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Quantum correlations and measurements in tri-partite quantum systems

Bahari bin Idrus

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Abstract

Correlations and entanglement in a chain of three oscillators A, B, C with nearest neighbour coupling is studied. Oscillators A, B and B, C are coupled but there is no direct coupling between oscillators A, C . Examples with initial factorizable states are considered, and the time evolution is calculated. It is shown that the dynamics of the tri-partite system creates correlations and entanglement among the three oscillators and in particular, between oscillators A, C which are not coupled directly. We have performed photon number selective and non-selective measurements on oscillator A and we investigated their effects on the correlations and entanglement. It is shown that, before the measurement, the correlations between oscillators A, C can be stronger than the correlations of oscillators A, B . Moreover, some entanglement witness shows that oscillators A, C are entangled but the oscillators A, B might or might not be entangled. By using quantum discord, which measures the quantumness of correlations, it is shown that there are quantum correlations between oscillators A, B and after the measurements in both cases of selective and non-selective measurements, oscillators A, B and A, C become classically correlated.

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Chapter 1

Introduction

1.1 Motivation

Quantum correlation or entanglement was introduced by Einstein, Podolsky and Rosen (EPR) and Schrodinger in 1935. It is the most spectacular and counter-intuitive manifestation of quantum mechanics [7]. It describes the way that particles such as photons, electrons or qubits correlate with each other regardless of how far apart they are. This type of correlation provides knowledge about the direction of the spin state of one particle, whether the spin is up or down, and enables one to know that the spin of its pair is in the opposite direction irrespective of how far the distance between the correlated particles.

Entanglement is a real phenomenon [6], which has been demonstrated repeatedly through experimentation [13, 52, 59, 74]. There is currently a lot of discussion about entanglement. A few good examples of review are discussed in [1, 35, 48, 75].

Recently, research focusing on how to harness the potential of entanglement like achieving higher levels of security in quantum cryptography [32], faster rates of information processing in computer [62], as a resource in quantum communication [9] and quantum computation. In [39, 55, 73], it has been shown that it is possible to perform a universal quantum computation.

There are a number of measures of entanglement for a bipartite system. These include the entanglement of formation, the entanglement of distillation [10], the concurrence [45] and relative entropy [71, 76]. Other methods are by using quantum mutual information, conditional entropy, and negativity. These methods have been adopted in [4, 80].

Quantum correlations lead to various counter-intuitive results. The violation of Bell inequalities [6], sudden death of entanglement [23] and the negative values of conditional entropies for entangled system [19, 20] are examples of this. The entanglement of more than two particles leads to a contradiction with the local hidden variable model (LHVM) for non-statistical predictions of quantum formalism have been proven by [40].

If a composite system is not entangled, normally we conclude it is separable. However, a composite system may contain other types of non-classical correlation, even if it is separable. The most popular measure of such correlations are the quantum discord introduced by [44, 64]. Quantum discord measures the non-classical correlations more generally than the entanglement, although entanglement is vanished. It has been shown that quantum discord is responsible for the efficiency of quantum computation in some quantum tasks [27, 28, 56].

In our research work, we investigate a chain of three oscillators A, B, C

with nearest neighbour coupling. Oscillators A, B and oscillators B, C are coupled. There is no direct coupling between oscillators A, C . In the initial state we consider a tri-partite with factorizable state. At time t , we evolved the state.

In this thesis, we also investigate the effects of measurement to the correlations and entanglement. Finally, we have studied quantum discord for the oscillators especially between A, B and A, C .

Objectives of the thesis:

1. To study how the dynamics creates correlations and entanglement between the three oscillators, and in particular, between A, C which are not coupled directly. The emphasis of our work is placed on counter-intuitive results.
2. To study the effects of quantum measurement to the entropy, correlation and entanglement between the three oscillators.
3. To study quantumness of correlations and quantum discord.

1.2 Outline of the thesis

This thesis is divided into six chapters. The first chapter briefly gives an overall picture of the thesis. The second chapter is about quantum harmonic oscillators and continues with the basic formalism of quantum mechanics that will be used in forthcoming chapter. Definition and some examples of bi-partite and multi-partite entanglement are also discussed in this chapter.

We also introduce an entanglement measures that we consider to witness the entanglement in the quantum systems.

Chapter 3 discusses the generalized concept of quantum measurement and then focuses on von Neumann measurement. We also discuss briefly about the positive operator-valued measure (POVM). Here we study two approaches of measurement called selective and non-selective measurement. Selective measurement means we instantaneously inform or relay the outcome or result classically to the other parties. Non-selective measurement means we have known the measurement has been made, but the outcome is not known. Section measurements and entropy give some background about the effects of measurements to entropy, correlation and entanglement. We added in this chapter, quantum discord, a measure of the quantumness of correlations.

In chapter 4, we have studied a chain of three oscillators A, B, C with nearest neighbour coupling. We introduce three different coupling cases of Hamiltonian where oscillators A, B and B, C are coupled, but there is no direct coupling between oscillator A and C . We started with strong coupling between oscillators A, B and B, C . Next, in case 2, we considered strong coupling between oscillators B, C and weak coupling between oscillators A, B . In the third case, there is weak coupling between oscillators A, B and B, C . Numerical results for correlations and entanglement for all cases are presented. It shows counter-intuitive results in correlation and entanglement, particularly in oscillators A, C which is not coupling.

Chapter 5, we continue our study with the same example and Hamiltonian system. We performed two types of measurements on A at $\omega_A t = 3$.

These are referred as selective and non-selective measurements. Comparative results are presented for entropy, correlations and entanglement in the case of without measurement, selective and non-selective measurement. Finally, we have presented the quantum discord of oscillators A, B and A, C for all cases of measurement.

Lastly, in chapter 6, we discussed our main conclusion and some ideas for further work.

Chapter 2

Quantum systems

In this chapter, we start with quantum harmonic oscillator. We define the Hamiltonian for the quantum harmonic oscillator in Dirac representation. Briefly, we discuss the displacement operator and coherent state. Next, we discuss briefly a few basic formalism of quantum mechanics which we will use in the forthcoming chapter. Entanglement of bi-partite and multi-partite are defined, and a few examples are given. Lastly, we introduce von Neumann entropy to use as a tool to measure entanglement by using conditional entropy and quantum mutual information. Before that, we explain Shannon entropy, a classical information theory as an analogy to the von Neumann entropy. We also describe negativity as an entanglement witness.

2.1 The quantum harmonic oscillator

The Hamiltonian for the quantum harmonic oscillator in one dimension is expressed as a total of operators corresponding to the kinetic (T) and potential energies (V) of a system in the form [14]

$$H = T + V \quad (2.1)$$

The potential operator is defined as

$$V = \frac{1}{2}kx^2 \quad (2.2)$$

where $k = m\omega^2$ with m as a mass of a particle, ω is the angular frequency and x is the position operator. The operator T corresponding to kinetic energy is defined as

$$T = \frac{p^2}{2m} \quad (2.3)$$

where p is the momentum operator. Therefore, the Hamiltonian for the quantum harmonic oscillator in one dimension can be re-expressed as [14, 41]

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \quad (2.4)$$

where $\hat{x} = x$ is the position operator and \hat{p} is the momentum operator given by

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \quad (2.5)$$

In order to find the energy levels, we must solve the time-independent

Schrödinger equation

$$\begin{aligned} \hat{H}\psi(x) &= E\psi(x) \\ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \psi(x) &= E\psi(x) \end{aligned} \quad (2.6)$$

The solution of this equation (Eq.(2.6)) by using a spectral method is

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \quad (2.7)$$

where n is a non-negative integer, H_n are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \quad (2.8)$$

Hence, the corresponding energy levels are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad (2.9)$$

Based on Eq.(2.9), the lowest energy is not zero but $E_0 = \frac{\hbar\omega}{2}$, which is called ground state energy.

By using, raising (a^\dagger) and lowering (a) operator or Dirac representation, the Hamiltonian of harmonic oscillator (Eq.(2.4)) can be written as [14, 41]

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \quad (2.10)$$

where a and a^\dagger are defined as

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) \quad (2.11)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right) \quad (2.12)$$

The properties of operator a and a^\dagger are [14, 33, 36]

$$a|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle \quad (2.13)$$

$$a^\dagger|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle \quad (2.14)$$

Concerning Eq.(2.13), the operator a , lowers the state $|\psi_n\rangle$ to $|\psi_{n-1}\rangle$, so with this situation, the operator is called annihilation or lowering operator. The operator a^\dagger (Eq.(2.14)) raises the state $|\psi_n\rangle$ to $|\psi_{n+1}\rangle$, so it is called creation or raising operator. The vacuum state $|0\rangle$ is defined as

$$a|0\rangle = 0 \quad (2.15)$$

Note that to prove Eq.(2.10), we consider

$$a + a^\dagger = \sqrt{\frac{2m\omega}{\hbar}} x \quad (2.16)$$

So, the position operator can be written as

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad (2.17)$$

The Hamiltonian contains the square of x . Squaring Eq.(2.17), we find

$$\begin{aligned} x^2 &= \frac{\hbar}{2m\omega}(a + a^\dagger)^2 \\ &= \frac{\hbar}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2) \end{aligned} \quad (2.18)$$

Now, we consider

$$a - a^\dagger = ip\sqrt{\frac{2}{m\omega\hbar}} \quad (2.19)$$

So, the momentum operator in terms of a and a^\dagger

$$p = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger) \quad (2.20)$$

Squaring Eq.(2.20) and divide by $2m$

$$\begin{aligned} \frac{p^2}{2m} &= \frac{-m\omega\hbar(a - a^\dagger)^2}{4m} \\ &= \frac{-\hbar m\omega}{4m}[a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2] \end{aligned} \quad (2.21)$$

Now insert Eq.(2.18) and Eq.(2.21) to Eq.(2.4)

$$\begin{aligned} H &= \frac{-\hbar m\omega}{4m}[a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2] + \frac{m\omega^2}{2} \frac{\hbar}{2m\omega}[a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2] \\ &= \frac{-\hbar\omega}{4}[a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2] + \frac{\hbar\omega}{4}[a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2] \\ &= \frac{\hbar\omega}{4}[-a^2 + aa^\dagger + a^\dagger a - (a^\dagger)^2 + a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2] \end{aligned}$$

$$= \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a) \quad (2.22)$$

Since the commutation relation

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1 \quad (2.23)$$

Based on Eq.(2.23), we can write

$$aa^\dagger = 1 + a^\dagger a \quad (2.24)$$

As a result, insert Eq.(2.24) to Eq.(2.22), we have

$$\begin{aligned} H &= \frac{\hbar\omega}{2}(1 + a^\dagger a + a^\dagger a) \\ &= \frac{\hbar\omega}{2}[1 + 2a^\dagger a] \\ &= \hbar\omega(a^\dagger a + \frac{1}{2}) \end{aligned}$$

Therefore, it is proven that the Hamiltonian of quantum harmonic oscillator in one dimension in Dirac representation.

If we define $N = a^\dagger a$ and operating it on the state $|\psi_n\rangle$, it becomes [33, 36]

$$\begin{aligned} N|\psi_n\rangle &= a^\dagger a|\psi_n\rangle \\ &= a^\dagger \sqrt{n}|\psi_{n-1}\rangle \\ &= \sqrt{n}(a^\dagger|\psi_{n-1}\rangle) \\ &= n|\psi_n\rangle \end{aligned}$$

Hence,

$$N|\psi_n\rangle = n|\psi_n\rangle \quad (2.25)$$

The $N = a^\dagger a$ is called the number operator. Number operator is very important in the quantum theories of radiation and solids. In the quantum theory of radiation, N gives the number of photons in the radiation field. The creation operator “creates” a photon by increasing the number of photons in the field by one, and conversely the annihilation operator “annihilates” a photon by decreasing the number of photons in the field by one.

Now, for the $D = 1, 2, \dots, d$ dimensional of quantum harmonic oscillator, we label the position x and momentum p as x_1, x_2, \dots, x_d and p_1, p_2, \dots, p_d correspondingly. Therefore, the Hamiltonian for the quantum harmonic oscillator in D dimension in x and p terms is:

$$H = \sum_{i=1}^d \left(\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2 \right) \quad (2.26)$$

In terms of Dirac representation, the given annihilation and creation operators in D dimension are

$$a_i = \sqrt{\frac{m\omega}{2\hbar}} \left(x_i + \frac{i}{m\omega} p_i \right) \quad (2.27)$$

$$a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x_i - \frac{i}{m\omega} p_i \right) \quad (2.28)$$

The Hamiltonian for the quantum harmonic oscillator in D dimension can be written as:

$$H = \hbar\omega \sum_{i=1}^d (a_i^\dagger a_i + \frac{1}{2}) \quad (2.29)$$

For once there is considered to exist an interaction between two or more oscillators in the systems. This interaction is called coupling. Coupling constant is used to measure the strength of the interaction between two oscillators. This interaction forms a linear chain in one dimension, or a regular lattice in two or more dimensions. The Hamiltonian of the total system is

$$H = \sum_{i=1}^d \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 \sum_{ij(nn)} (x_i - x_j)^2 \quad (2.30)$$

The potential energy is summed over “nearest-neighbour” pairs.

Another factor that needs consideration is resonance frequency of the system. The strength of the interaction between two oscillators depends on the resonance frequency. An oscillator of a system at its natural frequency of vibration will give strong coupling between the oscillators.

2.1.1 Displacement operator

The displacement operator in the $x-p$ phase space of the harmonic oscillator is defined as [54, 67, 78]

$$D(z) = e^{za^\dagger - z^*a} \quad (2.31)$$

where z is a complex number and can be written in terms of x and p as

$$z = \frac{x + ip}{\sqrt{2}} \quad (2.32)$$

The displacement operator can be re-expressed in terms of the position and momentum operators \hat{x}, \hat{p} as

$$D(x, p) = e^{ip\hat{x} - ix\hat{p}} \quad (2.33)$$

Moreover, it can be proved [12, 54, 67, 78] that the product of two displacement operators is given by

$$\begin{aligned} D(z_1)D(z_2) &= D(z_1 + z_2)e^{\frac{1}{2}(z_1z_2^* - z_1^*z_2)} \\ &= D(z_2)D(z_1)e^{z_1z_2^* - z_1^*z_2} \end{aligned} \quad (2.34)$$

2.1.2 Coherent state

The coherent states were introduced by Glauber [38]. Mathematically, they are defined as the eigenstates of the annihilation operator

$$a|z\rangle = z|z\rangle \quad (2.35)$$

where $|z\rangle$ is a coherent state, and z is a complex number.

Further, the coherent states of the oscillator can be written as

$$\begin{aligned} |z\rangle &= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{a^n}{\sqrt{n!}} |n\rangle \\ &= e^{-\frac{|z|^2}{2}} e^{za^\dagger} |0\rangle \\ &= e^{-\frac{|z|^2}{2}} e^{za^\dagger} e^{-z^*a} |0\rangle \end{aligned} \quad (2.36)$$

Using the relation of Campbell-Baker-Hausdorff operator theorem in [58]

$$e^{za^\dagger - z^*a} = e^{-\frac{|z|^2}{2}} e^{za^\dagger} e^{-z^*a} = D(z) \quad (2.37)$$

where $D(z)$ is a displacement operator, Eq.(2.36) can be written as

$$|z\rangle = D(z)|0\rangle \quad (2.38)$$

where $|0\rangle$ is the ground state. Eq.(2.38) shows that the coherent state is a displaced vacuum state in the phase space.

The average number of photons $\langle n \rangle$ in the coherent state is

$$\langle n \rangle = \langle z|n|z \rangle = |z|^2 \quad (2.39)$$

The important features of the coherent state are the position and momentum uncertainties are equal and kept in their minimum values

$$\Delta x = \Delta p = \frac{1}{\sqrt{2}}; \quad \Delta x \Delta p = \frac{1}{2} \quad (2.40)$$

and another important property is the resolution of the identity

$$\frac{1}{\pi} \int |z\rangle \langle z| d^2z = 1 \quad (2.41)$$

2.2 Basic formalism

In this section, we introduce and discuss the basic formalism of quantum mechanics such as density operator, reduced density operator or partial trace [53] and time evolution [14]. Most of the subject discussed in this section can be found in [62].

2.2.1 Density operator

The density operator or density matrix is an alternate formulation in quantum mechanics besides using state vectors ($|\psi\rangle$). This formulation is mathematically equivalent to the state vector approach. The density operator for the system is defined by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|; \quad 0 \leq p_i \leq 1; \quad \sum_i p_i = 1 \quad (2.42)$$

where $|\psi_i\rangle$ is a quantum state with respective probabilities p_i . If we know exactly $|\psi_i\rangle$, where a single $p_i = 1$, all others are zero. This is called pure state and the density operator can be written as

$$\rho = |\psi\rangle\langle\psi| \quad (2.43)$$

Otherwise, ρ is in a mixed state. For ρ in a mixed state, a collection of a different pure state with all the probability is given.

The following are the properties of the density operator [61, 62]

- the density operator is Hermitian, $\rho = \rho^\dagger$.

- the trace of any density matrix is equal to one, $\text{Tr}(\rho) = 1$.
- for a pure state, since $\rho^2 = \rho$, $\text{Tr}(\rho^2) = 1$.
- for a mixed state, $\text{Tr}(\rho^2) < 1$.
- the eigenvalues of a density operator satisfy $0 \leq \lambda_i \leq 1$.
- the expectation value of an operator A can be calculated using $\langle A \rangle = \text{Tr}(\rho A)$.

A state is called a completely mixed state when the probability of each state is equal to all others. For example

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \quad (2.44)$$

is a completely mixed state with a 50% probability of finding the system in the state $|0\rangle$ or $|1\rangle$.

2.2.2 Reduced density operator

One of the most important tools in the density operator is to describe the state of a composite system by using a reduced density operator. Consider a bi-partite system

$$|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B \quad (2.45)$$

The mathematical operation to compute reduced density operator is the partial trace. Partial trace for the first system denoted by ρ_A is defined as

$$\rho_A = \text{Tr}_B(\rho_{AB}) \quad (2.46)$$

where Tr_B is the partial trace over the subsystem B. Tr_B is defined as

$$\text{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2| \text{Tr}(|b_1\rangle\langle b_2|) \quad (2.47)$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A , $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B . Since

$$\text{Tr}(|b_1\rangle\langle b_2|) = \text{Tr}(|b_2\rangle\langle b_1|) = \langle b_2|b_1\rangle \quad (2.48)$$

Therefore,

$$\text{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2| \langle b_2|b_1\rangle \quad (2.49)$$

This approach is the same for the partial trace for the second system, ρ_B

$$\begin{aligned} \rho_B &= \text{Tr}_A(\rho_{AB}) \\ &= \text{Tr}_A(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \\ &= \langle a_2|a_1\rangle |b_1\rangle\langle b_2| \end{aligned} \quad (2.50)$$

Reduced density operator can be computed for the composite systems consisting of more than two systems in an analogous way. For tri-partite system, the reduced density operator can be computed by

$$\rho_{AB} = \text{Tr}_C(\rho_{ABC}) \quad (2.51)$$

$$\rho_{AC} = \text{Tr}_B(\rho_{ABC}) \quad (2.52)$$

$$\rho_{BC} = \text{Tr}_A(\rho_{ABC}) \quad (2.53)$$

$$\rho_A = \text{Tr}_{BC}(\rho_{ABC}) \quad (2.54)$$

$$\rho_B = \text{Tr}_{AC}(\rho_{ABC}) \quad (2.55)$$

$$\rho_C = \text{Tr}_{AB}(\rho_{ABC}) \quad (2.56)$$

Generally, the reduced density operator for tri-partite system is

$$\rho_{ij} = \text{Tr}_k(\rho_{ijk}); \quad \rho_i = \text{Tr}_{jk}(\rho_{ijk}); \quad i, j, k = A, B, C \quad (2.57)$$

2.2.3 Time evolution of the system

In this section, we discussed the dynamics of the quantum system - how the quantum state evolves in time. Here we introduce a postulate called *postulate 1*, describe the evolution of the quantum system [62].

Postulate 1

The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2 ,

$$|\psi'\rangle = U|\psi\rangle \quad (2.58)$$

This postulate explains how quantum states of a closed quantum system are related when they evolve in time. A matrix U are unitary if $UU^\dagger = U^\dagger U = \mathbf{I}$ where \mathbf{I} is a identity matrix.

Further, *postulate 2* refined the *postulate 1*, which describes the evolution of a quantum system in continuous time.

Postulate 2

The time evolution of the state of a closed quantum system is described by the Schrödinger equation as

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle \quad (2.59)$$

where \hbar is a Planck's constant, H is a fixed Hermitian operator known as the Hamiltonian of the closed system.

The solution to the Schrödinger equation, Eq.(2.59) is [14]

$$\begin{aligned} i\hbar \frac{\partial\psi(t)}{\partial t} &= H\psi(t) \\ \frac{\partial\psi(t)}{\psi(t)} &= \frac{H\partial t}{i\hbar} \\ \int_{t_1}^{t_2} \frac{\partial\psi(t)}{\psi(t)} &= \frac{-i}{\hbar} H \int_{t_1}^{t_2} \partial t \\ \ln \frac{\psi(t_2)}{\psi(t_1)} &= \frac{-i}{\hbar} H(t_2 - t_1) \\ \psi(t_2) &= e^{\frac{-i}{\hbar} H(t_2 - t_1)} \psi(t_1) \end{aligned} \quad (2.60)$$

If we set $\hbar = 1$, $t_2 = t$ and $t_1 = 0$, Eq.(2.60) becomes

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \quad (2.61)$$

Referring to *postulate 1*, U is a unitary operator. We define

$$U(t) = e^{-iHt} \quad (2.62)$$

As we have known from Eq.(2.42), the density operator $\rho = |\psi\rangle\langle\psi|$ then

$$\rho = e^{-iHt}|\psi(0)\rangle\langle\psi(0)|e^{iHt} \quad (2.63)$$

$$= e^{-iHt}\rho(0)e^{iHt} \quad (2.64)$$

$$= U(t)\rho(0)U^\dagger(t) \quad (2.65)$$

2.3 Entanglement

Entanglement was a mystery phenomenon in the early stage. However, with the advent of quantum information theory, it has become an important resource for quantum information processing. Extensive research has been undertaken on various aspects of this subject. A few applications of entanglement like quantum cryptography [32, 51], quantum teleportation [9] and superdense coding [11].

In this section, we give mathematical formulation and physical meaning of entanglement for bi-partite and tri-partite system. To make it clear, we present examples for every definition.

2.3.1 Bi-partite entanglement

Assume that we have two physical systems A and B with the first system in a state $|\psi\rangle_A \in \mathcal{H}_A$ and the second in a state $|\psi\rangle_B \in \mathcal{H}_B$. \mathcal{H}_A and \mathcal{H}_B in a Hilbert space. The Hilbert space of a bi-partite system consisting of subsystem \mathcal{H}_A and \mathcal{H}_B denoted by the tensor product

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \quad (2.66)$$

Therefore, bi-partite system in a state $|\psi\rangle_{AB}$ is defined as

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B \quad (2.67)$$

The pure states which can be represented in this form (Eq.(2.67)) are called factorizable states or product states. Otherwise, a pure state is entangled [3]. Physically, the product state means the state uncorrelated [35].

Here is an example of pure factorizable state

$$\begin{aligned} |\psi\rangle_{AB} &= \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |0\rangle_A \otimes |1\rangle_B) \\ &= \frac{1}{\sqrt{2}}[|0\rangle_A \otimes (|0\rangle_B + |1\rangle_B)] \end{aligned} \quad (2.68)$$

Hence, upon measuring the first subsystem, giving us 100% state $|0\rangle_A$. The state of the second subsystem becomes

$$\frac{1}{\sqrt{2}}|0\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_B \quad (2.69)$$

giving us an equal probability for a $|0\rangle_B$ and $|1\rangle_B$ [7, 70]. It seems there is no correlation between the two subsystem.

Examples for pure entangled states are the Bell states [16]

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B) \quad (2.70)$$

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B) \quad (2.71)$$

If we consider the entangled state [70]

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \quad (2.72)$$

Measuring the first subsystem gives us 50% of state $|0\rangle_A$ and 50% of the state $|1\rangle_A$. The second subsystem is always the same as the first, i.e. we get two subsystem values for the price of one measurement. This shows, there exists a correlation between two subsystems.

Now, consider two density matrix ρ_1 and ρ_2 , their convex combination

$$\rho = \alpha\rho_1 + (1 - \alpha)\rho_2 \quad (2.73)$$

with $\alpha \in [0, 1]$. Assume $p_i \geq 0$ with $\sum_i p_i = 1$ then the convex combination $\sum p_i \rho_i$ of some states is also a state. We call coefficients $p_i \geq 0$ with property $\sum_i p_i = 1$ as convex weight [35].

A mixed state is called separable if and only if it can be written as [81]

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \quad (2.74)$$

The state is called separable if there are convex weight p_i and product states $\rho_i^A \otimes \rho_i^B$ as Eq.(2.74) holds. Otherwise, the state is called entangled [3, 35, 48]. In fact, any density matrix that is “close enough” to the identity is separable [86].

Physically, this definition can be interpreted in three scenarios. First, a product state is an uncorrelated state, with A and B each considered as a separate state. In the case of non-product states, there are two scenarios

of correlations. Separable states are classically correlated. This means that the production of a separable state is only necessary by local operations and classical communication (LOCC). Otherwise, the state is entangled, the correlations do not originate from a classical procedure.

An example for a mixed separable state which is an uncorrelated state is [16]

$$\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \quad (2.75)$$

A Werner state [16]

$$\rho = (1 - p)\frac{1}{4}\mathbf{I} + p|\phi^+\rangle\langle\phi^+| \quad (2.76)$$

where \mathbf{I} is an identity matrix and given $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. This state (Eq.(2.76)) is an example for a mixed separable state with classical correlations if $p < \frac{1}{3}$ and mixed entangled state for $\frac{1}{3} < p \leq 1$.

2.3.2 Multi-partite entanglement

In this section, we discuss classification of entanglement for more than two parties. We limit our discussion to tri-partite quantum state with an assumption we can generally generate for the case of multi-partite entanglement.

We start our discussion with pure tri-partite states. We can divide our separability or entanglement classification in three cases. First, fully separable state which can be written as [35, 48]

$$|\psi^{fs}\rangle_{ABC} = |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \quad (2.77)$$

Second, bi-separable states where two of the three quantum states are grouped together as one party. The bi-separable states can be written as a product state in the bi-partite system. In this case, we have three possibilities of grouping two quantum states [35, 48]

$$|\psi^{bs}\rangle_{ABC} = |\psi\rangle_A \otimes |\phi\rangle_{BC} \quad (2.78)$$

$$|\psi^{bs}\rangle_{ABC} = |\psi\rangle_B \otimes |\phi\rangle_{AC} \quad (2.79)$$

$$|\psi^{bs}\rangle_{ABC} = |\psi\rangle_C \otimes |\phi\rangle_{AB} \quad (2.80)$$

where ϕ represent two party states which might be entangled.

Third, a pure state called a genuine tri-partite entangled. Two examples of genuine tri-partite entangled are called Greenberger-Horne-Zeilinger (GHZ) state [40]

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad (2.81)$$

and the so-called W state [29]

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \quad (2.82)$$

Physically, only in genuine tri-partite is an entangled state, all three parties interact. However, this is not the case in fully separable or bi-separable states.

