The Computation of Schur Multiplier and Capability of Pairs of Groups of Order p^4

Adnin Afifi Nawi,[1, a)] Nor Muhainiah Mohd Ali,[2, b)] and Nor Haniza Sarmin[2, c)]

[1)] *Department of Science and Mathematics, Centre for Diploma Studies, Universiti Tun Hussein Onn Malaysia, Hab Pendidikan Tinggi Pagoh, KM1, Jalan Panchor, 84600 Pagoh, Muar, Johor, Malaysia*
[2)] *Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia*

[a)] *Corresponding author: adnin@uthm.edu.my*
[b)] *Electronic mail: normuhainiah@utm.my*
[c)] *Electronic mail: nhs@utm.my*

Abstract. The homological functors of a group have its foundation in homotopy theory and algebraic K-theory. The Schur multiplier of a group is one of the homological functors while the Schur multiplier of pairs of groups is a continuation of the Schur multiplier of a group. Meanwhile, a pair of groups is capable if the precise center or epicenter of the pair of group is trivial. In this research, the Schur multiplier and capability of pairs of all abelian groups of order p^4 are computed.

INTRODUCTION

The Schur multiplier of a group G, denoted by $M(G)$, was first initiated by Schur [1] while studying projective representations of groups in 1904. Since then, many researchers intrigued to study on the Schur multiplier of a group. In 1998, Ellis [2] extended the notion of the Schur multiplier of groups to the Schur multiplier of a pair of groups.

The study of capability of groups was pioneered by Baer [3] in 1938 who determined all capable abelian groups. In 1996, Ellis [4] broadened the theory of the capability of groups to the theory of a pair of groups. He defined the capability of a pair of groups in terms of Loday's notion of a relative central extension.

Furthermore, Groups, Algorithms and Programming (GAP) software [5] is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. In this research, we use GAP software to compute the capable pairs of abelian groups of order p^4.

In our previous research, the Schur multiplier of pairs of groups of order p^2q have been computed in [6]. In this research, the classification of abelian groups of order p^4 in Theorem 1 was used to determine the Schur multiplier and capability of pairs of abelian groups of order p^4.

Theorem 1 **[7]** *Let G be an abelian group of order p^4 where p is an odd prime. Then exactly one of the following holds:*

$$G \cong \mathbb{Z}_{p^4}; \tag{1}$$

$$G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p; \tag{2}$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}; \tag{3}$$

$$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p; \tag{4}$$

$$G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p. \tag{5}$$

PRELIMINARIES

In this section some fundamental concepts and preparatory results that are are needed and useful to determine the Schur multiplier and capability of a pair of groups are presented.

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Corollary 2 [8] *Let $\phi : G \to H$ be a homomorphism of groups.*

(i) *ϕ is injective if and only if $\ker\phi = 1$.*

(ii) *$|G : \ker\phi| = |\phi(G)|$.*

Proposition 3 [9] *If $K \lhd G$ then any complementary subgroup K of G is isomorphic to $Q = {}^{G}\!/_{K}$.*

Theorem 4 [10] *Let $G \cong \mathbb{Z}_m$ and $H \cong \mathbb{Z}_n$ be cyclic groups that act trivially on each other. Then $G \otimes H \cong \mathbb{Z}_{(m,n)}$ where (m,n) is the greatest common divisor of m and n.*

Theorem 5 [11] *Let A, B and C be groups with given actions of A on B and C, and of B and C on A. Suppose that the latter actions*

(i) *commute: ${}^{bc}a = {}^{cb}a$, so that $B \times C$ acts on A,*

(ii) *induce the trivial action of B on $A \otimes C$: ${}^{b}(a \otimes c) = a \otimes c$, and*

(iii) *induce the trivial action of C on $A \otimes B$: ${}^{c}(a \otimes b) = a \otimes b$,*

for all $a \in A$, $b \in B$, $c \in C$. Then

$$A \otimes (B \times C) \cong (A \otimes B) \times (A \otimes C).$$

Theorem 6 [12] *Let G be a finite group and let $G \cong {}^{F}\!/_{R}$ where F is a free group of rank n. Then*

