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DOI
10.1007/BF00312668

Publication date
1996

Published in
Letters in Mathematical Physics

Letters in Mathematical Physics **37**: 363–374, 1996.

On Dijkgraaf–Witten-Type Invariants

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(Received: 25 February 1994; revised version: 3 November 1995)

Abstract. We explicitly construct a series of lattice models based upon the gauge group Z_p which have the property of subdivision invariance, when the coupling parameter is quantized and the field configurations are restricted to satisfy a type of mod-p flatness condition. The simplest model of this type yields the Dijkgraaf–Witten invariant of a 3-manifold and is based upon a single link, or 1-simplex, field. Depending upon the manifold's dimension, other models may have more than one species of field variable, and these may be based on higher-dimensional simplices.

Mathematics Subject Classification (1991): 55-XX, 81Rxx.

Key words: lattice models, Dijkgraaf–Witten invariant, Chern–Simons series, Boltzmann weights.

1. Introduction

An intriguing three-dimensional lattice model was constructed by Dijkgraaf and Witten in [1]. By general considerations in gauge theory, it was shown that three-dimensional Chern–Simons theories are classified by the cohomology classes in $H^4(BG, Z)$, where BG is the universal classifying space for the group G. In the case of a finite group, they showed that the Boltzmann weight of such a theory was a 3-cocycle in $H^3(BG, R/Z)$; the cocycle condition being equivalent to the equation which guaranteed subdivision invariance of the lattice model. Subdivision invariance is, roughly speaking, the analogue of metric independence of a continuum theory.

In this Letter, we will find a more concrete formulation for lattice models which have some features similar to the Dijkgraaf–Witten theory; their theory will appear as the simplest example. Extensions of that model to all odd dimensions, which was implicit in their formulation, appear as one series of models in our construction. The Chern–Simons-type series just mentioned is based on dynamical variables associated only to links of the lattice, and is the closest to standard gauge theory. We also find other theories in our approach which have a superficial resemblance to the continuum U(1) theory introduced by Schwarz [2], which was related to Ray–Singer, and equivalently, Franz–Reidemeister, torsion. These theories will also involve lattice variables associated to higher-dimensional simplices. Additional

* Supported by Stichting voor Fundamenteel Onderzoek der Materie (FOM).

models which do not really lie within either of these two categories will also be formulated. Generically, this construction falls outside of the scope of [1] which is rooted in link based gauge theory.

An important component of our construction is a subtle variation of the usual cup product of simplicial cochains. The models we construct measure a new type of intersection of certain cohomology classes. For example, if K denotes a four-dimensional simplicial complex, then the intersection is between $H^1(K, Z_p)$ and $H^2(K, Z_q)$ where p and q can differ.

We work exclusively with the gauge group Z_p. Subdivision invariance follows naturally in each model when the field configurations are restricted to satisfy a type of mod-p flatness condition. While in three dimensions subdivision invariance of the partition function is sufficient to conclude that one has a topological invariant, the situation is more delicate in higher dimensions. There, subdivision invariance yields a combinatorial invariant of the piecewise linear structure. This situation is analogous to the continuum model phenomenon where metric independence allows one to conclude immediately that one has a diffeomorphism invariant, though further considerations may show that the theory is topological.

2. General Formalism

A lattice model is based on a simplicial complex which combinatorially encodes the topological structure of some manifold. Let us recall some of the essential ingredients that are required in such a formulation; we refer the reader to [3–5] for a more complete account.

Let $V = \{v_i\}$ denote a finite set of N_0 points which we will refer to as the vertices of a simplicial complex. An ordered k-simplex is an array of $k + 1$ distinct vertices which we denote by,

$$[v_0, \ldots, v_k]. \tag{1}$$

It will usually be convenient to use simply the indices themselves to label a given vertex when no confusion will arise, so the above simplex is denoted more economically by $[0, \ldots, k]$. Pictorially, a k-simplex should be regarded as a point, line segment, triangle, or tetrahedron for k equals zero through three, respectively. A simplex which is spanned by any subset of the vertices is called a face of the original simplex. An orientation of a simplex is a choice of ordering of its vertices, where we identify orderings that differ by an even permutation, but for the models described here we will require an ordering of all vertices. One then checks that the invariant we compute is actually independent of the choice made in vertex ordering.

