p-adic Integration on Hyperelliptic Curves of Bad Reduction: Algorithms and Applications

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Abstract

For curves over the field of p-adic numbers, there are two notions of p-adic integration: Berkovich–Coleman integrals which can be performed locally, and Vologodsky integrals with desirable number-theoretic properties. These integrals have the advantage of being insensitive to the reduction type at p, but are known to coincide with Coleman integrals in the case of good reduction. Moreover, there are practical algorithms available to compute Coleman integrals.

Berkovich–Coleman and Vologodsky integrals, however, differ in general. In this thesis, we give a formula for passing between them. To do so, we use combinatorial ideas informed by tropical geometry. We also introduce algorithms for computing Berkovich–Coleman and Vologodsky integrals on hyperelliptic curves of bad reduction. By covering such a curve by basic wide open spaces, we reduce the computation of Berkovich–Coleman integrals to the known algorithms on hyperelliptic curves of good reduction. We then convert the Berkovich–Coleman integrals into Vologodsky integrals using our formula.

As an application, we provide an algorithm for computing Coleman–Gross p-adic heights on Jacobians of bad reduction hyperelliptic curves, whose definition relies on Vologodsky integration. This algorithm, for instance, can be used in the quadratic Chabauty method to find rational points on hyperelliptic curves of genus at least two.

Uittreksel

Voor krommen over het lichaam van p-adische getallen bestaan er twee noties van p-adische integratie: Berkovich–Colemanintegralen, die locaal bepaald worden, en Vologodsky-integralen met goede getaltheoretische eigenschappen. Deze integralen hebben het voordeel dat ze ongevoelig zijn voor het reductietype bij p, maar het is bekend dat ze samenvallen met Colemanintegralen in het geval van goede reductie. Daarnaast zijn er praktische algoritmen beschikbaar voor het berekenen van Colemanintegralen.

Berkovich–Coleman- en Vologodsky-integralen zijn echter in het algemeen verschillend. In deze thesis geven we een formule om tussen de twee heen en weer te gaan. Hiervoor gebruiken we combinatorische ideeën vanuit de tropische meetkunde. Daarnaast introduceren we algoritmen voor het berekenen van Berkovich–Coleman- en Vologodsky-integralen op hyperelliptische krommen van slechte reductie. Door zo'n kromme met basale wijde open ruimtes te overdekken, reduceren we de berekening van Berkovich–Colemanintegralen tot de al bekende algoritmen voor hyperelliptische krommen met goede reductie. Vervolgens zetten we de Berkovich–Colemanintegralen om tot Vologodsky-integralen met behulp van onze formule.

Als een toepassing geven we een algoritme voor het berekenen van Coleman-Gross p-adische hoogtes op Jacobianen van hyperelliptische krommen met slechte reductie, waarvan de definitie is gebaseerd op Vologodsky-integratie. Dit algoritme kan bijvoorbeeld gebruikt worden in de kwadratische Chabautymethode om rationale punten op krommen van geslacht minstens twee te vinden.

Babam için

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Contents

Chapter 1

Introduction

1.1 p-adic Integration

Coleman integration [Col82, CdS88, Bes02] is a method for defining p-adic iterated integrals on rigid analytic spaces associated to varieties with good reduction at p, where p is from now on a fixed prime number. Vologodsky integration [Vol03] also produces such integrals on varieties, but it does not require that the varieties under consideration have good reduction at p. These two integration theories are surprisingly path-independent and they are known to be the *same* in the case of good reduction. Therefore, Vologodsky integration is, in a sense, a generalization of Coleman integration.

On the other hand, by a cutting-and-pasting procedure, one can define in a natural way integrals on curves of arbitrary reduction type[1]. More precisely, one can cover such a curve by *basic wide open spaces*, certain rigid analytic spaces, each of which can be embedded into a curve of good reduction, and by performing the Coleman integrals there, one can piece together an integral. This naive integral, which will be referred to as the Berkovich–Coleman integral [CdS88, Ber07], is generally path-dependent and hence disagrees with the Vologodsky integral.

These integration theories serve as p-adic analogues of the classical (real-valued) line integral. However, from the point of view of classical analysis, their existence is quite surprising. The main reason is that in the p-adic world one can integrate differential forms locally, but due to the fact that the p-adic topology is totally disconnected, naive analytic continuation is impossible.

[1] In what follows we will only consider curves, not higher-dimensional varieties.

1.2 Why p-adic Integration?

Coleman and Vologodsky integration have numerous applications in arithmetic and Diophantine geometry, such as[2]

- determining rational points on curves [Col85a, LT02, MP12, BBCF$^+$19, BD18, BD21, BDM$^+$19, BBB$^+$21],
- determining torsion points on curves lying on an abelian variety [Col85b],
- obtaining uniform bounds for the number of rational points on curves [Sto19, KRZB16],
- computing p-adic height pairings on Jacobians of curves [CG89, Bes17], and
- computing p-adic regulators in K-theory [CdS88, BDJ08, Bes00a, Bes00b, Bes21a].

Some of these applications rely heavily on explicitly computing integrals. Coleman's construction is quite suitable for machine computation, and, when p is odd, there are practical algorithms due to Balakrishnan and others: see [BBK10, BB12, Bal15, Bes19] (single integrals on hyperelliptic curves), [Bal13, Bal15] (double integrals on hyperelliptic curves), [Bes21b] (single integrals on superelliptic curves), and [BT20] (single integrals on smooth curves). However, due to the highly abstract nature of Vologodsky's construction, Vologodsky integration has been, so far, difficult to compute. The present thesis aims to take the first steps in this direction.

1.3 Why Bad Primes?

Why, one might ask, are primes of bad reduction interesting or useful? It would be hopeless to give a complete answer to this philosophical question here; however, we make a few comments.

Techniques developed in order to deal with p-adic problems (i.e., problems whose objects are defined over p-adic fields) depend, in general, on the *nature* of the prime p. One striking example of this phenomenon is the p-adic analogue of the conjecture of Birch and Swinnerton–Dyer for elliptic curves [MTT86, BPR93].

[2] Our list is quite far from complete.

For different reduction types, this conjecture is, in general, of a quite different nature and of interest in its own right. See also [SW13].

In another direction, we sometimes study a problem over $\mathbb{Q}$ by studying it over $\mathbb{Q}_p$ for a fixed prime p. A *good* prime is often more convenient to work with; however, for practical purposes, we may need to take p as small as possible and this additional condition might force us to work with a *bad* prime. As an illustration, we may look at [KZB13, Example 5.1], in which Katz–Zureick-Brown showed, for a certain curve $X/\mathbb{Q}$ with bad reduction at 5, that 5 is the only prime that allows to compute $X(\mathbb{Q})$ using Coleman's upper bound for $\#X(\mathbb{Q})$ [Col85a] and/or its refinements [LT02, Sto06, KZB13].

A related topic is the uniformity conjecture [CHM97]. This conjecture is one of the outstanding conjectures in Diophantine geometry, and asserts that there exists a constant $B(\mathbb{Q}, g)$ such that every smooth curve $X/\mathbb{Q}$ of genus $g \geq 2$ has at most $B(\mathbb{Q}, g)$ rational points. The first result along these lines is due to Stoll [Sto19], who proved uniform bounds for hyperelliptic curves of small rank; his result was later generalized by Katz, Rabinoff, and Zureick-Brown [KRZB16] to arbitrary curves of small rank. See also the paper of Kantor [Kan17]. Here, to get a bound *independent of the geometry of curves*, it is necessary to work with primes of bad reduction.

All in all, our slogan is: *Bad primes might be good!*

1.4 This Thesis

Our main goal in this thesis is to provide an algorithm for computing single Vologodsky integrals on bad reduction hyperelliptic curves for $p \neq 2$. We also discuss some immediate applications.

1.4.1 Integral Comparison

A key ingredient in our method is a comparison theorem for Vologodsky and Berkovich–Coleman integrals on curves, which reduces the computation of Vologodsky integrals to the computation of Berkovich–Coleman integrals. Here, to compare the two integration theories, we follow Katz–Rabinoff–Zureick-Brown [KRZB16] and Besser–Zerbes [BZ17] for holomorphic and meromorphic 1-forms, respectively. In both cases, the difference is essentially controlled by the *tropical* Abel–Jacobi map.

1.4.2 Main Algorithm

Our algorithm works by first computing the Berkovich–Coleman integral of a meromorphic 1-form ω and then correcting it to a Vologodsky integral. We consider a hyperelliptic curve X that is the projective normalization of the smooth plane affine curve of the form $y^2 = f(x)$, where $f(x)$ is a monic separable polynomial with integral coefficients in some finite extension of $\mathbb{Q}_p$. We write $\pi\colon X \to \mathbb{P}^1$ for the hyperelliptic double cover. By examining the *roots* of $f(x)$ and making use of a Newton polygon argument, we are able to cover $\mathbb{P}^{1,\mathrm{an}}$, the projective rigid line, by open subspaces $\{U_i\}$ such that for each U_i one can find a good reduction hyperelliptic curve $\tilde{X}_i$ into which $\pi^{-1}(U_i)$ embeds as the complement of finitely many closed discs. In the process of finding the covering $\{\pi^{-1}(U_i)\}$, we determine the dual graph Γ of the special fiber of a semistable model of X and therefore its tropicalization.

We expand $\omega|_{\pi^{-1}(U_i)}$ as a power series in certain meromorphic 1-forms on $\tilde{X}_i$. We pick a set of meromorphic 1-forms on $\tilde{X}_i$ that descend to a basis of the odd part of the de Rham cohomology of $\pi^{-1}(U_i)$. By a pole-reduction argument similar to work of Tuitman [Tui16, Tui17], we rewrite the terms in the power series as the sum of an exact form and a linear combination of 1-forms in our basis. Then, one is able to perform the integral using the techniques of [BBK10, BB12, Bal15]. This allows us to integrate ω between points of $\pi^{-1}(U_i)$.

The Berkovich–Coleman integral is path-dependent but invariant under fixed endpoint homotopy. The homotopy class of a path in the Berkovich analytification X^{an} of X can be specified by its endpoints together with a path in Γ between the tropicalizations of the endpoints. From our knowledge of the intersections of the U_i's and thus of the dual graph, we are able to perform the Berkovich–Coleman integral along any path in X^{an}. In particular, we can integrate meromorphic 1-forms along closed paths to determine the Berkovich–Coleman periods. Using them, together with a description of the tropical Abel–Jacobi map, we can correct the Berkovich–Coleman integral along a path to the Vologodsky integral between its endpoints.

Our algorithm works in great generality. In specific situations, we have implementations of various steps of this algorithm in Sage [The20] and/or Magma [BCP97], and using these, one can produce several interesting examples. In fact, we have mostly used Sage, because it includes implementations of the Coleman integration algorithms in [BBK10, BB12, Bal15], and such algorithms have only recently been implemented in Magma, see [BT20, BM]. Another reason for using

Sage is that it is currently the only option to compute double Coleman integrals. This will be important for future research, see Subsection 5.4.2.

It would be of great interest to provide a general implementation of our algorithm in Sage. This, however, seems out of reach at present, even if we take the base field to be $\mathbb{Q}_p$. There are two main reasons for this:

- Sage implementations of the integration algorithms in [BBK10, BB12, Bal15] assume that the base field is $\mathbb{Q}_p$. However, the hyperelliptic curves $\tilde{X}_i$ above are generally defined over non-trivial extensions of $\mathbb{Q}_p$.

- In our approach, it might be necessary to work with several extensions of $\mathbb{Q}_p$ at the same time. In Sage, Eisenstein and unramified extensions are implemented; however, neither passing between these extensions nor general extensions are available.

We note that, when these obstacles are overcome, it should be possible to implement our algorithm in Sage.

1.4.3 Applications

Let $X/\mathbb{Q}$ be a smooth, proper, and geometrically connected curve of genus $g \geq 1$, and let p be an odd prime.

Chabauty–Coleman method. When the genus g is at least 2, thanks to Faltings' theorem [Fal83], the set $X(\mathbb{Q})$ is known to be finite; however, his proof is ineffective and at present no general algorithm for the computation of $X(\mathbb{Q})$ is known. The method of Chabauty–Coleman [MP12] is a p-adic method that attempts to *determine* $X(\mathbb{Q})$ under the condition that the Mordell–Weil rank r of the Jacobian of X is less than g. This method relies on the construction of an *annihilating* differential, which exists by the hypothesis $r < g$. Such a construction requires, among other things, computing Vologodsky integrals of regular 1-forms on X. In the special case that X has good reduction at p, this can be achieved using the algorithms developed by Balakrishnan–Tuitman [BT20]. In fact, explicit Chabauty–Coleman computations have been, so far, restricted to the good reduction case. Our algorithms allow one to carry out this method on hyperelliptic curves at primes of bad reduction; see Example 4.8.4 for the first such example[3].

[3]The nature of [KZB13, Example 5.1] is slightly different.

p-adic height pairings. When the curve X has good reduction at p, Coleman and Gross [CG89] gave a construction of a p-adic height pairing on X which is, by definition, a sum of local height pairings at each prime number. The local components away from p are described using arithmetic intersection theory, and the local component at p is given in terms of the Coleman integral of a non-holomorphic differential. In another direction, replacing Coleman integration in this recipe by Vologodsky integration, Besser [Bes17, Definition 2.1] gave an extended definition of the Coleman–Gross pairing on X without any assumptions on the reduction at p. In the case that X is a hyperelliptic curve with good reduction at p, an algorithm to compute the local height pairing at p was provided in Balakrishnan–Besser [BB12]. The techniques of the current thesis make it possible to remove the good reduction assumption from this setting; as a second application, we provide such an algorithm.

We note that there are several definitions of p-adic height pairings. Moreover, algorithms for computing p-adic heights are required to carry out the quadratic Chabauty method, a generalization of the Chabauty–Coleman method, and can be used to compute p-adic regulators, some of which fit into p-adic versions of Birch and Swinnerton–Dyer conjecture.

1.5 Outline

The following is a brief outline of the individual chapters.

Chapter 2 is devoted to preliminaries for all chapters.

In Chapter 3, we review Vologodsky and Berkovich–Coleman integration theories. We also summarize the known Coleman integration algorithms.

Chapters 4 and 5, the heart of this thesis, are devoted to our integration algorithms. These algorithms were first developed for holomorphic 1-forms in joint work with Katz [KK20], and then extended to all meromorphic 1-forms by the author [Kay20]. Following this development, we describe the algorithms for holomorphic 1-forms in Chapter 4, and explain in Chapter 5 how to extend them in a natural way to also cover meromorphic 1-forms.