In the case of mixed states, the classification of separability or entanglement is also the same as pure tri-partite states where three cases are considered, fully separable, bi-separable and fully entangled. The definition for classification of mixed states similar with bi-partite entanglement, i.e. by

a convex combination [2, 35]. In the case of fully separable, a mixed state ρ_{ABC} can be written as a convex combination of fully separable pure states, i.e. if there are convex weight p_i and fully separable states $|\psi\rangle_{ABC}$ with

$$\rho_{ABC}^{fs} = \sum_i p_i |\psi^{fs}\rangle_{ABC} \langle \psi^{fs}|_{ABC} \quad (2.83)$$

In the case of bi-separable, ρ_{ABC}^{bs} can be written as a convex combination of bi-separable pure states

$$\rho_{ABC}^{bs} = \sum_i p_i |\psi^{bs}\rangle_{ABC} \langle \psi^{bs}|_{ABC} \quad (2.84)$$

Lastly, in the case of fully entangled mixed states, there are two classes, W class and GHZ class. If ρ_{ABC} can be written as a convex combination of W-type pure states

$$\rho_{ABC} = \sum_i p_i |\psi^W\rangle_{ABC} \langle \psi^W|_{ABC} \quad (2.85)$$

it is W class. Otherwise, it is GHZ class.

2.4 Entanglement measure

In the previous section, we have classified entanglement for bi-partite and tri-partite system. However, given a quantum state, to identify the entanglement is not an easy task. One approach is by looking at the separability criterion [35, 47, 69, 83], if it is not separable then it is entangled. In quantifying entanglement, a number of measures have been proposed [46, 71].

Entanglement witnesses have been proven an effect for the detection of entanglement [75, 77]. Entropy provides a tool, which can be used to quantify entanglement. Before we discussed entropy in a quantum state, we introduced the entropy of classical information theory called Shannon entropy. This is because quantum information theory parallels with classical information theory, but it is based entirely on density matrices, rather than probability distribution for the description of quantum ensembles [18]. A few methods of measure entanglement are discussed in this section. These include conditional entropy, quantum mutual information and negativity.

2.4.1 Shannon entropy

Entropy is a key concept in quantum information theory [62]. Entropy was introduced by Shannon and therefore, became known as the Shannon entropy

$$H(X) \equiv H(p) \equiv - \sum_x p_x \log_2 p_x \quad (2.86)$$

where p is a probability distribution, and we define $0 \log_2 0 = 0$. It measures an uncertainty of a random variable X , or as quantify the information in a source X that produces messages x_i with probabilities p_i [22, 44].

As an example, if we consider binary entropy, the entropy of two outcomes random variable as

$$H(X) = -p \log p - (1 - p) \log(1 - p) \quad (2.87)$$

where p and $1 - p$ are the probabilities of the outcomes. Assume $H(X) = 0$

when $p = 0$ or 1 , the variable is not random and there is no uncertainty. Intuitively, we expect the entropy equal to zero when it is certain and become a maximum when both variables equal likely.

The relative entropy measures the closeness of the two probability distributions, $p(x)$ and $q(x)$ from the same source X is defined as [44, 62]

$$H(p(x)||q(x)) \equiv \sum_x p(x) \log_2 \frac{p(x)}{q(x)} = -H(X) - \sum_x p(x) \log_2 q(x) \quad (2.88)$$

Suppose we have two random variables X and Y . The joint entropy of X and Y can be defined as

$$H(X, Y) \equiv - \sum_{x,y} p(x, y) \log_2 p(x, y) \quad (2.89)$$

measures the total uncertainty of the pair (X, Y) .

The entropy of X conditional on knowing Y is called conditional entropy and is defined by

$$H(X|Y) \equiv H(X, Y) - H(Y) \quad (2.90)$$

$H(X|Y)$ measure uncertainty on average about the value of X , knowing the value of Y .

Correlations between two different random variables X and Y or measure how much information X and Y have in common is measured by the mutual information. It is defined as [44, 62]

$$H(X : Y) = H(X) + H(Y) - H(X, Y) \quad (2.91)$$

It is also a special case of the relative entropy because it measured how distinguishable a joint probability distribution p_{ij} from the completely uncorrelated pair of distribution $p_i p_j$ [44]

$$H(p_{ij} \parallel p_i p_j) = H(p_i) + H(p_j) - H(p_{ij}) \quad (2.92)$$

2.4.2 Von Neumann entropy

In the quantum state, the entropy defined in a similar way as the Shannon entropy is called the von Neumann entropy. It replaces the probability distribution with the density operator, ρ . Here, we generalize the von Neumann entropy based on Shannon entropy. The von Neumann entropy of a quantum state ρ is defined as [62, 68]

$$S(\rho) = -\text{Tr}(\rho \log_2 \rho) \quad (2.93)$$

If λ_x are the eigenvalues of ρ , the von Neumann entropy can be re-express as

$$S(\rho) = -\sum_x \lambda_x \log_2 \lambda_x \quad (2.94)$$

Assume ρ and σ are two density operators. The relative entropy of ρ to σ is defined as

$$S(\rho \parallel \sigma) \equiv \text{Tr}(\rho \log_2 \rho) - \text{Tr}(\rho \log_2 \sigma) \quad (2.95)$$

From Klein's inequality, the quantum relative entropy is non-negative

$$S(\rho \parallel \sigma) \geq 0 \quad (2.96)$$

with equality if and only if $\rho = \sigma$.

If we consider a composite system consisting of two components A and B , by analogy with the Shannon entropy, we can define the joint entropy $S(\rho_{AB})$, conditional entropy, $E(A|B)$ and quantum mutual information, I_{AB} . The joint entropy is defined as

$$S(\rho_{AB}) = -\text{Tr}(\rho_{AB} \log_2 \rho_{AB}) \quad (2.97)$$

where ρ_{AB} is the density operator of the composite system AB .

Conditional entropy is defined as

$$E(A|B) = S(\rho_{AB}) - S(\rho_B) \quad (2.98)$$

Quantum mutual information is defined as [1, 65]

$$I_{AB} = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \quad (2.99)$$

Now, we look further into conditional entropy, $E(A|B)$. If we defined classical conditional entropy $H(X|Y)$ by using the conditional probability $p(i|j)$ and the joint probability $p(i, j)$

$$H(A|B) = - \sum_{ij} p(i, j) \log_2 p(i|j) \quad (2.100)$$

Analogy Eq.(2.100) in a quantum state, quantum conditional entropy becomes [19]

$$E(A|B) = -\text{Tr}_{AB}[\rho_{AB} \log_2(\rho_{A|B})] \quad (2.101)$$

As we know $p(i|j)$ is a probability distribution, $0 \leq p(i|j) \leq 1$. Hence, its quantum analog $\rho_{A|B}$ is not a density operator [18]. Based on the properties of the density operator (refer section 2.2.1), eigenvalues of a density operator satisfy $0 \leq \lambda_i \leq 1$. Thus, the eigenvalue can be larger than one. As a result, $E(A|B)$ can be negative

$$E(A|B) = S(\rho_{AB}) - S(\rho_B) < 0 \quad (2.102)$$

Therefore,

$$S(\rho_{AB}) < S(\rho_B) \quad (2.103)$$

This means, the entropy of the entire system AB is less than the entropy of the subsystems B which violates the classical information theory where $H(X, Y) \geq H(X)$. This happens, for example, in the case of entanglement between A and B .

The concavity of $E(A|B)$ in ρ_{AB} implies that [19]

$$E(A|B) \geq \sum_i w_i S(\rho_A^{(i)}) \geq 0 \quad (2.104)$$

As a result of the concavity of $E(A|B)$, any separable state is associated with non-negative $E(A|B)$. The converse is not true. Therefore, $E(A|B) \geq 0$ is only a necessary condition for separability [18, 19, 20].

Next, we look at the quantum mutual information, I_{AB} . This I_{AB} is used to measure the total correlations between the two subsystems [42, 44] and it is semipositive [31], $I_{AB} \geq 0$. Positive values indicate that there exist classical and quantum correlation between two subsystem A and B . Zero, if

and only if subsystem A and B is a product state, $\rho_{AB} = \rho_A \otimes \rho_B$.

2.4.3 Negativity

The partial transpose of a density operator ρ of a bi-partite state with respect to subsystem A is denoted as ρ^{TA} . If we define the density operator in $i - j$ element

$$\rho_{ij} = \langle \varphi_i | \rho | \phi_j \rangle = \langle \varphi_{i,A} | \langle \varphi_{i,B} | \rho | \phi_{j,A} \rangle | \phi_{j,B} \rangle \quad (2.105)$$

Then ρ^{TA} in $i - j$ is given by

$$\rho_{ij}^{TA} = \langle \phi_{j,A} | \langle \varphi_{i,B} | \rho | \varphi_{i,A} \rangle | \phi_{j,B} \rangle \quad (2.106)$$

In an easier way, we can say the $mn - pq$ elements of ρ , $\langle m | \langle n | \rho | p \rangle | q \rangle$ is mapped to the $pn - mq$ elements of ρ^{TA} , $\langle p | \langle n | \rho^{TA} | m \rangle | q \rangle$ for all the possible integer m, n, p, q where $|m\rangle, |p\rangle$ are basis vectors of the subsystem A and $|n\rangle, |q\rangle$ are basis vectors of the subsystem B . For example, we consider ρ as

$$\rho = \frac{1}{2}(|010\rangle\langle 010| + |010\rangle\langle 111| + |111\rangle\langle 010| + |111\rangle\langle 111|) \quad (2.107)$$

Therefore, ρ^{TA} is

$$\rho^{TA} = \frac{1}{2}(|010\rangle\langle 010| + |011\rangle\langle 110| + |110\rangle\langle 011| + |111\rangle\langle 111|) \quad (2.108)$$

Hence, the negativity is a computable measure of entanglement defined as [77]

$$\mathcal{N}(\rho) = \frac{\|\rho^{TA}\|_1 - 1}{2} \quad (2.109)$$

where $\|\rho^{TA}\|_1$ is a trace norm of a partial transpose with respect to subsystem A .

The trace norm of any Hermitian operator A is defined as

$$\|A\|_1 \equiv \text{Tr}\sqrt{A^\dagger A} \quad (2.110)$$

which is equal to the sum of the absolute value of the eigenvalue of A . Therefore, from Eq.(2.110), Eq.(2.109) becomes

$$\mathcal{N}(\rho) = \frac{\text{Tr}\sqrt{\rho^{TA^\dagger}\rho^{TA}} - 1}{2} \quad (2.111)$$

where $\mathcal{N}(\rho) = 0$ for separable and unentangled states.

2.5 Software Tools

All the numerical calculation in this thesis is calculating by using MATLAB [43, 72], the language of technical computing. This software gives a lot of built-in functions that we can use. A few examples are

- trace - to calculate trace of a matrix
- kron - to calculate kronecker tensor product
- log2 - to calculate base 2 logarithm
- expm - to calculate matrix exponential
- eig - to calculate eigenvalues and eigenvectors

It is also easy to build our own functions and use in the main program. Another advantage using MATLAB is we can get a few functions from the forum (Matlab Newsgroup) and embedded in our program.

In this thesis, we use MATLAB code to simulate the proposed models by plotting the figures. We use MATLAB figures to analyze the results. Alternatively, Mathematica is another tool that can be used. The main advantage of this software is easy to visualize any applications.

2.6 Summary

In this chapter, we have explained the Hamiltonian of quantum harmonic oscillator and how the quantum state evolves in time. We give the definition and examples of bi-partite and multi-partite entanglement. To measure the entanglement, we start the discussion with a Shannon entropy and then analogy to the von Neumann entropy. Quantum mutual information was introduced to quantify the classical and quantum correlation between the subsystems, $I_{AB} \geq 0$. The conditional entropy to confirm entanglement is described as $E(A|B) < 0$. We introduced another tool to measure the entanglement, negativity ($\mathcal{N}(\rho) > 0$), to make sure the subsystem is entangled.

Chapter 3

Measurements on quantum system

In this section, we discussed about quantum measurements. We started the discussion with generalized measurements and then focused on the von Neumann measurements. The positive operator-valued measure (POVM) also briefly discussed in this section. In the section of measurements and entropy, we explained some background about the effects of measurements on entropy and entanglement. Finally, we introduced quantum discord, a measure of the quantumness of correlations.

3.1 Generalized measurements

In quantum mechanics, the act of measurement generally changes the state of the system [85]. Measurement on a quantum system is given by a postulate called postulate 3 [8, 62].

Postulate 3

A generalized quantum measurement is described by a collection $\{M_m\}$ of measurement operators. The index m refers to the measurement outcomes that may occur. These measurement operators satisfy the completeness equation

$$\sum_m M_m^\dagger M_m = \mathbf{I} \quad (3.1)$$

If the state of the quantum system before the measurement is $|\psi\rangle$ then the state of the system after the measurement is

$$|\psi'_m\rangle = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \quad (3.2)$$

where $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$ there is the probability that result m occurs.

We note that the completeness equation assures that the probability sums up to one

$$\sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = \sum_m p(m) = 1 \quad (3.3)$$

To implement *Postulate 3*, consider the quantum state

$$|\psi\rangle = \frac{\sqrt{2}}{3}|0\rangle + \frac{\sqrt{3}}{3}|1\rangle + \frac{2}{3}|2\rangle \quad (3.4)$$

where $|0\rangle, |1\rangle, |2\rangle$ are orthonormal basis and $|\psi\rangle$ is normalized. Therefore the measurement operators

$$M_0 = |0\rangle\langle 0| \quad (3.5)$$

$$M_1 = |1\rangle\langle 1| \quad (3.6)$$

$$M_2 = |2\rangle\langle 2| \quad (3.7)$$

Each measurement operator is Hermitian $M_m^\dagger = M_m$. Thus, $M_0^2 = (|0\rangle\langle 0|)(|0\rangle\langle 0|) = |0\rangle\langle 0| = M_0$, similarly, $M_1^2 = M_1$ and $M_2^2 = M_2$. It also fulfills the completeness equation, $M_0 + M_1 + M_2 = \mathbf{I}$.

Then, the probability of outcome 0

$$\begin{aligned} p(0) &= \langle \psi | M_0^\dagger M_0 | \psi \rangle \\ &= \langle \psi | M_0 | \psi \rangle \\ &= \left(\frac{\sqrt{2}}{3} \langle 0 | + \frac{\sqrt{3}}{3} \langle 1 | + \frac{2}{3} \langle 2 | \right) | 0 \rangle \left(\frac{\sqrt{2}}{3} | 0 \rangle + \frac{\sqrt{3}}{3} | 1 \rangle + \frac{2}{3} | 2 \rangle \right) \\ &= \frac{2}{9} \end{aligned}$$

With the same approach, the probability to have outcome 1 and 2 will be $p(1) = \frac{3}{9}$ and $p(2) = \frac{4}{9}$. As a result, the sum of the probability is 1

$$p(0) + p(1) + p(2) = \frac{2}{9} + \frac{3}{9} + \frac{4}{9} = 1 \quad (3.8)$$

The state after the measurement for outcome 0 is

$$|\psi'_0\rangle = \frac{M_0|\psi\rangle}{\sqrt{p(0)}} = \frac{\frac{\sqrt{2}}{3}|0\rangle}{\sqrt{\frac{2}{9}}} = |0\rangle \quad (3.9)$$

Similarly for outcome 1 and 2, $|\psi'_1\rangle = |1\rangle$ and $|\psi'_2\rangle = |2\rangle$.

3.2 Von Neumann measurements

The von Neumann measurement or projective measurement is a special case of *postulate 3* or generalized measurement. The projection operators Π_m are orthogonal projectors, that is Hermitian $\Pi_m^\dagger = \Pi_m$ and $\Pi_m\Pi_n = \delta_{mn}\Pi_m$ [8, 60, 62].

The von Neumann measurement is described by a measurement operator M , a Hermitian operator on the state of the system being observed. If

$$M = \sum_m m\Pi_m \quad (3.10)$$

where m is an eigenvalue of M with the corresponding projector Π_m . The possible outcome of the measurement corresponds to the eigenvalues m given by [57, 66]

$$p(m) = \text{Tr}[\Pi_m\rho] \quad (3.11)$$

where ρ is the state of the system just before the measurement.

The state of the system immediately after the measurement can be divided to selective or non-selective measurement [66]. Selective measurement means instantaneously we are informed about the result classically. **Non-selective measurement, physically describes the density matrix of the system, when it is known that a measurement has been made, but the result or outcome is not known.** The state for the selective

measurement immediately after the measurement is given by

$$\rho' = \frac{\Pi_m \rho \Pi_m}{p(m)} \quad (3.12)$$

where ρ is a state just before the measurement. The state after measurement for non-selective is

$$\tilde{\rho} = \sum_m \Pi_m \rho \Pi_m \quad (3.13)$$

The relation between selective and non-selective measurement is total over all possible outcomes. Selective measurement with corresponding probabilities will give non-selective measurement [79]

$$\begin{aligned} \sum_{m=0}^{d-1} \rho' \text{Tr}(\rho \Pi_m) &= \sum_{m=0}^{d-1} \frac{\Pi_m \rho \Pi_m}{\text{Tr}(\rho \Pi_m)} \text{Tr}(\rho \Pi_m) \\ &= \sum_{m=0}^{d-1} \Pi_m \rho \Pi_m \\ &= \tilde{\rho} \end{aligned}$$

3.3 Positive operator-valued measure (POVM)

A positive operator-valued measure (POVM) is a set of non-negative Hermitian operators E_m on a Hilbert space \mathcal{H} that sum to unity or the completeness relation is obeyed as [8, 62, 68]

$$\sum_m E_m = \mathbf{I} \quad (3.14)$$

The index m labels the possible outcome of a measurement. If the measurement is performed on a state $|\psi\rangle$, the probability of the outcome is

$$p(m) = \langle \psi | E_m | \psi \rangle \quad (3.15)$$

The POVM formalism is needed for the case of a projective measurement on a larger system which is not projective measurement on the subsystem alone. Another difference of POVM from the projective measurement is that the elements of a POVM are not necessarily orthogonal and a POVM is not repeatable.

A POVM can distinguish between two non-orthogonal states. A good example of this is presented in [62] which assumes that Alice sends one of the two states to Bob

$$|\psi_1\rangle = |0\rangle \quad (3.16)$$

or

$$|\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad (3.17)$$

It is impossible for Bob to identify the state he received. However, it is possible by performing the POVM measurement to distinguish the state.

We consider three elements of POVM

$$E_1 = \frac{\sqrt{2}}{1 + \sqrt{2}} |1\rangle\langle 1| \quad (3.18)$$

$$E_2 = \frac{\sqrt{2}}{1 + \sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2} \quad (3.19)$$

$$E_3 = \mathbf{I} - E_1 - E_2 \quad (3.20)$$

It is clear that E_1, E_2 and E_3 are positive operator and

$$E_1 + E_2 + E_3 = \mathbf{I} \quad (3.21)$$

Therefore, the completeness relation is obeyed. As a result $\{E_1, E_2, E_3\}$ form a POVM.

Assume that Bob received the state $|\psi_1\rangle = |0\rangle$ and performs a measurement described by the POVM elements E_1, E_2 and E_3 . The probability for Bob to obtain outcome m , provided he received the state $|\psi_i\rangle (i = 1, 2)$ is

$$p(m|i) = \langle \psi_i | E_m | \psi_i \rangle \quad (3.22)$$

In the case that Bob wants to observe the result outcome E_1 , the probability is

$$p(1|1) = \langle \psi_1 | E_1 | \psi_1 \rangle = 0 \quad (3.23)$$

Therefore, there is zero probability for outcome E_1 and he can conclude that he received state $|\psi_2\rangle$. Similar case if Bob received the state $|\psi_2\rangle$. Assume, he wants to observe the outcome E_2 . Then, the probability of the outcome will be

$$p(2|2) = \langle \psi_2 | E_2 | \psi_2 \rangle = 0 \quad (3.24)$$

Hence, he can conclude that the state $|\psi_1\rangle$ was sent to him. Sometimes, if Bob obtains the measurement outcome is E_3 , he cannot conclude anything. The important is, although he cannot always distinguish the state he received, by taking advantage of POVM, he will never make a mistake identifying the

state sent to him.

3.4 Measurements and entropy

This section describes the relation between measurement and entropy. The effect of entropy of a quantum system when a measurement is performed to the subparts of the quantum systems. One theorem is given by [62]

Theorem (von Neumann measurement increase entropy)

Suppose Π_N is a complete set of orthogonal projectors and ρ is a density operator. Then the entropy of the state $\tilde{\rho} = \sum_N \Pi_N \rho \Pi_N$ of the system after the measurement is at least as great as the original entropy

$$S(\tilde{\rho}) \geq S(\rho) \quad (3.25)$$

with equality if and only if $\rho = \tilde{\rho}$.

In [4], Zukarnain et al studied the time evolution of a system comprising three oscillators A, B and C . The system is described with the Hamiltonian where the oscillator B and C interact with each other and no interaction between oscillator A . The initial state at a time $t = 0$, is

$$\rho_{ABC}(0) = \rho_{AB}(0) \otimes \rho_C(0)$$

Zukarnain et al [4] considered two examples of $\rho_{AB}(0)$, i.e. separable state and entangled state. After the evolution, authors found in both cases, oscillators A, C become entangled.

Further, they also considered measurement after the evolution in oscilla-

tor C whose result is communicated classically to A. It is shown for both cases, effects of the measurements to the separable and entangled states are identical. The measurement effects the density matrix ρ_A of the oscillator A. In many cases, ρ_A becomes a pure state and in some cases it becomes a mixed state.

Another work done by [79]. It investigated the bi-partite system which performed the measurement on first subsystem and calculate the entropy of the second subsystem. The entropy is calculated for the three cases. The first case which we refer to as without measurement, calculates the entropy before the measurement, $S(\rho_2)$. The second case known as the non-selective measurement, calculates the entropy after the measurement without knowing the outcome, $S(\tilde{\rho}_2)$. The last case which we refer to as selective measurement calculates the entropy after the measurement with informed outcome, $S(\rho'_2(N))$ with N is the different outcome.

Various cases of state have been considered. These include pure states, separable states, entangled states and entangled states after the symplectic transformation [80]. The results show that

- pure states

$$S(\rho_2) = S(\tilde{\rho}_2) = S(\rho'_2(N)) = 0$$

All the entropy is equal to zero for every case, without measurement, non-selective and selective measurement.

- separable states

$$S(\tilde{\rho}_2) = S(\rho_2)$$

The entropy for the case of non-selective measurement is equal to the entropy of without measurement.

$$S(\rho'_2(N = 0, 3, 4)) > S(\rho_2)$$

$$S(\rho'_2(N = 1, 2)) < S(\rho_2)$$

The entropy for the case of selective measurement might be increased or decreased.

- entangled states

$$S(\tilde{\rho}_2) = S(\rho_2)$$

The entropy for the case of non-selective measurement is equal to the entropy of without measurement.

$$S(\rho'_2(N = 0, 1, 2, 3, 4)) = 0$$

The entropy for the case of selective measurement is equal to zero. It is shown that the entangled state became a pure state.

- entangled states after symplectic transformations

$$S(\tilde{\rho}_2) = S(\rho_2)$$

The entropy for the case of non-selective measurement is equal to the entropy of without measurement.

$$S(\rho'_2(N = 0, 1, 2, 3, 4)) < S(\rho_2)$$

The entropy for the case of selective measurement is less than the entropy of without measurement.

3.5 Quantumness and quantum discord

Entanglement is a special kind of quantum correlation [28]. If we measure the composite quantum system, generally we will have entangled state or non-entangled state. Non-entangled state is normally defined as a separable state. Separable state is often regarded as a classical state. Recently, it was pointed out that separability of the density matrix (i.e. the absence of entanglement) does not imply classicality [64].

The quantumness of the state is the correlations between two quantum systems, which have a quantum as well as a classical nature [30]. It is interesting to know how ‘non-classical’ or how ‘quantum’ a given state is. Very non-classical states, might be more useful in applications of quantum information processing than states which are only slightly non-classical. The state of maximal quantumness is called ‘Queen of quantum states’ [37].

Quantum discord, the most popular measure of quantumness of correla-

tions, can capture the non-classical correlations more general than the entanglement [17]. It captures all the non-classical correlations present in the system, including entanglement [28]. The positive values of quantum discord indicate the presence of non-classical correlation, even if they are separable. Vanishing the quantum discord is a criterion to be preferred effectively as classical correlations are called classical quantum state [25, 44, 64].

To calculate the quantum discord [17, 28, 44, 64], we consider the mutual information

$$J(S : M) = H(S) - H(S|M) \quad (3.26)$$

where $H(S)$ denotes the Shannon entropy and $H(S|M) = H(S, M) - H(M)$ is the conditional entropy. If we replaced $H(S|M)$ in Eq.(3.26), this leads to another classically equivalent expression for the mutual information

$$H(S : M) = H(S) + H(M) - H(S, M) \quad (3.27)$$

In the case of quantum systems, Eq.(3.27) becomes

$$I_{SM} = S(\rho_S) + S(\rho_M) - S(\rho_{SM}) \quad (3.28)$$

where $S(\rho) = -\text{Tr}[\rho \log(\rho)]$, the von Neumann entropy.

In the case of Eq.(3.26), to generalize in quantum systems, we need to consider the conditional entropy. The conditional entropy, $H(S|M)$ measures the ignorance about S that remains if we make measurements to determine M . If M is in a quantum system, information that we can extract about it depends on the measurement. Assuming that we restrict the measurement

on a complete set of orthogonal projectors Π_j , corresponding to outcome j , the state of S after the measurement is

$$\rho_{S|\Pi_j} = \frac{\Pi_j \rho_{SM} \Pi_j}{p_j} \quad (3.29)$$

where $p_j = \text{Tr}(\Pi_j \rho_{SM})$ is the probability corresponding to outcome j .

As a result, a quantum analogue for the conditional entropy can be defined as

$$E(S|\Pi_j) = \sum_j p_j S(\rho_{S|\Pi_j}) \quad (3.30)$$

Then, $J(S : M)$ can be generalized for quantum systems as

$$J_{SM} = S(\rho_S) - E(S|\Pi_j) \quad (3.31)$$

The value of J_{SM} depends on the choice of Π_j . If we want to quantify all the classical correlations by maximizing the J_{SM} over all Π_j , then

$$J_{SM} = S(\rho_S) - \tilde{E}(S|M) \quad (3.32)$$

where $\tilde{E}(S|M) = \min_{\Pi_j} \sum_j p_j S(\rho_{S|\Pi_j})$. Hence, the quantum discord is the difference between I_{SM} and J_{SM}

$$\delta_{SM} = I_{SM} - J_{SM} \quad (3.33)$$

$$= S(\rho_M) - S(\rho_{SM}) + \tilde{E}(S|M) \quad (3.34)$$

A few properties of quantum discord are given below:

- quantum discord is always positive, i.e. $\delta_{SM} \geq 0$ [26, 64].
- $\delta_{SM} > 0$ indicates the presence of non-classical correlations [17, 28, 64].
- $\delta_{SM} = 0$ for the states with only classical correlation [24, 26, 64].
- $\delta_{SM} = 0 \Leftrightarrow \rho_{SM} = \sum_j \Pi_j \rho_{SM} \Pi_j$ where Π_j is a complete set of orthogonal projectors with outcome j [25, 26, 64].

A state with vanishing discord is a necessarily separable state with positive partial transpose (PPT) [17]. This is the analog of Peres-Horodecki criterion, i.e. if a state is not PPT, then it must be entangled. [17] also provides a witness for quantum discord where it is proved that all classical correlations $2 \times N$ states are the necessarily strong positive partial transpose (SPPT). SPPT was introduced in [21]. It is a subclass of PPT which guarantees the positivity of their partial transposition by using the canonical factorization of the original density operator. Therefore, if a $2 \times N$ state is not SPPT, the state must be non-classical, measured by quantum discord. Moreover, it has been shown that almost all quantum states have non-vanishing discord [34]. These mean that generally a composed quantum system does contain non-classical correlation. Note that [34] gives a simple necessary criterion for vanishing discord: if $\delta_{SM} = 0$ then $[\rho_{SM}, \mathbf{1}_S \otimes \rho_M] = 0$. Thus, if ρ_{SM} does not commute with $\mathbf{1}_S \otimes \rho_M$, it means quantum discord is positive and hence, ρ_{SM} is a non-classical correlation.

3.6 Summary

In this chapter, we have presented quantum measurement and quantum discord. In the quantum measurement, we started the discussion with generalized measurement given by a postulate. Next, we focused our discussion on von Neumann measurement in which we introduced definitions and formulations of selective and non-selective measurement.