The boundary operator ∂ on the ordered simplex $\sigma = [v_0, \ldots, v_k]$ is defined by

$$\partial\,\sigma = \sum_{i=0}^{k} (-1)^i\, [v_0, \ldots, \hat{v}_i, \ldots, v_k], \tag{2}$$

where the 'hat' indicates a vertex which has been omitted. It is easy to show that the composition of boundary operators is zero; $\partial^2 = 0$.

We model a closed n-dimensional manifold as a collection $K = \{\sigma_i\}$ of n-simplices constructed from the set of vertices V, subject to a few technical conditions. Most importantly, every $(n-1)$-face of any given n-simplex appears as an $(n-1)$-face of precisely two different n-simplices in the collection K. One thinks of the n-simplices then as glued together along $(n-1)$-faces. There is an additional restriction on the 'link' of a vertex for the the simplicial complex to represent a manifold, but this condition will not play a role in the sequel and we refer the reader to [5] for a more complete discussion.

The dynamical variables in the theories we construct will be objects which assign an element in the cyclic group $Z_p = Z/pZ$, which we represent as the set of integers

$$\{0, \ldots, p-1\}, \tag{3}$$

to ordered simplices of some specified dimension. We call these dynamical variables k-colours with coefficients in Z_p, and denote the evaluation of some k-colour $B^{(k)}$ on the ordered k-simplex $[0, \ldots, k]$ by

$$\langle B^{(k)}, [0, \ldots, k] \rangle = B_{0\ldots k} \in Z_p. \tag{4}$$

The superscript (k) will usually be omitted when its value is clear from the context. It is important to note that we are assigning a Z_p element in a way which depends on the ordering of vertices in the simplex; we do not have the rule $B^{(1)}_{01} = -B^{(1)}_{10}$, for example. Instead, we shall assume that

$$B^{(1)}_{10} = -B^{(1)}_{01} \bmod p, \tag{5}$$

and similarly extend this to a k-colour for odd permutations of the vertices. The case closest to conventional lattice gauge theory is where a 1-colour variable is assigned to every 1-simplex in the complex.

The coboundary operator δ acts on the dynamical variables as follows. Given a $(k-1)$-colour, an application of the coboundary operator produces an integer in Z, when evaluated on an ordered k-simplex, namely

$$\begin{aligned} &\langle \delta B^{(k-1)}, [0, \cdots, k] \rangle \\ &= \langle B, \partial[0, \cdots, k] \rangle \\ &= B_{123\ldots k} - B_{023\ldots k} + B_{013\ldots k} - \cdots . \end{aligned} \tag{6}$$

We must emphasize that the above sum of integers is not taken with modular p arithmetic; it is simply an element in Z. In cases where we will need to take some combination mod-p, we will put those terms between square brackets, so, for example,

$$[a+b] = a+b \bmod p. \tag{7}$$

There is also a cup product operation on colours which takes a k-colour $B^{(k)}$ and a l-colour $C^{(l)}$ and gives an integer in Z when evaluated on a $(k+l)$-ordered simplex:

$$\langle B \cup C, [0,\ldots,k+l]\rangle = B_{0\ldots k} \cdot C_{k\ldots k+l}. \tag{8}$$

Note once again that this product is in Z and the value is not taken mod-p; it is therefore different from the usual cup product of simplicial cochains. In fact, it makes sense to consider the product of a Z_p-valued colour B and a Z_q-valued colour C.

