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On superconformal characters and partition functions in three dimensions

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Possible short and semishort positive energy, unitary representations of the $Osp(2N|4)$ superconformal group in three dimensions are discussed. Corresponding character formulas are obtained, consistent with character formulas for the SO(3,2) conformal group, revealing long multiplet decomposition at unitarity bounds in a simple way. Limits, corresponding to reduction to various $Osp(2N|4)$ subalgebras, are taken in the characters that isolate contributions from fewer states, at a given unitarity threshold, leading to considerably simpler formula. Via these limits, applied to partition functions, closed formula for the generating functions for numbers of BPS operators in the free field limit of superconformal $U(n) \times U(n)$ $\mathcal{N}=6$ Chern–Simons theory and its supergravity dual are obtained in the large n limit. Partial counting of semishort operators is performed and consistency between operator counting for the free field and supergravity limits with long multiplet decomposition rules is explicitly demonstrated. Partition functions counting certain protected scalar primary semishort operators, and their superconformal descendants, are proposed and computed for large n. Certain chiral ring partition functions are discussed from a combinatorial perspective. © *2010 American Institute of Physics.* [doi:10.1063/1.3211091]

I. INTRODUCTION

With the resurgence of interest recently in three dimensional superconformal field theory, due largely to the discovery by Bagger and Lambert[1] of a new superconformal $\mathcal{N}=8$ Chern–Simons theory and, more recently, by Aharony *et al.*[2] of a superconformal $\mathcal{N}=6$ Chern–Simons theory, with $U(n) \times U(n)$ gauge symmetry, much attention has been devoted to uncovering dualities,[3,2] investigating integrability,[4,5] spectra,[6–8] etc. One issue that has hitherto perhaps not been explored in very much detail is the representation theory, underlying these theories, for positive energy, unitary representations of $Osp(\mathcal{N}|4)$. This paper is an attempt to address in some detail this issue and focuses on the case of even $\mathcal{N}$.[9]

For $\mathcal{N}=4$ superconformal symmetry in four dimensions such a detailed study of the representation theory[11] proved fruitful in many ways, e.g., investigating the operator product expansion, in terms of conformal partial wave expansions of four point functions, see Ref. 12, for example, in partially determining the operator spectrum of $\mathcal{N}=4$ super Yang Mills. In this context, superconformal characters provide for an alternative and arguably more straightforward and powerful way of organizing the sometimes detailed multiplet structure that arises. They easily lead to the rules for long multiplet decomposition into short/semishort multiplets at unitarity bounds, formulas for decompositions in terms of subalgebra representations, and indicate how partition functions may be computed for decoupled sectors.

The outline of this paper is as follows. In Sec. II the superconformal algebra for $Osp(2N|4)$, i.e., for $\mathcal{N}=2N$, is given in detail. Mainly for the purposes of discussing shortening conditions,

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constructing Verma module characters and finding subalgebras, it proves easiest to use the ortho-normal basis for the SO(2N) R-symmetry algebra. Section II gives a brief account of the possible shortening conditions for semishort representations, denoted by (N,A,n), $n=1,\ldots,N-1$, and $(N,A,\pm)$, and short or BPS representations, denoted by (N,B,n) and $(N,B,\pm)$, as well as for the conserved current multiplet, denoted $(N,\text{cons.})$. Here, highest weight states in the (N,A,n), re-spectively, $(N,A,\pm)$, (N,B,n), $(N,B,\pm)$, representations are annihilated by $n/4N$, respectively 1/4, $n/2N$, 1/2 of the supercharges. [Hence, the $(N,B,\pm)$ representations are referred to here as half BPS. For $N=4$, it is well known that there are two types of half BPS representation, (see Refs. 13 and 14 for a discussion.)

In Sec. IV the characters for positive energy, unitary representations of the Osp(2N|4) super-conformal group are obtained through use of Verma module characters and the Weyl symmetrizer for the maximal compact subgroup, U(1)$\otimes$SU(2)$\otimes$SO(2N). BPS and conserved conserved cur-rent multiplet characters are decomposed in terms of SO(3,2)$\otimes$SO(2N) characters. The decom-position rules for long multiplets at the unitarity bounds are found. In particular, they imply that in any $\mathcal{N}=2N$ superconformal field theory in three dimensions, operators in all (N,B,n), $n>1$, as well as in $(N,B,\pm)$ and certain $(N,B,1)$, BPS multiplets must have protected conformal dimen-sions.

In Sec. V, various limits are taken in the characters that lead to nonvanishing formulas only for subsets of the characters corresponding to different short/semishort multiplets. These are shown, from Appendix A, to match with various Osp(2N|4) subalgebra characters and, in terms of partition functions, isolate different sectors of operators.[15] For the subset being $\{(N,B,m),m\geq n,(N,B,+),(N,B,-)\}$ the subalgebra is U(1)$\otimes$SO(2$N-2m$) while for the subsets $\{(N,B,\pm)\}$ there are two U(1) subalgebras. Similarly, for the subset being $\{(N,A,m),\ m\geq n,\ (N,A,+),\ (N,A,-),\ (N,B,l),\ 1\leq l\leq N,\ (N,B,+),\ (N,B,-),\ (N,\text{cons.})\}$, where (N,B,l), $l<m$, multiplets have particular SO(2N) R-symmetry eigenvalues, the subalgebra is U(1)$\otimes$Osp(2$N-2m$|2). Fi-nally, for the subsets $\{(N,A,\pm),\ (N,B,l),\ 1\leq l\leq N,\ (N,B,+),\ (N,B,-),\ (N,\text{cons.})\}$, where (N,B,l), $l<m$, $(N,B,\mp)$ multiplets have particular SO(2N) R-symmetry eigenvalues, there are two U(1)$\otimes$SU(1,1) subalgebras. The limits are shown to be consistent with long multiplet de-composition rules and also with the index of Ref. 6.

In Sec. VI, the general free field multiparticle partition function for $\mathcal{N}=6$ superconformal Chern–Simons theory is written in detail in terms of superconformal characters and, for large n, computed using symmetric polynomial techniques. Also, the corresponding partition function is given in detail for the supergravity limit.

In Sec. VII, operator counting for short and certain semishort multiplets is investigated using the limits taken in Sec. V, and other symmetric polynomial techniques, described in Appendix B, and shown to be consistent with expectations from long multiplet decomposition rules.

In the conclusion, partition functions counting certain protected semishort operators are dis-cussed. Some related comments are made about $\mathcal{N}=4$ super Yang Mills, explained in more detail in Appendix C, where a certain class of chiral ring partition functions are computed using the Polya enumeration theorem.

II. THE SUPERCONFORMAL ALGEBRA IN THREE DIMENSIONS

In d dimensions, the standard nonzero commutators of the conformal group SO(d,2) are given by, for $\eta_{ab}=\text{diag.}(-1,1,\ldots,1)$, $a,b=0,1,\ldots,d-1$,

$$[M_{ab},P_c]=i(\eta_{ac}P_b-\eta_{bc}P_a), \quad [M_{ab},K_c]=i(\eta_{ac}K_b-\eta_{bc}K_a),$$

$$[M_{ab},M_{cd}]=i(\eta_{ac}M_{bd}-\eta_{bc}M_{ad}-\eta_{ad}M_{bc}+\eta_{bd}M_{ac}),$$

$$[D,P_a]=P_a, \quad [D,K_a]=-K_a, \quad [K_a,P_b]=-2iM_{ab}+2\eta_{ab}D, \tag{2.1}$$

where the generators of translations are P_a, those of special conformal transformations are K_a, those of SO($d-1$,1) are $M_{ab}=-M_{ba}$, while that of scale transformations is D.

For later application it is convenient to write the $Sp(4,\mathbb{R})/\mathbb{Z}_2 \simeq SO(3,2)$ algebra for three dimensions in the spinor basis so that for[16]

$$P_{\alpha\beta} = (\gamma^a)_{\alpha\beta} P_a, \quad K^{\alpha\beta} = (\bar{\gamma}^a)^{\alpha\beta} K_a, \quad M_\alpha{}^\beta = \frac{i}{2}(\gamma^a \bar{\gamma}^b)_\alpha{}^\beta M_{ab}, \tag{2.2}$$

the algebra (2.1) becomes

$$[M_\alpha{}^\beta, P_{\gamma\delta}] = \delta_\gamma{}^\beta P_{\alpha\delta} + \delta_\delta{}^\beta P_{\alpha\gamma} - \delta_\alpha{}^\beta P_{\gamma\delta},$$

$$[M_\alpha{}^\beta, K^{\gamma\delta}] = -\delta_\alpha{}^\gamma K^{\beta\delta} - \delta_\alpha{}^\delta K^{\beta\gamma} + \delta_\alpha{}^\beta K^{\gamma\delta},$$

$$[M_\alpha{}^\beta, M_\gamma{}^\delta] = -\delta_\alpha{}^\delta M_\gamma{}^\beta + \delta_\gamma{}^\beta M_\alpha{}^\delta, \quad [D, P_{\alpha\beta}] = P_{\alpha\beta}, \quad [D, K^{\alpha\beta}] = -K^{\alpha\beta},$$

$$[K^{\alpha\beta}, P_{\gamma\delta}] = 4\delta_{(\gamma}{}^{(\alpha} M_{\delta)}{}^{\beta)} + 4\delta_{(\gamma}{}^\alpha \delta_{\delta)}{}^\beta D. \tag{2.3}$$

For supercharges and their superconformal extensions the nonzero (anti-)commutators are given by,

$$\{Q_{r\alpha}, Q_{s\beta}\} = 2\delta_{rs} P_{\alpha\beta}, \quad \{S_r{}^\alpha, S_s{}^\beta\} = 2\delta_{rs} K^{\alpha\beta},$$

$$[K^{\alpha\beta}, Q_{r\gamma}] = i(\delta_\gamma{}^\alpha S_r{}^\beta + \delta_\gamma{}^\beta S_r{}^\alpha), \quad [P_{\alpha\beta}, S_r{}^\gamma] = -i(\delta_\alpha{}^\gamma Q_{r\beta} + \delta_\beta{}^\gamma Q_{r\alpha}),$$

$$[M_\alpha{}^\beta, Q_{r\gamma}] = \delta_\gamma{}^\beta Q_{r\alpha} - \tfrac{1}{2}\delta_\alpha{}^\beta Q_{r\gamma}, \quad [M_\alpha{}^\beta, S_r{}^\gamma] = -\delta_\alpha{}^\gamma S_r{}^\beta + \tfrac{1}{2}\delta_\alpha{}^\beta S_r{}^\gamma,$$

$$[D, Q_{r\alpha}] = \tfrac{1}{2}Q_{r\alpha}, \quad [D, S_r{}^\alpha] = -\tfrac{1}{2}S_r{}^\alpha,$$

$$[R_{rs}, Q_{t\alpha}] = i(\delta_{rt} Q_{s\alpha} - \delta_{st} Q_{r\alpha}), \quad [R_{rs}, S_t{}^\alpha] = i(\delta_{rt} S_s{}^\alpha - \delta_{st} S_r{}^\alpha), \tag{2.4}$$

along with

$$\{Q_{r\alpha}, S_s{}^\beta\} = 2i(M_\alpha{}^\beta \delta_{rs} - i\delta_\alpha{}^\beta R_{rs} + \delta_\alpha{}^\beta \delta_{rs} D), \tag{2.5}$$

where the $SO(n)$ R-symmetry generators $R_{rs} = -R_{sr} = R_{rs}{}^\dagger$ satisfy

$$[R_{rs}, R_{tu}] = i(\delta_{rt} R_{su} - \delta_{st} R_{ru} - \delta_{ru} R_{st} + \delta_{su} R_{rt}). \tag{2.6}$$

In order to discuss highest weight states and shortening conditions, (2.6) is now rewritten in terms of generators in the orthonormal basis of $SO(2N)$, which has rank N. In this basis, the Cartan subalgebra H_n, and raising/lowering operators $E_{mn}^{+\pm}/E_{mn}^{-\pm}$, $m < n$, are given by, for $m,n = 1, \ldots, N$,[17]

$$H_n = R_{2n-1\ 2n}, \quad E_{mn}^{+\pm} = R_{2m-1\ 2n-1} + iR_{2m\ 2n-1} \pm iR_{2m-1\ 2n} \mp R_{2m\ 2n},$$

$$E_{mn}^{-\pm} = E_{mn}^{+\mp\dagger}, \quad E_{mn}^{-\pm} = -E_{nm}^{\pm-}, \quad E_{mn}^{+\pm} = -E_{nm}^{\pm+}, \tag{2.7}$$

so that the nonzero commutators from (2.6) are, for $l = 1, \ldots, N$,

$$[H_l, E_{mn}^{+\pm}] = (\delta_{lm} \pm \delta_{ln})E_{mn}^{+\pm}, \quad [H_l, E_{mn}^{-\pm}] = (-\delta_{lm} \pm \delta_{ln})E_{mn}^{-\pm},$$

$$[E_{mn}^{+\pm}, E_{mn}^{-\mp}] = 4(H_m \pm H_n),$$

$$[E_{lm}^{+\pm}, E_{ln}^{-\pm}] = 2iE_{mn}^{\pm\pm}, \quad [E_{lm}^{+\pm}, E_{lm}^{-\mp}] = 2iE_{mn}^{\pm\mp}, \tag{2.8}$$

where $m \neq n$ in the last line.

In this basis a convenient choice for the supercharges and their superconformal extensions is given by

$$\mathcal{Q}_{n\alpha} = \frac{1}{\sqrt{2}}(Q_{2n-1\ \alpha} + iQ_{2n\ \alpha}), \quad \bar{\mathcal{Q}}_{n\alpha} = \frac{1}{\sqrt{2}}(Q_{2n-1\ \alpha} - iQ_{2n\ \alpha}),$$

$$\mathcal{S}_n^{\ \alpha} = \frac{1}{\sqrt{2}}(S_{2n-1}^{\ \alpha} + iS_{2n}^{\ \alpha}), \quad \bar{\mathcal{S}}_n^{\ \alpha} = \frac{1}{\sqrt{2}}(S_{2n-1}^{\ \alpha} - iS_{2n}^{\ \alpha}). \tag{2.9}$$

Trivially, from (2.4),

$$[K^{\alpha\beta}, \mathcal{Q}_{n\gamma}] = i(\delta_\gamma^{\ \alpha}\mathcal{S}_n^{\ \beta} + \delta_\gamma^{\ \beta}\mathcal{S}_n^{\ \alpha}), \quad [P_{\alpha\beta}, \mathcal{S}_n^{\ \gamma}] = -i(\delta_\alpha^{\ \gamma}\mathcal{Q}_{n\beta} + \delta_\beta^{\ \gamma}\mathcal{Q}_{n\alpha}),$$

$$[M_\alpha^{\ \beta}, \mathcal{Q}_{n\gamma}] = \delta_\gamma^{\ \beta}\mathcal{Q}_{n\alpha} - \tfrac{1}{2}\delta_\alpha^{\ \beta}\mathcal{Q}_{n\gamma}, \quad [M_\alpha^{\ \beta}, \mathcal{S}_n^{\ \gamma}] = -\delta_\alpha^{\ \gamma}\mathcal{S}_n^{\ \beta} + \tfrac{1}{2}\delta_\alpha^{\ \beta}\mathcal{S}_n^{\ \gamma},$$

$$[D, \mathcal{Q}_{n\alpha}] = \tfrac{1}{2}\mathcal{Q}_{n\alpha}, \quad [D, \mathcal{S}_n^{\ \alpha}] = -\tfrac{1}{2}\mathcal{S}_n^{\ \alpha}, \tag{2.10}$$

along with identical equations for $\mathcal{Q}_{n\alpha}, \mathcal{S}_n^{\ \alpha} \to \bar{\mathcal{Q}}_{n\alpha}, \bar{\mathcal{S}}_n^{\ \alpha}$. The nonzero anticommutators among the supercharges and their superconformal extensions are

$$\{\mathcal{Q}_{m\alpha}, \bar{\mathcal{Q}}_{n\beta}\} = 2\delta_{mn}P_{\alpha\beta}, \quad \{\mathcal{S}_m^{\ \alpha}, \bar{\mathcal{S}}_n^{\ \beta}\} = 2\delta_{mn}K^{\alpha\beta},$$

$$\{\mathcal{Q}_{m\alpha}, \bar{\mathcal{S}}_m^{\ \beta}\} = 2i(M_\alpha^{\ \beta} + \delta_\alpha^{\ \beta}D - \delta_\alpha^{\ \beta}H_m),$$

$$\{\bar{\mathcal{Q}}_{m\alpha}, \mathcal{S}_m^{\ \beta}\} = 2i(M_\alpha^{\ \beta} + \delta_\alpha^{\ \beta}D + \delta_\alpha^{\ \beta}H_m),$$

$$\{\mathcal{Q}_{m\alpha}, \mathcal{S}_n^{\ \beta}\} = \delta_\alpha^{\ \beta}E_{mn}^{++}, \quad \{\bar{\mathcal{Q}}_{m\alpha}, \bar{\mathcal{S}}_n^{\ \beta}\} = \delta_\alpha^{\ \beta}E_{mn}^{--},$$

$$\{\mathcal{Q}_{m\alpha}, \bar{\mathcal{S}}_n^{\ \beta}\} = \delta_\alpha^{\ \beta}E_{mn}^{+-}, \quad \{\bar{\mathcal{Q}}_{m\alpha}, \mathcal{S}_n^{\ \beta}\} = \delta_\alpha^{\ \beta}E_{mn}^{-+}, \tag{2.11}$$

where $m \neq n$ in the last two lines. Finally, under the action of the generators (2.7) the nonzero commutators are

$$[H_m, \mathcal{Q}_{n\alpha}] = \delta_{mn}\mathcal{Q}_{n\alpha}, \quad [H_m, \mathcal{S}_n^{\ \alpha}] = \delta_{mn}\mathcal{S}_n^{\ \alpha},$$

$$[H_m, \bar{\mathcal{Q}}_{n\alpha}] = -\delta_{mn}\bar{\mathcal{Q}}_{n\alpha}, \quad [H_m, \bar{\mathcal{S}}_n^{\ \alpha}] = -\delta_{mn}\bar{\mathcal{S}}_n^{\ \alpha},$$

$$[E_{lm}^{-+}, \mathcal{Q}_{n\alpha}] = -[E_{ml}^{+-}, \mathcal{Q}_{n\alpha}] = 2i\delta_{ln}\mathcal{Q}_{m\alpha}, \quad [E_{lm}^{-+}, \mathcal{S}_n^{\ \alpha}] = -[E_{ml}^{+-}, \mathcal{S}_n^{\ \alpha}] = 2i\delta_{ln}\mathcal{S}_m^{\ \alpha},$$

$$[E_{lm}^{--}, \mathcal{Q}_{n\alpha}] = 2i(\delta_{ln}\bar{\mathcal{Q}}_{m\alpha} - \delta_{mn}\bar{\mathcal{Q}}_{l\alpha}), \quad [E_{lm}^{--}, \mathcal{S}_n^{\ \alpha}] = 2i(\delta_{ln}\bar{\mathcal{S}}_m^{\ \alpha} - \delta_{mn}\bar{\mathcal{S}}_l^{\ \alpha}),$$

$$[E_{lm}^{+-}, \bar{\mathcal{Q}}_{n\alpha}] = -[E_{ml}^{-+}, \bar{\mathcal{Q}}_{n\alpha}] = 2i\delta_{ln}\bar{\mathcal{Q}}_{m\alpha}, \quad [E_{lm}^{+-}, \bar{\mathcal{S}}_n^{\ \alpha}] = -[E_{ml}^{-+}, \bar{\mathcal{S}}_n^{\ \alpha}] = 2i\delta_{ln}\bar{\mathcal{S}}_m^{\ \alpha},$$

$$[E_{lm}^{++}, \bar{\mathcal{Q}}_{n\alpha}] = 2i(\delta_{ln}\mathcal{Q}_{m\alpha} - \delta_{mn}\mathcal{Q}_{l\alpha}), \quad [E_{lm}^{++}, \bar{\mathcal{S}}_n^{\ \alpha}] = 2i(\delta_{ln}\mathcal{S}_m^{\ \alpha} - \delta_{mn}\mathcal{S}_l^{\ \alpha}). \tag{2.12}$$

Suppressing spinor indices, the action of the linearly independent set of lowering operators on $\mathcal{Q}_n$ may be expressed by the following diagram:

$$
\begin{array}{ccccccccccccc}
& & & & & & \mathcal{Q}_N & & & & & & \\
& & & & & {\scriptstyle E^{-+}_{N-1\,N}}\nearrow & & \searrow {\scriptstyle E^{--}_{N-1\,N}} & & & & & \\
\mathcal{Q}_1 \underset{E^{-+}_{12}}{\longrightarrow} \mathcal{Q}_2 \underset{E^{-+}_{23}}{\longrightarrow} \mathcal{Q}_3 \longrightarrow \cdots \longrightarrow \mathcal{Q}_{N-1} & & & & & & & & \bar{\mathcal{Q}}_{N-1} \longrightarrow \cdots \longrightarrow \bar{\mathcal{Q}}_3 \underset{E^{-+}_{23}}{\longrightarrow} \bar{\mathcal{Q}}_2 \underset{E^{-+}_{12}}{\longrightarrow} \bar{\mathcal{Q}}_1 \\
& & & & & {\scriptstyle E^{--}_{N-1\,N}}\searrow & & \nearrow {\scriptstyle E^{-+}_{N-1\,N}} & & & & & \\
& & & & & & \bar{\mathcal{Q}}_N & & & & & &
\end{array}
$$

$$(2.13)$$

Of course, an identical diagram applies to $\mathcal{S}_n, \bar{\mathcal{S}}_n$. (This diagram is helpful later for discussing the shortening conditions and the computation of superconformal characters.)

The spin generators in terms of SU(2) generators may be expressed as follows:

$$
[M_\alpha{}^\beta] = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix}, \quad [J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \tag{2.14}
$$

where $(\mathcal{Q}_{n1}, \mathcal{Q}_{n2})$, $(\bar{\mathcal{Q}}_{n1}, \bar{\mathcal{Q}}_{n2})$, $(\mathcal{S}_n{}^2, -\mathcal{S}_n{}^1)$, $(\bar{\mathcal{S}}_n{}^2, -\bar{\mathcal{S}}_n{}^1)$ transform as usual spin $\frac{1}{2}$ doublets, each with J_3 eigenvalues $(\frac{1}{2}, -\frac{1}{2})$.

III. SHORTENING CONDITIONS FOR UNITARY MULTIPLETS

For physical applications, we require states with positive real conformal dimensions, non-negative half integer spin eigenvalues, and in finite dimensional irreducible representations of the R-symmetry group SO(2N). Hence it is sufficient to consider superconformal representations defined by highest weight states $|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}}$ with

$$
(K^{\alpha\beta}, \mathcal{S}_n{}^\alpha, \bar{\mathcal{S}}_n{}^\alpha, J_+, E^{+\pm}_{mn})|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \quad 1 \leq m < n \leq N,
$$

$$
(D, J_3, H_m)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = (\Delta, j, r_m)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}},
$$

$$
\mathrm{r} = (r_1, \ldots, r_N), \tag{3.1}
$$

where Δ is the conformal dimension, $j \in \frac{1}{2}\mathbb{N}$ is the spin, and with SO(2N) Dynkin labels expressed in terms of $r_j \in \frac{1}{2}\mathbb{Z}$ required to satisfy

$$
[r_1 - r_2, \ldots, r_{N-2} - r_{N-1}, r_{N-1} + r_N, r_{N-1} - r_N] \in \mathbb{N}^N, \tag{3.2}
$$

so that, in particular, $r_1 \geq r_2 \geq \cdots \geq r_{N-1} \geq |r_N|$.

For a representation space with basis $V_{(\Delta;j;\mathrm{r})}$, the states are given by

$$
V_{(\Delta;j;\mathrm{r})}\left\{ \prod_{\substack{\alpha,\beta,\gamma,\delta=1,2 \\ \gamma \leq \delta, 1 \leq n,m,r,s \leq N}} (\mathcal{Q}_{n\alpha})^{\kappa_{n\alpha}}(\bar{\mathcal{Q}}_{m\beta})^{\bar{\kappa}_{m\beta}}(P_{\gamma\delta})^{k_{\gamma\delta}}(J_-)^K(E^{-\pm}_{rs})^{K_{rs}}|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} \right\} \tag{3.3}
$$

for $\kappa_{n\alpha}$, $\bar{\kappa}_{m\beta} \in \{0, 1\}$ and $k_{\gamma\delta}$, K, $K_{rs} \in \mathbb{N}$.