For the effects of measurement, we discussed in measurements and entropy. From this section, we can conclude

- the entropy of the state after non-selective measurement is at least as great as entropy before the measurement.
- effects of measurement to separable and entangled state are identical.
- the entropy of pure state before and after the measurement is equal to zero.
- the entropy of separable state for non-selective measurement is equal with entropy without measurement.
- the entropy of separable state for selective measurement might increase or decrease with entropy without measurement.
- the entropy of entangled state for non-selective measurement is equal with entropy without measurement.
- the entropy of entangled state for selective measurement becomes zero.
- the entropy of entangled state after symplectic transformations for non-selective measurement is equal with entropy without measurement.

- the entropy of entangled state after symplectic transformations for selective measurement is less than the entropy without measurement.

Lastly, we introduced quantum discord, the most popular measure of quantumness of correlations. The positive values of quantum discord indicate the presence of non-classical correlations, even if the state is separable. Separable state (not entangled) does not mean classicality. Vanishing quantum discord is a criterion for the classical correlations called classical quantum state.

Chapter 4

Counter-intuitive results in correlations of tri-partite systems

In this chapter, we discuss about correlations in a chain of three oscillators A, B and C with nearest neighbour coupling [49]. We considered oscillators A, B and oscillators B, C are couplings. There is no direct coupling between oscillators A, C . We have considered three cases of Hamiltonian. Case 1, strong coupling between oscillators A, B and B, C ; Case 2, strong coupling existing between oscillators B, C and weak coupling existing between oscillators A, B and case 3, weak coupling existing among all the three oscillators. The initial state of the system is a factorizable state with no coupling for all the three oscillators. At time t , we evolve the state. We demonstrate the existence of counter-intuitive results.

4.1 Hamiltonian and notations

The tri-partite system that is considered in this study consists of three oscillators, A , B and C with the Hamiltonian:

$$\begin{aligned}
H = & \omega_A(a_A^\dagger a_A \otimes \mathbf{1} \otimes \mathbf{1}) + \omega_B(\mathbf{1} \otimes a_B^\dagger a_B \otimes \mathbf{1}) + \omega_C(\mathbf{1} \otimes \mathbf{1} \otimes a_C^\dagger a_C) \\
& + \lambda_{AB}[a_A \otimes (a_B^\dagger)^2 \otimes \mathbf{1}] + \lambda_{AB}^*(a_A^\dagger \otimes a_B^2 \otimes \mathbf{1}) \\
& + \lambda_{BC}(\mathbf{1} \otimes a_B^\dagger \otimes a_C^2) + \lambda_{BC}^*[\mathbf{1} \otimes a_B \otimes (a_C^\dagger)^2] \quad (4.1)
\end{aligned}$$

where ω_A, ω_B and ω_C are frequencies of the three oscillators. λ_{AB} and λ_{BC} are coupling constants for oscillators A, B and oscillators B, C , correspondingly.

With the initial state at $t = 0$, we assume the density matrix is $\rho_{ABC}(0)$. Then at time t the density matrix evolves with

$$\rho_{ABC}(t) = e^{iHt} \rho_{ABC}(0) e^{-iHt} \quad (4.2)$$

We also calculate the partial trace by

$$\rho_{ij} = \text{Tr}_k(\rho_{ijk}); \quad \rho_i = \text{Tr}_{jk}(\rho_{ijk}); \quad i, j, k = A, B, C \quad (4.3)$$

Entropy of the density matrix is calculated by von Neumann entropy as

$$S(\rho) = -\text{Tr}[\rho \log \rho] \quad (4.4)$$

In this Eq.(4.4), we used logarithm with base two, so all the entropic quantities are in bits. If we assume λ_i is the eigenvalue of ρ then we can re-expressed

the von Neumann entropy as

$$S(\rho) = - \sum_i \lambda_i \log(\lambda_i) \quad (4.5)$$

In our calculation, we used Eq.(4.5) to calculate the entropy with log base of 2.

Quantum mutual information, I_{ij} is used to measure classical and quantum correlation between two subsystems

$$I_{ij} = S(\rho_i) + S(\rho_j) - S(\rho_{ij}) \quad (4.6)$$

If $I_{ij} \geq 0$, it indicates existing classical and quantum correlation between two subsystems.

Entanglement is measured by using an entanglement witness called conditional entropy, $E(i|j)$

$$E(i|j) = S(\rho_{ij}) - S(\rho_j) \quad (4.7)$$

Negative values of $E(i|j)$ show there exist entanglements between two subsystems whilst positive values give an inconclusive witness result. In addition to the conditional entropy, in confirming the result of entanglement, we calculate the negativity as [77]

$$N[\rho_{12}] = \frac{\text{Tr}[\sigma_{12}^\dagger \sigma_{12}]^{\frac{1}{2}} - 1}{2} \quad (4.8)$$

where ρ_{12} is a density matrix of a bipartite system; the σ_{12} is a partial trans-

pose of ρ_{12} with respect to the first subsystem; the $\text{Tr}[\sigma_{12}^\dagger \sigma_{12}]^{\frac{1}{2}}$, calculates the sum of the singular values of σ_{12} . If $N[\rho_{12}]$ is zero, this means the subsystem is not entangled and for $N[\rho_{12}] > 0$, the subsystem is entangled.

4.2 Numerical Results

4.2.1 Example of tri-partite system

In our study, we consider an example of a pure state at $t = 0$

$$|\psi\rangle = \frac{1}{2}(|0\rangle_A + |1\rangle_A) \otimes |1\rangle_B \otimes (|1\rangle_C + |2\rangle_C) \quad (4.9)$$

This example is a factorizable state, i.e. $\rho_A \otimes \rho_B \otimes \rho_C = \rho_{ABC}$. We assume at $t = 0$, all the three oscillators not coupling with each other. At time t , we evolve the state by using the time evolution as Eq.(4.2).

In the numerical calculation, the Hilbert space was truncated. The Hilbert space, spanned by the number state $|N_A, N_B, N_C\rangle$ with $N_A, N_B, N_C = 0, 1, \dots, (K - 1)$. To choose a sufficient truncation, two tests of accuracy were used. First, the traces of all density matrices should be greater than 0.98. In the full Hilbert space, these traces are equal to one. Second, the quantum mutual information, I_{ij} was computed for the various number of truncation. Refer to Table 4.1 where it is indicated that $K = 9$ is sufficiently truncated. This is a very good approximation. The rest of the calculation in this section is based on $K = 9$.

If we refer at the Hamiltonian system that we considered in Eq.(4.1), the interaction term between A and B ($(a_A \otimes (a_B^\dagger)^2 \otimes \mathbf{1})$ and $(a_A^\dagger \otimes a_B^2 \otimes \mathbf{1})$) and

Table 4.1: The quantum mutual informations I_{AB} , I_{AC} and I_{BC} for various values of the truncation dimension, K for $\omega_A t = 10$.

	K=8	K=9	K=10	K=11
I_{AB}	0.1471	0.5147	0.5147	0.5147
I_{AC}	0.8990	1.4775	1.4775	1.4775
I_{BC}	2.0649	1.8832	1.8832	1.8832

B and C ($\mathbf{1} \otimes a_B^\dagger \otimes a_C^2$ and $\mathbf{1} \otimes a_B \otimes (a_C^\dagger)^2$). It is shown that the energy can be preserved only if $\omega_A = 2\omega_B$ and $\omega_B = 2\omega_C$. This situation can create resonant coupling between the two subsystems. As a result, in our numerical calculation we consider three cases that we denote as case 1, case 2 and case 3.

CASE 1

We consider the Hamiltonian of Eq.(4.1) with frequencies and coupling constant correspondingly as

$$\omega_A = 2\omega_B = 4\omega_C = 1 \quad (4.10)$$

$$\lambda_{AB} = \lambda_{BC} = 1 \quad (4.11)$$

This shows that

$$\omega_A = 2\omega_B \quad (4.12)$$

$$2\omega_B = 4\omega_C \Rightarrow \omega_B = 2\omega_C \quad (4.13)$$

which is creating the resonant coupling between the subsystems A, B and B, C . We also assign strong coupling constant, $\lambda_{AB} = \lambda_{BC} = 1$ where we expect strong coupling between oscillators A, B and also oscillators B, C .

CASE 2

We consider the Hamiltonian of Eq.(4.1) with

$$\omega_A = \omega_B = 2\omega_C = 1 \quad (4.14)$$

$$\lambda_{AB} = \lambda_{BC} = 1 \quad (4.15)$$

In this case, it shows that there exists resonant coupling ($\omega_B = 2\omega_C$) between subsystems B and C . No resonant coupling exists between subsystems A and B . We also consider strong coupling constant, which is $\lambda_{AB} = \lambda_{BC} = 1$. From this assumption, we expect strong coupling between oscillators B, C and weak coupling between oscillators A, B .

CASE 3

We consider the Hamiltonian of Eq.(4.1) with

$$\omega_A = \omega_B = \omega_C = 1 \quad (4.16)$$

$$\lambda_{AB} = \lambda_{BC} = 0.1 \quad (4.17)$$

Here, there is no resonant coupling between all the oscillators, and we assign weak coupling constant. With this condition, we expect weak coupling among all three oscillators.

4.2.2 Classical and quantum correlations

The quantum mutual information, I_{ij} as Eq.(4.6), indicates there exists classical and quantum correlations between two subsystems if I_{ij} is non-negative. Reference to Fig.(4.1), Fig.(4.2) and Fig.(4.3) clearly shows that in cases 1, 2 and 3 there exists in all the subsystems, classical and quantum correlations ($I_{ij} \geq 0$). If we refer to Fig.(4.3), it is shown that $I_{AC} < I_{AB}$ and $I_{AC} < I_{BC}$. This result is expected because oscillators A, B and oscillators B, C are couplings. There is no direct coupling between oscillators A, C . In Fig.(4.1) and Fig.(4.2), it has been shown that $I_{AC} > I_{AB}$. This is the first counter-intuitive result that we have. In order to confirm our counter-intuitive result, we compute the ratio I_{AB}/I_{AC} as a function of ω_B and ω_C at $\omega_A t = 5$ and $\lambda_{AB} = \lambda_{BC} = 0.1$. This is shown in Fig.(4.4) that I_{AC} is greater than I_{AB} (white area) for some values of parameters ω_B and ω_C .

4.2.3 Entanglement

The entanglement witness that we used is a conditional entropy ($E(i|j)$). Negative value of $E(i|j)$ indicates an entanglement between subsystems [18, 19]. In Fig.(4.7), for case 3, it shows that $E(A|B)$ and $E(C|B)$ are negative and $E(A|C)$ and $E(C|A)$ are positive. This means that oscillators A, B and oscillators B, C are entangled where as the result is inconclusive for oscillators A, C . This result is expected due to no direct coupling between oscillators A, C .

In Fig.(4.5) and Fig.(4.6), we observe that $E(A|C)$, $E(B|C)$ and $E(C|B)$ are negative whilst $E(A|B)$ and $E(B|A)$ are positive. This means, oscillators

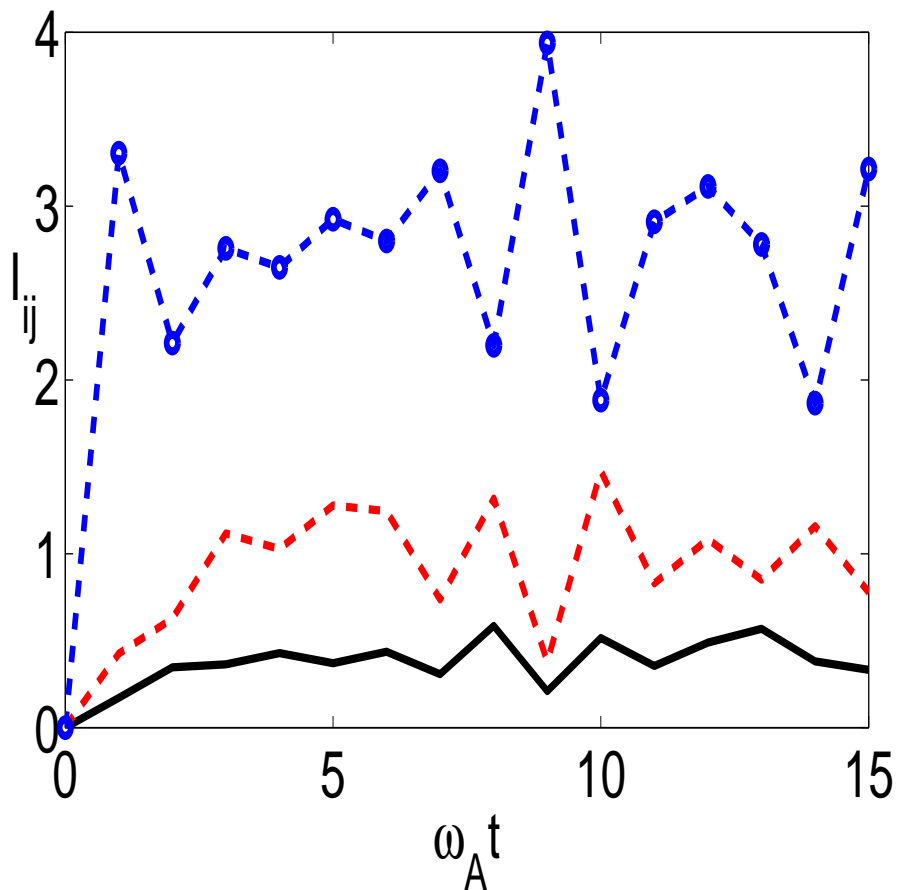


Figure 4.1: The quantum mutual information I_{ij} as in Eq.(4.6) for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). I_{AB} (solid line), I_{AC} (dashed line), and I_{BC} (dashed line and circle) are plotted as functions of time ($\omega_A t$) for Case 1 (pg. 54). $I_{ij} \geq 0$ indicates the existence of classical and quantum correlations between two subsystems.

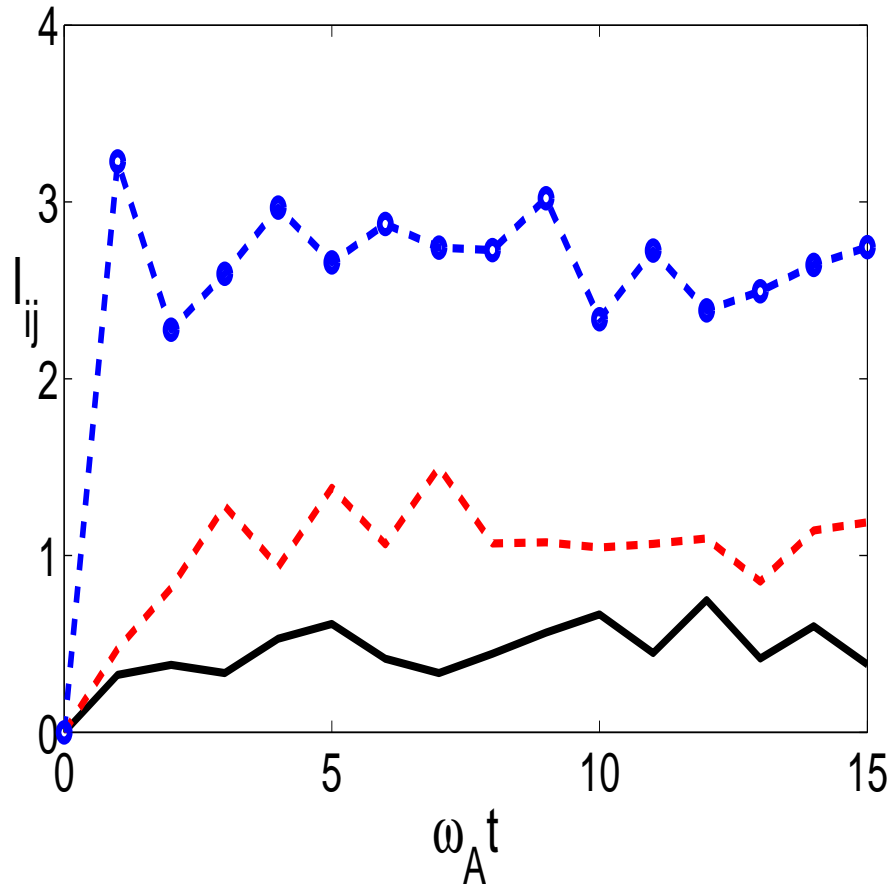


Figure 4.2: The quantum mutual information I_{ij} as in Eq.(4.6) for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). I_{AB} (solid line), I_{AC} (dashed line), and I_{BC} (dashed line and circle) are plotted as functions of time ($\omega_A t$) for Case 2(pg. 55). $I_{ij} \geq 0$ indicates the existence of classical and quantum correlations between two subsystems.

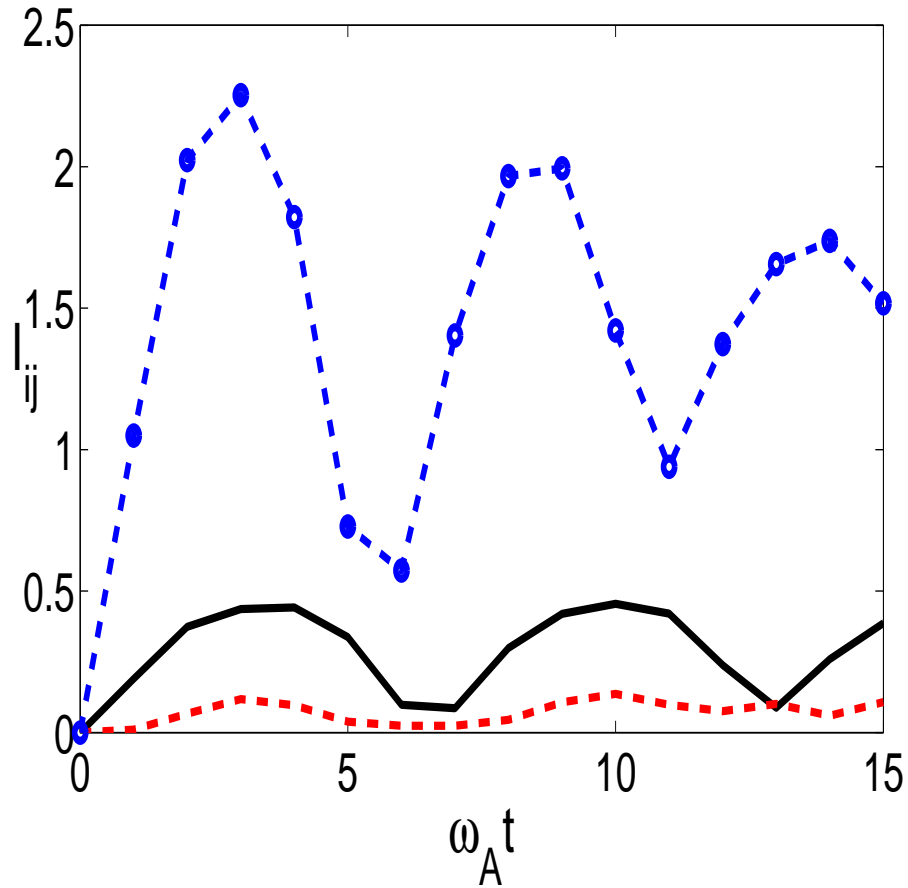


Figure 4.3: The quantum mutual information I_{ij} as in Eq.(4.6) for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). I_{AB} (solid line), I_{AC} (dashed line), and I_{BC} (dashed line and circle) are plotted as functions of time ($\omega_A t$) for Case 3 (pg. 55). $I_{ij} \geq 0$ indicates the existence of classical and quantum correlations between two subsystems.

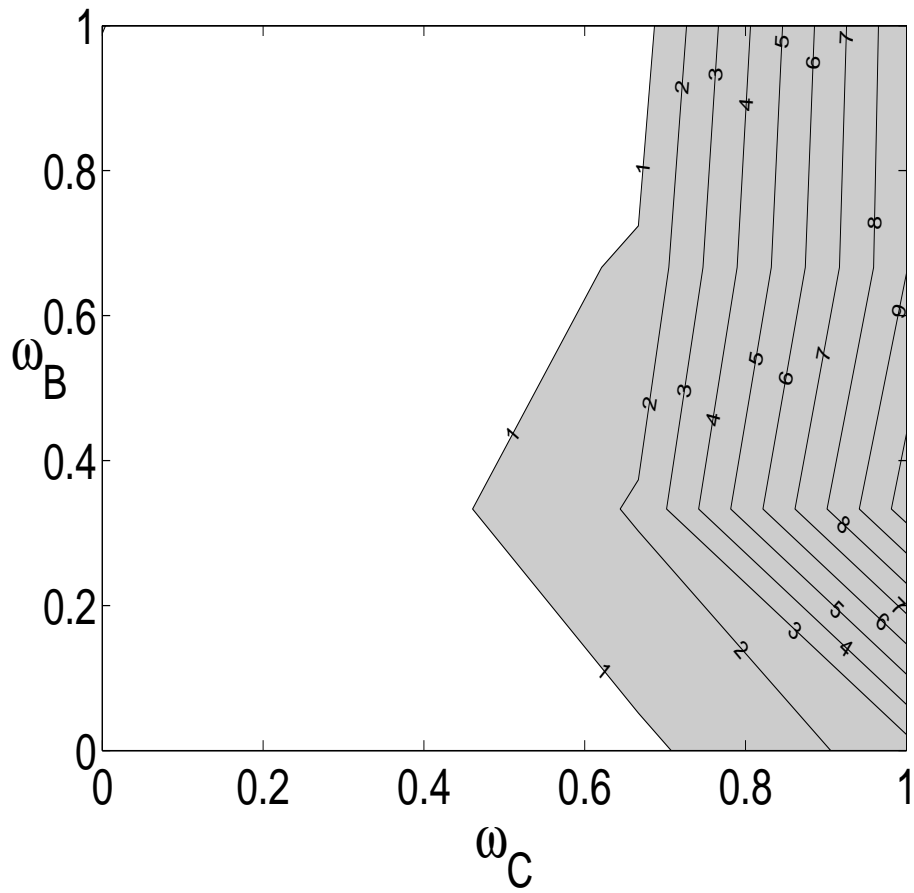


Figure 4.4: The ratio of I_{AB}/I_{AC} as functions of ω_B and ω_C for $\omega_A t = 5$ and $\lambda_{AB} = \lambda_{BC} = 0.1$. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered. The ratio is smaller/greater than one, in the white/grey area correspondingly.

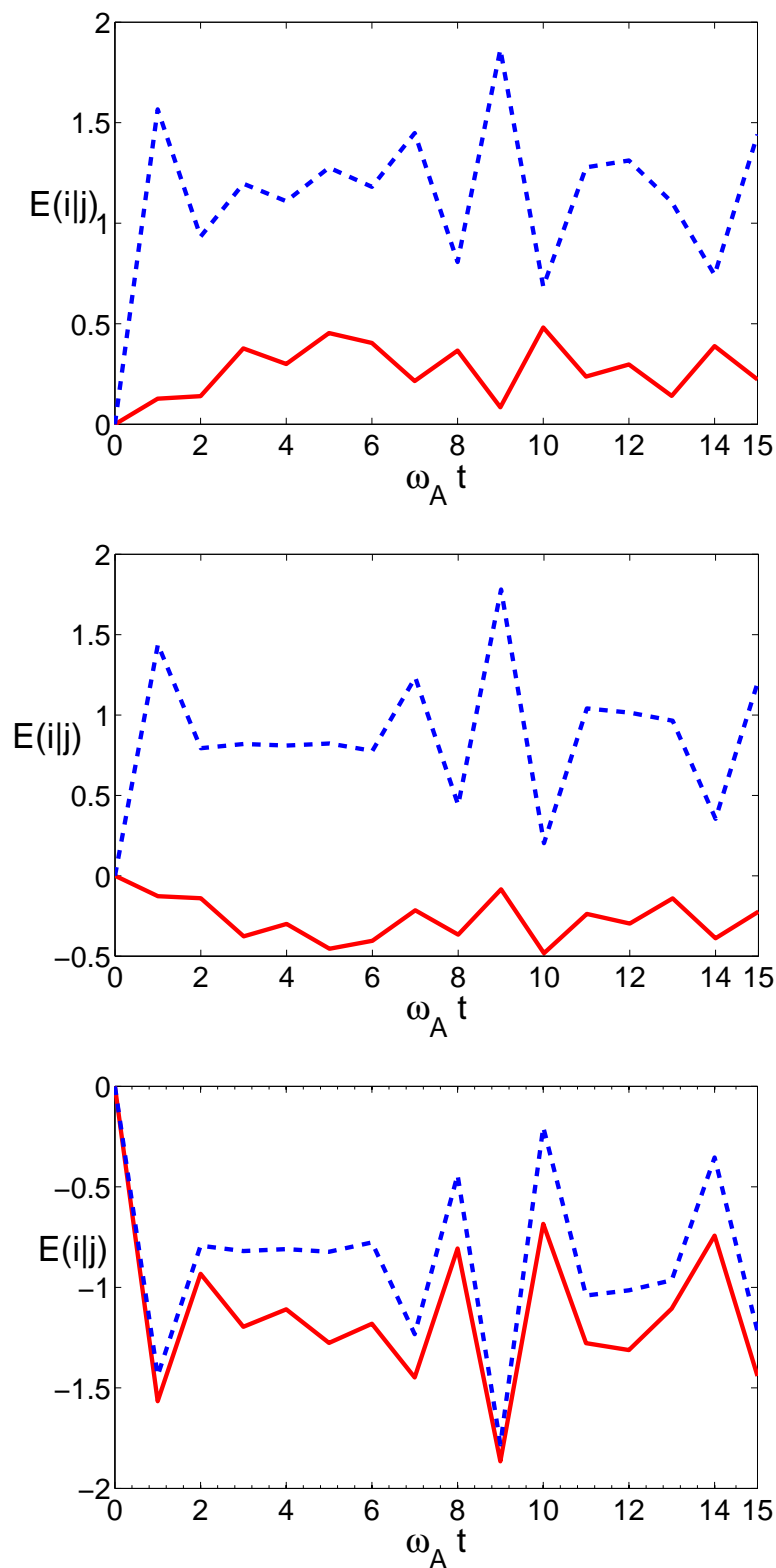


Figure 4.5: The conditional entropy $E(i|j)$ as Eq.(4.7) as function of time $\omega_A t$ for case 1 (pg. 54). The Hamiltonian of Eq.(4.1) with initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered. In the top figure we show $E(A|B)$ (solid line) and $E(B|A)$ (dashed line); in the middle figure we show $E(A|C)$ (solid line) and $E(C|A)$ (dashed line); in the bottom figure we show $E(B|C)$ (solid line) and $E(C|B)$ (dashed line). $E(i|j) < 0$ shows existence of entanglement between two subsystems whilst positive values give an inconclusive result.

