Fixed-point logics on trees

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Publication date
2010

Chapter 4

Interpolation for Linear Temporal Languages

4.1 Introduction

Craig's interpolation theorem in classical model theory dates back from the late fifties [44]. It states that if a first-order formula φ (semantically) entails another first-order formula ψ, then there is an *interpolant* first-order formula θ, such that every non-logical symbol in θ occurs both in φ and ψ, φ entails θ and θ entails ψ. The key idea of the Craig interpolation theorem is to relate different logical theories via their common non-logical vocabulary. In his original paper, Craig presents his work as a generalization of Beth's definability theorem, according to which implicit (semantic) definability is equivalent to explicit (syntactic) definability. Indeed, Beth's definability theorem follows from Craig's interpolation theorem, but the latter is more general.

From the point of view of applications in computer science, interpolation is often a desirable property of a logic. For instance, in fields such as automatic reasoning and software development, interpolation is related to modularization [2, 103], a property which allows systems or specifications to be developed efficiently by first building component subsystems (or modules). Interpolation for temporal logics is also an increasingly important topic. Temporal logics in general are widely used in systems and software verification, and interpolation has proven to be useful for building efficient model-checkers [45]. This is particularly true of a strong form of Craig interpolation known as uniform interpolation, which is quite rare in modal logic, but that the modal μ-calculus satisfies (see [46]), whereas most temporal logics lack even Craig interpolation (see [107]).

We study Craig interpolation for fragments and extensions of propositional linear temporal logic (LTL). We use the framework of [12] and work with a general notion of *abstract temporal language* which allows us to consider a general notion of extension of such languages. We consider different sets of temporal connectives and, for each, identify the smallest extension of the fragment of LTL with these

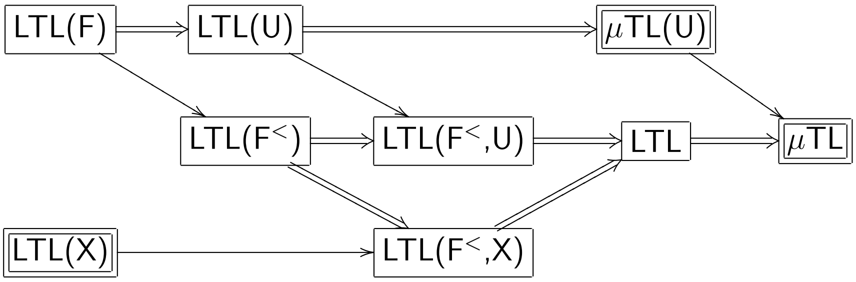

Figure 4.1: Hierarchy of temporal languages

temporal connectives that has Craig interpolation. Depending on the set of temporal connectives, the resulting logic turns out to be either the fragment of LTL with only the Next operator, or the extension of LTL with a fixed-point operator μ (known as linear time μ-calculus), or the fixed-point extension of the fragment of LTL with only the Until operator (which we will show to be the stutter-invariant fragment of the linear time μ-calculus). The diagram in Figure 4.1 summarizes our results. A simple arrow linking two languages means that the first one is an extension of the second one and a double arrow means that, furthermore, every extension of the first one having Craig interpolation is an extension of the second one. Temporal languages with Craig interpolation (in fact, uniform interpolation) are represented in a double frame. Thus we have for instance that $\mu\mathsf{TL}(\mathsf{U})$ is the least expressive extension of $\mathsf{LTL}(\mathsf{F})$ with Craig interpolation.

Outline of the chapter: In Section 4.2, we introduce a general notion of *abstract temporal language*. We then introduce LTL, some of its natural fragments and its fixed-point extension known as linear time μ-calculus ($\mu\mathsf{TL}$) as samples of abstract temporal languages.

Section 4.3 contains some technical results that are used in subsequent sections. One of these relates projective definability in LTL to definability in the fixed-point extension $\mu\mathsf{TL}$. Another result relates in a similar way $\mathsf{LTL}(\mathsf{U})$ and $\mu\mathsf{TL}(\mathsf{U})$. Along the way, we show that $\mu\mathsf{TL}(\mathsf{U})$ is the stutter invariant fragment of $\mu\mathsf{TL}$. Stutter-invariance is a property that is argued by some authors [101] to be natural and desirable for a temporal logic. Roughly, a temporal logic is stutter-invariant if it cannot detect the addition of identical copies of a state.

In Section 4.4, we give three positive interpolation results. Among the fragments of LTL obtained by restricting the set of temporal operators, we show that only one (the "Next-only" fragment) has Craig interpolation. In fact, this fragment satisfies a stronger form of interpolation, called uniform interpolation. The logics $\mu\mathsf{TL}$ and $\mu\mathsf{TL}(\mathsf{U})$ also have uniform interpolation.

Section 4.5 completes the picture by showing that $\mu\mathsf{TL}$ and $\mu\mathsf{TL}(\mathsf{U})$ are the least extensions of $\mathsf{LTL}(F)$ and $\mathsf{LTL}(F^{<})$, respectively, with Craig interpolation.

4.2 Preliminaries

4.2.1 Abstract Temporal Languages

We will be dealing with a variety of temporal languages. They are all interpreted in structures consisting of a set of worlds (or, time points), a binary relation intuitively representing temporal precedence, and a valuation of proposition letters. In this section, we give an *abstract model-theoretic* definition of temporal languages (on the general topic of abstract model theory, we refer to [12]).

Let us recall that a *flow of time*, or *frame*, is a structure $\mathcal{T} = (W, <)$, where W is a non-empty set of worlds and $<$ is a binary relation on W. We will focus here on $\mathbf{T}_\omega$, the class of linear orders of order type ω, i.e., frames $(D, <)$ that are isomorphic to $(\mathbb{N}, <)$, where $\mathbb{N}$ is the set of natural numbers with the natural ordering. We will also freely use $\leq$ to denote the reflexive closure of $<$.

By a *propositional signature* we mean a finite non-empty set of propositional letters $\sigma = \{p_i \mid i \in I\}$. A *pointed σ-structure* is a structure $\mathfrak{M} = (\mathcal{T}, V, w)$ where $\mathcal{T} = (W, R)$ is a frame, $V : \sigma \to \wp(W)$ a valuation and $w \in W$ a world. The class of all pointed σ-structures is denoted by $Str[\sigma]$ and we call them σ-structures for short. Furthermore, for any class of frames $\mathbf{T}$, $Str_{\mathbf{T}}[\sigma]$ will denote the class of σ-structures of which the underlying frame belongs to $\mathbf{T}$. Let $\sigma \subseteq \tau$ be propositional signatures. Given a τ-structure $\mathfrak{M} = (\mathcal{T}, V, w)$, we define its *$\sigma$-reduct* $\mathfrak{M} \upharpoonright \sigma$ as the σ-structure $(\mathcal{T}, V \upharpoonright \sigma, w)$ where $V \upharpoonright \sigma$ is the restriction of the valuation to the propositional letters in σ. We call $\mathfrak{M}$ a *τ-expansion* of $\mathfrak{M} \upharpoonright \sigma$. We also write $K \upharpoonright \sigma$ for $\{\mathfrak{M} \upharpoonright \sigma \mid \mathfrak{M} \in K\}$. Let $(\mathcal{T}, V, w)$ be a σ-structure and $A \subseteq W$ a subset of its domain. By $V[A/p]$, we will refer to the valuation V extended with $V(p) = A$ (p being a fresh proposition letter). We will refer to the corresponding $\sigma \cup \{p\}$-expansion of $(\mathcal{T}, V, w)$ by $(\mathcal{T}, V[A/p], w)$.

Definition 4.2.1 (Abstract temporal language)**.** An abstract temporal language (*temporal language* for short) is a pair $\mathcal{L} = (\mathcal{L}, \models_{\mathcal{L}})$, where $\mathcal{L} : \sigma \mapsto \mathcal{L}[\sigma]$ is a map from propositional signatures to sets of objects that we call formulas and $\models_{\mathcal{L}}$ is a relation between formulas and pointed structures satisfying the following conditions, for all propositional signatures σ, τ:

1. **Expansion property.** If $\sigma \subseteq \tau$ then $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$. Furthermore, for all $\varphi \in \mathcal{L}[\sigma]$ and $\mathfrak{M} \in Str[\tau]$, $\mathfrak{M} \models_{\mathcal{L}} \varphi$ iff $\mathfrak{M} \upharpoonright \sigma \models_{\mathcal{L}} \varphi$. If $\mathfrak{M} \in Str[\sigma]$ and $\mathfrak{M} \models_{\mathcal{L}} \varphi$, then $\varphi \in \mathcal{L}[\sigma]$.

2. **Closure under uniform substitution.** For all $\psi \in \mathcal{L}[\sigma]$, $p \notin \sigma$ and $\varphi \in \mathcal{L}[\sigma \cup \{p\}]$, there is a formula of $\mathcal{L}[\sigma]$, which we will denote by $\varphi[p/\psi]$, such that for every $(\mathcal{T}, V, w) \in Str[\sigma]$ the following holds:

$$(\mathcal{T}, V, w) \models_{\mathcal{L}} \varphi[p/\psi] \text{ iff } (\mathcal{T}, V', w) \models_{\mathcal{L}} \varphi$$

   where $V' = V[\{w \mid (\mathcal{T}, V, w) \models_{\mathcal{L}} \psi\}/p]$.

3. **Negation property.** For each $\varphi \in \mathcal{L}[\sigma]$ there is a formula of $\mathcal{L}[\sigma]$, which we will denote by $\neg\varphi$, s.t. for all $\mathfrak{M} \in Str[\sigma]$, $\mathfrak{M} \models_{\mathcal{L}} \neg\varphi$ iff $\mathfrak{M} \not\models_{\mathcal{L}} \varphi$.