Let us now put these ingredients together and define our theories. First, we must be given some oriented simplicial complex K which we take to represent a manifold of dimension n. One then has some collection of n-simplices defined up to orientation. Take the vertex set of this complex and give it an ordering. This is done arbitrarily and we will have to show that our construction is independent of this choice. Now we can write down an ordered collection of the n-simplices; each of the simplices is written in ascending order and a sign in front of that simplex indicates whether that ordering is positively or negatively oriented with respect to the orientation of the complex K. Let us denote this ordered set of n-simplices by K^n,

$$K^n = \sum_i \epsilon_i \, \sigma_i, \tag{9}$$

where the index i runs over the ordered n-simplices σ_i and ϵ_i is a sign which indicates the orientation. We will assign a Boltzmann weight $W[K^n]$ to K^n by taking a product of factors, one for every n-simplex,

$$W[K^n] = \prod_i W[\sigma_i]^{\epsilon_i}. \tag{10}$$

Each of the individual factors is a nonzero complex number and will be some function of the colours. The details of which colours we use and how the function is defined will depend on the particular model. Finally, the partition function, which we will require to be a combinatorial invariant, is defined to be a quantity which is proportional to the sum of the Boltzmann weights over all colourings,

$$Z = \frac{1}{|G|^{f(N)}} \sum_{\text{colours}} W[K^n]. \tag{11}$$

Here $f(N)$ is a function of the number of simplices of various dimensions, and $|G|$ is the order of the gauge group. This scale factor depends on the specific model under consideration and is required to achieve subdivision invariance; it is fixed by scaling considerations. In a theory based entirely on a single 1-colour field in three dimensions, for example, $f(N) = N_0$, the number of 0-simplices in the complex. Let us make all of this very explicit by defining some specific models.

3. The Dijkgraaf–Witten Invariant

The simplest model of the type we are describing will lead to the Dijkgraaf–Witten invariant of 3-manifolds [1]. Further analysis of this model has been presented in [6–8]. So, let us be given a simplicial complex of dimension 3 and an ordering of vertices as described above. This model will be constructed out of a single 1-colour (with values in Z_p) denoted by A. The weight assigned to some ordered 3-simplex $[0, 1, 2, 3]$ is:

$$\begin{aligned} W[[0,1,2,3]] &= \exp\{\beta\,\langle A \cup \delta A, [0,1,2,3]\rangle\} \\ &= \exp\{\beta\, A_{01}\,(A_{12} + A_{23} - A_{13})\}. \end{aligned} \tag{12}$$

Here β is a complex number which at this stage is unrestricted. Clearly, our motivation for taking this particular structure is to try and mimic the action of a continuum Chern–Simons theory. We will now see that the requirement of subdivision invariance will quantize this coupling parameter.

Consider the subdivision of a specific ordered 3-simplex $[0, 1, 2, 3]$ obtained by installing a new vertex c at the center and linking it to the other four vertices; symbolically,

$$[0,1,2,3] \to [c,1,2,3] - [c,0,2,3] + [c,0,1,3] - [c,0,1,2]. \tag{13}$$

Let us declare this new vertex to be the first in the total ordering of all vertices. It is a simple exercise to show that

$$\begin{aligned} &W[[0,1,2,3]\ \exp\{-\beta\,\langle \delta A \cup \delta A, [c,0,1,2,3]\rangle\} \\ &\quad = W[[c,1,2,3]]\, W[[c,0,2,3]]^{-1} W[[c,0,1,3]]\, W[[c,0,1,2]]^{-1}. \end{aligned} \tag{14}$$

Thus, we see that our Boltzmann weight is not generally invariant under the replacement of the original Boltzmann factor of $W[[0, 1, 2, 3]]$ by the four factors on the right hand side of (14); there is this added 'insertion' which somehow must be trivialized. While one might imagine other more complicated suggestions, the conditions that lead to the Dijkgraaf–Witten invariant are to impose a restriction on the sum over colourings and on the parameter β. Those conditions are to take $s = e^{\beta}$ to be a p^2 root of unity ($s^{p^2} = 1$) and to restrict the sum over colourings to those which satisfy

$$\delta\, A = 0 \bmod p, \tag{15}$$

for all 2-simplices in the complex K. This restriction shall be termed a 'flatness' condition. For example on the 2-simplex $[0,1,2]$, we have the restriction