Chapter 6 concerns p-adic heights. We, in particular, introduce our algorithm for computing Coleman–Gross p-adic heights on hyperelliptic curves of bad reduction.

Chapter 2

Background and Definitions

Let us first discuss some preliminaries. We introduce notation and assumptions used throughout this thesis.

As usual, $\mathbb{Z}$ denotes the ring of integers, and $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the field of rational, real and complex numbers, respectively. For any prime ℓ, we let $\mathbb{Z}_\ell$ denote the ring of ℓ-adic integers and $\mathbb{Q}_\ell$ denote the field of ℓ-adic numbers.

We work with a fixed odd prime number p. Let $\mathbb{C}_p$ denote the completion of an algebraic closure of $\mathbb{Q}_p$. Let v_p be the valuation on $\mathbb{C}_p$, normalized such that $v_p(p) = 1$; it corresponds to the absolute value $\|\cdot\|_p$ where $\|\cdot\|_p = p^{-v_p(\cdot)}$. The field $\bar{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$, is the residue field of $\mathbb{C}_p$.

Let K be a complete subfield of $\mathbb{C}_p$ with ring of integers $\mathcal{O}_K$ and residue field $\Bbbk$.

2.1 Graphs

Let Γ be a finite and connected graph. We denote by $V(\Gamma)$ and $E(\Gamma)$, respectively, the set of vertices and oriented edges. For $e \in E(\Gamma)$, we write $i(e)$ and $t(e)$ for the initial and terminal point of e, respectively; and we denote by $-e$ the same edge as e with the reverse orientation.

Let A be one of $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}_p$. The free A-module generated by $V(\Gamma)$ is denoted by $C_0(\Gamma, A)$, and its elements are called 0**-chains** with coefficients in A. The A-module generated by $E(\Gamma)$ subject to the relations

$$e + (-e) = 0 \text{ for each edge } e$$

is denoted by $C_1(\Gamma, A)$; its elements are called 1-**chains** with coefficients in A. As $C_0(\Gamma, A)$ is canonically isomorphic to its dual $C_0(\Gamma, A)^*$, we may identify a 0-chain $C = \sum_v c_v v$ with the function $C\colon V(\Gamma) \to A$ defined by $v \mapsto c_v$; a similar remark applies to $C_1(\Gamma, A)$. The boundary map

$$d\colon C_1(\Gamma, A) \to C_0(\Gamma, A)$$

is given by $e \mapsto t(e) - i(e)$. We set $H_1(\Gamma, A) = \ker(d)$, its elements are called the 1-**cycles**.

2.2 Differential Forms on Curves

Here, we follow [CG89, Section 2] and [BB12, Section 2].

Let X/K be a smooth, proper, geometrically connected curve of genus g and let ω be a meromorphic 1-form on X over K. We say ω is

- of the **first kind** if ω is regular,
- of the **second kind** if all residues of ω are zero, and
- of the **third kind** if ω is regular, except possibly for simple poles with integer residues.

Exact differentials, i.e., those of the form df for $f \in K(X)$, are of the second kind. Logarithmic differentials, i.e., those of the form df/f for $f \in K(X)^\times$, are of the third kind.

The **first algebraic de Rham cohomology group** of X/K, denoted by $H^1_{\mathrm{dR}}(X/K)$, is the first hypercohomology group of the de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1_{X/K} \to 0.$$

The set

$$\{\text{differentials of the second kind}\}/\{\text{exact differentials}\}$$

form a $2g$-dimensional K-vector space and it is canonically isomorphic to $H^1_{\mathrm{dR}}(X/K)$. Using the Hodge filtration

$$0 \to H^0(X, \Omega^1_{X/K}) \to H^1_{\mathrm{dR}}(X/K) \to H^1(X, \mathcal{O}_X) \to 0,$$

we identify the space $H^0(X, \Omega^1_{X/K})$ with its image and denote it by $H^{1,0}_{\mathrm{dR}}(X/K)$.

Let $T(K)$ denote the subgroup of differentials of the third kind and $\mathrm{Div}^0(X)$ the group of divisors of degree zero on X over K. The residue divisor homomorphism

$$\mathrm{Res}\colon T(K) \to \mathrm{Div}^0(X), \quad \nu \mapsto \sum_P \mathrm{Res}_P \nu \cdot (P)$$

where the sum is taken over closed points of X, gives the following exact sequence:

$$0 \to H^{1,0}_{\mathrm{dR}}(X/K) \to T(K) \to \mathrm{Div}^0(X) \to 0. \tag{2.1}$$

Let

$$\cup\colon H^1_{\mathrm{dR}}(X/K) \times H^1_{\mathrm{dR}}(X/K) \to K$$

denote the algebraic cup product pairing. This pairing is non-degenerate and alternating, and may be computed as follows. Suppose μ_1 and μ_2 are two differentials of the second kind, with classes $[\mu_1]$ and $[\mu_2]$ in $H^1_{\mathrm{dR}}(X/K)$. For each point P of X, pick a formal primitive λ_P of μ_1. Then

$$[\mu_1] \cup [\mu_2] = \sum_P \mathrm{Res}_P \left(\lambda_P \cdot \mu_2\right).$$

Note that this is well-defined since μ_2 has no residues.

We denote the subgroup of $T(K)$ consisting of the logarithmic differentials by $T_l(K)$. By [CG89, Proposition 2.5], there is a canonical homomorphism

$$\Psi\colon T(K)/T_l(K) \to H^1_{\mathrm{dR}}(X/K) \tag{2.2}$$

which is the identity on differentials of the first kind. We extend the domain of Ψ to all meromorphic differentials on X/K as follows. Let ω be a meromorphic 1-form on X over K. We express ω as a sum $\sum c_i\nu_i + \mu$, where c_i is an element of $\bar{K}$, ν_i is of the third kind and μ is of the second kind, and define

$$\Psi(\omega) = \sum a_i \Psi(\nu_i) + [\mu].$$

Note that this rule maps a form of the second kind to its class.

2.3 Rigid Analytic Spaces

We will make use of several concepts from rigid analytic geometry. Some standard references are the book of Bosch–Güntzer–Remmert [BGR84] and the book of Fresnel–van der Put [FvdP04] (see also Schneider [Sch98] for a quick introduction).

We use the marker "an" to denote rigid analytification; in particular, $\mathbb{A}^{1,\mathrm{an}}$ (resp. $\mathbb{P}^{1,\mathrm{an}}$) denotes the rigid affine (resp. projective) line. We write

$$\begin{aligned} B(a,r) &= \{z \in \mathbb{P}^{1,\mathrm{an}} \mid \|z-a\|_p < r\}, \quad r > 0; \\ B(\infty,r) &= \{z \in \mathbb{P}^{1,\mathrm{an}} \mid \|1/z\|_p < r\}, \quad r > 0. \end{aligned}$$

We write $\overline{B}(a,r)$ for those sets when we replace the strict inequality with the nonstrict one.

Rigid analytic spaces are built by gluing affinoids. Given an affinoid of good reduction V, let

$$\mathrm{red}\colon V \to V_{\mathbb{k}}$$

be its reduction map. The preimage of a closed point under red is a **residue disc**. In the case of $\mathbb{P}^{1,\mathrm{an}}$, the residue disc about $a \in \mathbb{A}^1(\mathbb{C}_p)$ is $B(a,1)$ while the residue disc about ∞ is $B(\infty,1)$. A map $\phi\colon V \to V$ is called **a lift of Frobenius** if it induces the Frobenius map on $V_{\mathbb{k}}$.

Definition 2.3.1. A **wide open** is a rigid analytic space isomorphic to the complement of finitely many closed discs in a connected smooth complete curve.

Definition 2.3.2. A **basic wide open** is a rigid analytic space isomorphic to the complement of finitely many closed discs in a connected good reduction complete curve, each of which is contained in a distinct residue disc.

In these two definitions, by abuse of terminology, by a "curve" we mean the rigid analytic space associated to an algebraic curve. We will freely make use of this convention.

In this thesis, two types of basic wide opens will be particularly important: rational and hyperelliptic. A basic wide open is called **rational** (resp. **hyperelliptic**) if it lies in the rigid analytic space associated to $\mathbb{P}^1$ (resp. a hyperelliptic curve).

We have the following elementary examples of (rational basic) wide opens. First, $\mathbb{P}^{1,\mathrm{an}}$ with one closed disc removed is called an **open disc**, and such a space is isomorphic to $B(0,1)$, the **standard** open disc. Similarly, $\mathbb{P}^{1,\mathrm{an}}$ with two disjoint closed discs removed is called an **open annulus**; such a space is isomorphic to a **standard** open annulus, i.e., a space of the form

$$A(r,1) = \{z \in \mathbb{A}^{1,\mathrm{an}} \mid r < \|z\|_p < 1\}, \quad 0 < r < 1.$$

Let U be a wide open space. For an affinoid subdomain V of U, we write $\mathrm{CC}(U \setminus V)$ for the set of the connected components of $U \setminus V$. The **ends** of U are the elements of the set

$$\varprojlim \mathrm{CC}(U \setminus V)$$

where V runs over all affinoid subdomains of U.

Definition 2.3.3. An **underlying affinoid** of a basic wide open U is an affinoid subdomain $V \subset U$ such that the elements of $\mathrm{CC}(U \setminus V)$ are annuli and are in bijective correspondence with the ends of U. These annuli are called **boundary annuli**.

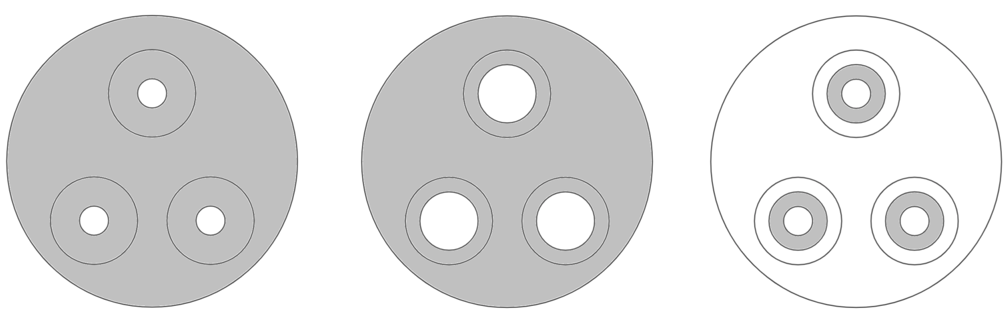
Note that underlying affinoids are necessarily of good reduction. We will need to consider underlying affinoids within rational basic wide opens. Let

$$U = \mathbb{P}^{1,\mathrm{an}} \setminus (\cup_i D_i)$$

be a rational basic wide open. For each closed disc D_i, we may pick a slightly larger open disc D_i' (still contained in a residue disc) containing D_i. Then

$$V = U \setminus (\cup_i D_i') = \mathbb{P}^{1,\mathrm{an}} \setminus (\cup_i D_i')$$

is an underlying affinoid [Col89, Corollary 3.5a] and $U \setminus V$ is a finite union of annuli. An illustration is given in the following figure, where the pictures represent U, V and $U \setminus V$, respectively.



2.4 Berkovich Analytic Spaces

In general, we will use the language of rigid spaces but will freely invoke Berkovich spaces when convenient. The canonical references are Berkovich's book

[Ber90] and the paper of Baker–Payne–Rabinoff [BPR13]; see also [KRZB16, KRZB18].

We abuse notation and use the marker "an" also to denote Berkovich analytification in the sense of [Ber90]; the intention should be clear from the context. In fact, we sometimes identify a Berkovich analytic space with its corresponding rigid analytic space.

Let X be a smooth, proper, geometrically connected K-curve. Let X^{an} denote the Berkovich analytification of X. The curve X admits a semistable $\mathcal{O}_K$-model $\mathfrak{X}$ after making a finite extension of the ground field K. There is a metric graph $\Gamma_{\mathfrak{X}}$ attached to $\mathfrak{X}$, the **skeleton** of $\mathfrak{X}$, and a retraction map $\tau\colon X^{\mathrm{an}} \to \Gamma_{\mathfrak{X}}$. The skeleton $\Gamma_{\mathfrak{X}}$ is naturally identified with the dual graph of the special fiber $\mathfrak{X}_{\Bbbk}$. In general, a **skeleton** of X is a skeleton $\Gamma = \Gamma_{\mathfrak{X}}$ for some semistable $\mathcal{O}_K$-model $\mathfrak{X}$ of X.

2.5 Hyperelliptic Curves

We will consider a hyperelliptic curve X which is the projective normalization of the smooth plane affine curve given by

$$y^2 = f(x)$$

for a separable polynomial $f(x) \in K[x]$ of degree $d \geq 3$. The curve X is of genus $\lfloor \frac{d-1}{2} \rfloor$. We write $\pi\colon X \to \mathbb{P}^1$ for the hyperelliptic double cover.

If d is odd, then there is a single point lying over ∞; this point is always K-rational. If d is even, then there are two distinct points lying over ∞; these points are K-rational precisely when the leading coefficient of $f(x)$ is a square in K. When d is even and $f(x)$ is monic, we denote the points $(1 : 1 : 0)$ and $(1 : -1 : 0)$ at infinity simply by ∞^+ and ∞^-, respectively.

The curve X has a hyperelliptic involution extending $w(x, y) = (x, -y)$. The fixed points of the involution are the **Weierstrass** points. If d is odd, the point at infinity is Weierstrass; if d is even, the points at infinity are non-Weierstrass.

On X, a residue disc is said to be **Weierstrass** (resp. **non-Weierstrass**) if it corresponds to a Weierstrass (resp. non-Weierstrass) point. We say that a residue disc is **infinite** if it contains a point at infinity; a residue disc is called **finite** if it is not infinite.

Chapter 3

p-adic Integration Theories

In this chapter, we review Vologodsky and Berkovich–Coleman integration. We also summarize the known p-adic integration algorithms.

Choose once and for all a branch of the p-adic logarithm, i.e., a homomorphism

$$\mathrm{Log}\colon \mathbb{C}_p^\times \to \mathbb{C}_p$$

given by the Mercator series

$$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

when $\|z\|_p < 1$. A branch is determined by specifying $\mathrm{Log}(p)$ in $\mathbb{C}_p$.

3.1 Vologodsky Integration

Recall that K is a complete subfield of $\mathbb{C}_p$. For a smooth, geometrically connected algebraic K-variety X, we let $Z^1_{\mathrm{dR}}(X)$ denote the space of closed 1-forms on X.