For any class of frames $\mathbf{T}$, $\models_{\mathcal{L},\mathbf{T}}$ will denote the restriction of $\models_{\mathcal{L}}$ to pointed structures based on $\mathbf{T}$. For $\varphi \in \mathcal{L}[\sigma]$, we will use $Mod^{\sigma}(\varphi)$ as shorthand for $\{\mathfrak{M} \in Str[\sigma] \mid \mathfrak{M} \models_{\mathcal{L},\mathbf{T}} \varphi\}$ and $Mod^{\sigma}_{\mathbf{T}}(\varphi)$ when restricting to a frame class $\mathbf{T}$. Whenever this is clear from the context, we will be omitting superscript and subscripts in $Mod^{\sigma}_{\mathbf{T}}(\varphi)$ and $\models_{\mathcal{L},\mathbf{T}}$. We say that a class of pointed structures $\mathbf{K} \subseteq Str_{\mathbf{T}}[\sigma]$ is *definable* in an abstract temporal language $\mathcal{L}$ (relative to the frame class $\mathbf{T}$) if there is a $\mathcal{L}$-formula φ such that for every $(\mathcal{T}, V, w) \in Str_{\mathbf{T}}[\sigma]$, $(\mathcal{T}, V, w) \models \varphi$ iff $(\mathcal{T}, V, w) \in \mathbf{K}$.

Definition 4.2.2 (Extension of a temporal language)**.** Let $\mathcal{L}_1 = (\mathcal{L}_1, \models_{\mathcal{L}_1})$, $\mathcal{L}_2 = (\mathcal{L}_2, \models_{\mathcal{L}_2})$ be temporal languages. $\mathcal{L}_2$ extends $\mathcal{L}_1$ (notation: $\mathcal{L}_1 \subseteq \mathcal{L}_2$) if for all σ, for all $\varphi \in \mathcal{L}_1[\sigma]$, there exists $\varphi^* \in \mathcal{L}_2[\sigma]$ such that $Mod_{\sigma}(\varphi) = Mod_{\sigma}(\varphi^*)$. Also, whenever $\mathcal{L}_1 \subseteq \mathcal{L}_2$, we say that $\mathcal{L}_1$ is a *fragment* of $\mathcal{L}_2$. Whenever restricting attention to a frame class $\mathbf{T}$ we write $\mathcal{L}_1 \subseteq_{\mathbf{T}} \mathcal{L}_2$.

The following notion is related to existential second-order quantification over propositional letters. Allowing such a form of quantification in a given temporal language indeed amounts to considering its projective classes. It is a classical notion in abstract modal theory and it will be useful in the context of Δ-interpolation (see Definition 4.5.2).

Definition 4.2.3 (Projective class)**.** Let σ be a propositional signature, $\mathbf{T}$ a frame class and let $K \subseteq Str_{\mathbf{T}}[\sigma]$. Then K is a projective class of a temporal language $\mathcal{L}$ relative to $\mathbf{T}$ if there is a $\varphi \in \mathcal{L}[\tau]$ with $\tau \supseteq \sigma$ a propositional signature, such that $K = Mod(\varphi) \upharpoonright \sigma$.

Lemma 4.2.4. *Let $\mathbf{T}$ be a frame class. If $\mathcal{L}_1 \subseteq_{\mathbf{T}} \mathcal{L}_2$, then every projective class of $\mathcal{L}_1$ relative to $\mathbf{T}$ is also a projective class of $\mathcal{L}_2$ relative to $\mathbf{T}$.*

Proof. Let K be a projective class of $\mathcal{L}_1$ relative to a frame class $\mathbf{T}$. So there is $\varphi \in \mathcal{L}_1[\tau]$ with $\tau \subseteq \sigma$ a propositional signature, such that $K = Mod^{\tau}_{\mathcal{L},\mathbf{T}}(\varphi) \upharpoonright \sigma$. As $\mathcal{L}_1 \subseteq \mathcal{L}_2$, there is also $\varphi^* \in \mathcal{L}_2[\tau]$ such that $Mod^{\tau}_{\mathcal{L}_1}(\varphi) = Mod^{\tau}_{\mathcal{L}_2}(\varphi^*)$. It follows that $K = Mod^{\tau}_{\mathcal{L}_2}(\varphi^*) \upharpoonright \sigma$. □

Definition 4.2.5 (Entailment)**.** Let $\mathcal{L}$ be a temporal language, σ a propositional signature, $\mathbf{T}$ a frame class and $\varphi, \psi \in \mathcal{L}[\sigma]$. We say that φ entails ψ in $\mathcal{L}$ over $\mathbf{T}$ and write $\varphi \models_{\mathcal{L},\mathbf{T}} \psi$ if for any $(\mathcal{T}, V, w) \in Str_{\mathbf{T}}[\sigma]$, whenever $(\mathcal{T}, V, w) \models_{\mathcal{L},\mathbf{T}} \varphi$, then also $(\mathcal{T}, V, w) \models_{\mathcal{L},\mathbf{T}} \psi$.

4.2.2 Propositional Linear Temporal Logic

Recall that $\mathbf{T}_{\omega}$ denotes the linear orders of order type ω. We now recall the syntax and semantics of LTL, following the terminology of [59].

Definition 4.2.6 (LTL)**.** Let σ be a propositional signature. The set of formulas $\mathsf{LTL}[\sigma]$ is defined inductively, as follows:

$$\varphi, \psi := At \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \varphi \vee \psi \mid \mathsf{X}\varphi \mid \mathsf{F}\varphi \mid \mathsf{F}^{<}\varphi \mid \varphi \mathsf{U} \psi$$

where $At \in \sigma$. We use G and $\mathsf{G}^{<}$ as shorthand for respectively $\neg\mathsf{F}\neg$ and $\neg\mathsf{F}^{<}\neg$. The relation $\models_{\mathsf{LTL}}$ between LTL-formulas and structures $(\mathcal{T}, V, w)$ is defined as follows (we only list the clauses of the temporal operators, the others are as in the case of classical propositional logic):

- $(\mathcal{T}, V, w) \models_{\mathsf{LTL}} \mathsf{X}\varphi$ iff there exists w' such that $w < w'$, there is no w'' such that $w < w'' < w'$ and $(\mathcal{T}, V, w') \models \varphi$
- $(\mathcal{T}, V, w) \models_{\mathsf{LTL}} \mathsf{F}\varphi$ iff there exists w' such that $w \leq w'$ and $(\mathcal{T}, V, w') \models \varphi$
- $(\mathcal{T}, V, w) \models_{\mathsf{LTL}} \mathsf{F}^{<}\varphi$ iff there exists w' such that $w < w'$ and $(\mathcal{T}, V, w') \models \varphi$
- $(\mathcal{T}, V, w) \models_{\mathsf{LTL}} \varphi\mathsf{U}\psi$ iff there exists w' such that $w \leq w'$, $(\mathcal{T}, V, w') \models \psi$ and for all w'' such that $w \leq w'' < w'$, $(\mathcal{T}, V, w'') \models \varphi$

While the above definition in principle applies to arbitrary pointed structures, the intended semantics will be, of course, in terms of structures based on frames in $\mathbf{T}_\omega$, and in what follows we will always restrict attention to such frames.

We define fragments $\mathsf{LTL}(\mathcal{O})$ of LTL by allowing in their syntax only a subset $\mathcal{O} \subseteq \{\mathsf{X}, \mathsf{F}^{<}, \mathsf{F}, \mathsf{U}\}$ of temporal operators. Note that $\mathsf{LTL}(\mathsf{U},\mathsf{X})$ has the same expressive power as LTL, because $\mathsf{F}\varphi$ can be defined as $\top\mathsf{U}\varphi$ and $\mathsf{F}^{<}\varphi$ as $\mathsf{X}(\top\mathsf{U}\varphi)$. The same holds of $\mathsf{LTL}(\mathsf{F}^{<},\mathsf{X})$ and $\mathsf{LTL}(\mathsf{F}^{<},\mathsf{X},\mathsf{F})$, as $\mathsf{F}\varphi$ can be defined as $\varphi \vee \mathsf{F}^{<}\varphi$. Nevertheless, it is known (see [93]), that $\varphi\mathsf{U}\psi$ can be defined neither in $\mathsf{LTL}(\mathsf{F})$ nor in $\mathsf{LTL}(\mathsf{F}^{<},\mathsf{X})$. Also $\mathsf{X}\varphi$ and $\mathsf{F}^{<}\varphi$ can be defined neither in $\mathsf{LTL}(\mathsf{U})$ nor in $\mathsf{LTL}(\mathsf{F})$ (we will see why later on in this chapter, once we introduce the notion of stutter-invariance).

4.2.3 Linear Time μ-Calculus

A way of increasing the expressive power of temporal languages is to add a fixed-point operator. On arbitrary structures, adding to LTL the least fixed-point operator μ gives the μ-calculus (see for instance [46]). Here, the class of intended structures for μ-calculus is restricted to those based on $\mathbf{T}_w$ and the resulting restricted temporal language is called $\mu\mathsf{TL}$ (see for instance [92]). We also recall here its syntax and semantics.