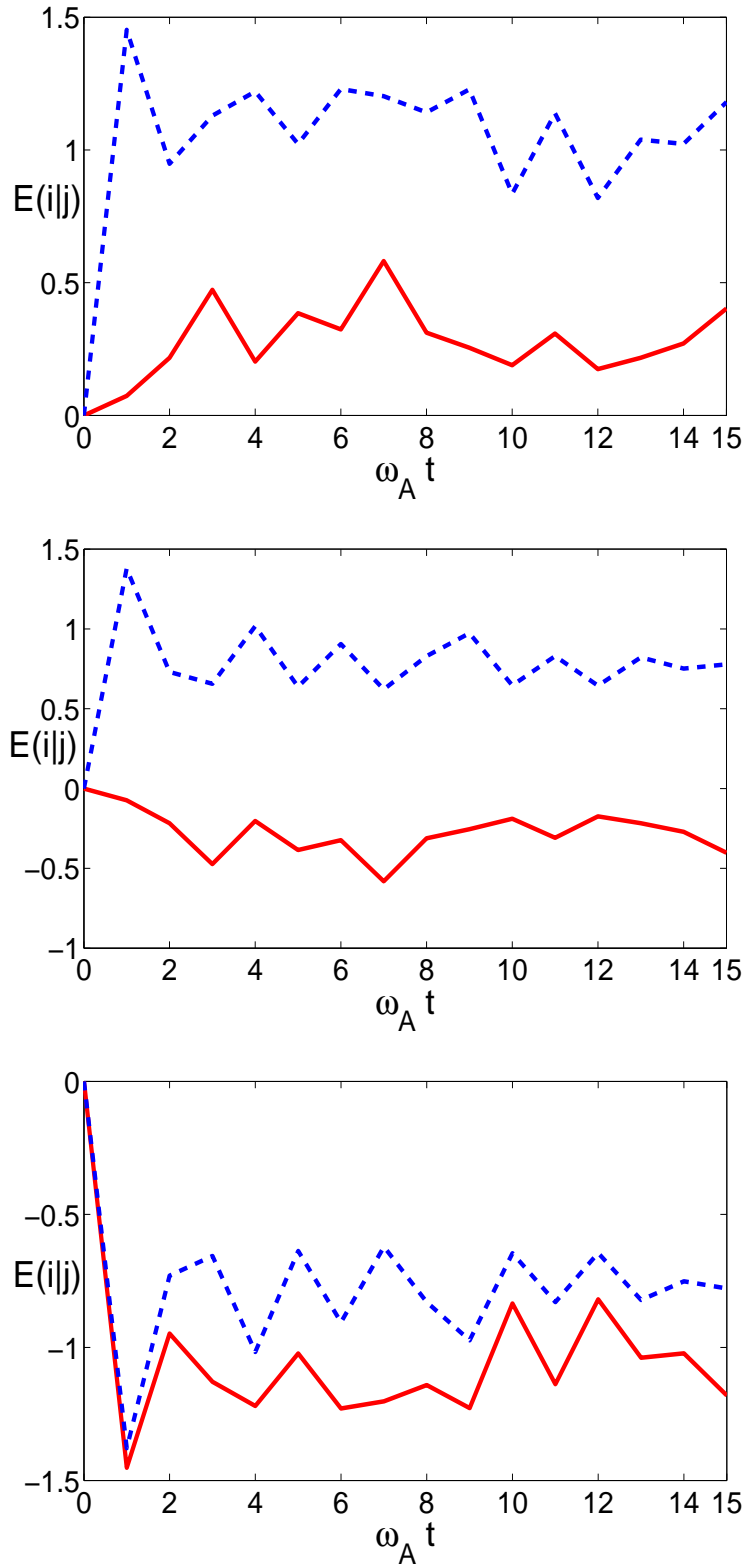


Figure 4.6: The conditional entropy $E(i|j)$ as Eq.(4.7) as function of time $\omega_A t$ for case 2 (pg. 55). The Hamiltonian of Eq.(4.1) with initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered. In the top figure we show $E(A|B)$ (solid line) and $E(B|A)$ (dashed line); in the middle figure we show $E(A|C)$ (solid line) and $E(C|A)$ (dashed line); in the bottom figure we show $E(B|C)$ (solid line) and $E(C|B)$ (dashed line). $E(i|j) < 0$ shows existence of entanglement between two subsystems whilst positive values give an inconclusive result.

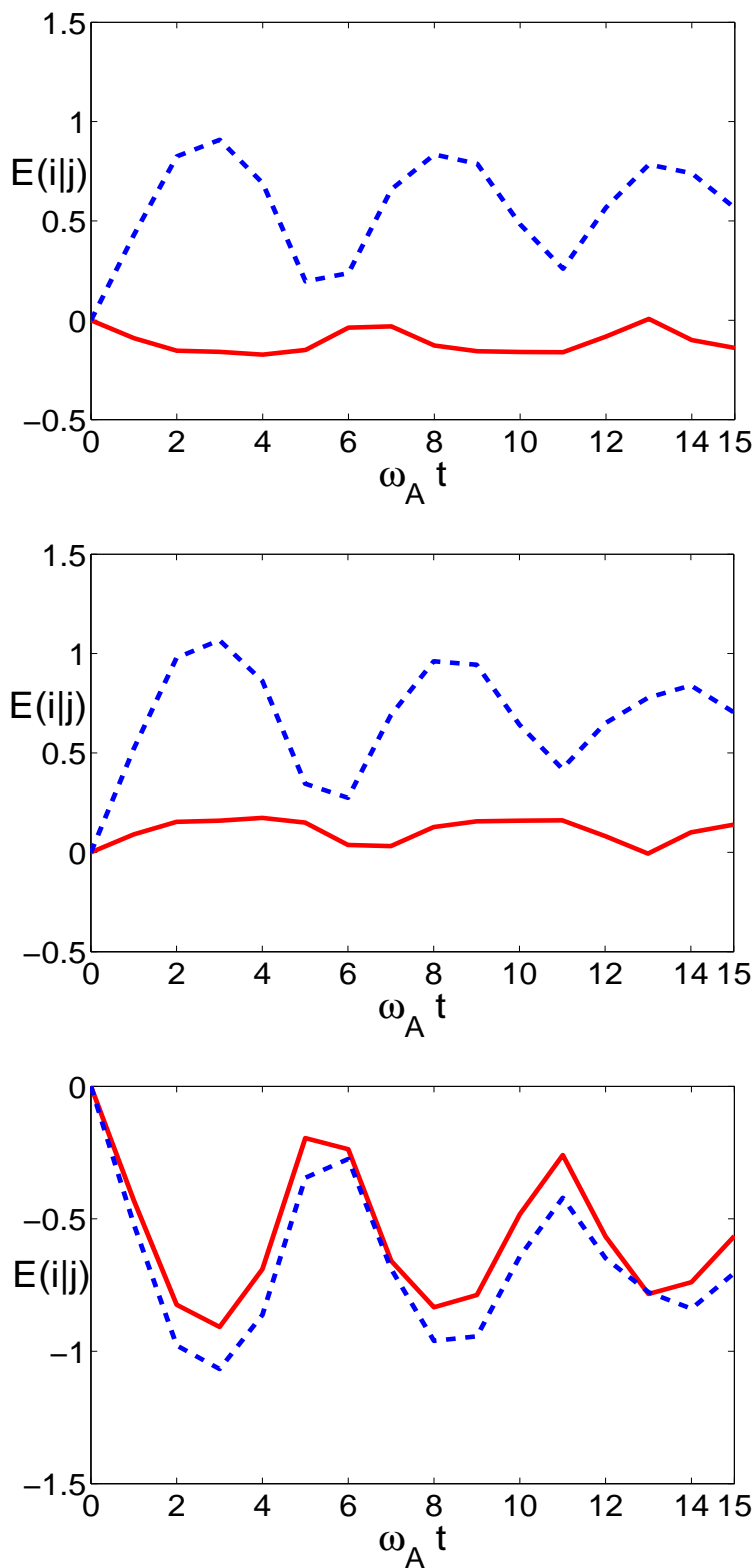


Figure 4.7: The conditional entropy $E(i|j)$ as Eq.(4.7) as function of time $\omega_A t$ for case 3 (pg. 55). The Hamiltonian of Eq.(4.1) with initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered. In the top figure we show $E(A|B)$ (solid line) and $E(B|A)$ (dashed line); in the middle figure we show $E(A|C)$ (solid line) and $E(C|A)$ (dashed line); in the bottom figure we show $E(B|C)$ (solid line) and $E(C|B)$ (dashed line). $E(i|j) < 0$ shows existence of entanglement between two subsystems whilst positive values give an inconclusive result.

A, C and oscillators B, C are entangled but not inconclusive for oscillators A, B . This is the second counter-intuitive result that we have. To confirm our counter-intuitive result, we introduced parameter R . $R = 1$, for area where $E(A|C)$ or $E(C|A)$ are negative and both of $E(A|B)$ and $E(B|A)$ are positive. $R = 0$, for the other cases. We plot $R = 1$ (white) and $R = 0$ (grey) as a function of ω_B and ω_C for $\omega_A = 1$ at $t = 5$ and $\lambda_{AB} = \lambda_{BC} = 0.1$. This is present in Fig.(4.8), where the values of ω_B and ω_C , indicates that oscillators A, C are entangled but not sure about oscillators A, B .

In order to confirm our calculation, we used negativity (N_{ij}) to measure the entanglement. If N_{ij} is vanished, it means no entanglement. Based on Fig.(4.9) and Fig.(4.10), we plot the negativity. Fig.(4.9), shows strong entanglement for oscillators A, C and very weak entanglement for oscillators A, B whereas Fig.(4.10), indicates a consistent pattern of entanglement for oscillator A, C and oscillators A, B when compared with Fig.(4.8). White area in Fig.(4.8), gives $N_{AB} < N_{AC}$ in Fig.(4.10). Negativity result gives strong entanglement for oscillators A, C and a weak one at oscillators A, B .

4.3 Discussions

In this chapter, the correlation between the three oscillators A, B and C have been studied. The Hamiltonian system that we considered vividly described the coupling of the oscillators. Oscillators A, B and oscillators B, C are coupled but there is no direct coupling between oscillators A, C . At $t = 0$, we considered a factorizable state as in Eq.(4.9) and observed no coupling among all the oscillators. At time t with density matrix evolved by Eq.(4.2), the

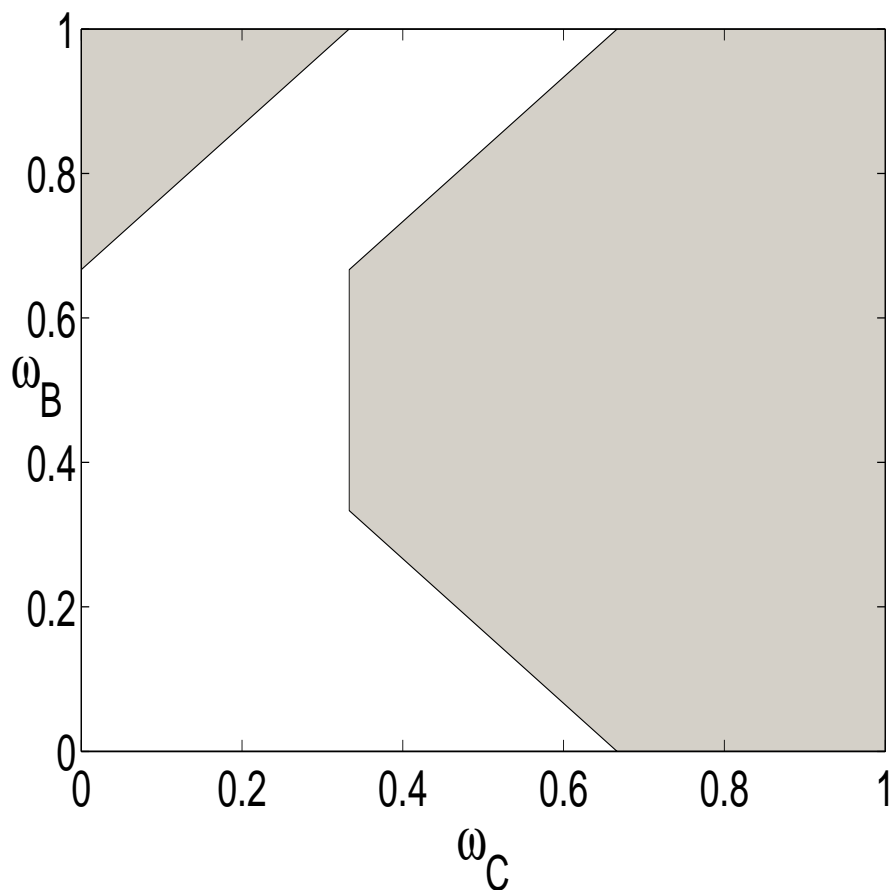


Figure 4.8: The parameter R as functions of ω_B, ω_C for $\omega_A = 1$ and $\lambda_{AB} = \lambda_{BC} = 0.1$ at time $\omega_A t = 5$. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered. $R = 1$ and $R = 0$ in the white and grey area, correspondingly. $R = 1$ (white) for area where $E(A|C)$ or $E(C|A)$ are negative and both of $E(A|B)$ and $E(B|A)$ are positive. $R = 0$ (grey) for the other cases.

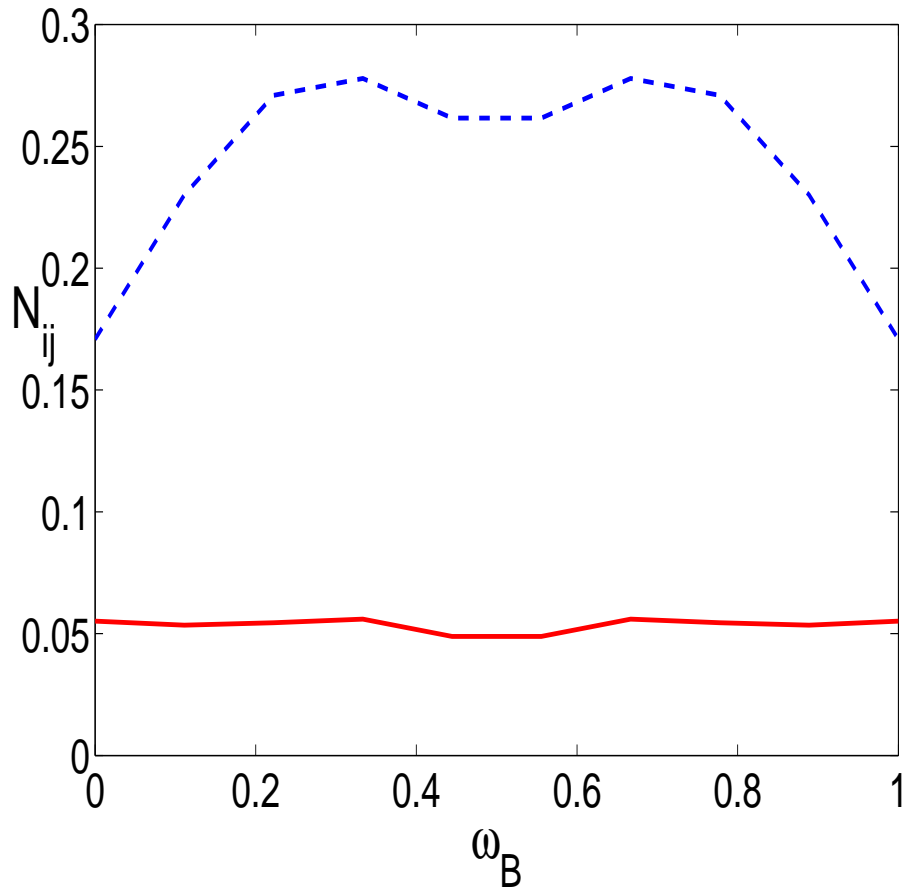


Figure 4.9: The negativity N_{ij} of Eq.(4.8) for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). N_{AB} (solid line) and N_{AC} (dashed line) are plotted as a function of ω_B for $\omega_A = 1, \omega_C = 0.25$ and $\lambda_{AB} = \lambda_{BC} = 0.1$ at time $\omega_A t = 5$. $N_{ij} > 0$ indicates the subsystem is entangled and $N_{ij} = 0$ that it is not entangled.

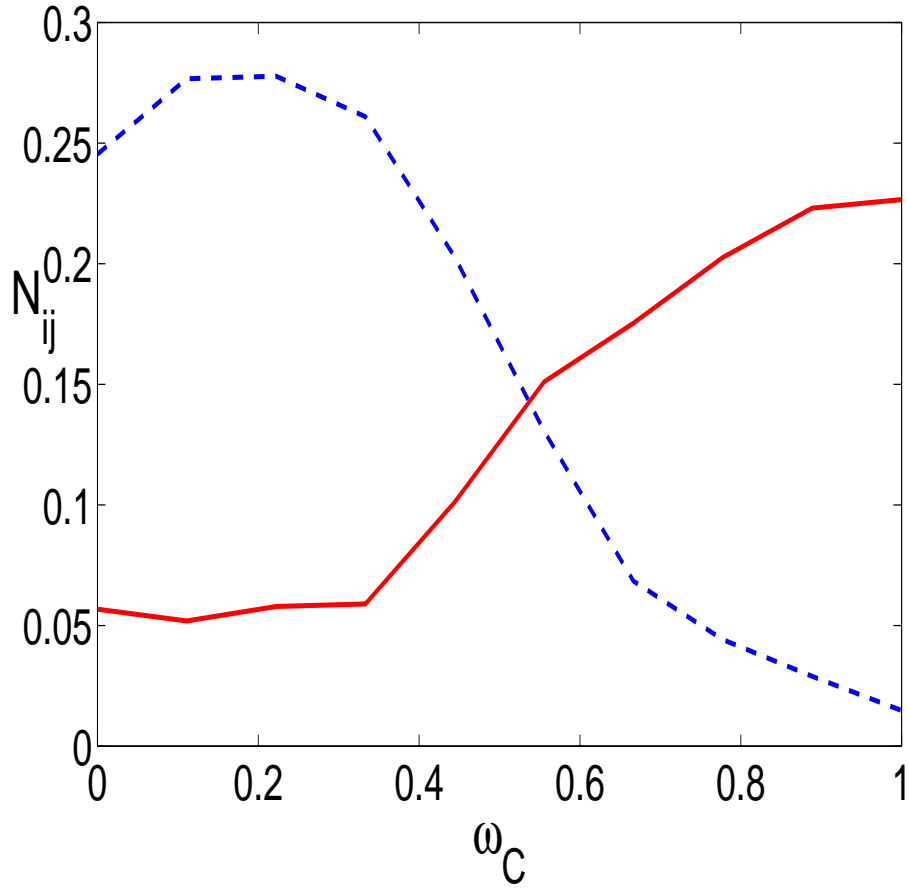


Figure 4.10: The negativity N_{ij} of Eq.(4.8) for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). N_{AB} (solid line) and N_{AC} (dashed line) are plotted as a function of ω_C for $\omega_A = 1, \omega_B = 0.25$ and $\lambda_{AB} = \lambda_{BC} = 0.1$ at time $\omega_A t = 5$. $N_{ij} > 0$ indicates the subsystem is entangled and $N_{ij} = 0$ that it is not entangled.

aim was to study correlation and entanglement among the three oscillators using the Hamiltonian system.

Based on the numerical results, we calculate a quantum mutual information (I_{ij}) to identify the existence of classical and quantum correlations. Conditional entropy, $E(i|j)$ was computed to indicate the existence of entanglement for the subsystem. In Fig.(4.3) and Fig.(4.7), our test proved that $I_{AC} < I_{AB}$ and oscillators A, B entangled with an inconclusive result for oscillators A and C . This is expected result because oscillators A, B formed coupling whilst oscillators A, C did not. We had shown from Fig.(4.1), Fig.(4.2), Fig.(4.5) and Fig.(4.6) that quantum mechanics violate this expected results. In our counter-intuitive test results, we observed the first $I_{AC} > I_{AB}$ and second, oscillators A, C entangled but were not sure about oscillators A, B . In order to make it clear, we plot Fig.(4.4) to confirm that in some values of ω_B and ω_C (white area) I_{AC} is greater than I_{AB} . For the entanglement, we confirmed by calculating the negativity that shows (Fig.(4.9) and Fig.(4.10)) oscillators A and C entangled, but we were not sure about oscillators A, B .

Chapter 5

Von Neumann measurements in tri-partite systems

In this chapter, we extend our discussion from Chapter 4. By using the same example as in the previous Chapter with the focus on case 1 (strong coupling between oscillators A, B and B, C), we continue our studies with measurements. We want to study the effects of measurement to the correlations and entanglement of the quantum system. To identify the effects, we have calculated the entropy, $S(\rho)$, quantum mutual information, I_{ij} and conditional entropy, $E(i|j)$ of the system after the measurement. Further, we have investigated the quantumness of the system, especially on inconclusive correlations. Our objectives are to study how the collapse state due to the measurement on A affects the whole state of the systems and to look whether we still can get a counter-intuitive result after the measurement.

5.1 Measurements on A

Two counter-intuitive results found in Chapter 4. First, $I_{AC} > I_{AB}$ and another one is the oscillators A, C entangled and not being sure about oscillators A, B , although there is coupling between oscillators A, B and no direct coupling between oscillators A, C . Next, we study the measurement. We are interested in investigating the effects of the measurement to the correlations and entanglement of the tri-partite quantum system and whether we still can get the counter-intuitive results. We performed measurements of the photon number N by A at $\omega_A t = 3$.

Projection operators Π_N which is performed on the oscillator A is

$$\Pi_N = \Pi_N^A \otimes \mathbf{1}_B \otimes \mathbf{1}_C \equiv |N\rangle\langle N| \otimes \mathbf{1}_B \otimes \mathbf{1}_C \quad (5.1)$$

where N is a photon number or outcomes and $\mathbf{1}_i$ is an identity matrix for the subsystems.

We consider two types of von Neumann measurement as discussed in section 3.2, selective and non-selective measurements. With selective measurement, the results are communicated instantaneously with classical methods to oscillators B and C . However, with non-selective measurement, it is known that a measurement has been made, but the results or outcome are not known.

In the case of the selective measurement, the state immediately known

after the measurement

$$\rho'_{ABC}(t, N) = \frac{\Pi_N \rho_{ABC}(t) \Pi_N}{\text{Tr}[\rho_{ABC}(t) \Pi_N]} \quad (5.2)$$

where $\rho_{ABC}(t)$, the state just before the measurements at a time $\omega_A t$ and

$$prob(t, N) = \text{Tr}[\rho_{ABC}(t) \Pi_N] \quad (5.3)$$

is the probability outcome of the measurement corresponding to photon number N .

The partial trace for the selective measurement is

$$\rho'_{ij}(t, N) = \text{Tr}_k[\rho'_{ijk}(t, N)] \quad (5.4)$$

$$\rho'_i(t, N) = \text{Tr}_{jk}[\rho'_{ijk}(t, N)] \quad (5.5)$$

where $i, j, k = A, B, C$.

The von Neumann entropy can be calculated as

$$S(\rho'_i(t, N)) = -\text{Tr}[\rho'_i(t, N) \log_2(\rho'_i(t, N))] \quad (5.6)$$

$$S(\rho'_{ij}(t, N)) = -\text{Tr}[\rho'_{ij}(t, N) \log_2(\rho'_{ij}(t, N))] \quad (5.7)$$

The quantum mutual information, $I'_{ij}(t, N)$ is calculated to indicate the existence of classical and quantum correlations between the two subsystem after the selective measurement.

$$I'_{ij}(t, N) = S(\rho'_i(t, N)) + S(\rho'_j(t, N)) - S(\rho'_{ij}(t, N)) \geq 0 \quad (5.8)$$

Entanglement after the selective measurement is identified by calculating the conditional entropy

$$E'(i|j)(t, N) = S(\rho'_{ij}(t, N)) - S(\rho'_j(t, N)) \quad (5.9)$$

In the case of non-selective measurement, the state immediately after the measurement is

$$\widetilde{\rho_{ABC}}(t) = \sum_{i=0}^{d-1} \Pi_i \rho_{ABC}(t) \Pi_i \quad (5.10)$$

where $\rho_{ABC}(t)$ is the state just before the non-selective measurement.

We noted that, in the case of non-selective measurement, the approach is same as selective measurement, and we have calculated the partial trace as

$$\widetilde{\rho_{ij}}(t) = \text{Tr}_k[\widetilde{\rho_{ijk}}(t)] \quad (5.11)$$

$$\widetilde{\rho_i}(t) = \text{Tr}_{jk}[\widetilde{\rho_{ijk}}(t)] \quad (5.12)$$

where $i, j, k = A, B, C$.

The von Neumann entropy can be calculated as

$$S(\widetilde{\rho_i}(t)) = -\text{Tr}[\widetilde{\rho_i}(t) \log_2(\widetilde{\rho_i}(t))] \quad (5.13)$$

$$S(\widetilde{\rho_{ij}}(t)) = -\text{Tr}[\widetilde{\rho_{ij}}(t) \log_2(\widetilde{\rho_{ij}}(t))] \quad (5.14)$$

The quantum mutual information has been calculated as

$$\widetilde{I_{ij}}(t) = S(\widetilde{\rho_i}(t)) + S(\widetilde{\rho_j}(t)) - S(\widetilde{\rho_{ij}}(t)) \geq 0 \quad (5.15)$$

The conditional entropy is calculated by

$$E(\widetilde{i|j})(t) = S(\widetilde{\rho_{ij}}(t)) - S(\widetilde{\rho_j}(t)) \quad (5.16)$$

5.2 Quantum discord of oscillators A, B and A, C

We are interested in studying the quantumness of oscillators A, B and oscillators A, C for all cases, without measurement, selective measurement and non-selective measurement. In this section, we have applied the approach to calculate the quantum discord based on the quantum discord discussed in [17, 28, 64].

In the case of without measurement, we have calculated the partial trace of A and AB

$$\rho_A(t) = \text{Tr}_{BC}(\rho_{ABC}(t)) \quad (5.17)$$

$$\rho_{AB}(t) = \text{Tr}_C(\rho_{ABC}(t)) \quad (5.18)$$

where $\rho_{ABC}(t)$ is a state after the evolution as Eq.(4.2). We have also calculated the entropy

$$S(\rho_A(t)) = -\text{Tr}[\rho_A(t) \log(\rho_A(t))] \quad (5.19)$$

$$S(\rho_{AB}(t)) = -\text{Tr}[\rho_{AB}(t) \log(\rho_{AB}(t))] \quad (5.20)$$

We assume that we performed the measurement on A at $\omega_A t = 3$ with

projection operators

$$\Pi_j^A = |j\rangle\langle j| \otimes \mathbf{1}_B \quad (5.21)$$

where j is labelled as a different outcome for this measurement and $\mathbf{1}_B$ is an identity matrix for subsystem B . The state of B after the outcome corresponding to Π_j^A

$$\rho_{B|\Pi_j^A}(t) = \frac{\Pi_j^A \rho_{AB}(t) \Pi_j^A}{\text{prob}_j(t)} \quad (5.22)$$

with

$$\text{prob}_j(t) = \text{Tr}(\Pi_j^A \rho_{AB}(t)) \quad (5.23)$$

The entropy, $S(\rho_{B|\Pi_j^A}(t))$ has been calculated by

$$S(\rho_{B|\Pi_j^A}(t)) = -\text{Tr}[\rho_{B|\Pi_j^A}(t) \log(\rho_{B|\Pi_j^A}(t))] \quad (5.24)$$

Then, we calculated the conditional entropy

$$E(B|\{\Pi_j^A\})(t) = \sum_j \text{prob}_j(t) S(\rho_{B|\Pi_j^A}(t)) \quad (5.25)$$

Quantum discord is the difference between two identical expressions of mutual information, thus the quantum discord

$$\delta_{AB}(t) = I_{AB}(t) - J_{AB}(t) \quad (5.26)$$

$$= S(\rho_A(t)) - S(\rho_{AB}(t)) + E(B|\{\Pi_j^A\})(t) \quad (5.27)$$

In our case, we consider only measurement on A and Π_j^A in Eq.(5.21) which defines a measurement that is optimal for the conditional entropy Eq.(5.25).

We also calculated the quantum discord of oscillators A, C . In this case, we used the same approach to calculate the quantum discord of oscillators A, B which replaced the notation B with notation C in all our calculations. Therefore, the quantum discord for oscillators A, C

$$\delta_{AC}(t) = I_{AC}(t) - J_{AC}(t) \quad (5.28)$$

$$= S(\rho_A(t)) - S(\rho_{AC}(t)) + E(C|\{\Pi_j^A\})(t) \quad (5.29)$$

where

$$E(C|\{\Pi_j^A\})(t) = \sum_j \text{prob}_j(t) S(\rho_{C|\Pi_j^A}(t)) \quad (5.30)$$

with $S(\rho_{C|\Pi_j^A}(t))$ is the entropy of $\rho_{C|\Pi_j^A}(t)$

$$\rho_{C|\Pi_j^A}(t) = \frac{\Pi_j^A \rho_{AC}(t) \Pi_j^A}{\text{prob}_j(t)} \quad (5.31)$$

where

$$\text{prob}_j(t) = \text{Tr}(\Pi_j^A \rho_{AC}(t)) \quad (5.32)$$

and

$$\Pi_j^A = |j\rangle\langle j| \otimes \mathbf{1}_C. \quad (5.33)$$

Details of the approach or algorithms to calculate the quantum discord for oscillators A, B and oscillators A, C for the case of without measurement given in Appendix A and Appendix B correspondingly.