Theorem 3.1.1. *There is a unique way to construct, for every smooth, geometrically connected algebraic K-variety X, every $\omega \in Z^1_{\mathrm{dR}}(X)$ and every pair of points $x, y \in X(K)$, an integral*

$$\int_x^y \omega \in K$$

such that the following are true:

1. *If $x_0 \in X(K)$, then*

$$F\colon X(K) \to K, \quad x \mapsto \int_{x_0}^{x} \omega$$

is a locally analytic function satisfying $dF = \omega$.

2. *If $\omega_1, \omega_2 \in Z^1_{\mathrm{dR}}(X)$ and $c_1, c_2 \in K$, then*

$$\int_x^y (c_1\omega_1 + c_2\omega_2) = c_1 \int_x^y \omega_1 + c_2 \int_x^y \omega_2.$$

3. *If $x, y, z \in X(K)$, then*

$$\int_x^z \omega = \int_x^y \omega + \int_y^z \omega.$$

4. *If f is a rational function on X, then*

$$\int_x^y df = f(y) - f(x)$$

provided that f is defined on the endpoints.

5. *If f is a non-zero rational function on X, then*

$$\int_x^y \frac{df}{f} = \mathrm{Log}\left(\frac{f(y)}{f(x)}\right)$$

provided that f is defined and non-zero on the endpoints.

6. *If $h\colon X \to Y$ is a K-morphism of smooth, geometrically connected algebraic K-varieties, $x, y \in X(K)$ and $\omega \in Z^1_{\mathrm{dR}}(Y)$, then*

$$\int_x^y h^*\omega = \int_{h(x)}^{h(y)} \omega.$$

This theorem was proved by Colmez [Col98, Théorème 1], and, when K is finite over $\mathbb{Q}_p$, a generalization involving iterated integrals was given later by Vologodsky [Vol03, Theorem B]. Following Besser and Zerbes [BZ17, Bes17], we call this integral the **Vologodsky integral** and denote it by ${}^{\mathrm{Vol}}\!\!\int$.

Remark 3.1.2. The notion of (p-adic) **abelian** integration is well known in the literature. It is a special case of Vologodsky integration. More precisely, Vologodsky integrals of holomorphic forms are often called abelian integrals.

The construction of Vologodsky integration is highly sophisticated in nature. Therefore, computing Vologodsky integrals directly from the definition does not seem feasible, even for curves; one needs to take an indirect approach. In Chapters 4 and 5, we will explain how to compute these integrals on hyperelliptic curves in practice. A key ingredient in our (indirect) approach is a comparison theorem, which reduces the computation of Vologodsky integrals to the computation of Berkovich–Coleman integrals; the latter is comparatively easy to handle.

3.2 Berkovich–Coleman Integration

In order to study Berkovich–Coleman integration, we will follow Berkovich's language; in principle this can be avoided by using intersection theory on semistable curves, however, the analytic framework seems much more natural.

For a smooth $\mathbb{C}_p$-analytic space X, we let $Z^1_{\mathrm{dR}}(X)$ denote the space of closed analytic 1-forms on X, and we let $\mathcal{P}(X)$ denote the set of paths $\gamma\colon [0,1] \to X$ with ends in $X(\mathbb{C}_p)$. The following is a special case of a theorem of Berkovich [Ber07, Theorem 9.1.1].

Theorem 3.2.1. *There is a unique way to construct, for every smooth $\mathbb{C}_p$-analytic space X, every $\omega \in Z^1_{\mathrm{dR}}(X)$ and every $\gamma \in \mathcal{P}(X)$, an integral*

$$\int_\gamma \omega \in \mathbb{C}_p$$

such that the following are true:

1. *If $\omega_1, \omega_2 \in Z^1_{\mathrm{dR}}(X)$ and $c_1, c_2 \in \mathbb{C}_p$, then*

$$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

2. *If $\gamma_1, \gamma_2 \in \mathcal{P}(X)$ with $\gamma_1(1) = \gamma_2(0)$, then*

$$\int_{\gamma_1\gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$

where $\gamma_1\gamma_2$ is the concatenation.

3. *If* $\gamma_1, \gamma_2 \in \mathcal{P}(X)$ *are homotopic with fixed endpoints, then*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

4. *If* f *is a rational function on* X*, then*

$$\int_{\gamma} df = f(\gamma(1)) - f(\gamma(0))$$

   provided that f *is defined on the ends of* γ*.*

5. *If* f *is a non-zero rational function on* X*, then*

$$\int_{\gamma} \frac{df}{f} = \mathrm{Log}\left(\frac{f(\gamma(1))}{f(\gamma(0))}\right)$$

   provided that f *is defined and non-zero on the ends of* γ*.*

6. *If* $h\colon X \to Y$ *is a morphism of smooth* $\mathbb{C}_p$*-analytic spaces,* $\gamma \in \mathcal{P}(X)$ *and* $\omega \in Z^1_{\mathrm{dR}}(Y)$*, then*

$$\int_{\gamma} h^*\omega = \int_{h\circ\gamma} \omega.$$

This integration was first developed for curves by Coleman [Col82, Col85b] and Coleman–de Shalit [CdS88] (using the language of rigid analytic spaces). Following Katz, Rabinoff, and Zureick-Brown [KRZB16, KRZB18], we call this integral the **Berkovich–Coleman integral** and denote it by ${}^{\mathrm{BC}}\!\!\int$. We call a Berkovich–Coleman integral along a closed path a **Berkovich–Coleman period**.

We note that even if the ends of γ belong to $X(K)$ and ω is defined over K, the integral ${}^{\mathrm{BC}}\!\!\int_\gamma \omega$ is not necessarily an element of K; see Example 4.8.3.

Unfortunately, the Berkovich–Coleman integral is generally path-dependent: ${}^{\mathrm{BC}}\!\!\int_\gamma$ depends on γ, and not just on its endpoints. However, when X is simply-connected, ${}^{\mathrm{BC}}\!\!\int_\gamma$ is path-independent by the homotopy invariance; in this case, we simply write ${}^{\mathrm{BC}}\!\!\int_x^y = {}^{\mathrm{BC}}\!\!\int_\gamma$ for any path γ from x to y.

Recall that the Vologodsky integral on an algebraic variety X depends only on the endpoints and not on a path between them. According to [Ber07, Remark 9.1.3 (ii)], the main reason is that this integral is functorial with respect to morphisms in the category of algebraic varieties and this category is too *coarse* to recognize a nontrivial homotopy type of X^{an}. Therefore, the Berkovich–Coleman integration theory seems a more natural analogue of the classical (i.e., complex) abelian integration theory.

Remark 3.2.2. In contrast with the Vologodsky integral, the Berkovich–Coleman integral is *local*, i.e., if $U \subset X$ is an open subdomain and $\gamma([0,1]) \subset U$, then the integral ${}^{\mathrm{BC}}\!\int_\gamma \omega$ is determined by U, $\omega|_U$ and γ; see [KRZB18, Section 4.2].

Thanks to this key property, there is good reason to believe Berkovich–Coleman integrals to be practically computable in general. In Chapters 4 and 5, we will discover how to compute these integrals on hyperelliptic curves; this will allow us to compute Vologodsky integrals on such curves.

3.2.3 Berkovich–Coleman Integration on Basic Wide Opens

The Berkovich–Coleman integration has a useful characterization on basic wide opens [CdS88, Section 2]. On a basic wide open U, the Berkovich–Coleman integral is univalent: given $\omega \in \Omega^1(U)$, the rigid analytic differential 1-forms on U, there is a locally analytic function f_ω unique up to a global constant such that

$$ {}^{\mathrm{BC}}\!\!\int_\gamma \omega = f_\omega(\gamma(1)) - f_\omega(\gamma(0)) \text{ for all } \gamma. $$

We describe such an f_ω. Fix a basic wide open U and $\omega \in \Omega^1(U)$.

Let V be an underlying affinoid of U and let $\mathcal{A}$ denote the set of annuli which are the connected components of $U \setminus V$. A **Frobenius neighborhood** of V in U is a pair consisting of a basic wide open W with $V \subset W \subset U$ and a rigid analytic morphism $\phi\colon W \to U$ restricting to a lift of Frobenius on V; such a pair always exists, see [CdS88, Section 2.2].

For an open subdomain $U' \subset U$, let $A(U')$ be the ring of rigid analytic functions on U', and set

$$ A_{\mathrm{Log}}(U') = A(U')[\{\mathrm{Log}(f) \mid f \in A(U')^\times\}]. $$

Write $\mathcal{L}(U)$ for the ring of locally analytic functions on U. Note that

$$ A_{\mathrm{Log}}(E) \subset \mathcal{L}(E) \text{ for each } E \in \mathcal{A}. $$

Let $P(T)$ be a polynomial without roots-of-unity as roots, such that (after possibly shrinking W to ensure $P(\phi^*)$ is well-defined) $P(\phi^*)$ annihilates $(\Omega^1(U)|_W)/dA(W)$. Such a polynomial exists by the proof of the Weil conjectures for curves.

The Berkovich–Coleman integral has the following characterization on basic wide open spaces.

Lemma 3.2.4. ([CdS88, Proposition 2.4.1]) *The locally analytic function $f_\omega \in \mathcal{L}(U)$ is characterized up to addition of a global constant by the following properties:*

1. $f_\omega|_E \in A_{\mathrm{Log}}(E)$ *for each* $E \in \mathcal{A}$,
2. $df_\omega = \omega$, *and*
3. $P(\phi^*)f_\omega|_W \in A(W)$.

Functions that satisfy the properties of the above lemma are said to be a **Coleman primitive** of ω.

The following result, which interchanges limits and integration, follows from applying the characterization to $\lim f_{\omega_i}$.

Proposition 3.2.5. *For a basic wide open U and an underlying affinoid V in U, let $\mathcal{A}$, W and ϕ be as above. If $\{f_{\omega_i}\}$ is a sequence of locally uniformly convergent Coleman primitives on U such that*

1. $\{f_{\omega_i}|_E\}$ *converges uniformly in* $A_{\mathrm{Log}}(E)$ *for each* $E \in \mathcal{A}$,
2. $\{\omega_i\}$ *converges uniformly in* $\Omega^1(U)$, *and*
3. $\{P(\phi^*)f_{\omega_i}|_W\}$ *converges uniformly on* W.

Then the locally analytic limit $\lim f_{\omega_i}$ *is a Coleman primitive of* $\lim \omega_i$ *on* U.

This result will play a crucial role in our integration algorithms.

3.3 Summary of Coleman Integration Algorithms

When the variety under consideration has good reduction, Vologodsky and Berkovich–Coleman integration coincide with Coleman integration; see [Vol03, Section 1.18] and [Ber07, Remark 9.1.3 (ii)]. Moreover, there are practical algorithms for computing Coleman integrals on curves; here, we summarize these algorithms.

3.3.1 Hyperelliptic Curves

Algorithms for odd degree hyperelliptic curves were developed in Balakrishnan–Bradshaw–Kedlaya [BBK10]. By drawing on Kedlaya's algorithm for computing the zeta function of hyperelliptic curves [Ked01], one is able to choose an explicit lift of Frobenius ϕ and write down its action on Monsky–Washnitzer cohomology, a form of de Rham cohomology on affinoid spaces. For an open affinoid subset U of the underlying hyperelliptic curve, one considers $H^1_{\mathrm{dR}}(U)^-$, the odd subspace of $H^1_{\mathrm{dR}}(U)$, that is, the (-1)-eigenspace of the hyperelliptic involution. Given a basis $\omega_1, \dots, \omega_k$ of $H^1_{\mathrm{dR}}(U)^-$, one writes

$$\phi^*\omega_i = df_i + \sum_j M_{ij}\omega_j$$

for constants M_{ij} and meromorphic functions f_i. From knowledge of M_{ij}, f_i, and the action of ϕ on points R and S, one is able to solve for $\int_S^R \omega_i$. We note that Best [Bes19] has improved the complexity of the integration algorithms introduced in [BBK10].

The paper [BBK10] came with certain restrictions. One is only able to integrate meromorphic 1-forms whose poles are in residue discs around Weierstrass points. However, for some applications (e.g., computation of p-adic heights on curves as in [CG89]), it is necessary to perform more general integrals. Such an algorithm was provided in Balakrishnan–Besser [BB12] based on the theory of local symbols [Bes00b].

Another restriction in [BBK10] is that the authors only deal with odd degree models; this restriction is inherited from [Ked01] where it is used out of convenience. Building on [Har12], in which Harrison adapted Kedlaya's algorithm to the even degree case, Balakrishnan [Bal15] extended the integration algorithms in [BBK10] to even degree models of hyperelliptic curves.

Remark 3.3.2. Explicit methods in this thesis only deal with hyperelliptic curves; therefore, we will only make use of the integration algorithms described in [BBK10, BB12, Bal15]. A more detailed summary of these algorithms will be given in Subsection 4.4.2.

We also note that Balakrishnan [Bal13, Bal15] developed algorithms for computing double Coleman integrals on hyperelliptic curves.

3.3.3 Beyond Hyperelliptic Curves

Algorithms to carry out integration on more general curves were developed in Balakrishnan–Tuitman [BT20] based on the work of Tuitman [Tui16, Tui17] that generalizes Kedlaya's algorithm to this setting. We note that these algorithms work only for meromorphic 1-forms that are holomorphic away from *very bad* points (see [BT20, Definition 2.8]).

See also the work of Best [Bes21b] on Coleman integration on superelliptic curves.

Chapter 4

Explicit Abelian Integration for Hyperelliptic Curves

Our main goal here is to give an algorithm for computing Vologodsky integrals of holomorphic forms on bad reduction hyperelliptic curves. As discussed in Remark 3.1.2, Vologodsky integration of holomorphic forms is referred to as abelian integration, whence the name "abelian" in the chapter title. This chapter is joint work with Katz; it is the article [KK20] with some minor changes.

For a fixed hyperelliptic curve X with affine model $y^2 = f(x)$, the algorithm roughly proceeds in the following steps:

1. Based on a comparison formula, we reduce the problem to the computation of certain Berkovich–Coleman integrals on X^{an} (Section 4.2).

2. By viewing X as a double cover of the projective line $\mathbb{P}^1$ and examining the *roots* of $f(x)$, we construct a covering $\mathcal{D}$ of X^{an} by basic wide open spaces (Section 4.3).[4]

3. Using additivity of Berkovich–Coleman integrals under concatenation of paths, we reduce the computation of the integrals in Step (1) to the computation of various Coleman integrals on certain elements of $\mathcal{D}$. In order to compute these integrals, we first embed the elements of $\mathcal{D}$ of interest into good reduction hyperelliptic curves. Then, by a pole reduction argument, we rewrite the corresponding differentials in simplified forms. Finally, we make use of the known Coleman integration algorithms (Sections 4.4-4.5-4.6-4.7.).