(i) *$M(G) \cong {}^{(R \cap [F,F])}\!/_{[R,F]}$,*

(ii) *${}^{(R \cap [F,F])}\!/_{[R,F]}$ is the torsion subgroup of ${}^{R}\!/_{[R,F]}$ and torsion-free factor ${}^{R}\!/_{([F,F] \cap R)}$ is free abelian of rank n. In particular, ${}^{R}\!/_{[R,F]}$ is a finitely generated abelian group.*

Theorem 7 [12] *Let G be a finite group then*

(i) *$M(G)$ is a finite group, whose elements have order dividing the order of G,*

(ii) *$M(G) = 1$ if G is cyclic.*

Theorem 8 [12] *Let G_1 and G_2 be finite groups. Then*

$$M(G_1 \times G_2) \cong M(G_1) \times M(G_2) \times (G_1 \otimes G_2).$$

Theorem 9 [13] *Let G be a group of order p^2 where p is a prime number. Then exactly one of the following holds:*

$$M(G) = \begin{cases} 1 & ; \quad \text{if } G \text{ is } \mathbb{Z}_{p^2}, \\ \mathbb{Z}_p & ; \quad \text{if } G \text{ is } \mathbb{Z}_p \times \mathbb{Z}_p. \end{cases} \tag{6}$$

Theorem 10 [14] *Let G be an abelian group of order p^4 where p is an odd prime. Then exactly one of the following holds:*

$$M(G) \cong \begin{cases} 1 & ; \quad \text{if } G \text{ is of Type (1)}, \\ \mathbb{Z}_p & ; \quad \text{if } G \text{ is of Type (2)}, \\ \mathbb{Z}_{p^2} & ; \quad \text{if } G \text{ is of Type (3)}, \\ (\mathbb{Z}_p)^3 & ; \quad \text{if } G \text{ is of Type (4)}, \\ (\mathbb{Z}_p)^6 & ; \quad \text{if } G \text{ is of Type (5)}. \end{cases} \tag{7}$$

Definition 11 [2] The Schur Multiplier of a Pair of Groups
The Schur multiplier of a pair of groups is said to be a functorial abelian groups and is denoted as $M(G,N)$ whose principal feature is the following natural exact sequence

$$H_3(G) \xrightarrow{\eta} H_3\left({}^{G}/_{N}\right) \rightarrow M(G,N) \rightarrow M(G) \xrightarrow{\mu} M\left({}^{G}/_{N}\right) \rightarrow {}^{N}/_{[N,G]} \rightarrow (G)^{ab} \xrightarrow{\alpha} \left({}^{G}/_{N}\right)^{ab} \rightarrow 1 \tag{8}$$

in which $H_3(-)$ denotes some finiteness-preserving functor from groups to abelian groups (to be precise, $H_3(-)$ is the third homology of a group with integer coefficients). The homomorphisms η, μ, α are those due to the functorial of $H_3(-)$, $M(-)$ and $(-)^{ab}$.

From the natural exact sequence in Definition 11, we obtain the following proposition.

Proposition 12 *If $M\left({}^{G}/_{N}\right)$ is trivial then $M(G,N) \cong M(G)$.*
Proof If $M\left({}^{G}/_{N}\right)$ is trivial then the exact sequence $1 \rightarrow M(G,N) \xrightarrow{\sigma} M(G) \xrightarrow{\mu} M\left({}^{G}/_{N}\right) = 1$ shows that ${}^{M(G,N)}/_{\ker(\sigma)} \cong M(G)$. Since σ is injective then by Corollary 2, $\ker(\sigma) = 1$. Therefore, $M(G,N) \cong M(G)$.

Proposition 13 *If N is normal in a group G, then $M(G,N)$ is a subgroup of $M(G)$.*
Proof From Theorem 6 (i), $M(G) \cong {}^{(R\cap[F,F])}/_{[R,F]}$ if $G \cong {}^{F}/_{R}$ where F is a free group of rank n. Following Moghaddam *et al.* [15], $M(G,N) \cong {}^{(R\cap[S,F])}/_{[R,F]}$ if S is a normal subgroup of F such that $N \cong {}^{S}/_{R}$. Since S is a normal subgroup of F, then clearly $M(G,N)$ is a subgroup of $M(G)$.

