

$$H_2 = \sum_{i \neq j} \left(\frac{z_i z_j}{z_{ij} z_{ji}} \right) (P_{ij} - 1), \quad (2.12)$$

where P_{ij} is a permutation operator that exchanges the states at sites i and j , and $z_{ij} = z_i - z_j$. We choose the complex parameters $\{z_j\}$ as $z_j = \omega^j$, with $\omega = \exp(2\pi i/L)$, so that the exchange described by (2.12) is proportional to the inverse-square of the chord distance between the sites. It was found in Ref. [19] that the hamiltonian H_2 commutes with the following operators:

$$Q_0^a = \sum_i J_i^a, \quad Q_1^a = \frac{1}{4} \sum_{i \neq j} \frac{(z_i + z_j)}{z_{ij}} f_{bc}^a J_i^b J_j^c, \quad (2.13)$$

where the J_i^a are associated to the action of $su(n)$ on the n basis states at site i . Furthermore, the operators Q_0^a , Q_1^a satisfy the defining relations (Y1)–(Y4) of the yangian $Y(sl_n)$. This remarkable result has been understood to be at the basis of the integrability of these spin chains. Indeed, it is possible [19,20] to define mutually commuting integrals of motion H_p , $p \geq 3$, which commute with the hamiltonian H_2 and with the yangian generators.

We thus see that the $\widehat{SU}(n)_1$ WZW model (which is a quantum field theory) and the $su(n)$ HS chain (which is a quantum mechanical many body system) share a common algebraic structure. This structure was first uncovered for the HS chains [19,21], and later established for the CFT counterparts [3,4].

We shall now briefly summarize the solution of the HS chain as given in [1,19,21]. In this work, eigenstates of the hamiltonian (2.12) are characterized via ‘motifs’, which are sequences of $L+1$ digits ‘0’ or ‘1’, beginning and ending with ‘0’, and containing at the most $n-1$ consecutive ‘1’. The motifs can be parametrized by ‘rapidity’ sequences $l_j \in \{1, 2, \dots, L-1\}$ that indicate the positions of the ‘1’. Every motif corresponds to a degenerate set of eigenstates, which together form an irreducible representation of the yangian symmetry. These states have H_2 eigenvalue

$$H_2 = - \sum_j l_j (L - l_j) \quad (2.14)$$

and crystal momentum $K = -\left(\frac{2\pi}{L} H_1\right) \bmod 2\pi$, where

$$H_1 = - \sum_j l_j. \quad (2.15)$$

The CFT analogue of the operator H_1 will be the Virasoro zero-mode L_0 , which is interpreted as ‘energy’ in the CFT setting. One easily finds that the lowest possible value of H_1 on the L -site chain is

$$H_1^{(0)} = - \left(\frac{n-1}{2n} \right) L^2 + \frac{1}{2} |A_k|^2, \quad (2.16)$$

where $L \equiv k \bmod n$ with $k = 0, 1, \dots, n-1$ and $|A_k|^2 = k(n-k)/n$.

For $n > 2$, the $su(n)$ content of yangian irreducible representations is not known in general. The $su(n)$ content of the motif-related yangian irreducible representations was first studied [19,22], where the following procedure was proposed. One replaces every ‘0’ by ‘)’ (a ‘(’ for the first ‘0’ and ‘)’ for the last), so that the motif breaks up as a string of elementary motifs of the form $(1 \dots 1)$ with at most $n-1$ ‘1’s. These elementary motifs correspond to the singlet ($n-1$ ‘1’s) and to the $n-1$ fundamental irreducible representations of $su(n)$. For example, for $su(3)$ one has that ‘(11)’ is a singlet, ‘(1)’ is a $\bar{3}$ and ‘()’ is a 3. The tensor product of the elementary motifs gives an upper bound to the total $su(n)$ content of the motif. The precise content is obtained from here by taking into account certain reductions. For $su(2)$ these reductions amount to the total symmetrization of adjacent doublets, i.e. ‘() ()’ represents a triplet rather than the product of two doublets. The reductions that are needed for $su(3)$ have been exemplified in Ref. [22], but until now have not been given in closed form. In Ref. [22] an indirect characterization of the motif-related yangian representations was given using ‘squeezed strings’. This was used to show (for $n = 2$ and numerically for n up to 6 and small L) that the motif prescription is complete in the sense that it accounts for the correct number of n^L eigenstates of the HS hamiltonian.

In Section 3.1 below we shall give a very simple and general characterization of the $su(n)$ content of the motif-related yangian irreducible representations, and from there derive a simple formula for the HS character $\text{ch}_{\text{HS}}^L(q, z)$, defined as

$$\text{ch}_{\text{HS}}^L(q, z) \equiv \text{tr}_{\text{HS}} (q^{L_0} z^A), \quad (2.17)$$

where (see Eq. (2.16))

$$L_0 = H_1 - H_1^{(0)} + \frac{1}{2} |A_k|^2. \quad (2.18)$$

3. $\widehat{SU(n)}_1$ CFT as a limit of the $su(n)$ HS chain

3.1. Yangian irreducible representations, I

In this subsection we give a closed form result for the $su(n)$ content of yangian irreducible representation that corresponds to a general motif of the L -site HS chain. We encode the motif by an ordered set $1 \leq l_1 < l_2 < \dots < l_s \leq L-1$, the l_j denoting the positions of the ‘1’ in the motif. We shall express the yangian character in the characters of $su(n)$ irreducible representations with Dynkin labels $(m \ 0 \dots 0)$, which we write as $\chi_m(z)$.

By working out some examples, one quickly discovers that the motif ‘00...0’ ($L+1$ zero’s) represents a completely symmetrized tensor product $(\mathbf{n}^{\otimes L})_s$ with character $\chi_L(z)$. Replacing one of the ‘0’ by ‘1’, i.e. introducing a single non-zero l , represents the product $\chi_{L-l}(z)\chi_l(z)$ minus the completely symmetric term $\chi_L(z)$.

This pattern can quickly be generalized to a general motif with ‘1’ at positions $l_1 < l_2 < \dots < l_s$, leading to an alternating sum over ordered subsets $S' = \{l_{i_1}, l_{i_2}, \dots, l_{i_t}\}$ of $S = \{l_1, l_2, \dots, l_s\}$. We therefore propose the following closed expression for the character $\chi_{\{l_j\}}^Y(z)$ of the motif-associated yangian irreducible representations:

$$\chi_{\{l_j\}}^Y(z) = \sum_{S' \subset S} (-1)^{s-t} \chi_{L-l_{i_t}}(z) \chi_{l_{i_t}-l_{i_{t-1}}}(z) \dots \chi_{l_{i_1}}(z) \quad (3.1)$$

(cf. Ref. [23], Proposition 4.6, for $Y(sl_2)$).

Notice that this expression has the effect of anti-symmetrizing the tensor power $\mathbf{n}^{\otimes(p+1)}$ of adjacent n -plets separated by p ‘1’s. For $p = n-1$ this leads to a singlet since

$$\sum_{S' \subset \{1, 2, \dots, n-1\}} (-1)^{n-1-t} \chi_{n-l_{i_t}}(z) \chi_{l_{i_t}-l_{i_{t-1}}}(z) \dots \chi_{l_{i_1}}(z) = \chi_0(z) = 1, \quad (3.2)$$

while for $p \geq n$ consecutive ‘1’ the result is identically zero. This then confirms the observation of Ref. [19] that non-vanishing motifs can have no more than $n-1$ consecutive ‘1’.