Unitarity requires that,[18,19]

$$
\Delta \geq \begin{cases} r_1 + j + 1 & \text{for } j > 0 \\ r_1 & \text{for } j = 0. \end{cases} \tag{3.4}
$$

The superconformal multiplets may be truncated by various shortening conditions which are considered now. For the BPS shortening condition,

$$
\mathcal{Q}_{n\alpha}|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \quad \alpha = 1, 2, \tag{3.5}
$$

then this leads to the following equations, using (2.11):

$$(E^{+\pm}_{nm}, J_\pm, D \pm J_3 - H_n)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \quad 1 \leqslant m \leqslant N, \quad m \neq n, \tag{3.6}$$

so that, using (2.8), (2.14), and (3.1),

$$\Delta = r_n, \quad r_1 = r_2 = \cdots = r_n, \quad j = 0. \tag{3.7}$$

Clearly, (3.5) with (3.6) implies that $\mathcal{Q}_{m\alpha}|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0$, $m < n$.

Imposing the constraint,

$$\bar{\mathcal{Q}}_{N\alpha}|\Delta, 0, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \quad \alpha = 1, 2, \tag{3.8}$$

leads to, using (2.11),

$$(E^{\pm -}_{mN}, J_\pm, D \pm J_3 + H_N)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \quad 1 \leqslant m < N, \tag{3.9}$$

so that, using (2.8), (2.14), and (3.1),

$$\Delta = -r_N, \quad r_1 = r_2 = \cdots = -r_N, \quad j = 0. \tag{3.10}$$

We may also consider the semishort multiplet condition,

$$\left(\mathcal{Q}_{n2} - \frac{1}{2j}\mathcal{Q}_{n1}J_-\right)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \tag{3.11}$$

whereby applying $\bar{\mathcal{S}}_l^2$ and using (2.11), we obtain

$$\left(E^{+\pm}_{nm}, D - J_3 - H_n - \frac{1}{2j}J_+J_-\right)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \quad 1 \leqslant m \leqslant N, \quad m \neq n, \tag{3.12}$$

so that, using (2.8), (2.14), and (3.1),

$$\Delta = r_n + j + 1, \quad r_1 = r_2 = \cdots = r_n. \tag{3.13}$$

We may also consider

$$\left(\bar{\mathcal{Q}}_{N2} - \frac{1}{2j}\bar{\mathcal{Q}}_{N1}J_-\right)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \tag{3.14}$$

whereby applying $\mathcal{S}_l^2$ and using (2.11) then

$$\left(E^{\pm -}_{mN}, D - J_3 + H_N - \frac{1}{2j}J_+J_-\right)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \quad 1 \leqslant m < N, \tag{3.15}$$

so that, using (2.8), (2.14), and (3.1),

$$\Delta = -r_N + j + 1, \quad r_1 = r_2 = \cdots = -r_N. \tag{3.16}$$

Imposing both,

$$\left(\mathcal{Q}_{N2} - \frac{1}{2j}\mathcal{Q}_{N1}J_-\right)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \quad \left(\bar{\mathcal{Q}}_{N2} - \frac{1}{2j}\bar{\mathcal{Q}}_{N1}J_-\right)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0 \tag{3.17}$$

leads to a conservation condition on the highest weight state, using (2.11),

$$((2j-1)(2jP_{22} - 2P_{12}J_-) + P_{11}J_-^2)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0, \tag{3.18}$$

and also to, using (3.13), for $n = N$, and (3.16),

$$\Delta = j + 1, \quad \mathrm{r} = 0. \tag{3.19}$$

TABLE I. Osp $(2N|4)$ unitary, positive energy irreps.

Type	Δ	r	Omitted	Denoted
Long	$\geqslant r_1+j+1$	$r_1 \geqslant \cdots \geqslant |r_N|$	None	$(N, A, 0)$
Semishort	r_1+j+1	$r_1 = \cdots = r_n$	$\{\mathcal{Q}_{i2}\}_{i=1}^n$	(N, A, n) for $n < N$
Semishort	r_1+j+1	$r_1 = \cdots = r_N$	$\{\mathcal{Q}_{i2}\}_{i=1}^N$	$(N, A, +)$
Semishort	r_1+j+1	$r_1 = \cdots = -r_N$	$\{\mathcal{Q}_{i2}, \bar{\mathcal{Q}}_{N2}\}_{i=1}^{N-1}$	$(N, A, -)$
BPS	r_1	$r_1 = \cdots = r_n$	$\{\mathcal{Q}_{i\alpha}\}_{i=1}^n$	(N, B, n) for $n < N$
$\frac{1}{2}$ BPS	r_1	$r_1 = \cdots r_N$	$\{\mathcal{Q}_{i\alpha}\}_{i=1}^N$	$(N, B, +)$
$\frac{1}{2}$ BPS	r_1	$r_1 = \cdots = -r_N$	$\{\mathcal{Q}_{i\alpha}, \bar{\mathcal{Q}}_{N\alpha}\}_{i=1}^{N-1}$	$(N, B, -)$
Cons. current	$j+1$	$r_i = 0$	$\{\mathcal{Q}_{i2}, \bar{\mathcal{Q}}_{N2}, P_{22}\}_{i=1}^N$	$(N, \text{cons.})$

It is easy to see that imposing BPS or semishortening conditions, other than the above, using other $\bar{\mathcal{Q}}_{n\alpha}$, $n \neq N$, leads to violations of (3.4), implying that corresponding multiplets are nonunitary. These are not considered here.

For the truncated supermultiplets $\mathcal{M}$, the Verma modules $V_{(\Delta;j;\mathrm{r})} \to \mathcal{V}_{(\Delta;j;\mathrm{r})}^{\mathcal{M}}$ are generated by a subset of the generators in (3.3) so that it is sufficient to set some $\kappa_{n\alpha}$, $\bar{\kappa}_{m\beta}$, $k_{\gamma\delta}$ to zero.

Using the information encapsulated in (2.13), then the condition (3.5), for BPS multiplets, entails omitting $\mathcal{Q}_{j\alpha}$, $j=1,\ldots,n$ from (3.3), so that $\kappa_{j\alpha}=0$. Similarly, for (3.8), then $\kappa_{j\alpha}, \bar{\kappa}_{N\alpha}=0$, $j=1,\ldots,N-1$, while for the semishortening conditions, (3.11), $\kappa_{j2}=0$, $j=1,\ldots,n$, and (3.14), κ_{j2}, $\bar{\kappa}_{N2}=0$, $j=1,\ldots,N-1$. Corresponding to (3.17) with (3.18), then $\kappa_{j2}, \bar{\kappa}_{N2}=0$, $j=1,\ldots,N$, $k_{22}=0$ for the multiplet of conserved currents. This information along with notation is summarised in Table I.

IV. SUPERCONFORMAL CHARACTERS FOR SO(2N) R-SYMMETRY

A procedure for computing conformal characters for higher than two dimensions and $\mathcal{N}=4$ superconformal characters for four dimensions has been explained in detail elsewhere.[20,21] The procedure is also closely related to that in Ref. 11 for constructing supermultiplets by employing the Racah–Speiser algorithm. We proceed by analogy with Refs. 20 and 21.

Introducing variables s, x, $\mathrm{y}=(y_1,\ldots,y_N)$, we may write the character corresponding to the restricted Verma module, (3.3), for $V_{(\Delta;j;\mathrm{r})} \to \mathcal{V}_{(\Delta;j;\mathrm{r})}^{\mathcal{M}}$, as a formal trace,

$$C_{(\Delta;j;\mathrm{r})}^{\mathcal{M}}(s,x,\mathrm{y}) = \widetilde{\mathrm{Tr}}_{\mathcal{V}_{(\Delta;j;\mathrm{r})}^{\mathcal{M}}} \left(s^{2D} x^{2J_3} y_1^{H_1} \cdots y_N^{H_N} \right) = s^{2\Delta} C_{2j}(x) C_{\mathrm{r}}^{(N)}(\mathrm{y}) \times \sum_{k_{\gamma\delta} \in \mathbb{N}} (s^2 x^2)^{k_{11}} s^{2k_{12}} (s^2 x^{-2})^{k_{22}}$$

$$\times \sum_{\kappa_{n\alpha}, \bar{\kappa}_{m\beta} \in \{0,1\}} (sy_n x)^{\kappa_{n1}} (sy_n x^{-1})^{\kappa_{n2}} (sy_m^{-1} x)^{\bar{\kappa}_{m1}} (sy_m^{-1} x^{-1})^{\bar{\kappa}_{m2}}, \qquad (4.1)$$

where the sum over $k_{\gamma\delta}$ gives the contributions of $P_{\gamma\delta}$ and that over $\kappa_{n\alpha}, \bar{\kappa}_{m\beta}$ gives those of $\mathcal{Q}_{n\alpha}, \bar{\mathcal{Q}}_m{}^\beta$, and where

$$C_j(x) = \frac{x^{j+1}}{x - x^{-1}},$$

$$C_{\mathrm{r}}^{(N)}(\mathrm{y}) = \prod_{j=1}^N y_j^{r_j+j-1} / \Delta(\mathrm{y} + \mathrm{y}^{-1}), \quad \Delta(\mathrm{y}) = \prod_{1 \leqslant i < j \leqslant N} (y_i - y_j), \qquad (4.2)$$

are the Verma module characters for SU(2) and SO(2N), giving contributions from the J_-, $E_{rs}^{-\pm}$ generators, and the highest weight state, in (3.3).

Once the correct generators are omitted from (3.3), so that various $\kappa_{n\alpha}$, $\bar{\kappa}_{m\beta}$, $k_{\gamma\delta}$ are zero in (4.1), the prescription for finding the characters of corresponding unitary irreducible representations $\mathcal{R}^{\mathcal{M}}_{(\Delta;j;r)}$ is simply given by[22]

$$\chi^{\mathcal{M}}_{(\Delta;j;r)}(s,x,\mathrm{y}) = \mathrm{Tr}_{\mathcal{R}^{\mathcal{M}}_{(\Delta;j;r)}}(s^{2D}x^{2J_3}y_1{}^{H_1}\cdots y_N{}^{H_N}) = \mathfrak{W}^{(N)}C^{\mathcal{M}}_{(\Delta;j;r)}(s,x,\mathrm{y}), \tag{4.3}$$

where $\mathfrak{W}^{(N)} = \mathfrak{W}^{\mathcal{S}_2}\mathfrak{W}^{\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}}$ is the Weyl symmetrizer for the maximal compact subgroup of the superconformal group, $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SO}(2N)$. Here $\mathcal{S}_2$ and $\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}$ are the Weyl symmetry groups for $\mathrm{SU}(2)$ and $\mathrm{SO}(2N)$ and, for some functions $f(x)$, $f(\mathrm{y}) = f(y_1, \ldots, y_N)$, the action of the relevant Weyl symmetrizers is given by

$$\mathfrak{W}^{\mathcal{S}_2} f(x) = f(x) + f(x^{-1}),$$

$$\mathfrak{W}^{\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}} f(\mathrm{y}) = \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_N=\pm 1 \\ \Pi_i \varepsilon_i = 1}} \sum_{\sigma \in \mathcal{S}_N} f(y_{\sigma(1)}{}^{\varepsilon_1}, \ldots, y_{\sigma(N)}{}^{\varepsilon_N}). \tag{4.4}$$

It is important to realize that the resulting characters may be expanded in terms of $\mathrm{SU}(2) \times \mathrm{SO}(2N)$ characters using

$$\chi_j(x) = \mathfrak{W}^{\mathcal{S}_2} C_j(x) = \frac{x^{j+1} - x^{-j-1}}{x - x^{-1}},$$

$$\chi_\mathrm{r}^{(N)}(\mathrm{y}) = \mathfrak{W}^{\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}} C_\mathrm{r}^{(N)}(\mathrm{y}) = (\det[y_i^{r_j+N-j} + y_i^{-r_j-N+j}] + \det[y_i^{r_j+N-j} - y_i^{-r_j-N+j}])/2\Delta(\mathrm{y} + \mathrm{y}^{-1}), \tag{4.5}$$

the usual Weyl character formulas for $\mathrm{SU}(2)$ and $\mathrm{SO}(2N)$ finite dimensional, irreducible representations.

Defining

$$P(s,x) = \frac{1}{(1-s^2)(1-s^2x^2)(1-s^2x^{-2})},$$

$$\mathcal{Q}_n(\mathrm{y},x) = \prod_{j=n+1}^{N} (1 + y_j x), \quad \bar{\mathcal{Q}}_n(\mathrm{y},x) = \prod_{j=1}^{n} (1 + y_j^{-1}x), \tag{4.6}$$

then this prescription leads to the following character formulas for the unitary irreducible representations, using (4.1) with (4.3) and with the notation of Table I:

$$\chi^{(N,i,n)}_{(\Delta;j;r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,\mathrm{y}) = \mathfrak{W}^{(N)} C^{(N,i,n)}_{(\Delta;j;r_1,\ldots,r_1,r_{n+1},\ldots r_N)}(s,x,\mathrm{y})$$

$$= s^{2\Delta}P(s,x)\mathfrak{W}^{(N)}\bigg(C_{2j}(x)C_\mathrm{r}^{(N)}(\mathrm{y})\mathcal{R}^{(N,i,n)}(s,x,\mathrm{y})\prod_{\varepsilon=\pm 1}\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y},x^\varepsilon)\bigg),$$

$$n < N,$$

$$\chi^{(N,i,\pm)}_{(\Delta;j;r,\ldots,r,\pm r)}(s,x,\mathrm{y}) = \mathfrak{W}^{(N)} C^{(N,i,\pm)}_{(r+j+1;j;r,\ldots r,\pm r)}(s,x,\mathrm{y})$$

$$= s^{2\Delta}P(s,x)\mathfrak{W}^{(N)}\bigg(C_{2j}(x)C_\mathrm{r}^{(N)}(\mathrm{y})\mathcal{R}^{(N,i,\pm)}(s,x,\mathrm{y})\prod_{\varepsilon=\pm 1}\bar{\mathcal{Q}}_{N-1}(s^{-1}\mathrm{y},x^\varepsilon)\bigg), \tag{4.7}$$

where

$$\mathcal{R}^{(N,i,n)}(s,x,\mathrm{y}) = \begin{cases} \mathcal{Q}_0(s\mathrm{y},x)\mathcal{Q}_n(s\mathrm{y},x^{-1}) & \text{for } i=A \\ \displaystyle\prod_{\varepsilon=\pm1} \mathcal{Q}_n(s\mathrm{y},x^\varepsilon) & \text{for } i=B, \end{cases}$$

$$\mathcal{R}^{(N,i,\pm)}(s,x,\mathrm{y}) = \begin{cases} \mathcal{Q}_0(s\mathrm{y},x)(1+s\mathrm{y}_N^{-1}x)(1+s\mathrm{y}_N^{\mp1}x^{-1}) & \text{for } i=A \\ \displaystyle\prod_{\varepsilon=\pm1} (1+s\mathrm{y}_N^{\mp1}x^\varepsilon) & \text{for } i=B, \end{cases} \tag{4.8}$$

and appropriate Δ, j are as given in Table I. (The conserved current multiplet is discussed separately below.)

Using invariance of $\Pi_{\varepsilon=\pm1}\mathcal{Q}_0(s\mathrm{y},x^\varepsilon)\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y},x^\varepsilon)$ under $\mathfrak{W}^{(N)}$ then the long multiplet character is given by

$$\chi_{(\Delta;j;\mathrm{r})}^{(N,\mathrm{long})}(s,x,\mathrm{y}) = \chi_{(\Delta;j;\mathrm{r})}^{(N,A,0)}(s,x,\mathrm{y}) = s^{2\Delta}P(s,x)\chi_{2j}(x)\chi_{\mathrm{r}}^{(N)}(\mathrm{y})\prod_{\varepsilon=\pm1}\mathcal{Q}_0(s\mathrm{y},x^\varepsilon)\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y},x^\varepsilon). \tag{4.9}$$

This may be expanded in terms of $SU(2)\times SO(2N)$ characters using the identities

$$P(s,x) = \frac{1}{1-s^4}\sum_{n=0}^{\infty} s^{2n}\chi_{2n}(x), \tag{4.10}$$

along with, for later use,

$$\prod_{j=1}^{N}(1+t\mathrm{y}_j)(1+t\mathrm{y}_i^{-1}) = \sum_{n=0}^{N-1}(t^n+t^{2N-n})\chi_{(1^n,0^{N-n})}^{(N)}(\mathrm{y}) + t^N\chi_{(1^N)}^{(N)}(\mathrm{y}) + t^N\chi_{(1^{N-1},-1)}^{(N)}(\mathrm{y}). \tag{4.11}$$

A. Simplification of BPS multiplet characters

Half BPS characters may be simplified by first writing, easily obtained from (4.7),

$$\chi_{(\mathrm{r};0;\mathrm{r},\dots,\mathrm{r},\pm\mathrm{r})}^{(N,B,\pm)}(s,x,\mathrm{y}) = s^{2\mathrm{r}}P(s,x)\sum_{a_1,\dots,a_N=0}^{2} s^{a_1+\cdots+a_N}\chi_{j_{a_1}}(x)\cdots\chi_{j_{a_N}}(x)\chi_{(\mathrm{r}-a_1,\mathrm{r}-a_2,\dots,\pm\mathrm{r}\mp a_N)}^{(N)}(\mathrm{y}), \tag{4.12}$$

where, $j_a = \frac{1}{2}(1-(-1)^a)$ so that

$$j_0 = j_2 = 0, \quad j_1 = 1. \tag{4.13}$$

Some further manipulation shows that (4.12) may be simlified further to

$$\chi_{(\mathrm{r};0;\mathrm{r},\dots,\mathrm{r},\pm\mathrm{r})}^{(N,B,\pm)}(s,x,\mathrm{y}) = s^{2\mathrm{r}}P(s,x)\sum_{0\leqslant a_1\leqslant\cdots\leqslant a_N\leqslant2} s^{a_1+\cdots+a_N}\chi_{j_{a_1,\dots,a_N}}(x)\chi_{(\mathrm{r}-a_1,\mathrm{r}-a_2,\dots,\pm\mathrm{r}\mp a_N)}^{(N)}(\mathrm{y}), \tag{4.14}$$

where

$$j_{a_1,\dots,a_N} = \frac{1}{2}(N-(-1)^{a_1}-\cdots-(-1)^{a_N}). \tag{4.15}$$

Another consistency check is for $N=4$, or $SO(8)$ R-symmetry, whereby this formula leads directly to

$$
\begin{aligned}
\chi^{(4,B,-)}_{(r;0;r,r,r,-r)}(s,x,y) = s^{2r}P(s,x)\big(&\chi^{(4)}_{(r,r,r,-r)}(y) + s\chi_1(x)\chi^{(4)}_{(r,r,r,1-r)}(y) + s^2\chi^{(4)}_{(r,r,r,2-r)}(y) \\
&+ s^2\chi_2(x)\chi^{(4)}_{(r,r,r-1,1-r)}(y) + s^3\chi_1(x)\chi^{(4)}_{(r,r,r-1,2-r)}(y) + s^3\chi_3(x)\chi^{(4)}_{(r,r-1,r-1,1-r)}(y) \\
&+ s^4\chi^{(4)}_{(r,r,r-2,2-r)}(y) + s^4\chi_2(x)\chi^{(4)}_{(r,r-1,r-1,2-r)}(y) + s^4\chi_4(x)\chi^{(4)}_{(r-1,r-1,r-1,1-r)}(y) \\
&+ s^5\chi_1(x)\chi^{(4)}_{(r,r-1,r-2,2-r)}(y) + s^5\chi_3(x)\chi^{(4)}_{(r-1,r-1,r-1,2-r)}(y) + s^6\chi^{(4)}_{(r,r-2,r-2,2-r)}(y) \\
&+ s^6\chi_2(x)\chi^{(4)}_{(r-1,r-1,r-2,2-r)}(y) + s^7\chi_1(x)\chi^{(4)}_{(r-1,r-2,r-2,2-r)}(y) + s^8\chi^{(4)}_{(r-2,r-2,r-2,2-r)}(y)\big)
\end{aligned}
$$

which corresponds exactly to the graviton spectrum derived in Ref. 23.

Further simplifications to the half BPS character (4.14) occur for $r < 2$. For $r = 1$ then,

$$
\chi^{(N,B,\pm)}_{(1;0;1,\dots,1,\pm1)}(s,x,y) = \mathcal{A}_{(1,0)}(s,x)\chi^{(N)}_{(1^{N-1},\pm1)}(y) + \mathcal{A}_{(3/2,1/2)}(s,x)\chi^{(N)}_{(1^{N-1},0)}(y) + \mathcal{A}_{(2,0)}(s,x)\chi^{(N)}_{(1^{N-1},\mp1)}(y)
$$
$$
+ \sum_{n=0}^{N-2} \mathcal{D}_{1/2(N-n)}(s,x)\chi^{(N)}_{(1^n,0^{N-n})}(y), \tag{4.16}
$$

where

$$
\mathcal{A}_{(\Delta,j)}(s,x) = s^{2\Delta}\chi_{2j}(x)P(s,x), \quad \mathcal{D}_j(s,x) = s^{2j+2}(\chi_{2j}(x) - s^2\chi_{2j-2}(x))P(s,x) \tag{4.17}
$$

are the characters for unitary irreducible representations of the conformal group in three dimensions, SO(3,2),[24]—see also Ref. 20—whereby $\mathcal{A}_{(\Delta,j)}$ corresponds to an unconstrained spin j field, conformal dimension $\Delta \geq j = 1$, while $\mathcal{D}_j$ corresponds to a conserved current with spin j, conformal dimension $j+1$, including all their conformal descendants (or derivatives acting on fields).

Similarly, for $r = \frac{1}{2}$ then,

$$
\chi^{(N,B,\pm)}_{(1/2;0;1/2,\dots,1/2,\pm1/2)}(s,x,y) = \mathcal{D}_{\text{Rac}}(s,x)\chi^{(N)}_{(1/2,\dots,1/2,\pm1/2)}(y) + \mathcal{D}_{\text{Di}}(s,x)\chi^{(N)}_{(1/2,\dots,1/2,\mp1/2)}(y), \tag{4.18}
$$

where

$$
\mathcal{D}_{\text{Rac}}(s,x) = \frac{s+s^3}{(1-s^2x^2)(1-s^2x^{-2})}, \quad \mathcal{D}_{\text{Di}}(s,x) = \frac{s^2(x+x^{-1})}{(1-s^2x^2)(1-s^2x^{-2})} \tag{4.19}
$$

are characters for the free field representations of SO(3,2), the so called "Di," respectively, "Rac," singleton representations,[25] corresponding to a free spin $\frac{1}{2}$, respectively, scalar, field with conformal dimension 1, respectively, $\frac{1}{2}$, and all its descendants.

Finally, from (4.14) it may be shown that

$$
\chi^{(N,B,\pm)}_{(0;0;0,\dots,0)}(s,x,y) = 1, \tag{4.20}
$$

the character for the identity representation.

For other BPS characters, from (4.7), we may write, similarly to (4.12) with (4.13),

$$
\chi^{(N,B,n)}_{(r_1;0;r_1,r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,y) = s^{2r_1}P(s,x) \sum_{\substack{a_1,\dots,a \\ \bar{a}_{n+1},\dots,\bar{a}_N \\ =0}}^{2} s^{a_1+\cdots+a_N+\bar{a}_{n+1}+\cdots+\bar{a}_N} \prod_{i=1}^{N} \chi_{j_{a_i}}(x) \prod_{i=n+1}^{N} \chi_{j_{a_i}}(x)
$$
$$
\times \chi^{(N)}_{(r_1-a_1,\dots,r_1-a_n,r_{n+1}+\bar{a}_{n+1}-a_{n+1},\dots,r_N+\bar{a}_N-a_N)}(y), \tag{4.21}
$$

and, similarly to (4.14) with (4.15), using that $r_1 = r_2 = \cdots = r_n$,

$$\chi_{(r_1;0;r_1,r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}^{(N,B,n)}(s,x,y) = s^{2r_1}P(s,x) \times \sum_{0 \le a_1 \le \cdots \le a_n \le 2} \sum_{\substack{a_{n+1},\ldots,a_N \\ \bar{a}_{n+1},\ldots,\bar{a}_N = 0}}^{2} s^{a_1+\cdots+a_N+\bar{a}_{n+1}+\cdots+\bar{a}_N} \chi_{j_{a_1,\ldots a_2,\ldots,a_n}}(x)$$

$$\times \prod_{i=n+1}^{N} \chi_{j_{a_i}}(x) \chi_{j_{a_i}}(x) \times \chi_{(r_1-a_1,\ldots,r_1-a_n,r_{n+1}+\bar{a}_{n+1}-a_{n+1},\ldots,r_N+\bar{a}_N-a_N)}^{(N)}(y),$$

$$(4.22)$$

where,

$$j_{a_1,\ldots,a_n} = \tfrac{1}{2}(n - (-1)^{a_1} - \cdots - (-1)^{a_n}). \tag{4.23}$$

Without further restrictions on r_i, or s,x,y, there appear to be no further simplifications to (4.22).