[4]This is essentially equivalent to constructing a semistable model of X.

In this chapter, we assume that our field K has finite residue degree over $\mathbb{Q}_p$; as we will see, this finiteness assumption will be important. Unless otherwise noted, K will be a finite extension of $\mathbb{Q}_p$.

4.1 Abelian Integration

Abelian integration on a curve is defined via the p-adic Lie theory of its Jacobian. This was done in great generality by Zarhin [Zar96] and Colmez [Col98]. Recall that for all complete subfields of $\mathbb{C}_p$, we have the Vologodsky (hence abelian) integration theory. Following Katz–Rabinoff–Zureick-Brown [KRZB16], we restrict our attention to $\mathbb{C}_p$.

Let A be an abelian variety over $\mathbb{C}_p$. Recall that every regular 1-form on A is translation-invariant. In other words,

$$\Omega^1(A) = \Omega^1_{\text{inv}}(A).$$

This allows us to identify naturally $\Omega^1(A)$ with the dual of $\mathrm{Lie}(A)$; write $\langle \cdot\,, \cdot \rangle$ for the pairing between $\mathrm{Lie}(A)$ and $\Omega^1(A)$. The **abelian logarithm** on A is the unique homomorphism of $\mathbb{C}_p$-Lie groups

$$\log_{A(\mathbb{C}_p)} \colon A(\mathbb{C}_p) \to \mathrm{Lie}(A)$$

whose linearization

$$d\log_{A(\mathbb{C}_p)} \colon \mathrm{Lie}(A) \to \mathrm{Lie}(\mathrm{Lie}(A)) = \mathrm{Lie}(A)$$

is the identity map. See [Zar96] for the existence and uniqueness of $\log_{A(\mathbb{C}_p)}$. For $x \in A(\mathbb{C}_p)$ and $\omega \in \Omega^1(A)$, we define

$$^{\mathrm{Ab}}\!\!\int_0^x \omega = \langle \log_{A(\mathbb{C}_p)}(x), \omega \rangle.$$

For $x, y \in A(\mathbb{C}_p)$, we set

$$^{\mathrm{Ab}}\!\!\int_x^y \omega = {}^{\mathrm{Ab}}\!\!\int_0^y \omega - {}^{\mathrm{Ab}}\!\!\int_0^x \omega.$$

We call $^{\mathrm{Ab}}\!\!\int$ the **abelian integral** on A.

We may define an integration theory on a smooth, proper, connected curve X over $\mathbb{C}_p$ by pulling back the abelian integral from its Jacobian J by the Abel–Jacobi map $\iota\colon X \to J$ with respect to a base-point $x_0 \in X(\mathbb{C}_p)$.

See [KRZB16, KRZB18] for further details.

Remark 4.1.1. Because $\mathrm{Lie}(J)$ is torsion free, if x, y are points in $X(\mathbb{C}_p)$ such that $[y] - [x]$ represents a torsion point of $J(\mathbb{C}_p)$, then

$$^{\mathrm{Ab}}\!\!\int_x^y \omega = 0 \text{ for all } \omega \in \Omega^1(X).$$

We will observe this vanishing numerically to test the correctness of our algorithm.

4.2 Integral Comparison

In this section, we compare the Berkovich–Coleman and abelian integrals following [KRZB16].

4.2.1 Comparison on Abelian Varieties

Let A be an abelian variety over $\mathbb{C}_p$ and let $\pi\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$ be the topological universal cover of A^{an}. We have the **Raynaud uniformization cross**,

$$\begin{tikzcd} & T^{\mathrm{an}} \arrow[d] & \\ M' \arrow[r] & E^{\mathrm{an}} \arrow[r, "\pi"] \arrow[d] & A^{\mathrm{an}} \\ & B^{\mathrm{an}} & \end{tikzcd}$$

with exact row and column where M' is canonically isomorphic to $\pi_1(A^{\mathrm{an}}) = H_1(A^{\mathrm{an}}; \mathbb{Z})$, T is a torus and B is an abelian variety with good reduction. Let M be the character lattice of T, so $T = \mathrm{Spec}(\mathbb{C}_p[M])$.

Let $N = \mathrm{Hom}(M, \mathbb{Z})$. There is a surjective group homomorphism, the **tropicalization map**

$$\mathrm{trop}\colon E(\mathbb{C}_p) \to N_{\mathbb{Q}} = \mathrm{Hom}(M, \mathbb{Q}).$$

The restriction of trop to $M' \subset E(\mathbb{C}_p)$ is injective, and its image $\operatorname{trop}(M') \subset N_{\mathbb{Q}}$ is a full-rank lattice in the real vector space $N_{\mathbb{R}} = \operatorname{Hom}(M, \mathbb{R})$. We define the real torus

$$\Sigma = N_{\mathbb{R}}/\operatorname{trop}(M')$$

to be the **skeleton** of A. The tropicalization map $\bar{\tau}\colon A^{\mathrm{an}} \to \Sigma$ is defined as the quotient of trop and fits into the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\operatorname{trop}", "\simeq"'] & E^{\mathrm{an}} \arrow[r] \arrow[d, "\operatorname{trop}"] & A^{\mathrm{an}} \arrow[r] \arrow[d, "\bar{\tau}"] & 0 \\
0 \arrow[r] & \operatorname{trop}(M') \arrow[r] & N_{\mathbb{R}} \arrow[r] & \Sigma \arrow[r] & 0
\end{tikzcd}$$

The torus Σ is a deformation retract of A^{an}.

To compare the two integrals, we first define logarithms $\operatorname{Log}_{\mathrm{BC}}, \operatorname{Log}_{\mathrm{Ab}}$ from $E(\mathbb{C}_p)$ to $\operatorname{Lie}(E)$ as follows. Using the isomorphisms,

$$\operatorname{Lie}(E) \cong \Omega^1_{\mathrm{inv}}(E)^* \cong \Omega^1_{\mathrm{inv}}(A)^* \cong \Omega^1(A)^*,$$

we define

$$\begin{aligned} \operatorname{Log}_{\mathrm{BC}}\colon E(\mathbb{C}_p) &\to \operatorname{Lie}(E) \\ x &\mapsto \left[\omega \mapsto {}^{\mathrm{BC}}\!\!\int_0^x \omega\right] \end{aligned} \qquad \begin{aligned} \operatorname{Log}_{\mathrm{Ab}}\colon E(\mathbb{C}_p) &\to \operatorname{Lie}(E) \\ x &\mapsto \left[\omega \mapsto {}^{\mathrm{Ab}}\!\!\int_0^{\pi(x)} \omega\right] \end{aligned}$$

recalling that $\pi\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$ is the topological universal cover of A^{an}.

Proposition 4.2.2. ([KRZB16, Proposition 3.16]) *The difference between the two logarithms*

$$\operatorname{Log}_{\mathrm{BC}} - \operatorname{Log}_{\mathrm{Ab}}\colon E(\mathbb{C}_p) \to \operatorname{Lie}(E)$$

factors as

$$E(\mathbb{C}_p) \xrightarrow{\operatorname{trop}} N_{\mathbb{Q}} \xrightarrow{L} \operatorname{Lie}(E)$$

where L is a $\mathbb{Q}$-linear map.

Using the identification $H_1(A^{\mathrm{an}}; \mathbb{Z}) \cong M' \cong \operatorname{trop}(M')$ and the inclusion $\operatorname{trop}(M') \subset N_{\mathbb{Q}}$, we have the following:

Lemma 4.2.3. *The map L is characterized by the following property: for any $C \in H_1(A^{\mathrm{an}}; \mathbb{Z})$, one has*

$$L(C) = \left[\omega \mapsto {}^{\mathrm{BC}}\!\!\int_\gamma \omega\right]$$

where γ is any loop in $\mathcal{P}(A^{\mathrm{an}})$ whose homology class is equal to C.

Proof. Because the abelian logarithm is defined on $A(\mathbb{C}_p)$ (and not just on its universal cover $E(\mathbb{C}_p)$), we see that $\mathrm{Log}_{\mathrm{Ab}}(M') = 0$. Consequently, $L(C) = \mathrm{Log}_{\mathrm{BC}}(\tilde{\gamma}(1))$ where $\tilde{\gamma}$ is the lift of γ in E^{an} based at the identity element in $E^{\mathrm{an}}(\mathbb{C}_p)$. $\square$

4.2.4 Tropical Integration

We will need to pull back the comparison between integrals to a curve X via its Abel–Jacobi map $\iota\colon X \to J$. To do so, we will make use of the *tropical Abel–Jacobi map* which was described using tropical integration by Mikhalkin–Zharkov [MZ08] (see also [BR15, Section 3]) together with some results of Baker–Rabinoff [BR15]. The statement of the comparison result is different from that given in [KRZB16].

Let Γ be a finite connected graph (usually taken to be a graph structure on the skeleton of a curve X^{an}). We parameterize each oriented edge $e = vw$ by $[0,1]$ using the coordinate t such that v corresponds to $t = 0$ and w corresponds to $t = 1$. By flipping the orientation of the edge, we change the parameterization by $t' = 1 - t$. We take each edge of Γ to be of length 1.

Definition 4.2.5. A **tropical 1-form** on Γ is a function $\eta\colon E(\Gamma) \to \mathbb{R}$ such that

1. $\eta(-e) = -\eta(e)$, and
2. η satisfies the *harmonicity* condition: for each $v \in V(\Gamma)$,

$$\sum_e \eta(e) = 0$$

where the sum is over edges adjacent to v directed away from v.

Denote the space of tropical 1-forms on Γ by $\Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{R})$.

To an oriented edge $e = vw$ of Γ, let η_e be the function $E(\Gamma) \to \mathbb{R}$ that is 0 away from e and takes the value 1 on e with the given orientation (and -1 on $-e$). For a cycle $C = \sum_e a_e e \in H_1(\Gamma; \mathbb{R})$, define

$$\eta_C = \sum_e a_e \eta_e.$$

It is easily seen that η_C is a tropical 1-form.

Given a path γ specified as a sequence of directed edges $\gamma = e_1 e_2 \dots e_\ell$, we define the **tropical integral** of a tropical 1-form η on γ by

$$\int_\gamma^t \eta := \sum_{i=1}^{\ell} \eta(e_i).$$

Moreover, we may extend the tropical integral to paths between points on Γ. To a path between points p and q contained in an edge e, we define

$$\int_p^{t\,q} \eta := \eta(e)(q-p)$$

where we identify e with $[0,1]$ by use of the orientation on e. Then, we extend tropical integration to arbitrary paths by additivity of integrals under concatenation of paths.

For a closed path γ, this integral is seen to depend only on $[\gamma] \in H_1(\Gamma;\mathbb{R})$. Therefore, tropical integration gives a map

$$\begin{aligned} \mu\colon H_1(\Gamma;\mathbb{R}) &\to \Omega^1_{\mathrm{trop}}(\Gamma,\mathbb{R})^* \\ C &\mapsto \left[\eta \mapsto \int_C^t \eta\right]. \end{aligned}$$

Recall that the cycle pairing $\langle \cdot,\cdot \rangle$ on $H_1(\Gamma;\mathbb{R}) \subset C_1(\Gamma;\mathbb{R})$ is the pairing induced from the inner product on $C_1(\Gamma;\mathbb{R})$ making the set of edges (oriented in some way) into an orthonormal basis. In other words, this pairing takes cycles C and D to the length of their oriented intersection.

The following is easily verified.

Proposition 4.2.6. *Tropical integration is equal to the cycle pairing in the following sense: for $C, D \in H_1(\Gamma;\mathbb{R})$,*

$$\int_C^t \eta_D = \langle C, D\rangle.$$

This proposition implies that the map μ is an isomorphism because the cycle pairing is nondegenerate on $H_1(\Gamma;\mathbb{R})$.

Corollary 4.2.7. *Any cohomology class in $H^1(\Gamma;\mathbb{R})$ can be represented by a tropical 1-form: for any $c \in H^1(\Gamma;\mathbb{R})$, there is a tropical 1-form η such that*

$$c(D) = \int_D^t \eta$$

for any $D \in H_1(\Gamma;\mathbb{R})$.

Corollary 4.2.8. *There exists a basis $C_1, \ldots, C_h$ of $H_1(\Gamma; \mathbb{R})$ and a basis $\eta_1, \ldots, \eta_h$ of $\Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{R})$ such that*

$$\prescript{t}{}{\int_{C_i}} \eta_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

4.2.9 Comparison on Curves

Now, let X be a smooth, proper, connected curve over $\mathbb{C}_p$ with skeleton Γ. Note that Γ is a deformation retract of X^{an}. Let J be its Jacobian. We identify the real torus Σ from Γ.

Proposition 4.2.10. *There is an isomorphism of real tori*

$$\Sigma \cong \Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{R})^* / \mu(H_1(\Gamma; \mathbb{Z})).$$

Proof. Because the Abel–Jacobi map ι induces an isomorphism $\iota_* \colon H_1(X^{\mathrm{an}}; \mathbb{R}) \to H_1(J^{\mathrm{an}}; \mathbb{R})$, we have a sequence of isomorphisms,

$$N_{\mathbb{R}} \cong \mathrm{trop}(M') \otimes \mathbb{R} \cong H_1(J^{\mathrm{an}}; \mathbb{R}) \cong H_1(X^{\mathrm{an}}; \mathbb{R}) \cong H_1(\Gamma; \mathbb{R}) \cong \Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{R})^*.$$

Under the composition, $\mathrm{trop}(M')$ is mapped to $\mu(H_1(\Gamma; \mathbb{Z}))$. □

Let P_0 be a base-point of Γ. Now, if we let $\tilde{\Gamma}$ denote the universal cover of Γ with a base-point $\tilde{P}_0$ over P_0, tropical integration gives a map

$$\begin{aligned} \tilde{\beta} \colon \tilde{\Gamma} &\to \Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{R})^* \\ \tilde{Q} &\mapsto \left[\eta \mapsto \prescript{t}{}{\int_{\tilde{P}_0}^{\tilde{Q}}} \eta := \prescript{t}{}{\int_{\gamma}} \eta \right] \end{aligned}$$

where γ is the image in Γ of the unique path in $\tilde{\Gamma}$ from $\tilde{P}_0$ to $\tilde{Q}$. The map $\tilde{\beta}$ descends to quotients giving the **tropical Abel–Jacobi map**

$$\beta \colon \Gamma \to \Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{R})^* / \mu(H_1(\Gamma; \mathbb{Z})) \cong \Sigma.$$

The tropical Abel–Jacobi map is equal to the tropicalization of the Abel–Jacobi map in the following sense. Let $\iota \colon X \to J$ be the Abel–Jacobi map with respect to $x_0 \in X(\mathbb{C}_p)$. Let $\tau \colon X^{\mathrm{an}} \to \Gamma$ be the tropicalization map and set $P_0 = \tau(x_0)$.