3.2. HS characters

A direct consequence of the result (3.1) is that the sum over all motifs of the associated yangian characters simply gives $(\chi_1)^L$, i.e. the $su(n)$ character of the tensor product of L spins in the \mathbf{n} of $su(n)$. We can easily do better than this, and evaluate the sum over all motifs keeping track of the eigenvalues H_1 , meaning that we are evaluating the character (2.17). Consider $su(3)$, $L=3$ as an example. We have

Motif	q^{H_1+3}	Character	$su(3)$ content
0110	q^0	$\chi_1^3 - \chi_2\chi_1 - \chi_1\chi_2 + \chi_3$	1
0010	q^1	$\chi_1\chi_2 - \chi_3$	8
0100	q^2	$\chi_2\chi_1 - \chi_3$	8
0000	q^3	χ_3	10

The q -powers can be written as $q^{\sum l'_j}$ where l'_j are the positions of the '0' ($\neq 0, L$) in the motif. We can write them as $\prod_j (1 - (1 - q^{l'_j}))$ and then perform the sum over all motifs. In the example this gives

$$((1 - q)(1 - q^2)\chi_3 - (1 - q^2)\chi_2\chi_1 - (1 - q)\chi_1\chi_2 + \chi_1^3). \quad (3.3)$$

In the general case, one finds that the total coefficient of

$$\chi_{L-l_s}(z)\chi_{l_s-l_{s-1}}(z)\dots\chi_{l_1}(z) \quad (3.4)$$

in the character equals

$$q^{\frac{n-1}{2n}L^2 - \frac{1}{2}L(L-1)}(-1)^{L-1-s} \prod_{i=1}^s (1 - q^{l'_i}), \quad (3.5)$$

where the set $\{l'_j\}$ is the complement of $\{l_j\}$ in $\{1, 2, \dots, L-1\}$. This gives

$$\begin{aligned} \text{ch}_{\text{HS}}^L(q, z) &= q^{\frac{n-1}{2n}L^2 - \frac{1}{2}L(L-1)} \\ &\times \sum_{1 \leq l_1 < \dots < l_s \leq L-1} (-1)^{L-1-s} \frac{(q)^{L-1}}{\prod_{i=1}^s (1 - q^{l_i})} \chi_{L-l_s}(z)\chi_{l_s-l_{s-1}}(z)\dots\chi_{l_1}(z), \end{aligned} \quad (3.6)$$

where

$$(q)_M \equiv \prod_{l=1}^M (1 - q^l). \quad (3.7)$$

Expanding this result in powers of q^{-1} leads to a representation of the character in terms of occupation numbers $m_k = 0, 1, \dots$ of 'orbitals' of 'energy' $k = 0, 1, 2, \dots$

$$\text{ch}_{\text{HS}}^L(q, z) = (-1)^L q^{\frac{n-1}{2n}L^2 - \frac{1}{2}L(L+1)} (q)_L \sum_{\sum_k m_k = L} q^{-\sum_k k m_k} \prod \chi_{m_k}(z). \quad (3.8)$$

One would like to view the HS character as a q -deformation of a formula that gives the decomposition of the $su(n)$ character of the full Hilbert space in terms of irreducible representations of $su(n)$. To write such a formula, we need to introduce a bit of notation.

We denote by ϵ^i , $i = 1, 2, \dots, n$ the weights of the fundamental representation $L(\Lambda_1)$ (the \mathbf{n}) of $su(n)$, with inner product $(\epsilon^i, \epsilon^j) = \delta^{ij} - 1/n$. By P we denote the weight space of $su(n)$ and $P^{(k)}$ denotes the weights in the conjugacy class k , i.e. if $\lambda = \sum_i m_i \Lambda_i$, then $\lambda \in P^{(k)}$ iff $\sum_i m_i \equiv k \pmod n$. By multinomial expansion we have

$$(\chi_1(z))^L = \sum_{\lambda \in P^{(L \bmod n)}} z^\lambda M_\lambda^{L,n}(q=1), \quad (3.9)$$

where, for $\lambda \in P^{(L \bmod n)}$,

$$M_\lambda^{L,n}(q) \equiv \left[\begin{matrix} L \\ \frac{L}{n} + (\lambda, \epsilon^1), \dots, \frac{L}{n} + (\lambda, \epsilon^n) \end{matrix} \right]_q \quad (3.10)$$

with the q -multinomial defined as

$$\left[\begin{matrix} m_1 + m_2 + \dots + m_n \\ m_1, m_2, \dots, m_n \end{matrix} \right]_q = \frac{(q)_{m_1+m_2+\dots+m_n}}{(q)_{m_1}(q)_{m_2}\dots(q)_{m_n}}. \quad (3.11)$$

As announced, we can write the HS character as a natural q -deformation of (3.9). From (3.8) we derive

$$\text{ch}_{\text{HS}}^L(q, z) = \sum_{\lambda \in P^{(L \bmod n)}} z^\lambda q^{\frac{n-1}{2n}L^2} M_\lambda^{L,n}(q^{-1}) \quad (3.12)$$

and it follows that

$$\text{ch}_{\text{HS}}^L(q, z) = \sum_{\lambda \in P^{(L \bmod n)}} z^\lambda q^{\frac{1}{2}|\lambda|^2} M_\lambda^{L,n}(q). \quad (3.13)$$

Note that the correct ground state energy (2.16) follows directly from this character. For $n = 2$, a formula of this type was already given in Ref. [24].

3.3. The limit $L \rightarrow \infty$: CFT characters

Having obtained explicit results for the HS characters, we can take the limit $L \rightarrow \infty$. In this limit, the combination L_0 in (2.18) corresponds to the Virasoro zero-mode L_0 of the CFT, and the HS characters directly turn into CFT characters. For on the string function $c_\lambda^A(q)$ ($A = A_{L \bmod n}$), we find

$$c_\lambda^A(q) = \lim_{L \rightarrow \infty} q^{\frac{1}{2}|\lambda|^2} M_\lambda^{L,n}(q) = \frac{q^{\frac{1}{2}|\lambda|^2}}{(q)_{\infty}^{n-1}}, \quad (3.14)$$

in agreement with the known result.

Closely related to the string functions are the generating functionals $\Phi_{\text{HS},\lambda}^L(q)$ defined as

$$\Phi_{\text{HS},\lambda}^L(q) = \sum_{L_0} (\text{mult. of irrep } L(\lambda) \text{ at } L_0) q^{L_0} \quad (3.15)$$

(with L_0 as in (2.18)) for which we derive from (3.13) (see Ref. [8] for details).

$$\begin{aligned}\Phi_{\text{HS},\lambda}^L(q) &= \frac{(q)_L}{\prod_{i=1}^n (q)_{\frac{L}{n} + (\lambda, \epsilon^i) + (n-i)}} q^{\frac{1}{2}|\lambda|^2} \prod_{\alpha \in \Delta_+} (1 - q^{(\lambda+\rho, \alpha)}) \\ &= q^{-\frac{L(L-n)}{2n}} K_{(\frac{L}{n} + (\lambda, \epsilon^1), \dots, \frac{L}{n} + (\lambda, \epsilon^n), (1^L)}(q),\end{aligned}\quad (3.16)$$

where $K_{\lambda, \mu}(q)$ is the so-called Kostka polynomial (see, e.g., Ref. [25] and references therein). In this expression Δ_+ stands for the positive root lattice and $\rho = \frac{1}{2} \sum_{\Delta_+} \alpha$.

In the limit $L \rightarrow \infty$, the $\Phi_{\text{HS},\lambda}^L(q)$ go to the generating functionals $\Phi_\lambda^A(q)$ of the $\widehat{SU(n)}_1$ WZW CFT, where $A = A_{L \bmod n}$. For $n = 2$, $\Phi_\lambda^A(q)$ are irreducible characters of a $c = 1$ Virasoro algebra, while for $n > 2$ the $\Phi_\lambda^A(q)$ are irreducible characters of the \mathcal{W}_n algebra at $c = n - 1$. Indeed, in the limit $L \rightarrow \infty$ (3.16) reduce to

$$\lim_{L \rightarrow \infty} \Phi_{\text{HS},\lambda}^L = \frac{1}{(q)_\infty^{n-1}} q^{\frac{1}{2}|\lambda|^2} \prod_{\alpha \in \Delta_+} (1 - q^{(\lambda+\rho, \alpha)}), \quad (3.17)$$

in agreement with the known character formulas for the \mathcal{W}_n algebra at $c = n - 1$ [16].

Note that Eq. (3.17) is a special case of the statement that the $\widehat{sl_n}/sl_n$ characters (at arbitrary level) can be written as limits of $q^{-kL(L-n)/2n}$ times Kostka polynomials. This result was conjectured in Ref. [25] and recently proven in Ref. [26].

4. N -spinon states in the $\widehat{SU(n)}_1$ CFT

4.1. Spinons

In the previous section we have seen that the $\widehat{SU(n)}_1$ WZW CFT can be viewed as the limit $L \rightarrow \infty$ of the HS quantum chain. Under this correspondence, the CFT primary field transforming as the $\bar{\mathbf{n}}$ of $su(n)$ corresponds to the elementary ‘spinon’ excitation over the HS ground state.