Theorem 14 [2] *Let $N = 1$, then $M(G,1) = 1$.*

Theorem 15 [2] *Let $N = G$, then $M(G,G) = M(G)$.*

Theorem 16 [2] *Let (G,N) be an arbitrary pair of finite groups.*

(i) *Then $M(G,N)$ is a finite abelian group with exponent of the Schur multiplier of pairs of group,* $\exp(M(G,N))$, *dividing the order of G.*

(ii) *Also,* $\exp(M(G,N))$ *divides the order of N.*

Theorem 17 [16] *Let (G,N) be a pair of groups and K be the complement of N in G. Then*

$$|M(G,N)| = |M(N)||N^{ab} \otimes K^{ab}|.$$

Theorem 18 [17] *Every nontrivial cyclic group is not capable.*

Definition 19 [4]
Given a normal subgroup N in G, a relative central extension of the pair (G,N) consists of a group homomorphism $d : M \rightarrow G$ and action $(g,m) \rightarrow {}^{g}m$ of G on M satisfying:

(i) *$d({}^{g}m) = gd(m)g^{-1}$ for g in G and m in M;*

(ii) *$mm'm^{-1} = {}^{d(m)}m$ for m and m' in M;*

(iii) *$N = Image(d)$;*

(iv) *the action of G on M is such that G acts trivially on the kernel of d.*

The pair of groups (G,N) is said to be capable if it admits a relative central extension with the property that $Ker(d)$ consists precisely of those elements in M on which G acts trivially.

Definition 20 [4]
Let N be an arbitrary normal subgroup of G. The exterior G-center of N, $Z_G^{\wedge}(N)$ is defined to be

$$Z_G^{\wedge}(N) = \{n \in N : 1 = g \wedge n \in G \wedge N \text{ for all } g \in G\}.$$

Theorem 21 [4] *The pair of group, (G,N) is capable if and only if the exterior G-center of N, $Z_G^{\wedge}(N)$ is trivial.*

THE SCHUR MULTIPLIER OF PAIRS OF ABELIAN GROUPS OF ORDER p^4

In this section, the computation of the Schur multiplier of pairs of all abelian groups of order p^4 are discussed. The following theorem gives the Schur multiplier of pairs of these groups.

Theorem 22 *Let G be an abelian group of order p^4 where p is an odd prime and (G,N) be an arbitrary pair of finite groups where N is a normal subgroup of G. Then exactly one of the following holds:*

$$M(G,N) \cong \begin{cases} 1 & ; \textit{ if } G \textit{ is of Type (1) or} \\ & \ G \textit{ is of Type (2) to Type (5) when } N \cong 1, \\ \mathbb{Z}_p & ; \textit{ if } G \textit{ is of Type (2) when } N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \mathbb{Z}_{p^3}, \mathbb{Z}_p \textit{ or } G, \\ \mathbb{Z}_{p^2} & ; \textit{ if } G \textit{ is of Type (3) when } N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \mathbb{Z}_{p^2} \textit{ or } G, \\ (\mathbb{Z}_p)^2 & ; \textit{ if } G \textit{ is of Type (2) when } N \cong \mathbb{Z}_{p^2}, \\ & \ G \textit{ is of Type (3) when } N \cong \mathbb{Z}_p, \\ & \ G \textit{ is of Type (4) when } N \cong \mathbb{Z}_{p^2} \textit{ or } \mathbb{Z}_p, \\ (\mathbb{Z}_p)^3 & ; \textit{ if } G \textit{ is of Type (2) when } N \cong (\mathbb{Z}_p)^2, \\ & \ G \textit{ is of Type (4) when } N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p, (\mathbb{Z}_p)^3, (\mathbb{Z}_p)^2 \textit{ or } G, \\ (\mathbb{Z}_p)^5 & ; \textit{ if } G \textit{ is of Type (3) when } N \cong (\mathbb{Z}_p)^2, \\ & \ G \textit{ is of Type (5) when } N \cong (\mathbb{Z}_p)^2, \\ (\mathbb{Z}_p)^6 & ; \textit{ if } G \textit{ is of Type (5) when } N \cong (\mathbb{Z}_p)^3 \textit{ or } G. \end{cases} \tag{9}$$