In the case of non-selective measurement, the same approach is applied.

We calculated the quantum discord for oscillators A, B

$$\widetilde{\delta_{AB}}(t) = \widetilde{I_{AB}}(t) - \widetilde{J_{AB}}(t) \quad (5.34)$$

$$= S(\widetilde{\rho_A}(t)) - S(\widetilde{\rho_{AB}}(t)) + E(B|\{\Pi_j^A\})(t) \quad (5.35)$$

where

$$E(B|\{\Pi_j^A\})(t) = \sum_j prob_j(t) S(\widetilde{\rho_{B|\Pi_j^A}}(t)) \quad (5.36)$$

with $S(\widetilde{\rho_{B|\Pi_j^A}}(t))$ is the entropy of $\widetilde{\rho_{B|\Pi_j^A}}(t)$

$$\widetilde{\rho_{B|\Pi_j^A}}(t) = \frac{\Pi_j^A \widetilde{\rho_{AB}}(t) \Pi_j^A}{prob_j(t)} \quad (5.37)$$

where

$$prob_j(t) = \text{Tr}(\Pi_j^A \widetilde{\rho_{AB}}(t)) \quad (5.38)$$

and

$$\Pi_j^A = |j\rangle\langle j| \otimes \mathbf{1}_B \quad (5.39)$$

By using the same approach, quantum discord for oscillators A, C is

$$\widetilde{\delta_{AC}}(t) = \widetilde{I_{AC}}(t) - \widetilde{J_{AC}}(t) \quad (5.40)$$

$$= S(\widetilde{\rho_A}(t)) - S(\widetilde{\rho_{AC}}(t)) + E(C|\{\Pi_j^A\})(t) \quad (5.41)$$

where

$$E(C|\{\Pi_j^A\})(t) = \sum_j prob_j(t) S(\widetilde{\rho_{C|\Pi_j^A}}(t)) \quad (5.42)$$

with $S(\widetilde{\rho_{C|\Pi_j^A}(t)})$ is the entropy of $\widetilde{\rho_{C|\Pi_j^A}(t)}$

$$\widetilde{\rho_{C|\Pi_j^A}(t)} = \frac{\Pi_j^A \widetilde{\rho_{AC}(t)} \Pi_j^A}{\text{prob}_j(t)} \quad (5.43)$$

where

$$\text{prob}_j(t) = \text{Tr}(\Pi_j^A \widetilde{\rho_{AC}(t)}) \quad (5.44)$$

and

$$\Pi_j^A = |j\rangle\langle j| \otimes \mathbf{1}_C \quad (5.45)$$

Details of the procedure are given in Appendix C and Appendix D correspondingly.

In the case of selective measurement, we calculated the quantum discord for oscillators A, B

$$\delta'_{AB}(t, N) = I'_{AB}(t, N) - J'_{AB}(t, N) \quad (5.46)$$

$$= S(\rho'_A(t, N)) - S(\rho'_{AB}(t, N)) + E'(B|\{\Pi_N^A\})(t, N) \quad (5.47)$$

where

$$E'(B|\{\Pi_N^A\})(t, N) = \text{prob}(t, N) S(\rho'_{B|\Pi_N^A}(t, N)) \quad (5.48)$$

with $S(\rho'_{B|\Pi_N^A}(t, N))$ as the entropy of $\rho'_{B|\Pi_N^A}(t, N)$

$$\rho'_{B|\Pi_N^A}(t, N) = \frac{\Pi_N^A \rho'_{AB}(t, N) \Pi_N^A}{\text{prob}(t, N)} \quad (5.49)$$

where

$$\text{prob}(t, N) = \text{Tr}(\Pi_N^A \rho'_{AB}(t, N)) \quad (5.50)$$

and

$$\Pi_N^A = |N\rangle\langle N| \otimes \mathbf{1}_B \quad (5.51)$$

and we also calculated the quantum discord for oscillators A, C .

$$\delta'_{AC}(t, N) = I'_{AC}(t, N) - J'_{AC}(t, N) \quad (5.52)$$

$$= S(\rho'_A(t, N)) - S(\rho'_{AC}(t, N)) + E'(C|\{\Pi_N^A\})(t, N) \quad (5.53)$$

where

$$E'(C|\{\Pi_N^A\})(t, N) = \text{prob}(t, N) S(\rho'_{C|\Pi_N^A}(t, N)) \quad (5.54)$$

with $S(\rho'_{C|\Pi_N^A}(t, N))$ as the entropy of $\rho'_{C|\Pi_N^A}(t, N)$

$$\rho'_{C|\Pi_N^A}(t, N) = \frac{\Pi_N^A \rho'_{AC}(t, N) \Pi_N^A}{\text{prob}(t, N)} \quad (5.55)$$

where

$$\text{prob}(t, N) = \text{Tr}(\Pi_N^A \rho'_{AC}(t, N)) \quad (5.56)$$

and

$$\Pi_N^A = |N\rangle\langle N| \otimes \mathbf{1}_C \quad (5.57)$$

The details of the approach to calculate the quantum discord is given in Appendix E and Appendix F.

5.3 Numerical Results

In this area of study, we have considered the same examples as the ones in section 4.2.1. However, we only focused on case 1. We considered the

Hamiltonian system the same as Eq.(4.1). To make sure that we choose a sufficient truncation, two tests of accuracy are considered. First test, to make sure all the traces of all density matrices should be greater than 0.98. In our calculation, we found that all the traces of density matrices are greater than 0.98. In the second test, we computed the quantum mutual information after the measurement for the various numbers of truncation (K). Base on Table 5.1, it is shown that $K = 9$ is a sufficient truncation to give a good approximation. After this, all the calculation is based on truncation $K = 9$.

Table 5.1: The quantum mutual informations for non-selective measurement ($\widetilde{I}_{AB}, \widetilde{I}_{AC}, \widetilde{I}_{BC}$) as Eq.(5.15) and selective measurement ($I'_{AB}, I'_{AC}, I'_{BC}$) as Eq.(5.8) for various values of the truncation dimension, K at $\omega_A t = 8$.

	K=8	K=9	K=10	K=11
\widetilde{I}_{AB}	0.0977	0.0627	0.0627	0.0627
\widetilde{I}_{AC}	0.5431	0.4288	0.4288	0.4288
\widetilde{I}_{BC}	2.5142	2.1986	2.1986	2.1986
$I'_{AB}(N = 0, 1, 2)$	0	0	0	0
$I'_{AC}(N = 0, 1, 2)$	0	0	0	0
$I'_{BC}(N = 0)$	3.0733	2.8857	2.8857	2.8857
$I'_{BC}(N = 1)$	2.2677	2.4246	2.4246	2.4246
$I'_{BC}(N = 2)$	0	0	0	0

In this example, we performed a photon number measurement on A and the measurement takes place at $\omega_A t = 3$. We investigated three cases, without measurement, selective measurement and non-selective measurement. Without measurement means that we do not perform any measurement on density matrix after the evolution. Then we investigated the correlation and entanglement of the tri-partite quantum systems for all cases by calculating the entropy, quantum mutual information and conditional entropy. Further,

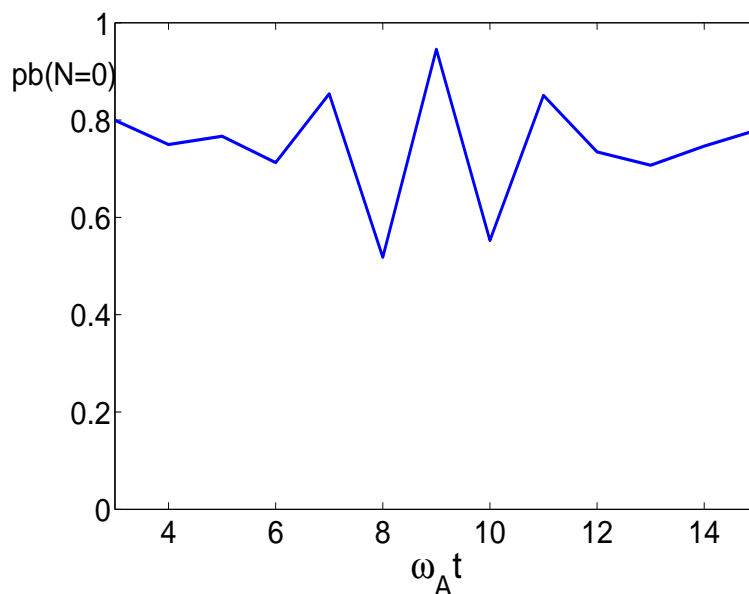


Figure 5.1: The probability outcome of the measurement of the photon number as in Eq.(5.3) for $N = 0$ as a function of time $\omega_A t$, for the selective measurement. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered.

we also investigated the quantumness of oscillators A, B and oscillators A, C by calculating the quantum discord.

5.3.1 Entropy

In the numerical calculation of selective measurement, we found that for the photon number $N = 0, 1, 2$, the probability of the outcome of the photon number N are shown in Fig.(5.1), Fig.(5.2) and Fig.(5.3). Meanwhile, for the case of photon number $N = 3, 4, \dots, 9$, the probability is zero, $prob(t, N = 3, 4, \dots, 9) = 0$.

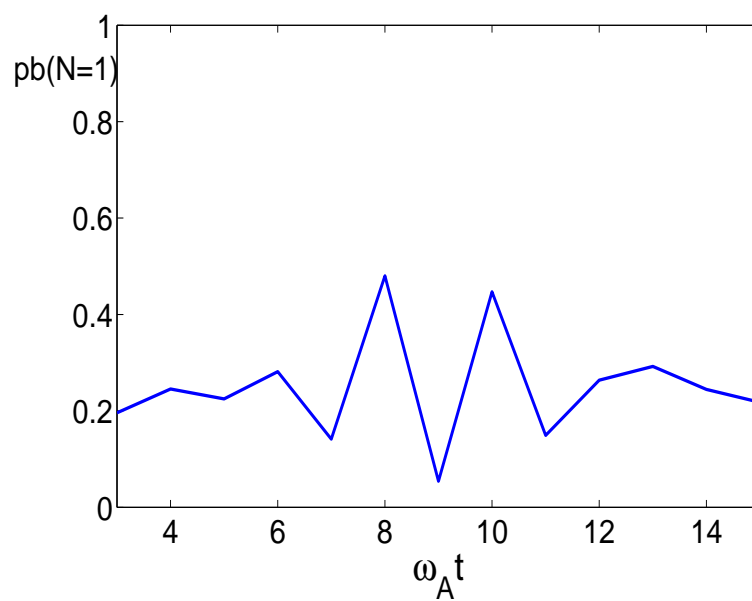


Figure 5.2: The probability outcome of the measurement of the photon number as in Eq.(5.3) for $N = 1$ as a function of time $\omega_A t$, for the selective measurement. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered.

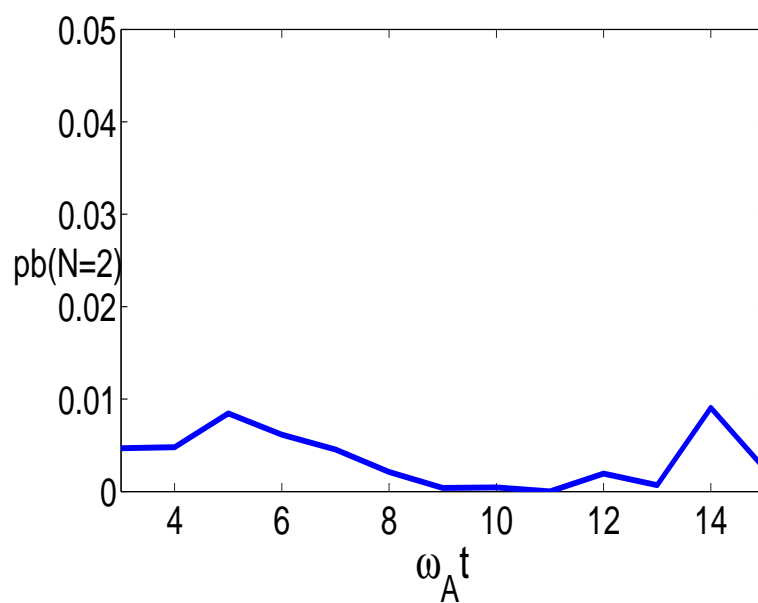


Figure 5.3: The probability outcome of the measurement of the photon number as in Eq.(5.3) for $N = 2$ as a function of time $\omega_A t$, for the selective measurement. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered.

Referring to Fig.(5.4), it has shown that

$$S(\widetilde{\rho_A(t)}) \geq S(\rho_A(t)) \quad (5.58)$$

the entropy of density matrix A of non-selective measurement is greater than or equal to the entropy of density matrix A without measurement.

For selective measurement

$$S(\rho'_A(t, N = 0, 1, 2)) = 0 \quad (5.59)$$

it has been shown that state A becomes a pure state after the selective measurement.

Base on Fig.(5.5) and Fig.(5.6), it is shown that

$$S(\widetilde{\rho_B(t)}) = S(\rho_B(t)) \quad (5.60)$$

$$S(\widetilde{\rho_C(t)}) = S(\rho_C(t)) \quad (5.61)$$

the entropy of oscillator B and oscillator C for non-selective measurement are equal with entropy of oscillator B and oscillator C without measurement.

This equality can happen if and only if [62]

$$\widetilde{\rho_B(t)} = \rho_B(t) \quad (5.62)$$

$$\widetilde{\rho_C(t)} = \rho_C(t) \quad (5.63)$$

the density matrix after non-selective measurement equals with density matrix before the measurement.

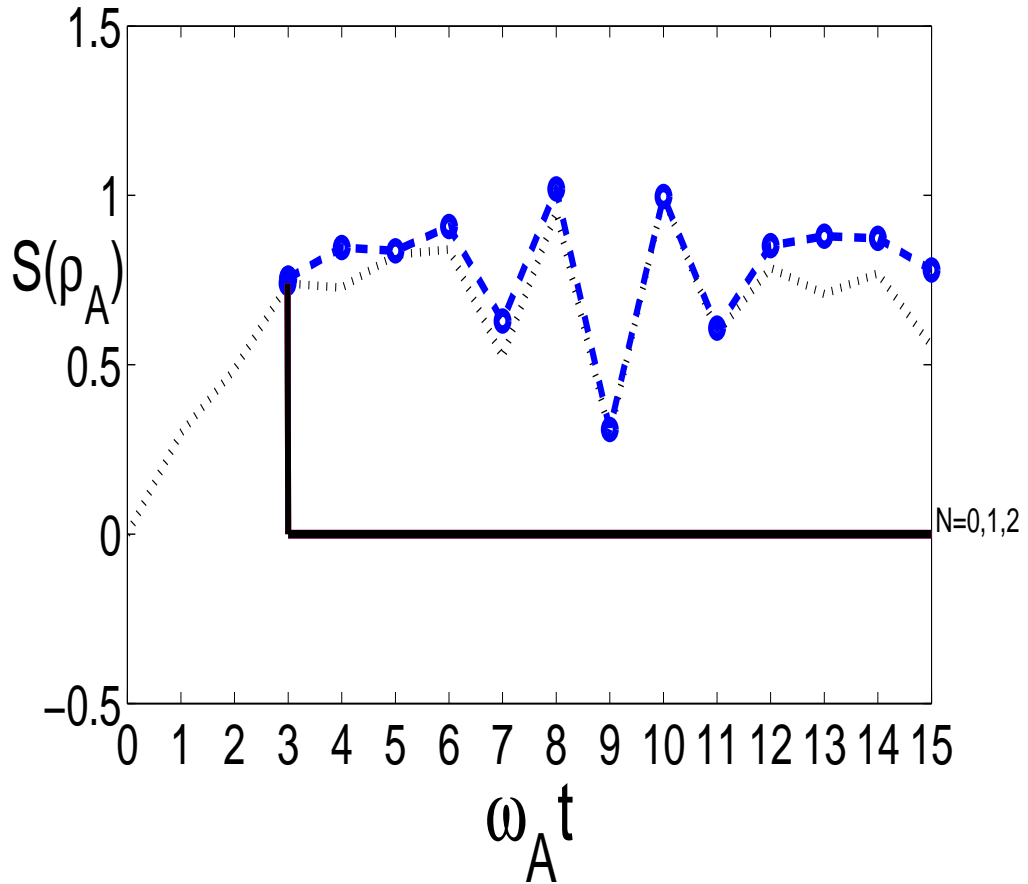


Figure 5.4: The entropy of oscillator A $S(\rho_A)$, as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $S(\rho'_A(t, N))$ (solid line) for the case of selective measurement with $N = 0, 1, 2$ as in Eq.(5.6); $S(\rho_A(t))$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.4); $S(\rho_A(t))$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.13). The measurement takes place at $\omega_A t = 3$.

Meanwhile for selective measurement

$$S(\rho'_B(t, N = 0, 1)) \leq S(\rho_B(t)) \quad (5.64)$$

$$S(\rho'_C(t, N = 0, 1)) < S(\rho_C(t)) \quad (5.65)$$

it shown that for the outcome $N = 0, 1$, the selective measurement decrease the entropy of oscillators B, C . Interesting results for outcome $N = 2$

$$S(\rho'_B(t, N = 2)) = 0 \quad (5.66)$$

$$S(\rho'_C(t, N = 2)) = 0 \quad (5.67)$$

the state of oscillators B and C become a pure state.

5.3.2 Correlation

Looking up to Fig.(5.7), Fig.(5.8) and Fig.(5.9), it has revealed that

$$\widetilde{I_{ij}(t)} \geq 0; \quad i, j = A, B, C \quad (5.68)$$

It indicates that there exists classical and quantum correlation among all the oscillators A, B and C . It also has shown that

$$\widetilde{I_{ij}(t)} \leq I_{ij}(t) \quad (5.69)$$

means that non-selective measurement decrease the correlation but the values of the $\widetilde{I_{AC}(t)} > \widetilde{I_{AB}(t)}$ as shown in Table (5.2).

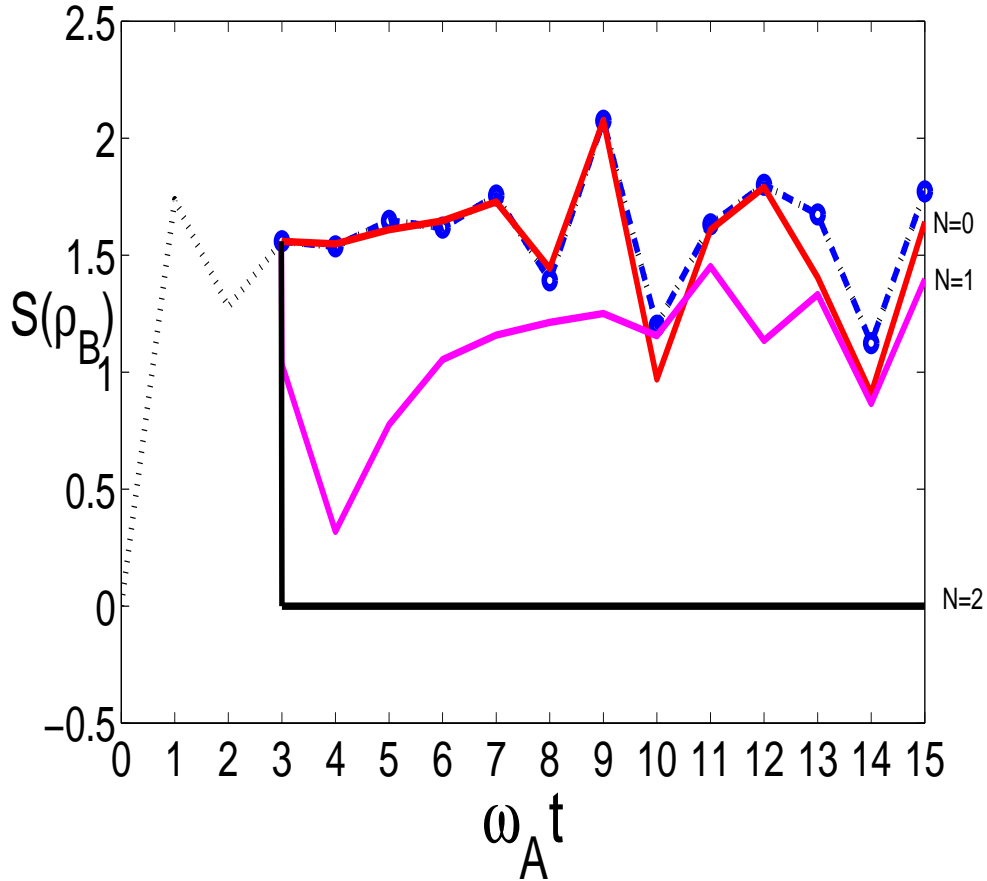


Figure 5.5: The entropy of oscillator B $S(\rho_B)$, as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $S(\rho'_B(t, N))$ (solid line) for the case of selective measurement with $N = 0, 1, 2$ as in Eq.(5.6); $S(\rho_B(t))$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.4); $S(\rho_B(t))$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.13). The measurement takes place at $\omega_A t = 3$.

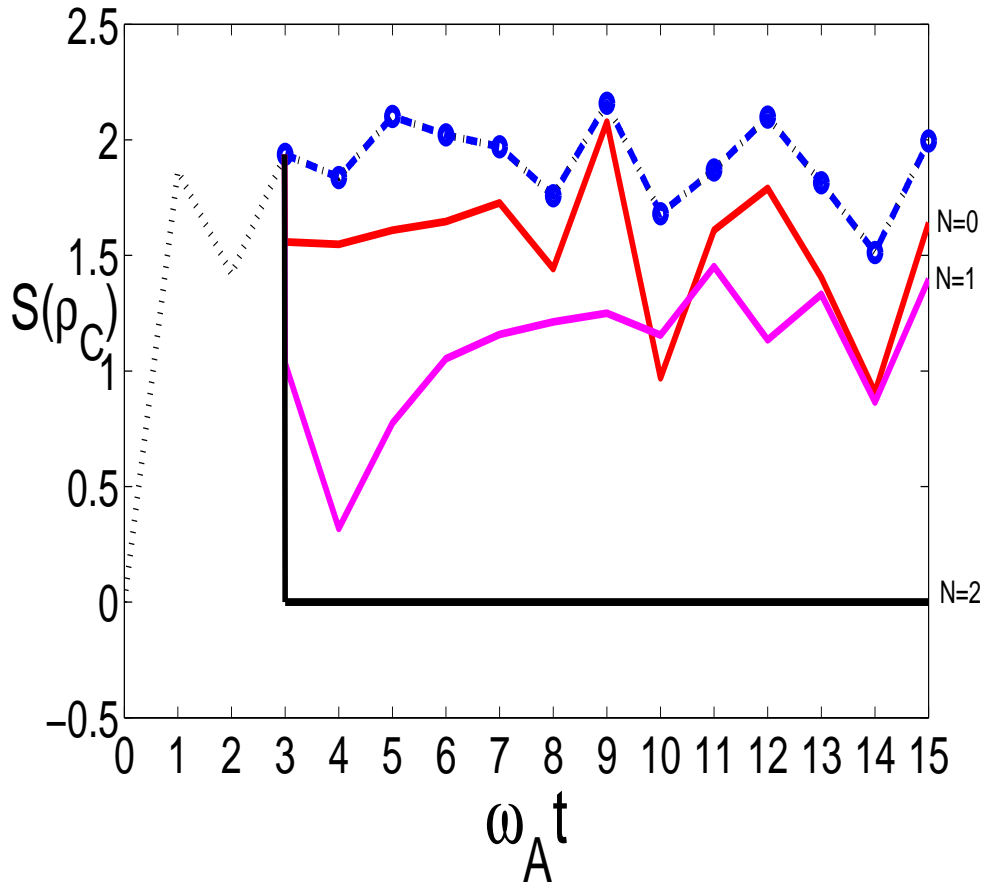


Figure 5.6: The entropy of oscillator C $S(\rho_C)$, as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $S(\rho'_C(t, N))$ (solid line) for the case of selective measurement with $N = 0, 1, 2$ as in Eq.(5.6); $S(\rho_C(t))$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.4); $S(\rho_C(t))$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.13). The measurement takes place at $\omega_A t = 3$.

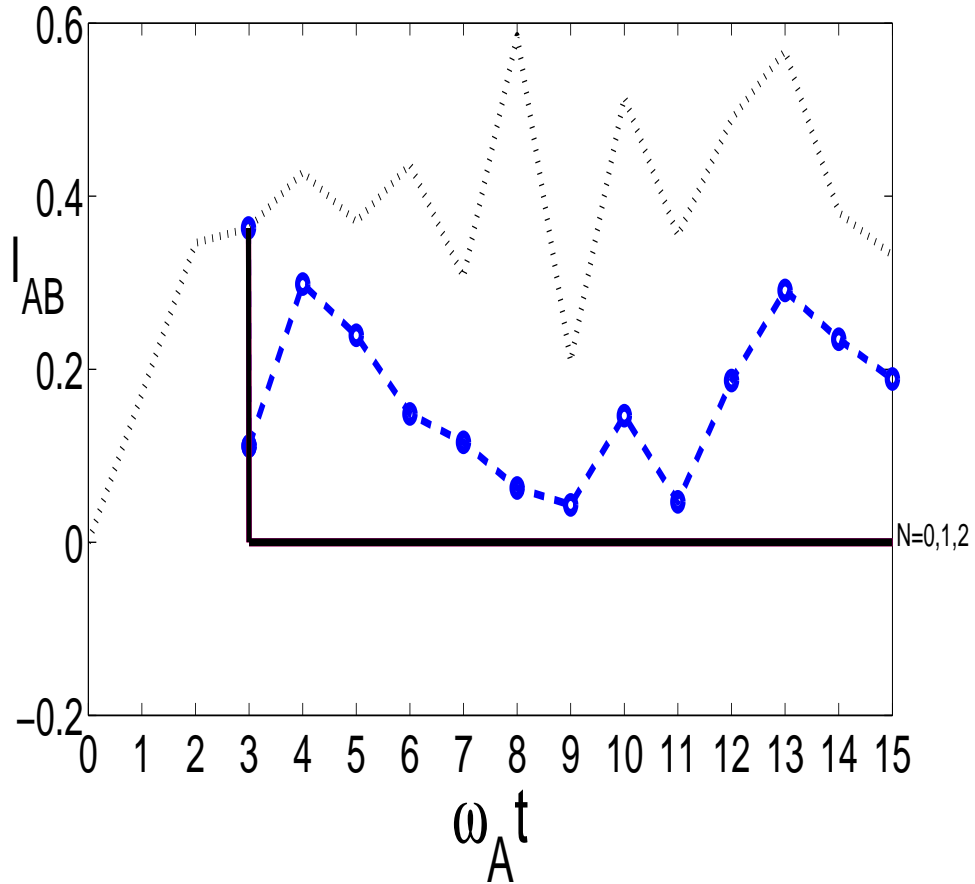


Figure 5.7: The quantum mutual information I_{AB} as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $I'_{AB}(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.8) with $N = 0, 1, 2$; $I_{AB}(t)$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.6); $I_{AB}(t)$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.15). The measurement takes place at $\omega_A t = 3$.