Let us explain the nature of these spinons. For the purpose of illustration, we put $n = 3$ and consider the $su(3)$ HS chain on L sites, where we assume that L is a multiple of 3. The motif

$$0110110 \dots 110 \rightarrow (11)(11) \dots (11) \quad (4.1)$$

represents the ground state, which is a singlet under $su(3)$. The idea is that the fundamental spins (each carrying a $\mathbf{3}$ of $su(3)$) have bound into groups of three that each carry the singlet representation, which is the fully anti-symmetrized product of the constituent $\mathbf{3}$. The lowest excited states are obtained by removing a single ‘1’ from the motif and, possibly, shifting one or more of the remaining ‘1’. This typically leads to a motif containing three times ‘(1)’ and for the rest ‘(11)’. The elementary motif ‘(1)’ represents the anti-symmetrized product of two $\mathbf{3}$, which is a $\bar{\mathbf{3}}$. The fact that removing a single ‘1’ from a motif leads to *three* independent excitations ‘(1)’ can be seen as a manifestation of the fact that the elementary excitation ‘(1)’ carries ‘fractional statistics’

with parameter $\theta = \pi/3$. The generic 3-spinon excitation is of the form (the dots stand for groups ‘(11)’)

$$\dots (1) \dots (1) \dots (1) \dots \quad (4.2)$$

and has $su(3)$ content $\bar{3}^{\otimes 3} = 1 \oplus 8_2 \oplus \bar{10}$. Two ‘(1)’ motifs may merge into ‘()’ giving

$$\dots (1) \dots () \dots, \quad \text{or} \quad \dots () \dots (1) \dots \quad (4.3)$$

with $su(3)$ content $3 \otimes \bar{3} = 1 \oplus 8$ for the case where the ‘(1)’ and ‘()’ are separated by at least one singlet group ‘(11)’. When they are adjacent there is a further reduction to

$$\dots (10) \dots, \quad \text{or} \quad \dots (01) \dots \quad (4.4)$$

with $su(3)$ content $3 \otimes \bar{3} - 1 = 8$. Note that we have used the notation ‘(10)’ for ‘(1)()’ and ‘(01)’ for ‘() (1)’ (compare with Appendix C).

The fact that the total number of $3^3 = 27$ possible 3-spinon states gets reduced when the three individual spinons occupy nearby orbitals follows from a Generalized Pauli Principle, see Section 4.3. For $n = 2$ there is only one non-trivial elementary motif, the ‘()’ representing an $su(2)$ doublet, and the Generalized Pauli Principle amounts to symmetrizing the product $2^{\otimes l}$ of l adjacent ‘()’.

For general n , the elementary excitations ‘(11...1)’ ($n - 2$ times ‘1’), which we shall call *spinons*, have the two characteristic features which we already mentioned in our introduction: they transform in the $su(n)$ representation \bar{n} and carry fractional statistics of angle $\theta = \frac{\pi}{n}$. A generic N -spinon state will carry the $su(n)$ representation $\bar{n}^{\otimes N}$ but in many cases the actual $su(n)$ content is reduced according to a Generalized Pauli Principle. The $n = 3$ example shows that the correct rules for general n are rather subtle. A detailed description shall be given in Section 4.3 below.

4.2. Multi-spinon states

In the $\widehat{SU(n)}_1$ WZW CFT the spinon excitations are represented by the primary field $\phi^\alpha(z)$, $\alpha = 1, 2, \dots, n$, transforming in the \bar{n} of $su(n)$ and having conformal dimension $h = \frac{n-1}{2n}$. The CFT ground state corresponds to a half-infinite sequence of vacuum motifs $(11\dots 1)(11\dots 1)\dots$. Single-spinon excitations are created by applying a positive-energy mode of the spinon field $\phi^\alpha(z)$ to the vacuum. Such modes are defined by

$$\phi^\alpha(z) = \sum_{m \in \mathbb{Z}} \phi_{-\frac{n-2k-1}{2n}-m}^\alpha z^{m-\frac{k}{n}}, \quad (4.5)$$

for modes acting on the HWM (‘sector’) with highest weight Λ_0 for $k = 0$ and Λ_{n-k} for $k = 1, \dots, n - 1$.

The idea is now that the entire (chiral) Hilbert space of the CFT is spanned by *multi-spinon* excitations, which are created by repeatedly acting with modes of the fundamental spinon fields. A general N -spinon state is written as

$$\phi_{-\frac{n-(2N-1)}{2n}-n_N}^{\alpha_N} \cdots \phi_{-\frac{n-5}{2n}-n_3}^{\alpha_3} \phi_{-\frac{n-3}{2n}-n_2}^{\alpha_2} \phi_{-\frac{n-1}{2n}-n_1}^{\alpha_1} |0\rangle,$$

$$\text{with } n_N \geq n_{N-1} \geq \dots \geq n_2 \geq n_1 \geq 0. \quad (4.6)$$

The fact that states with different ordering for the n_i can be expressed in terms of states (4.6) follows from the Generalized Commutation Relations (GCR) of the spinon modes, see below. The energy (eigenvalue of the Virasoro zero-mode L_0) of these states is

$$L_0 = -\frac{N(N-n)}{2n} + \sum_{i=1}^N n_i. \quad (4.7)$$

The yangian generators Q_i^a and the conserved charge H_2 have a simple action on one-spinon states. Having assigned the $su(n)$ representation \bar{n} to the fundamental spinons, we should take the lower sign in (2.9). (This choice is such that the space of N -spinon states will be invariant under the action of the yangian.) We then find

$$\begin{aligned} Q_0^a \phi_{-\frac{n-1}{2n}-n_1}^{\alpha} |0\rangle &= (t^a)^{\alpha}_{\beta} \phi_{-\frac{n-1}{2n}-n_1}^{\beta} |0\rangle, \\ Q_1^a \phi_{-\frac{n-1}{2n}-n_1}^{\alpha} |0\rangle &= \left[-n n_1 - \frac{n-2}{4} \right] (t^a)^{\alpha}_{\beta} \phi_{-\frac{n-1}{2n}-n_1}^{\beta} |0\rangle \end{aligned} \quad (4.8)$$

and

$$H_2 \phi_{-\frac{n-1}{2n}-n_1}^{\alpha} |0\rangle = E_2(n_1) \phi_{-\frac{n-1}{2n}-n_1}^{\alpha} |0\rangle, \quad (4.9)$$

where

$$E_2(n_1) = \left[\frac{(n-1)(n-2)}{6n} + n n_1^2 + (n-1) n_1 \right]. \quad (4.10)$$

In these formulas, the matrices $(t^a)^{\alpha}_{\beta}$ describe the representation \bar{n} of $su(n)$ and we have

$$t^a t^b = \frac{1}{n} d^{ab} - \frac{1}{2} f^{ab}{}_c t^c - \frac{1}{2} d^{ab}{}_c t^c. \quad (4.11)$$

In technical terms, the one-spinon states constitute an *evaluation representation* of the yangian $Y(sl_n)$.

The action (4.8) of the yangian can be extended without problem to the level of multi-spinon states of the form (4.6). One then finds that the yangian generators preserve the spinon number, i.e. they map an N -spinon state of the form (4.6), with modes $\{n_i\}$, $i = 1, 2, \dots, N$, onto similar states with modes $\{n'_i\}$ that are equal to or smaller than (in the sense of a natural partial ordering \preceq) the original modes $\{n_i\}$. A similar result is found for the action of H_2 . As a direct consequence, one finds that one may define linear combinations of states of the form (4.6), with labels $\{n'_i\} \preceq \{n_i\}$, such that on the new states H_2 is diagonal with eigenvalue

$$H_2 = \sum_{i=1}^N E_2 \left(n_i - \frac{i-1}{n} \right). \quad (4.12)$$

The eigenspaces of H_2 naturally carry a representation of the yangian, and in this manner we arrive at a decomposition of the space of N -spinon states in terms of irreducible representations of the yangian $Y(sl_n)$. For $n = 2$ this procedure has been worked out in great detail in [3] and in our papers [4,5] with A.W.W. Ludwig. See also Ref. [27] where a number of results for general n were given.

Spinon fields are not free fields, and we already remarked that they carry statistics which are neither bosonic nor fermionic. To illustrate this point we remark that the Operator Product Expansion of spinon fields $\phi(z)$ and $\phi(w)$ has the form

$$\phi(z)\phi(w) = (z-w)^{-\frac{1}{n}} [\phi'(w) + \dots], \quad (4.13)$$

where $\phi'(z)$ is the primary field in the representation $L(\Lambda_{n-2})$ of $su(n)$ (for $n = 2$ this is the identity). It is the value $-\frac{1}{n}$ of the exponent which leads to statistics of angle π/n . Note that it is *not* the conformal dimension $h = \frac{n-1}{2n}$ which determines the statistics of the spinons. Indeed, while for $n \rightarrow \infty$ this dimension approaches the value $h = \frac{1}{2}$ of a free fermion, the statistics approach those of a bosonic theory.