$$[A_{01} + A_{12} - A_{02}] = 0. \tag{16}$$

We remind the reader that the brackets denote a sum which is to be taken mod-p, so this particular equation can also be written as

$$[A_{01} + A_{12}] = A_{02}. \tag{17}$$

With only these flat field configurations, the product $\delta A \cup \delta A$ is clearly a multiple of p^2 and the above insertion becomes unity. The resulting identity (14) shall be referred to as the $5W$ identity. It should be remarked that subdivision invariance is achieved without the necessity of summing over the additional colour fields attached to the vertex c, and this will be a general feature of the models presented here. The above subdivision is known as a move of type $(1,4)$. In order to complete the proof of subdivision invariance, one is required to establish invariance with respect to a complete set of (k,l) moves [9]. In the present case, invariance under the remaining $(2,3)$ move follows immediately from the $5W$ identity.

Notice also that the Boltzmann weight of $[0,1,2,3]$ becomes

$$\exp\left\{\frac{2\pi i k}{p^2} A_{01}(A_{12} + A_{23} - [A_{12} + A_{23}])\right\}, \tag{18}$$

with $k \in \{0,\ldots,p-1\}$. This is precisely the well-known representation of a 3-cocycle for the group cohomology of Z_p with coefficients in Z_p (or U(1)).

As discussed in [1], one can now check that the Boltzmann weight is gauge invariant for a closed manifold. This property, together with a verification that the partition function is independent of the chosen vertex ordering, follows immediately from the $5W$ identity.

4. Another Model in Three Dimensions

Having illuminated the general formalism, which in the case of a single 1-colour yields the Dijkgraaf–Witten model, we can immediately consider generalizations. In three dimensions, we have the obvious choice of a theory with two independent 1-colour fields. Let us now treat this theory is some detail. The Boltzmann weight of an ordered 3-simplex $[0,1,2,3]$ is defined as

$$\begin{aligned} W[[0,1,2,3]] &= s^{\langle B \cup \delta A,\, [0,1,2,3]\rangle} \\ &= s^{B_{01}\,(A_{12} + A_{23} - A_{13})}, \end{aligned} \tag{19}$$

where the two independent 1-colour fields are denoted by B and A.

Our first duty is to consider the behaviour of the theory under the subdivision of eqn. (13), and we find

$$\begin{aligned} &W[[0,1,2,3]]\, s^{-\langle \delta B \cup \delta A,\, [c,0,1,2,3]\rangle} \\ &\quad = W[[c,1,2,3]]\, W[[c,0,2,3]]^{-1}\, W[[c,0,1,3]]\, W[[c,0,1,2]]^{-1}. \end{aligned} \tag{20}$$

In this case, we see that invariance under subdivision can be achieved by again quantizing the coupling scale s to be a p^2 root of unity, and restricting the sum over colourings to those which satisfy the 'flatness' conditions:

$$[\delta B] = [\delta A] = 0. \tag{21}$$

The Boltzmann weight of a single ordered 3-simplex then assumes the form

$$W[[0,1,2,3]] = \exp\left\{\frac{2\pi i k}{p^2} B_{01}\left(A_{12} + A_{23} - [A_{12} + A_{23}]\right)\right\}, \tag{22}$$

where $k \in \{0, \cdots, p-1\}$ as before.

Let us now address the issue of gauge invariance on closed manifolds. We wish to show that the Boltzmann weight (22) is invariant under independent gauge transformations of the A and B colour fields. Consider the A and B colour fields defined on the ordered 1-simplex $[0,1]$; then the gauge transformations of those fields are defined as

$$\begin{aligned} A'_{01} &= [A - \delta\, k]_{01} = [A_{01} + k_0 - k_1], \\ B'_{01} &= [B - \delta\, l]_{01} = [B_{01} + l_0 - l_1]. \end{aligned} \tag{23}$$