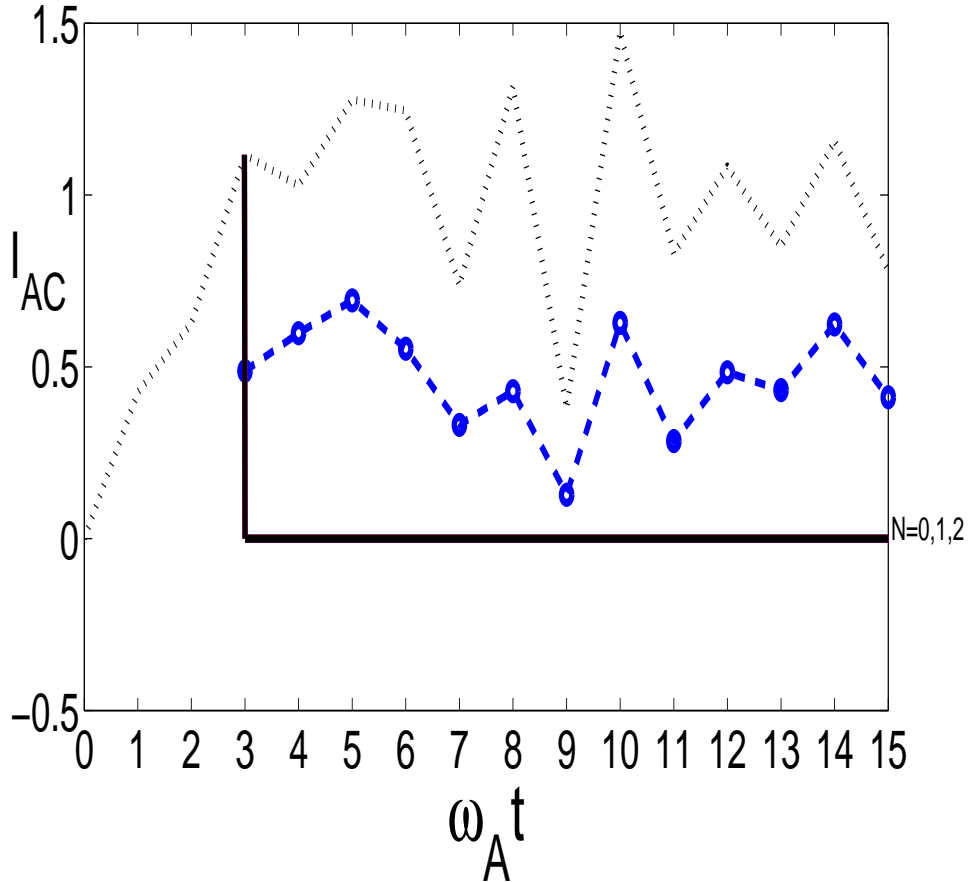


Figure 5.8: The quantum mutual information I_{AC} as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $I'_{AC}(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.8) with $N = 0, 1, 2$; $I_{AC}(t)$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.6); $\widetilde{I_{AC}}(t)$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.15). The measurement takes place at $\omega_A t = 3$.

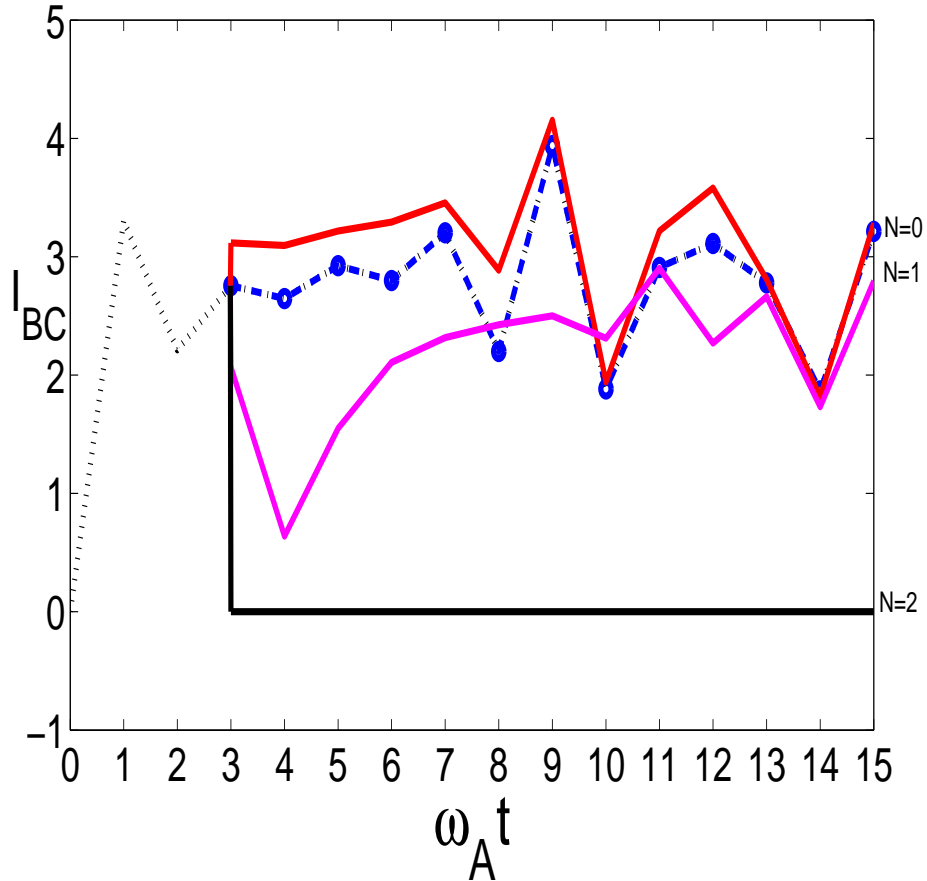


Figure 5.9: The quantum mutual information I_{BC} as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $I'_{BC}(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.8) with $N = 0, 1, 2$; $I_{BC}(t)$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.6); $\widetilde{I_{BC}}(t)$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.15). The measurement takes place at $\omega_A t = 3$.

Table 5.2: Comparative values of the quantum mutual information for non-selective measurement $\widetilde{I}_{AC}(t)$ and $\widetilde{I}_{AB}(t)$ as Eq.(5.15) at $\omega_A t = 3, 4, \dots, 15$. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered.

$\omega_A t$	$\widetilde{I}_{AC}(t)$	$\widetilde{I}_{AB}(t)$
3	0.4882	0.1112
4	0.5980	0.2985
5	0.6931	0.2393
6	0.5524	0.1482
7	0.3307	0.1156
8	0.4288	0.0627
9	0.1274	0.0431
10	0.6276	0.1462
11	0.2840	0.0468
12	0.4841	0.1870
13	0.4320	0.2911
14	0.6235	0.2347
15	0.4115	0.1887

For the case of selective measurement,

$$I'_{AB}(t, N = 0, 1, 2) = 0 \quad (5.70)$$

$$I'_{AC}(t, N = 0, 1, 2) = 0 \quad (5.71)$$

it means that there is nothing common between oscillator A, B and oscillators A, C . We can conclude that no correlations between oscillators A, B and oscillators A, C . An interesting result for oscillators B, C

$$I'_{BC}(t, N = 0) \geq I_{BC}(t) \quad (5.72)$$

$$I'_{BC}(t, N = 1) < I_{BC}(t) \quad (5.73)$$

This result applies for most of the time except at $\omega_A t = 10$ and $\omega_A t = 11$.

The I'_{BC} might increase or decrease after the selective measurement.

It is different for outcome $N = 2$

$$I'_{BC}(t, N = 2) = 0 \quad (5.74)$$

no correlation between oscillators B, C .

5.3.3 Entanglement

Based on our second counter-intuitive result as discussed in Chapter 4, study shows that without measurement, oscillators A, C are entangled but we cannot conclude whether oscillators A, B are entangled or not. Now, by performing a measurement on A , we investigated whether it will affect the results.

Referring to Fig.(5.10), Fig.(5.11), Fig.(5.12) and Fig.(5.13) it shows that after non-selective measurement

$$E(\widetilde{A|B})(t) > 0; \quad E(\widetilde{B|A})(t) > 0 \quad (5.75)$$

$$E(\widetilde{A|C})(t) > 0; \quad E(\widetilde{C|A})(t) > 0 \quad (5.76)$$

oscillators A, B and oscillators A, C are not entangled. Meanwhile, Fig.(5.14) and Fig.(5.15) show that

$$E(\widetilde{B|C})(t) < 0 \quad (5.77)$$

$$E(\widetilde{C|B})(t) < 0 \quad (5.78)$$

oscillators B, C are entangled.

For selective measurement

$$E'(A|B)(t, N = 0, 1, 2) = 0; \quad E'(B|A)(t, N = 0, 1, 2) \geq 0 \quad (5.79)$$

$$E'(A|C)(t, N = 0, 1, 2) = 0; \quad E'(C|A)(t, N = 0, 1, 2) \geq 0 \quad (5.80)$$

it shows that, the oscillators A, B and oscillators A, C are not entangled.

Next, by looking at Fig.(5.14) and Fig.(5.15) we assume that the more negative the conditional entropy, it will give more entanglement. This shows that $E'(B|C)(N = 0)$ is more entangled then $E'(B|C)(N = 1)$. Meanwhile, for outcome $N = 2$ where

$$E'(B|C)(t, N = 2) = E'(C|B)(t, N = 2) = 0 \quad (5.81)$$

it shows that at $N = 2$, the oscillators B, C not entangled.

5.3.4 Quantum discord

Before the measurement, we have known that oscillators A, C are entangled, but not sure about oscillators A, B . By calculating the quantum discord of oscillators A, B and oscillators A, C , Fig. (5.16), shows that

$$\delta_{AB} > 0 \quad (5.82)$$

$$\delta_{AC} > 0 \quad (5.83)$$

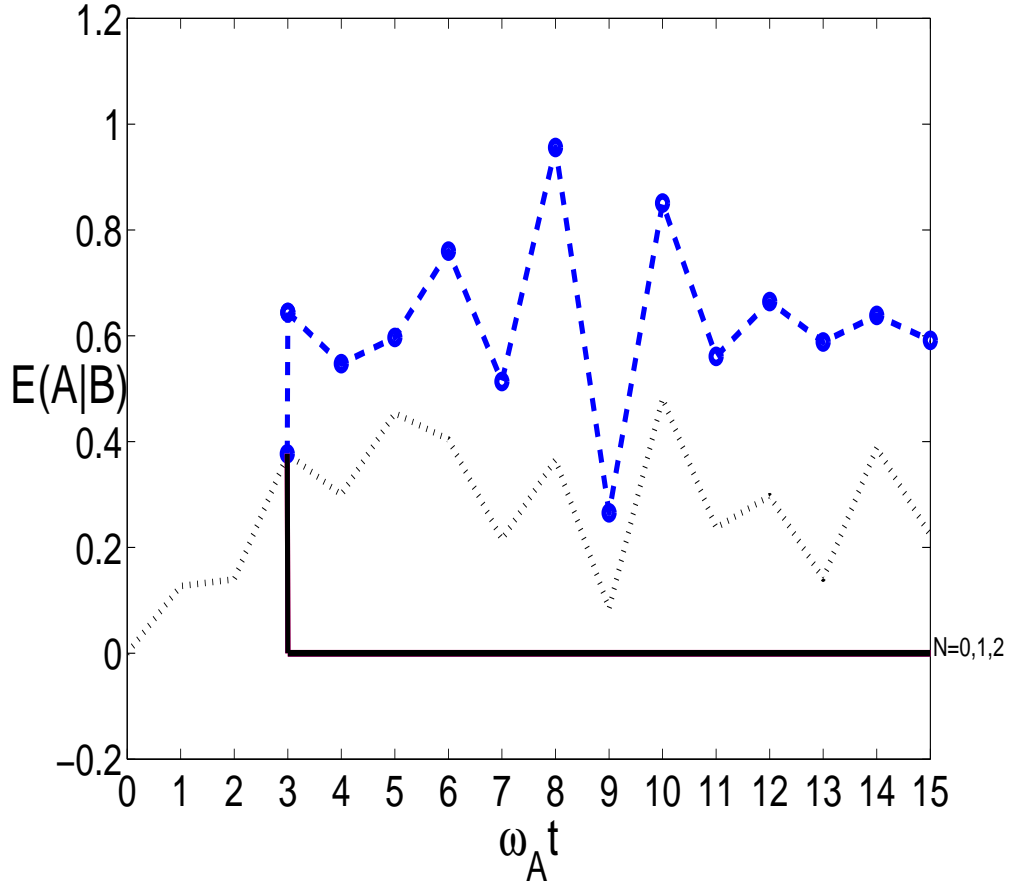


Figure 5.10: The conditional entropy $E(A|B)$ as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $E'(A|B)(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.9) with $N = 0, 1, 2$; $\widetilde{E}(A|B)(t)$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.7); $E(A|B)(t)$ (dashed line and circle) for the case of non-selective measurement as Eq.(5.16). The measurement takes place at $\omega_A t = 3$.

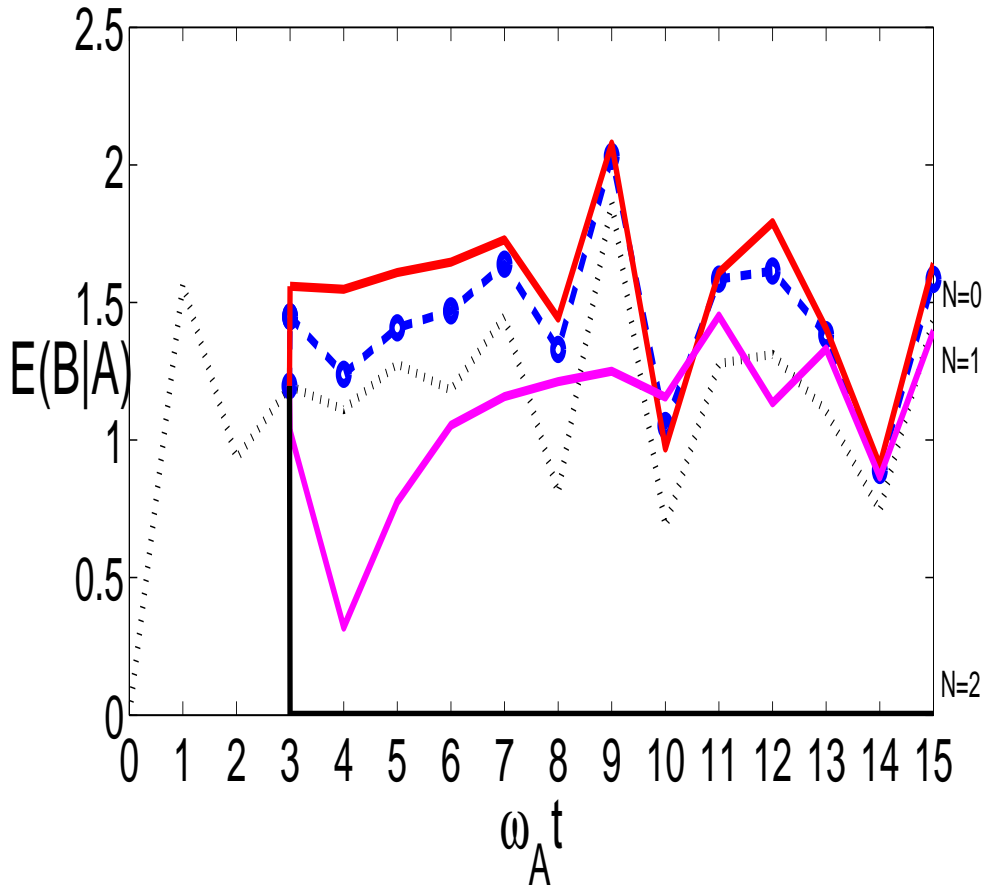


Figure 5.11: The conditional entropy $E(B|A)$ as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $E'(B|A)(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.9) with $N = 0, 1, 2$; $\widetilde{E(B|A)}(t)$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.7); $\widetilde{E(B|A)}(t)$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.16). The measurement takes place at $\omega_A t = 3$.

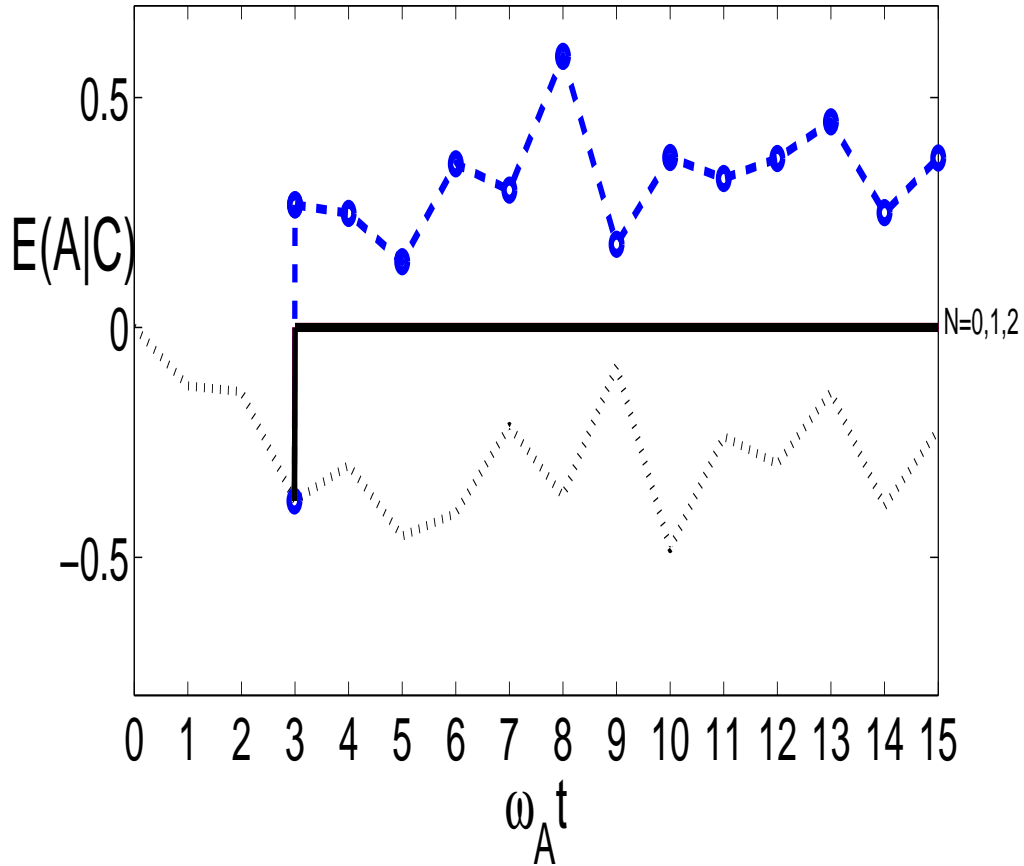


Figure 5.12: The conditional entropy $E(A|C)$ as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $E'(A|C)(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.9) with $N = 0, 1, 2$; $\widetilde{E}(A|C)(t)$ (dotted line) for the case of 'no measurement' as in Eq.(4.7); $\widetilde{E}(A|C)(t)$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.16). The measurement takes place at $\omega_A t = 3$.

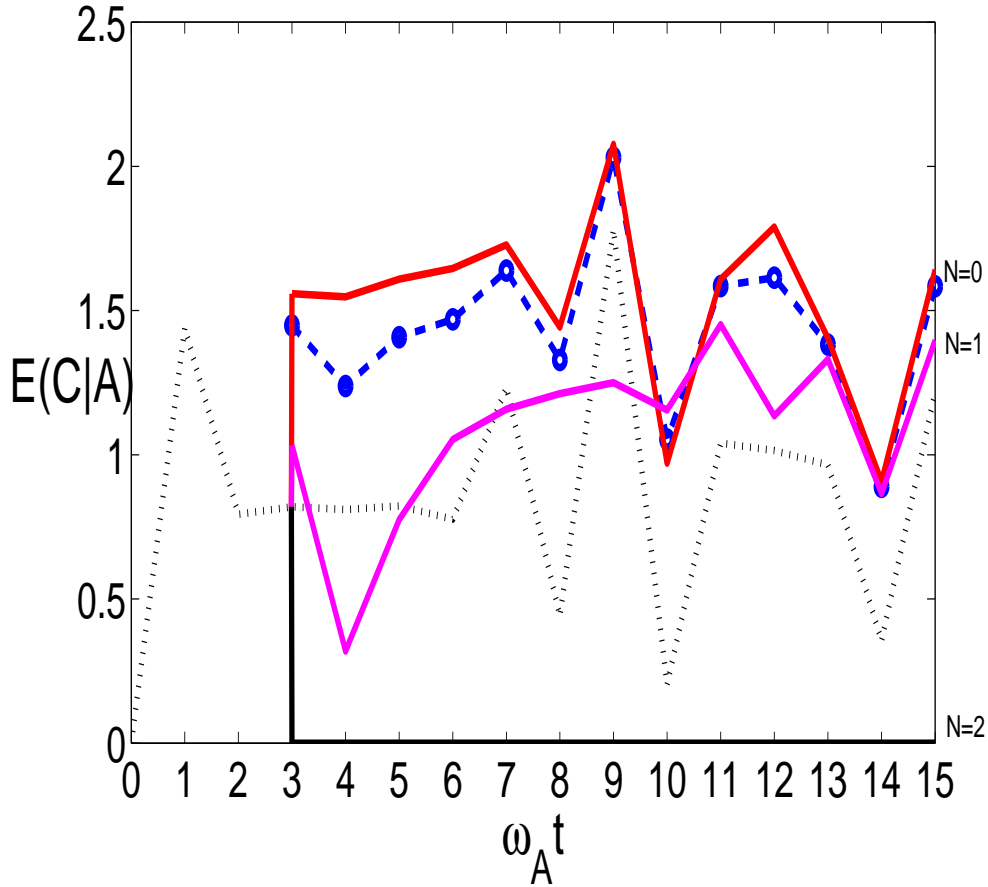


Figure 5.13: The conditional entropy $E(C|A)$ as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $E'(C|A)(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.9) with $N = 0, 1, 2$; $\widetilde{E}(C|A)(t)$ (dotted line) for the case of 'no measurement' as in Eq.(4.7); $\underline{E}(C|A)(t)$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.16). The measurement takes place at $\omega_A t = 3$.

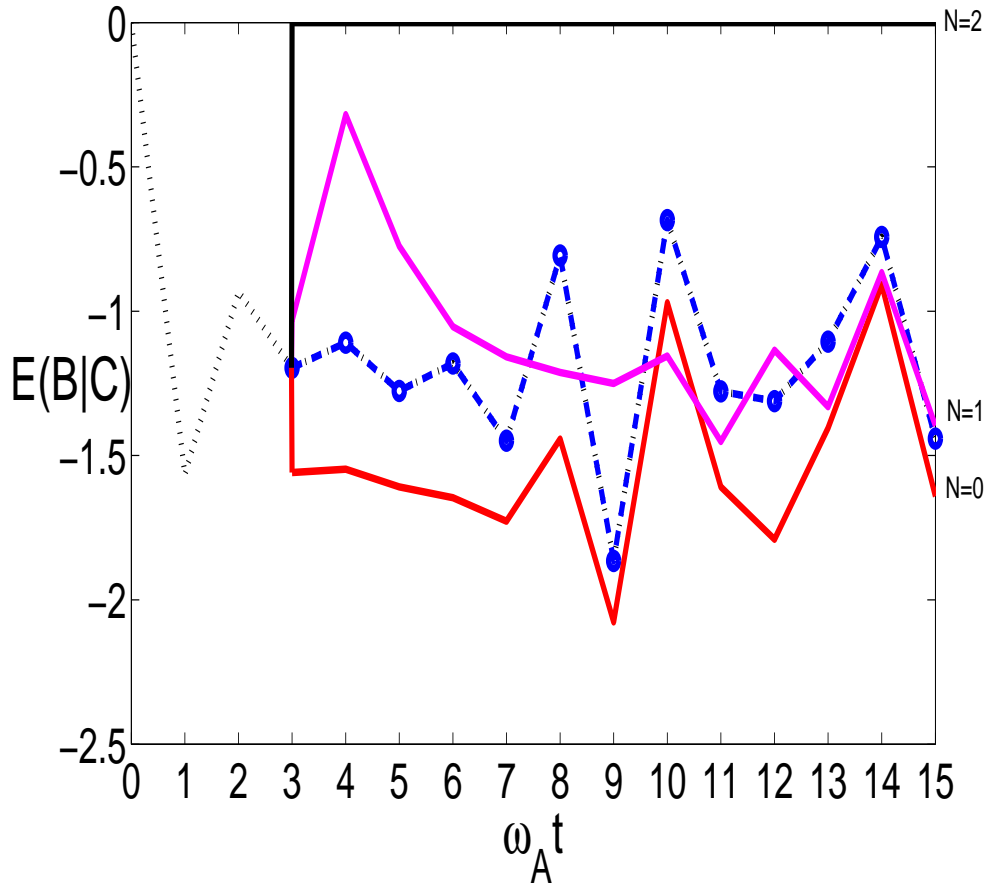


Figure 5.14: The conditional entropy $E(B|C)$ as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $E'(B|C)(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.9) with $N = 0, 1, 2$; $E(B|C)(t)$ (dotted line) for the case of ‘no measurement’ as in Eq.(4.7); $\widetilde{E(B|C)}(t)$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.16). The measurement takes place at $\omega_A t = 3$.

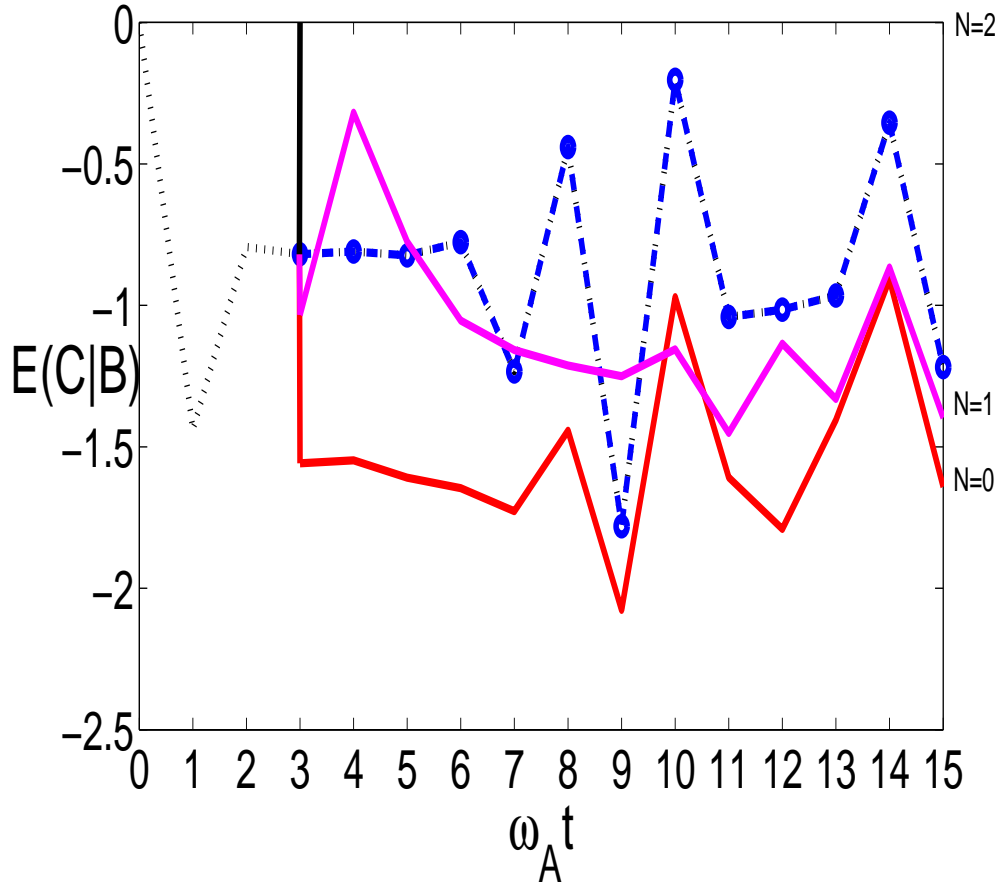


Figure 5.15: The conditional entropy $E(C|B)$ as a function of time $\omega_A t$, for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $E'(C|B)(t, N)$ (solid line) for the case of selective measurement as in Eq.(5.9) with $N = 0, 1, 2$; $\widetilde{E}(C|B)(t)$ (dotted line) for the case of 'no measurement' as in Eq.(4.7); $\widetilde{E}(C|B)(t)$ (dashed line and circle) for the case of non-selective measurement as in Eq.(5.16). The measurement takes place at $\omega_A t = 3$.

and

$$\delta_{AC} > \delta_{AB} \quad (5.84)$$

In Fig.(5.17), Fig.(5.18) and Fig.(5.19), we found the results

$$\widetilde{\delta}_{AB} = \delta'_{AB} = 0 \quad (5.85)$$

$$\widetilde{\delta}_{AC} = \delta'_{AC} = 0 \quad (5.86)$$

A summary of all the results, before and after the measurements are illustrated in Fig.(5.20), Fig.(5.21) and Fig.(5.22) where \longleftrightarrow indicate coupling, $\leftarrow\text{-----}\rightarrow$ indicate no direct coupling, E indicate entangled state, \check{E} indicate inconclusive state, $\delta > 0$ indicate non-classical correlations and $\delta = 0$ indicate classical correlations.