B. The conserved current multiplet character

The conserved current multiplet (which has been excluded so far from the above discussion) corresponding to the semishortening condition (3.17), has character

$$\chi_{(j+1;j;0,\ldots,0)}^{(N,\text{cons.})}(s,x,y) = s^{2j+2}P(s,x)\mathfrak{W}^{(N)}(C_{2j}(x)C_0^{(N)}(y)(1 - s^2 x^{-2})\mathcal{Q}_0(sy,x)\mathcal{Q}_N(s^{-1}y,x)\bar{\mathcal{Q}}_{N-1}(s^{-1}y,x^{-1})), \tag{4.24}$$

which ensures that the contributions from the supercharges $\mathcal{Q}_{n2}$, $n=1,\ldots,N$, $\tilde{\mathcal{Q}}_{N2}$ along with P_{22} are omitted from the Verma module character (4.1).

To simplify, it is useful to observe that $\mathcal{Q}_0(sy,x)\bar{\mathcal{Q}}_N(s^{-1}y,x)$ is invariant under action by $\mathfrak{W}^{\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}}$ and, further, that $\mathfrak{W}^{\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}}C_0(y)\bar{\mathcal{Q}}_{N-1}(y,t)=1$ so that

$$\chi_{(j+1;j;0,\ldots,0)}^{(N,\text{cons.})}(s,x,y) = s^{2j+2}P(s,x)\mathfrak{W}^{\mathcal{S}_2}(C_{2j}(x)(1 - s^2 x^{-2})\mathcal{Q}_0(sy,x)\dot{\mathcal{Q}}_N(s^{-1}y,x)) = \sum_{n=0}^{N-1} (\mathcal{D}_{j+(1/2)n}(s,x)$$

$$+ \mathcal{D}_{j+N-(1/2)n}(s,x))\chi_{(1^n,0^{N-n})}^{(N)}(y) + \mathcal{D}_{j+(1/2)N}(s,x)(\chi_{(1^N)}^{(N)}(y) + \chi_{(1^{N-1},-1)}^{(N)}(y)), \tag{4.25}$$

using (4.11), to expand in x, and subsequently the expression for the conserved current character in (4.17).

C. Long multiplet decompositions

Using linearity of $\mathfrak{W}^{(N)}$ we may easily obtain, for $j \ge \frac{1}{2}$, $r_1 > r_2$,

$$\chi_{(r_1+j+1;j;r_1,r_2,\ldots,r_N)}^{(N,\text{long})}(s,x,y) = \mathfrak{W}^{(N)}(1 + sy_1 x^{-1})C_{(r_1+j+1;j;r_1,r_2,\ldots,r_N)}^{(N,A,1)}(s,x,y) = \chi_{(r_1+j+1;j;r_1,r_2,\ldots,r_N)}^{(N,A,1)}(s,x,y)$$

$$+ \chi_{(r_1+j+3/2;j-1/2;r_1+1,r_2,\ldots,r_N)}^{(N,A,1)}(s,x,y), \tag{4.26}$$

which expresses the reducibility of a long multiplet with $\Delta = r_1+j+1$ into a sum of semishort multiplets.

For semishort multiplet characters it may be shown that, for $r_1=r_2=\cdots=r_n>r_{n+1}$, $r>0$,

$$\chi_{(r_1+j+1;j;r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}^{(N,A,1)}(s,x,y) = \chi_{(r_1+j+1;j;r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}^{(N,A,n)}(s,x,y),$$

$$\chi_{(r+j+1,j,r,\ldots,r,\pm r)}^{(N,A,1)}(s,x,y) = \chi_{(r+j+1,j,r,\ldots,r,\pm r)}^{(N,A,\pm)}(s,x,y). \tag{4.27}$$

This employs

$$\chi^{(N,A,1)}_{(r_1+j+1;j;r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,y) = \mathfrak{W}^{(N)}\prod_{i=2}^{n}(1+sy_ix^{-1})C^{(N,A,n)}_{(r_1+j+1;j;r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,y),$$

along with the identity $\chi^{(N)}_{r_w}(y)=(-1)^{|w|}\chi^{(N)}_r(y)$, $w \in \mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}$, where $r^w=w(r+\rho)-\rho$, with $w \in \mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}$, the Weyl group, and $\rho=(N-1,N-2,\ldots,0)$, being the Weyl vector. In particular, $\chi^{(N)}_{(r_1,\ldots,r_1,r_1+1,\ldots,r_N)}(y)=0$ which implies $\chi^{(N,A,n)}_{(\Delta';j';r_1,\ldots,r_1,r_1+1,\ldots,r_N)}(s,x,y)=0$. The $(N,A,\pm)$ cases may be found similarly.

Similarly, for the conserved current multiplet character,

$$\chi^{(N,A,1)}_{(j+1;j;0,\ldots,0)}(s,x,y) = \chi^{(N,\text{cons.})}_{(j+1;j;0,\ldots,0)}(s,x,y). \tag{4.28}$$

In order to avoid ambiguity, it is assumed in what follows that $r_n > r_{n+1}$ for (N,A,n) semishort multiplet characters, unless otherwise stated.

From (4.26) with (4.27) and (4.28), then we have, for $j \geq \frac{1}{2}$, $r_1 > r_{n+1}$, $n < N$,

$$\chi^{(N,\text{long})}_{(r_1+j+1;j;r_1,r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,y) = \chi^{(N,A,n)}_{(r_1+j+1;j;r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,y)$$
$$+ \chi^{(N,A,1)}_{(r_1+j+3/2;j-1/2;r_1+1,r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,y), \tag{4.29}$$

along with, for $r > 0$,

$$\chi^{(N,\text{long})}_{(r+j+1;j;r,\ldots,r,\pm r)}(s,x,y) = \chi^{(N,A,\pm)}_{(r+j+1;j;r,\ldots,r,\pm r)}(s,x,y) + \chi^{(N,A,1)}_{(r_1+j+3/2;j-1/2;r+1,r,\ldots,r,\pm r)}(s,x,y) \tag{4.30}$$

and

$$\chi^{(N,\text{long})}_{(j+1;j;0,\ldots,0)}(s,x,y) = \chi^{(N,\text{cons.})}_{(j+1;j;0,\ldots,0)}(s,x,y) + \chi^{(N,A,1)}_{(j+3/2;j-1/2;1,0,\ldots,0)}(s,x,y). \tag{4.31}$$

It is also easily seen that

$$\chi^{(N,A,1)}_{(r_1+1/2;-1/2;r_1,r_2,\ldots,r_N)}(s,x,y) = \mathfrak{W}^{(N)}(1+sy_1x)C^{(N,B,1)}_{(r_1+1/2;-1/2;r_1,r_2,\ldots,r_N)}(s,x,y) = \chi^{(N,B,1)}_{(r_1+1;0;r_1+1,r_2,\ldots,r_N)}$$
$$\times(s,x,y), \tag{4.32}$$

using $\mathfrak{W}^{\mathcal{S}_2}C_{-1}(x)=0$, $\mathfrak{W}^{\mathcal{S}_2}xC_{-1}(x)=\mathfrak{W}^{\mathcal{S}_2}C_0(x)=1$, so that, for $r_1 > r_2$,

$$\chi^{(N,\text{long})}_{(r_1+1;0;r_1,r_2,\ldots,r_N)}(s,x,y) = \mathfrak{W}^{(N)}(1+sy_1x^{-1})C^{(N,A,1)}_{(r_1+1;0;r_1,r_2,\ldots,r_N)}(s,x,y) = \chi^{(N,A,1)}_{(r_1+1;0;r_1,r_2,\ldots,r_N)}(s,x,y)$$
$$+ \chi^{(N,B,1)}_{(r_1+2;0;r_1+2,r_2,\ldots,r_N)}(s,x,y), \tag{4.33}$$

which expresses the reducibility of a long multiplet with $\Delta = r_1+1$ into a sum of a $(N,B,1)$ BPS and a semishort multiplet.

Thus, from (4.33) with (4.27) and (4.28), we have, for $r_1 = r_n > r_{n+1}$,

$$\chi^{(N,\text{long})}_{(r_1+1;0;r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,y) = \chi^{(N,A,n)}_{(r_1+1;0;r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,y) + \chi^{(N,B,1)}_{(r_1+2;0;r_1+2,r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,y), \tag{4.34}$$

along with, for $r > 0$,

$$\chi^{(N,\text{long})}_{(r+1;0;r,\ldots,r,\pm r)}(s,x,y) = \chi^{(N,A,\pm)}_{(r+1;0;r,\ldots,r,\pm r)}(s,x,y) + \chi^{(N,B,1)}_{(r+2;0;r+2,r,\ldots,r,\pm r)}(s,x,y) \tag{4.35}$$

and

$$\chi^{(N,\text{long})}_{(1;0;0,\ldots,0)}(s,x,y) = \chi^{(N,\text{cons.})}_{(1;0;0,\ldots,0)}(s,x,y) + \chi^{(N,B,1)}_{(2;0;2,0,\ldots,0)}(s,x,y). \tag{4.36}$$

(4.26), (4.29)–(4.31), and (4.33)–(4.36) appear to exhaust all possibilities for long multiplet decompositions (and are also consistent with limits, as discussed in Sec. V) and have important consequences for any superconformal field theory in three dimensions. In particular, they imply

that all $(N,B,\pm)$ and (N,B,n), $N > n > 1$, short multiplet operators as well as certain $(N,B,1)$ short multiplet operators in $\mathcal{R}_{(r_1+1,r_2,\ldots,r_N)}$, $r_1 \geqslant r_2$, SO(2N) R-symmetry representations must remain protected against gaining anomalous dimensions. The decomposition formulas (4.26) and (4.33) have essentially appeared in Ref. 10, where also comments about the protectedness of certain operators in three dimensional superconformal field theories were made. [Note that all the decomposition formulas here are consequences of the basic formula (4.26). This is not surprising as a similar thing happens for decomposition formulas of long multiplets for $\mathcal{N}=4$ superconformal symmetry in four dimensions.[20]]

V. REDUCTION TO SUBALGEBRA CHARACTERS

Here are described certain limits that can be taken in the previous BPS and semishort multiplet characters that isolate contributions from fewer states in each multiplet and hence lead to significant simplifications. These limits are equivalent to reductions in the characters to those for various subgroups of the superconformal group, as explained in more detail in Appendix A.

A. BPS limits

1. The U(1)⊗SO(2N−2m) sector

By considering,

$$\chi_{(\Delta;j;\mathrm{r})}^{\mathcal{M}}(\delta u^{1/2},x,(\delta^{-2}u)^{1/m}\hat{y},\bar{y}) = \mathrm{Tr}_{\mathcal{R}_{(\Delta;j;\mathrm{r})}^{\mathcal{M}}}(\delta^{2\mathcal{H}_{\hat{m}}}u^{\mathcal{I}_m}x^{2J_3}\hat{y}_1{}^{H_1}\cdots\hat{y}_m{}^{H_m}\bar{y}_1{}^{H_{\hat{m}+1}}\cdots\bar{y}_{N-m}{}^{H_N}),$$

$$\hat{y} = (\delta^{-2}u)^{-1/m}(y_1,\ldots,y_m), \quad \prod_{\hat{m}=1}^{m}\hat{y}_{\hat{m}} = 1, \quad \bar{y} = (y_{m+1},\ldots,y_N),$$

$$\mathcal{H}_m = D - \frac{1}{m}\sum_{\hat{m}=1}^{m}H_m, \quad \mathcal{I}_m = D + \frac{1}{m}\sum_{\hat{m}=1}^{m}H_m, \tag{5.1}$$

in the limit $\delta \to 0$, it is clear that only those states in $\mathcal{R}_{(\Delta;j;\mathrm{r})}^{\mathcal{M}}$ for which $\mathcal{H}_m$ has zero eigenvalue contribute. In particular, this applies to the highest weight state in the (N,B,m) BPS multiplet.

It can be shown that

$$\lim_{\delta\to\infty}\chi_{(\Delta;j;\mathrm{r})}^{\mathcal{M}}(\delta u^{1/2},x,(\delta^{-2}u)^{1/m}\hat{y},\bar{y}) = \chi_{(R;\bar{\mathrm{r}})}^{\mathrm{U}(1)\otimes\mathrm{SO}(2N-2m)}(u,\bar{y}) = u^R\chi_{\bar{\mathrm{r}}}^{(N-m)}(\bar{y}), \tag{5.2}$$

in terms of $\mathrm{U}(1)_{\mathcal{I}_m}\otimes\mathrm{SO}(2N-2m)$ characters, for appropriate $R,\bar{\mathrm{r}}$, see Appendix A.

An identity which is useful for simplifying the limits considered here is the following, for $\mathrm{r} = (r_1,r_2,\ldots,r_N)$:

$$\chi_{\mathrm{r}}^{(N)}(\delta^{-2/m}\hat{u},\bar{y}) \sim \delta^{-2/m(r_1+\cdots+r_m)}s_{\hat{r}}(\hat{u})\chi_{\bar{\mathrm{r}}}^{(N-m)}(\bar{y}),$$

$$\hat{\mathrm{r}} = (r_1,\ldots,r_m), \quad \bar{\mathrm{r}} = (r_{m+1},\ldots,r_N), \tag{5.3}$$

giving the leading behavior as $\delta \to 0$, where $s_{\hat{r}}(\hat{u})$ is a Schur polynomial,

$$s_{\hat{r}}(\hat{u}) = \det[\hat{u}_i^{r_j+n-j}]/\Delta(\hat{u}), \quad s_{(r,\ldots,r)}(\hat{u}) = \prod_{i=1}^{m}\hat{u}_i^r. \tag{5.4}$$

The identity (5.3) may be obtained by noting that, for small δ,

$$C_{\mathrm{r}}^{(N)}(\delta^{-2/m}\hat{u},\bar{y}) \sim \bar{C}_{\hat{r}}^{(m)}(\delta^{-2/m}\hat{u})C_{\bar{\mathrm{r}}}^{(N-m)}(\bar{y}), \quad \widetilde{C}_{\hat{r}}^{(m)}(\hat{u}) = \prod_{i=1}^{m}\hat{u}_i^{\hat{r}_i+m-i}/\Delta(\hat{u}),$$

$$\mathfrak{W}^{\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}} C_{\mathrm{r}}^{(N)}(\delta^{-2/m}\hat{u}, \overline{y}) \sim \mathfrak{W}^{\mathcal{S}_m} \mathfrak{W}^{\mathcal{S}_{(N-m)} \ltimes (\mathcal{S}_2)^{N-m-1}} \widetilde{C}_{\overline{\mathrm{r}}}^{(m)}(\delta^{-2/m}\hat{u}) C_{\overline{\mathrm{r}}}^{(N-m)}(\overline{y}).$$

Here, $\widetilde{C}_{\hat{r}}^{(m)}(\hat{u})$ is equivalent to the U(m) Verma module character while $\mathfrak{W}^{\mathcal{S}_m}$ is the Weyl symmetrizer for U(m), acting on $\hat{u}$, so that for any $f(\hat{u}) = f(\hat{u}_1, \ldots, \hat{u}_m)$, $\mathfrak{W}^{\mathcal{S}_m}(\hat{u}) = \Sigma_{\sigma \in \mathcal{S}_m} f(\hat{u}_{\sigma(1)}, \ldots, \hat{u}_{\sigma(m)})$. Hence, $\mathfrak{W}^{\mathcal{S}_m} \widetilde{C}_{\hat{r}}^{(m)}(\hat{u}) = s_{\hat{r}}(\hat{u})$, the U($m$) Weyl character. Here also, $\mathfrak{W}^{\mathcal{S}_{N-m} \ltimes (\mathcal{S}_2)^{N-m-1}}$ acts on $\overline{y}$, so that $\mathfrak{W}^{\mathcal{S}_{N-m} \ltimes (\mathcal{S}_2)^{N-m-1}} C_{\overline{\mathrm{r}}}^{(N-m)}(\overline{y}) = \chi_{\overline{\mathrm{r}}}^{(N-m)}(\overline{y})$.

Using (5.3), we may obtain from (4.7) that the limit taken in (5.1) gives the following U(1) $\otimes$ SO($2N-2m$) characters, for $n \geq m$, see Appendix A:

$$\chi_{(r_1;\overline{\mathrm{r}})}^{\mathrm{U}(1) \otimes \mathrm{SO}(2N-2m)}(u, \overline{y}) = \lim_{\delta \to \infty} \chi_{(r_1;0;\mathrm{r})}^{(N,B,n)}(\delta u^{1/2}, x, (\delta^{-2}u)^{1/m}\hat{y}, \overline{y}) = u^{2r_1} \chi_{\overline{\mathrm{r}}}^{(N-m)}(\overline{y}),$$

$$\chi_{(r;r,\ldots,r,\pm r)}^{\mathrm{U}(1) \otimes \mathrm{SO}(2N-2m)}(u, \overline{y}) = \lim_{\delta \to \infty} \chi_{(r;0;r,\ldots,r,\pm r)}^{(N,B,\pm)}(\delta u^{1/2}, x, (\delta^{-2}u)^{1/m}\hat{y}, \overline{y}) = u^{2r} \chi_{(r,\ldots,r,\pm r)}^{(N-m)}(\overline{y}). \tag{5.5}$$

For $m=1$, we have in addition to (5.5) that

$$\chi_{(r_1+2;\overline{\mathrm{r}})}^{\mathrm{U}(1) \otimes \mathrm{SO}(2N-2)}(u, \overline{y}) = \lim_{\delta \to 0} \chi_{(r_1+1;0;\mathrm{r})}^{(N,\mathrm{long})}(\delta u^{1/2}, x, \delta^{-2}u, \overline{y}), \tag{5.6}$$

while other long multiplet characters, for $\Delta \geq r_1 + j + 1$, $j \neq 0$, along with semishort multiplet characters vanish, consistent with (4.33). Since, in the limit taken in (5.2), long multiplet characters vanish for $m > 1$, this provides further evidence, apart from long multiplet decomposition formulas listed in Sec. IV, that (N, B, m), $m > 1$ BPS operators must remain protected in any three dimensional superconformal field theory.

2. The U(1) sectors

Similarly, by considering

$$\chi_{(\Delta;j;r)}^{\mathcal{M}}(\delta u^{1/2}, x, (\delta^{-2}u)^{1/N}\hat{y}_1, \ldots, (\delta^{-2}u)^{1/N}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm 1/N}\hat{y}_N^{\pm 1})$$
$$= \mathrm{Tr}_{\mathcal{R}_{(\Delta;j;r)}^{\mathcal{M}}}(\delta^{2\mathcal{H}_{\pm}} u^{\mathcal{I}_{\pm}} x^{2J_3} \hat{y}_1^{H_1} \cdots \hat{y}_N^{\pm H_N}),$$

$$\mathcal{H}_{\pm} = D - \frac{1}{N}(H_1 + \cdots + H_{N-1} \pm H_N), \quad \mathcal{I}_{\pm} = D + \frac{1}{N}(H_1 + \cdots + H_{N-1} \pm H_N), \tag{5.7}$$

separately in the limit $\delta \to 0$, isolating states for which $\mathcal{H}_{\pm}$ has zero eigenvalues, only the corresponding $\frac{1}{2}$ BPS character is nonzero, giving

$$\chi_{(r)}^{\mathrm{U}(1)}(u) = \lim_{\delta \to 0} \chi_{(r;0;r,\ldots,r,\pm r)}^{(N,B,\pm)}(\delta u^{1/2}, x, (\delta^{-2}u)^{1/N}\hat{y}_1, \ldots, (\delta^{-2}u)^{1/N}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm 1/N}\hat{y}_N^{\pm 1}) = u^{2r}.$$

$$\tag{5.8}$$

B. Semishort limits
1. The U(1) $\otimes$ Osp($2N-2m$|2) sector

Considering, for $\hat{y}, \overline{y}$ as in (5.1),

$$\chi_{(\Delta;j;r)}^{\mathcal{M}}(\delta \overline{s}^{1/2}, \delta^{-1}\overline{s}^{1/2}, (\delta^{-2}u)^{1/m}\hat{y}, \overline{y}) = \mathrm{Tr}_{\mathcal{R}_{(\Delta;j;r)}^{\mathcal{M}}}(\delta^{2\mathcal{J}_m} u^{\mathcal{K}_m} \overline{s}^{\overline{D}} \hat{y}_1^{H_1} \cdots \hat{y}_m^{H_m} \overline{y}_1^{H_{m+1}} \cdots \overline{y}_{N-m}^{H_N}),$$

$$\mathcal{J}_m = D - J_3 - \frac{1}{m}\sum_{\hat{m}=1}^{m} H_m, \quad \mathcal{K}_m = \frac{1}{m}\sum_{\hat{m}=1}^{m} H_m, \quad \bar{D} = D + J_3, \tag{5.9}$$

in the limit $\delta \to 0$, again, it is clear that only those states in $\mathcal{R}_{(\Delta;j;r)}^{\mathcal{M}}$ for which the eigenvalue of $\mathcal{J}_m$ is zero contribute.

It can be shown that

$$\lim_{\delta \to 0} \chi_{(\Delta;j;r)}^{\mathcal{M}}(\delta \bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/m}, (\delta^{-2}u)^{1/m}\hat{y}, \bar{y}) = \chi_{(R;\Delta;\bar{r})}^{(\mathrm{U}(1)\otimes\mathrm{Osp}(2N-2m|2),i)}(u,\bar{s},\bar{y}) = u^R \chi_{(\bar{\Delta},\bar{r})}^{(\mathrm{Osp}(2N-2m|2),i)}(\bar{s},\bar{y}),$$
$$\tag{5.10}$$

in terms of $\mathrm{U}(1)_{\mathcal{K}_m} \otimes \mathrm{Osp}(2N-2m|2)$ characters, for appropriate i, R, $\bar{\Delta}$, see Appendix A, for notation.