Proof Let G be an abelian group of order p^4 where p is an odd prime. Suppose N is a normal subgroup of G, then the Schur multiplier of pairs of G is computed by using the classification in Theorem 1. By Lagrange's theorem, the order of N must be equal to 1, p, p^2, p^3 or p^4, which implies $N \cong 1, \mathbb{Z}_p, \mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ or G. First, we have the following:

(i) If $N \cong 1$, then by Theorem 14, $M(G,N) = 1$.

(ii) If $N \cong G$, then by Theorem 15, $M(G,N) = M(G)$.

Next, we have the following cases:

Case 1: Let G be group of Type (1) which is $G \cong \mathbb{Z}_{p^4}$. Then by Theorem 10, $M(G) = 1$. By Proposition 13, $M(G,N)$ is a subgroup of $M(G)$. Therefore, for all $N \triangleleft G$, $M(G,N) \leq M(G) = 1$, and hence $M(G,N) = 1$.

Case 2: Let G be group of Type (2) which is $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p$. Then by Theorem 10, $M(G) \cong \mathbb{Z}_p$.

(i) If $N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ or $N \cong \mathbb{Z}_{p^3}$, then ${}^{G}/_{N} \cong \mathbb{Z}_p$. Then by Theorem 7, $M\left({}^{G}/_{N}\right) \cong 1$. Thus by Proposition 12, $M(G,N) \cong M(G)$. Therefore, $M(G,N) \cong \mathbb{Z}_p$.

(ii) If $N \cong \mathbb{Z}_{p^2}$, then ${}^{G}/_{N} \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}/_{N}$. Since $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p \cap \mathbb{Z}_{p^2} = 1$, $Q \cong {}^{G}/_{N} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 17, Theorem 9, Theorem 5 and Theorem 4, $|M(G,N| = p^2$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

(iii) If $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then ${}^{G}/_{N} \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}/_{N}$. Since $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p \cap \mathbb{Z}_{p^2} = 1$, $Q \cong {}^{G}/_{N} \cong \mathbb{Z}_{p^2}$. By Theorem 17, Theorem 9, Theorem 5 and Theorem 4, $|M(G,N| = p^3$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

(iv) If $N \cong \mathbb{Z}_p$, then ${}^{G}/_{N} \cong \mathbb{Z}_{p^3}$ or $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}/_{N}$. Since $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^3} \cap \mathbb{Z}_p = 1$, $Q \cong {}^{G}/_{N} \cong \mathbb{Z}_{p^3}$. By Theorem 17, Theorem 7 and Theorem 4, $|M(G,N| = p$. Therefore, $M(G,N) \cong \mathbb{Z}_p$.

Case 3: Let G be group of Type (3) which is $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$. Then by Theorem 10, $M(G) \cong \mathbb{Z}_{p^2}$.

(i) If $N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, then ${}^{G}\!/_{N} \cong \mathbb{Z}_p$. Then by Theorem 7, $M\left({}^{G}\!/_{N}\right) \cong 1$. Thus by Proposition 12, $M(G,N) \cong M(G)$. Therefore, $M(G,N) \cong \mathbb{Z}_{p^2}$.

(ii) If $N \cong \mathbb{Z}_{p^2}$, then $\left|{}^{G}\!/_{N}\right| = p^2$. Since $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$, ${}^{G}\!/_{N} \cong \mathbb{Z}_{p^2}$. Then by Theorem 9, $M\left({}^{G}\!/_{N}\right) \cong 1$. Thus by Proposition 12, $M(G,N) \cong M(G)$. Therefore, $M(G,N) \cong \mathbb{Z}_{p^2}$.