By a result of Baker–Rabinoff [BR15, Proposition 6.1], the following diagram commutes:

$$\begin{tikzcd} X^{\mathrm{an}} \arrow[r, "\iota"] \arrow[d, "\tau"'] & J^{\mathrm{an}} \arrow[d, "\bar{\tau}"] \\ \Gamma \arrow[r, "\beta"'] & \Sigma. \end{tikzcd}$$

Now, we give a comparison theorem for Berkovich–Coleman and abelian integrals.

Theorem 4.2.11. *Let X^{an} be a connected, smooth, compact analytic curve over $\mathbb{C}_p$ with skeleton Γ and retraction $\tau\colon X^{\mathrm{an}} \to \Gamma$. Let $x_0 \in X(\mathbb{C}_p)$ be a base-point and set $P_0 = \tau(x_0)$. Let $C_1, \ldots, C_h$ and $\eta_1, \ldots, \eta_h$ be as in Corollary 4.2.8. Let $\gamma_1, \ldots, \gamma_h$ be loops in X^{an} whose homology classes are $C_1, \ldots, C_h$, respectively. The following formula holds: for $x \in X(\mathbb{C}_p)$, pick a path γ in X^{an} with $\gamma(0) = x_0$ and $\gamma(1) = x$, then*

$$\,^{\mathrm{BC}}\!\!\int_{\gamma} \omega - \,^{\mathrm{Ab}}\!\!\int_{x_0}^{x} \omega = \sum_i \left(\,^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega \right) \left(\,^{t}\!\!\int_{\tau(\gamma)} \eta_i \right)$$

for every holomorphic 1-form ω.

See Theorem 5.1.6 for a generalization of this theorem that applies to arbitrary meromorphic 1-forms.

Proof. Let $\widetilde{X^{\mathrm{an}}}$ be the topological universal cover of X^{an}. We have a commutative diagram

$$\begin{tikzcd} \widetilde{X^{\mathrm{an}}}(\mathbb{C}_p) \arrow[r, "\tilde{\iota}"] \arrow[d, "\tau"] & \widetilde{J^{\mathrm{an}}}(\mathbb{C}_p) = E(\mathbb{C}_p) \arrow[r, "\mathrm{Log}_{\mathrm{BC}} - \mathrm{Log}_{\mathrm{Ab}}"] \arrow[d, "\bar{\tau}"] & \mathrm{Lie}(E) \\ \tilde{\Gamma} \arrow[r, "\tilde{\beta}"] & \tilde{\Sigma} = N_{\mathbb{R}} \arrow[ur, "L"] & \end{tikzcd}$$

Now, consider the image of the lift $\tilde{\gamma}(1)$ of x under the maps in the top row, evaluated on ω. It suffices to show that $L \circ \tilde{\beta}\colon \tilde{\Gamma} \to \mathrm{Lie}(E) = \Omega^1(X^{\mathrm{an}})^*$ is given by

$$\tilde{Q} \mapsto \left[\omega \mapsto \sum_i \left(\,^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega \right) \left(\,^{t}\!\!\int_{\tilde{P}_0}^{\tilde{Q}} \eta_i \right) \right].$$

Under the identification $\Omega^1_{\mathrm{trop}}(\Gamma)^* \cong H_1(\Gamma;\mathbb{R})$, we claim that for $\tilde{Q} \in \tilde{\Gamma}$, we have

$$\tilde{\beta}(\tilde{Q}) = \sum_i \left({}^t\!\!\int_{\tilde{P}_0}^{\tilde{Q}} \eta_i \right) C_i.$$

This is true after evaluating by $\eta_j \in \Omega^1_{\mathrm{trop}}(\Gamma,\mathbb{R}) \cong H^1(\Gamma;\mathbb{R}) \cong H_1(\Sigma;\mathbb{R})^* \cong M \otimes \mathbb{R}$:

$$\eta_j(\tilde{\beta}(\tilde{Q})) = {}^t\!\!\int_{\tilde{P}_0}^{\tilde{Q}} \eta_j = \sum_i \left({}^t\!\!\int_{\tilde{P}_0}^{\tilde{Q}} \eta_i \right) \eta_j(C_i).$$

Because the η_j's form a basis for $\Omega^1_{\mathrm{trop}}(\Gamma,\mathbb{R})$, the claim follows.

Applying L, we see

$$\begin{aligned} L(\tilde{\beta}(\tilde{Q})) &= \left[\omega \mapsto \sum_i \left({}^t\!\!\int_{\tilde{P}}^{\tilde{Q}} \eta_i \right) \left(L(C_i)(\omega) \right) \right] \\ &= \left[\omega \mapsto \sum_i \left({}^t\!\!\int_{\tilde{P}}^{\tilde{Q}} \eta_i \right) \left({}^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega \right) \right] \end{aligned}$$

by Lemma 4.2.3. □

Remark 4.2.12. The integral ${}^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega$ is well-defined as γ_i is unique up to fixed endpoint homotopy.

Remark 4.2.13. The Abelian and Berkovich–Coleman integrals coincide when Γ is a tree, or equivalently, when J has good reduction; and, in particular, when X has good reduction.

4.3 Coverings of Curves

In our comparison theorem, we used a semistable model. Alternatively, we could use a *semistable covering*; an equivalent rigid analytic notion introduced in [CM88]. Surprisingly, constructing a semistable covering is often easier in practice than constructing a semistable model[5], and this is what we will do.

[5]This fact has been used by several people in order to determine semistable models of some modular curves, see [MC10, CM06, Tsu15, IT17, Wei16].

4.3.1 Semistable Coverings

Given a finite set of $\mathbb{C}_p$-points S on $\mathbb{P}^1$, we define a covering of $\mathbb{P}^{1,\mathrm{an}}$ by rational basic wide opens with respect to S. This allows us to define a covering of the hyperelliptic curve $y^2 = f(x)$ by hyperelliptic basic wide opens when we set S to be the set of *roots* of $f(x)$.

We follow [CM88, CdS88] in using the notion of semistable covering.

Definition 4.3.2. Let Y be a smooth, compact, connected analytic curve over $\mathbb{C}_p$. A **covering** $\mathcal{C}$ of Y is an admissible finite covering by distinct wide open subspaces of Y. The **dual graph** $\Gamma(\mathcal{C})$ of the covering is a finite graph whose vertices correspond to elements of $\mathcal{C}$ such that the edges between U and V correspond to components of $U \cap V$ while the self-edges at U correspond to ordinary double-points in the reduction of U. The covering is said to be **semistable** if, in addition,

1. If $U, V, W \in \mathcal{C}$ then U is disconnected from every component of $V \cap W$,
2. If $U \in \mathcal{C}$ then $U^\circ = U \setminus \bigcup_{V \neq U} V$ is a non-empty affinoid subdomain in U whose reduction $U^\circ_{\Bbbk}$ is absolutely irreducible, reduced, and has no singularities except ordinary double-points, and
3. The genus of Y obeys
$$g(Y) = \sum_{U \in \mathcal{C}} g(U^\circ_{\Bbbk}) + b_1(\Gamma(\mathcal{C}))$$
where $b_1(\Gamma(\mathcal{C}))$ is the first Betti number of $\Gamma(\mathcal{C})$.

We say that an element $U \in \mathcal{C}$ is **good** with respect to a subset $S \subset Y(\mathbb{C}_p)$ if there is an embedding into a compact good reduction curve, $\iota\colon U \to Y_U$ such that the points of $\iota(S \cap U(\mathbb{C}_p))$ lie in distinct residue discs. We say $\mathcal{C}$ is **good** with respect to S if each element of S belongs to at most one element of $\mathcal{C}$ and each $U \in \mathcal{C}$ is good with respect to S. The dual graph $\Gamma(\mathcal{C}, S)$ of the covering with respect to S is obtained from $\Gamma(\mathcal{C})$ by attaching half-open edges corresponding to elements of S to the vertices corresponding to the elements of $\mathcal{C}$ containing them.

4.3.3 Rational Coverings

We discuss the existence of good semistable coverings of $\mathbb{P}^{1,\mathrm{an}}$ with respect to a given set of points $S \subset \mathbb{P}^1(\mathbb{C}_p)$. If S contains only one element, then this is clear. From now on, let us assume that S has at least two elements.

Theorem 4.3.4. *Let $S \subset \mathbb{P}^1(\mathbb{C}_p)$ be a finite set. There is a good semistable covering $\mathcal{C}$ of $\mathbb{P}^{1,\mathrm{an}}$ with respect to S.*

We prove Theorem 4.3.4 by an inductive argument making use of Lemma 4.3.5. We produce a dual graph attached to the covering as we proceed. To do so, we introduce semistable coverings of open discs by rational basic wide opens. They are defined as in Definition 4.3.2 except condition (3) is replaced by the condition

$$g(Y) = \sum_{U \in \mathcal{C}} g(U) + b_1(\Gamma(\mathcal{C}))$$

where $g(U) = 0$ for a rational basic wide open U. This mandates that $\Gamma(\mathcal{C})$ is a tree. The embeddings $\iota\colon U \to \mathbb{P}^{1,\mathrm{an}}$ will be linear fractional transformations.

Lemma 4.3.5. *Let $R \in \|\mathbb{C}_p^*\|_p$ and $\beta \in \mathbb{A}^1(\mathbb{C}_p)$. Set $Y = B(\beta, R)$. For any non-empty finite subset S of $Y(\mathbb{C}_p)$, there is a good semistable covering $\mathcal{C}_Y$ of Y with respect to S. The dual graph of the covering respect to S is a rooted tree T_Y.*

Proof. Write $S = \{\alpha_1, \dots, \alpha_n\}$. By translating by $-\alpha_1$, we may assume that $\alpha_1 = 0$. If the set S has at least two elements, by scaling, we may assume that $\max_{i \neq 1}(\|\alpha_i\|_p) = 1$.

We induct on n. If $n \leq 2$, then all the points of S are in distinct residue discs, and we may let $\mathcal{C}_Y = \{Y\}$. The tree attached to this covering is a single vertex with a half-open edge for each element of S.

Let $n > 2$. Not all elements of S are in a single residue disc. Let $I_1, \dots, I_m$ be the partition of S according to which residue disc a point belongs. For each i such that $|I_i| \geq 2$ pick a point $\beta_i \in I_i$. Let R_i be the largest element of $\|\mathbb{C}_p^*\|_p$ such that $B(\beta_i, R_i) \cap S = I_i$ (so that $\overline{B}(\beta_i, R_i)$ contains some point of $S \setminus I_i$). Set $Y_i = B(\beta_i, R_i)$ and $S_i = I_i$. Because $|I_i| < n$, Y_i has a good semistable covering $\mathcal{C}_{Y_i}$ with respect to S_i. Now, let

$$U = Y \setminus \bigsqcup_{i : |I_i| \geq 2} \overline{B}(\beta_i, r_i)$$

where r_i is the smallest element of $\|\mathbb{C}_p^*\|_p$ such that $\overline{B}(\beta_i, r_i) \cap S = I_i$. The covering $\mathcal{C}_Y$ is defined as

$$\{U\} \cup \bigcup_{i : |I_i| \geq 2} \mathcal{C}_{Y_i}.$$

Because there is at most one element of S in every residue disc of U, U is good with respect to S. Moreover, since each element of S is either contained in U or in exactly one element of the covering $\mathcal{C}_{Y_i}$ for some i, the covering $\mathcal{C}_Y$ is good with respect to S.

Denote the rooted tree corresponding to the covering $\mathcal{C}_{Y_i}$ by T_{Y_i}. Consider the tree whose root is U and where U is connected to the roots of T_{Y_i} for each i with $|I_i| \geq 2$. To obtain T_Y from this tree, attach to U half-open edges corresponding to I_i with $|I_i| = 1$. These half-open edges correspond to the points of S that are contained in U. □

Proof of Theorem 4.3.4. Write $S = \{\alpha_1, \ldots, \alpha_n\}$. Let S' be the elements of S contained in $\mathbb{A}^1(\mathbb{C}_p)$, and let r be the maximum of their p-adic absolute values. Pick $R \in \|\mathbb{C}_p^*\|_p$ with $R > r$ and set $Y = B(0, R)$. Using Lemma 4.3.5, find a good semistable covering $\mathcal{C}_Y$ of Y with respect to S' and its rooted tree T_Y. This covering together with $U = \mathbb{P}^{1,\mathrm{an}} \setminus \overline{B}(0, r)$ is our desired covering $\mathcal{C}$.

The dual graph $\Gamma(\mathcal{C}, S)$ is obtained by adjoining the vertex corresponding to U to the root of T_Y and then attaching the half-open edge corresponding to ∞ to U if $\infty \in S$. □

One can see from general considerations or by examining the above construction that the intersection of two distinct elements of the semistable cover is either empty or an annulus.

Notice that the dual graph $\Gamma(\mathcal{C}, S)$ is also a tree. From now on, we denote this tree by T.

Remark 4.3.6. The covering and graph can be constructed intrinsically using Berkovich spaces [BPR13]. The tree T consists of the type II and type III points of $\mathbb{P}^{1,\mathrm{an}}$ corresponding to discs of the form $\overline{B}(\alpha, r)$ for $\alpha \in \mathbb{P}^1(\mathbb{C}_p)$ such that at least one point of S is contained in each of $\overline{B}(\alpha, r)$ and $\mathbb{P}^{1,\mathrm{an}} \setminus B(\alpha, r)$.

The covering can be obtained by taking a semistable vertex set as in [BPR13]. It consists of the type II points of the form $\overline{B}(\beta_i, r_i)$ as constructed in the above algorithm.

Lemma 4.3.7. *The dual graph T is a graph structure on the skeleton of* $\mathbb{P}^{1,\mathrm{an}} \setminus S$.

Remark 4.3.8. The above lemma can be proved in several ways. One can use the semistable vertex set to identify the skeleton as in [BPR13, Section 3]. Alternatively, one can construct a semistable model from the semistable covering [CM88, Theorem 1.2] and obtain the skeleton by [BPR13, Section 4].