5.4 Discussion

Investigating quantum measurement in various components of a multipartite quantum system has been an interesting research, especially when it changes the state of the system. In this research, we investigated the quantum measurement of three oscillators A , B and C . We considered the strong coupling between the oscillators A, B and oscillators B, C . There are no direct coupling between oscillator A, C . We performed the photon number measurement on A at $\omega_A t = 3$ and calculated the entropy, quantum mutual information, conditional entropy and quantum discord before and after the measurements.

In the numerical calculation, it is shown that the entropy of oscillators

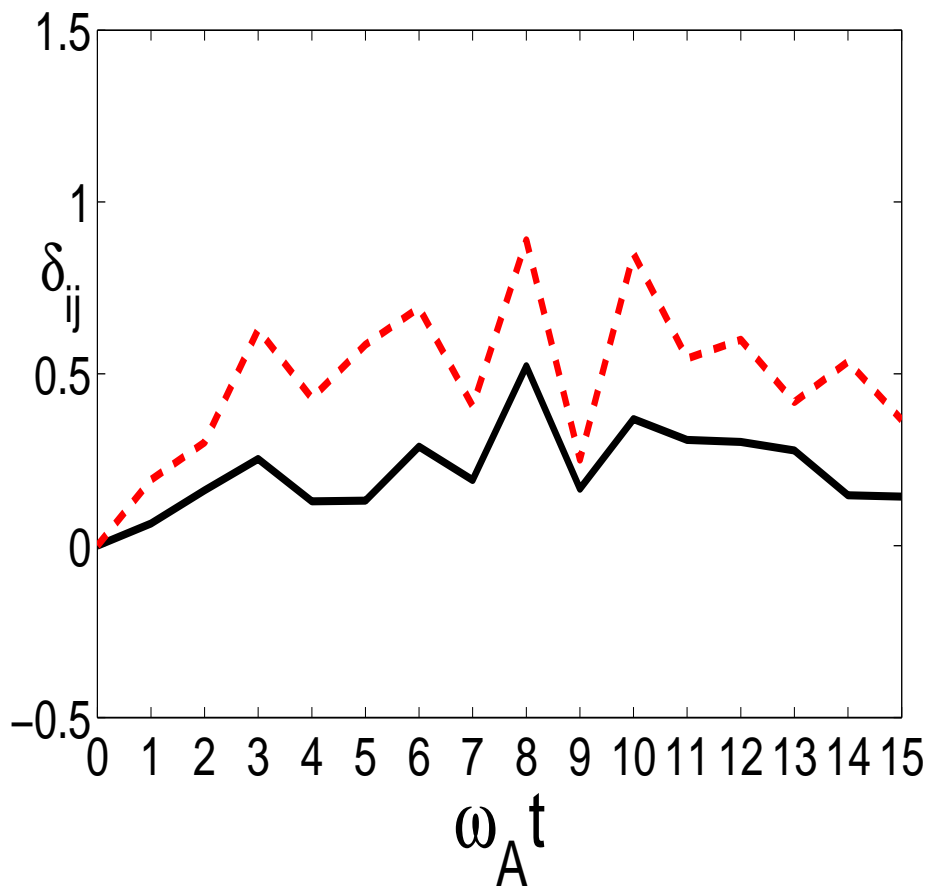


Figure 5.16: The quantum discord δ_{ij} as in Eq.(3.34) for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show δ_{AB} (solid line) as in Eq.(5.27) and δ_{AC} (dashed line) as in Eq.(5.29) as functions of time $\omega_A t$, for the case of ‘no measurement’. The fact that $\delta_{ij} > 0$ shows that the oscillators A, B and A, C are correlated quantum mechanically.

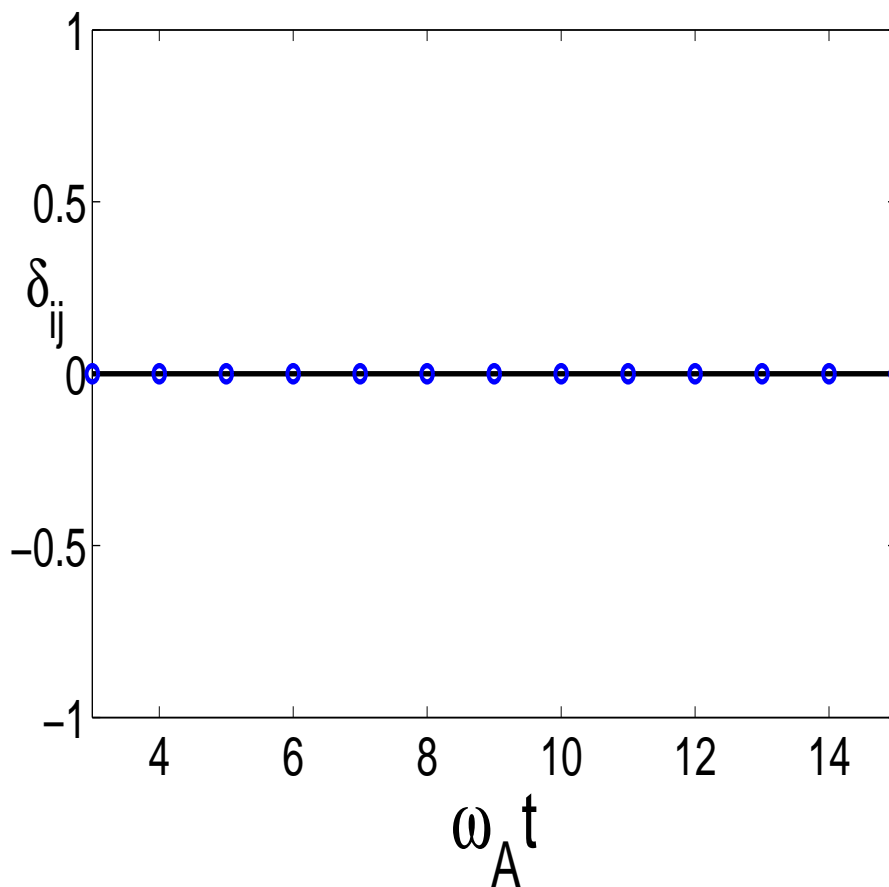


Figure 5.17: The quantum discord $\widetilde{\delta}_{ij}$ for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $\widetilde{\delta}_{AB}$ (solid line) as in Eq.(5.35) and $\widetilde{\delta}_{AC}$ (cycle) as in Eq.(5.29) as a function of time $\omega_A t$, for the case of non-selective measurements. The measurement takes place at $\omega_A t = 3$. The fact that $\widetilde{\delta}_{AB} = \widetilde{\delta}_{AC} = 0$, shows that the oscillators A, B and A, C are classically correlated. (Note that, the quantum discord $\widetilde{\delta}_{AB}$ and $\widetilde{\delta}_{AC}$ lines are superimposing each other.)

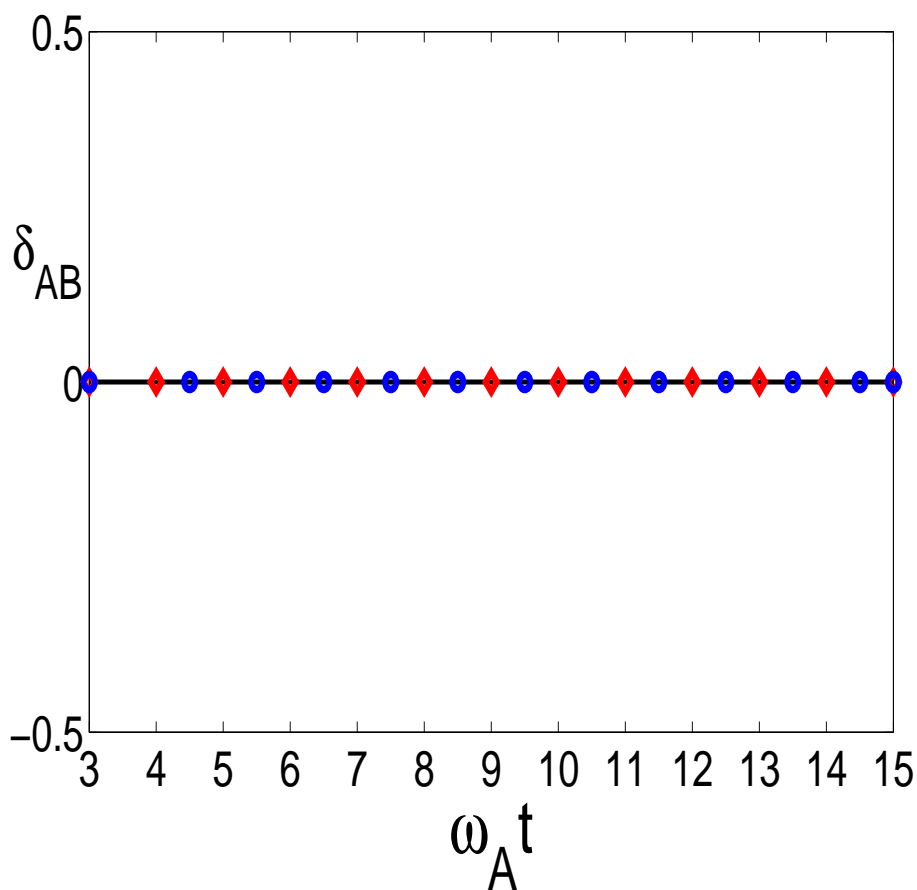


Figure 5.18: The quantum discord δ'_{AB} as in Eq.(5.47) for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $\delta'_{AB}(N = 0)$ (solid line); $\delta'_{AB}(N = 1)$ (circle) and $\delta'_{AB}(N = 2)$ (diamond) as a function of time $\omega_A t$, for the case of selective measurements. The measurement takes place at $\omega_A t = 3$. The fact that $\delta'_{AB}(N = 0, 1, 2) = 0$ shows that the oscillators A, B at $N = 0, 1, 2$ are classically correlated. (Note that, the quantum discord $\delta'_{AB}(N = 0)$, $\delta'_{AB}(N = 1)$ and $\delta'_{AB}(N = 2)$ lines are superimposing each other.)

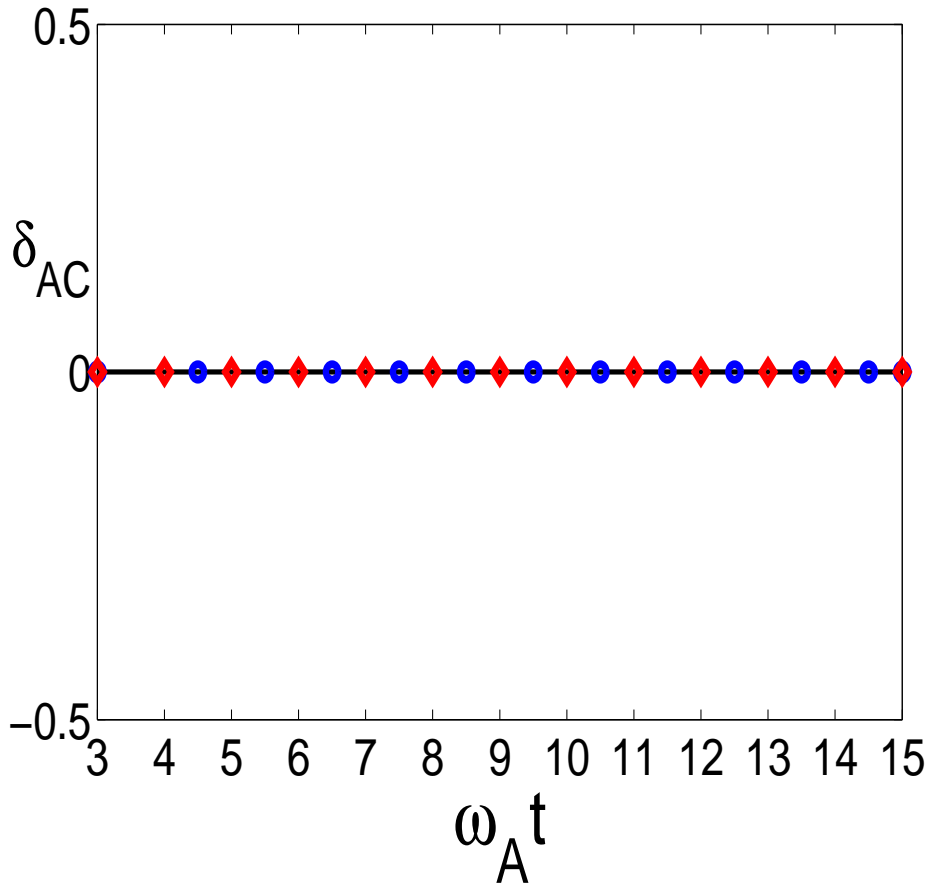


Figure 5.19: The quantum discord δ'_{AC} as in Eq.(5.53) for the Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9). We show $\delta'_{AC}(N=0)$ (solid line); $\delta'_{AC}(N=1)$ (circle) and $\delta'_{AC}(N=2)$ (diamond) as a function of time $\omega_A t$, for the case of selective measurements. The measurement takes place at $\omega_A t = 3$. The fact that $\delta'_{AC}(N=0, 1, 2) = 0$ shows that the oscillators A, C at $N=0, 1, 2$ are classically correlated. (Note that, the quantum discord $\delta'_{AC}(N=0)$, $\delta'_{AC}(N=1)$ and $\delta'_{AC}(N=2)$ lines are superimposing each other.)

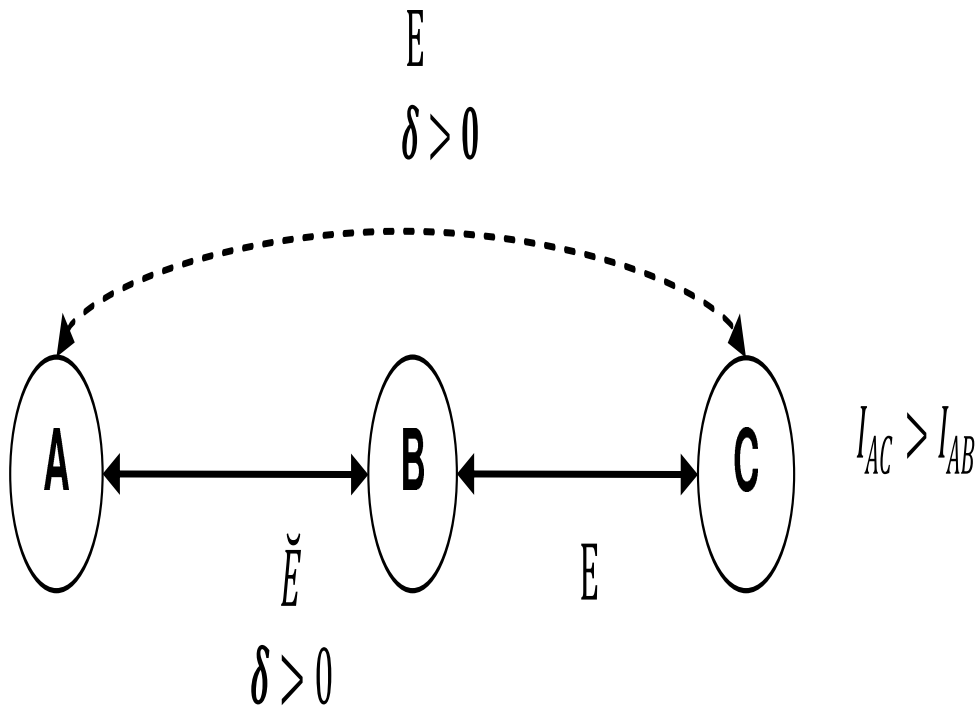


Figure 5.20: The summary of the case of ‘no measurement’ for a chain of three oscillators A, B, C . A solid arrow indicates coupling, dashed arrow indicates no direct coupling. E indicates entangled state, and \check{E} indicates that the entanglement witness does not lead to any conclusion. $\delta > 0$ indicates non-classical correlations. It is shown that after the evolution, oscillators A, C and B, C are entangled and oscillators A, B might or might not be entangled. Correlations between oscillators A, C are stronger than the correlations between the oscillators A, B . Quantum discord indicates that oscillators A, B and A, C are non-classically correlated.

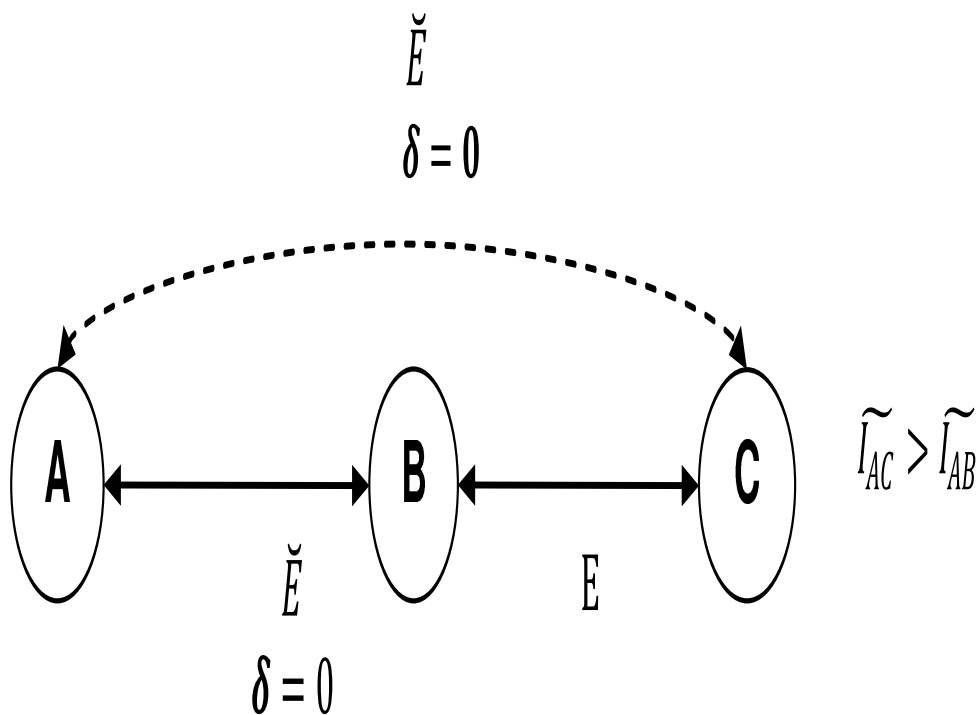


Figure 5.21: The summary of the case of non-selective measurement for a chain of three oscillators A, B, C . A solid arrow indicates coupling, dashed arrow indicates no direct coupling, E indicates entangled state, and \check{E} indicates that the entanglement witness does not lead to any conclusion. $\delta = 0$ indicates classically correlated system. It is shown that after the measurements, oscillators B, C are entangled and oscillators A, B and A, C might or might not be entangled. Correlations between oscillators A, C are stronger than the correlations between the oscillators A, B . Quantum discord indicates that oscillators A, B and A, C are classically correlated.

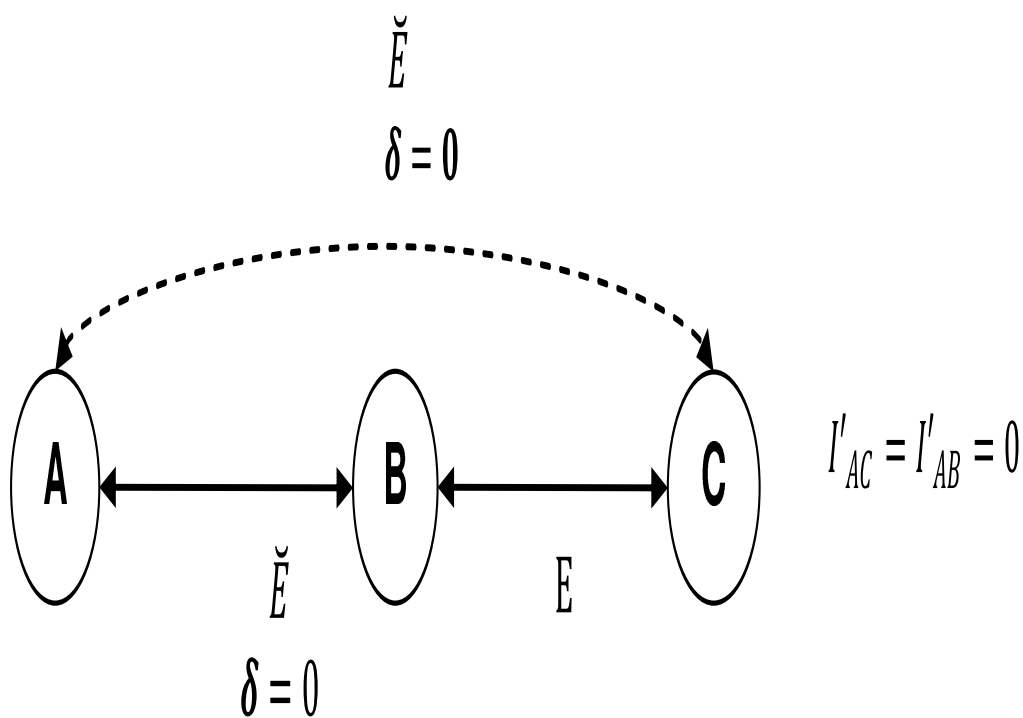


Figure 5.22: The summary of the case of selective measurement for a chain of three oscillators A, B, C . A solid arrow indicates coupling, dashed arrow indicates no direct coupling, E indicates entangled state, and \check{E} indicates that the entanglement witness does not lead to any conclusion. $\delta = 0$ indicates classically correlated system. It is shown that after the measurements, oscillators B, C are entangled and oscillators A, B and A, C might or might not be entangled. Quantum discord indicates that oscillators A, B and A, C are classically correlated.

A, B and C after the non-selective measurement is non-negative

$$S(\widetilde{\rho_i(t)}) \geq 0; \quad i = A, B, C \quad (5.87)$$

and

$$S(\widetilde{\rho_i(t)}) \geq S(\rho_i(t)); \quad i = A, B, C \quad (5.88)$$

This is an expected result because the theorem in section 3.4 mentions that, the entropy after the non-selective measurement is at least as great as the original entropy.

In the case of selective measurement, we found the oscillator A become a pure state for all outcomes $N = 0, 1, 2$ due to the measurement on A . In spite of that, the entropy of B and C decrease after the measurement for $N = 0, 1$ and an interestingly for the outcome $N = 2$, both entropy, B and C become zero.

We have also calculated the quantum mutual information I_{ij} for the case of after measurements. In many cases, the results prove that there exist classical and quantum correlation among all the oscillators A, B and C . Non-selective measurement proves that the measurement decrease the correlations of oscillators A, B and oscillators A, C (Refer to Fig.(5.7) and Fig.(5.8)). We also note that

$$\widetilde{I}_{AC} > \widetilde{I}_{AB} \quad (5.89)$$

Below we have some interesting results on selective measurement

$$I'_{AB}(N = 0, 1, 2) = I'_{AC}(N = 0, 1, 2) = 0 \quad (5.90)$$

Table 5.3: The entropy of oscillator B in the case of selective measurement, $S(\rho'_B(t, N))$ as Eq.(5.6) for $N = 0, 1, 2$ at $\omega_A t = 3, 4, \dots, 15$. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered.

	N=0	N=1	N=2
3	1.5586	1.0334	0
4	1.5477	0.3191	0
5	1.6087	0.7749	0
6	1.6461	1.0534	0
7	1.7275	1.1581	0
8	1.4429	1.2123	0
9	2.0764	1.2503	0
10	0.9705	1.1555	0
11	1.6086	1.4514	0
12	1.7900	1.1348	0
13	1.4048	1.3313	0
14	0.9076	0.8659	0
15	1.6411	1.3968	0

These results show that there is nothing common between oscillators A, B and oscillators A, C after the measurement. We can conclude that, after the selective measurement, oscillators A, B and oscillators A, C become factorizable. Meanwhile, for oscillators B, C , we found the results might decrease ($N = 0$) or increase ($N = 1$) after the measurement. This did not happen to an outcome $N = 2$, where $I'_{BC}(t, N = 2) = 0$.

Further, to investigate the results in Eq.(5.90), refer to Table (5.3), (5.4), (5.5) and (5.6), this shows that all the entropy is equal, $S(\rho'_B(t, N = 0, 1, 2)) = S(\rho'_C(t, N = 0, 1, 2)) = S(\rho'_{AB}(t, N = 0, 1, 2)) = S(\rho'_{AC}(t, N = 0, 1, 2))$.

As we know, to calculate the classical and quantum correlation as Eq.(5.8), therefore

$$I'_{AB} = S(\rho'_A) + S(\rho'_B) - S(\rho'_{AB}) \quad (5.91)$$

Table 5.4: The entropy of oscillator C in the case of selective measurement, $S(\rho'_C(t, N))$ as Eq.(5.6) for $N = 0, 1, 2$ at $\omega_A t = 3, 4, \dots, 15$. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered.

	N=0	N=1	N=2
3	1.5586	1.0334	0
4	1.5477	0.3191	0
5	1.6087	0.7749	0
6	1.6461	1.0534	0
7	1.7275	1.1581	0
8	1.4429	1.2123	0
9	2.0764	1.2503	0
10	0.9705	1.1555	0
11	1.6086	1.4514	0
12	1.7900	1.1348	0
13	1.4048	1.3313	0
14	0.9076	0.8659	0
15	1.6411	1.3968	0

Table 5.5: The entropy of oscillators A, B in the case of selective measurement, $S(\rho'_{AB}(t, N))$ as Eq.(5.7) for $N = 0, 1, 2$ at $\omega_A t = 3, 4, \dots, 15$. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered.

	N=0	N=1	N=2
3	1.5586	1.0334	0
4	1.5477	0.3191	0
5	1.6087	0.7749	0
6	1.6461	1.0534	0
7	1.7275	1.1581	0
8	1.4429	1.2123	0
9	2.0764	1.2503	0
10	0.9705	1.1555	0
11	1.6086	1.4514	0
12	1.7900	1.1348	0
13	1.4048	1.3313	0
14	0.9076	0.8659	0
15	1.6411	1.3968	0

Table 5.6: The entropy of oscillators A, C in the case of selective measurement, $S(\rho'_{AC}(t, N))$ as Eq.(5.7) for $N = 0, 1, 2$ at $\omega_A t = 3, 4, \dots, 15$. The Hamiltonian of Eq.(4.1) with the initial state $\rho_{ABC}(0)$ of Eq.(4.9) is considered.