Similar to above, using (4.7) with (5.3), and also

$$\chi_{2j}(\delta^{-1}\bar{s}^{1/2}) \sim \delta^{-2j}\bar{s}^{j}, \tag{5.11}$$

giving the leading behavior as $\delta \to 0$, then we have, for semishort and conserved multiplet cases, for $n \geq m$ and with $\bar{r}$ as in (5.3),

$$\chi_{(r_1+1;r_1+2j+m+1;\bar{r})}^{(\mathrm{U}(1)\otimes\mathrm{Osp}(2N-2m|2),\mathrm{long})}(u,\bar{s},\bar{y}) = \lim_{\delta \to 0} \chi_{(r_1+j+1;j;r)}^{(N,A,n)}(\delta \bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/m}\hat{y}, \bar{y})$$

$$= u^{r_1+1}\frac{\bar{s}^{r_1+2j+m+1}}{1-\bar{s}^2}\chi_{\bar{r}}^{(N-m)}(\bar{y})\prod_{\varepsilon=\pm 1}\prod_{i=1}^{N-m}(1+\bar{y}_i^{\varepsilon}\bar{s}),$$

$$\chi_{(r+1;r+2j+m+1;r,\ldots,r,\pm r)}^{(\mathrm{U}(1)\otimes\mathrm{Osp}(2N-2m|2),\mathrm{long})}(u,\bar{s},\bar{y}) = \lim_{\delta \to 0} \chi_{(r+j+1;j;r,\ldots,r,\pm r)}^{(N,A,\pm)}(\delta \bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/m}\hat{y}, \bar{y})$$

$$= u^{r+1}\frac{\bar{s}^{r+2j+m+1}}{1-\bar{s}^2}\chi_{(r,\ldots,r,\pm r)}^{(N-m)}(\bar{y})\prod_{\varepsilon=\pm 1}\prod_{i=1}^{N-m}(1+\bar{y}_i^{\varepsilon}\bar{s}),$$

$$\chi_{(1;2j+m+1;0,\ldots,0)}^{(\mathrm{U}(1)\otimes\mathrm{Osp}(2N-2m|2),\mathrm{long})}(u,\bar{s},\bar{y}) = \lim_{\delta \to 0} \chi_{(j+1;j;0,\ldots,0)}^{(N,\mathrm{cons.})}(\delta \bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/m}\hat{y}, \bar{y})$$

$$= u\frac{\bar{s}^{2j+m+1}}{1-\bar{s}^2}\prod_{\varepsilon=\pm 1}\prod_{i=1}^{N-m}(1+\bar{y}_i^{\varepsilon}\bar{s}). \tag{5.12}$$

For (N,B,n) BPS multiplet characters there are a number of cases to consider. For $n < m$ we have, for $r_n = \cdots = r_1$ and $r_{n+1} = \cdots = r_m = r_1 - 1 \geq r_{m+1} \geq \cdots \geq |r_N|$,

$$\chi_{(r_1;r_1+m-n;\bar{r})}^{(\mathrm{U}(1)\otimes\mathrm{Osp}(2N-2m|2),\mathrm{long})}(u,\bar{s},\bar{y}) = \lim_{\delta \to 0} \chi_{(r_1;0;r)}^{(N,B,n)}(\delta \bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/m}\hat{y}, \bar{y})$$

$$= u^{r_1}\frac{\bar{s}^{r_1+m-n}}{1-\bar{s}^2}\chi_{\bar{r}}^{(N-m)}(\bar{y})\prod_{\varepsilon=\pm 1}\prod_{i=1}^{N-m}(1+\bar{y}_i^{\varepsilon}\bar{s}), \tag{5.13}$$

while for $n = m$, so that $r_1 = \cdots = r_n > r_{n+1} \geq \cdots \geq |r_N|$,

$$\chi_{(r_1;r_1;\bar{r})}^{(\mathrm{U}(1)\otimes\mathrm{Osp}(2N-2m|2),\mathrm{long})}(u,\bar{s},\bar{y}) = \lim_{\delta \to 0} \chi_{(r_1;0;r)}^{(N,B,m)}(\delta \bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/m}\hat{y}, \bar{y})$$

$$= u^{r_1}\frac{\bar{s}^{r_1}}{1-\bar{s}^2}\chi_{\bar{r}}^{(N-m)}(\bar{y})\prod_{\varepsilon=\pm 1}\prod_{i=1}^{N-m}(1+\bar{y}_i^{\varepsilon}\bar{s}). \tag{5.14}$$

For $n > m$ we have, using results from Appendix A,

$$\chi_{(r_1;r_1;\bar{r})}^{(U(1)\otimes Osp(2N-2m|2),n)}(u,\bar{s},\bar{y}) = \lim_{\delta\to 0}\chi_{(r_1;0;r)}^{(N,B,n)}(\delta\bar{s}^{1/2},\delta^{-1}\bar{s}^{1/2},(\delta^{-2}u)^{1/m}\hat{y},\bar{y})$$

$$= u^{r_1}\frac{\bar{s}^{r_1}}{1-\bar{s}^2}\mathfrak{W}^{\mathcal{S}_{N-m}\ltimes(S_2)^{N-m-1}}\left(C_{\bar{r}}^{(N-m)}(\bar{y})\prod_{i=n-m+1}^{N-m}(1+\bar{y}_i\bar{s})\prod_{i=1}^{N-m}(1+\bar{y}_i^{-1}\bar{s})\right),$$

(5.15)

so that

$$\chi_{(r_1;r_1;\bar{r})}^{(U(1)\otimes Osp(2N-2m|2),n)}(u,\bar{s},\bar{y}) = u^{r_1}\frac{\bar{s}^{r_1}}{1-\bar{s}^2}\sum_{0\leqslant a_1\leqslant\cdots\leqslant a_{n-m}\leqslant 1}\sum_{\substack{a_{n-m+1},\ldots,a_{N-m}=0 \\ a_{n-m+1},\ldots,a_{N-m}}}^{1} s^{a_1+\cdots+a_{N-m}+\bar{a}_{n-m+1}+\cdots+\bar{a}_{N-m}}$$

$$\times \chi_{(r_1-a_1,\ldots,r_1-a_{n-m},r_{n+1}+\bar{a}_{n-m+1}-a_{n-m+1},\ldots,r_N+\bar{a}_{N-m}-a_{N-m})}^{(N-m)}(\bar{y}). \qquad (5.16)$$

Finally, for the $\frac{1}{2}$ BPS cases,

$$\chi_{(r;r;r,\ldots,\bar{r},\pm r)}^{(U(1)\otimes Osp(2N-2m|2),\pm 1)}(u,\bar{s},\bar{y}) = \lim_{\delta\to\infty}\chi_{(r;0;r,\ldots,r,\pm r)}^{(N,B,\pm)}(\delta\bar{s}^{1/2},\delta^{-1}\bar{s}^{1/2},(\delta^{-2}u)^{1/m}\hat{y},\bar{y})$$

$$= u^r\frac{\bar{s}^r}{1-\bar{s}^2}\sum_{0\leqslant a_1\leqslant\cdots\leqslant a_{N-m}\leqslant 1}\bar{s}^{a_1+\cdots+a_{N-m}}\chi_{(r-a_1,\ldots,r-a_{N-m-1},\pm r\mp a_{N-m})}^{(N-m)}(\bar{y}).$$

(5.17)

In particular, this implies

$$\chi_{(1/2;1/2;1/2,\ldots,1/2,\pm 1/2)}^{(U(1)\otimes Osp(2N-2m|2),\pm)}(u,\bar{s},\bar{y}) = \lim_{\delta\to 0}\chi_{(1/2;0;1/2,\ldots,1/2,\pm 1/2)}^{(N,B,\pm 1)}(\delta\bar{s}^{1/2},\delta^{-1}\bar{s}^{1/2},(\delta^{-2}u)^{1/m}\hat{y},\bar{y})$$

$$= u^{1/2}\frac{\bar{s}^{1/2}}{1-\bar{s}^2}(\chi_{(1/2,\ldots,1/2,\pm 1/2)}^{(N-m)}(\bar{y})+\bar{s}\chi_{(1/2,\ldots,1/2,\mp,1/2)}^{(N-m)}(\bar{y})), \quad (5.18)$$

agreeing with (4.18), in this limit, using (5.3) with (5.4) for $r=\frac{1}{2}$.

For $m>1$, we have for long multiplet characters,

$$\chi_{(r_1+1;r_1+2j+m+1;r_{m+1},\ldots,r_N)}^{(U(1)\otimes Osp(2N-2m|2),\text{long})}(u,\bar{s},\bar{y}) = \lim_{\delta\to 0}\chi_{(r_1+j+1;j;r_1,\ldots,r_1,r_{m+1},\ldots,r_N)}^{(N,\text{long})}(\delta\bar{s}^{1/2},\delta^{-1}\bar{s}^{1/2},(\delta^{-2}u)^{1/m}\hat{y},\bar{y}),$$

(5.19)

which is in accord with (4.26) with (4.27). For $m=1$, we have that

$$(1+u)\chi_{(r_1+1;r_1+2j+2;r_2,\ldots,r_N)}^{(U(1)\otimes Osp(2N-2|2),\text{long})}(u,\bar{s},\bar{y}) = \lim_{\delta\to 0}\chi_{(r_1+j+1;j;r_1,\ldots,r_N)}^{(N,\text{long})}(\delta\bar{s}^{1/2},\delta^{-1}\bar{s}^{1/2},\delta^{-2}u,\bar{y}), \quad (5.20)$$

compatible with (4.26).

2. The $\mathbf{U(1)\otimes SU(1,1)}$ sectors

Similarly, by considering,

$$\chi_{(\Delta;j;r)}^{\mathcal{M}}(\delta\bar{s}^{1/2},\delta^{-1}\bar{s}^{1/2},(\delta^{-2}u)^{1/N}\hat{y}_1,\ldots,(\delta^{-2}u)^{1/N}\hat{y}_{N-1},(\delta^{-2}u)^{\pm 1/N}\hat{y}_N^{\pm 1})$$

$$= \mathrm{Tr}_{\mathcal{R}_{(\Delta;j;r)}^{\mathcal{M}}}(\delta^{2\mathcal{J}_\pm}u^{\mathcal{K}_\pm}\bar{s}^{\bar{D}}\hat{y}_1^{H_1}\cdots\hat{y}_N^{\pm H_N}),$$

$$\mathcal{J}_\pm = D - J_3 - \frac{1}{N}(H_1+\cdots+H_{N-1}\pm H_N), \quad \mathcal{K}_\pm = \frac{1}{N}(H_1+\cdots+H_{N-1}\pm H_N), \qquad (5.21)$$

separately in the limit $\delta\to 0$, we have for the corresponding $(N,A,\pm)$ multiplet character,

$$\chi^{\mathrm{U}(1)\otimes\mathrm{SU}(1,1)}_{(r+1;r+2j+1+N)}(u,\bar{s})$$

$$= \lim_{\delta\to 0} \chi^{(N,A,\pm)}_{(r+j+1;j;r,\dots,r,\pm r)}(\delta\bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/N}\hat{y}_1, \dots, (\delta^{-2}u)^{1/N}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm 1/N}\hat{y}_N^{\pm 1})$$

$$= u^{r+1}\frac{\bar{s}^{r+2j+N+1}}{1-\bar{s}^2}, \tag{5.22}$$

and, for the conserved current character, in either limit in (5.21),

$$\chi^{\mathrm{U}(1)\otimes\mathrm{SU}(1,1)}_{(1;2j+1+N)}(u,\bar{s}) = \lim_{\delta\to 0} \chi^{(N,\mathrm{cons.})}_{(j+1;j;0,\dots,0)}(\delta\bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/N}\hat{y}_1, \dots, (\delta^{-2}u)^{1/N}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm 1/N}\hat{y}_N^{\pm 1})$$

$$= u\frac{\bar{s}^{2j+N+1}}{1-\bar{s}^2}. \tag{5.23}$$

For (N,B,n), $n\leq N-1$, BPS characters with $r_n=\cdots=r_1$ and $r_{n+1}=\cdots=r_N=\pm r_1\pm 1$ and taking corresponding limit in (5.21), then

$$\chi^{\mathrm{U}(1)\otimes\mathrm{SU}(1,1)}_{(r_1;r_1+N-n)}(u,\bar{s}) = \lim_{\delta\to 0} \chi^{(N,B,n)}_{(r_1;0;r)}(\delta\bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/N}\hat{y}_1, \dots, (\delta^{-2}u)^{1/N}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm 1/N}\hat{y}_N^{\pm 1})$$

$$= u^{r_1}\frac{\bar{s}^{r_1+N-n}}{1-\bar{s}^2}. \tag{5.24}$$

For the $(N,B,\pm)$ half BPS cases and in the corresponding limit in (5.21), we have

$$\chi^{\mathrm{U}(1)\otimes\mathrm{SU}(1,1)}_{(r;r)}(u,\bar{s}) = \lim_{\delta\to 0} \chi^{(N,B,n)}_{(r;0;r,\dots,r,\pm r)}(\delta\bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/N}\hat{y}_1, \dots, (\delta^{-2}u)^{1/N}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm 1/N}\hat{y}_N^{\pm 1})$$

$$= u^r\frac{\bar{s}^r}{1-\bar{s}^2}, \tag{5.25}$$

along with

$$\chi^{\mathrm{U}(1)\otimes\mathrm{SU}(1,1)}_{(1/2;3/2)}(u,\bar{s})$$

$$= \lim_{\delta\to\infty} \chi^{(N,B,n)}_{(1/2;0;1/2,\dots,1/2,\mp 1/2)}(\delta\bar{s}^{1/2}, \delta^{-1}\bar{s}^{1/2}, (\delta^{-2}u)^{1/N}\hat{y}_1, \dots, (\delta^{-2}u)^{1/N}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm 1/N}\hat{y}_N^{\pm 1})$$

$$= u^{1/2}\frac{\bar{s}^{3/2}}{1-\bar{s}^2}. \tag{5.26}$$

All other characters, apart from the long multiplet one corresponding to (4.35), in the limit (5.21) vanish.

C. The superconformal index

The (single particle) superconformal indices[10,26] may be computed by taking the limit $u\to -1$ in the $\mathrm{U}(1)\otimes\mathrm{Osp}(2N-2|2)$ characters above, i.e., for $m=1$ in the semishort limits above.[27]

In particular, from (5.20), this limit ensures that $\mathrm{Osp}(2N|4)$ long multiplet characters vanish, and hence do not contribute in the decomposition of partition functions, in the same limit, in terms of $\mathrm{Osp}(2N|4)$ characters. Thus, partition functions evaluated in this limit receive contributions from protected operators only. It should be noted, however, that the magnitude of the numbers obtained for counting operators by expansion in terms of characters, in this limit, provide only a lower bound on the numbers of actual protected operators due to the $(-1)^F$ sign in the index, for fermion number F.

VI. PARTITION FUNCTIONS FOR $\mathcal{N}=6$ SUPERCONFORMAL CHERN–SIMONS THEORY

In Ref. 2, a class of $\mathcal{N}=6$ superconformal Chern–Simons theories, with gauge group $U(n)$ $\times U(n)$, was proposed. For levels $\pm k$ in the Chern–Simons terms, these theories admit a dual description in terms of M theory compactified on $AdS_4 \times S^7/\mathbb{Z}_k$.

Here, the free field partition function of the theory, where the effective 't Hooft coupling $n/k \rightarrow 0$, so that $k \rightarrow \infty$, in the large $n < k$ limit, is computed using appropriate superconformal characters.

The supergravity partition function, obtained using the duality proposed in Ref. 2 in the $n/k \rightarrow \infty$ limit, for $n > k \rightarrow \infty$, is also computed using appropriate superconformal characters.

A. Free field theory

For $\mathcal{N}=6 \, U(n) \times U(n)$ superconformal Chern–Simons theory, the dynamical field content consists of free scalars ϕ_i, $\tilde{\phi}_i$, $i=1,\ldots,4$, and free spin half fermions $\psi_{i\alpha}$, $\tilde{\psi}_{i\alpha}$, $\alpha=1,2$, where ϕ_i, $\tilde{\psi}_{i\alpha}$ belong to the fundamental representation of $SO(6) \simeq SU(4)$, with $\tilde{\phi}_i$, $\psi_{i\alpha}$ belonging to the antifundamental, while ϕ_i, $\psi_{i\alpha}$ transform in the $(n,\bar{n})$ bifundamental representation of $U(n) \times U(n)$, with $\tilde{\phi}_i$, $\tilde{\psi}_{i\alpha}$ transforming in the $(\bar{n},n)$ representation. Under the action of $SU(6)$ lowering operators,

$$\phi_1 \rightarrow \phi_2 \rightarrow \phi_3 \rightarrow \phi_4, \quad \psi_{1\alpha} \rightarrow \psi_{2\alpha} \rightarrow \psi_{3\alpha} \rightarrow \psi_{4\alpha},$$

$$\tilde{\phi}_1 \rightarrow \tilde{\phi}_2 \rightarrow \tilde{\phi}_3 \rightarrow \tilde{\phi}_4, \quad \tilde{\psi}_{1\alpha} \rightarrow \tilde{\psi}_{2\alpha} \rightarrow \tilde{\psi}_{3\alpha} \rightarrow \tilde{\psi}_{4\alpha}, \tag{6.1}$$

where $\psi_{j1} \rightarrow \psi_{j2}$, $\tilde{\psi}_{j1} \rightarrow \tilde{\psi}_{j2}$ under the action of the $SU(2)$ lowering operator. The detailed fields are given below.

Here the $SU(6)$ orthonormal basis labels (r,q,p) are related to $SU(4)$ Dynkin labels $[a,b,c]$ by

$$(r,q,p) \rightarrow [q+p, r-q, q-p], \tag{6.2}$$

so that $\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) \rightarrow [1,0,0]$, $\left(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right) \rightarrow [0,0,1]$.

Notice that the derivatives give rise to a factor $P(s,x)$ as in (4.6) multiplying the character for each conformal primary field in Table II, however, the condition that the fields be free gives rise to an effective factor $(1-s^4)$ for the scalar cases and $(1-s^2)$ for the fermion cases so that the conformal group character contributions are given, respectively, by the "Rac" and "Di" characters given in (4.19). [See Ref. 20 for details of the derivation of $SO(d,2)$ conformal group characters.]

As may be evident from (4.18), ϕ and ψ, respectively, $\tilde{\phi}$ and $\tilde{\psi}$, belong to the half BPS multiplet $(3,B,+)$, respectively, $(3,B,-)$. [The scalar ϕ_1, respectively, $\tilde{\phi}_1$, is the primary field of the $(3,B,+)$, respectively, $(3,B,-)$, multiplet.]

For free field theory, the single particle partition function is given by

$$Y_{\text{free}}(s,x,y,u,v) = \text{Tr}(s^{2D}x^{2J_3}y_1^{H_1}y_2^{H_2}y_3^{H_3}u_1^{L_1}\cdots u_n^{L_n}v_1^{M_1}\cdots v_n^{M_n}), \tag{6.3}$$

where the trace is over states corresponding to ϕ_1, $\tilde{\phi}_1$, and all their superconformal descendants, of the form (3.3), and s, x, y, u, v are "fugacities" with $L_1,\ldots,L_n$, $M_1,\ldots,M_n$ being usual $U(n)$ $\times U(n)$ Cartan subalgebra generators.

Thus we may write, in terms of characters,

$$Y_{\text{free}}(s,x,y,u,v) = f_+(s,x,y)p_1(u)p_1(v^{-1}) + f_-(s,x,y)p_1(u^{-1})p_1(v), \tag{6.4}$$

defining, for subsequent use,

TABLE II. Dynamical field content

Field	$(\Delta;j;r_1,r_2,r_3)$	$s^{2\Delta}x^{2j_3}y_1^{H_1}y_2^{H_2}y_3^{H_3}$	$U(n)\times U(n)$
ϕ_1	$(\frac{1}{2};0;\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$sy_1^{1/2}y_2^{1/2}y_3^{1/2}$	$(n,\bar{n})$
ϕ_2	$(\frac{1}{2};0;\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$sy_1^{1/2}y_2^{-1/2}y_3^{-1/2}$	$(n,\bar{n})$
ϕ_3	$(\frac{1}{2};0;-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$sy_1^{-1/2}y_2^{1/2}y_3^{-1/2}$	$(n,\bar{n})$
ϕ_4	$(\frac{1}{2};0;-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$sy_1^{-1/2}y_2^{-1/2}y_3^{1/2}$	$(n,\bar{n})$
$\psi_{1\alpha}$	$(1;\pm\frac{1}{2};\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$s^2x^{\pm1}y_1^{1/2}y_2^{1/2}y_3^{-1/2}$	$(n,\bar{n})$
$\psi_{2\alpha}$	$(1;\pm\frac{1}{2};\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$s^2x^{\pm1}y_1^{1/2}y_2^{-1/2}y_3^{1/2}$	$(n,\bar{n})$
$\psi_{3\alpha}$	$(1;\pm\frac{1}{2};-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$s^2x^{\pm1}y_1^{-1/2}y_2^{1/2}y_3^{1/2}$	$(n,\bar{n})$
$\psi_{4\alpha}$	$(1;\pm\frac{1}{2};-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$s^2x^{\pm1}y_1^{-1/2}y_2^{-1/2}y_3^{-1/2}$	$(n,\bar{n})$
$\widetilde{\phi}_1$	$(\frac{1}{2};0;\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$sy_1^{1/2}y_2^{1/2}y_3^{-1/2}$	$(\bar{n},n)$
$\widetilde{\phi}_2$	$(\frac{1}{2};0;\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$sy_1^{1/2}y_2^{-1/2}y_3^{1/2}$	$(\bar{n},n)$
$\widetilde{\phi}_3$	$(\frac{1}{2};0;-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$sy_1^{-1/2}y_2^{1/2}y_3^{1/2}$	$(\bar{n},n)$
$\widetilde{\phi}_4$	$(\frac{1}{2};0;-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$sy_1^{-1/2}y_2^{-1/2}y_3^{-1/2}$	$(\bar{n},n)$
$\widetilde{\psi}_{1\alpha}$	$(1;\pm\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$s^2x^{\pm1}y_1^{1/2}y_2^{1/2}y_3^{1/2}$	$(\bar{n},n)$
$\widetilde{\psi}_{2\alpha}$	$(1;\pm\frac{1}{2};\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$s^2x^{\pm1}y_1^{1/2}y_2^{-1/2}y_3^{-1/2}$	$(\bar{n},n)$
$\widetilde{\psi}_{3\alpha}$	$(1;\pm\frac{1}{2};-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$s^2x^{\pm1}y_1^{-1/2}y_2^{1/2}y_3^{-1/2}$	$(\bar{n},n)$
$\widetilde{\psi}_{4\alpha}$	$(1;\pm\frac{1}{2};-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	$s^2x^{\pm1}y_1^{-1/2}y_2^{-1/2}y_3^{1/2}$	$(\bar{n},n)$
∂_{11}	$(1;1;0,0,0)$	s^2x^2	$(1,1)$
∂_{12}	$(1;0;0,0,0)$	s^2	$(1,1)$
∂_{22}	$(1;-1;0,0,0)$	s^2x^{-2}	$(1,1)$

$$p_j(\mathrm{u}) = \sum_{i=1}^{n} u_i^j, \quad p_j(\mathrm{u}^{-1}) = p_{-j}(\mathrm{u}), \tag{6.5}$$

so that $p_1(\mathrm{u})p_1(\mathrm{v}^{-1})$, $p_1(\mathrm{u}^{-1})p_1(\mathrm{v})$ correspond to the characters for the $(n,\bar{n})$, respectively, $(\bar{n},n)$ representations of $U(n)\times U(n)$, and where

$$f_{\pm}(s,x,y) = \chi^{(3,B,\pm)}_{(1/2;0;1/2,1/2,\pm1/2)}(s,x,y) = (y_1y_2y_3)^{\pm1/2}\left(\sum_{j=1}^{3} y_j^{+1} + (y_1y_2y_3)^{+1}\right)\mathcal{D}_{\mathrm{Rac}}(s,x)$$

$$+ (y_1y_2y_3)^{\pm1/2}\left(\sum_{j=1}^{3} y_j^{\mp1} + (y_1y_2y_3)^{\mp1}\right)\mathcal{D}_{\mathrm{Di}}(s,x), \tag{6.6}$$

being half BPS characters (4.18) for SU(6) R-symmetry.

The multiparticle partition function, receiving contributions from only gauge invariant operators, is given by the usual integral over the gauge group, namely,

$$Z^{(n)}_{\mathrm{free}}(s,x,y) = \int_{U(n)} d\mu(\mathrm{u})\int_{U(n)} d\mu(\mathrm{v})\exp\left(\sum_{j=1}^{\infty}\frac{1}{j}Y_{\mathrm{free}}(s^j,(-1)^{j+1}x^j,y^j,\mathrm{u}^j,\mathrm{v}^j)\right), \tag{6.7}$$

where the signs on x take account of particle statistics.

The integral may be evaluated in the large n limit by using a method described in Ref. 28 (see also Ref. 29 for other applications). We may first expand

$$Z^{(n)}_{\mathrm{free}}(s,x,y) = \sum_{\underline{\lambda},\underline{\rho}}\frac{1}{z_{\underline{\lambda}}z_{\underline{\rho}}}f_{+\underline{\lambda}}(s,x,y)f_{-\underline{\rho}}(s,x,y)\int_{U(n)}d\mu(\mathrm{u})\int_{U(n)}d\mu(\mathrm{v})p_{\underline{\lambda}}(\mathrm{u})p_{\underline{\rho}}(\mathrm{u}^{-1})p_{\underline{\rho}}(\mathrm{v})p_{\underline{\lambda}}(\mathrm{v}^{-1}), \tag{6.8}$$

in terms of partitions,

$$\underline{\lambda} = (\lambda_1, \ldots, \lambda_j, \ldots), \quad \sum_{j \geq 1} j\lambda_j = |\underline{\lambda}| \in \mathbb{N},$$

$$\underline{\rho} = (\rho_1, \ldots, \rho_j, \ldots), \quad \sum_{j \geq 1} j\rho_j = |\underline{\rho}| \in \mathbb{N}, \tag{6.9}$$

where for $\underline{\sigma} = (\sigma_1, \ldots, \sigma_j, \ldots)$,

$$z_{\underline{\sigma}} = \prod_{j \geq 1} \sigma_j! j^{\sigma_j}, \quad f_{\pm\underline{\sigma}}(s,x,y) = \prod_{j \geq 1} f_\pm(s^j, (-1)^{j+1}x^j, y^j)^{\sigma_j}, \quad p_{\underline{\sigma}}(x) = \prod_{j \geq 1} p_j(x)^{\sigma_j}. \tag{6.10}$$

Using the orthogonality relation for power symmetric polynomials,

$$\int_{\mathrm{U}(n)} d\mu(\mathrm{u}) p_{\underline{\lambda}}(\mathrm{u}) p_{\underline{\rho}}(\mathrm{u}^{-1}) = z_{\underline{\lambda}} \delta_{\underline{\lambda}\underline{\rho}}, \quad |\underline{\lambda}|, |\underline{\rho}| \leq n, \tag{6.11}$$

then we trivially obtain in the large n limit,

$$Z_{\mathrm{free}}^{(n)}(s,x,y) \sim Z_{\mathrm{free}}(s,x,y) = \sum_{\underline{\lambda}} f_{+\underline{\lambda}}(s,x,y) f_{-\underline{\lambda}}(s,x,y) = \prod_{j \geq 1} \sum_{\lambda_j \geq 0} (f_+(s^j, (-1)^{j+1}x^j, y^j) f_-(s^j, ($$

$$-1)^{j+1}x^j, y^j))^{\lambda_j} = \prod_{j \geq 1} \frac{1}{1 - f_+(s^j, (-1)^{j+1}x^j, y^j) f_-(s^j, (-1)^{j+1}x^j, y^j)}. \tag{6.12}$$

This result was also derived in Ref. 6 by using saddle point methods, in the context of showing superconformal index matching.

B. Supergravity limit

In the strong coupling limit, large n/k, $n > k \to \infty$, as explained in Ref. 2, using results of Ref. 30, the single particle states (gravitons) effectively belong to scalar superconformal representations with conformal dimensions r and in the $\mathcal{R}_{(r,r,0)}$ SU(6) representations, for $r \in \mathbb{N}$, $r > 0$, so that, in terms of notation here, they belong to $(3, B, 2)$ BPS multiplets.