Definition 4.2.7 ($\mu\mathsf{TL}$)**.** Let σ be a propositional signature and $\mathcal{V} = \{x_1, x_2, \ldots\}$ a disjoint countably infinite stock of *propositional variables*. We define $\mu\mathsf{TL}[\sigma]$ as the set of all formulas *without free variables* that are generated by the following inductive definition:

$$\varphi, \psi, \xi := At \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \mathsf{X}\varphi \mid \mathsf{F}\varphi \mid \mathsf{F}^{<}\varphi \mid \varphi\mathsf{U}\psi \mid \mu x_i.\xi$$

where $At \in \sigma \cup \mathcal{V}$ and, in the last clause, x_i occurs only positively in ξ (i.e., within the scope of an even number of negations). We will use $\varphi \to \psi$ as shorthand in the usual way and $\nu x_i.\varphi(x_i)$ as shorthand for $\neg\mu x_i.\neg\varphi(\neg x_i)$. The relation $\models_{\mu\mathsf{TL}}$ is defined between $\mu\mathsf{TL}$-formulas and pointed structures $(\mathcal{T}, V, w)$ where $\mathcal{T} \in \mathbf{T}_\omega$. In order to define it inductively, we use an auxiliary assignment to interpret formulas with free variables. The assignment g maps each free variable of φ to a set of worlds. We let $g[x \mapsto A]$ be the assignment which differ from g only by assigning A to x and we only recall:

- $(\mathcal{T}, V, w) \models_{\mu\mathsf{TL}} x_i\ [g]$ iff $w \in g(x_i)$
- $(\mathcal{T}, V, w) \models_{\mu\mathsf{TL}} \mu x.\varphi\ [g]$ iff $\forall A \subseteq W$, if $\{v \mid (\mathcal{T}, V, v) \models_{\mu\mathsf{TL}} \varphi\ [g[x \mapsto A]]\} \subseteq A$, then $w \in A$

It is easy to see that, for formulas without free variables, the assignment is irrelevant, and therefore $\models_{\mu\mathsf{TL}}$ defines a binary relation between (the set of sentences of) $\mu\mathsf{TL}$ and pointed structures. In this way, $\mu\mathsf{TL}$ is an abstract modal language in the sense of Definition 4.2.1.

As before, we define a fragment $\mu\mathsf{TL}(\mathcal{O})$ for each $\mathcal{O} \subseteq \{\mathsf{X}, \mathsf{F}^<, \mathsf{F}, \mathsf{U}\}$. $\mu\mathsf{TL}(\mathsf{X})$ already as the full expressive power of TL, since $\varphi\mathsf{U}\psi$ can be defined by $\mu y.(\psi \vee (\varphi \wedge \mathsf{X}y))$, $\mathsf{F}^<\varphi$ by $\mu y.(X\varphi \vee \mathsf{X}y)$ and $\mathsf{F}\varphi$ by $\mu y.(\varphi \vee \mathsf{X}y)$. Another fragment of particular interest will be $\mu\mathsf{TL}(\mathsf{U})$. In $\mu\mathsf{TL}(\mathsf{U})$, we can still define $\mathsf{F}\varphi$ in the usual way by $\top\mathsf{U}\varphi$, but we will see that $\mathsf{X}\varphi$ and $\mathsf{F}^<\varphi$ are not definable.

4.3 Projective Definability versus Definability with Fixed-Points

In this section, we discuss two results that relate projective definability in languages without fixed-point operators to explicit definability in the corresponding language with fixed-point operators. Along the way, we also show that $\mu\mathsf{TL}(\mathsf{U})$ is the stutter-invariant fragment of $\mu\mathsf{TL}$. These results will be put to use in Section 4.4 and 4.5.

We first state a general result relating projective definability in LTL and definability in $\mu\mathsf{TL}$. It will be convenient to consider also definability in MSO and definability by a Büchi automaton (for background on Büchi-automata and on MSO, we refer to Chapter 2). In order to be fully precise, we first provide the following definition:

Definition 4.3.1. Let $\sigma = \{p_1, \ldots, p_n\}$ be a propositional signature. We define $\Sigma = \wp(\sigma)$ as the *corresponding alphabet over ω-words* and $\sigma_{\mathsf{FO}} = \{<, P_1, \ldots, P_n\}$ as the *corresponding FO signature over* $\mathbf{T}_\omega$. Now let $\mathcal{T} = (D, <) \in \mathbf{T}_{fin}$ with $D = \{w_0, w_1, \ldots\}$ and $w_i < w_{i+1}$ for all $i \geq 0$. Given a σ-structure $(\mathcal{T}, V, w_j)$, we define the *corresponding ω-word* $(\mathcal{T}, V)^{w_j}$ in signature Σ and the *corresponding relational structure* $(\mathcal{T}, V)^{w_j}_{\mathsf{FO}}$ in signature σ_{FO} in the following way:

- let $w_i^V = \{p \in \sigma \mid w_i \in V(p)\}$, we define $(\mathcal{T}, V)^{w_j}$ as the word $w_j^V w_{j+1}^V \dots$ (i.e., w_j^V is the first letter of the word and for every $i \geq j$, w_{i+1}^V is the letter immediately following w_i^V)
- $(\mathcal{T}, V)_{\mathsf{FO}}^{w_j}$ is the relational structure $(D^{w_j}, <^{w_j}, P_1^{w_j}, \dots, P_n^{w_j})$ in signature σ_{FO} with
  - a domain $D^{w_j} = \{w_i \in D \mid i \geq j\}$
  - a binary relation $<^{w_j} = < \restriction \{w_i \in D \mid i \geq j\}$ (i.e., $<^{w_j}$ is the restriction of the relation $<$ to the points in D that are greater or equal in $<$ to w_j)
  - for every $l \geq 1$, a unary relation $P_l^{w_j} = \{w_i \mid w_i \in V(p_l) \text{ and } i \geq j\}$

Now we can state the general result we are interested in.

Theorem 4.3.2. *Let σ be a propositional signature. For any $K \subseteq Str_{\mathbf{T}_\omega}[\sigma]$, the following are equivalent:*

1. *there is an MSO sentence φ in signature σ_{FO} such that*
$$K = \{(\mathcal{T}, V, w) \mid (\mathcal{T}, V)_{FO}^{w} \models_{MSO} \varphi\}$$
2. *there is a Büchi automata $\mathcal{A}$ over the alphabet $\wp(\sigma)$ such that*
$$K = \{(\mathcal{T}, V, w) \mid (\mathcal{T}, V)^{w} \text{ is accepted by } \mathcal{A}\}$$
3. *K is a projective class of* $\mathsf{LTL}(\mathsf{F}^<, \mathsf{X})$ *relative to $\mathbf{T}_\omega$*
4. *there is a μ*TL *sentence φ such that*
$$K = \{(\mathcal{T}, V, w) \mid (\mathcal{T}, V, w) \models_{\mu\mathsf{TL}} \varphi\}$$

Proof.

$1 \Rightarrow 2$ This is a known result (see [127]).

$2 \Rightarrow 3$ Let $\mathcal{A} = (Q, \Sigma = \wp(P_1, \dots, P_m), \Delta, q_0, Acc)$ be a Büchi automaton. Assume $Q = \{q_0, \dots, q_k\}$ and let $r_1, \dots, r_k$ be pairwise distinct propositional letters not in σ. We will construct a $\mathsf{LTL}(\mathsf{F}^<, \mathsf{X})$-formula which holds in a $\sigma \cup \{r_0, \dots, r_k\}$-structure $(\mathcal{T}, V, w)$ (with $\mathcal{T} \in \mathbf{T}_\omega$) if and only if $(\mathcal{T}, V \restriction \sigma)^w$ is accepted by $\mathcal{A}$. Given an ω-word $(\mathcal{T}, V \restriction \sigma)^w \in L(\mathcal{A})$ of the form $\alpha(0)\alpha(1)\dots$, the sentence will state the existence of a successful run $\rho(0), \rho(1), \dots$ of $\mathcal{A}$, i.e., with $\rho(0) = q_0$ $(\rho(i), \alpha(i), \rho(i+1)) \in \Delta$ for $i \geq 0$, and $Inf(\rho) \cap F = \emptyset$. We introduced new propositional letters because we can code such a state sequence by a tuple of propositional letters $(r_0, \dots, r_k)$

of pairwise disjoint subsets of $\{0, 1, \ldots\}$ such that r_i contains those positions of $\alpha(0)\alpha(1)\ldots$ where state q_i is assumed. The automaton should be able to reach a final state infinitely often. For every $\alpha \in \wp(P_1, \ldots, P_m)$ let α^* be $\bigwedge_{p_i \in \alpha} p_i \wedge \bigwedge_{p_i \notin \alpha} \neg p_i$. Thus, $\mathcal{A}$ accepts the nonempty word $(\mathcal{T}, V \upharpoonright \sigma)^{\geq w}$ iff

$$(\mathcal{T}, V, w) \models (r_0 \wedge \bigwedge_{i>0} \neg r_i)$$

(r_0 contains the first position in w, i.e., r_0 is true at the first node of w)

$$\wedge (\bigwedge_{i \neq j} \neg \mathsf{F}^{<}(r_i \wedge r_j))$$

(all other r_i positions are pairwise different, i.e., if r_i contains a position in w, then if $i \neq j$, r_j does not contain this position)

$$\wedge (\mathsf{G}^{<} \bigvee_{(q_i, \alpha, q_j) \in \Delta} (r_i \wedge \alpha^* \wedge \mathsf{X} r_j))$$

(the next position is consistent with Δ)

$$\wedge \bigvee_{q_j \in F} G^{<}\mathsf{F}^{<} r_j$$

(some state in F occurs infinitely often)

$3 \Rightarrow 1$ Let K be projectively definable relative to $\mathbf{T}_\omega$ by a $\mathsf{LTL}(\mathsf{F}^{<}, \mathsf{X})$-formula φ in an extension σ' of σ. Construct the *standard translation* of φ (this is a FO formula in signature σ', see Chapter 2) and call it φ^*. Now consider $p_1, \ldots, p_n \in \sigma' \backslash \sigma$ and replace uniformly in φ^* the corresponding FO predicates $P_1, \ldots, P_n \in \sigma'_{\mathsf{FO}} \backslash \sigma_{\mathsf{FO}}$ by set variables $X_1, \ldots, X_n$. We obtain the formula $\varphi^*[X_1/P_1, \ldots, X_n/P_n]$ that we can now prefix with existential set quantifiers over $X_1, \ldots, X_n$. The obtained formula

$$\exists X_1 \ldots \exists X_n \varphi^*[X_1/P_1, \ldots, X_n/P_n]$$

is in signature σ_{FO} and has the desired property.