Here, the k and l fields are 0-colours defined on the vertices of the complex. The model is gauge invariant on a closed oriented simplicial complex when the field configurations are restricted by the flatness conditions and the coupling parameter is quantized, though this symmetry is not manifest. Under a B transformation, one easily sees that

$$s^{B' \cup \delta A} = s^{B \cup \delta A}\, s^{-\delta l \cup \delta A} = s^{B \cup \delta A}\, s^{-\delta(l \cup \delta A)}, \tag{24}$$

where the first equality uses the fact that δA is proportional to p, and that s is a p^2-root of unity. The total Boltzmann weight is therefore invariant up to a boundary term which vanishes on a closed oriented simplicial complex. For gauge group Z_2, one does not actually require orientation of the complex, and it is sufficient that the 3-simplices be glued pairwise along 2-faces. Similarly, one establishes invariance under A field transformations.

As we have noted, the Boltzmann weight is defined with an arbitrary choice of ordering of the vertex set V, and one needs to establish that the partition function is actually independent of that choice. It is not difficult to show that invariance under vertex permutations actually follows from Alexander type 1 subdivision invariance [10]; this in turn is a consequence of invariance under the (k, l) subdivision moves which we have already established.

At this point, we have shown that to achieve subdivision invariance, we must restrict the sum over colourings to those which satisfy the 'flatness' conditions on each 2-simplex in the simplicial complex. Thus, the subdivision invariant Boltzmann weight is one which contains an insertion of delta functions which impose these flatness restrictions. As we shall now see, the true subdivision invariant partition function is obtained by including a certain scaling factor (see (11)). This takes into account the scaling behaviour of the delta functions under (k,l) subdivision.

Let us denote by ΔN_i the increase in the number of i-simplices due to a (k,l) move. For the $(1,4)$ move we have

$$\Delta N_0 = 1, \qquad \Delta N_1 = 4, \qquad \Delta N_2 = 6, \qquad \Delta N_3 = 3, \tag{25}$$

while the $(2,3)$ move has

$$\Delta N_0 = 0, \qquad \Delta N_1 = 1, \qquad \Delta N_2 = 2, \qquad \Delta N_3 = 1. \tag{26}$$

It is a simple matter to check that the assembly of delta functions for the combined A and B sectors scale with a factor of $|G|^2$ under the $(1,4)$ move, and do not scale under the $(2,3)$ move. Hence, if the partition function (11) is taken with $f(N) = 2N_0$, then it defines a subdivision invariant quantity.

Since the Boltzmann weight and the delta function restrictions are gauge invariant objects, one has the freedom to gauge fix arbitrarily the values of a certain number of the colour configurations. In the case of a 1-colour field, the maximal allowable gauge fixing is called a maximal tree. A simple argument shows that a maximal tree is specified by the requirement that it should contain no closed 2-simplices. Given the vertex set of N_0 elements, it is clear that an ordering exists such that the maximal tree contains $N_0 - 1$ links. In this way, the partition function can be reduced to a sum over all gauge inequivalent flat colourings (denoted as flat′), with a normalization as follows:

$$Z = \frac{1}{|G|^2} \sum_{\text{flat}'} W[K^n]. \tag{27}$$

Therefore, we note that the normalization coincides with that used in the definition of the Dijkgraaf–Witten theory, where the partition function is defined as a sum over all inequivalent flat connections, $\mathrm{Hom}(\pi_1(K), G)$.

From a practical point of view, the freedom to perform this gauge fixing facilitates the evaluation of the partition function, to which we now turn. For the case of the 3-sphere, S^3, a suitable simplicial complex is provided by the boundary of a single 4-simplex. An easy calculation then shows that there is only a single gauge inequivalent flat colouring, for both the A and B field. The subdivision invariant value of the partition function is therefore

$$Z(S^3) = \frac{1}{|G|^2}, \tag{28}$$

for all groups $G = Z_p$, and all roots of unity s.

An equally simple calculation establishes the result,

$$Z(S^2 \times S^1) = 1, \tag{29}$$

for all Z_p, and all roots of unity s. Both these results yield the square of the value obtained in the Z_p Dijkgraaf–Witten theory. This will not be the case in the next example.