(iii) If $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then ${}^{G}\!/_{N} \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}\!/_{N}$. Since $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$, $Q \cong {}^{G}\!/_{N} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 17, Theorem 9, Theorem 5 and Theorem 4, $|M(G,N| = p^5$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

(iv) If $N \cong \mathbb{Z}_p$, then $\left|{}^{G}\!/_{N}\right| = p^3$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}\!/_{N}$. Since $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$, $Q \cong {}^{G}\!/_{N} \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. By Theorem 17, Theorem 7, Theorem 5 and Theorem 4, $|M(G,N| = p^2$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

Case 4: Let G be group of Type (4) which is $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$. Then by Theorem 10, $M(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

(i) If $N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ or $N \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, then ${}^{G}\!/_{N} \cong \mathbb{Z}_p$. Then by Theorem 7, $M\left({}^{G}\!/_{N}\right) \cong 1$. Thus by Proposition 12, $M(G,N) \cong M(G)$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

(ii) If $N \cong \mathbb{Z}_{p^2}$, then $\left|{}^{G}\!/_{N}\right| = p^2$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}\!/_{N}$. Since $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$, $Q \cong {}^{G}\!/_{N} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 17, Theorem 9, Theorem 5 and Theorem 4, $|M(G,N| = p^2$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

(iii) If $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $\left|{}^{G}\!/_{N}\right| = p^2$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}\!/_{N}$. Since $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$, $Q \cong {}^{G}\!/_{N} \cong \mathbb{Z}_{p^2}$. By Theorem 17, Theorem 9, Theorem 5 and Theorem 4, $|M(G,N| = p^3$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

(iv) If $N \cong \mathbb{Z}_p$ then $\left|{}^{G}\!/_{N}\right| = p^3$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}\!/_{N}$. Since $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$, $Q \cong {}^{G}\!/_{N} \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. By Theorem 17, Theorem 7, Theorem 5 and Theorem 4, $|M(G,N| = p^2$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

Case 5: Let G be group of Type (5) which is $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Then by Theorem 10, $M(G) = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

(i) If $N \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, then ${}^{G}\!/_{N} \cong \mathbb{Z}_p$. Then by Theorem 7, $M\left({}^{G}\!/_{N}\right) \cong 1$. Thus by Proposition 12, $M(G,N) \cong M(G)$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

(ii) If $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $\left|{}^{G}\!/_{N}\right| = p^2$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}\!/_{N}$. Since $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $Q \cong {}^{G}\!/_{N} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 17, Theorem 9, Theorem 5 and Theorem 4, $|M(G,N| = p^5$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

(iii) If $N \cong \mathbb{Z}_p$ then $\left|{}^{G}\!/_{N}\right| = p^3$. By Proposition 3, the complementary subgroup of N in G, $Q \cong {}^{G}\!/_{N}$. Since $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, ${}^{G}\!/_{N} \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. By Theorem 17, Theorem 7, Theorem 5 and Theorem 4, $|M(G,N| = p^3$. Therefore, $M(G,N) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ since by Theorem 16, $\exp(M(G,N))$ is a factor of $|G|$ and $|N|$.

THE CAPABILITY OF PAIRS OF ABELIAN GROUPS OF ORDER p^4

In this section the GAP algorithms for determining the capability of pairs of all abelian groups of order p^4 are presented. For the group $G \cong \mathbb{Z}_{p^4}$, the GAP algorithms are given as follows:

Firstly, we define $p = 3$ to generate cyclic group of order 81.

```
p=3
gap> G:=CyclicGroup(3^4);
<pc group of size 81 with 4 generators>
gap> Size(G);
81
gap> StructureDescription(G);
"C81"
```

Next, we compute the center of G, $Z(G)$ and commutator subgroup of G, G'.

```
gap> C:=Center(G);;
gap> StructureDescription(C);
"C81"
gap> D:=DerivedSubgroup(G);;
gap> StructureDescription(D);
"1"
```

The answer given by GAP for the center of G, $Z(G)$, is cyclic group of order 81, that is the group G itself while the commutator subgroup of G, G', is trivial.