Let $f(x)$ be a non-constant polynomial with coefficients in K. We define the **roots** of $f(x)$ to be the usual zeroes of $f(x)$ together with ∞ if $f(x)$ has odd degree and write the set of roots as S_f; these are the usual roots of the degree $2g+2$ homogenization of $f(x)$. Lemma 4.3.5 and Theorem 4.3.4 can be turned into an algorithm for constructing a good covering of $\mathbb{P}^{1,\mathrm{an}}$ with respect to S_f. A priori, it looks as it would be necessary to solve exactly for the roots of $f(x)$. However, this can be avoided. First, we approximate a root and use a translation to put it in $B(0,1)$. Then, we find the valuation of the roots by using the theory of Newton polygons. We rescale x by an element of $\mathbb{C}_p$ to make sure that the largest absolute value of the roots is 1. Indeed, the polynomial $f(x)$ has a root of valuation s if and only if its Newton polygon has a segment of slope $-s$. This segment corresponds to roots of p-adic absolute values equal to p^{-s}. After these reduction steps, the methods in the proof of Lemma 4.3.5 are still applicable. Algorithm 1 produces a good semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ with respect to the roots of $f(x)$, by following the proof of Theorem 4.3.4.

4.3.9 Hyperelliptic Coverings

Let $\pi : X^{\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$ be the analytification of the proper hyperelliptic curve defined by $y^2 = f(x)$. Using ideas similar to those of Stoll in [Sto19], we will show that any good semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ with respect to the roots of $f(x)$ induces a good semistable covering of X^{an} with respect to the Weierstrass points by taking inverse images. This covering will have a nice combinatorial structure whose dual graph Γ is a double cover of the dual graph T of the covering of $\mathbb{P}^{1,\mathrm{an}}$.

Write $S_f = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Recall that, if $f(x)$ is of odd degree, we follow the convention of counting ∞ as one of its roots. In any case, $f(x)$ has an even number of roots. We will need the following observation:

Lemma 4.3.10. *Let $R \in \|K^*\|_p$. Let $h(x) \in K[x]$ be a polynomial. Let $D_1, \ldots, D_m$ be disjoint closed discs in $\overline{B}(0,R)$ the union of whose interior contains $S_h \cap \overline{B}(0,R)$. Suppose that an even number of roots of $h(x)$ is contained in each D_i. Then $h(x)$ has an analytic square root on $B(0,R) \setminus \left(\bigcup_i D_i\right)$.*

Proof. We work with one disc of $D_1, \ldots, D_m$ at a time, beginning with a disc D. Write $D = \overline{B}(\gamma, r)$. Let $\alpha_1, \ldots, \alpha_{2\ell}$ be the roots of $h(x)$ contained in D. Then,

$$\left(\prod_{i=1}^{2\ell}(x-\alpha_i)\right)^{1/2} = (x-\gamma)^{\ell} \prod_{i=1}^{2\ell}\left(1 - \frac{\alpha_i - \gamma}{x-\gamma}\right)^{1/2}$$

Algorithm 1: Covering of $B(0,R)$ with respect to the roots of a polynomial

Input:

- A polynomial $f \in K[x]$ of degree at least 2 whose finite roots are in K.
- A disc $B(0,R)$.

Output: A good semistable covering $\mathcal{C}$ of $B(0,R)$ with respect to the roots of $f(x)$ contained in $B(0,R)$ together with its rooted tree T.

1. Find $\alpha \in B(0,R)$ sufficiently close to a root of $f(x)$ such that the polynomial $f(x-\alpha)$ has roots of at least two different p-adic absolute values in $B(0,R)$ (this can be verified by checking that the Newton polygon of $f(x-\alpha)$ has at least two segments of distinct slope less than or equal to $\log_p(R)$). Replace $f(x)$ by $f(x-\alpha)$.

2. Compute the slopes of the Newton polygon of $f(x)$ and set λ to be the maximum of the slopes less than or equal to $\log_p(R)$.

3. Pick $c \in K$ satisfying $v_p(c) = -\lambda$. Replace $f(x)$ by $f(cx)$ and R by $\frac{R}{p^\lambda}$ so that the maximum of p-adic absolute value of roots in the disc is 1.

4. Multiply $f(x)$ by a power of a uniformizer of K to ensure that the minimum of the valuation of the coefficients of $f(x)$ is 0. Factor the polynomial $f(x)$ mod p to determine the partition $\{I_i\}$ of the set of roots of $f(x)$ in $B(0,R)$ according to which residue disc the root belongs.

5. For each i with $|I_i| \geq 2$, pick a point β_i in the same residue disc as the points in I_i. Set $f_i(x) = f(x-\beta_i)$. Let λ_i be the largest negative slope of the Newton polygon of f_i, and let Λ_i be the smallest positive slope of the Newton polygon of f_i. Set $r_i = p^{\lambda_i}$ and $R_i = p^{\Lambda_i}$. Apply this algorithm to $f_i(x)$ and the disc $B(0,R_i)$ to find a good covering $\mathcal{C}_i$ of $B(\beta_i, R_i)$ with respect to I_i together with its rooted tree T_i.

6. Set $U = B(0,R) \setminus \bigcup_{i:|I_i|\geq 2} \overline{B}(\beta_i, r_i)$. Combine U with the coverings $\mathcal{C}_i$ found in the previous step to obtain the covering $\mathcal{C}$ of $B(0,R)$.

7. Let U be the root of the tree T; for each i, do the following: if $|I_i| \geq 2$, attach the root of T_i to U by an edge; if $|I_i| = 1$, attach a half-open edge corresponding to the unique element of I_i.

8. Return $\mathcal{C}$ and T.

converges away from D. Now, if $\alpha_1, \dots, \alpha_\ell \in \mathbb{C}_p$ are the roots of $h(x)$ in $\mathbb{A}^{1,\mathrm{an}} \setminus \overline{B}(0, R)$, then

$$\left(\prod_{i=1}^{\ell}(x - \alpha_i)\right)^{1/2} = \prod_{i=1}^{\ell}\left((-\alpha_i)^{1/2}\left(1 - \frac{x}{\alpha_i}\right)^{1/2}\right)$$

converges on $B(0, R)$. By multiplying these functions, we get the desired square root of $h(x)$. $\square$

Proposition 4.3.11. *Let $U \subset \mathbb{P}^{1,\mathrm{an}}$ be a rational basic wide open that is good with respect to S_f, then $\pi^{-1}(U)$ is the union of at most two basic wide opens.*

Proof. We will find a new coordinate $\tilde{y}$ and a polynomial $g(x)$ whose roots lie in distinct residue discs such that

$$\pi^{-1}(U) \simeq \{(x, \tilde{y}) \mid x \in U, \tilde{y}^2 = g(x)\}.$$

We view U as a subset of $\mathbb{P}^{1,\mathrm{an}}$ where we have made a fractional linear transformation to ensure that the roots of $f(x)$ contained in U are in distinct residue discs. We can suppose that either $U = \mathbb{P}^{1,\mathrm{an}}$ or that $U \subset \overline{B}(0, R)$ for some $R \in \|K^*\|_p$ with $R \geq 1$. If $U = \mathbb{P}^{1,\mathrm{an}}$, then $\pi^{-1}(U) = \tilde{X}$ is a good reduction curve. Otherwise, write $D_1, \dots, D_m$ for the closed discs contained in $\overline{B}(0, R)$ (each contained in a distinct residue disc) in which $f(x)$ has multiple roots in their interior.

We factor $f(x) = g(x)h(x)$ where

1. the polynomial $g(x)$ only has roots in $\overline{B}(0, R)$,
2. the polynomial $g(x)$ has at most one root in each residue disc, and
3. the polynomial $h(x)$ has an even number of roots in each D_i.

Let $\ell(x)$ be a square root of $h(x)$ on U which exists by Lemma 4.3.10, and set $\tilde{y} = \frac{y}{\ell(x)}$. Because $\ell(x)$ is non-vanishing on U, the map

$$(x, y) \mapsto (x, \tilde{y})$$

is invertible for x in U hence gives the desired isomorphism.

We now consider the complete curve $\tilde{\pi} \colon \tilde{X} \to \mathbb{P}^1$ given by compactifying $\tilde{y}^2 = g(x)$. Because the roots of $g(x)$ lie in distinct residue discs, $\tilde{X}$ has a smooth model over $\mathcal{O}_K$, the valuation ring of some field K. Since $\tilde{X}$ has good reduction, we need only show that $\pi^{-1}(U)$ is a basic wide open. If $g(x)$ is of degree 0, $\tilde{X}$ is the union of two copies of $\mathbb{P}^1$. In this case, $\pi^{-1}(U)$ is isomorphic to two copies of U. If $g(x)$ is of positive degree, we must identify $\pi^{-1}(U)$.

Write $U = \mathbb{P}^{1,\mathrm{an}} \setminus \left(\bigcup_{i=1}^m D_i \cup D_\infty\right)$ for closed discs $D_1, \ldots, D_m$, each contained in a residue disc where D_∞ is a disc of the form $\mathbb{P}^{1,\mathrm{an}} \setminus B(0, R')$ for some $R' > 1$. We need to identify $\tilde{\pi}^{-1}(D_i)$. As we discussed, the polynomial $g(x)$ has at most one root in each D_i. Consider the case where $g(x)$ has no roots in D_i. Then $g(x)$ has an analytic square root on D_i, and $\tilde{\pi}^{-1}(D_i)$ is the union of two disjoint closed discs, each isomorphic to D_i. Now, consider the case where $g(x)$ has exactly one root in D_i. By a fractional linear transformation, we may suppose that $g(x) = x$. Then the closed disc D_i is of the form $\{x \mid \|x\|_p \leq r\}$ for some $r \in G$. Consequently,

$$\pi^{-1}(D_i) = \{(x, \tilde{y}) \mid x \in D_i, \tilde{y}^2 = x\} = \{\tilde{y} \mid \|y\|_p \leq r^{1/2}\}$$

is a closed disc. This lies in a residue disc in the model over $\mathcal{O}_K$. A similar argument applies to D_∞. It follows that

$$\pi^{-1}(U) = \tilde{X} \setminus \left(\bigcup_{i=1}^m \tilde{\pi}^{-1}(D_i) \cup \pi^{-1}(D_\infty)\right)$$

is a basic wide open. $\square$

Observe that in the above, $\tilde{X}$ has either one or two components according to where $g(x)$ is degree 0 or not. We immediately see that $\pi^{-1}(U)$ is disconnected exactly when $f(x)$ has no roots in U and has an even number of roots in each deleted disc D_i. In this case, we say that U is **even**. Otherwise, we say that it is **odd**.

The double cover of an annulus (such as one arising as a component of the intersection of two elements of a semistable covering) given by $y^2 = f(x)$ is the following.

Lemma 4.3.12. *Let A be an annulus in $\mathbb{A}^{1,\mathrm{an}} \subset \mathbb{P}^{1,\mathrm{an}}$. Suppose that S_f is disjoint from A. Then $\pi^{-1}(A) = \{(x, y) \mid x \in A, y^2 = f(x)\}$ is*

1. *the union of two disjoint annuli if S_f has an even number of elements in each component of $\mathbb{P}^{1,\mathrm{an}} \setminus A$,*

2. *an annulus if S_f has an odd number of elements in each component of $\mathbb{P}^{1,\mathrm{an}} \setminus A$.*

Proof. By a fractional linear transformation, we may reduce to the case where $A = \{x \mid 1 < \|x\|_p < r\}$ for some $r > 1$. As in the proof of Proposition 4.3.11, we can reduce to the case where $\pi^{-1}(A) = \{(x, \tilde{y}) \mid \tilde{y}^2 = g(x), x \in A\}$ where $g(x)$ is of degree at most 1. If $g(x)$ is of degree 0, then we are in case (1). If $g(x)$ is of degree 1, we can reduce to the case where $g(x) = x$. Then, $\pi^{-1}(A)$ is given by $\{\tilde{y} \in \mathbb{A}^{1,\mathrm{an}} \mid 1 < \|y\|_p < r^{1/2}\}$, and we are in case (2). □

We refer to an annulus A as **even** or **odd** according to whether $\pi^{-1}(A)$ is disconnected or connected.

If $\mathcal{C}$ is a semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ that is good with respect to S_f, we can produce a semistable covering $\mathcal{D}$ of X^{an} that is good with respect to the set of Weierstrass points W. We let $\mathcal{D}$ be the set of components of $\pi^{-1}(U)$ as U ranges over elements of $\mathcal{C}$. In the case that $U \in \mathcal{C}$ is even, $\pi^{-1}(U)$ have two components which gives two elements of the covering of X^{an}. Let Y_1 and Y_2 be two distinct elements of $\mathcal{D}$; put $U_i = \pi(Y_i)$ for $i = 1, 2$. There are three possibilities for the intersection $Y_1 \cap Y_2$; it is

1. empty if $U_1 = U_2$ or $U_1 \cap U_2$ is empty,
2. an annulus if $U_1 \cap U_2$ is an odd annulus,
3. the union of two disjoint annuli if $U_1 \cap U_2$ is an even annulus.

Let $\Gamma(\mathcal{D}, W)$ be the dual graph of the covering $\mathcal{D}$ with respect to W as in Definition 4.3.2. We give a description of $\Gamma(\mathcal{D}, W)$ in terms of T similar to [Sto19, Section 6]. The dual graph has both closed edges and half-open edges. Unless noted otherwise, edges are taken to be closed. We first designate half-open edges, edges and vertices of T as even or odd. All half-open edges of T are odd. An edge of T is even exactly when the corresponding annulus is even. A vertex of T is even exactly when all of its adjacent edges are even. For a vertex v, its **genus** is the integer $g(v)$ satisfying

$$2g(v) - 2 = -4 + n_o(v)$$

where $n_o(v)$ are the number of odd edges (including half-open edges) adjacent to v. Observe that even vertices have genus equal to -1 (corresponding to a disjoint union of two $\mathbb{P}^1$'s). By the Riemann–Hurwitz formula, $g(v)$ is the genus of the good reduction curve in which $\pi^{-1}(U)$ is embedded.

Definition 4.3.13. To T, we attach a graph Γ. Let Γ be the graph whose vertex set consists of

1. one vertex $\tilde{v}$ for each odd vertex v of T; and
2. two vertices $\tilde{v}_+$, $\tilde{v}_-$ for each even vertex v of T

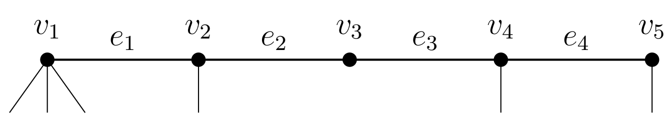
whose edge set is

1. one edge $\tilde{e}$ for each odd edge e of T;
2. two edges $\tilde{e}_+, \tilde{e}_-$ for each even edge e of T; and
3. one half-open edge $\tilde{e}$ for each half-open edge e of T.