An interesting case to consider is provided by the real projective 3-space, RP^3, and we shall deal here with the gauge group Z_2. We refer to [11], where a convenient simplicial complex in terms of a small number of vertices is provided. One should bear in mind, however, that attention must be paid to the relative orientation of the simplices in the triangulation of ref. [11], so that the boundary of the complex is zero. The relevant flatness conditions can then be solved, and one finds that each of the independent 1-colour fields A and B has two gauge inequivalent flat solutions. When a nontrivial 4th root of unity is taken for s, only one of the four total field configurations has a Boltzmann weight different from 1, and the result is

$$Z(RP^3) = \tfrac{1}{4}(1+1+1-1) = \tfrac{1}{2}. \tag{30}$$

The point to note here is that this value differs from the calculation in the Z_2 Dijkgraaf–Witten theory, where a value of zero is obtained. It is more meaningful, however, to compare the $B\delta A$ model with the $Z_2 \times Z_2$ Dijkgraaf–Witten theory. One nontrivial way to represent a group cocycle in that case is to take the action to be a sum of two independent Chern–Simons type terms

$$A \cup \delta A + B \cup \delta B. \tag{31}$$

The partition function simply factorizes and one merely has to square the Z_p result. Once again, a value of zero is obtained for RP^3 when a nontrivial 4th root of unity is taken for s. However, the $B\delta A$ model we have been discussing has a Boltzmann weight which can be regarded as a function from $G \times G \times G$ to Z_p (where $G = Z_p \times Z_p$) which satisfies the equation for subdivision invariance. This follows from associating one copy of Z_p to each of the A and B variables. Since this equation is equivalent to the group cocycle condition, this $B\delta A$ theory is presumably a representation of a different inequivalent 3-cocycle in the $Z_p \times Z_p$ Dijkgraaf–Witten model. This is interesting since normally in gauge theory the only possibility when writing down an action for a model based on a direct product group is to take a sum of terms, one for each factor, as in (31).

5. DW Models in Higher Dimensions

An immediate question at this point is whether the higher-dimensional extensions of the Dijkgraaf–Witten model can also be interpreted within the formalism we have been discussing. In $n = 2m$ dimensions, the action one would take, based on a single 1-colour field, is clearly a $\cup$-product of m copies of δA. In terms of the Boltzmann weight, one has

$$W[\sigma] = \exp\{\beta\,\langle \delta A \cup \cdots \cup \delta A, \sigma\rangle\}. \tag{32}$$

Since this structure is a 'total derivative', the Boltzmann weight is always 1 on a closed $2m$-manifold, and no interesting phases can result. While the group cohomology of Z_p with U(1) coefficients is trivial in even dimensions, this is not so with Z_p coefficients. In fact, a simple application of the universal coefficient theorem [4],

$$H^n(X,G) = H^n(X,Z) \otimes G \, \oplus \, \mathrm{Tor}(H^{n+1}(X,Z),G) \tag{33}$$

to the result

$$H^0(BZ_p,Z) = Z, \qquad H^{\mathrm{even}}(BZ_p,Z) = Z_p \quad \text{and} \quad H^{\mathrm{odd}}(BZ_p,Z) = 0,$$

shows that

$$H^n(BZ_p,Z_p) = Z_p \tag{34}$$

for all nonnegative n. In particular, for $n = 4$, when the flatness condition is imposed and $s^{p^3} = 1$, Equation (32) provides a representation of the 4-cocycle. In this particular model, the trouble is that when one multiplies together all the W factors for a closed complex, the total Boltzmann weight is 1. Since the Boltzmann weights are actually Z_p valued, it would be fascinating if they could be realized in some other lattice model in even dimension.

For $2m + 1$ dimensions, one easily writes the higher-dimensional analogue of the three-dimensional Chern-Simons term. One takes the Boltzmann weight

$$W[\sigma] = \exp\{\beta \, \langle A \cup \delta A \cdots \cup \delta A, \sigma \rangle\}, \tag{35}$$

where one has m factors of δA in the action. The same analysis that we have given earlier goes through without difficulty, and one finds a subdivision invariant model when the factor $s = e^\beta$ is a p^{m+1}-root of unity. These would be concrete realizations of the more abstract models implicit in [1].