$4 \Leftrightarrow 1$ This is a known result (see [111] and [5]).

□

Below, we will show a similar theorem linking projective definability in $\mathsf{LTL}(\mathsf{U})$ (which was shown in [116, 61] to be the stutter-invariant fragment of LTL) to definability in $\mu\mathsf{TL}(\mathsf{U})$, which we show here to be the stutter-invariant fragment of linear time μ-calculus. Before stating this second result, we first define stuttering.

Intuitively, a stuttering of a linearly ordered structure $\mathfrak{M}$ is a structure obtained from $\mathfrak{M}$ by replacing each world by a non-empty finite sequence of worlds, all satisfying the same proposition letters.

Definition 4.3.3 (Stuttering)**.** Let σ be a propositional signature and $\mathfrak{M} = ((W,<),V,w)$, $\mathfrak{M}' = ((W',<),V',w')$ be in $Str_{\mathbf{T}_\omega}[\sigma]$. We say that $\mathfrak{M}'$ is a stuttering of $\mathfrak{M}$ if and only if there is a surjective function $s : W' \to W$ such that

1. $s(w') = w$
2. for every $w_i, w_j \in W'$, $w_i < w_j$ implies $s(w_i) \leq s(w_j)$
3. for every $w_i \in W'$ and $p \in \sigma, w_i \in V'(p)$ iff $s(w_i) \in V(p)$

Some notation will be useful later on. For any $w \in W$, we let $s^{-1}(w) = \{w' \in W' \mid s(w') = w\}$. We also extend s and s^{-1} to subsets of W' in the following way: for any $A' \subseteq W', A \in W$, we let $s(A') = \{s(v') \mid v' \in A'\}$ and $s^{-1}(A) = \bigcup_{v \in A} s^{-1}(v)$.

Lemma 4.3.4. *Let $\mathfrak{M} = ((W,<),V,w)$, $\mathfrak{M}' = ((W',<),V',w')$ be in $Str_{\mathbf{T}_\omega}[\sigma]$ and $\mathfrak{M}'$ be a stuttering of $\mathfrak{M}$, then the following hold:*

1. *$\forall v' \in W', \forall A' \subseteq W'$ such that $v' \in A'$ implies $s^{-1}(s(v')) \subseteq A'$:*

   $$((W',<),V'[A'/p],v') \textit{ is a stuttering of } ((W,<),V[s(A')/p],s(v'))$$

2. *$\forall v \in W, \forall A \subseteq W, \forall v' \in s^{-1}(v)$:*

   $$((W',<),V'[s^{-1}(A)/p],v') \textit{ is a stuttering of } ((W,<),V[A/p],v)$$

Definition 4.3.5 (Stutter-Invariant Class of Pointed Structures)**.** Let σ be a propositional signature and $\mathbf{K} \subseteq Str_{\mathbf{T}_\omega}[\sigma]$. Then $\mathbf{K}$ is a stutter-invariant class relative to $\mathbf{T}_\omega$ iff for every $\mathfrak{M} \subseteq Str_{\mathbf{T}_\omega}[\sigma]$ and for every stuttering $\mathfrak{M}'$ of $\mathfrak{M}$, $\mathfrak{M} \in \mathbf{K} \Leftrightarrow \mathfrak{M}' \in \mathbf{K}$.

Definition 4.3.6 (Stutter-free Pointed Structure)**.** We say that a pointed structure $\mathfrak{M}$ is *stutter-free* whenever for all $\mathfrak{M}'$ such that $\mathfrak{M}$ is a stuttering of $\mathfrak{M}'$, $\mathfrak{M}'$ is isomorphic to $\mathfrak{M}$.

Only stutter-invariant classes of structures in $Str_{\mathbf{T}_\omega}[\sigma]$ are definable in $\mathsf{LTL}(\mathsf{U})$ and $\mu\mathsf{TL}(\mathsf{U})$. This is known for $\mathsf{LTL}(\mathsf{U})$ (see [61, 116]), but it also holds for $\mu\mathsf{TL}(\mathsf{U})$.

Proposition 4.3.7. *Let σ be a propositional signature. For every $\mu\mathsf{TL}(\mathsf{U})$-sentence φ in signature σ, $Mod(\varphi)$ is stutter-invariant.*

Proof. By induction on the sentence complexity. For the sake of the induction, we can use expanded σ-structures as in classical model theory. Hence we consider two base cases, one for propositional letters and one for propositional variables. The propositional letter case is clear. We handle the propositional variable case x_i similarly, except that we use σ-models expanded with the value of x_i (i.e., models considered together with a partial auxiliary valuation, so that x_i can be seen as a sentence). The induction hypothesis says that for any propositional signature σ and $\mu\mathsf{TL}(\mathsf{U})$-sentence φ of complexity n in signature σ, $Mod(\varphi)$ is a stutter-invariant invariant class. Now consider the case were φ is of complexity $n+1$. We handle the Boolean connectives and the U operator as in the $\mathsf{LTL}(\mathsf{U})$ case. For the U case, suppose $\varphi :\approx \psi U \xi$. We want to show that for every $\mathfrak{M} = ((<, W), V, w) \subseteq Str_{\mathbf{T}}[\sigma]$ and for every stuttering $\mathfrak{M}' = ((<, W'), V', w')$ of $\mathfrak{M}$:

$$\mathfrak{M} \in Mod^{\sigma}_{\mu\mathsf{LTL}(\mathsf{U}),\mathbf{T}}(\psi U \xi) \Leftrightarrow \mathfrak{M}' \in Mod^{\sigma}_{\mu\mathsf{LTL}(\mathsf{U}),\mathbf{T}}(\psi U \xi)$$

$\Rightarrow$ Suppose $((W, <), V, w) \models \psi U \xi$, i.e., there exists w_i such that $w = w_i$ or $w < w_i$, $((W, <), V, w_i) \models \xi$ and for all w_j such that $w < w_j < w_i$, $(\mathcal{T}, V, w_j) \models \psi$. Let w_i be the first point such that $(\mathcal{T}, V, w_i) \models \xi$, then all points w_j before it are such that $(\mathcal{T}, V, w_j) \models \psi$. It follows from the definition of stuttering that the minimal point $s \in s^{-1}(w_j)$ is the first point such that $(\mathcal{T}, V, s) \models \xi$ and all points $s' \in s^{-1}(w_j)$ (with w_j before w_i) are such that $(\mathcal{T}, V, s') \models \psi$, i.e., $((W', <), V', w') \models \psi U \xi$.

$\Leftarrow$ The reasoning is similar.

Now for the fixed-point case, suppose $\varphi :\approx \mu x.\psi(x)$. We want to show that for every $\mathfrak{M} \subseteq Str_{\mathbf{T}}[\sigma]$ and for every stuttering $\mathfrak{M}'$ of $\mathfrak{M}$:

$$\mathfrak{M} = ((<, W), V, w) \in Mod(\mu x.\psi(x)) \Leftrightarrow \mathfrak{M}' = ((<, W'), V', w') \in Mod(\mu x.\psi(x))$$

For the left to right direction, suppose $((W, <), V, w) \models \mu x.\psi(x)$, i.e., $\forall A \subseteq W$, if $\{v \mid ((W, <), V[A/p], v) \models \psi(p)\} \subseteq A$, then $w \in A$. Consider $A' \subseteq W'$ such that $\{v \mid ((W', <), V'[A'/p], v) \models \psi(p)\} \subseteq A'$. We want to show that $w' \in A'$. Let us first show that $v' \in A'$ implies $s^{-1}(s(v')) \subseteq A'$. For every $v' \in A'$, we have that $((W', <), V'[A'/p], v') \models \psi(p)$. Now by induction hypothesis for any $v \in s^{-1}(s(v'))$, $((W', <), V'[A'/p], v) \models \psi(p)$ and by hypothesis on A', $v \in A'$. It follows from this property of A' that $\mathfrak{M}'$ being a stuttering of $\mathfrak{M}$, by Lemma 4.3.4 for any $v' \in W'$, $((<, W'), V'[A'/p], v')$ is also a stuttering of $((<, W), V[s(A')/p], s(v'))$ and by induction hypothesis:

$$((W', <), V'[A'/p], v') \models \psi(p) \text{ iff } ((<, W), V[s(A')/p], s(v')) \models \psi(p)$$

Hence $\{v \mid ((W, <), V[s(A')/p], v) \models \psi(p)\} \subseteq s(A')$. But $\mathfrak{M} \models \mu x.\psi(x)$. It follows that $w \in S(A')$, so $s(w) \in A'$, i.e., $w' \in A'$.