Following Ref. [28], we may interpret the non-trivial π/n statistics of the spinons as ‘exclusion statistics’. As such, they manifest themselves in the motif description of the spectrum (‘creating a single hole leads to an n -spinon excitation’, see Section 4.1) or, more directly, in the fact that the minimal allowed energy difference between adjacent spinon modes is $\frac{1}{n}$ (compare with (4.6)).

Due to the statistics aspect, it is non-trivial to compare different multi-spinon states and to select a set that forms a basis of the chiral Hilbert space (the \mathcal{W}_n modules, say) of our CFT. A direct but cumbersome way to work out relations among multi-spinon states is by using ‘Generalized Commutation Relations’ (GCR) for the modes of the fundamental spinon fields. In our papers [4,5] with A.W.W. Ludwig we derived the fundamental GCR for the case $n = 2$, $k = 1$ and worked out the ensuing relations among multi-spinon states.

In this paper we shall proceed at a more abstract level and proceed by giving a relation between the N -spinon states of the $\widehat{SU(n)}_1$ WZW model and associated $N - nl$ -site ($l = 0, 1, \dots$) HS models with fundamental spins in the \bar{n} of $su(n)$. In this way, we shall be able to characterize a basis of multi-spinon states and compute N -spinon cuts of a variety of CFT characters.

4.3. Yangian irreducible representations, II

The $su(n)$ content for a multi-spinon yangian multiplet is quickly found if we remember that the CFT can be viewed as the $L \rightarrow \infty$ limit of an L -site HS chain. Using the following rule we can construct a motif of the $L = \infty$ HS chain from the modes sequence $n_1 \leq n_2 \leq \dots \leq n_N$ of a multi-spinon state.

Table 1
3-Spinon states in the $\widehat{SU}(3)_1$ theory

Modes	Motif	$su(3)$ content
$n_1 < n_2 < n_3$	$\dots(1)\dots(1)\dots(1)\dots$	$1 \oplus 8_2 \oplus \overline{10}$
$n_1 = n_2 < n_3$	$\dots() \dots(1)\dots$	$1 \oplus 8$
$n_1 < n_2 = n_3$	$\dots(1)\dots() \dots$	$1 \oplus 8$
$n_1 = n_2 = n_3$	$\dots(01)\dots$	8
$n_1 = n_2 = n_3$	$\dots(10)\dots$	8

We restrict to sequences n_i with at most $n - 1$ consecutive n_i equal (the other ones are either zero or equal to a state of lower spinon number). Consider the integers $k = 0, 1, \dots$ and denote by m_k the number of n_i equal to k . Next, replace m_k by the elementary motif $(11 \dots 1)$ having $n - 1 - k$ times ‘1’, and replace ‘)’ (‘) and the initial ‘(’ by ‘0’. In this way, one finds a half-infinite motif of the $L = \infty$ chain, which differs from the vacuum motif (in the $\widehat{L}(\Lambda_{-N \bmod n})$ sector) in finitely many places. The $su(n)$ content of the multi-spinon yangian multiplet is then given by (3.1), restricted to the (finite) non-trivial part of the motif.

As an example, we again put $n = 3$ and consider the 3-spinon states with modes n_1, n_2, n_3 . One quickly finds the result shown in Table 1¹ (compare with Section 4.1).

Before coming to N -spinon characters, we wish to rephrase the $su(n)$ structure of the yangian multiplet with mode sequence $n_1 \leq n_2 \leq \dots \leq n_N$ in a way that is more convenient for the CFT setting. The intuition for this rephrasing comes from the structure of the GCR for spinon modes ϕ_r^α , which roughly speaking tell us two things. First of all, they imply that we should antisymmetrize over neighboring spinons with equal mode-indices, and secondly they imply that in certain cases the singlet combination (‘trace’) of n spinons has to be subtracted.

We start from a mode sequence $0 \leq n_1 \leq n_2 \leq \dots \leq n_N$. We denote by

$$\widehat{\chi}_{\{n_i\}}(z) \quad (4.14)$$

the character of $\overline{\mathfrak{n}}^{\otimes N}$, antisymmetrized over equal $n_i = n_{i+1} = \dots = n_{i+k}$. Next, we have a prescription (see Appendix B) for reducing this character further by placing l ‘hooks’, which each eliminate n modes n_i from the sequence. In the character, placing a hook means that the corresponding n spinons have combined into a singlet and no longer contribute. We write the reduced character obtained by placing l hooks in all possible positions as

$$\widehat{\chi}_{\{n_i\}}^{(l)}(z). \quad (4.15)$$

We claim the following result for the N -spinon yangian character $\widehat{\chi}_{\{n_i\}}^V(z)$:

¹ We use the notation $a \stackrel{i}{=} b$ for $b - a = i$ and $a \stackrel{i}{<} b$ for $b - a > i$.

$$\widehat{\chi}_{\{n_i\}}^Y(z) = \sum_l (-1)^l \widehat{\chi}_{\{n_i\}}^{(l)}(z). \quad (4.16)$$

Clearly, the two steps in this little construction reflect the two aspects ('antisymmetrization' and 'trace subtraction') of the GCR.

One may view the rule encompassed in the formula (4.16) as a *Generalized Pauli Principle* for the construction of multi-spinon states in the $\widehat{SU}(n)_1$ CFT. Returning to the example $su(3)$, $N = 3$ (3 spinons), we can see this Generalized Pauli Principle in action. Focusing on $(n_1, n_2, n_3) = (m, m, m+1)$, corresponding to the motif ' $\dots(01)\dots$ ', we have

$$\begin{aligned} \widehat{\chi}_{m,m,m+1}(z) &= (\chi_1 \chi_1 - \chi_2)(z) \chi_1(z), \\ \widehat{\chi}_{m,m,m+1}^{(0)}(z) &= (\chi_1^3 - \chi_2 \chi_1)(z), \quad \widehat{\chi}_{m,m,m+1}^{(1)}(z) = \widehat{\chi}_{\underbrace{m,m,m+1}}(z) = \chi_0(z), \\ \widehat{\chi}_{m,m,m+1}^Y(z) &= (\chi_1^3 - \chi_2 \chi_1 - \chi_0)(z), \end{aligned} \quad (4.17)$$

confirming that the $su(3)$ content of this yangian multiplet is a single **8**.

One may note that the leading term (with hook number $l = 0$) of (4.16) overlaps with states at lower spinon number. This redundancy is corrected for by subtractions at higher hook numbers. One quickly finds that the lowest N -spinon state in the CFT spectrum, for N of the form $N = p(n-1) + q$ with $q \in \{0, 1, \dots, n-2\}$, has Dynkin labels $m_1 = p$ and (for $q \neq 0$) $m_{n-q} = 1$ and energy

$$L_0^{(\min, N)} = \frac{1}{2} |pA_1 + A_{n-q}|^2 = \frac{N^2}{2n(n-1)} + \frac{q(n-1-q)}{2(n-1)}. \quad (4.18)$$

In Appendix A we illustrate the spinon decomposition of the WZW spectrum for the case $\widehat{SU}(3)_1$.

4.4. N -spinon characters

Having come this far, we are finally ready to evaluate the N -spinon contributions to the various characters of the $\widehat{SU}(n)_1$ CFT. We write the $\widehat{su}(n)_1$ characters as

$$\text{ch}_{\widehat{L}(\Lambda_{n-k})}^N(q, z) = \sum_{N \equiv k \bmod n} \text{ch}_{\text{WZW}}^N(q, z), \quad (4.19)$$

where $\text{ch}_{\text{WZW}}^N(q, z)$ denotes the total contribution of the N -spinon states to the chiral spectrum of the CFT. It can be written as

$$\text{ch}_{\text{WZW}}^N(q, z) = \sum_{0 \leq n_1 \leq n_2 \leq \dots \leq n_N} q^{-\frac{N(N-n)}{2n} + \sum_i n_i} \widehat{\chi}_{n_1, \dots, n_N}^Y(z). \quad (4.20)$$

Using (4.16), we can write this summation as an alternating sum over a hook number l . If we then evaluate the character at fixed l , we find the following beautiful result:

$$\begin{aligned} \text{ch}_{\text{WZW}}^{N,l}(q, z) &= \sum_{n_1 \leq n_2 \leq \dots \leq n_N} q^{-\frac{N(N-n)}{2n} + \sum_i n_i} \widehat{\chi}_{n_1, \dots, n_N}^{(l)}(z) \\ &= \frac{q^{\frac{1}{2}l(l-1)}}{(q)_l} \frac{1}{(q)_{N-nl}} \text{ch}_{\text{HS}}^{N-nl}(q, z), \end{aligned} \quad (4.21)$$

so that

$$\text{ch}_{\text{WZW}}^N(q, z) = \sum_{l \geq 0} (-1)^l \frac{q^{\frac{1}{2}l(l-1)}}{(q)_l} \frac{1}{(q)_{N-nl}} \text{ch}_{\text{HS}}^{N-nl}(q, z). \quad (4.22)$$

Here $\text{ch}_{\text{HS}}^{N-nl}(q, z)$ denotes the character of a *conjugate* HS model, with fundamental spins in the \bar{n} of $su(n)$, on $L = N - nl$ sites.