Taking account of superconformal descendants, we may thus write for the single particle partition function,

$$Y_{\mathrm{sugra}}(s,x,y) = \sum_{r=1}^\infty \chi_{(r;0;r,r,0)}^{(3,B,2)}(s,x,y), \tag{6.13}$$

where using (4.7), for $\mathfrak{W}^{\mathcal{S}_3 \times (\mathcal{S}_2)^2}$ acting on $y = (y_1, y_2, y_3)$,

$$\chi_{(r;0;r,r,0)}^{(3,B,2)}(s,x,y) = s^{2r} P(s,x) \mathfrak{W}^{\mathcal{S}_3 \ltimes (\mathcal{S}_2)^2} \left(C_{(r,r,0)}^{(3)}(y) \prod_{\varepsilon = \pm 1} (1 + s y_3 x^\varepsilon) \prod_{j=1}^3 (1 + s y_i^{-1} x^\varepsilon) \right). \tag{6.14}$$

Thus, using the definition of the SU(6) Verma module character (4.2), for $N = 3$, and summing over r in (6.13) with (6.14), we may write

$$Y_{\mathrm{sugra}}(s,x,y) = s^2 P(s,x) \mathfrak{W}^{\mathcal{S}_3 \ltimes (\mathcal{S}_2)^2} \left(\frac{y_1^{\,3} y_2^{\,2} \prod_{\varepsilon = \pm 1} (1 + s y_3 x^\varepsilon) \prod_{i=1}^3 (1 + s y_i^{-1} x^\varepsilon)}{(1 - s^2 y_1 y_2) \prod_{1 \leq j < k \leq 3} (y_j^{-1} - y_k^{-1})(1 - y_j y_k)} \right). \tag{6.15}$$

Simplifying the $\mathcal{S}_3$ part of the action of $\mathfrak{W}^{\mathcal{S}_3 \times (\mathcal{S}_2)^2}$ in the latter we end up with the more succinct formula,

$$Y_{\text{sugra}}(s,x,\text{y}) = f(s,x,y_1,y_2,y_3) + f\left(s,x,\frac{1}{y_1},\frac{1}{y_2},y_3\right) + f\left(s,x,\frac{1}{y_1},y_2,\frac{1}{y_3}\right) + f\left(s,x,y_1,\frac{1}{y_2},\frac{1}{y_3}\right) - 1,$$

(6.16)

where

$$f(s,x,y_1,y_2,y_3) = P(s,x)\frac{\prod_{\varepsilon=\pm 1}(1 + s^3 y_1 y_2 y_3 x^\varepsilon)\prod_{i=1}^{3}(1 + sy_i x^\varepsilon)}{\prod_{1\leqslant j<k\leqslant 3}(1 - s^2 y_j y_k)(1 - y_j^{-1} y_k^{-1})}.$$

(6.17)

We then have the multiparticle (free graviton gas) partition function given by the usual plethystic exponential, also taking into account particle statistics,

$$Z_{\text{sugra}}(s,x,\text{y}) = \exp\left(\sum_{j=1}^{\infty}\frac{1}{j}Y_{\text{sugra}}(s^j,(-1)^{j+1}x^j,y_1^{\ j},y_2^{\ j},y_3^{\ j})\right).$$

(6.18)

VII. COUNTING MULTITRACE GAUGE INVARIANT OPERATORS

The limits in $\text{Osp}(2N|4)$ characters discussed in Sec. V, giving reductions to subalgebra characters, are equivalent to decoupling limits that isolate sectors of operators in partition functions. By taking such limits, for $N=3$, in (6.12) and (6.18), and decomposing in terms of characters in the same limits, we are able to count corresponding multitrace gauge invariant operators in the free field, supergravity limits. For free field theory, the counting numbers obtained provide an upper bound on the numbers of protected operators in the interacting theory, while for the supergravity limit they are expected to count actually protected ones.

For $f_\pm(s,x,\text{y}) \rightarrow f_\pm^G(u,\bar{\text{y}})$, $Y_{\text{sugra}}(s,x,\text{y}) \rightarrow Y_{\text{sugra}}^G(u,\bar{\text{y}})$, in the BPS limits, where G is the corresponding subgroup, and $f_\pm(s,x,\text{y}) \rightarrow f_\pm^H(u,\bar{s},\bar{\text{y}})$, $Y_{\text{sugra}}(s,x,\text{y}) \rightarrow Y_{\text{sugra}}^H(u,\bar{s},\bar{\text{y}})$, in the semishort limits, where H is the corresponding subgroup, the limits generically give

$$f_\pm^G(u,\bar{\text{y}}) = \chi^{(3,B,\pm)}_{(1/2;0;1/2,1/2,\pm 1/2)}(u,\bar{\text{y}}), \quad Y_{\text{sugra}}^G(u,\bar{\text{y}}) = \sum_{r=1}^{\infty}\chi^{(3,B,2)}_{(r;0;r,r,0)}(u,\bar{\text{y}}),$$

$$f_\pm^H(u,\bar{s},\bar{\text{y}}) = \chi^{(3,B,\pm)}_{(1/2;0;1/2,1/2,\pm 1/2)}(u,\bar{s},\bar{\text{y}}), \quad Y_{\text{sugra}}^H(u,\bar{s},\bar{\text{y}}) = \sum_{r=1}^{\infty}\chi^{(3,B,2)}_{(r;0;r,r,0)}(u,\bar{s},\bar{\text{y}}),$$

(7.1)

where, for the superconformal characters,

$$\chi^{\mathcal{M}}_{(\Delta;j;\text{r})}(s,x,\text{y}) \rightarrow \chi^{\mathcal{M}}_{(\Delta;j;\text{r})}(u,\bar{\text{y}}), \quad \chi^{\mathcal{M}}_{(\Delta;j;\text{r})}(s,x,\text{y}) \rightarrow \chi^{\mathcal{M}}_{(\Delta;j;\text{r})}(u,\bar{s},\bar{\text{y}}),$$

(7.2)

in the same limits. An important point for the corresponding multiparticle partition functions is properly taking into account particle statistics which differs in both cases. For $Z_{\text{free}}(s,x,\text{y}) \rightarrow Z_{\text{free}}^G(u,\bar{\text{y}})$, $Z_{\text{sugra}}(s,x,\text{y}) \rightarrow Z_{\text{sugra}}^G(u,\bar{\text{y}})$, in the BPS limits and $Z_{\text{free}}(s,x,\text{y}) \rightarrow Z_{\text{free}}^H(u,\bar{s},\bar{\text{y}})$, $Z_{\text{sugra}}(s,x,\text{y}) \rightarrow Z_{\text{sugra}}^H(u,\bar{s},\bar{\text{y}})$, in the semishort limits, consistency requires[1,31]

$$Z_{\text{free}}^G(u,\bar{\text{y}}) = \prod_{j=1}^{\infty}\frac{1}{1 - f_+^G(u^j,\bar{\text{y}}^j)f_-^G(u^j\bar{\text{y}}^j)},$$

$$Z_{\text{sugra}}^G(u,\bar{\text{y}}) = \exp\left(\sum_{j=1}^{\infty}\frac{1}{j}Y_{\text{sugra}}^G(u^j,\bar{\text{y}}^j)\right),$$

$$Z_{\text{free}}^H(u,\overline{s},\overline{y}) = \prod_{j=1}^{\infty} \left. \frac{1}{1 - f_+^H(u^j\alpha_j, \overline{s}^j/\alpha_j, \overline{y}^j) f^H(u^j\alpha_j, \overline{s}^j/\alpha_j, \overline{y}^j)} \right|_{\alpha_j = (-1)^{j+1}},$$

$$Z_{\text{sugra}}^H(u,\overline{s},\overline{y}) = \exp\left(\sum_{j=1}^{\infty} \frac{1}{j} Y_{\text{sugra}}^H(u^j\alpha_j, \overline{s}^j/\alpha_j, \overline{y}^j) \bigg|_{\alpha_j = (-1)^{j+1}} \right). \tag{7.3}$$

A. BPS cases

1. The U(1) sectors

Corresponding to (5.8) we consider the $(3,B,+)$ limit, whereby

$$\chi_{(r;0;r,r,\pm r)}^{(3,B,+)}(u) = \chi_{(r)}^{U(1)}(u) = u^{2r}, \tag{7.4}$$

so that $f_+^{U(1)}(u) = u$, $f_-^{U(1)}(u) = 0$. The set of fields that contribute from Table II is

$$(\phi_1)_i^{\overline{\iota}}, \tag{7.5}$$

where $i, \overline{\iota} = 1, \ldots, n$ are $U(n) \times U(n)(n, \overline{n})$ bifundamental indices. It is also easily seen that $Y_{\text{sugra}}^{U(1)}(u) = 0$ in the limit (5.8). Thus, from (7.3),

$$Z_{\text{free}}^{U(1)}(u) = Z_{\text{sugra}}^{U(1)}(u) = 1, \tag{7.6}$$

so that, clearly, there are no $(3,B,+)$ BPS gauge invariant operators in the free field or supergravity spectrum, apart from the identity operator. Of course this case is trivial since there is no way of forming $SU(n) \times SU(n)$ gauge invariants involving solely $\phi_{1i}{}_i^{\overline{\iota}}$.

The same result applies to $(3,B,-)$ BPS operators.

2. The U(1)⊗U(1) sector

Corresponding to (5.5) for $N=3$, $m=2$, we have that

$$\chi_{(r;0;r,r,q)}^{(3,B,2)}(u,y) = \chi_{(r;q)}^{U(1)\otimes U(1)}(u,y) = u^{2r}y^q,$$

$$\chi_{(r;0;r,r,\pm r)}^{(3,B,\pm)}(u,y) = \chi_{(r;\pm r)}^{U(1)\otimes U(1)}(u,y) = u^{2r}y^{\pm r}, \tag{7.7}$$

so that $f_\pm^{U(1)\otimes U(1)}(u,y) = uy^{\pm 1/2}$.

In this case the set of fundamental fields that contribute from Table II is

$$(\phi_1)_i^{\overline{\iota}}, \quad (\tilde{\phi}_1)_{\overline{\jmath}}^j, \tag{7.8}$$

where $j, \overline{\jmath} = 1, \ldots, n$ are $U(n) \times U(n)(\overline{n}, n)$ bifundamental indices.

We also have that

$$Y_{\text{sugra}}^{U(1)\otimes U(1)}(u,y) = \sum_{r=1}^{\infty} u^{2r} = \frac{u^2}{1 - u^2}. \tag{7.9}$$

Thus, from (7.3),

$$Z_{\text{free}}^{U(1)\otimes U(1)}(u,y) = Z_{\text{sugra}}^{U(1)\otimes U(1)}(u,y) = \prod_{j \geq 1} \frac{1}{1 - u^{2j}}. \tag{7.10}$$

Thus, expanding over the characters in (7.7),

$$Z_{\text{free}}^{\text{U}(1)\otimes\text{U}(1)}(u,y) = Z_{\text{sugra}}^{\text{U}(1)\otimes\text{U}(1)}(u,y) = 1 + \sum_{r=1}^{\infty} N_r^{(3,B,2)} \chi_{(r;0;r,r,0)}^{(3,B,2)}(u,y),$$ (7.11)

where

$$N_r^{(3,B,2)} = p(r) = 1,2,3,5,\ldots, \quad r = 1,2,3,4,\ldots,$$ (7.12)

where $p(r)$ is the usual partition number for r. The agreement (7.10) is expected as $(3,B,2)$ operators are protected due to long multiplet decomposition rules discussed in Sec. IV.

This counting is simple to see in terms of (7.8) as the gauge invariants may be written as, for $n_k \in \mathbb{N}$,

$$\text{Tr}(Z^{n_1}) \cdots \text{Tr}(Z^{n_r}), \quad Z_i^{\ j} = (\phi_1)_i^{\ \bar{\iota}} (\tilde{\phi}_1)_{\bar{\iota}}^{\ j}, \quad n_1 \leqslant \cdots \leqslant n_r \leqslant r, \quad \sum_{k=1}^{r} n_k = r,$$ (7.13)

of which, for a given r, there are $p(r)$.

3. The U(1)⊗SO(4) sector

Due to (4.33), we expect disagreement between counting of $(3,B,1)$ operators in the free field and supergravity limits and thus we split the discussion here. Corresponding to (5.5) for $N=3$, $m=1$, we have that

$$\chi_{(r;0;r,q,p)}^{(3,B,1)}(u,u_+,u_-) = \chi_{(r;q,p)}^{\text{U}(1)\otimes\text{SO}(4)}(u,u_+,u_-) = u^{2r}\chi_{q+p}(u_+)\chi_{q-p}(u_-),$$

$$\chi_{(r;0;r,r,p)}^{(3,B,2)}(u,u_+,u_-) = \chi_{(r;r,p)}^{\text{U}(1)\otimes\text{SO}(4)}(u,u_+,u_-) = u^{2r}\chi_{r+p}(u_+)\chi_{r-p}(u_-),$$

$$\chi_{(r;0;r,r,\pm r)}^{(3,B,\pm)}(u,u_+,u_-) = \chi_{(r;r,\pm r)}^{\text{U}(1)\otimes\text{SO}(4)}(u,u_+,u_-) = u^{2r}\chi_{2r}(u_\pm),$$ (7.14)

in terms of SU(2) characters in (4.5), using $\text{SO}(4) \simeq \text{SU}(2)\otimes\text{SU}(2)$, so that, for SO(4) characters,

$$\chi_{(q,p)}^{(2)}(\bar{y}_1,\bar{y}_2) = \chi_{q+p}(u_+)\chi_{q-p}(u_-), \quad \bar{y}_1 = u_+ u_-, \quad \bar{y}_2 = u_+/u_-.$$ (7.15)

We thus have $f_\pm^{\text{U}(1)\otimes\text{SO}(4)}(u) = u\chi_1(u_\pm)$, where $\chi_1(u_\pm) = u_\pm + u_\pm^{-1}$.

In this case the fundamental fields that contribute from Table II are

$$(\phi_1)_i^{\ \bar{\iota}}, \quad (\phi_2)_i^{\ \bar{\iota}}, \quad (\tilde{\phi}_1)_J^{\ j}, \quad (\tilde{\phi}_2)_J^{\ j}.$$ (7.16)

Then from (7.3),

$$Z_{\text{free}}^{\text{U}(1)\otimes\text{SO}(4)}(u,u_+,u_-) = \prod_{j=1}^{\infty} \frac{1}{1 - u^{2j}\chi_1(u_+^{\ j})\chi_1(u_-^{\ j})}.$$ (7.17)

The numbers of multitrace $(3,B,1)$ BPS operators are then determined by expansions over the characters (7.14), using (7.12),

$$Z_{\text{free}}^{\text{U}(1)\otimes\text{SO}(4)}(u,u_+,u_-) = 1 + \sum_{r=1}^{\infty} p(r)\chi_{(r;0;r,r,0)}^{(3,B,2)}(u,u_+,u_-) + \sum_{r,q,|p|\geqslant 0}^{\infty} N_{\text{free},(r,q,p)}^{(3,B,1)}\chi_{(r;0;r,q,p)}^{(3,B,1)}(u,u_+,u_-),$$

(7.18)

for which formulas are obtained in Appendix B. We may determine for the first few cases, for $r = 0,1,2,\ldots$,

$$N^{(3,B,1)}_{\text{free},(r+1,r,\pm1)} = \sum_{j=0}^{r} p(j) - p(r+1) = 0, 1, 2, 5, 8 \dots \tag{7.19}$$

and

$$N^{(3,B,1)}_{\text{free},(r+2,r,0)} = 2\sum_{j=1}^{r}\sum_{k=0}^{j} p(k) - \sum_{j=0}^{r+1} p(j) + p(r+2) = 2, 5, 12, 23, 44, \dots, \tag{7.20}$$

along with

$$N^{(3,B,1)}_{\text{free},(r+4,r+2,\pm2)} = 2\sum_{j=1}^{r}\sum_{k=0}^{[j/2]} p\left(\frac{1}{2}(1-(-1)^j)+2k\right) + \sum_{j=0}^{r+2} p(j) - p(r+3) = 3, 6, 15, 26, 49, \dots . \tag{7.21}$$

Note that generally, from the results in Appendix B, $N^{(3,B,1)}_{\text{free},(r,q,p)}$ is a potentially nonzero integer only for $N^{(3,B,1)}_{\text{free},(r,r-s-t,s-t)}$, $r \in \mathbb{N}$, $s,t=0,\dots,[r/2]$, $st \neq 0$.

[From (6.2), $(r, r-s-t, t-s) \to [r-2s, s+t, r-2t]$, in terms of SU(4) Dynkin labels.]

For the supergravity limit we may determine

$$Y^{\text{U(1)}\otimes\text{SO(4)}}_{\text{sugra}}(u, u_+, u_-) = \sum_{r=1}^{\infty} \chi^{(3,B,2)}_{(r;0,r,r,0)}(u, u_+, u_-) = \frac{1-u^4}{\prod_{\varepsilon,\eta=\pm1}(1-u^2 u_+{}^{\varepsilon} u_-{}^{\eta})} - 1, \tag{7.22}$$

so that, from (7.3),

$$Z^{\text{U(1)}\otimes\text{SO(4)}}_{\text{sugra}}(u, u_+, u_-) = \prod_{j=1}^{\infty} \frac{1}{\prod_{k,l=0}^{j}(1-u^{2j} u_+{}^{2k-j} u_-{}^{2l-j})}. \tag{7.23}$$

Again we may expand

$$Z^{\text{U(1)}\otimes\text{SO(4)}}_{\text{sugra}}(u, u_+, u_-) = 1 + \sum_{r=1}^{\infty} p(r)\chi^{(3,B,2)}_{(r;0,r,r,0)}(u, u_+, u_-) + \sum_{r,q,|p|\geqslant 0} N^{(3,B,1)}_{\text{sugra},(r,q,p)}\chi^{(3,B,1)}_{(r;0;r,q,p)}(u, u_+, u_-) \tag{7.24}$$

to find for the first few cases, using results from Appendix B, for $r=0,1,2,\dots,$

$$N^{(3,B,1)}_{\text{sugra},(r+1,r,\pm1)} = N^{(3,B,1)}_{\text{free},(r+1,r,\pm1)}, \tag{7.25}$$

while

$$N^{(3,B,1)}_{\text{sugra},(r+2,r,0)} = \sum_{j=0}^{r}\sum_{k=0}^{j} p(k) - \sum_{j=0}^{r+1} p(j) + p(r+2) = 1, 2, 5, 9, 18, \dots, \tag{7.26}$$

along with

$$N^{(3,B,1)}_{\text{sugra},(r+4,r+2,\pm2)} = \sum_{j=1}^{r}\sum_{k=0}^{[j/2]} p\left(\frac{1}{2}(1-(-1)^j)+2k\right) + \sum_{j=0}^{r+2} p(j) - p(r+3) = 2, 4, 10, 17, 32, \dots . \tag{7.27}$$

As emphasized, the matching (7.25) is expected from (4.33) along with the free field restrictions implied for general r, q, p in $N^{(3,B,1)}_{\text{free},(r,q,p)}$ mentioned above.

TABLE III. BPS primary operators with conformal dimensions Δ belonging to SO(6) representations $\mathcal{R}_{(r,q,p)}$, as obtained from expansion of partition functions. (For the free field case, the extra operators appear in the rightmost column.)

	BPS primary operators	
Δ	Supergravity limit	Remaining operators
1	$\mathcal{R}_{(1,1,0)}$	
2	$2\mathcal{R}_{(2,2,0)}, \mathcal{R}_{(2,0,0)}$	$\mathcal{R}_{(2,0,0)}$
3	$3\mathcal{R}_{(3,3,0)}, \mathcal{R}_{(3,2,\pm1)}, 2\mathcal{R}_{(3,1,0)}$	$3\mathcal{R}_{(3,1,0)}$
4	$5\mathcal{R}_{(4,4,0)}, 2\mathcal{R}_{(4,3,\pm1)}, 2\mathcal{R}_{(4,2,\pm2)}, 5\mathcal{R}_{(4,2,0)}$	$\mathcal{R}_{(4,2,\pm2)}, 7\mathcal{R}_{(4,2,0)}$
	$\mathcal{R}_{(4,1,\pm1)}, 3\mathcal{R}_{(4,0,0)}$	$3\mathcal{R}_{(4,1,\pm1)}, 6\mathcal{R}_{(4,0,0)}$

The first few numbers of operators may be easily obtained by performing series expansions to low orders in u and using the orthogonality relation for SU(2) characters in Appendix B and are listed in Table III.

Note that more generally, as may be easily argued from results in Appendix B, $N_{\text{sugra},(rq,p)}^{(3,B,1)}$ is potentially nonvanishing only for $N_{\text{sugra},(r,r-s-t,s-t)}^{(3,B,1)}$, $r \in \mathbb{N}$, $s,t=0,\ldots,[r/2]$, $st \neq 0$, consistent with the free field theory result.

B. Semishort cases

1. The U(1)⊗SU(1,1) sectors

In these sectors, where the relevant limits in characters are given by (5.21), for $N=3$, fermion contributions become important in multiparticle partition functions.

The surviving characters, for the $(N, A, +)$ limit, are given by

$$\chi_{(1/2;0;1/2,-1/2)}^{(3,B,-)}(u,\bar{s}) = \frac{u^{1/2}\bar{s}^{3/2}}{1-\bar{s}^2}, \quad \chi_{(r;0;r,r)}^{(3,B,+)}(u,\bar{s}) = \frac{u^r \bar{s}^r}{1-\bar{s}^2},$$

$$\chi_{(r;0;r,r-1)}^{(3,B,2)}(u,\bar{s}) = \frac{u^r \bar{s}^{r+1}}{1-\bar{s}^2}, \quad \chi_{(r;0;r-1,r-1)}^{(3,B,1)}(u,\bar{s}) = \frac{u^r \bar{s}^{r+2}}{1-\bar{s}^2},$$

$$\chi_{(j+1;j;0,0,0)}^{(3,\text{cons.})}(u,\bar{s}) = \frac{u\bar{s}^{2j+4}}{1-\bar{s}^2}, \quad \chi_{(r+j+1;j;r,r,r)}^{(3,A,+)}(u,\bar{s}) = \frac{u^{r+1}\bar{s}^{r+2j+4}}{1-\bar{s}^2}. \tag{7.28}$$

Thus, we have that $f_+^{\text{U}(1)\otimes\text{SU}(1,1)}(u,\bar{s}) = (u\bar{s})^{1/2}/(1-\bar{s}^2)$, $f_-^{\text{U}(1)\otimes\text{SU}(1,1)}(u,\bar{s}) = u^{1/2}\bar{s}^{3/2}/(1-\bar{s}^2)$. In this case the fields that contribute from Table II are

$$(\phi_1)_i^{\bar{\iota}}, \quad (\bar{\psi}_{11})_J^j, \quad \partial_{11}. \tag{7.29}$$

Then from (7.3),

$$Z_{\text{free}}^{\text{U}(1)\otimes\text{SU}(1,1)}(u,\bar{s}) = \prod_{j=1}^{\infty} \frac{1}{1 + (-1)^j u^j \bar{s}^{2j}(1-\bar{s}^{2j})^{-2}}. \tag{7.30}$$

Expanding in u we find

$$Z_{\text{free}}^{\text{U}(1)\otimes\text{SU}(1,1)}(u,\bar{s}) = 1 + \frac{us^2}{(1-s^2)^2} + \frac{4u^2s^6}{(1-s^2)^2(1-s^4)^2} + O(u^3,s^6) = 1 + \chi_{(1;0;1,1,0)}^{(3,B,2)}(u,\bar{s})$$

$$+ \sum_{j=0}^{\infty} \chi_{(j+1;j;0,0,0)}^{(3,\text{cons.})}(u,\bar{s}) + \sum_{r=1}^{\infty} \sum_{\substack{j=r/2-1 \\ j\geqslant 0}}^{\infty} N_{\text{free},(r,j)}^{(3,A,+)} \chi_{(r+j+1;j;r,r,r)}^{(3,A,+)}(u,\bar{s}). \tag{7.31}$$

It appears nontrivial to determine $N_{\text{free},(r,j)}^{(3,A,+)}$ generally, however, we may easily determine,

$$N_{\text{free},(1,j)}^{(3,A,+)} = 2\left[\frac{1}{2}j+1\right]\left[\frac{1}{2}j+2\right] = 4, 4, 12, 12, \dots, \quad j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots,$$

$$N_{\text{free},(r,r/2-1)}^{(3,A,+1)} = 1, \quad r = 2, 3, 4, \dots, \quad N_{\text{free},(r,r/2-1+n)}^{(3,A,+)} > 1, \quad n = 1, 2, 3, \dots . \tag{7.32}$$

In this limit, the supergravity single particle partition function reduces to

$$Y_{\text{sugra}}^{\text{U}(1)\otimes\text{SU}(1,1)}(u,\bar{s}) = \chi_{(1;0;1,1,0)}^{(3,B,2)}(u,\bar{s}) = \frac{u\bar{s}^2}{1-\bar{s}^2}, \tag{7.33}$$

so that, from (7.3),

$$Z_{\text{sugra}}^{\text{U}(1)\otimes\text{SU}(1,1)}(u,\bar{s}) = \prod_{j=1}^{\infty} \left(1 + u\bar{s}^{2j}\right), \tag{7.34}$$

which is a reflection of $Y_{\text{sugra}}^{\text{U}(1)\otimes\text{SU}(1,1)}(u,\bar{s})$ receiving purely fermionic operator contributions [it is odd under $(u,\bar{s}) \rightarrow -(u,\bar{s})$].