Now for the right to left direction, suppose $((W',<),V',w') \models \mu x.\psi(x)$, i.e., $\forall A' \subseteq W'$, if $\{v \mid (W',<),V'[A'/p],v \models \psi(p)\} \subseteq A'$, then $w' \in A'$. Consider $A \subseteq W$ such that $\{v \mid (W,<),V[A/p],v \models \psi(p)\} \subseteq A$. We want to show that $w \in A$. $\mathfrak{M}'$ being a stuttering of $\mathfrak{M}$, by Lemma 4.3.4, for any $v \in W$, $v' \in s^{-1}(v)$, $((<,W'),V'[s^{-1}(A)/p],v')$ is also a stuttering of $((<,W),V[A/p],v)$ and by induction hypothesis, for any $v \in W, v' \in s^{-1}(v)$:

$$((W',<),V'[s^{-1}(A)/p],v') \models \psi(p) \text{ iff } ((W,<),V[A/p],v) \models \psi(p)$$

Hence $\{v \mid ((W',<),V'[s^{-1}(A)/p],v) \models \psi(p)\} \subseteq s^{-1}(A)$. But $\mathfrak{M}' \models \mu x.\psi(x)$. It follows that $w' \in s^{-1}(A)$, so $s^{-1}(w') \subseteq A$, i.e., $w \in A$. □

Corollary 4.3.8. *Let $K \subseteq Str_{\mathbf{T}_\omega}[\sigma]$ be stutter-invariant and let $\varphi \in \mu\mathsf{TL}(\mathsf{U})[\sigma]$ be a sentence such that for each stutter-free $\mathfrak{M} \in Str_{\mathbf{T}_\omega}[\sigma]$, $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{M} \in K$. Then φ defines K.*

We now show that (over $\mathbf{T}_\omega$) $\mu\mathsf{TL}(\mathsf{U})$ is the stutter-invariant fragment of $\mu\mathsf{TL}$. The proof is a variant of [116], where Peled and Wilke show that stutter-invariant LTL properties are expressible without X. We give it in detail, as the construction procedure below will be useful again later on in the chapter.

Lemma 4.3.9. *Let σ be a modal vocabulary. For every $\mu\mathsf{TL}$ sentence φ in vocabulary σ, there exists a $\mu\mathsf{TL}(\mathsf{U})$ sentence φ^* in vocabulary σ that agrees with φ on all stutter-free σ-structures over $\mathbf{T}_\omega$:*

$$\mathfrak{M} \models \varphi \leftrightarrow \varphi^* \textit{ for all stutter free pointed structures } \mathfrak{M} \in Str_{\mathbf{T}_\omega}[\sigma]$$

Proof. Assume $\sigma = \{p_0,\dots,p_{n-1}\}$. The proof goes by induction on the structure of φ. For convenience, we use expanded structures. The base case is clear: $p^* = p$ for any propositional variable or letter p. Now as regards the induction step, we can set $(\neg\psi)^* = \neg\psi^*$, $(\psi \wedge \xi)^* = \psi^* \wedge \tau^*$, $(\psi\mathsf{U}\xi)^* = \psi^*\mathsf{U}\xi^*$, $(\mu x.\psi)^* = \mu x.\psi^*$. If φ is of the form $\mathsf{X}\psi$, we let B be the set of all possible valuations $\sigma \to \{\bot,\top\}$, and for each $g \in B$, we let β_g be the formula $\alpha_0 \wedge \ldots \wedge \alpha_{n-1}$ where $\alpha_j = p_j$ if $g(p_j) = \top$ and $\alpha_j = \neg p_j$ if $g(p_j) = \bot$. Now observe that if $g, g' \in B$ are such that $g \neq g'$, then

$$\mathfrak{M}, w \models \beta_g \wedge \mathsf{X}\beta_{g'} \leftrightarrow \beta_g\mathsf{U}\beta_{g'} \text{ for } \mathfrak{M} \in Str_{\mathbf{T}}[\sigma] \text{ stutter-free}$$

We have $\mathfrak{M}, w \models \mathsf{X}\psi$ if and only if every point in it satisfies the same set of proposition letters and $\mathfrak{M}, w \models \psi$, or the valuation function does not send the same set of proposition letters to w and to its immediate successor w' and $\mathfrak{M}, w' \models \varphi$. Thus we can set:

$$(\mathsf{X}\psi)^* = \bigvee_{g\in G}((\mathsf{G}\beta_g \wedge \psi^*) \vee \bigvee_{g\neq g'}(\beta_g \wedge \beta_g\mathsf{U}(\beta_{g'} \wedge \psi^*)))$$

□

Theorem 4.3.10. *Let $\varphi \in \mu\mathsf{TL}[\sigma]$ be a sentence such that $Mod_\sigma(\varphi)$ is stutter-invariant. Then there exists $\varphi^* \in \mu\mathsf{TL}(\mathsf{U})[\sigma]$ such that $Mod_\sigma(\varphi) = Mod_\sigma(\varphi^*)$.*

Proof. Follows from Lemma 4.3.9 and Corollary 4.3.8. □

Following [61], we now introduce a variant of the notion of projective class, that we call *harmonious projective class*, which preserves stutter-invariance. Before we define it, we first introduce the notion of a *harmonious expansion*. For any propositional signature σ and worlds w, w', we write $w \equiv_\sigma w'$ if w and w' satisfy the same propositions in σ.

Definition 4.3.11 (Harmonious expansion)**.** Let $\sigma \subseteq \tau$ be propositional signatures and $\mathfrak{M} \in Str_{\mathbf{T}_\omega}[\tau]$. We say that $\mathfrak{M}$ is a harmonious expansion of $\mathfrak{M} \restriction \sigma$ whenever $\forall w, w' \in W$ such that w' is a direct successor of w, $w \equiv_\sigma w'$ implies $w \equiv_\tau w'$.

Definition 4.3.12 (Harmonious projective class)**.** Let σ be a propositional signature and $K \subseteq Str_{\mathbf{T}_\omega}[\sigma]$. Then K is a *harmonious projective class* of a temporal language $\mathcal{L}$ relative to $\mathbf{T}_\omega$ whenever there is $\varphi \in \mathcal{L}[\tau]$ with $\tau \supseteq \sigma$ such that for all $\mathfrak{M} \in Str_{\mathbf{T}_\omega}[\sigma]$: $\mathfrak{M} \in K$ iff there is a harmonious τ-expansion $\mathfrak{M}^+$ of $\mathfrak{M}$ such that $\mathfrak{M}^+ \models \varphi$.

We will be using the following proposition in order to show Theorem 4.3.14. It refers to the notion of ω-regular language (cf. [127], an ω-regular language is a language of ω-words which is definable in MSO or, equivalently, which is recognizable by a Büchi automata). The proof of the proposition in [61] uses a notion of stutter-invariant ω-automata.

Proposition 4.3.13 ([61])**.** *On $\mathbf{T}_\omega$, harmonious projective classes of $\mathsf{LTL}(\mathsf{U})$ define exactly the stutter-invariant ω-regular languages.*

Now we are able to show the following theorem:

Theorem 4.3.14. *Let σ be a propositional signature. For any $K \subseteq Str_{\mathbf{T}}[\sigma]$, the following are equivalent:*

1. *K is a harmonious projective class of $\mathsf{LTL}(\mathsf{U})$ relative to $\mathbf{T}_\omega$*
2. *K is definable by a $\mu\mathsf{TL}(\mathsf{U})$-sentence φ relative to $\mathbf{T}_\omega$*

Proof. Follows from Theorem 4.3.2 and Proposition 4.3.13, because by [61, 116], $\mathsf{LTL}(\mathsf{U})$ is the stutter-invariant fragment of LTL and by Theorem 5.2.9, $\mu\mathsf{TL}(\mathsf{U})$ is the stutter-invariant fragment of $\mu\mathsf{TL}$. □

4.4 Temporal Languages with Craig Interpolation

In this section, we show that three of the temporal languages previously discussed have Craig interpolation.

Definition 4.4.1 (Craig interpolation property)**.** Let $\mathcal{L}$ be a temporal language and $\mathbf{T}$ a frame class. Then $\mathcal{L}$ has the Craig interpolation property over $\mathbf{T}$ whenever the following holds. Let $\varphi \in \mathcal{L}[\sigma]$, $\psi \in \mathcal{L}[\sigma']$. Whenever $\varphi \models_{\mathcal{L},\mathbf{T}} \psi$, then there exists $\theta \in \mathcal{L}[\sigma \cap \sigma']$ such that $\varphi \models_{\mathcal{L},\mathbf{T}} \theta$ and $\theta \models_{\mathcal{L},\mathbf{T}} \psi$.

They even satisfy a stronger form of interpolation called uniform interpolation. Intuitively if a temporal language has uniform interpolation, it means that the interpolant can be constructed so that it depends only on the signature of the antecedent and its intersection with the signature of the consequent.

Definition 4.4.2 (Uniform Interpolation)**.** Let $\mathcal{L}$ be a temporal language and $\mathbf{T}$ a frame class. $\mathcal{L}$ has the *uniform interpolation* property over $\mathbf{T}$ if, for all signatures $\sigma \subseteq \tau$ and for each formula $\varphi \in \mathcal{L}[\tau]$ there is a formula $\theta \in \mathcal{L}[\sigma]$ such that $\varphi \models_{\mathcal{L}} \theta$ and for each formula $\psi \in \mathcal{L}[\tau']$ with $\tau \cap \tau' \subseteq \sigma$, if $\varphi \models_{\mathcal{L}} \psi$ then $\theta \models_{\mathcal{L}} \psi$.