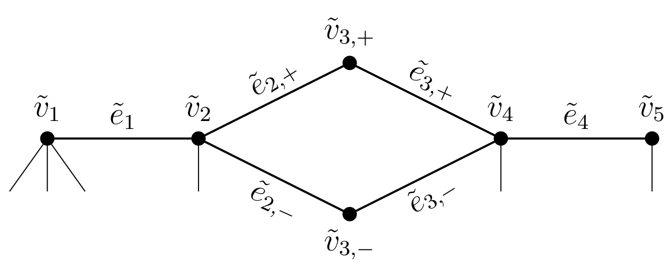
For each adjacent pair (v, e) of T with v and e odd, we declare $\tilde{v}$ and $\tilde{e}$ adjacent. If v is odd and e is even, we declare $\tilde{v}$ and $\tilde{e}_\sigma$ adjacent for $\sigma = +, -$. If v and e are even, we declare $\tilde{v}_\sigma$ and $\tilde{e}_\sigma$ adjacent for $\sigma = +, -$. Because a half-open edge e is only attached to an odd vertex v of T, the corresponding half-open edge $\tilde{e}$ is attached to the vertex $\tilde{v}$ of Γ.

There is a natural map $\pi \colon \Gamma \to T$ taking $\tilde{v}$ or $\tilde{v}_+, \tilde{v}_-$ to v and $\tilde{e}$ or $\tilde{e}_+, \tilde{e}_-$ to e.

Example 4.3.14. As an illustration, consider the following tree T:



The edges e_2, e_3 and the vertex v_3 are even; all the others are odd. Here is the corresponding graph Γ:



By unwinding the description of Γ, we have the following proposition (from which one sees that Γ is a graph structure on the skeleton of $X^{\mathrm{an}} \setminus W$ by reasoning identical to that of Remark 4.3.8):

Proposition 4.3.15. *The dual graph* $\Gamma(\mathcal{D}, W)$ *is equal to* Γ.

Let T_e be the union of even edges of T and let V_o be the odd vertices of T that are adjacent to even edges. We will describe the first homology group of Γ in terms of the relative homology group $H_1(T_e, V_o; \mathbb{R})$ which is given as the kernel of the map

$$\partial \colon C_1(T_e; \mathbb{R}) \to C_0(T_e; \mathbb{R})/C_0(V_o; \mathbb{R}).$$

Define a map $\iota \colon C_1(T_e; \mathbb{R}) \to C_1(\Gamma; \mathbb{R})$ by $e \mapsto \tilde{e}_+ - \tilde{e}_-$.

Proposition 4.3.16. *The map* ι *induces an isomorphism* $\iota \colon H_1(T_e, V_o; \mathbb{R}) \to H_1(\Gamma; \mathbb{R})$.

Proof. Define a map $\kappa \colon C_1(\Gamma; \mathbb{R}) \to C_1(T_e; \mathbb{R})$ by

$$\kappa(\tilde{e}) = 0, \ \kappa(\tilde{e}_+) = e, \ \kappa(\tilde{e}_-) = 0.$$

We first show that κ maps $H_1(\Gamma; \mathbb{R})$ to $H_1(T_e, V_o; \mathbb{R})$. Let $C \in H_1(\Gamma; \mathbb{R})$. For an even vertex v, π is an simplicial homeomorphism of the open star of $\tilde{v}_+$ onto its image. Because the coefficient of $\tilde{v}_+$ in ∂C is zero, the coefficient of v in $\partial(\kappa(C))$ is also zero. Consequently, we have $\kappa(C) \in H_1(T_e, V_o; \mathbb{R})$.

Now, we claim that κ and ι, considered as maps between $H_1(\Gamma; \mathbb{R})$ and $H_1(T_e, V_o; \mathbb{R})$, are inverses of one another. Clearly $\kappa \circ \iota$ is the identity. We claim $\iota \circ \kappa$ is the identity. If $C = \sum_e a_e e$ is a cycle in Γ, then $\pi_*(C) = 0$ in $H_1(T; \mathbb{R})$. The only way that this can occur is if $a_{\tilde{e}} = 0$ for all edges $\tilde{e}$ and $a_{\tilde{e}_+} = -a_{\tilde{e}_-}$ for all pairs $(\tilde{e}_+, \tilde{e}_-)$ above an even edge e. From this we conclude that $C = \iota(\kappa(C))$. $\square$

The following relation between the cycle pairing on (T_e, V_o) and that on Γ is straightforward:

Proposition 4.3.17. *Let* $C, D \in H_1(T_e, V_o; \mathbb{R})$. *Then* $\langle \iota(C), \iota(D) \rangle = 2\langle C, D \rangle$.

We conclude this section with a discussion on what makes our coverings (or rather, our models) special. This is something to wonder about, since there are several other methods in the literature to determine semistable models of hyperelliptic curves; see, for example, the recent works of Dokchitser–Dokchitser–Maistret–Morgan [DDMM19, DDMM18] based on the notion of a *cluster picture*. In Proposition 4.3.11, we have worked with reciprocals of infinite series in order to pass to good reduction curves (this idea will play a significant role in what follows). For a "generic" semistable model, these series can be divergent and/or may take the value zero, but, in such a case, our method certainly fails; our models are constructed in such a way that these issues do not arise. Nevertheless, it would be interesting to compare our method with other methods, especially with the cluster picture approach, since semistable models of curves and their dual graphs play a central role in algebraic geometry and in various other fields.

4.4 Integrals on Hyperelliptic Basic Wide Opens

4.4.1 1-forms on Hyperelliptic Basic Wide Opens

Let X be a hyperelliptic curve defined by $y^2 = f(x)$. In Subsection 4.3.9, we explained how to construct a semistable covering of X^{an} by hyperelliptic basic wide opens. In this section, we summarize Berkovich–Coleman integration algorithms on these spaces. We note that these are ordinary Coleman integrals; in particular, they are path-independent.

We fix a covering as above and consider an element Y of this covering. Let ω be an odd holomorphic 1-form on Y. Recall that odd means that the hyperelliptic involution acts on ω as multiplication by -1.

If the space Y is isomorphic to the standard open disc (resp. a standard open annulus) with parameter t, then ω pulls back as $F(t)dt$ where $F(t)$ is a power (resp. Laurent) series. In this case one can compute the integral by antidifferentiating.

For other spaces, in order to make use of the existing explicit methods, we need to pass to a good reduction curve. By the proof of Proposition 4.3.11, the space Y is isomorphic to a basic wide open space Z inside the good reduction curve $\tilde{X}^{\mathrm{an}}$ given by $y^2 = g(x)$ for some polynomial $g(x)$ of degree d. Note that if d is odd, then $d = 2g + 1$, and if d is even, $d = 2g + 2$ where g is the genus of $\tilde{X}$. We will suppose that $g(x) \in K[x]$ for some finite extension K of $\mathbb{Q}_p$. As we will discuss in Section 4.6, the form ω pulls back to Z as an odd 1-form that can be expressed as a convergent series of odd 1-forms. By Proposition 3.2.5, we can interchange the order of summation and integration. Thus we need to integrate terms in this series. Let η denote such a term. Using the change-of-variables property for Berkovich–Coleman integrals, it suffices to compute the integral of η on Z. On the other hand, we will see that the form η extends to $\tilde{X}$ as a meromorphic form with poles outside of Z and by Remark 3.2.2 we can perform this integral on the complete curve $\tilde{X}$.

If we write $Z = \tilde{X}^{\mathrm{an}} \setminus (\bigcup_{i=1}^{r} D'_i)$ for closed discs $D'_1, \ldots, D'_r$ (which arise as preimages of discs closed in $\mathbb{P}^{1,\mathrm{an}}$), then by [Col89, Propositions 4.3, 4.4] (see also the discussion in [KRZB16, Theorem 2.24]) the sequence

$$0 \to H^1_{\mathrm{dR}}(\tilde{X}) \to H^1_{\mathrm{dR}}(Z) \xrightarrow{\bigoplus \mathrm{Res}} \bigoplus_{i=1}^{r} \mathbb{C}_p \xrightarrow{\Sigma} \mathbb{C}_p \to 0$$

is exact where Res takes the residue around the D_i''s and Σ is summation. If we let the superscript "$-$" denote the (-1)-eigenspace of the maps induced by the hyperelliptic involution, we have the short exact sequence

$$0 \to H^1_{\mathrm{dR}}(\tilde{X})^- \to H^1_{\mathrm{dR}}(Z)^- \xrightarrow{\bigoplus \mathrm{Res}} \left(\bigoplus_{i=1}^{r} \mathbb{C}_p\right)^- \to 0 \tag{4.1}$$

which says that in order to obtain a spanning set for the odd part of the first de Rham cohomology of Z, we only need to adjoin 1-forms with poles in the D_i''s to a basis for the de Rham cohomology of $\tilde{X}$. Here the hyperelliptic involution acting on the last factor exchanges the residues around hyperelliptically conjugate discs and acts as the identity on residues around discs containing a Weierstrass point.

We now consider the case where $d = \deg(g(x)) \geq 3$ so that the curve $\tilde{X}$ is hyperelliptic. Extend the field K so that it contains the roots of $g(x)$.

By our construction, the D_i''s arise as components of the preimages under $\pi\colon \tilde{X}^{\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$ of some closed discs $D_1, \dots, D_n$ in $\mathbb{A}^{1,\mathrm{an}}$, each contained in a distinct residue disc and possibly also of a disc D_∞ around ∞. Such a disc is called **Weierstrass** if it contains a root of $g(x)$. Suppose that we have ordered the discs such that $D_1, \dots, D_k$ are the non-Weierstrass discs and $D_{k+1}, \dots, D_n$ are the Weierstrass discs. Observe that D_∞ is Weierstrass if and only if $g(x)$ is of odd degree.

Let $\beta_1, \dots, \beta_n$ be elements of $\mathbb{P}^1(K)$ contained in $D_1, \dots, D_n$. We choose β_i to be a root of $g(x)$ if D_i is a Weierstrass disc contained in $\mathbb{A}^{1,\mathrm{an}}$. For D_∞, choose $\beta_\infty = \infty$. Define the forms

$$\left\{\nu_j = \frac{dx}{(x-\beta_j)2y}\right\}_{j=1,\dots,k}$$

where the form ν_j has simple poles at the hyperelliptically conjugate points $\pi^{-1}(\beta_j)$.

For an integer i, define the 1-form

$$\omega_i = x^i \frac{dx}{2y}.$$

In both the odd and even degree cases, $\{\omega_0, \dots, \omega_{d-2}, \nu_1, \dots, \nu_k\}$ forms a spanning set for $H^1_{\mathrm{dR}}(Z)^-$. Consequently,

$$\eta = dF + \sum_{i=0}^{d-2} c_i \omega_i + \sum_{j=1}^{k} d_j \nu_j$$

holds for an analytic function F on Z and $c_i, d_j \in K$. For points $R, S \in Z(\mathbb{C}_p)$, the equality above gives

$$^{\mathrm{BC}}\!\!\int_S^R \eta = F(R) - F(S) + \sum_{i=0}^{d-2} c_i \,^{\mathrm{BC}}\!\!\int_S^R \omega_i + \sum_{j=1}^{k} d_j \,^{\mathrm{BC}}\!\!\int_S^R \nu_j.$$

Below, we explain how to compute the integrals on the right.

4.4.2 Summary of Integration Algorithms

We first state the algorithms when $g(x)$ is of odd degree (the case where the algorithms are most fully developed). There is partial work in the even degree case, and one can apply a fractional linear transformation to $\mathbb{P}^1$ to transform the even degree case to the odd degree case.

We start with the integrals $^{\mathrm{BC}}\!\!\int_S^R \omega_i$. The paper [BBK10] describes a method for computing Coleman integrals of those meromorphic forms whose poles all belong to Weierstrass residue discs.

If the points R and S lie in the same residue disc, in which case we refer to the integral as a *tiny integral*, we may use the following lemma.

Lemma 4.4.3. ([BBK10, Algorithm 8]) *For points $R, S \in \tilde{X}(\mathbb{C}_p)$ in the same residue disc, neither equal to the point at infinity, we have*

$$^{\mathrm{BC}}\!\!\int_S^R \omega_i = \int_0^1 \frac{x(t)^i}{2y(t)} \frac{dx(t)}{dt} dt$$

where $(x(t), y(t))$ is a linear interpolation from S to R in terms of a local coordinate t. We can similarly integrate any form that is holomorphic in the residue disc containing the endpoints.

If the points R and S lie in distinct non-Weierstrass residue discs, the method of tiny integrals is not available. Coleman's idea was to extend the notion of integration by *analytic continuation along Frobenius*. Let ϕ be the lift of Frobenius constructed in [BBK10, Algorithm 10]. This map is rigid analytic; moreover it maps a $\mathbb{Q}_p$-point into its residue disc. By the change-of-variables formula with respect to ϕ, we have the following theorem.

Theorem 4.4.4. ([BBK10, Algorithm 10, Algorithm 11, Remark 13]) *Let M denote the matrix over K such that*

$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j \tag{4.2}$$

for all $i = 0, 1, \ldots, 2g-1$. Then, for points $R, S \in \tilde{X}(\mathbb{Q}_p)$ in distinct non-Weierstrass residue discs, we have the equality

$$\sum_{j=0}^{2g-1}(M-I)_{ij} \,{}^{\mathrm{BC}}\!\!\int_S^R \omega_j = f_i(S) - f_i(R) - {}^{\mathrm{BC}}\!\!\int_S^{\phi(S)} \omega_i - {}^{\mathrm{BC}}\!\!\int_{\phi(R)}^R \omega_i.$$

Moreover, the matrix $M - I$ is invertible (see [Ked01, Section 2]), and we can solve this linear system to obtain the integrals ${}^{\mathrm{BC}}\!\int_S^R \omega_i$.

Thanks to this theorem, beyond evaluating primitives, computing tiny integrals and solving a linear system, the matrix of Frobenius is the only data that is needed to compute Coleman integrals between endpoints in distinct non-Weierstrass residue discs.

Suppose now that R' and S' are points, at least one of which is Weierstrass, lying in different residue discs. Recall that w is the hyperelliptic involution. The following lemma will be useful.

Lemma 4.4.5. ([BBK10, Lemma 16]) *Let ω be an odd meromorphic 1-form on $\tilde{X}$. For points $R', S' \in \tilde{X}(\mathbb{C}_p)$ which are not poles of ω, such that S' is a Weierstrass point, we have*

$$ {}^{\mathrm{BC}}\!\!\int_{S'}^{R'} \omega = \frac{1}{2}\,{}^{\mathrm{BC}}\!\!\int_{w(R')}^{R'} \omega.$$

In particular, if R' is also a Weierstrass point, then ${}^{\mathrm{BC}}\!\int_{S'}^{R'} = 0$.