Using the identities,

$$\prod_{j=1}^{\infty} (1+zq^j) = \sum_{n=0}^{\infty} \frac{q^{(1/2)n(n+1)}z^n}{\prod_{j=1}^{n}(1-q^j)}, \quad \prod_{j=2}^{n} \frac{1}{1-q^j} = \sum_{m=0}^{\infty} P_n(m)q^m, \tag{7.35}$$

where $P_n(m)$ is the number of partitions of m into no more than n parts, each part $\geqslant 2$, we may write

$$Z_{\text{sugra}}^{\text{U}(1)\otimes\text{SU}(1,1)}(u,\bar{s}) = 1 + \chi_{(1;0;1,1,0)}^{(3,B,2)}(u,\bar{s}) + \sum_{r=1}^{\infty} \sum_{j=1/2r^2+r-1}^{\infty} N_{\text{sugra},(r,j)}^{(3,A,+)} \chi_{(r+j+1;j;r,r,r)}^{(3,A,+)}(u,\bar{s}), \tag{7.36}$$

where

$$N_{\text{sugra},(r,j)}^{(3,A,+)} = P_{r+1}\left(j+1-\frac{1}{2}r^2-r\right). \tag{7.37}$$

Using the foregoing we may find relatively simple formulas for at least the first few cases,

$$N_{\text{sugra},(1,j)}^{(3,A,+)} = \begin{cases} 1, & j-\frac{1}{2}=0 \bmod 2, \\ \\ 0, & j-\frac{3}{2}=0 \bmod 2, \end{cases} \quad j = \frac{1}{2}, \frac{3}{2}, \dots,$$

$$N^{(3,A,+)}_{\text{sugra},(2,j)} = 1,0,1,1,\ldots = \begin{cases} \left[\dfrac{1}{6}j - \dfrac{1}{2}\right], & j = 4 \bmod 6, \\[3mm] \left[\dfrac{1}{6}j + \dfrac{1}{2}\right], & \text{otherwise,} \end{cases} \quad j = 3,4,5,6,\ldots,$$

$$N^{(3,A,+)}_{\text{sugra},(r,(1/2)r^2+r-1)} = 1. \tag{7.38}$$

Assuming that the partition functions in the supergravity limit receive contributions from only protected operators, then a number of observations from (7.31), (7.32), (7.36), and (7.38) are evident.

First, there are no $\chi^{(3,B,1)}_{(r;0;r,r-1,r-1)}(u,\bar{s})$ contributions and one contribution from $\chi^{(3,B,2)}_{(r;0;r,r,r-1)}(u,\bar{s})$, for $r=1$, both consistent with previous analysis for BPS cases. Second, the conserved current contributions in (7.31) disappear from the spectrum in the strong coupling limit, as happens for $\mathcal{N}=4$ super Yang Mills. Third, there is one scalar $(3,A,+)$ semishort primary operator in the SU(6) representation $\mathcal{R}_{(2,2,2)}$, and its conformal descendants, contributing in the free field theory limit, due to $N^{(3,A,+)}_{\text{free},(2,0)}=1$ in (7.31), and no such contribution in the supergravity limit, from (7.36). In the interacting theory, these operators, from the long multiplet decomposition formulas (4.35), must then pair up with a $(3,B,1)$ BPS primary operator in the SU(6) representation $\mathcal{R}_{(4,2,2)}$, and its descendants. This is consistent with Table III as there is precisely one such remaining $(3,B,1)$ primary operator. (Similarly, the scalar conserved current, and its descendants, contributing to (7.31) pairs with the single $(3,B,1)$ BPS primary operator in the SU(6) representation $\mathcal{R}_{(2,0,0)}$, and its descendants, listed in Table III.)

Note that the analysis for $(3,A,-)$ semishort operators is very similar and gives the same result for counting of BPS and conserved current multiplets along with

$$N^{(3,A,-)}_{\text{free},(r,j)} = N^{(3,A,+)}_{\text{free},(r,j)}, \quad N^{(3,A,-)}_{\text{sugra},(r,j)} = N^{(3,A,+)}_{\text{sugra},(r,j)}. \tag{7.39}$$

In particular, again there is one scalar $(3,A,-)$ primary semishort operator contributing in the free field theory limit, absent from the supergravity spectrum, that pairs up with a $(3,B,1)$ BPS primary operator in the $\mathcal{R}_{(4,2,-2)}$ SU(6) representation, consistent with Table III.

The U(1)⊗Osp(2|2) sector

In the limit (5.9), for $N=3$, $m=2$, the nonvanishing characters are, defining $\mathcal{Q}(\bar{s},y)=(1+\bar{s}y)$ $\times(1+\bar{s}y^{-1})/(1-\bar{s}^2)$,

$$\chi^{(3,B,\pm)}_{(r;0;r,r,\pm r)}(u,\bar{s},y) = \frac{u^r\bar{s}^r}{1-\bar{s}^2}y^{\pm r}(1+\bar{s}y^{\mp 1}), \quad \chi^{(3,B,2)}_{(r;0;r,r,q)}(u,\bar{s},y) = (u,\bar{s})^r y^q Q(\bar{s},y),$$

$$\chi^{(3,B,1)}_{(r;0;r,r-1,q)}(u,\bar{s},y) = u^r\bar{s}^{r+1}y^q Q(\bar{s},y), \quad \chi^{(3,\text{cons.})}_{(j+1;j;0,0,0)}(u,\bar{s},y) = u\bar{s}^{2j+3}Q(\bar{s},y),$$

$$\chi^{(3,A,\pm)}_{(r+j+1;j;r,r,\pm r)}(u,\bar{s},y) = u^{r+1}s^{r+2j+3}y^{\pm r}Q(\bar{s},y),$$

$$\chi^{(3,A,2)}_{(r+j+1;j;r,r,q)}(u,\bar{s},y) = u^{r+1}\bar{s}^{r+2j+3}y^q Q(\bar{s},y). \tag{7.40}$$

Thus $f^{\text{U}(1)\otimes\text{Osp}(2|2)}_{\pm}(u,\bar{s},y) = (u\bar{s})^{1/2}(y^{\pm 1/2}+\bar{s}y^{\mp 1/2})/(1-\bar{s}^2)$.

The fields that contribute from Table II in this case are

$$(\phi_1)^{\bar{\iota}}_i, \quad (\psi_{11})^{\bar{\iota}}_i, \quad (\tilde{\phi}_1)^j_J, \quad (\tilde{\psi}_{11})^j_J, \quad \partial_{11}. \tag{7.41}$$

From (7.3) we have

$$Z_{\text{free}}^{\text{U}(1)\otimes\text{Osp}(2|2)}(u,\overline{s},y) = \prod_{j=1}^{\infty} \frac{1}{1 - u^j\overline{s}^j(1-(-1)^j\overline{s}^jy^j)(1-(-1)^j\overline{s}^jy^{-j})(1-\overline{s}^{2j})^{-2}}. \tag{7.42}$$

Using the previous results for counting of BPS operators, we may determine

$$W_{\text{free}}^{(3,B,2)}(u,\overline{s},y) = \sum_{r=1}^{\infty} p(r)\chi_{(r;0;r,r,0)}^{(3,B,2)}(u,\overline{s},y) = \left(\prod_{k=1}^{\infty} \frac{1}{1-(u\overline{s})^j} - 1\right)Q(\overline{s},y) \tag{7.43}$$

for contributions from $(3,B,2)$ multiplets, using (7.12), and

$$W_{\text{free}}^{(3,B,1)}(u,\overline{s},y) = \sum_{r=1}^{\infty} (N_{\text{free},(r,r-1,1)}^{(3,B,1)}\chi_{(r;0;r,r-1,1)}^{(3,B,1)}(u,\overline{s},y) + N_{\text{free},(r,r-1,-1)}^{(3,B,1)}\chi_{(r;0;r,r-1,-1)}^{(3,B,1)}(u,\overline{s},y)) = \overline{s}(y+y^{-1})$$

$$\times\left(\frac{2u\overline{s}-1}{1-u\overline{s}}\prod_{k=1}^{\infty}\frac{1}{1-(u\overline{s})^j}+1\right)Q(\overline{s},y) \tag{7.44}$$

for contributions of $(3,B,1)$ multiplets, using the form of the corresponding characters in (7.40), along with (7.19). Similarly, using the free field results for the $\text{U}(1)\otimes\text{SU}(1,1)$ sector in (7.31), we may determine

$$W_{\text{free}}^{(3,\text{cons.})}(u,\overline{s},y) = \sum_{j=0}^{\infty} \chi_{(j+1;j;0,0,0)}^{(3,\text{cons.})}(u,\overline{s},y) = \frac{u\overline{s}^3}{1-\overline{s}^2}Q(\overline{s},y) \tag{7.45}$$

for conserved current multiplet contributions and

$$W_{\text{free}}^{(3,A,\pm)}(u,\overline{s},y) = \sum_{r=1}^{\infty}\sum_{\substack{j=r/2-1 \\ j\geqslant 0}}^{\infty} N_{\text{free},(r,j)}^{(3,A,\pm)}\chi_{(r+j+1;j;r,r,\pm r)}^{(3,A,\pm)}(u,\overline{s},y) = y^{\mp 1}\overline{s}^{-1}(1+\overline{s}y)(1+\overline{s}y^{-1})$$

$$\times\left(\prod_{k=1}^{\infty}\frac{1}{1+(-1)^j(uy^{\pm 1}\overline{s}^2)^j(1-\overline{s}^{2j})^{-2}} - \frac{uy^{\pm 1}\overline{s}}{(1-\overline{s}^2)^2} - 1\right) \tag{7.46}$$

for $(3,A,\pm)$ semishort multiplets, evident by using (7.31) and (7.39), and the form of the corresponding characters in (7.40). We may then determine numbers $N_{\text{free},(r,j,q)}^{(3,A,2)}$ of $(3,A,2)$ semishort operators from

$$Z_{\text{free}}^{\text{U}(1)\otimes\text{Osp}(2|2)}(u,\overline{s},y) = 1 + W_{\text{free}}^{(3,B,2)}(u,\overline{s},y) + W_{\text{free}}^{(3,B,1)}(u,\overline{s},y) + W_{\text{free}}^{(3,\text{cons.})}(u,\overline{s},y) + W_{\text{free}}^{(3,A,+)}(u,\overline{s},y)$$

$$+ W_{\text{free}}^{(3,A,-)}(u,\overline{s},y) + \sum_{\substack{r>1,2j>0 \\ 0\leqslant|q|<r}} N_{\text{free},(r,j,q)}^{(3,A,2)}\chi_{(r+j+1;j;r,r,q)}^{(3,A,2)}(u,s,y). \tag{7.47}$$

Finding general formulas for $N_{\text{free},(r,j,q)}^{(3,A,2)}$ is nontrivial, however, series expansion of (7.47), using MATHEMATICA, suggests the following results for particular cases, for $r=0,1,2,\dots$:

$$N_{\text{free},(r+1,0,0)}^{(3,A,2)} = 2\sum_{j=0}^{r}\sum_{k=0}^{j}p(k) + \sum_{j=0}^{r+1}p(j) = 4,10,21,40,71,\dots, \tag{7.48}$$

along with

$$N_{\text{free},(r+3,0\pm 2)}^{(3,A,2)} = 2\sum_{j=0}^{r+1}\sum_{k=0}^{[j/2]}p\left(\frac{1}{2}(1-(-1)^j)+2k\right) - \sum_{j=0}^{r+3}p(j) + p(r+4) = 2,5,10,19,33,\dots. \tag{7.49}$$

For the $\mathrm{U}(1) \otimes \mathrm{Osp}(2|2)$ sector, the supergravity single particle partition function reduces to

$$Y_{\mathrm{sugra}}^{\mathrm{U}(1) \otimes \mathrm{Osp}(2|2)}(u,\bar{s},y) = \sum_{r=1}^{\infty} \chi_{(r;0;r,r,0)}^{(3,B,2)}(u,\bar{s},y) = \frac{u\bar{s}(1+\bar{s}y)(1+\bar{s}y^{-1})}{(1-\bar{s}^2)(1-u\bar{s})}, \tag{7.50}$$

so that, from (7.3),

$$Z_{\mathrm{sugra}}^{\mathrm{U}(1) \otimes \mathrm{Osp}(2|2)}(u,\bar{s},y) = \prod_{j,k=1}^{\infty} \frac{(1+u^j\bar{s}^{j+2k-1}y)(1+u^j\bar{s}^{j+2k-1}y^{-1})}{(1-u^j\bar{s}^{j+2k-2})(1-u^j\bar{s}^{j+2k})}. \tag{7.51}$$

This time, we may determine

$$W_{\mathrm{sugra}}^{(3,B,2)}(u,\bar{s},y) = W_{\mathrm{free}}^{(3,B,2)}(u,\bar{s},y),$$

$$W_{\mathrm{sugra}}^{(3,B,1)}(u,\bar{s},y) = W_{\mathrm{free}}^{(3,B,1)}(u,\bar{s},y), \quad W_{\mathrm{sugra}}^{(3,\mathrm{cons.})}(u,\bar{s},y) = 0 \tag{7.52}$$

due to the relevant sector of BPS operators remaining protected and the conserved current multiplet operators disappearing from the spectrum, as shown previously. Similarly, using the supergravity limit results for the $\mathrm{U}(1) \otimes \mathrm{SU}(1,1)$ sector in (7.36), we may determine

$$W_{\mathrm{free}}^{(3,A,\pm)}(u,\bar{s},y) = \sum_{r=1}^{\infty} \sum_{\substack{j=r/2-1 \\ j \geqslant 0}}^{\infty} N_{\mathrm{sugra},(r,j)}^{(3,A,\pm)} \chi_{(r+j+1;j;r,r,\pm r)}^{(3,A,\pm)}(u,\bar{s},y) = y^{\mp 1}\bar{s}^{-1}(1+\bar{s}y)(1+\bar{s}y^{-1}) \Bigg(\prod_{k=1}^{\infty} (1$$

$$+ uy^{\pm 1}\bar{s}^{2j}) - \frac{uy^{\pm 1}\bar{s}}{(1-\bar{s}^2)^2} - 1 \Bigg) \tag{7.53}$$

for $(3,A,\pm)$ semishort multiplets, evident by using also (7.39), and the form of the corresponding characters in (7.40). We may then determine numbers $N_{\mathrm{sugra},(r,j,q)}^{(3,A,2)}$ of $(3,A,2)$ semishort operators from

$$Z_{\mathrm{sugra}}^{\mathrm{U}(1) \otimes \mathrm{Osp}(2|2)}(u,\bar{s},y) = 1 + W_{\mathrm{sugra}}^{(3,B,2)}(u,\bar{s},y) + W_{\mathrm{sugra}}^{(3,B,1)}(u,\bar{s},y) + W_{\mathrm{sugra}}^{(3,A,+)}(u,\bar{s},y) + W_{\mathrm{sugra}}^{(3,A,-)}(u,\bar{s},y)$$

$$+ \sum_{\substack{r \geqslant 1, 2j \geqslant 0 \\ 0 \leqslant |q| < r}} N_{\mathrm{sugra},(r,j,q)}^{(3,A,2)} \chi_{(r+j+1;j;r,r,q)}^{(3,A,2)}(u,\bar{s},y). \tag{7.54}$$

Again, general formulas for $N_{\mathrm{sugra},(r,j,q)}^{(3,A,2)}$ seem nontrivial to obtain, however, using MATHEMATICA suggests, for particular cases, for $r=0,1,2,\ldots$,

$$N_{\mathrm{sugra},(1,j,0)}^{(3,A,2)} = 1, \quad j = 0,1,2,\ldots, \tag{7.55}$$

and

$$N_{\mathrm{sugra},(r+1,0,0)}^{(3,A,2)} = \sum_{j=0}^{r} \sum_{k=0}^{j} p(k) = 1,3,7,14,26,\ldots, \tag{7.56}$$

along with

$$N_{\mathrm{sugra},(r+3,0,\pm2)}^{(3,A,2)} = \sum_{j=0}^{r+1} \sum_{k=0}^{[j/2]} p\left(\frac{1}{2}(1-(-1)^j)+2k\right) - \sum_{j=0}^{r+3} p(j) + p(r+4) = 0,0,1,2,5,\ldots. \tag{7.57}$$

A consistency check is provided by (4.34) which implies that the unprotected scalar $(3,A,2)$ semishort primary operators in SU(6) representation $\mathcal{R}_{(r,r,0)}$, respectively, $\mathcal{R}_{(r,r,\pm2)}$, and their con-

formal descendants, counted above, should pair with unprotected $(3, B, 1)$ BPS primary operators in the SU(6) representation $\mathcal{R}_{(r+2,2,0)}$, respectively, $\mathcal{R}_{(r+2,r,\pm2)}$, and their descendants, counted previously. Using (7.20), (7.26), (7.48), and (7.56), along with (7.21), (7.27), (7.48), and (7.57), we find

$$N^{(3,B,1)}_{\text{free},(r+2,r,0)} - N^{(3,B,1)}_{\text{sugra},(r+2,r,0)} = N^{(3,A,2)}_{\text{free},(r,0,0)} - N^{(3,A,2)}_{\text{sugra},(r,0,0)} = \sum_{j=0}^{r} \sum_{k=0}^{j} p(k),$$

$$N^{(3,B,1)}_{\text{free},(r+2,r,\pm2)} - N^{(3,B,1)}_{\text{sugra},(r+2,r,\pm2)} = N^{(3,A,2)}_{\text{free},(r,0,\pm2)} - N^{(3,A,2)}_{\text{sugra},(r,0,\pm2)} = \sum_{j=0}^{r-2} \sum_{k=0}^{[j/2]} p\left(\frac{1}{2}(1-(-1)^j) + 2k\right),$$

$$(7.58)$$

expressing perfect agreement with this expectation.

3. The $\mathrm{U}(1) \otimes \mathrm{Osp}(4|2)$ sector

This sector is nontrivial to analyze in similar terms as above due to necessary and nontrivial expansions over SO(4), characters, as done for the $\mathrm{U}(1) \otimes \mathrm{SO}(4)$ sector above, so here are simply given formulas for (7.1). Using (5.15), for $N=3$, $m=1$, with (7.15), we have

$$\chi^{(3,B,2)}_{(r;0,;r,0)}(u,\bar{s},u_+,u_-) = \chi^{(\mathrm{U}(1)\otimes\mathrm{Osp}(2|2),2)}_{(r;r;r,0)}(u,\bar{s},u_+,u_-)$$

$$= \frac{(u\bar{s})^r}{1-\bar{s}^2} \sum_{\varepsilon,\eta=\pm1} \frac{(u_+)^{\varepsilon r}(u_-)^{\eta r}(1+\bar{s}u_+^{\varepsilon}u_-^{-\eta})(1+\bar{s}u_+^{-\varepsilon}u_-^{\eta})(1+\bar{s}u_+^{-\varepsilon}u_-^{-\eta})}{(1-u_+^{-2\varepsilon})(1-u_-^{-2\eta})},$$

$$(7.59)$$

so that, with (5.18),

$$f^{\mathrm{U}(1)\otimes\mathrm{Osp}(4|2)}_{\pm}(u,\bar{s},u_+,u_-) = \frac{(u\bar{s})^{1/2}}{1-\bar{s}^2}(\chi_1(u_{\pm}) + \bar{s}\chi_1(u_{\mp})),$$

$$Y^{\mathrm{U}(1)\otimes\mathrm{Osp}(4|2)}_{\text{sugra}}(u,\bar{s},u_+,u_-) = \frac{1}{1-\bar{s}^2} \sum_{\varepsilon=\pm1} \frac{(1+u\bar{s}^2u_+^{2\varepsilon})(1+\bar{s}(u_+u_-)^{-\varepsilon})(1+\bar{s}(u_+/u_-)^{-\varepsilon})}{(1-u_+^{-2\varepsilon})(1-u\bar{s}(u_+u_-)^{\varepsilon})(1-u\bar{s}(u_+/u_-)^{\varepsilon})} - 1.$$

$$(7.60)$$

(7.60) is consistent with (6.16) in the limit (5.9) and the formula in the second line can also be shown to be symmetric under exchange of u_+, u_-, as is necessary.

In terms of the fields in Table II, the contributions are from

$$(\phi_1)_i^{\bar{\imath}}, \quad (\phi_2)_i^{\bar{\imath}}, \quad (\psi_{11})_i^{\bar{\imath}}, \quad (\psi_{21})_i^{\bar{\imath}},$$

$$(\tilde{\phi}_1)_j^{j}, \quad (\tilde{\phi}_2)_j^{j}, \quad (\tilde{\psi}_{11})_j^{j}, \quad (\tilde{\psi}_{21})_j^{j}, \quad \partial_{11}. \tag{7.61}$$

VIII. CONCLUSION

While much progress has been made recently in terms of determining the spectra of superconformal field theories, there are many open questions, for instance, for the new superconformal Chern–Simons theories or $\mathcal{N}=4$ super Yang Mills.

Focusing on the former, while the spectra of the $\mathcal{N}=6$ superconformal Chern–Simons theory at zero (effective) 't Hooft coupling and in the large n, k limits has been partially addressed here by use of character methods, it may be interesting to investigate operator counting for finite n, k.

For large n, k, the results here provide extra confirmation of expectations that the primary operators dual to Kaluza Klein modes, and multitraces of these operators, in $[r,0,r]$ SU(4) representations, conformal dimension r, are protected, as argued in another way in Ref. 2. Also, the counting here implies that these are the only gauge invariant multitrace primary operators in the $(3,B,2)$ superconformal representation.

For the $(3,B,1)$ representations, the only gauge invariant primary operators belong to $[r-2s,s+t,r-2t]$, $s,t=0,\dots,[r/2]$, $st \neq 0$, SU(4) representations, for which generating functions for counting are given in Appendix B. Furthermore, in accord with long multiplet decomposition rules, counting of $(s,t)=(1,0)$ and $(0,1)$ cases shows matching between the free field and supergravity limits, providing a consistency check of the character procedure used here and further evidence, perhaps, in favor of the duality proposed in Ref. 2.

Turning to semishort cases, conserved current multiplet operators disappear from the spectrum in the supergravity limit while the primary operators for $(3,A,\pm)$ semishort operators, belonging to $[2r,0,0]$, $[0,0,2r]$ SU(4) representations, have spins $j \geq \frac{1}{2}r-1$ in the free field limit, while for the supergravity limit, $j \geq \frac{1}{2}r^2+r-1$, a reflection of the very simple partition function obtained in (7.51).

While only partial counting is obtained here for semishort operators, the numbers of primary operators obtained are consistent with the following formula, implied by (4.34)–(4.36),

$$N^{\mathcal{M}}_{\text{prot},(r,q)} = N^{\mathcal{M}}_{\text{free},(r,q)} - N^{(3,B,1)}_{\text{free},(r+2,r,q)} + N^{(3,B,1)}_{\text{prot},(r+2,r,q)}, \tag{8.1}$$

where $N^{\mathcal{M}}_{\text{prot},(r,q)}$, $N^{\mathcal{M}}_{\text{free},(r,q)}$ denote numbers of corresponding protected/free *scalar* semishort primary operators, in appropriate $\mathcal{R}_{(r,r,q)}$ SU(6) representations [so that for $\mathcal{M}=(3,\text{cons.})$, then $r,q=0$, for $\mathcal{M}=(3,A,\pm)$ then $r=\pm q \neq 0$ and for $\mathcal{M}=(3,A,2)$ then $r>|q|>0$], while $N^{(3,B,1)}_{\text{prot},(r+2,r,q)}$, $N^{(3,B,1)}_{\text{free},(r+2,r,q)}$ denote numbers of protected/free primary operators in $(3,B,1)$ BPS multiplets, in $\mathcal{R}_{(r+2,r,q)}$ SU(6) representations.

Note also that, for free field theory in the large n limit, r,q are further restricted to $r=2$, $q=\pm 2$ for the $\mathcal{M}=(3,A,\pm)$ cases and $r \in \mathbb{N}$, $q=0,\pm 2$, $r>|q|$, for the $\mathcal{M}=(3,A,2)$ cases.

(8.1) implies that it is possible to compute the partition function corresponding to such protected operators using only free field theory and the knowledge of which $(3,B,1)$ BPS operators remain protected.