We also remark that, as in three dimensions, we have the freedom to consider the $2m+1$-dimensional model, with an array of different 1-colour fields. For example, in five dimensions, we obviously can define models with the following Boltzmann weights

$$\begin{aligned} W[\sigma] &= \exp\{\beta\langle B^{(1)} \cup \delta A^{(1)} \cup \delta A^{(1)}, \sigma\rangle\}, \\ W[\sigma] &= \exp\{\beta\langle B^{(1)} \cup \delta B^{(1)} \cup \delta A^{(1)}, \sigma\rangle\}, \\ W[\sigma] &= \exp\{\beta\langle A^{(1)} \cup \delta B^{(1)} \cup \delta C^{(1)}, \sigma\rangle\}. \end{aligned} \tag{36}$$

The expectation would be that such models are related in some way to the single 1-colour model for product groups.

6. General Models

Given the preceding framework and its correspondence to known models in three dimensions, it is natural to consider potentially interesting generalizations in higher dimensions. Let us define a series of models in n dimensions with the Boltzmann weight given by

$$W[\sigma] = \exp\{\beta\langle B^{(r)} \cup \delta A^{(n-r-1)}, \sigma\rangle\}, \tag{37}$$

where $\sigma = [0, 1, \ldots, n]$ is an n-simplex. In this case, the colour degrees are r and $(n - r - 1)$, respectively, and again subdivision invariance requires that $s = e^\beta$ is a p^2 root of unity, with field configurations being restricted by the flatness conditions:

$$[\delta B^{(r)}] = [\delta A^{(n-r-1)}] = 0. \tag{38}$$

A more thorough analysis of these general models can be found in [13].

At this point, it is worth remarking that nontrivial solutions to these flat conditions will generically exist, and these are enumerated by the relevant cohomology groups, $H^r(K, Z_p)$ and $H^{(n-r-1)}(K, Z_p)$, of the complex K.

In $2m + 1$ dimensions, we can also construct a model with Boltzmann weight

$$W[\sigma] = \exp\{\beta\langle B^{(m)} \cup \delta B^{(m)}, \sigma\rangle\}, \tag{39}$$

where $\sigma = [0, 1, \cdots, 2m+1]$, and $B^{(m)}$ is a m-colour field, which, as usual, will be restricted by the relevant flatness condition. The important point to note here is that this model has a structure distinct from the higher-dimensional Chern–Simons-type theories of the previous section, which were based only on 1-colour fields.

It is also possible to consider extensions of these models in which the B and A fields take values in different groups, Z_p and Z_q, say, and with the scale parameter being chosen as $s^{pq} = 1$. Gauge and subdivision invariance follow in the same way as before. Here one is really considering a type of intersection of cohomology classes which belong to different coefficient groups.

To summarize, it is clear that when the scale parameter $s = 1$ the theories described above reduce simply to a sum over all gauge inequivalent solutions to the flatness conditions. Such an invariant is itself nontrivial, and thus the even-dimensional models presented above certainly differ from the Franz–Reidemeister torsion, which is trivial in those dimensions. Our main interest, of course, is in obtaining more subtle behaviour at the nontrivial roots of unity. One should note that in all the theories described, the central identity obtained involves a product of $(n + 2)$ factors of the Boltzmann weight. In [14], a variation of the cup product was used to define a subdivision invariant lattice model in four dimensions. In that case, a similar identity involving six factors of the Boltzmann weight allowed one to establish triviality of the invariant. The reason for this is that the model was defined with an assignment of arbitrary group elements to each link, without the imposition of flatness restrictions. Perhaps, it is worth mentioning the possibility that expectation values of gauge invariant observables, beyond the partition function, may also yield some interesting structures, but we leave that for the future.

Acknowledgements

D.B. would like to thank R. Dijkgraaf, and M. de Wild Propitius, for discussions.

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