The result (4.21) can be explained as follows. Group theoretically, an N -spinon state with l hooks corresponds to a combination of $N - nl$ spinons that each transform in the \bar{n} of $su(n)$. This is the same kinematics as that of the conjugate HS model on $N - nl$ sites, and it can be shown that the systematics of how the fundamental spins combine is identical in both cases. The identity (4.21) hinges on the following relation between the yangian character $\overline{\chi}_{\{b_i\}}^Y(z)$ for a motif '0 $b_{N-1} \dots b_2 b_1 0$ ' (with $b_i = 0$ or 1) for the conjugate HS chain on N sites and the anti-symmetrized N -spinon character $\widehat{\chi}_{\{n_i\}}(z)$ (4.14). Using (3.1) we derive

$$\widehat{\chi}_{\{n_i\}}(z) = \sum_{b_1, b_2, \dots, b_{N-1}} \overline{\chi}_{\{b_i\}}^Y(z), \quad (4.23)$$

where $b_i = 0, 1$ if $n_i \neq n_{i+1}$ and $b_i = 1$ if $n_i = n_{i+1}$.

The q -factors in the expression for $\text{ch}_{\text{WZW}}^{N,l}(q, z)$ can be understood in the following way. First of all, the minimal sequence n_1, n_2, \dots, n_N that allows placing l hooks is obtained from the one for $l - 1$ hooks by incrementing $l - 1$ of the n_i by one, leading to a total added energy of $\frac{1}{2}l(l - 1)$ and a factor $q^{\frac{1}{2}l(l-1)}$. Secondly, the term with l hooks involves a factor

$$\frac{1}{(q)_l (q)_{N-nl}}, \quad (4.24)$$

which arises by taking into account a number of combinatorial factors. For $l = 0$ this factor equals $1/(q)_N$ and arises from summing over the $\{n_i\}$ with given ordering. For an example with non-zero l , consider the $l = 1$ term for $N = n$. It involves a factor

$$\frac{1}{(q)_n} (1 - q)(1 - q^2) \dots (1 - q^{n-1}), \quad (4.25)$$

expressing that the n hooked spinons can no longer propagate independently and a factor

$$1 + q + q^2 + \dots + q^{n-1} = \frac{1 - q^n}{(q)_1} \quad (4.26)$$

coming from the fact that there are n distinct possibilities for placing a single hook (see Appendix B). Together, the factors (4.25) and (4.26) produce the desired factor (4.24). For general N, l , the factors (4.24) can be established using induction arguments.

We have thus reduced the general expression for the N -spinon contribution to the CFT partition function to an expression in terms of the characters of a conjugate HS model and we can use the explicit results of Section 3.2 to simplify the formulas. Writing

$$\text{ch}_{\text{WZW}}^N(q, z) = \sum_{\lambda} z^{\lambda} c_{\lambda}^{A, N}(q), \quad (4.27)$$

where $\lambda = \lambda_{N \bmod n}$ and $\lambda \in P^{(N \bmod n)}$, we obtain from (3.13) the following result for the N -spinon cut of the $\widehat{su(n)}_1$ string functions $c_{\lambda}^A(q)$:

$$\begin{aligned} c_{\lambda}^{A, N}(q) &= q^{\frac{1}{2}|\lambda|^2} \sum_{l \geq 0} (-1)^l \frac{q^{\frac{1}{2}l(l-1)}}{(q)_l} \frac{1}{(q)_{N-nl}} M_{\lambda}^{N-nl, n}(q) \\ &= q^{\frac{1}{2}|\lambda|^2} \sum_{l \geq 0} (-1)^l \frac{q^{\frac{1}{2}l(l-1)}}{(q)_l} \frac{1}{(q)_{A_1-l} \cdots (q)_{A_n-l}}, \end{aligned} \quad (4.28)$$

where $A_i = \frac{N}{n} - (\lambda, \epsilon^i)$ (we used that the weights of the representation \bar{n} are $-\epsilon^i$). It follows that

$$\Phi_{\lambda}^{A, N}(q) = \sum_{l \geq 0} (-1)^l \frac{q^{\frac{1}{2}l(l-1)}}{(q)_l} \frac{q^{-\frac{(N-nl)(N-nl-n)}{2n}}}{(q)_{N-nl}} K_{(\{A_i-l\}), (1^{N-nl})}(q) \quad (4.29)$$

with Kostka polynomials $K_{\lambda, \mu}(q)$ as in (3.16). Finally

$$\text{ch}_{\text{WZW}}^N(q, z) = q^{-\frac{N^2}{2n}} \sum_{\sum_i j_i = N} \sum_{l \geq 0} (-1)^l \frac{q^{\frac{1}{2}l(l-1)}}{(q)_l} \frac{q^{\frac{1}{2} \sum_i j_i^2}}{\prod_i (q)_{j_i-l}} z^{-\sum_i j_i \epsilon^i}. \quad (4.30)$$

4.5. Discussion

The formula (4.28) allows for the following interpretation. The $l=0$ contribution to the string function $c_{\lambda}^{A, N}$, which starts at level $L_0 = \frac{1}{2}|\lambda|^2$ irrespective of N , describes N -spinon states modded out by the homogeneous part of the GCR. The inhomogeneous GCR are taken into account by the alternating sum over $l \geq 0$. Due to this structure, some of the features of the N -spinon spectrum are not manifest in (4.28); for example, the energy $L_0^{\min, N}$ (see (4.18)) of the lowest N -spinon states is hidden from the eye.

In order to get a more transparent formula, we have tried to trade the alternating sum (over l) for a sum (over a set l_i) without alternating signs. For this purpose we had to give up the manifest $su(n)$ symmetry and choose a preferred direction in the weight lattice P . We then obtained

$$\begin{aligned} c_{\lambda}^{A, N}(q) &= q^{\frac{1}{2}|\lambda|^2} \\ &\times \sum_{l_1, \dots, l_{n-2}} \frac{1}{\prod (q)_{l_i}} \frac{q^{A_n l_1 + (A_{n-1}-l_1)l_2 + \dots + (A_2-(l_1+\dots+l_{n-2}))(A_1-(l_1+\dots+l_{n-2}))}}{(q)_{A_n} (q)_{A_{n-1}-l_1} \cdots (q)_{A_2-(l_1+\dots+l_{n-2})} (q)_{A_1-(l_1+\dots+l_{n-2})}}. \end{aligned} \quad (4.31)$$

For $n = 2$ (4.31) reduces to [29]

$$c_{\lambda}^{A,N}(q) = \frac{q^{\frac{1}{4}N^2}}{(q)^{\frac{N}{2}+\frac{N}{2}}(q)^{\frac{N}{2}-\frac{N}{2}}} \quad (4.32)$$

for $\lambda = m\Lambda_1$. One clearly sees that the lowest N -spinon states have energy $\frac{1}{4}N^2$ and have $su(2)$ character $\chi_N(z)$ since

$$\chi_N(z) = \sum_{-N \leq m \leq N} z^{m\Lambda_1}. \quad (4.33)$$

The lowest N -spinon states are of the form (4.6) with $n_1 = 0, n_2 = 1, \dots, n_N = N - 1$.