In particular, this applies to the $\mathcal{N}=6$ Chern–Simons theory, having finite n and large levels k, in which case allowable r,q in (8.1), deriving from free field theory, may change. The generating functions,

$$F_q(z) = \sum_{r \geq 0} N^{\mathcal{M}}_{\text{free},(r,q)} z^r,$$

$$G_q(z) = \sum_{r \geq 0} (N^{(3,B,1)}_{\text{free},(r+2,r,q)} - N^{(3,B,1)}_{\text{prot},(r+2,r,q)}) z^r, \tag{8.2}$$

may then be determined using the free field partition functions $Z^{(n),\text{U}(1) \otimes \text{Osp}(2|2)}_{\text{free}}(u,\bar{s},y)$, $Z^{(n),\text{U}(1) \otimes \text{SO}(4)}_{\text{free}}(u,u_+,u_-)$, denoting (6.7) evaluated in the relevant $H \subset \text{Osp}(6|4)$ sector, and $Z^{(n),\text{U}(1) \otimes \text{SO}(4)}_{\text{prot}}(u,u_+,u_-)$, the partition function for protected $(3,B,1)$ operators evaluated in the U(1) $\otimes$ SO(4) sector. It is evident from the form of the characters in (7.40) that

$$F_q(z) = \frac{1}{(2\pi i)^2} \oint \oint \frac{1-x^2}{x^3 y^q(x+y)(1+xy)} (Z^{(n),\text{U}(1) \otimes \text{Osp}(2|2)}_{\text{free}}(z/x,x,y) - 1) \mathrm{d}x \mathrm{d}y$$

in terms of a double contour integral, where the contribution from the identity operator is subtracted. (Depending on free field constraints for finite n it may also be necessary to subtract contributions from other short multiplet representations as well.) $G_q(z)$ may, in principle, be computed in terms of a double contour integral similar to that in Appendix B, where $f(u,x,y)$

$= Z_{\text{free}}^{(n),\text{U}(1)\otimes\text{SO}(4)}(u^{1/2},x,y) - Z_{\text{prot}}^{\text{U}(1)\otimes\text{SO}(4)}(u^{1/2},x,y)$ in (B1). (Presumably the latter partition function is equivalent to the chiral ring partition function, for finite n.)

These generating functions may then be used to write, using also (4.7), (4.27), and (4.28) for $N=3$,

$$Z_q(s,x,y_1,y_2,y_3) = \sum_{r\geqslant 0} N_{\text{prot},(r,q)}^{\mathcal{M}} \chi_{(r+1;0;r,r,q)}^{(3,A,1)}(s,x,y_1,y_2,y_3) = \mathfrak{W}^{(3)}(H_q(s^2 y_1 y_2) y_3{}^q C_{(1;0;0,0,0)}^{(3,A,1)}$$

$$\times (s,x,y_1,y_2,y_3)),$$

$$H_q(z) = F_q(z) - G_q(z) \tag{8.3}$$

for the partition function restricted to relevant protected scalar primary operators and their super-conformal descendants.

Such partition functions should be consistent with the large n result where, using that there are no relevant scalar (3, cons.), $(3,A,\pm)$ primary operators in the supergravity spectrum and that $N_{\text{prot},(r,q)}^{(3,A,2)} = N_{\text{sugra},(r,0,q)}^{(3,A,2)}$ with (7.56) and (7.57), then

$$H_0(z) = \sum_{r=1}^{\infty} N_{\text{sugra}(r,0,0)}^{(3,A,2)} z^r = \frac{z}{(1-z)^2} \prod_{k=1}^{\infty} \frac{1}{1-z^k},$$

$$H_{\pm 2}(z) = \sum_{r=3}^{\infty} N_{\text{sugra},(r,0,\pm 2)}^{(3,A,2)} z^r = \left(\frac{z^2}{(1-z)(1-z^2)} - \frac{1}{1-z} + \frac{1}{z} \right) \prod_{k=1}^{\infty} \frac{1}{1-z^k} - \frac{1}{z}, \tag{8.4}$$

with other $H_q(z)$ vanishing.

Turning to $\mathcal{N}=4$ super-Yang Mills, with gauge $\text{U}(N)$, there are still many outstanding questions regarding operator spectra that perhaps may be more easily answered for $\mathcal{N}=6$ superconformal Chern–Simons theories due to the latter having fewer decoupled sectors, as demonstrated here. One concerns finite N counting for protected operators. (Addressing this question may help answer the difficult black hole entropy/microstate counting problem.)

Recently, the latter issue received some attention in Ref. 32, where, among other things, the problem of counting certain gauge invariant operators, consisting of products of single trace chiral ring operators acted on by derivatives, was considered. (These operators should remain protected and certainly enumerating them is interesting and worthwhile.)

The simpler problem of counting chiral ring operators has been addressed from the perspective of computing Hilbert series for the ring of symmetric polynomials in Ref. 33 (equivalently, the latter issue has been investigated from a perhaps more formal perspective in Ref. 34), whereby the nontrivial part of the computation is in taking account of syzygies (or trace identities in terms of matrices). The same difficulty applies to counting operators of the sort, referred to above, considered in Ref. 32 and perhaps could be circumvented by a judicious choice of basis for the operators.

In Appendix C, a more combinatorial approach is described using a natural basis for multi-trace operators involving commuting matrices, the simplest sort of chiral ring that is relevant for $\mathcal{N}=4$ super-Yang Mills. The technique involves employing the Polya enumeration theorem, applied to counting graph colorings where the relevant graphs, in this case, have a particular wreath product group automorphism symmetry. (This is similar to the approach used for counting single trace operators, in the large N limit, for $\mathcal{N}=4$ SYM,[35,36] where the relevant graphs there were necklaces with cyclic group automorphism symmetry.) The result obtained by this approach (equivalent to the Hilbert series mentioned above) is naturally expressed in terms of cycle index polynomials for the symmetric permutation group. (For alternative polynomial expansions of chiral ring partition functions, useful in the context of asymptotics, see, for example, Ref. 37.)

It may be worth trying to find other combinatorial methods of counting chiral ring operators more generally, perhaps using known extensions of the Polya enumeration theorem, taking into

account symmetry in colors,[38] for example. In any case, Hilbert series appear to have very interesting connections with counting graph colorings that perhaps may lead to even other extensions of the Polya enumeration theorem.

ACKNOWLEDGMENTS

APPENDIX A: Osp($2N|4$) SUBALGEBRAS

Corresponding to the shortening conditions (3.5), (3.8), (3.11), and (3.14), with notation as in Table I, there are subalgebras which are now discussed. The characters for these subalgebras lead to the limits considered in Sec. V.

Corresponding to (3.5) for (N,B,m) short multiplets, we have

$$\text{Osp}(2N|4) \supset (\text{SU}(2|m) \otimes \text{SO}(2N-2m)) \ltimes \text{U}(1)_{\mathcal{I}_m}. \tag{A1}$$

The generators of SU($2|m$) are $M_\alpha{}^\beta$, $\mathcal{Q}_{\hat{m}\alpha}$, $\bar{\mathcal{S}}_{\hat{n}}{}^\beta$, $\hat{m},\hat{n}=1,\ldots,m$, of Sec. II, along with $\mathcal{H}_m$ as in (5.1), and $T_{\hat{m}\hat{n}}$, generators of SU(m)$\subset$SO($2m$), with, in terms of the generators in (2.7),

$$T_{\hat{m}\hat{m}} = H_{\hat{m}} - \frac{1}{m}\sum_{\hat{n}=1}^{m} H_{\hat{n}}, \quad T_{\hat{m}\hat{n}} = \begin{cases} \dfrac{i}{2}E_{\hat{m}\hat{n}}^{+-}, & \hat{m} < \hat{n} \\[2mm] -\dfrac{i}{2}E_{\hat{n}\hat{m}}^{-+}, & \hat{n} < \hat{m} \end{cases}, \tag{A2}$$

so that $\Sigma_{\hat{m}=1}^{m} T_{\hat{m}\hat{m}}=0$ and, for $\hat{k},\hat{l}=1,\ldots,m$,

$$[T_{\hat{m}\hat{n}}, T_{\hat{k}\hat{l}}] = \delta_{\hat{n}\hat{k}} T_{\hat{m}\hat{l}} - \delta_{\hat{m}\hat{l}} T_{\hat{n}\hat{k}}. \tag{A3}$$

SU($2|m$) has usual algebra with, in particular,

$$\{\mathcal{Q}_{\hat{m}\alpha}, \bar{\mathcal{S}}_{\hat{n}}{}^\beta\} = 2i(M_\alpha{}^\beta \delta_{\hat{m}\hat{n}} - \delta_\alpha{}^\beta T_{\hat{m}\hat{n}} + \delta_\alpha{}^\beta \delta_{\hat{m}\hat{n}} \mathcal{H}_m) \tag{A4}$$

and

$$[\mathcal{H}_m, \mathcal{Q}_{\hat{m}\alpha}] = \left(\frac{1}{2} - \frac{1}{m}\right)\mathcal{Q}_{\hat{m}\alpha}, \quad [\mathcal{H}_m, \bar{\mathcal{S}}_{\hat{n}}{}^\beta] = -\left(\frac{1}{2} - \frac{1}{m}\right)\bar{\mathcal{S}}_{\hat{n}}{}^\beta. \tag{A5}$$

For $m=2$, $\mathcal{H}_m$ is evidently then a central extension.

The SO($2N-2m$) subgroup in (A1) is generated by $R_{\bar{r}\bar{s}}$ $\bar{r},\bar{s},\bar{t},\bar{u}=2m+1,\ldots,2N$,

$$[R_{\bar{r}\bar{s}}, R_{\bar{t}\bar{u}}] = i(\delta_{\bar{r}\bar{t}} R_{\bar{s}\bar{u}} - \delta_{\bar{s}\bar{t}} R_{\bar{r}\bar{u}} - \delta_{\bar{r}\bar{u}} R_{\bar{s}\bar{t}} + \delta_{\bar{s}\bar{u}} R_{\bar{r}\bar{t}}), \tag{A6}$$

while $\mathcal{I}_m$ in (5.1) is an external automorphism with

$$[\mathcal{I}_m, \mathcal{Q}_{\hat{m}\alpha}] = \left(\frac{1}{2} + \frac{1}{m}\right)\mathcal{Q}_{\hat{m}\alpha}, \quad [\mathcal{I}_m, \bar{\mathcal{S}}_{\hat{n}}{}^\beta] = -\left(\frac{1}{2} + \frac{1}{m}\right)\bar{\mathcal{S}}_{\hat{n}}{}^\beta. \tag{A7}$$

Corresponding to (3.5) for $n=N$ for $(N,B,+)$ multiplets and, separately, (3.8) for $(N,B,-)$ multiplets, we have

$$\text{Osp}(2N|4) \supset \text{SU}(2|N) \ltimes \text{U}(1)_{\mathcal{I}_\pm}. \tag{A8}$$

The generators of SU($2|N$) are $M_\alpha{}^\beta$, $\mathcal{Q}_{\hat{m}\alpha}$, $\bar{\mathcal{S}}_{\hat{n}}{}^\beta$, for $\hat{m},\hat{n}=1,\ldots,N-1,+$ for the $(N,B,+)$ case and $\hat{m},\hat{n}=1,\ldots,N-1,-$ for the $(N,B,-)$ case, where we define

$$\mathcal{Q}_{+\alpha} = \mathcal{Q}_{N\alpha}, \quad \mathcal{Q}_{-\alpha} = \bar{\mathcal{Q}}_{N\alpha}, \quad \bar{S}_+{}^\alpha = \bar{S}_N{}^\alpha, \quad \bar{S}_-{}^\alpha = S_N{}^\alpha, \tag{A9}$$

along with $\mathcal{H}_\pm$ as in (5.7), and $T_{\hat{m}\hat{n}}$, generators of SU(N) given by, for $\hat{m} < N$,

$$T_{\hat{m}\hat{m}} = H_{\hat{m}} - \frac{1}{N}(H_1 + \cdots + H_{N-1} \pm H_N),$$

$$T_{\pm\pm} = \pm H_N - \frac{1}{N}(H_1 + \cdots + H_{N-1} \pm H_N),$$

$$T_{\hat{m}\pm} = \frac{i}{2}E_{\hat{m}N}^{+\pm}, \quad T_{\pm\hat{m}} = -\frac{i}{2}E_{\hat{m}N}^{-\pm}, \quad T_{\hat{m}\hat{n}} = \begin{cases} \dfrac{i}{2}E_{\hat{m}\hat{n}}^{+-}, & \hat{m} < \hat{n} < N \\[2ex] -\dfrac{i}{2}E_{\hat{n}\hat{m}}^{-+}, & \hat{n} < \hat{m} < N, \end{cases} \tag{A10}$$

satisfying the same commutation relation (A3) for $\hat{m}, \hat{n}, \hat{k}, \hat{l} = 1, \ldots, N-1, \pm$.

The $\mathcal{Q}_{\hat{m}\alpha}$, $\bar{S}_{\hat{n}}{}^\beta$ generators satisfy (A4) for $\mathcal{H}_m$ replaced by $\mathcal{H}_\pm$, as appropriate, and $\hat{m}, \hat{n} = 1, \ldots, N-1, \pm$, and with

$$[\mathcal{H}_\pm, \mathcal{Q}_{\hat{m}\alpha}] = \left(\frac{1}{2} - \frac{1}{N}\right)\mathcal{Q}_{\hat{m}\alpha}, \quad [\mathcal{H}_\pm, \bar{S}_{\hat{n}}{}^\beta] = -\left(\frac{1}{2} - \frac{1}{N}\right)\bar{S}_{\hat{n}}{}^\beta, \tag{A11}$$

so that for R-symmetry group SO(4), i.e., $N=2$, $\mathcal{H}_\pm$ is a central extension.

$\mathcal{I}_\pm$ in (5.7) acts and as external automorphism with

$$[\mathcal{I}_\pm, \mathcal{Q}_{\hat{m}\alpha}] = \left(\frac{1}{2} + \frac{1}{N}\right)\mathcal{Q}_{\hat{m}\alpha}, \quad [\mathcal{I}_\pm, \bar{S}_{\hat{n}}{}^\beta] = -\left(\frac{1}{2} + \frac{1}{N}\right)\bar{S}_{\hat{n}}{}^\beta. \tag{A12}$$

The expression (5.2) may be understood as follows. By taking the limit $\delta \to 0$, only (N, B, n), $n \geq m$, and $(N, B, \pm)$ BPS multiplets have states, including the highest weight state, with zero $\mathcal{H}_m$ eigenvalues. Characters for the $(\text{SU}(2|m) \otimes \text{SO}(2N-2m)) \ltimes \text{U}(1)_{\mathcal{I}_m}$ subalgebra, when restricted to these representations, reduce to $\text{U}(1)_{\mathcal{I}_m} \otimes \text{SO}(2N-2m)$ characters, as these representations are trivial representations of the $\text{SU}(2|m)$ subalgebra, evident from Sec. III.

The limit in (5.8) may be understood similarly to other BPS cases whereby the characters reduce in the half BPS cases to $\text{U}(1)_{\mathcal{I}_\pm}$ characters.

Corresponding to (3.11) for (N, A, m) semishort multiplets, we have

$$\text{Osp}(2N|4) \supset (\text{SU}(1|m) \otimes \text{Osp}(2N-2m|2)) \ltimes \text{U}(1)_{\mathcal{K}_m}. \tag{A13}$$

The generators of SU($1|m$) are $\mathcal{Q}_{\hat{m}2}$, $\bar{S}_{\hat{n}}{}^2$, $T_{\hat{m}\hat{n}}$, $\hat{m}, \hat{n} = 1, \ldots, m$, along with $\mathcal{J}_m$ as in (5.9), where SU($1|m$) has usual algebra with, in particular,

$$\{\mathcal{Q}_{\hat{m}2}, \bar{S}_{\hat{n}}{}^2\} = 2i(-T_{\hat{m}\hat{n}} + \delta_{\hat{m}\hat{n}}\mathcal{J}_m), \tag{A14}$$

with

$$[\mathcal{J}_m, \mathcal{Q}_{\hat{m}2}] = \left(1 - \frac{1}{m}\right)\mathcal{Q}_{\hat{m}2}, \quad [\mathcal{J}_m, \bar{S}_{\hat{n}}{}^2] = -\left(1 - \frac{1}{m}\right)\bar{S}_{\hat{n}}{}^2. \tag{A15}$$

For $m=1$, $\mathcal{J}_m$ is then a central extension.

The generators of $\text{Osp}(2N-2m|2)$ are $\text{Sp}(2, \mathbb{R}) \simeq \text{SU}(1,1)$ generators P_{11}, K^{11} of Sec. II, along with, $\bar{D} = D + J_3$ in (5.9) satisfying

$$[\bar{D},P_{11}]=2P_{11},\quad [\bar{D},K^{11}]=-2K^{11},\quad [K^{11},P_{11}]=4\bar{D}, \tag{A16}$$

along with $Q_{\bar{r}1}$, $S_{\bar{s}}^{\ 1}$, $\bar{r},\bar{s}=2m+1,\dots,2N$, of Sec. II, and SO($2N-2m$) generators $R_{\bar{r}\bar{s}}$ as before, satisfying

$$[\bar{D},Q_{\bar{r}1}]=Q_{\bar{r}1},\quad [\bar{D},S_{\bar{s}}^{\ 1}]=-S_{\bar{s}}^{\ 1},$$

$$[K^{11},Q_{\bar{r}1}]=2iS_{\bar{r}}^{\ 1},\quad [P_{11},S_{\bar{s}}^{\ 1}]=-2iQ_{\bar{s}1}, \tag{A17}$$

and

$$\{Q_{\bar{r}1},Q_{\bar{s}1}\}=2\delta_{\bar{r}\bar{s}}P_{11},\quad \{S_{\bar{r}}^{\ 1},S_{\bar{s}}^{\ 1}\}=2\delta_{\bar{r}\bar{s}}K^{11},\quad \{Q_{\bar{r}1},S_{\bar{s}}^{\ 1}\}=2i\delta_{\bar{r}\bar{s}}\bar{D}+2R_{\bar{r}\bar{s}}. \tag{A18}$$

In (A13), $\mathcal{K}_m$ as in (5.9) is an outer automorphism with

$$[\mathcal{K}_m,\mathcal{Q}_{\hat{m}2}]=\frac{1}{m}\mathcal{Q}_{\hat{m}2},\quad [\mathcal{K}_m,\bar{\mathcal{S}}_{\hat{n}}^{\ 2}]=-\frac{1}{m}\bar{\mathcal{S}}_{\hat{n}}^{\ 2},\quad [\mathcal{K}_m,\mathcal{Q}_{\bar{r}1}]=[\mathcal{K}_m,S_{\bar{s}}^{\ 1}]=0. \tag{A19}$$

Corresponding to (3.11) for $n=N$ for $(N,A,+)$ multiplets and, separately, (3.14) for $(N,A,-)$ semishort multiplets, we have

$$\text{Osp}(2N|4)\supset(\text{SU}(1|N)\otimes\text{SU}(1,1))\ltimes\text{U}(1)_{\mathcal{K}_\pm}. \tag{A20}$$

The generators of SU($1|N$) are $\mathcal{Q}_{\hat{m}2}$, $\bar{\mathcal{S}}_{\hat{n}}^{\ 2}$, $T_{\hat{m}\hat{n}}$ for $\hat{m},\hat{n}=1,\dots,N-1,+$, for the $(N,A,+)$ case, and $\hat{m},\hat{n}=1,\dots,N-1,-$ for the $(N,A,-)$ case, with the definitions (A9) for $\alpha=2$, with $T_{\hat{m}\hat{n}}$ as in (A10) and $\mathcal{J}_\pm$ as in (5.21).

The supercharges satisfy (A14) with $\mathcal{J}_m$ replaced by $\mathcal{J}_\pm$, as appropriate, and $\hat{m},\hat{n}=1,\dots,N-1,\pm$, with

$$[\mathcal{J}_\pm,\mathcal{Q}_{\hat{m}2}]=\left(1-\frac{1}{N}\right)\mathcal{Q}_{\hat{m}2},\quad [\mathcal{J}_\pm,\bar{\mathcal{S}}_{\hat{n}}^{\ 2}]=-\left(1-\frac{1}{N}\right)\bar{\mathcal{S}}_{\hat{n}}^{\ 2}, \tag{A21}$$

so that, for R symmetry group SO(2), i.e., $N=1$, $\mathcal{J}_\pm$ is a central extension. The SU($1,1$) generators are P_{11}, K^{11}, and $\bar{D}$, as above, satisfying (A16). Also, $\mathcal{K}_\pm$ as given in (5.21) is an outer automorphism with

$$[\mathcal{K}_\pm,\mathcal{Q}_{\hat{m}2}]=\frac{1}{N}\mathcal{Q}_{\hat{m}2},\quad [\mathcal{K}_\pm,\bar{\mathcal{S}}_{\hat{n}}^{\ 2}]=-\frac{1}{N}\bar{\mathcal{S}}_{\hat{n}}^{\ 2}. \tag{A22}$$

(5.10) can be understood similarly to BPS limits. Again by taking the limit $\delta\to0$, only those states in relevant semishort multiplets and various BPS multiplets with zero $\mathcal{J}_m$ eigenvalue contribute to corresponding characters. Note, however, that for semishort multiplets, the states do not include highest weight states, but some superconformal descendants. Such characters, when restricted to the $(\text{SU}(1|m)\otimes\text{Osp}(2N-2m|2))\ltimes\text{U}(1)_{\mathcal{K}_m}$ subalgebra, reduce to $\text{U}(1)_{\mathcal{K}_m}\otimes\text{Osp}(2N-2m|2)$ characters as these representations are trivial representations of the SU($1|m$) subalgebra.

To see the equivalence in terms of Osp($2N-2m|2$) characters, we may, as in Sec. II, make a change of basis for $Q_{\bar{r}1}$, $S_{\bar{s}}^{\ 1}$, $R_{\bar{r}\bar{s}}$ to the orthonormal basis, as in (2.8) and (2.9), to $\{\mathcal{Q}_{\bar{m}1},\bar{\mathcal{Q}}_{\bar{m}1},\mathcal{S}_{\bar{n}}^{\ 1},\bar{\mathcal{S}}_{\bar{n}}^{\ 1},H_{\bar{m}},E_{\bar{m}\bar{n}}^{\pm\pm},E_{\bar{m}\bar{n}}^{\pm\mp},\bar{m},\bar{n}=m+1,\dots,N\}$, in an obvious way. Denoting highest weight states by $|\bar{\Delta};\bar{\mathbf{r}}\rangle^{\text{h.w.}}$, where

$$(K^{11},S_{\bar{s}}^{\ 1})|\bar{\Delta};\bar{\mathbf{r}}\rangle^{\text{h.w.}}=0,\quad (\bar{D},H_{\bar{m}})|\bar{\Delta};\bar{\mathbf{r}}\rangle^{\text{h.w.}}=(\bar{\Delta},r_{\bar{m}})|\bar{\Delta};\bar{\mathbf{r}}\rangle^{\text{h.w.}}, \tag{A23}$$

unitarity requires $\bar{\Delta}\geq r_{m+1}$. Compatible shortening conditions are given by, for $\bar{n}=m+1,\dots,N$,

$$\mathcal{Q}_{\bar{n}1}|\bar{\Delta};\vec{r}\rangle^{\text{h.w.}} = 0 \Rightarrow \bar{\Delta} = r_{m+1} = \cdots = r_{\bar{n}},$$

$$\bar{\mathcal{Q}}_{N1}|\bar{\Delta};\vec{r}\rangle^{\text{h.w.}} = 0 \Rightarrow \bar{\Delta} = r_{m+1} = \cdots = r_{N-1} = -r_N. \tag{A24}$$

We may follow a similar procedure as for $\text{Osp}(2N|4)$ characters in Sec. IV to find $\text{Osp}(2N|2)$ characters for representations $\mathcal{R}_{(\bar{\Delta},\vec{r})}$. The corresponding characters for irreducible representations are given by[39]

$$\chi^{(\text{Osp}(2N-2m|2),i)}_{(\bar{\Delta},\vec{r})}(\bar{s},\bar{y}) = \text{Tr}_{\mathcal{R}_{(\bar{\Delta},\vec{r})}}(\bar{s}^D \bar{y}_1^{H_{m+1}} \cdots \bar{y}_{N-m}^{H_N}) = \frac{\bar{s}^{\bar{\Delta}}}{1-\bar{s}^2} \mathfrak{W}^{\mathcal{S}_{N-m} \ltimes (\mathcal{S}_2)^{N-m-1}} \left(C^{(N-m)}_{\bar{r}}(\bar{y}) \mathcal{R}^{(i)} \right.$$

$$\left. \times (\bar{s},\bar{y}) \prod_{j=1}^{N-m-1} (1+\bar{s}y_j^{-1}) \right), \tag{A25}$$

where, corresponding to the action of supercharges on the highest weight state for long and short representations $\mathcal{R}_{(\bar{\Delta},\vec{r})}$,

$$\mathcal{R}^{(i)}(\bar{s},\bar{y}) = \begin{cases} (1+\bar{s}(\bar{y}_{N-m})^{-1})\prod_{j=1}^{N-m}(1+\bar{s}y_j) & \text{for } i = \text{long} \\ (1+\bar{s}(\bar{y}_{N-m})^{-1})\prod_{j=\bar{n}+1-m}^{N-m}(1+\bar{s}y_j) & \text{for } i = \bar{n} \\ (1+\bar{s}(\bar{y}_{N-m})^{\mp 1}) & \text{for } i = \pm, \end{cases} \tag{A26}$$

where for $i=\text{long}$ then $\bar{\Delta} \geqslant r_{m+1}$, for long multiplets, while for $i=\bar{n}$ then $\bar{\Delta}=r_{m+1}=\cdots=r_{\bar{n}}>r_{\bar{n}+1}$ and for $i=\pm$ then $\bar{\Delta}=r_{m+1}=\cdots=r_{N-1}=\pm r_n$. Here the maximal compact subgroup is $\text{U}(1) \otimes \text{U}(1) \otimes \text{SO}(2N-2m)$ so that the relevant Weyl symmetrizer, acting on Verma module characters, is $\mathfrak{W}^{\mathcal{S}_{N-m} \ltimes (\mathcal{S}_2)^{N-m-1}}$.