Theorem 4.4.3. $\mu\mathsf{TL}$ *has uniform interpolation over* $\mathbf{T}_\omega$*.*

Proof. MSO has uniform interpolation (for monadic predicates) on any class of structures (so in particular on $\mathbf{T}_\omega$) because it has set quantifiers (see [45]). By [111, 5], $\mu\mathsf{TL}$ is expressively complete for MSO. Hence $\mu\mathsf{TL}$ uniform interpolants can always be obtained via translation into MSO and back. □

Theorem 4.4.4. $\mu\mathsf{TL}(\mathsf{U})$ *has uniform interpolation over* $\mathbf{T}_\omega$*.*

Proof. Let $\sigma \subseteq \tau$ be modal signatures and let $\varphi \in \mu\mathsf{TL}(\mathsf{U})[\tau]$. By Theorem 4.4.3, there exists $\theta \in \mu\mathsf{TL}[\sigma]$ such that $\varphi \models \theta$ and for each formula $\psi \in \mu\mathsf{TL}[\tau']$ with $\tau \cap \tau' \subseteq \sigma$, if $\varphi \models \psi$, then $\theta \models \psi$. Now let $\theta^* \in \mu\mathsf{TL}(\mathsf{U})$ be the formula that agrees with θ on all stutter-free structures based on $\mathbf{T}_\omega$ (by Lemma 4.3.9, such a formula exists). We want to show that $\varphi \models \theta^*$ and that for each formula $\psi \in \mu\mathsf{TL}(\mathsf{U})[\tau']$ with $\tau \cap \tau' \subseteq \sigma$, if $\varphi \models \psi$, then $\theta^* \models \psi$. Let $SMod(\varphi)$ denote the set of stutter free structures in $Mod(\varphi)$. As $Mod(\varphi) \subseteq Mod(\theta)$, $SMod(\varphi) \subseteq SMod(\theta)$. Now by construction of θ^* also $SMod(\varphi) \subseteq SMod(\theta^*)$. $Mod(\varphi)$ and $Mod(\theta^*)$ are both stutter-invariant classes. It follows from Corollary 4.3.8 that the closure under stuttering of $SMod(\varphi)$ is included in the closure under stuttering of $SMod(\theta^*)$, i.e., $Mod(\varphi) \subseteq Mod(\theta^*)$, i.e., $\varphi \models \theta^*$. The argument for $\theta^* \models \psi$ is similar. □

Theorem 4.4.5. $\mathsf{LTL}(\mathsf{X})$ *has uniform interpolation over* $\mathbf{T}_\omega$*.*

Proof. We will show something much stronger, namely that every projective class of $\mathsf{LTL}(\mathsf{X})$ is definable by a $\mathsf{LTL}(\mathsf{X})$-formula.
Let $\varphi \in \mathsf{LTL}(\mathsf{X})[\sigma \cup \tau]$ with $\tau = \{p_1, \ldots, p_l\}$. We will show how to construct a formula $\psi \in \mathsf{LTL}(\mathsf{X})[\sigma]$ that defines the class of σ-reducts of models of φ.
We first show that for every $\sigma \cup \tau$-pointed structure $\mathfrak{M}, w$, there exists $\varphi^S \in \mathsf{LTL}(\mathsf{X})[\sigma]$ such that $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M} \restriction \sigma \models \varphi^S$ and for every σ-pointed structure $\mathfrak{N}, v$, $\mathfrak{N}, v \models \varphi^S$ implies that there exists a $\sigma \cup \tau$-expansion $\mathfrak{N}^+$ of $\mathfrak{N}$ such that $\mathfrak{N}^+, v \models \varphi$. Let $md(\varphi) = n$ be the modal depth of φ, i.e., the maximal nesting depth of X-operators in φ. Intuitively, φ can only talk about the first n worlds in the pointed structure (starting from the designated world w). For each p_i, we can represent the valuation of p_i in $\mathfrak{M}$ at these n first worlds by a set $S_i \subseteq \{0, \ldots, n\}$, where $k \in S_i$ represents that p_i is true at the k-th world starting from w. We denote by $S = (S_1, \ldots, S_l)$ the ordered sequence of all the S_i. Now we define φ^S as follows: we replace each occurrence of p_i in φ that is in the scope of $k \leq n$ X-operators by $\top$ if $k \in S_i$ and $\bot$ otherwise. We can now show by induction on $md(\varphi)$ that for every $\sigma \cup \tau$-pointed structure $\mathfrak{M}, w$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M} \restriction \sigma, w \models \varphi^S$ and for every σ-pointed structure $\mathfrak{N}, v$, $\mathfrak{N}, v \models \varphi^S$ implies that there exists a $\sigma \cup \tau$-expansion $\mathfrak{N}^+$ of $\mathfrak{N}$ such that $\mathfrak{N}^+, v \models \varphi$. Whenever $md(\varphi) = 0$, then we are just in the propositional case and the property immediately follows. Now assume the property holds for all formulas ψ with $md(\psi) = n$ and consider φ with $md(\varphi) = n + 1$. φ is equivalent to a Boolean combination of formulas which are either of modal depth $\leq n$ (and to which the inductive hypothesis applies directly), or which are of the form $\mathsf{X}\xi$ with $md(\xi) = n$. Let w' be the first successor of w. For every such ξ, by induction hypothesis $\mathfrak{M}, w' \models \xi$ iff $\mathfrak{M} \restriction \sigma, w' \models \xi^{S'}$ and for every σ-pointed structure $\mathfrak{N}, v$, $\mathfrak{N}, v \models \xi^{S'}$ implies that there exists a $\sigma \cup \tau$-expansion $\mathfrak{N}^+$ of $\mathfrak{N}$ such that $\mathfrak{N}^+, v \models \xi$, where S' encodes the valuation of the proposition letters in τ at each of the n first states starting from w'. By the semantics of the X-operator, it follows that $\mathfrak{M}, w \models \mathsf{X}\xi$ iff $\mathfrak{M} \restriction \sigma, w \models \mathsf{X}(\xi^{S'})$. Also, assuming there is a state v' in $\mathfrak{N}$ which is the immediate predecessor of v, $\mathfrak{N}, v' \models \mathsf{X}\xi^{S'}$ implies that there exists a $\sigma \cup \tau$-expansion $\mathfrak{N}^+$ of $\mathfrak{N}$ such that $\mathfrak{N}^+, v' \models \mathsf{X}\xi$. Now it is enough to remark that $\mathsf{X}(\xi^{S'})$ and $(\mathsf{X}\xi)^S$ denote one and the same formula. Hence $\mathfrak{M}, w \models \mathsf{X}\xi$ iff $\mathfrak{M} \restriction \sigma, w \models \mathsf{X}\xi^S$ and $\mathfrak{N}, v' \models \mathsf{X}\xi^{S'}$ iff $\mathfrak{N}, v' \models \mathsf{X}\xi^S$. So the property also follows for φ.
Finally, the number of proposition variables in τ being finite, we can quantify over the finite number of all such possible valuations S and we let $\psi = \bigvee_S \varphi^S$. Assume ψ holds in a pointed σ-structure $\mathfrak{M}, w$. Then for some S there is φ^S such that $\mathfrak{M}, w \models \varphi^S$, i.e., $\mathfrak{M}^+, w \models \varphi$ where $\mathfrak{M}^+$ is a $\sigma \cup \tau$-expansion of $\mathfrak{M}, w$ in which the valuation of the p_i's is as described by S. Now assume $\mathfrak{M}, w$ has a $\sigma \cup \tau$-expansion satisfying φ. Then the valuation of the p_i's in the first n worlds after w can be represented by some S and $\mathfrak{M}, w \models \varphi^S$, which yields $\mathfrak{M}, w \models \psi$. This means that ψ holds in a pointed σ-structure $\mathfrak{M}, w$ iff $\mathfrak{M}, w$ has a $\sigma \cup \tau$-expansion satisfying φ , i.e., ψ defines the class of σ-reducts of models of φ. $\square$

4.5 Interpolation Closure Results for Temporal Languages

In this section, we look at the fragments of LTL that do not have Craig interpolation, and we address the question how much expressive power must be added in order to regain interpolation. We will phrase our main results in terms of the notion of interpolation closure, which we define by taking inspiration from abstract model theory (see [12]):

Definition 4.5.1 (Interpolation Closure)**.** Let $\mathbf{T}$ be a frame class. $\mathcal{L}_2$ is the interpolation closure of $\mathcal{L}_1$ over $\mathbf{T}$ if $\mathcal{L}_1 \subseteq_{\mathbf{T}} \mathcal{L}_2$, $\mathcal{L}_2$ has interpolation over $\mathbf{T}$, and for every abstract temporal language $\mathcal{L}_3$, if $\mathcal{L}_1 \subseteq \mathcal{L}_3$ and $\mathcal{L}_3$ has Craig interpolation on $\mathbf{T}$, then $\mathcal{L}_2 \subseteq_{\mathbf{T}} \mathcal{L}_3$.

4.5.1 The Interpolation Closure of LTL($\mathsf{F}^{<}$)

A useful tool (see [12]) for proving interpolation closure results is the following lemma:

Definition 4.5.2 (Δ-interpolation property)**.** Let $\mathcal{L}$ be a temporal language and $\mathbf{T}$ a frame class. Then $\mathcal{L}$ has the Δ-interpolation property over $\mathbf{T}$ whenever the following holds: let σ be a propositional signature and $K \subseteq Str_{\mathbf{T}}[\sigma]$, if both K and $\bar{K}$ are projective classes of $\mathcal{L}$ relative to $\mathbf{T}$, there is a $\mathcal{L}$-formula φ such that $K = Mod^{\sigma}_{\mathbf{T}}(\varphi)$.

Lemma 4.5.3. *Let $\mathcal{L}$ be a temporal language with Craig interpolation on $\mathbf{T}_\omega$. Then $\mathcal{L}$ has Δ-interpolation over $\mathbf{T}_\omega$.*

Lemma 4.5.4 (Δ-interpolation follows from Craig interpolation)**.** *Let $\mathcal{L}$ be a temporal language with Craig interpolation on some frame class $\mathbf{T}$. Then $\mathcal{L}$ has Δ-interpolation over $\mathbf{T}$.*

Proof. Let $\mathbf{K} \subseteq Str_{\mathbf{T}}[\sigma]$ such that both $\mathbf{K}$ and $Str_{\mathbf{T}}[\sigma] \backslash \mathbf{K}$ are projective classes of $\mathcal{L}$ relative to $\mathbf{T}$. We want to show that there is a $\xi \in \mathcal{L}[\sigma]$ such that $\mathbf{K} = Mod_{\mathcal{L},\mathbf{T}}(\xi)$.