For $n > 2$ the interpretation of (4.31) is similar but somewhat more involved. Putting $n = 3$ and choosing $\lambda = m_1\Lambda_1 + m_2\Lambda_2$, $\Lambda = \Lambda_{2m_1+m_2 \bmod 3}$, and $N = 3m_0 + 2m_1 + m_2$, we have

$$c_{\lambda}^{A,N}(q) = q^{\frac{N^2}{12}} \sum_{l \geq 0} \frac{1}{(q)_l} \frac{q^{\frac{1}{4}(m_2-m_0+2l)^2}}{(q)_{m_0+m_1+m_2}(q)_{m_0+m_1-l}(q)_{m_0-l}}. \quad (4.34)$$

Note that λ can also be written as $\lambda = -\sum_{i=1}^3 j_i \epsilon^i$ where $j_1 = m_0, j_2 = m_0 + m_1, j_3 = m_0 + m_1 + m_2$. Let us assume that N is even and look at the lowest level ($L_0 = \frac{N^2}{12}$) contributions to the spectrum. These arise from a sum over $l = 1, 2, \dots, \frac{N}{2}$. For given l , the weights λ that contribute are selected by $m_2 - m_0 + 2l = 0, m_0 + m_1 - l \geq 0$ and $m_0 - l \geq 0$. Together, these weights constitute the weights of the $su(3)$ representation with Dynkin labels $(\frac{N}{2}, 0)$, as expected. This last result is the case $m_2 = 0$ of the identity (we write $\chi_{m_1, m_2}(z)$ for the character of the irreducible $su(3)$ representation with Dynkin labels (m_1, m_2))

$$\chi_{m_1, m_2}(z) = \sum_{l=0}^{m_1} \sum_{l'=0}^{m_2} \sum_{\substack{j_1 = 2m_1 + m_2 \\ j_1 + j_2 - j_3 = -m_2 + 2(l+l') \\ j_1 \geq l, j_2 \geq l}} z^{-\sum_i j_i \epsilon^i}. \quad (4.35)$$

For $n = 3, N$ odd, this same identity with $m_2 = 1$ can be used to show that the leading $su(3)$ representation, at energy $\frac{N^2+3}{12}$, is indeed the one with Dynkin labels $(\frac{N-1}{2}, 1)$. For general n , a similar analysis is easily performed.

There is a good reason for the fact that the $n > 2$ character formulas are quite a bit more complicated than the ones for $n = 2$. That reason is the way the Generalized Pauli Principle works out for the lowest N -spinon excitations over the ground state. For $n = 2$ the Generalized Pauli Principle tells us to symmetrize over N ‘adjacent’ spinons, leading to the simple character formula (4.32). A naive generalization to $n > 2$ would have led to a lowest N -spinon state which is a symmetric combination of N adjacent spinons, but this is not the correct result. Instead, we have seen that the lowest state with, say, $N = p(n-1)$ spinons of $su(n)$, is obtained by (i) antisymmetrizing over groups of $n-1$ spinons with equal mode indices n_i , giving a representation \mathbf{n} for each group, and (ii) symmetrizing over p adjacent groups, giving a representation with Dynkin

labels $(p\ 0\ \dots)$. The corresponding character formulas can be viewed as affinizations of identities of the type (4.35).

5. Alternative formulations

5.1. More spinons

It will be clear that in our analysis so far a very special role was played by products

$$\dots (\phi_{x_{i+k}-n}^{\alpha_{i+k}} \dots \phi_{x_{i+1}-n}^{\alpha_{i+1}}) \dots \quad (5.1)$$

($x_j = -\frac{n-(2j-1)}{2n}$) of k consecutive spinon modes with equal index n . The Generalized Pauli Principle of Section 4.3 tells us that in this situation the $su(n)$ indices $\alpha_{i+1}, \dots, \alpha_{i+k}$ should be anti-symmetrized, so that the ‘block’ of k bound spinons carries $su(n)$ representation $L(\Lambda_{n-k})$. It is then natural to consider a formulation that has independent spinon fields $\phi^{(j)}$, $j = 1, 2, \dots, n-1$, transforming in $L(\Lambda_j)$ of $su(n)$. Of course, these spinons simply correspond to the $n-1$ independent primary fields of the $\widehat{SU(n)}_1$ WZW theory. (Such a description was also suggested in [30] by analysis of the $SU(n)$ Heisenberg chain in the infinite chain limit.)

We can thus take a point of view where the chiral Hilbert space of a WZW theory is built by acting with the modes of a set of $n-1$ independent spinon fields, modded out by Generalized Commutation Relations and/or null fields. In this section we focus on $su(3)$ to illustrate such an approach. The $su(n)$ case will be discussed in Ref. [8].

In Section 5.2 we shall reformulate the approach based on the yangian $Y(sl_3)$ in terms of spinons $\phi^{(1)}$ and $\phi^{(2)}$ with mode indices that are related to parameters $m_1^1, \dots, m_{M_1}^1$ and $m_1^2, \dots, m_{M_2}^2$, which are the zero’s of Drinfel’d polynomials $P_1(u)$ and $P_2(u)$.

Clearly, $\phi^{(1)}$ and $\phi^{(2)}$, which transform in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $su(3)$, respectively, are related by a discrete symmetry (the conjugation symmetry of $su(3)$). However, the yangian generators do not respect this symmetry and it follows that in the yangian formulation the spinons $\phi^{(1)}$ and $\phi^{(2)}$ are not on equal footing. In Section 5.3 we present alternative character formulas that do respect the symmetry between $\phi^{(1)}$ and $\phi^{(2)}$, meaning that conjugate pairs of physical states are labeled by conjugate sets of parameters. Depending on the application one has in mind, one may prefer either the ‘yangian formulation’ or the ‘symmetric formulation’.

5.2. Yangian irreducible representations, III

In Section 4 we characterized a motif-related yangian representation in the spectrum of the $\widehat{SU(n)}_1$ CFT by a set of mode indices $\{n_i\}$. Following our observation in Section 5.1, we may trade pairs of equal indices, $n_i = n_{i+1}$ for parameters m_j^1 and single n_i for parameters m_j^2 . We then obtain a parametrization of motifs in terms of parameters $\{m_j^1\}$, $\{m_j^2\}$, which have the interpretation of zero’s of Drinfel’d polynomials $P_1(u)$ and $P_2(u)$.

The m_j^1 are integer while the m_j^2 are half-odd-integer. Placing them in increasing order, they should satisfy the following. The smallest is of the form $m_1^1 = 1 + 3n$ or $m_1^2 = \frac{3}{2} + 3n$ with n a non-negative integer. The increment between adjacent m_j^1 is of the form $1 + 3n$, between adjacent m_j^1 and m_j^2 it is $\frac{3}{2} + 3n$, and between adjacent m_j^2 it is $2 + 3n$.

For given $\{m_j^1\}$, $\{m_j^2\}$, the corresponding motif is reconstructed as follows. Every m_j^1 is replaced by ‘()’, every m_j^2 by ‘(1)’. If two m_j^i are separated by the minimal increment plus $3n$ we place n times ‘(11)’ between the corresponding motifs, and we place ‘(11)(11) ...’ to the right of the largest m_j^i . Finally, we replace the initial ‘(’ and all ‘)’ by ‘0’. This connection was first explained in [21].

Translating the yangian character formula (4.16) to this new formulation, we find the following. We first define a character

$$\widehat{\chi}_{\{m_j^1\}, \{m_j^2\}}(z), \quad (5.2)$$

which is obtained by writing a factor $\chi_{l,0}(z)$ for a group of l adjacent m_j^1 with minimal increment 1 and a factor $\chi_{0,1}(z)$ for every m_j^2 . If the increment between adjacent m_j^1 and m_j^2 is minimal, i.e. equal to $\frac{3}{2}$, we allow a contraction, meaning that we replace a product $\chi_{0,1}(z)\chi_{l,0}(z)$ (or $\chi_{l,0}(z)\chi_{0,1}(z)$) by the trace $\chi_{l-1,0}(z)$. The yangian character is then written as

$$\chi_{\{m_j^1\}, \{m_j^2\}}^Y(z) = \sum_{l \geq 0} (-1)^l \widehat{\chi}_{\{m_j^1\}, \{m_j^2\}}^{(l)}(z), \quad (5.3)$$

where $\widehat{\chi}_{\{m_j^1\}, \{m_j^2\}}^{(l)}(z)$ is the character (5.2) with l contractions. Let us stress that the character formula (5.3) is only applicable to those Drinfel’d polynomials that occur in the yangian decomposition of the HS and WZW modules and does not describe the most general irreducible representations of $Y(sl_3)$.

As an example of (5.3), consider $m_1^1 = 1$, $m_1^2 = \frac{5}{2}$, which corresponds to the motif ‘001011011 ...’. We have

$$\chi_{\{1\}, \{\frac{5}{2}\}}^Y(z) = \chi_{1,0}(z)\chi_{0,1}(z) - \chi_{0,0}(z) \quad (5.4)$$

in agreement with (4.17). Note that a single character $\chi_{l,0}$ may be contracted twice, as in

$$\chi_{\{3,4\}, \{\frac{3}{2}, \frac{11}{2}\}}^Y(z) = (\chi_{0,1}\chi_{2,0}\chi_{0,1} - \chi_{1,0}\chi_{0,1} - \chi_{0,1}\chi_{1,0} + \chi_{0,0})(z), \quad (5.5)$$

which is the character for the motif ‘010001011011 ...’.