APPENDIX B: CHARACTER EXPANSIONS

Defining

$$f(u,x,y) = \sum_{r,s,t \geqslant 0} N_{rst} u^r \chi_s(x) \chi_t(y), \tag{B1}$$

and using usual the orthogonality relation for SU(2) characters in (4.5),

$$-\frac{1}{4\pi i} \oint \frac{dz}{z}(z-z^{-1})^2 \chi_s(z)\chi_t(z) = \delta_{st}, \tag{B2}$$

which may be equivalently expressed by, due to $\chi_j(z)=\chi_j(z^{-1})$,

$$-\frac{1}{2\pi i} \oint \chi_s(z)\chi_t(z)(z-z^{-1})dz = \delta_{st}, \tag{B3}$$

we have that

$$N_{rst} = \frac{1}{(2\pi i)^3} \oint \oint \oint f(u,x,y)u^{-r-1}\chi_s(x)\chi_t(y)(x-x^{-1})(y-y^{-1})dudxdy, \tag{B4}$$

where each contour is the relevant unit circle. Using $\chi_s(z)=-\chi_{-s-2}(z)$, we have

$$N_{rst} = -N_{r-s-2t} = -N_{rs-t-2} = N_{r-s-2-t-2}. \tag{B5}$$

It is convenient below to construct generating functions

$$F_{rs}(z) = \sum_{r=0}^{\infty} N_{rr-2sr-2t} z^r, \tag{B6}$$

so that using (B4), summing over r, and performing the subsequent contour integral over u for $|z| < |xy|$,

$$F_{rs}(z) = \frac{1}{(2\pi i)^2} \oint \oint (x^{1-2s}y^{1-2t}f(zxy,x,y) - x^{1-2s}y^{-1+2t}f(zx/y,x,y) - x^{-1+2s}y^{1-2t}f(zy/x,x,y)$$
$$+ x^{-1+2s}y^{-1+2t}f(z/(xy),x,y)) \mathrm{d}x \mathrm{d}y. \tag{B7}$$

Note that for application to the main text, in terms of the notation used below,

$$N_{\mathrm{free},(r,r-s-t,t-s)} = N_{\mathrm{F},r\ r-2s\ r-2t}, \quad N_{\mathrm{sugra},(r,r-s-t,t-s)} = N_{\mathrm{S},r\ r-2s\ r-2t}. \tag{B8}$$

1. The free field case

For the free field case, due to (7.17) with (7.18), we consider

$$f_{\mathrm{F}}(u,x,y) = \prod_{j\geq 1} \frac{1}{1 - u^j \chi_1(x^j)\chi_1(y^j)} = \sum_{r,s,t\geq 0} N_{\mathrm{F},rst} u^r \chi_s(x)\chi_t(y), \tag{B9}$$

which may be expanded as

$$f_{\mathrm{F}}(u,x,y) = \sum_{\underline{\lambda}} u^{|\underline{\lambda}|} p_{\underline{\lambda}}(x,x^{-1}) p_{\underline{\lambda}}(y,y^{-1}), \tag{B10}$$

in terms of power symmetric polynomials $p_{\underline{\lambda}}(z)$, (6.10) with (6.5). Using an expansion formula for power symmetric polynomials in terms of Schur polynomials, we may expand the latter in terms of SU(2) characters via

$$p_{\underline{\lambda}}(z,z^{-1}) = \sum_{\substack{m,n\geq 0 \\ m+2n=|\underline{\lambda}|}} \omega_{\underline{\lambda}}^{(m,n)} \chi_m(z), \tag{B11}$$

where $\omega_{\underline{\lambda}}^{(m,n)}$ are symmetric group characters. Introducing the notation, $\binom{\underline{\lambda}}{\underline{\rho}} = \Pi_{j\geq 1}\binom{\lambda_j}{\rho_j}$ then it is easily determined from (B11) that (other formulas for these characters may be found in Ref. 40 but the following is the most useful for purposes here, and is perhaps simpler)

$$\omega_{\underline{\lambda}}^{(|\underline{\lambda}|,0)} = 1, \quad \omega_{\underline{\lambda}}^{(|\underline{\lambda}|+2n,n)} = \sum_{\substack{\underline{\rho} \\ |\underline{\rho}|=n}} \binom{\underline{\lambda}}{\underline{\rho}} - \sum_{\substack{\underline{\rho} \\ |\underline{\rho}|=n-1}} \binom{\underline{\lambda}}{\underline{\rho}}, \quad n = 1, \ldots, [|\underline{\lambda}|/2], \tag{B12}$$

with $\omega_{\underline{\lambda}}^{(n,m)}$ being otherwise zero. Thus, using (B3) and (B4) for (B10) with (B11),

$$N_{\mathrm{F},r\ r-2s\ r-2t} = \sum_{\substack{\underline{\lambda} \\ |\underline{\lambda}|=r}} \omega_{\underline{\lambda}}^{(r-2s,s)} \omega_{\underline{\lambda}}^{(r-2t,t)}, \quad s,t = 0, \ldots, [r/2]. \tag{B13}$$

Thus, $N_{\mathrm{F},r\ r-2s\ r-2t}$ is a potentially nonzero, non-negative integer only for $r,s=0, \ldots, [r/2]$.

A useful identity, which may be easily generalized, is the following, namely, for

$$n_{rij} = \sum_{\substack{\underline{\lambda} \\ |\underline{\lambda}|=r}} \binom{\lambda_j}{i},$$

(B14)

then[41]

$$g_{ij}(z) = \sum_{r=0}^{\infty} n_{rij}z^r = \frac{z^{ij}}{(1-z^j)^i}\prod_{k=1}^{\infty}\frac{1}{1-z^k}.$$

(B15)

This may be seen in a simple way by first introducing, $h(x,z)=[1/(1-xz)]\prod_{\substack{k=1 \\ k \neq j}}^{\infty} 1/(1-z^k)$, so that, in a series expansion, $h(x,z)=\Sigma_{\underline{\lambda}}x^{\lambda_j}z^{|\underline{\lambda}|}$. We then have $\lim_{x\to 1}(1/i!)(d^i/dx^i)h(x,z)=g_{ij}(z)$ using $(1/i!)(d^i/dx^i)[1/(1-xz^j)]=z^{ij}/(1-xz^j)^{i+1}$.

The simplest case is for $s=t=0$, whereby for (B6) we find, using (B12) along with (B15),

$$F_{\mathrm{F},00}(z) = g_{00}(z) = \prod_{k=1}^{\infty}\frac{1}{1-z^k},$$

(B16)

and thus have, giving (7.12),

$$N_{\mathrm{F},rrr} = n_{r00} = p(r).$$

(B17)

Similarly, for (B6) using (B12) along with (B15),

$$F_{\mathrm{F},10}(z) = F_{\mathrm{F},01}(z) = g_{11}(z) - g_{00}(z) = \frac{2z-1}{1-z}\prod_{k=1}^{\infty}\frac{1}{1-z^k},$$

(B18)

and so, agreeing with (7.19),

$$N_{\mathrm{F},r\ r-2\ r} = N_{\mathrm{F},r\ r\ r-2} = n_{r11} - p(r) = \sum_{j=0}^{r-1}p(j) - p(r).$$

(B19)

Similarly, for (B6) using (B12) along with (B15),

$$F_{\mathrm{F},11}(z) = 2g_{21}(z) - g_{11}(z) + g_{00}(z) = \frac{4z^2-3z+1}{(1-z)^2}\prod_{k=1}^{\infty}\frac{1}{1-z^k},$$

(B20)

and so, giving (7.20),

$$N_{\mathrm{F},rr-2r-2} = 2n_{r21} - n_{r11} + p(r).$$

(B21)

Notice that by a very similar argument, it may be shown that from (B12) with (B13), for $t=0$,

$$F_{\mathrm{F},s0}(z) = F_{\mathrm{F},0s}(z) = \sum_{\rho_1,\ldots,\rho_s \geqslant 0}(z^s\delta_{\rho_1+\cdots+s\rho_s,s} - z^{s-1}\delta_{\rho_1+\cdots+s\rho_s,s-1})\frac{1}{\prod_{j=1}^{s}(1-z^j)^{\rho_j}}\prod_{k=1}^{\infty}\frac{1}{1-z^k},$$

(B22)

enabling determination of $N_{\mathrm{F},r\ r-2s\ r-2t}$ for $st=0$.

2. The supergravity case

In this case we consider, due to (7.23) with (7.24),

$$f_S(u,x,y) = \prod_{j \geqslant 1} \frac{1}{\prod_{k,l=0}^{j}(1 - u^j x^{2k-j} y^{2l-j})} = \sum_{r,s,l \geqslant 0} N_{S,rst} u^r \chi_s(x) \chi_t(y),$$ (B23)

and we use (B7) and (B6) to determine generating functions for low s,t.

For application to (B6), we have the leading terms

$$f_S(zxy,x,y) = \prod_{k=1}^{\infty} \frac{1}{1-z^k} + \cdots,$$

$$f_S(zx/y,x,y) = \prod_{k=1}^{\infty} \frac{1}{(1-z^k)(1-z^k y^{-2})} + \cdots,$$

$$f_S(zy/x,x,y) = \prod_{k=1}^{\infty} \frac{1}{(1-z^k)(1-z^k x^{-2})} + \cdots,$$

$$f_S(z/(xy),xy) = \prod_{k=1}^{\infty} \frac{1}{(1-z^k)(1-z^k x^{-2})(1-z^k y^{-2})(1-z^k x^{-2} y^{-2})} + \cdots,$$ (B24)

where $\cdots$ denotes terms that in a series expansion in x,y involve powers $x^{2a}, y^{2b}, a,b \in \mathbb{Z}$, $a,b \neq 0,-1$, that do not contribute below.

We have, from (B24),

$$\frac{1}{(2\pi i)^2} \oint \oint xy f_S(zxy,x,y) \mathrm{d}x \mathrm{d}y = 0,$$

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{x}{y} f_S(zx/y,x,y) \mathrm{d}x \mathrm{d}y = 0,$$

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{y}{x} f_S(zy/x,x,y) \mathrm{d}x \mathrm{d}y = 0,$$

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{1}{xy} f_S(z/(xy),x,y) \mathrm{d}x \mathrm{d}y = \prod_{k=1}^{\infty} \frac{1}{1-z^k},$$ (B25)

so that for (B6) we have, from (B7),

$$F_{S,00}(z) = g_{00}(z).$$ (B26)

agreeing with (B16). Hence, $N_{S,rrr} = N_{F,rrr}$ in (B17), leading to the first equation in (7.25).

Similarly, using (B24) and

$$\prod_{k=1}^{\infty} \frac{1}{1-tz^k} = 1 + \frac{zt}{1-z} + O(t^2, z^2),$$ (B27)

we have

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{x}{y} f_S(zxy,x,y) \mathrm{d}x \mathrm{d}y = 0,$$

$$\frac{1}{(2\pi i)^2} \oint \oint yxf_S(zx/y,x,y)dxdy = 0,$$

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{1}{xy} f_S(zy/x,x,y)dxdy = \prod_{k=1}^{\infty} \frac{1}{1-z^k},$$

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{y}{x} f_S(z/(xy),x,y)dxdy = \frac{z}{1-z} \prod_{k=1}^{\infty} \frac{1}{1-z^k}, \tag{B28}$$

so that for (B6), from (B7),

$$F_{S,01}(z) = g_{11}(z) - g_{00}(z), \tag{B29}$$

agreeing with (B20), so that $N_{S,rrr-2} = N_{F,rrr-2}$ in (B21). It is easy to show that $F_{S,10}(z) = F_{S,01}(z)$ so that $N_{S,rr-2r} = N_{S,rrr-2}$. Thus we have (7.25).

Similarly, using (B24) with (B27),

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{1}{xy} f_S(zxy,x,y)dxdy = \prod_{k=1}^{\infty} \frac{1}{1-z^k},$$

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{y}{x} f_S(zx/y,x,y)dxdy = \frac{z}{1-z} \prod_{k=1}^{\infty} \frac{1}{1-z^k},$$

$$\frac{1}{(2\pi i)^2} \oint \oint \frac{x}{y} f_S(zy/x,x,y)dxdy = \frac{z}{1-z} \prod_{k=1}^{\infty} \frac{1}{1-z^k},$$

$$\frac{1}{(2\pi i)^2} \oint \oint xyf_S(z/(xy),x,y)dxdy = \left(\frac{z}{1-z} + \frac{z^2}{(1-z)^2} \right) \prod_{k=1}^{\infty} \frac{1}{1-z^k}, \tag{B30}$$

so that for (B6), from (B7),

$$F_{S,11}(z) = g_{21}(z) - g_{11}(z) + g_{00}(z) = \frac{3z^2 - 3z + 1}{(1-z)^2} \prod_{k=1}^{\infty} \frac{1}{1-z^k}, \tag{B31}$$

giving, from (B15),

$$N_{S,r,r-2r-2} = n_{r21} - n_{r11} + p(r), \tag{B32}$$

leading to (7.26).

APPENDIX C: COUNTING MULTITRACES OF COMMUTING MATRICES

In Ref. 34 it was shown that a basis for multiple traces of J commuting $N \times N$ matrices X_k, $k = 1, \ldots, J$, is provided by

$$\text{Tr}\,\mathcal{U}_1 \text{Tr}\,\mathcal{U}_2 \cdots \text{Tr}\,\mathcal{U}_N, \quad \mathcal{U}_i = \prod_{j=1}^{n} Y_{ij}, \quad Y_{ij} \in \{X_k, k = 0, \ldots, J\} \tag{C1}$$

for $n \to \infty$, where $X_0 = \mathbb{I}$, the identity matrix, which, of course, also commutes with X_k, $k = 1, \ldots, J$, so that initially we treat it on an equal footing with the other matrices. (C1) is linearly independent up to the action of the automorphism symmetry group described below.

Each $\mathcal{U}_i$ has an automorphism symmetry $\mathcal{S}_n$ since $\mathcal{U}_i = \Pi_{j=1}^n Y_{ij} = (\mathcal{U}_i)^\sigma = \Pi_{j=1}^n Y_{i\sigma(j)}$ for every $\sigma \in \mathcal{S}_n$ (since X_k commute). Hence each $\mathrm{Tr}\,\mathcal{U}_i$ corresponds to a graph with $\mathcal{S}_n$ symmetry K_n, where Y_{ij} to corresponds to the jth vertex. Here, K_n is taken to be the complete graph on n vertices.[42] Particular $\mathrm{Tr}\,\mathcal{U}_i$ corresponds to a coloring of the vertices of K_n, in colors c_k, $k = 0, \ldots, J$, with the exact value of $Y_{ij} \in \{X_k, k = 0, \ldots, J\}$ corresponding to the color of the jth vertex.

$\mathrm{Tr}\,\mathcal{U}_1 \mathrm{Tr}\,\mathcal{U}_2 \cdots \mathrm{Tr}\,\mathcal{U}_N$ corresponds to N copies of K_n, denoted K_n^N. This graph has a corresponding automorphism group, in graph theory known as the wreath product group, in this case given by the semidirect product $(\mathcal{S}_n)^N \rtimes \mathcal{S}_N$. For this group, for $\sigma = (\sigma_1, \ldots, \sigma_N) \in (\mathcal{S}_n)^N$, $\tau \in \mathcal{S}_N$, and defining $\sigma^\tau = (\sigma_{\tau^{-1}(1)}, \ldots, \sigma_{\tau^{-1}(N)})$, then for (σ, τ), $(\sigma', \tau') \in (\mathcal{S}_n)^N \rtimes \mathcal{S}_N$, group multiplication is defined by $(\sigma, \tau)(\sigma', \tau') = (\sigma \sigma'^{\tau^{-1}}, \tau \tau')$, so that, with $1_{(\mathcal{S}_n)^N \rtimes \mathcal{S}_N} = (1_{\mathcal{S}_n}, \ldots, 1_{\mathcal{S}_n}, 1_{\mathcal{S}_N})$, then the inverse $(\sigma, \tau)^{-1} = ((\sigma^{-1})^\tau, \tau^{-1})$.

The action of $(\mathcal{S}_n)^N \rtimes \mathcal{S}_N$ on (C1) is given by $(\mathrm{Tr}\,\mathcal{U}_1 \mathrm{Tr}\,\mathcal{U}_2 \cdots \mathrm{Tr}\,\mathcal{U}_N)^{(\sigma, \tau)} = \mathrm{Tr}(\mathcal{U}_{\tau(1)})^{\sigma_1} \mathrm{Tr}(\mathcal{U}_{\tau(2)})^{\sigma_2} \cdots \mathrm{Tr}(\mathcal{U}_{\tau(N)})^{\sigma_N}$, $(\mathcal{U}_i)^{\sigma_k} = \Pi_{j=1}^n Y_{i\sigma_k(j)}$, with elements of the basis related by such a group transformation being identical. To achieve a linearly independent basis, it is necessary to divide out by this automorphism symmetry.

Generically, depending on the graph G, with n vertices, having automorphism group symmetry Γ, the generating function for the number $N_{n_0, \ldots, n_j}$ of colorings of the vertices of G, in n_k colors c_k, $k = 0, \ldots, J$, $\Sigma_{k=0}^J n_k = n$, may be written as

$$C_\Gamma(x_0, \ldots, x_J) = \sum_{\substack{n_0, \ldots, n_J \geq 0 \\ n_0 + \cdots + n_J = n}} N_{n_0, \ldots, n_J} x_0^{n_0} \cdots x_J^{n_J}. \tag{C2}$$

We may use the Polya enumeration theorem to determine the generating function for the number of colorings of K_n^N, equivalent to the number of independent multiple trace operators, (C1), subject to the automorphism symmetry $(\mathcal{S}_n)^N \rtimes \mathcal{S}_N$, in terms of cycle indices for the symmetric permutation group.

The cycle index for the symmetric permutation group $\mathcal{S}_n$ is given by, with the notation of Sec. VI,

$$Z_{\mathcal{S}_n}(s_1, \ldots, s_n) = \sum_{\substack{\underline{\lambda} \\ |\underline{\lambda}| = n}} \prod_{j=1}^n \frac{1}{j^{\lambda_j} \lambda_j!} s_j^{\lambda_j} = \frac{1}{2\pi i} \oint \frac{\mathrm{d}z}{z^{n+1}} \exp\left(\sum_{j \geq 1} \frac{1}{j} z^j s_j\right), \tag{C3}$$

and the generating function for the number of colorings of K_n is, by the Polya enumeration theorem,

$$C_{\mathcal{S}_n}(x_0, x_1, \ldots, x_J) = Z_{\mathcal{S}_n}(s_1, \ldots, s_n), \quad s_i = p_i(x_0, \ldots, x_J),$$

$$= \frac{1}{2\pi i} \oint \frac{\mathrm{d}z}{z^{n+1}} \frac{1}{(1 - zx_0) \cdots (1 - zx_J)}, \tag{C4}$$

where $p_i(x) = \Sigma_{k=0}^J x_k^i$, a power symmetric polynomial. Similarly, for K_n^N, with automorphism group $(\mathcal{S}_n)^N \rtimes \mathcal{S}_N$, the generating function for the number of colorings is given by, using the Polya enumeration theorem applied to wreath product groups,[38]

$$C_{(\mathcal{S}_n)^N \rtimes \mathcal{S}_N}(x_0, x_1, \ldots, x_J) = Z_{\mathcal{S}_N}(\tilde{s}_1, \ldots, \tilde{s}_N), \tag{C5}$$

where

$$\tilde{s}_i = Z_{\mathcal{S}_n}(s_{1i}, \ldots, s_{ni}) = C_{\mathcal{S}_n}(x_0^i, \ldots, x_J^i), \quad s_{ji} = p_j(x_0^i, x_1^i, \ldots, x_J^i). \tag{C6}$$

In order that contributions to $\lim_{n \to \infty} C_{\mathcal{S}_n}(x_0, x_1, \ldots, x_J)$ from finite numbers of X_k, $k = 1, \ldots, J$ in $\mathrm{Tr}\,U_i$ not vanish then we must also take $x_0 \to 1$. Following from (C4),

$$\lim_{n \to \infty} C_{\mathcal{S}_n}(1, x_1, \ldots, x_J) = \lim_{s \to \infty} (1-s) \sum_{n=0}^{\infty} s^n C_{\mathcal{S}_n}(1, x_1, \ldots, x_J) = \frac{1}{(1-x_1) \cdots (1-x_J)}. \tag{C7}$$

Thus the generating function for the number of distinct basis elements of the form (C1) as $n \to \infty$ is

$$\lim_{n \to \infty} C_{(\mathcal{S}_n)^N \rtimes \mathcal{S}_N}(1, x_1, \ldots, x_J) = Z_{\mathcal{S}_N}(\tilde{s}_1, \ldots, \tilde{s}_N), \quad \tilde{s}_i = \frac{1}{(1-x_1^i) \cdots (1-x_J^i)},$$

$$= \frac{1}{2\pi i} \oint \frac{dz}{z^{N+1}} \exp\left(\sum_{n \geq 1} \frac{1}{n} z^n \tilde{s}_n \right) = \frac{1}{2\pi i} \oint \frac{dz}{z^{N+1}} \prod_{j_1, \ldots, j_J = 0}^{\infty} \frac{1}{(1 - z x_1^{j_1} \cdots x_J^{j_J})}, \tag{C8}$$

which matches exactly with the chiral ring partition function obtained using the plethystic approach.[33]

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[15] I thank Troels Harmark for pointing out to me that different sectors, along with the SU(2)×SU(2) one, were also briefly considered in Ref. 5, for $\mathcal{N}=6$ superconformal Chern Simons theory.

[16] The conventions used here for the gamma matrices are that $\gamma^0 = \mathbb{I}$, $\gamma^1 = \sigma_1$, $\gamma^2 = \sigma_3$, in terms of Pauli matrices σ_1, σ_3, so that γ^μ are real, symmetric matrices. We take $(\bar{\gamma}^a)^{\alpha\beta} = \varepsilon^{\alpha\gamma}(\gamma^a)_{\gamma\delta}\varepsilon^{\delta\beta}$, where $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon^{12} = 1$, $\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = -\delta_\alpha{}^\gamma \delta_\beta{}^\delta + \delta_\alpha{}^\delta \delta_\beta{}^\gamma$. We thus have $\gamma^a \bar{\gamma}^b + \gamma^b \bar{\gamma}^a = 2\eta^{ab}\mathbb{I}$, $\bar{\gamma}^a \gamma^b + \bar{\gamma}^b \gamma^a = 2\eta^{ab}\mathbb{I}$ and the completeness relation $(\gamma^a)_{\alpha\beta}(\bar{\gamma}_a)^{\gamma\delta} = \delta_\alpha{}^\gamma \delta_\beta{}^\delta + \delta_\alpha{}^\delta \delta_\beta{}^\gamma$.

[17] $E_{mn}^{+\pm}/E_{mn}^{-\pm}$ correspond to the positive/negative roots $e_m \pm e_n/-e_m \pm e_n$, $m < n$, where e_n, e_m are usual $\mathbb{R}^N$ orthonormal vectors. A linearly independent basis for raising/lowering operators is $\{E_{nn+1}^{+-}, E_{N-1N}^{++}\}/\{E_{nn+1}^{-+}, E_{N-1N}^{--}\}$, $n = 1, \ldots, N-1$, corresponding to the positive/negative simple roots.

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[42] This is the well known graph where any two vertices are joined by an edge. K_n is used here for illustrative purposes—we could, in fact, consider any graph with S_n automorphism symmetry, for instance, the graph complement of K_n, composed of just n vertices, no edges.