Since $\mathbf{K}$ and $Str_{\mathbf{T}}[\sigma] \backslash \mathbf{K}$ are projective classes, there are formulas $\varphi \in \mathcal{L}[\sigma \cup \tau]$ such that $\mathbf{K} = Mod_{\mathcal{L},\mathbf{T}}(\varphi) \restriction \sigma$ and $\psi \in \mathcal{L}[\sigma \cup \tau']$ such that $Str_{\mathbf{F}}[\sigma] \backslash \mathbf{K} = Mod_{\mathcal{L},\mathbf{T}}(\psi) \restriction \sigma$. It follows that $\varphi \models_{\mathcal{L},\mathbf{T}} \neg\psi$. Without loss of generality, we can assume that τ and τ' are disjoint. Indeed, suppose $\tau \cap \tau' = p$ (we consider only the case where $\tau \cap \tau'$ contains one single propositional letter, as the other cases only generalize this simpler one). Now, let q be a fresh propositional letter. By closure under uniform substitution of $\mathcal{L}$, for every $\mathcal{T} \in \mathbf{T}$ and $(\mathcal{T}, V, w) \in Str_{\mathbf{T}}[\sigma \cup \tau]$ the following holds:

$$(\mathcal{T}, V, w) \models \varphi[q/p] \text{ iff } (\mathcal{T}, V', w) \models \varphi$$

where V' extends V with $V(q) = V(p)$. Hence $\mathbf{K} = Mod_{\mathcal{L},\mathbf{T}}(\varphi) \restriction \sigma$ and so $\mathbf{K} = Mod_{\mathcal{L},\mathbf{T}}(\varphi[q/p]) \restriction \sigma$ and the intersection of the signatures of $\varphi[q/p]$ and ψ does not contain any propositional letter not in σ.

Since $\mathcal{L}$ has interpolation, there must be a $\theta \in \mathcal{L}[\sigma]$ such that $\varphi \models_{\mathcal{L},\mathbf{T}} \theta$ and $\theta \models_{\mathcal{L},\mathbf{T}} \neg\psi$. As a last step, we will show that $Mod_{\mathcal{L},\mathbf{T}}(\theta) = \mathbf{K}$.

Suppose $\mathfrak{M} \in \mathbf{K}$. Then $\mathfrak{M} = \mathfrak{N} \restriction \sigma$ for some $\mathfrak{N} \in Mod_{\mathcal{L},\mathbf{T}}(\varphi)$. Since $\varphi \models_{\mathcal{L},\mathbf{T}} \theta$, it follows that $\mathfrak{N} \models \theta$. By the expansion property, $\mathfrak{M} \models \theta$. Conversely, suppose $\mathfrak{M} \notin K$. Then $\mathfrak{M} = \mathfrak{N} \restriction \sigma$ for some $\mathfrak{N} \in Mod_{\mathcal{L},\mathbf{T}}(\psi)$. Since $\theta \models_{\mathcal{L},\mathbf{T}} \neg\psi$, it follows that $\mathfrak{N} \not\models \theta$. By the expansion property, $\mathfrak{M} \not\models \theta$. □

The proof of Lemma 4.5.3 given below is similar to the one given in [39] (we only need to remark that the substitution property assumed here of abstract temporal languages is stronger than the *renaming* property assumed in [39] of abstract modal languages).

Now we will show that $\mathsf{LTL}(\mathsf{F}^<,\mathsf{X})$ is contained in the interpolation closure of $\mathsf{LTL}(\mathsf{F}^<)$ over $\mathbf{T}_\omega$. As an intermediate step, we show that in every extension of $\mathsf{LTL}(\mathsf{F}^<)$ having Craig interpolation, the property $\mathsf{X}p$ is "definable". By this, we mean the following:

Lemma 4.5.5. *Let $\mathcal{L}$ be an extension of $\mathsf{LTL}(\mathsf{F}^<)$ with Craig-interpolation over $\mathbf{T}_\omega$. Then there is $\xi \in \mathcal{L}[\{p\}]$ such that $Mod(\xi) = Mod(\mathsf{X}p)$.*

Proof. Let q, r be new distinct propositional letters. Consider the two following projective classes of $\mathsf{LTL}(\mathsf{F}^<)$: $Mod(\mathsf{F}^<(p\wedge q)\wedge\neg\mathsf{F}^<\mathsf{F}^<q) \restriction \{p\}$ and $Mod((\mathsf{F}^<(\neg p\wedge r)\wedge\neg\mathsf{F}^<\mathsf{F}^<r)\vee\mathsf{G}^<\bot) \restriction \{p\}$. As $\mathsf{LTL}(\mathsf{F}^<) \subseteq \mathcal{L}$, these two classes are also projective classes of $\mathcal{L}$ (by Lemma 4.2.4). They also complement each other, as a $\{p\}$-structure belongs to the first class exactly when the first node of this structure has a successor node where p holds and it belongs to the second class in all other cases. By Δ-interpolation for $\mathcal{L}$ on $\mathbf{T}$, it follows that the first class is definable in $\mathcal{L}$ by means of some formula ξ in signature $\{p\}$, i.e., there is $\xi \in \mathcal{L}[\{p\}]$ such that $Mod(\mathsf{X}p) = Mod(\xi)$. □

Theorem 4.5.6. *Every extension of $\mathsf{LTL}(\mathsf{F}^<)$ with Craig interpolation over $\mathbf{T}_\omega$ is an extension of $\mathsf{LTL}(\mathsf{F}^<,\mathsf{X})$ over $\mathbf{T}_\omega$.*

Proof. Let $\mathcal{L}$ be an extension of $\mathsf{LTL}(\mathsf{F}^<)$ with Craig interpolation over $\mathbf{T}_\omega$ and σ a propositional signature. We show by induction on the complexity of φ (number of Boolean and temporal operators in φ) that for all $\varphi \in \mathsf{LTL}(\mathsf{F}^<,\mathsf{X})[\sigma]$, there exists $\varphi' \in \mathcal{L}[\sigma]$ such that $Mod(\varphi) = Mod(\varphi')$. The base case is clear. The induction hypothesis says that for all σ, for all $\varphi \in \mathsf{LTL}(\mathsf{F}^<,\mathsf{X})[\sigma]$ of complexity at most n, there exists $\varphi' \in \mathcal{L}[\sigma]$ such that $Mod(\varphi) = Mod(\varphi')$. Now let φ be of complexity $n+1$. If $\varphi := \mathsf{X}\psi$, by induction hypothesis there exists $\psi' \in \mathcal{L}[\sigma]$ such that $Mod(\psi) = Mod(\psi')$. Pick any $p \notin \sigma$. By Lemma 4.5.5 and the expansion property we know:

1. There is $\xi \in \mathcal{L}[\sigma \cup \{p\}]$ such that $Mod(\mathsf{X}p) = Mod(\xi)$.

We will define φ' as $\xi[p/\psi'] \in \mathcal{L}[\sigma]$ (by closure under uniform substitution of $\mathcal{L}$, such a formula exists). We need to show that $Mod(\mathsf{X}\psi) = Mod(\xi[p/\psi'])$. From 1 we can derive as a particular case:

2. For any $(\mathcal{T}, V, w) \in Str_{\mathbf{T}}[\sigma \cup \{p\}]$ where $V(p) = \{w_i \mid (F, V, w_i) \models \psi'\}$, $(\mathcal{T}, V, w) \models \xi$ iff there exists $w' \in D$ such that $w < w'$, there is no w'' such that $w < w'' < w'$ and $(\mathcal{T}, V, w') \models p$.

Now by closure under uniform substitution of $\mathcal{L}$, 2 is equivalent to the following:

3. For any $(\mathcal{T}, V, w) \in Str_{\mathbf{T}}[\sigma]$, $(F, V, w) \models \xi[p/\psi']$ iff there exists $w' \in D$ such that $w < w'$, there is no w'' such that $w < w'' < w'$ and $(F, V, w') \models p[p/\psi']$.

Finally, ψ' and $p[p/\psi']$ holding exactly in the same models, we can replace $p[p/\psi']$ by ψ' in the second member of the equivalence in 3. It follows that $Mod(\mathsf{X}\psi) = Mod(\xi[p/\psi'])$. We can use similar arguments for the operator $\mathsf{F}^<$ and for Boolean connectives. □

By putting Lemma 4.5.3 to use, we now improve Theorem 4.5.6 and identify the interpolation closure of $\mathsf{LTL}(\mathsf{F}^<)$.

Theorem 4.5.7. *$\mu\mathsf{TL}$ is the interpolation closure of $\mathsf{LTL}(\mathsf{F}^<,\mathsf{X})$ over $\mathbf{T}_\omega$.*

Proof. Let σ be a propositional signature. Now let $K \subseteq Str_{\mathbf{T}_\omega}[\sigma]$ be definable by a $\mu\mathsf{TL}$-sentence φ in signature σ. As $\mu\mathsf{TL}$ is closed under negation, there is a $\mu\mathsf{TL}$-sentence $\neg\varphi$ in signature σ, which defines the complement of K over $Str_{\mathbf{T}_\omega}[\sigma]$. It follows by Theorem 4.3.2 that both K and its complement are projective classes of $\mathsf{LTL}(\mathsf{F}^<,\mathsf{X})$. Now consider a temporal language $\mathcal{L} \supseteq \mathsf{LTL}(\mathsf{F}^<,\mathsf{X})$ with Craig interpolation over $\mathbf{T}_\omega$. By Lemma 4.2.4, K and its complement are also projective classes of $\mathcal{L}$ and by Lemma 4.5.3, it follows that K is definable in $\mathcal{L}$. □

4.5.2 The Interpolation Closure of $\mathsf{LTL}(\mathsf{F})$

For the case of the stutter-invariant languages $\mathsf{LTL}(\mathsf{F})$ and $\mathsf{LTL}(\mathsf{U})$, we need to refine the notion of Δ-interpolation, by considering harmonious projective classes.