The energy of the states labeled by $\{m_j^1\}$, $\{m_j^2\}$ comes out as

$$L_0 = -\frac{2M_1 + M_2}{6} + \frac{1}{3} \sum_j (2m_j^1 + m_j^2), \quad (5.6)$$

and we can write the following character formula for the states with ‘spinon numbers’ M_1 and M_2

$$\widehat{\text{ch}}_{\text{WZW}}^{M_1, M_2}(q, z) = \sum_{\substack{m_1^1, \dots, m_{M_1}^1 \\ m_1^2, \dots, m_{M_2}^2}} q^{-\frac{2M_1+M_2}{6} + \frac{1}{3} \sum_j (2m_j^1 + m_j^2)} \chi_{\{m_j^1\}, \{m_j^2\}}^Y(z). \quad (5.7)$$

By construction, this character has the property

$$\text{ch}_{\text{WZW}}^N(q, z) = \sum_{2M_1+M_2=N} \widehat{\text{ch}}_{\text{WZW}}^{M_1, M_2}(q, z). \quad (5.8)$$

We would like to repeat once again that the character (5.7) is based on a decomposition of the spectrum in terms of yangian multiplets, and does thereby not treat the conjugate spinons $\phi^{(1)}$ and $\phi^{(2)}$ on equal footing.

5.3. More character formulas

We already mentioned that it is natural to look for a formulation which respects the conjugation symmetry between the fields $\phi^{(1)}$ and $\phi^{(2)}$. Note that this means that we give up the connection with the yangian symmetry and are no longer working with eigenstates of operators such as H_2 (in Ref. [27] this point was not fully appreciated).

For $\widehat{su(3)}_1$, we have found a character formula with the expected structure. To write it, we first define a character $\text{ch}_{\text{F}}^{N_1, N_2}(q, z)$, which is to be viewed as the character of a ‘Fock space’ of a total of six (i.e. the components of $\phi^{(1)}$ and $\phi^{(2)}$) sets of spinon modes

$$\text{ch}_{\text{F}}^{N_1, N_2}(q, z) = \sum_{\sum_i j_i^1 = N_1, \sum_i j_i^2 = N_2} \frac{q^{\frac{1}{3}(N_1^2 + N_2^2 + N_1 N_2)}}{\prod_i (q)_{j_i^1} (q)_{j_i^2}} z^{\sum_i (j_i^1 - j_i^2) \epsilon^i}. \quad (5.9)$$

This character is an affinization of

$$\widehat{\chi}_{m_1, m_2}(z) = \sum_{\sum_i j_i^1 = m_1, \sum_i j_i^2 = m_2} z^{\sum_i (j_i^1 - j_i^2) \epsilon^i}, \quad (5.10)$$

which is related to the irreducible $su(3)$ character via

$$\chi_{m_1, m_2}(z) = \widehat{\chi}_{m_1, m_2}(z) - \widehat{\chi}_{m_1-1, m_2-1}(z). \quad (5.11)$$

The irreducible affine characters should be affinizations of the irreducible $su(3)$ characters and we therefore expect an alternating sum to make its appearance. In close analogy with (4.22) we define

$$\text{ch}_{\text{WZW}}^{N_1, N_2}(q, z) = \sum_{l \geq 0} (-1)^l \frac{q^{\frac{1}{3}l(l-1)}}{(q)_l} q^{l(N_1 + N_2 - l)} \text{ch}_{\text{F}}^{N_1-l, N_2-l}(q, z). \quad (5.12)$$

It can be proved that

$$\text{ch}_{\text{WZW}}^N(q, z) = \sum_{2N_1+N_2=N} \text{ch}_{\text{WZW}}^{N_1, N_2}(q, z) \quad (5.13)$$

so that

$$\widehat{\text{ch}}_{L(\Lambda_{3-k})}(q, z) = \sum_{2N_1+N_2 \equiv k \pmod{3}} \text{ch}_{\text{WZW}}^{N_1, N_2}(q, z). \quad (5.14)$$

We have thus succeeded in breaking down N -spinon characters into (N_1, N_2) -spinon characters with manifest conjugation symmetry.

In a follow-up paper [8], we shall extend the discussion of this section to general $su(n)$ and further investigate character formulas such as (5.12).

6. Conclusions

The results presented in this paper should be viewed as a first step towards a full-fledged multi-spinon formulation of these (and other) CFT's. As a next step, one would like to construct explicit multi-spinon bases for the Hilbert space, and deal with correlation functions and form factors. For the case $\widehat{SU}(2)_1$, part of such a program has already been completed [3–5].

At the mathematical level, there are several interesting aspects to the results presented in this paper. We have given a number of equivalent formulae for the characters of a large class of finite-dimensional irreducible representations of the yangian $Y(sl_n)$. Thus far, these characters were not known. Expressing them in terms of Schur polynomials we find a new interesting class of symmetric functions (see Ref. [8]). We believe that a proper understanding of the origin of these symmetric functions will ultimately lead to an understanding of all $Y(sl_n)$ irreducible finite-dimensional representations.

We have also seen an intricate connection between $\widehat{SU}(n)_1$ CFT's and the Haldane-Shastry model. On the one hand, the $\widehat{SU}(n)_1$ theory can be obtained as the $L \rightarrow \infty$ limit of an L -site HS model, while on the other, the N -spinon cuts of the affine characters can be expressed as an alternating sum over characters of the $(N - nl)$ -site ($l \geq 0$) conjugate HS model. This 'duality' deserves to be better understood.

Apart from the issues that remain for the $\widehat{SU}(n)_k$ WZW models, it is interesting to consider other groups. Of particular interest are the $\widehat{SO}(n)_1$ WZW models, which are related to n Majorana fermions via non-abelian bosonization and are strongly expected to show up in condensed matter systems. The challenge here is to give a formulation in terms of fundamental quasi-particles that are spinors (of conformal dimension $n/16$) of $so(n)$. The case $\widehat{so}(3)_1$ is formally the same as $\widehat{su}(2)_2$ and is thus covered by the analysis of Ref. [9]. Similarly, $\widehat{so}(6)_1$ is the same as $\widehat{su}(4)_1$, which was treated in this paper. Full results for general n will be given elsewhere.

There are other examples (other than WZW models) of CFT's that describe quasi-particles of fractional statistics (in a sense, any rational CFT has an interpretation of this type). Perhaps the simplest among these are $c = 1$ theories at radius $R^2 = p/q$. For $p = 2$, $q = 1$ this is nothing else than the $\widehat{SU}(2)_1$ WZW theory but for more general p, q these theories do not have Lie algebra symmetries. Choosing $p = 2m + 1$, $q = 1$ gives a theory describing one-component edge excitations of a fractional quantum Hall (FQH) sample of filling fraction $1/(2m + 1)$ (they are the case $n = 1$ of the n -

component edge theories mentioned in the introduction). A ‘multi-spinon formulation’ [31,32] of these CFT’s directly corresponds to a description of the FQH edge dynamics in terms of the fundamental edge degrees of freedom of charge $e/(2m+1)$ and statistics $\theta = \pi/(2m+1)$.

Note added

After submission of our paper we learned that the Haldane–Shastry characters and their yangian decomposition have also been discussed in [33]. From [34] we also learned that the yangian modules appearing in our work are the so-called tame modules investigated by [35].

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Appendix A. Illustration: $\widehat{SU(3)}_1$

We illustrate the spinon structure of the $\widehat{SU(3)}_1$ CFT in Table A.1.