Next, we compute the normal subgroups of G.

```
gap> List(NormalSubgroups(G),StructureDescription);
[ "C81", "C27", "C9", "C3", "1" ]
```

The normal subgroups of G are cyclic groups of order 81, 27, 9, 3 and trivial subgroup.

Next, we need to define cyclic groups of order 81, 27, 9, 3 as N1, N2, N3, N4, and trivial subgroup as N5.

```
gap> norm_subG:=NormalSubgroups(G);;
gap> N1:=norm_subG[1];
C81
gap> N2:=norm_subG[2];
C27
gap> N3:=norm_subG[3];
C9
gap> N4:=norm_subG[4];
C3
gap> N5:=norm_subG[5];
1
```

Lastly, we compute the order of the epicenters of pairs of G, namely epicenter of $(G,\text{N1})$, $(G,\text{N2})$, $(G,\text{N3})$, $(G,\text{N4})$ and $(G,\text{N5})$.

```
gap> Order(EpiCentre(G,N1));
81
gap> Order(EpiCentre(G,N2));
9
gap> Order(EpiCentre(G,N3));
1
gap> Order(EpiCentre(G,N4));
1
gap> Order(EpiCentre(G,N5));
1
```

The answer given by GAP showed that for G, the pair $(G,\text{N1})$ and $(G,\text{N2})$ are not capable. Furthermore, for G, the pair $(G,\text{N3})$, $(G,\text{N4})$ and $(G,\text{N5})$ are capable.

For the groups $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p$, $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$, $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ the similar GAP algorithms have been used.

Based on the results obtained by GAP for abelian groups of order p^4, two conjectures are constructed in the following, but before that the result obtained for the group of order p is presented first.

Theorem 23 *Let G be a group of prime order p and (G,N) be a pair of G where N is a normal subgroup of G. Then (G,N) is capable if N is trivial.*

Proof Let G be a group of prime order p. It follows that G is cyclic hence abelian. This implies that every subgroup of G is normal. By Lagrange's theorem, the order of N divides the order of G, thus the possible orders of N are 1 or p. If the order of N is 1 then N will be the trivial subgroup of G that is $N = 1$. By Definition 20, $Z_G^{\wedge}(N) = 1$. Thus, by Theorem 21, $(G,1)$ is capable. If the order of N is p then $N = G$. By Theorem 18, G is not capable which implies that $Z^{\wedge}(G) \neq 1$. It is known that $Z_G^{\wedge}(G) = Z^{\wedge}(G)$. Therefore, $Z_G^{\wedge}(G) \neq 1$. Thus, (G,G) is not capable.

Conjecture 24 *Let G be a cyclic group of order p^4 where p is an odd prime. Suppose that (G,N) is a pair of G where N is a normal subgroup of G. Then (G,N) is capable if $N = 1, \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$.*

Conjecture 25 *Let G be an abelian group of order p^4 where p is an odd prime. Suppose that (G,N) is a pair of G where N is a normal subgroup of G. Then (G,N) is capable if*

(i) *$G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ for $N = 1, \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.*

(ii) *$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ for all N of G.*

(iii) *$G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ for $N = 1, \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or $(\mathbb{Z}_p)^3$.*

(iv) *$G \cong (\mathbb{Z}_p)^4$ for all N of G.*

CONCLUSION

There are five abelian groups of order p^4. In this paper, we determined the Schur multiplier and capability of pairs of abelian groups of p^4. Our proofs show that $M(G,N)$ for those groups are equal to 1, $\mathbb{Z}_p$, $\mathbb{Z}_{p^2}$, $(\mathbb{Z}_p)^2$, $(\mathbb{Z}_p)^3$, $(\mathbb{Z}_p)^5$ or $(\mathbb{Z}_p)^6$ depending on the their normal subgroups. Meanwhile, all pairs of groups where G is isomorphic to the elementary abelian groups of order p^4 are capable. For other groups, only certain pairs of groups are capable depending on their normal subgroups.

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