Definition 4.5.8 (Harmonious Δ-interpolation property)**.** Let $\mathcal{L}$ be a temporal language. Then $\mathcal{L}$ has the harmonious Δ-interpolation property over $\mathbf{T}_\omega$ whenever the following holds. Let K be a class of $\mathcal{L}$-structures based on $\mathbf{T}_\omega$. If both K and $\bar{K}$ are harmonious projective classes of $\mathcal{L}$ relative to $\mathbf{T}_\omega$, there is a $\mathcal{L}$-formula φ such that $K = Mod_{\mathbf{T}_\omega}(\varphi)$.

Lemma 4.5.9. *If $\mathcal{L}_1 \subseteq \mathcal{L}_2$, then every harmonious projective class of $\mathcal{L}_1$ is also a harmonious projective class of $\mathcal{L}_2$.*

Definition 4.5.10 (Harmonious temporal language)**.** A temporal language $\mathcal{L}$ is *harmonious* for $\mathbf{T}_\omega$ if the following holds. For every $\sigma \subseteq \tau$ propositional signatures, there is a formula $\varphi \in \mathcal{L}[\tau]$ such that for every $\mathfrak{M} \in Str_{\mathbf{T}_\omega}[\tau]$, $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{M}$ is an harmonious expansion of $\mathfrak{M} \restriction \sigma$.

Proposition 4.5.11. $\mathsf{LTL(U)}$ *and its extensions are harmonious for* $\mathbf{T}_\omega$.

Proof. Fix $\sigma \subseteq \tau$ with $|\sigma| = n$, $|\tau \backslash \sigma| = m$. We can represent any valuation over σ by a finite conjunction of atoms and negations of atoms. Let $\{\sigma_i \mid i \in 2^n\}$ be the set of all such conjunctions. Also, for each σ_i, we define the corresponding set $\{\tau^i_j \mid j \in 2^m\}$ as the set of conjunctions representing all possible ways of extending to τ the valuation represented by σ_i. Now for every $\mathfrak{M} \in Str_{\mathbf{T}}[\tau]$,

$$\mathfrak{M} \models \bigwedge_{i,j \in 2^n} (\sigma_i \mathsf{U} \sigma_j \to \bigvee_{k,l \in 2^m} \tau^i_k \mathsf{U} \tau^j_l)$$

if and and only if $\mathfrak{M}$ is an harmonious expansion of $\mathfrak{M} \restriction \sigma$, i.e., $\mathsf{LTL(U)}$ is harmonious. It is immediate from definition 4.2.2 that every extension of a temporal language which is harmonious for $\mathbf{T}_\omega$ is also harmonious for $\mathbf{T}_\omega$. □

Lemma 4.5.12. *Let $\mathcal{L}$ be a temporal language which has Craig interpolation and is harmonious for $\mathbf{T}_\omega$. Then $\mathcal{L}$ has harmonious Δ-interpolation over $\mathbf{T}_\omega$.*

Proof. $\mathcal{L}$ being harmonious, we can use the formula φ in Definition 4.5.10 and appeal for the proof of Lemma 4.5.12 to the same classical argument as for Lemma 4.5.3. Let $\mathbf{K} \subseteq Str[\sigma]$ such that both $\mathbf{K}$ and $Str_{\mathbf{T}}[\sigma] \backslash \mathbf{K}$ are harmonious projective classes of $\mathcal{L}$ relative to $\mathbf{T}$. Then there is $\varphi \in Fml_{\mathcal{L}}[\tau]$ with $\tau \supseteq \sigma$ such that for all $\mathfrak{M} \in Str_{\mathbf{T}}[\tau]$, $\mathfrak{M} \restriction \sigma \in K$ iff $\mathfrak{M} \models \varphi$ and $\mathfrak{M}$ is an harmonious expansion of $\mathfrak{M} \restriction \sigma$. Also there is $\psi \in Fml_{\mathcal{L}}[\tau']$ with $\tau' \supseteq \sigma$ such that for all $\mathfrak{M} \in Str_{\mathbf{T}}[\tau']$, $\mathfrak{M} \restriction \sigma \in K$ iff $\mathfrak{M} \models \psi$ and $\mathfrak{M}$ is an harmonious expansion of $\mathfrak{M} \restriction \sigma$. As $\mathcal{L}$ is harmonious for $\mathbf{T}$, it follows that there is ξ such that $\varphi \wedge \xi \models_{\mathcal{L},\mathbf{T}} \neg(\psi \wedge \xi)$. The remaining of the proof is as in Theorem 4.5.3. □

Theorem 4.5.13. *Every extension of* $\mathsf{LTL(F)}$ *with Craig interpolation over* $\mathbf{T}_\omega$ *is an extension of* $\mathsf{LTL(U)}$ *over* $\mathbf{T}_\omega$.

Proof. The reasoning is similar as in the case of Lemma 4.5.6 and Theorem 4.5.6, but we consider $Mod(p\mathsf{U}q) = Mod(\mathsf{G}(\mathsf{F}r \to r) \wedge \mathsf{F}(q \wedge r) \wedge \mathsf{G}((r \wedge \neg q) \to p)) \restriction \{p, q\}$ and $Mod(\neg p\mathsf{U}q) = Mod(\mathsf{F}q \to (\mathsf{F}(\neg p \wedge r) \wedge \mathsf{G}(\mathsf{F}r \to \neg q))) \restriction \{p, q\}$. □

Theorem 4.5.14. $\mu\mathsf{TL(U)}$ *is the interpolation closure of* $\mathsf{LTL(U)}$ *over* $\mathbf{T}_\omega$.

Proof. Let σ be a modal signature. Now let $K \subseteq Str_{\mathbf{T}_\omega}[\sigma]$ be definable by a $\mu\mathsf{TL(U)}$-sentence φ in signature σ. As $\mu\mathsf{TL(U)}$ is closed under negation, there is a $\mu\mathsf{TL(U)}$-sentence $\neg\varphi$ in signature σ, which defines the complement $\bar{K} \subseteq Str_{\mathbf{T}_\omega}[\sigma]$ of K over $Str_{\mathbf{T}_\omega}[\sigma]$. By Theorem 4.3.14, both K and $\bar{K}$ are harmonious projective classes of $\mathsf{LTL}(\mathsf{U})$. Now consider a temporal language $\mathcal{L} \supseteq \mathsf{LTL}(\mathsf{U})$ with Craig interpolation over $\mathbf{T}$. By Lemma 4.5.9, K and $\bar{K}$ are also harmonious projective classes of $\mathcal{L}$. By Proposition 4.5.11, $\mathcal{L}$ is harmonious and by Lemma 4.5.12, it follows that K is definable in $\mathcal{L}$, i.e., $\mathcal{L} \supseteq \mu\mathsf{TL}(\mathsf{U})$. □

4.6 Finite Linear Orders

We restricted our attention to the frame class $\mathbf{T}_\omega$, but our results easily extend to finite linear orders. Let $\mathbf{T}_{\mathit{fin}}$ be the class of frames $(D, <)$ where D is a finite set and $<$ is a strict linear order on D. All the definitions and results that we gave relative to $\mathbf{T}_\omega$ also apply to $\mathbf{T}_{\mathit{fin}}$. An analogous of Theorem 4.3.2 for $\mathbf{T}_{\mathit{fin}}$ can be obtained by considering automata on finite words. The proof of Proposition 4.3.13 can similarly be adapted by considering stutter-invariant automata on finite words. In the proof of Lemma 4.3.9, we can define $(\mathsf{X}\psi)^*$ as $\bigvee_{g \neq g'}(\beta_g \mathsf{U}(\beta_{g'} \wedge \psi^*))$ (i.e., we keep only the second disjoint, as no finite stutter free linear order exhibits two successor points satisfying the same set of proposition letters). The remaining of our arguments do not need any further adjustment.

4.7 Conclusion

In this chapter, we studied the temporal fragments of linear time μ-calculus satisfying Craig interpolation, showing essentially that there are only three distinct such fragments: $\mu\mathsf{TL}$ itself, $\mu\mathsf{TL}(\mathsf{U})$, and $\mathsf{LTL}(\mathsf{X})$. These results reconfirm the robustness of (linear time) μ-calculus as compared to less expressive temporal logics. They also allow to identify $\mu\mathsf{TL}(\mathsf{U})$ as a particularly well-behaved linear-time logic which does not seem to have been studied before. In particular, complete axiomatizations were already known for $\mu\mathsf{TL}$ and $\mathsf{LTL}(\mathsf{X})$ (see Chapter 2), but this was not the case for $\mu\mathsf{TL}(\mathsf{U})$. In the next Chapter, we will study this logic further by providing such a complete axiomatization.

We are currently working on extending our interpolation results to other flows of time such as finite trees, infinite trees, and infinite linear orders other than the natural numbers (as in [34]). There are some important differences in these settings. For example, it is known (see [3]) that the branching time temporal logic with only Since and Until has Craig interpolation, while linear time fails to have this property. Also there is still no definitive consensus on the appropriate notion of stuttering for infinite branching time (see [81]). Finally, let us note that whether Propositional Dynamic Logic PDL (see [26]), which can be defined as a semantic fragment of the μ-calculus, satisfies some form of interpolation is still an open problem. It would be worth trying to obtain at least partial results for PDL on finite trees by using our methods.