Appendix B. Hook rules

We present the rules for placing hooks in the construction of Section 4.3 of the character (4.16) of an irreducible representation of $Y(sl_n)$. Starting point is a mode sequence n_1, n_2, \dots, n_N . The rules for placing a single hook are

(i) A hook

$$\boxed{n_i \dots n_{i+n-1}} \quad (\text{B.1})$$

may be placed in the following situations:

$$\begin{aligned} n_i &= n_{i+1} = \dots = n_{i+n-1}, \\ n_i &= \dots = n_{i+k-1} \stackrel{1}{=} n_{i+k} = \dots = n_{i+n-1}, \quad k = 1, \dots, n-1. \end{aligned} \quad (\text{B.2})$$

Table A.1

Spinon structure of the $L_0 \leq 3$ states in the spectrum of the $\widehat{SU(3)}_1$ WZW model. The eigenvalues of H_2 are as in (4.12) and the $su(3)$ structure is as explained in Section 4.3. Note the asymmetry in the description of conjugate states in the spectrum

L_0	H_2	N	$\{n_i\}$	Motif	$SU(3)$ content
0	0	0		011011011...	1
$\frac{1}{3}$	$\frac{1}{9}$	1	0	01011011...	$\bar{3}$
$\frac{1}{3}$	$-\frac{1}{9}$	2	0,0	0011011011...	3
1	1	3	0,0,1	001011011...	8
$\frac{4}{3}$	$\frac{46}{9}$	1	1	01101011...	$\bar{3}$
$\frac{4}{3}$	$\frac{10}{9}$	4	0,0,1,1	00011011...	6
$\frac{4}{3}$	$\frac{26}{9}$	2	0,1	0101011011...	3 + $\bar{6}$
2	4	3	0,1,1	010011011...	8
2	8	3	0,0,2	001101011...	1 + 8
$\frac{7}{3}$	$\frac{155}{9}$	1	2	01101101011...	$\bar{3}$
$\frac{7}{3}$	$\frac{55}{9}$	4	0,0,1,2	00101011011...	$\bar{3}$ + 6 + $\bar{15}$
$\frac{7}{3}$	$\frac{71}{9}$	2	1,1	0110011011...	3
$\frac{7}{3}$	$\frac{107}{9}$	2	0,2	0101101011...	3 + $\bar{6}$
$\frac{7}{3}$	$\frac{35}{9}$	5	0,0,1,1,2	0001011011...	15
3	21	3	0,0,3	001101101011...	1 + 8
3	11	3	0,1,2	010101011011...	1 + 8 ₂ + $\bar{10}$
3	5	6	0,0,1,1,2,2	000011011...	10

(ii) The hook

$$n_{i-1} \underbrace{n_i \dots n_{i+n-1}} \quad (\text{B.3})$$

is allowed if $\underbrace{n_i \dots n_{i+n-1}}$ is allowed and $n_{i-1} < n_i$.

(iii) The hook

$$\underbrace{n_i \dots n_{i+n-1}} n_{i+n} \quad (\text{B.4})$$

is allowed if $\underbrace{n_i \dots n_{i+n-1}}$ is allowed and $n_{i+n-1} < n_{i+n}$, with the following exceptions. The hook

$$\underbrace{n_i \dots n_{i+n-1}} n_{i+n} \dots n_{i+2n-1} \quad (\text{B.5})$$

is *not* allowed if $n_i = \dots = n_{i+n-1} \stackrel{1}{=} n_{i+n} = \dots = n_{i+2n-1}$, while (for $n > 2$) the hook

$$\underbrace{n_i \dots n_{i+n-1}} n_{i+n} \dots \quad (\text{B.6})$$

is allowed if $n_{i+n-1} = n_{i+n}$ and the total number of n_j equal to n_{i+n} is precisely $n - 1$.

To determine if additional hooks can be placed *to the right of* existing hooks, one removes all hooked n_i from the sequence and then applies the above rules to the remaining n_i .

Appendix C. Decomposition of motifs

In this appendix we clarify further the $su(n)$ content of irreducible yangian representations. Let us consider a motif given by a sequence of '0' and '1'.

In Ref. [19] it was proposed that the $su(n)$ content of a motif can be obtained as a (free) tensor product of a number of component motifs. For $su(3)$ this decomposition is obtained by (i) replacing the initial and final '0' by '(' and ')', respectively, (ii) replacing all '0110' by ')(11)(' and (iii) replacing all '101' by 1)(1'. For example

$$010110 \rightarrow (1)(11), \quad (C.1)$$

giving $su(3)$ content $\bar{3}$.

The same motif for $su(5)$ has content $75 \oplus 24$, which is different from the product $10 \otimes \bar{10}$ of the motifs (1) and (11) (hence the name 'incomplete multiplets' in Ref. [15]). This clearly indicates that the above rule needs to be changed for general n . The correct rule that replaces (ii) and (iii) is to replace all '0' that are adjacent (on either side) to a total of at least $n - 1$ '1' by ')(' (This is most easily proved by using the description of Section 4.3.) According to this rule, the motif '010110' is indecomposable for $n \geq 5$.

Note that our prescriptions in Sections 3.1, 4.3 and 5.2 do not make use of this decomposition rule.

References

- [1] F.D.M. Haldane, Phys. Rev. Lett. 66 (1991) 1529.
- [2] P. Fendley, A.W.W. Ludwig and H. Saleur, Phys. Rev. Lett. 74 (1995) 3005 (cond-mat/9408068).
- [3] D. Bernard, V. Pasquier and D. Serban, Nucl. Phys. B 428 (1994) 612 (hep-th/9404050).
- [4] P. Bouwknegt, A.W.W. Ludwig and K. Schoutens, Phys. Lett. B 338 (1994) 448 (hep-th/9406020).
- [5] P. Bouwknegt, A.W.W. Ludwig and K. Schoutens, in Proc. of the 1994 Trieste Summer School on High Energy Physics and Cosmology, Trieste, July 1994 ((hep-th/949412199).
- [6] B.L. Feigin and A.V. Stoyanovsky, Quasi-particle models for the representations of Lie algebras and geometry of flag manifolds (hep-th/9308079).
- [7] G. Georgiev, Combinatorial constructions of modules for infinite-dimensional Lie algebras, I. Principal subspace (hep-th/9412054); II. Parafermionic space (q-alg/9504024).
- [8] P. Bouwknegt and K. Schoutens, in preparation.
- [9] P. Bouwknegt, A.W.W. Ludwig and K. Schoutens, Phys. Lett. B 359 (1995) 304 (hep-th/9412108).
- [10] A. Nakayashiki and Y. Yamada, Comm. Math. Phys. 178 (1996) 179 (hep-th/9504052).
- [11] T. Arakawa, T. Nakanishi, K. Oshima and A. Tsuchiya, Spectral decomposition of path space in solvable lattice model (q-alg/9507025).
- [12] V.G. Kac, Infinite dimensional Lie algebras (Cambridge University Press, Cambridge, 1985).
- [13] V.G. Drinfel'd, Sov. Math. Dokl. 32 (1985) 254; in Proceedings of the International Congress of Mathematicians, Berkeley, California (1986).
- [14] V. Chari and A. Pressley, A guide to Quantum Groups (Cambridge University Press, Cambridge, 1994).
- [15] K. Schoutens, Phys. Lett. B 331 (1994) 335 (hep-th/9401154).

- [16] P. Bouwknegt and K. Schoutens, *Phys. Rep.* 223 (1993) 183 (hep-th/9210010).
- [17] F.D.M. Haldane, *Phys. Rev. Lett.* 60 (1988) 635.
- [18] B. Sriram Shastry, *Phys. Rev. Lett.* 60 (1988) 639.
- [19] F.D.M. Haldane, Z.N.C. Ha, J.C. Talstra, D. Bernard and V. Pasquier, *Phys. Rev. Lett.* 69 (1992) 2021.
- [20] J.C. Talstra and F.D.M. Haldane, Integrals of motion of the Haldane Shastry model (cond-mat/9411065).
- [21] D. Bernard, M. Gaudin, F.D.M. Haldane and V. Pasquier, *J. Phys. A* 26 (1993) 5219 (hep-th/9301084).
- [22] F.D.M. Haldane and Z.N.C. Ha, *Phys. Rev. B* 47 (1993) 12459.
- [23] V. Chari and A. Pressley, Yangians: their representations and characters (q-alg/9508004).
- [24] E. Melzer, *Lett. Math. Phys.* 31 (1994) 233 (hep-th/9312043).
- [25] A.N. Kirillov, Dilogarithm identities (hep-th/9408113).
- [26] A. Nakayashiki and Y. Yamada, Kostka polynomials and energy functions in solvable lattice models (q-alg/9512027).
- [27] C. Ahn and S. Nam, Yangian Symmetries in the $SU(N)_1$ WZW Model and the Calogero-Sutherland Model, preprint SNUTP/95-113 (hep-th/9510242).
- [28] F.D.M. Haldane, *Phys. Rev. Lett.* 67 (1991) 937.
- [29] R. Kedem, B. McCoy and E. Melzer, in *Proc. of the conference SMQFT, USC 1994*, ed. P. Bouwknegt et al. (World Scientific, Singapore, 1995) (hep-th/9304056).
- [30] N. Reshetikhin, *J. Phys. A* 24 (1991) 3299.
- [31] S. Iso, *Nucl. Phys. B* 443 (1995) 581 (hep-th/9411051).
- [32] K. Schoutens, unpublished.
- [33] K. Hikami, *Nucl. Phys. B* 441 (1995) 530.
- [34] A.N. Kirillov, A. Kuniba and T. Nakanishi, Skew Young diagram method in spectral decomposition of integrable lattice models (q-alg/9607027).
- [35] M. Nazarov and V. Tarasov, Representations of Yangians with Gelfan'd-Zetlin basis (q-alg/9502008).