	N=0	N=1	N=2
3	1.5586	1.0334	0
4	1.5477	0.3191	0
5	1.6087	0.7749	0
6	1.6461	1.0534	0
7	1.7275	1.1581	0
8	1.4429	1.2123	0
9	2.0764	1.2503	0
10	0.9705	1.1555	0
11	1.6086	1.4514	0
12	1.7900	1.1348	0
13	1.4048	1.3313	0
14	0.9076	0.8659	0
15	1.6411	1.3968	0

with $S(\rho'_A) = 0$ and $S(\rho'_B) = S(\rho'_{AB})$, thus $I'_{AB} = 0$.

It is also same for I'_{AC}

$$I'_{AC} = S(\rho'_A) + S(\rho'_C) - S(\rho'_{AC}) \quad (5.92)$$

with $S(\rho'_A) = 0$ and $S(\rho'_C) = S(\rho'_{AC})$, thus $I'_{AC} = 0$.

Next, we investigate factorizable of oscillators A, B and oscillators A, C . Based on the previous results in section 5.3.3, we know that oscillators B, C at $N = 0, 1$ is entangled, so as a control of calculation, we also test the factorizable of oscillators B, C . To identify the factorizable, we calculate the partial trace $\rho'_A(t, N = 0, 1, 2)$, $\rho'_B(t, N = 0, 1, 2)$ and $\rho'_{AB}(t, N = 0, 1, 2)$. Then we calculate the tensor product of $\rho'_A(t, N = 0, 1, 2)$ and $\rho'_B(t, N =$

Table 5.7: Factorizable state for the oscillators A, B and oscillators A, C in the case of selective measurement for $N = 0, 1, 2$ at $\omega_A t = 3, 4, \dots, 15$. 1 is indicated as a factorizable state and 0 as not a factorizable state.

	N=0	N=1	N=2
3	1	1	1
4	1	1	1
5	1	1	1
6	1	1	1
7	1	1	1
8	1	1	1
9	1	1	1
10	1	1	1
11	1	1	1
12	1	1	1
13	1	1	1
14	1	1	1
15	1	1	1

0, 1, 2). If

$$\rho'_A(t, N = 0, 1, 2) \otimes \rho'_B(t, N = 0, 1, 2) = \rho'_{AB}(t, N = 0, 1, 2) \quad (5.93)$$

therefore, we justify that the oscillators A, B is factorizable. Refer to Table (5.7) and Table (5.8) with 1 indicated as factorizable and 0 is not factorizable, it is shown that oscillators A, B and oscillators A, C are factorizable for $N = 0, 1, 2$. Meanwhile, oscillators B, C are not factorizable for $N = 0, 1$ but factorizable for $N = 2$.

We also prove that, based on conditional entropy calculation, after the measurements for the both cases (selective and non-selective), oscillators A, B and oscillators A, C are not entangled. Meanwhile, oscillators B, C maintain the entanglement. We note that for the outcome $N = 0$, oscillators B, C more

Table 5.8: Factorizable state for the oscillators B, C in the case of selective measurement for $N = 0, 1, 2$ at $\omega_A t = 3, 4, \dots, 15$. 1 is indicated as factorizable state and 0 as not a factorizable state.

	N=0	N=1	N=2
3	0	0	1
4	0	0	1
5	0	0	1
6	0	0	1
7	0	0	1
8	0	0	1
9	0	0	1
10	0	0	1
11	0	0	1
12	0	0	1
13	0	0	1
14	0	0	1
15	0	0	1

entangled after the selective measurement. But for the outcome $N = 2$, the oscillators B, C is not entangled.

In the case of selective measurement for $N = 2$, we found that $S(\rho'_B(t, N = 2)) = 0$ (Fig.(5.5)), $S(\rho'_C(t, N = 2)) = 0$ (Fig.(5.6)) and $I'_{BC}(t, N = 2) = 0$ (Fig.(5.9)). This show that oscillator B and oscillator C become pure state and are factorizable. Refer to Fig.(5.1), Fig.(5.2) and Fig.(5.3), they show that the probability of photon number $N = 2$ is very small, $0 < \text{prob}(N = 2) < 0.01$. We can conclude that, for $N = 2$ in selective measurement, oscillators B, C become pure and vacuum state.

We also support these results which calculate the quantum discord of oscillators A, B and oscillators A, C before the measurement and after the measurement. It is proven that before the measurement, δ_{AB} and δ_{AC} is not vanished. This means, there is a non-classical correlation between oscillators

A, B and oscillators A, C . It also can be seen that $\delta_{AB} < \delta_{AC}$ (Fig.(5.16)) which A, C is entangled and which A, B is not entangled [64].

After the measurements for both cases, non-selective and selective measurements

$$\widetilde{\delta}_{AB} = \widetilde{\delta}_{AC} = 0 \quad (5.94)$$

$$\delta'_{AB} = \delta'_{AC} = 0 \quad (5.95)$$

A state with vanishing quantum discord called classical quantum state is necessarily separable [17] and pure [64]. This is proven that after the measurement, oscillators A, B and oscillators A, C become a pure and classical quantum state.

5.5 Appendix

In this section, we list all the appendix for the detail of the approach or algorithms to calculate the quantum discord of oscillators A, B and A, C for all cases.

5.5.1 Appendix A: Quantum discord of oscillators A, B for the case of without measurements

This is an algorithm to compute quantum discord for oscillators A, B for the case of without measurements.

1. compute the the density matrix at time t

$$\rho_{ABC}(t) = e^{iHt} \rho_{ABC}(0) e^{-iHt}$$

where $\rho_{ABC}(0)$ is the initial state at $t = 0$.

2. compute the partial trace

$$\begin{aligned} \rho_{AB}(t) &= Tr_C(\rho_{ABC}(t)) \\ \rho_A(t) &= Tr_{BC}(\rho_{ABC}(t)) \end{aligned}$$

3. compute the entropy

$$\begin{aligned} S(\rho_{AB}(t)) &= -Tr[\rho_{AB}(t) \log(\rho_{AB}(t))] \\ S(\rho_A(t)) &= -Tr[\rho_A(t) \log(\rho_A(t))] \end{aligned}$$

4. compute the projection with measurement on A

$$\Pi_j^A = |j\rangle\langle j| \otimes \mathbf{1}_B$$

5. compute the probability

$$prob_j(t) = Tr(\Pi_j^A \rho_{AB}(t))$$

6. compute the state of B after the outcome corresponding to Π_j^A

$$\rho_{B|\Pi_j^A}(t) = \frac{\Pi_j^A \rho_{AB}(t) \Pi_j^A}{\text{prob}_j(t)}$$

7. compute the entropy, $S(\rho_{B|\Pi_j^A}(t))$

$$S(\rho_{B|\Pi_j^A}(t)) = -\text{Tr}[\rho_{B|\Pi_j^A}(t) \log(\rho_{B|\Pi_j^A}(t))]$$

8. compute the conditional entropy

$$E(B|\{\Pi_j^A\})(t) = \sum_j \text{prob}_j(t) S(\rho_{B|\Pi_j^A}(t))$$

9. compute quantum discord. We know:

$$I_{AB}(t) = S(\rho_B(t)) + S(\rho_A(t)) - S(\rho_{AB}(t))$$

$$J_{AB}(t) = S(\rho_B(t)) - E(B|\{\Pi_j^A\})(t)$$

Therefore:

$$\begin{aligned} \delta_{AB}(t) &= I_{AB}(t) - J_{AB}(t) \\ &= S(\rho_B(t)) + S(\rho_A(t)) - S(\rho_{AB}(t)) - S(\rho_B(t)) + E(B|\{\Pi_j^A\})(t) \\ &= S(\rho_A(t)) - S(\rho_{AB}(t)) + E(B|\{\Pi_j^A\})(t) \end{aligned}$$

5.5.2 Appendix B: Quantum discord of oscillators A, C for the case of without measurements

This is an algorithm to compute quantum discord for oscillators A, C for the case of without measurements.

1. compute the the density matrix at time t

$$\rho_{ABC}(t) = e^{iHt} \rho_{ABC}(0) e^{-iHt}$$

where $\rho_{ABC}(0)$ is the initial state at $t = 0$.

2. compute the partial trace

$$\begin{aligned} \rho_{AC}(t) &= Tr_B(\rho_{ABC}(t)) \\ \rho_A(t) &= Tr_{BC}(\rho_{ABC}(t)) \end{aligned}$$

3. compute the entropy

$$\begin{aligned} S(\rho_{AC}(t)) &= -Tr[\rho_{AC}(t) \log(\rho_{AC}(t))] \\ S(\rho_A(t)) &= -Tr[\rho_A(t) \log(\rho_A(t))] \end{aligned}$$

4. compute the projection with measurement on A

$$\Pi_j^A = |j\rangle\langle j| \otimes \mathbf{1}_C$$

5. compute the probability

$$prob_j(t) = Tr(\Pi_j^A \rho_{AC}(t))$$

6. compute the state of C after the outcome corresponding to Π_j^A

$$\rho_{C|\Pi_j^A}(t) = \frac{\Pi_j^A \rho_{AC}(t) \Pi_j^A}{prob_j(t)}$$

7. compute the entropy, $S(\rho_{C|\Pi_j^A}(t))$

$$S(\rho_{C|\Pi_j^A}(t)) = -Tr[\rho_{C|\Pi_j^A}(t) \log(\rho_{C|\Pi_j^A}(t))]$$

8. compute the conditional entropy

$$E(C|\{\Pi_j^A\})(t) = \sum_j prob_j(t) S(\rho_{C|\Pi_j^A}(t))$$

9. compute quantum discord. We know:

$$I_{AC}(t) = S(\rho_C(t)) + S(\rho_A(t)) - S(\rho_{AC}(t))$$

$$J_{AC}(t) = S(\rho_C(t)) - E(C|\{\Pi_j^A\})(t)$$

Therefore:

$$\begin{aligned} \delta_{AC}(t) &= I_{AC}(t) - J_{AC}(t) \\ &= S(\rho_C(t)) + S(\rho_A(t)) - S(\rho_{AC}(t)) - S(\rho_C(t)) + E(C|\{\Pi_j^A\})(t) \\ &= S(\rho_A(t)) - S(\rho_{AC}(t)) + E(C|\{\Pi_j^A\})(t) \end{aligned}$$

5.5.3 Appendix C: Quantum discord of oscillators A, B for the case of non-selective measurements

This is an algorithm to compute quantum discord for oscillators A, B for the case of non-selection measurements.

1. compute the the density matrix at time t

$$\rho_{ABC}(t) = e^{iHt} \rho_{ABC}(0) e^{-iHt}$$

2. measurement performed on A with projection operators

$$\Pi_N = \Pi_N^A \otimes \mathbf{1}_B \otimes \mathbf{1}_C \equiv |N\rangle\langle N| \otimes \mathbf{1}_B \otimes \mathbf{1}_C$$

3. compute the state after the non-selective measurements

$$\widetilde{\rho_{ABC}(t)} = \sum_{i=0}^{d-1} \Pi_i \rho_{ABC}(t) \Pi_i$$

where $\rho_{ABC}(t)$ is a state after the evolution.

4. compute the partial trace

$$\begin{aligned} \widetilde{\rho_{AB}(t)} &= Tr_C(\widetilde{\rho_{ABC}(t)}) \\ \widetilde{\rho_A(t)} &= Tr_{BC}(\widetilde{\rho_{ABC}(t)}) \end{aligned}$$

5. compute the entropy $S(\widetilde{\rho}_A(t))$ and $S(\widetilde{\rho}_{AB}(t))$

$$\begin{aligned} S(\widetilde{\rho}_{AB}(t)) &= -Tr[\widetilde{\rho}_{AB}(t) \log(\widetilde{\rho}_{AB}(t))] \\ S(\widetilde{\rho}_A(t)) &= -Tr[\widetilde{\rho}_A(t) \log(\widetilde{\rho}_A(t))] \end{aligned}$$

6. compute the projection with measurement on A

$$\Pi_j^A = |j\rangle\langle j| \otimes \mathbf{1}_B$$

7. compute the probability

$$prob_j(t) = Tr(\Pi_j^A \widetilde{\rho}_{AB}(t))$$

8. compute the state of B after the outcome corresponding to Π_j^A

$$\rho_{B|\Pi_j^A}(t) = \frac{\Pi_j^A \widetilde{\rho}_{AB}(t) \Pi_j^A}{prob_j(t)}$$

9. compute the entropy, $S(\rho_{B|\Pi_j^A}(t))$

$$S(\rho_{B|\Pi_j^A}(t)) = -Tr[\rho_{B|\Pi_j^A}(t) \log(\rho_{B|\Pi_j^A}(t))]$$

10. compute the conditional entropy

$$E(B|\{\Pi_j^A\})(t) = \sum_j prob_j(t) S(\rho_{B|\Pi_j^A}(t))$$

11. compute quantum discord. We know:

$$\begin{aligned}\widetilde{I}_{AB}(t) &= S(\widetilde{\rho}_A(t)) + S(\widetilde{\rho}_B(t)) - S(\widetilde{\rho}_{AB}(t)) \\ \widetilde{J}_{AB}(t) &= S(\widetilde{\rho}_B(t)) - E(B|\{\Pi_j^A\})(t)\end{aligned}$$

Therefore:

$$\begin{aligned}\widetilde{\delta}_{AB}(t) &= \widetilde{I}_{AB}(t) - \widetilde{J}_{AB}(t) \\ &= S(\widetilde{\rho}_A(t)) + S(\widetilde{\rho}_B(t)) - S(\widetilde{\rho}_{AB}(t)) - S(\widetilde{\rho}_B(t)) + E(B|\{\Pi_j^A\})(t) \\ &= S(\widetilde{\rho}_A(t)) - S(\widetilde{\rho}_{AB}(t)) + E(B|\{\Pi_j^A\})(t)\end{aligned}$$

5.5.4 Appendix D: Quantum discord of oscillators A, C for the case of non-selective measurements

This is an algorithm to compute quantum discord for oscillators A, C for the case of non-selective measurements.

1. compute the the density matrix at time t

$$\rho_{ABC}(t) = e^{iHt} \rho_{ABC}(0) e^{-iHt}$$

2. measurement performed on A with projection operators

$$\Pi_N = \Pi_N^A \otimes \mathbf{1}_B \otimes \mathbf{1}_C \equiv |N\rangle\langle N| \otimes \mathbf{1}_B \otimes \mathbf{1}_C$$

3. compute the state after the non-selective measurements

$$\widetilde{\rho_{ABC}(t)} = \sum_{i=0}^{d-1} \Pi_i \rho_{ABC}(t) \Pi_i$$

where $\rho_{ABC}(t)$ is a state after the evolution.

4. compute the partial trace

$$\begin{aligned} \widetilde{\rho_{AC}(t)} &= Tr_B(\widetilde{\rho_{ABC}(t)}) \\ \widetilde{\rho_A(t)} &= Tr_{BC}(\widetilde{\rho_{ABC}(t)}) \end{aligned}$$

5. compute the entropy $S(\widetilde{\rho_A(t)})$ and $S(\widetilde{\rho_{AC}(t)})$

$$\begin{aligned} S(\widetilde{\rho_{AC}(t)}) &= -Tr[\widetilde{\rho_{AC}(t)} \log(\widetilde{\rho_{AC}(t)})] \\ S(\widetilde{\rho_A(t)}) &= -Tr[\widetilde{\rho_A(t)} \log(\widetilde{\rho_A(t)})] \end{aligned}$$

6. compute the projection with measurement on A

$$\Pi_j^A = |j\rangle\langle j| \otimes \mathbf{1}_C$$

7. compute the probability

$$prob_j(t) = Tr(\Pi_j^A \widetilde{\rho_{AC}(t)})$$

8. compute the state of C after the outcome corresponding to Π_j^A

$$\widetilde{\rho_{C|\Pi_j^A}(t)} = \frac{\Pi_j^A \widetilde{\rho_{AC}(t)} \Pi_j^A}{\text{prob}_j(t)}$$

9. compute the entropy, $S(\widetilde{\rho_{C|\Pi_j^A}(t)})$

$$S(\widetilde{\rho_{C|\Pi_j^A}(t)}) = -\text{Tr}[\widetilde{\rho_{C|\Pi_j^A}(t)} \log(\widetilde{\rho_{C|\Pi_j^A}(t)})]$$

10. compute the conditional entropy

$$E(C|\{\Pi_j^A\})(t) = \sum_j \text{prob}_j(t) S(\widetilde{\rho_{C|\Pi_j^A}(t)})$$

11. compute quantum discord. We know:

$$\begin{aligned} \widetilde{I_{AC}(t)} &= S(\widetilde{\rho_A(t)}) + S(\widetilde{\rho_C(t)}) - S(\widetilde{\rho_{AC}(t)}) \\ \widetilde{J_{AC}(t)} &= S(\widetilde{\rho_C(t)}) - E(C|\{\Pi_j^A\})(t) \end{aligned}$$

Therefore:

$$\begin{aligned} \widetilde{\delta_{AC}(t)} &= \widetilde{I_{AC}(t)} - \widetilde{J_{AC}(t)} \\ &= S(\widetilde{\rho_A(t)}) + S(\widetilde{\rho_C(t)}) - S(\widetilde{\rho_{AC}(t)}) - S(\widetilde{\rho_C(t)}) + E(C|\{\Pi_j^A\})(t) \\ &= S(\widetilde{\rho_A(t)}) - S(\widetilde{\rho_{AC}(t)}) + E(C|\{\Pi_j^A\})(t) \end{aligned}$$

5.5.5 Appendix E: Quantum discord of oscillators A, B for the case of selective measurements

This is an algorithm to compute quantum discord for oscillators A, B for the case of selective measurements.

1. compute the the density matrix at time t

$$\rho_{ABC}(t) = e^{iHt} \rho_{ABC}(0) e^{-iHt}$$

2. measurement performed on A with projection operators

$$\Pi_N = \Pi_N^A \otimes \mathbf{1}_B \otimes \mathbf{1}_C \equiv |N\rangle\langle N| \otimes \mathbf{1}_B \otimes \mathbf{1}_C$$

3. compute the state after the selective measurements

$$\rho'_{ABC}(t, N) = \frac{\Pi_N \rho_{ABC}(t) \Pi_N}{\text{Tr}[\rho_{ABC}(t) \Pi_N]}$$

where $\rho_{ABC}(t)$ is a state after the evolution and N is the number of photons.

4. compute the partial trace

$$\begin{aligned} \rho'_{AB}(t, N) &= \text{Tr}_C(\rho'_{ABC}(t, N)) \\ \rho'_A(t, N) &= \text{Tr}_{BC}(\rho'_{ABC}(t, N)) \end{aligned}$$

5. compute the entropy $S(\rho'_A(t, N))$ and $S(\rho'_{AB}(t, N))$

$$S(\rho'_{AB}(t, N)) = -Tr[\rho'_{AB}(t, N) \log(\rho'_{AB}(t, N))]$$

$$S(\rho'_A(t, N)) = -Tr[\rho'_A(t, N) \log(\rho'_A(t, N))]$$

6. compute the projection with measurement on A

$$\Pi_N^A = |N\rangle\langle N| \otimes \mathbf{1}_B$$

7. compute the probability

$$prob(t, N) = Tr(\Pi_N^A \rho'_{AB}(t, N))$$

8. compute the state of B after the outcome corresponding to Π_N^A

$$\rho'_{B|\Pi_N^A}(t, N) = \frac{\Pi_N^A \rho'_{AB}(t, N) \Pi_N^A}{prob(t, N)}$$

9. compute the entropy, $S(\rho'_{B|\Pi_N^A}(t, N))$

$$S(\rho'_{B|\Pi_N^A}(t, N)) = -Tr[\rho'_{B|\Pi_N^A}(t, N) \log(\rho'_{B|\Pi_N^A}(t, N))]$$

10. compute the conditional entropy

$$E'(B|\{\Pi_N^A\})(t, N) = prob(t, N) S(\rho'_{B|\Pi_N^A}(t, N))$$

11. compute quantum discord. We know:

$$\begin{aligned} I'_{AB}(t, N) &= S(\rho'_A(t, N)) + S(\rho'_B(t, N)) - S(\rho'_{AB}(t, N)) \\ J'_{AB}(t, N) &= S(\rho'_B(t, N)) - E'(B|\{\Pi_N^A\})(t, N) \end{aligned}$$

Therefore:

$$\begin{aligned} \delta'_{AB}(t, N) &= I'_{AB}(t, N) - J'_{AB}(t, N) \\ &= S(\rho'_A(t, N)) - S(\rho'_{AB}(t, N)) + E'(B|\{\Pi_N^A\})(t, N) \end{aligned}$$

5.5.6 Appendix F: Quantum discord of oscillators A, C for the case of selective measurements

This is an algorithm to compute quantum discord for oscillators A, C for the case of selective measurements.

1. compute the the density matrix at time t

$$\rho_{ABC}(t) = e^{iHt} \rho_{ABC}(0) e^{-iHt}$$

2. measurement performed on A with projection operators

$$\Pi_N = \Pi_N^A \otimes \mathbf{1}_B \otimes \mathbf{1}_C \equiv |N\rangle\langle N| \otimes \mathbf{1}_B \otimes \mathbf{1}_C$$

3. compute the state after the selective measurements

$$\rho'_{ABC}(t, N) = \frac{\Pi_N \rho_{ABC}(t) \Pi_N}{Tr[\rho_{ABC}(t) \Pi_N]}$$

where $\rho_{ABC}(t)$ is a state after the evolution and N is the number of photons.

4. compute the partial trace

$$\begin{aligned}\rho'_{AC}(t, N) &= Tr_B(\rho'_{ABC}(t, N)) \\ \rho'_A(t, N) &= Tr_{BC}(\rho'_{ABC}(t, N))\end{aligned}$$

5. compute the entropy $S(\rho'_A(t, N))$ and $S(\rho'_{AC}(t, N))$

$$\begin{aligned}S(\rho'_{AC}(t, N)) &= -Tr[\rho'_{AC}(t, N) \log(\rho'_{AC}(t, N))] \\ S(\rho'_A(t, N)) &= -Tr[\rho'_A(t, N) \log(\rho'_A(t, N))]\end{aligned}$$

6. compute the projection with measurement on A

$$\Pi_N^A = |N\rangle\langle N| \otimes \mathbf{1}_C$$

7. compute the probability

$$prob(t, N) = Tr(\Pi_N^A \rho'_{AC}(t, N))$$

8. compute the state of C after the outcome corresponding to Π_N^A

$$\rho'_{C|\Pi_N^A}(t, N) = \frac{\Pi_N^A \rho'_{AC}(t, N) \Pi_N^A}{\text{prob}(t, N)}$$

9. compute the entropy, $S(\rho'_{C|\Pi_N^A}(t, N))$

$$S(\rho'_{C|\Pi_N^A}(t, N)) = -\text{Tr}[\rho'_{C|\Pi_N^A}(t, N) \log(\rho'_{C|\Pi_N^A}(t, N))]$$

10. compute the conditional entropy

$$E'(C|\{\Pi_N^A\})(t, N) = \text{prob}(t, N) S(\rho'_{C|\Pi_N^A}(t, N))$$

11. compute quantum discord. We know:

$$I'_{AC}(t, N) = S(\rho'_A(t, N)) + S(\rho'_C(t, N)) - S(\rho'_{AC}(t, N))$$

$$J'_{AC}(t, N) = S(\rho'_B(t, N)) - E'(C|\{\Pi_N^A\})(t, N)$$

Therefore:

$$\begin{aligned} \delta'_{AC}(t, N) &= I'_{AC}(t, N) - J'_{AC}(t, N) \\ &= S(\rho'_A(t, N)) - S(\rho'_{AC}(t, N)) + E'(C|\{\Pi_N^A\})(t, N) \end{aligned}$$

Chapter 6

Conclusions

6.1 Conclusions

In the present work, we have studied a chain of three oscillators A, B, C . We assume that oscillators A, B and oscillators B, C are coupled and there is no direct coupling between oscillator A and C . The goal of this study is to investigate the correlation among all the oscillators after the evolution. In particular, we looked for counter-intuitive results.

We introduce three cases, which we denote as case 1, case 2 and case 3. In case 3, we assume that the coupling is not resonant and that there is weak coupling between both oscillators A, B and oscillators B, C . In case 2, we assume strong coupling between oscillators B, C and weak coupling between oscillators A, B . Lastly, in case 1, we consider strong coupling between oscillators A, B and also between oscillators B, C .

In the numerical calculation, we used the quantum mutual information, I_{ij} to calculate the existence of classical and quantum correlations. Then, we

used conditional entropy, $E(i|j)$ as a witness of entanglement if the value is negative. We also confirm the entanglement using the negativity, $N(\rho_{ij})$.

Results from this investigation are very interesting. In case 3, we have shown that $I_{AB} > I_{AC}$ which means that oscillators A, B are strongly correlated but oscillators A, C are weakly correlated. We have also shown that the oscillators A, B are entangled, but we cannot conclude for oscillators A, C whether they are entangled or not. This is expected result, since there is no direct coupling between oscillators A, C . In cases 1 and 2, we found two counter-intuitive results. The first is $I_{AB} < I_{AC}$ i.e. the oscillators A, B which are directly coupled are weakly correlated, and the oscillators A, C which are indirectly coupled are strongly correlated. Moreover, oscillators A, B might or might not be entangled. In contrast, oscillators A, C are entangled.

In the second investigation, we performed a photon number measurement in the same tri-partite quantum system with the focus on case 1. We performed two types of von Neumann measurements called selective and non-selective measurements. In the first case, a selective measurement is performed by A and the result is communicated instantaneously with classical methods to oscillators B and C . In the second case, a non-selective measurement is performed by A and the fact that a measurement has been made with the projectors Π_N , is communicated to oscillators B, C . In this case the exact result is not known. In order to show the different effects before and after the measurement, we also consider a case of without measurement. The objective of this study is to investigate the effects of measurement on the correlations and entanglement of the quantum system.

In the non-selective measurements there exist classical and quantum cor-

relations between all oscillators A, B and C . The use of an entanglement witness shows that oscillators B, C are entangled, and it is inconclusive for the entanglement between the oscillators A, B and A, C . The use of the quantum discord gives zero results for the oscillators A, B and A, C and indicates that oscillators A, B and A, C are classically correlated. We can conclude that after the non-selective measurement, oscillators A, B and A, C become classically correlated and oscillators B, C still remain entangled.

In the case of selective measurement, for the examples considered, only the outcomes $N = 0, 1, 2$ of the photon number measurements on the oscillator A are possible. Based on quantum mutual information, it is shown that the density matrices ρ_{AB}, ρ_{AC} are factorizable, but ρ_{BC} is correlated. Conditional entropies show that oscillators A, B may or may not be entangled ($E'(A|B) \geq 0, E'(B|A) \geq 0$) for $N = 0, 1, 2$ and the same is true for A, C ($E'(A|C) \geq 0, E'(C|A) \geq 0$). Oscillators B, C are entangled ($E'(B|C) < 0, E'(C|B) < 0$) for $N = 0, 1$. We note that for the case of $N = 2$, the oscillators B, C are in a pure and vacuum state correspondingly. Furthermore, by using quantum discord, we found that oscillators A, B and A, C are classically correlated for $N = 0, 1, 2$. As a result, we can conclude that after the selective measurement, oscillators A, B and A, C become classical correlated and oscillators B, C are entangled for $N = 0, 1$ and pure and vacuum state for $N = 2$.

6.2 Further Work

In this thesis, we consider three harmonic oscillators A, B, C with non-linear couplings $a_i^m a_j^{\dagger n}$ and examples with a small average number of photons. We also performed a photon number measurement on one harmonic oscillator only i.e. A .

The work could be extended to different types of Hamiltonians with all types of couplings and large average number of photons. It is interesting to investigate these new Hamiltonians and to find whether they will give similar counter-intuitive results. It is also interesting to investigate longer chains [5, 15, 63, 82]. Phenomena related to two measurements simultaneously, for example, on B and C or A and B are also interesting.

Moreover, it is interesting to investigate the phenomena of sudden death and sudden birth of entanglement [23, 50, 84] in our case. Although it is different from the scheme that we considered, it is interesting to study sudden death and sudden birth of entanglement in the present context. It is also possible that if we allow the coupling constants to be turned on and off, we find interesting phenomena.

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