Proof. This follows from ${}^{\mathrm{BC}}\!\int_{S'}^{R'} \omega = {}^{\mathrm{BC}}\!\int_{S'}^{w(R')} (-\omega) = {}^{\mathrm{BC}}\!\int_{w(R')}^{S'} \omega$ and additivity in endpoints. $\square$

If S lies in a finite Weierstrass residue disc containing Weierstrass point S', Lemma 4.4.5 gives

$$^{\mathrm{BC}}\!\!\int_S^R \omega_i = {}^{\mathrm{BC}}\!\!\int_S^{S'} \omega_i + \frac{1}{2}\,{}^{\mathrm{BC}}\!\!\int_{w(R)}^R \omega_i.$$

If R does not belong to a Weierstrass residue disc, the second integral can be calculated using Theorem 4.4.4; if R also lies in a finite Weierstrass residue disc containing Weierstrass point R', then by Lemma 4.4.5 again, we have

$$^{\mathrm{BC}}\!\!\int_S^R \omega_i = {}^{\mathrm{BC}}\!\!\int_S^{S'} \omega_i + {}^{\mathrm{BC}}\!\!\int_{R'}^R \omega_i.$$

These tiny integrals can be computed using Lemma 4.4.3.

Now, we consider the integrals ${}^{\mathrm{BC}}\!\int_S^R \nu_j$. As we discussed before, the form ν_j has poles at the hyperelliptically conjugate points $\pi^{-1}(\beta_j)$. The above approach does not work for this case, however, the paper [BB12] provides a new method.

First, consider the case where R and S lie in the same residue disc. If the form ν_j is holomorphic in the disc, then we can compute its integral as in Lemma 4.4.3. Otherwise, we make use of the following lemma in which we decompose our form in the disc into the sum of a holomorphic form and a logarithmic differential.

Lemma 4.4.6. ([BBM16, Lemma 4.2]) *Let P be a non-Weierstrass point and set*

$$\nu = \frac{y(P)}{x - x(P)}\frac{dx}{y}.$$

For points R, S different from P but contained in the residue disc of P, we have

$$^{\mathrm{BC}}\!\!\int_S^R \nu = {}^{\mathrm{BC}}\!\!\int_S^R \frac{g(x(P)) - g(x)}{y(x - x(P))(y(P) + y)}dx + \mathrm{Log}\left(\frac{x(R) - x(P)}{x(S) - x(P)}\right)$$

where the integrand on the right side is holomorphic on the residue disc.

Now, we examine the case where R and S lie in distinct residue discs. As before, using Lemma 4.4.5, we may reduce to the case that the residue discs are non-Weierstrass. We have the following which we state for curves and points defined over $\mathbb{Q}_p$:

Theorem 4.4.7. ([BB12, Algorithm 4.8, Remark 4.9]) *Suppose the curve $\tilde{X}$ is defined over $\mathbb{Q}_p$ and the polynomial $g(x)$ is monic. Let P and ν be as in Lemma 4.4.6. For points $R, S \in Z(\mathbb{Q}_p)$ in distinct non-Weierstrass residue discs, not equal to P and $w(P)$, we have*

$$^{\mathrm{BC}}\!\!\int_S^R \nu = \frac{1}{1-p}\left(\Psi(\alpha)\cup\Psi(\beta)+\sum_{A\in\tilde{X}(\mathbb{C}_p)}\mathrm{Res}_A\left(\alpha\int\beta\right)-{}^{\mathrm{BC}}\!\!\int_{\phi(S)}^{S}\nu-{}^{\mathrm{BC}}\!\!\int_{R}^{\phi(R)}\nu\right)$$

where $\alpha = \phi^*\nu - p\nu$, β is a form with $\mathrm{Res}(\beta) = R - S$.

Remark 4.4.8. The generalization of Theorem 4.4.7 to even degree case will be discussed in Gajović's upcoming PhD thesis at the University of Groningen. Combining this with the techniques in [Bal15], which extend the algorithms in [BBK10] to even degree models, one should be able to do the computation above for the even degree case.

Now, we consider the case where $g(x)$ is of degree at most 2. The curve $\tilde{X}$ is rational and therefore $H^1_{\mathrm{dR}}(\tilde{X})$ is trivial. By the exact sequence (4.1), our form η is the sum of an exact form dF and forms with simple poles. Moreover, using the equation $y^2 = g(x)$, one can easily express the non-exact part as a sum of logarithmic differentials $c_i dF_i/F_i$ for constants c_i. This gives

$$^{\mathrm{BC}}\!\!\int \eta = F + \sum_i c_i \mathrm{Log}(F_i).$$

4.5 Decomposition of 1-forms with Specified Poles

We now consider 1-forms with poles in a specified set. Let X be the good reduction hyperelliptic curve defined by $y^2 = g(x)$ where $g(x)$ is of degree d. Moreover, we assume that the polynomial $g(x)$ is monic with integral coefficients in some finite extension of $\mathbb{Q}_p$; this assumption guarantees that the p-adic absolute value of roots of $g(x)$ are at most 1. Let Y be a basic wide open in X contained in $B(0, R)$ for some $R \in \|K^*\|_p$ with $R > 1$.

Let $T = \{\beta_1, \ldots, \beta_\ell\}$ be a subset of $\mathbb{A}^1(K)$ for some finite extension K of $\mathbb{Q}_p$. We will study 1-forms of the form

$$\eta = x^{n_\infty}\prod_{j=1}^{\ell}\frac{1}{(x-\beta_j)^{n_j}}\frac{dx}{2y}$$

for nonnegative integers $n_1, \ldots, n_\ell, n_\infty$. We will further suppose that $\|\beta_i\|_p \leq 1$ for all i. Below, we will make the following assumptions: for $i = 1, \ldots, k$, we have $\|g(\beta_i)\|_p = 1$; and for $i = k+1, \ldots, \ell$, we have $g(\beta_i) = 0$ and $\|g'(\beta_i)\|_p = 1$.

We will soon need to consider a series of 1-forms whose terms are of the above form. To integrate them, we interchange integration and summation using Proposition 3.2.5. We will provide an algorithm to express the 1-forms in terms of our given basis: the 1-form η can be written as

$$\eta = dF + \sum_{i=0}^{d-2} c_i \omega_i + \sum_{j=1}^{k} d_j \nu_j \tag{4.3}$$

where F is an analytic function on Y. Furthermore, we will find bounds on c_i, on d_j, and on the maximum value of the norm of F on Y in Proposition 4.5.9.

Remark 4.5.1. We will make use of two types of exact 1-forms.

1. For a positive integer m, consider
$$d\left(\frac{y}{(x-\beta)^m}\right) = \frac{(x-\beta)g'(x) - 2mg(x)}{(x-\beta)^{m+1}} \frac{dx}{2y}.$$
Such a form has poles at the points above β and possibly also at the point(s) at infinity. If β is a root of $g(x)$, the pole is of order $2m$ at $\pi^{-1}(\beta)$; in fact, if we write Symb for the monomial involving the highest order power of $(x-\beta)^{-1}$, we have
$$\mathrm{Symb}\left(\frac{(x-\beta)g'(x) - 2mg(x)}{(x-\beta)^{m+1}} \frac{dx}{2y}\right) = \mathrm{Symb}\left(\frac{(1-2m)g'(\beta)}{(x-\beta)^m} \frac{dx}{2y}\right).$$
If β is not a root of $g(x)$, there are poles of order $m+1$ at each of the points of $\pi^{-1}(\beta)$; in fact, we have
$$\mathrm{Symb}\left(\frac{(x-\beta)g'(x) - 2mg(x)}{(x-\beta)^{m+1}} \frac{dx}{2y}\right) = \mathrm{Symb}\left(\frac{-2mg(\beta)}{(x-\beta)^{m+1}} \frac{dx}{2y}\right).$$
The two cases differ because y is a uniformizer in one case, while $x - \beta$ is a uniformizer in the other.

2. For a nonnegative integer m, consider
$$d(x^m y) = \left(x^m g'(x) + 2mx^{m-1}g(x)\right)\frac{dx}{2y}.$$
Such a form has poles at the point(s) at infinity. Notice that the leading coefficient of $x^m g'(x) + 2mx^{m-1}g(x)$ is $d + 2m$ as the polynomial $g(x)$ is monic.

4.5.2 Principal Parts

We will write our 1-form as in (4.3) by subtracting off the exact 1-forms in Remark 4.5.1 to cancel the non-simple poles. To do so, we use the language of principal parts.

Definition 4.5.3. Let α be a smooth point of a curve X and pick a uniformizer t on X for α. For a meromorphic function h, the **principal part of** h near α is the polynomial in t^{-1} given by the negative degree terms in the Laurent expansion of h in t. Let ω be a meromorphic 1-form on X that is regular and non-vanishing at α. For a meromorphic 1-form η on X, $\frac{\eta}{\omega}$ is a meromorphic function defined in a punctured neighborhood of α. The **principal part** $\mathrm{PP}_{\omega,\alpha}(\eta)$ of η near α with respect to ω is the principal part of $\frac{\eta}{\omega}$ near α.

For $\beta \in \mathbb{A}^1(K)$ and $\alpha \in \pi^{-1}(\beta)$, we have convenient choices for coordinates and 1-forms. The 1-form $\omega_0 = \frac{dx}{2y}$ is regular and non-vanishing away from the point(s) at infinity. Let η be an odd 1-form; then $\frac{\eta}{\omega_0}$ is invariant under the hyperelliptic involution.

We first explain how to pick a uniformizer at Weierstrass points. Let β be a root of $g(x)$, then y is a uniformizer at $\alpha = \pi^{-1}(\beta)$. However, we can pick a slightly more convenient uniformizer. We know that $\frac{x-\beta}{g(x)}$ does not vanish in a neighborhood of β and so has an analytic square root $h(x)$ there. Then $w = yh(x)$ is a uniformizer at α. Because the meromorphic function $\frac{\eta}{\omega_0}$ is invariant under the hyperelliptic involution, it can be written near α as a Laurent series in $w^2 = x - \beta$. Therefore, the principal part of $\frac{\eta}{\omega_0}$ is a polynomial in $z = (yh(x))^{-2} = (x-\beta)^{-1}$.

If β is not a root of $g(x)$, then $\pi^{-1}(\beta) = \{\alpha_1, \alpha_2\}$ and $x - \beta$ is a uniformizer near both α_1 and α_2. Consequently, $\mathrm{PP}_{\omega_0,\alpha_1}(\eta) = \mathrm{PP}_{\omega_0,\alpha_2}(\eta)$. In this case, the principal part of $\frac{\eta}{\omega_0}$ is a polynomial in $z = (x-\beta)^{-1}$.

In any case, by using the Taylor expansion for $g(x)$ at β, we compute

$$\mathrm{PP}_{\omega_0,\alpha}\left(d\left(\frac{y}{(x-\beta)^m}\right)\right) = \sum_{k=0}^{m} \frac{-m-k}{(m-k)!} g^{(m-k)}(\beta) z^{k+1}.$$

By a straightforward argument obtained by writing $x^k = (x-\beta+\beta)^k$ and using the integrality of binomial coefficients, one sees that the p-adic absolute value of the coefficients of the principal part are bounded above by $\max(1, \|\beta\|_p^d)$. In particular, if $\|\beta\|_p \leq 1$, we have

$$\left\|\mathrm{PP}_{\omega_0,\alpha}\left(d\left(\frac{y}{(x-\beta)^m}\right)\right)\right\|_p \leq 1. \tag{4.4}$$

Here, for a polynomial $q(t)$, we define the value $\|q(t)\|_p$ as the maximum of the p-adic absolute value of its coefficients.

Recall that, for an integer i, ω_i is defined as $x^i \frac{dx}{2y}$. If η is an odd 1-form, $\frac{\eta}{\omega_i}$ is a meromorphic function on $\mathbb{P}^1$, and we may speak of its pole order at $\infty \in \mathbb{P}^1(K)$. In analogy with the above, we choose $\frac{1}{x}$ as a uniformizer at ∞. Write $\mathrm{PP}_{\omega_i,\infty}(\eta)$ for the principal part of $\frac{\eta}{\omega_i}$ considered as a meromorphic function on $\mathbb{P}^1$. Observe that for η regular on the finite part of Y, if

$$\frac{\eta}{\omega_0} = a_0 + a_1 x + \cdots + a_{d-2} x^{d-2}$$

then

$$\mathrm{PP}_{\omega_{-1},\infty}(\eta) = \frac{\eta}{\omega_{-1}} = a_0 \left(\frac{1}{x}\right)^{-1} + a_1 \left(\frac{1}{x}\right)^{-2} + \cdots + a_{d-2} \left(\frac{1}{x}\right)^{-(d-2)}$$

and

$$\eta = a_0\omega_0 + a_1\omega_1 + \cdots + a_{d-2}\omega_{d-2}.$$

We abuse notation and refer to the degree of the polynomial $\frac{\eta}{\omega_0}$ as the pole order at ∞. If the pole order is at most $d-2$, the above formula lets us determine the cohomology class of η.

4.5.4 Pole Reduction

Using ideas similar to those of Tuitman [Tui16, Tui17], we will subtract off exact 1-forms to lower the pole orders of η at the β's. We begin by cancelling the poles of η of order greater than 1 at non-Weierstrass points and the poles of η at Weierstrass points. Then, we cancel the simple poles at non-Weierstrass points by subtracting off multiples of ν_j. The remainder η' can be expressed in terms of the ω_i's by examining $\mathrm{PP}_{\omega_{-1},\infty}(\eta')$.

Define meromorphic 1-forms $\mu_{\beta,m}$ by

$$\mu_{\beta,m} = d\left(\frac{y}{(x-\beta)^m}\right).$$

We omit the proof of the following lemma (which is a computation in coordinates).

Lemma 4.5.5. *We have the following:*

1. *the pole order of* ∞ *of* $\mu_{\beta,m}/\omega_{-1}$ *is at most* $d+1-m$*,*

2. *the principal part of* $\mu_{\beta,m}$ *at* ∞ *obeys* $\left\|\mathrm{PP}_{\omega_{-1},\infty}(\mu_{\beta,m})\right\|_p \leq 1$*, and*

3. *the principal part of* $d(x^m y)$ *at* ∞ *obeys* $\|\mathrm{PP}_{\omega_{-1},\infty}(d(x^m y))\|_p \leq 1$.

Below, we will make use of Legendre's formula for the p-adic valuation of factorials for $p \neq 2$. We have the bounds

$$\frac{m}{p-1} - \lceil \log_p(m) \rceil \leq v_p(m!) \leq \frac{m}{p-1}.$$

From this, we obtain the following bound on odd factorials:

$$v_p((2m-1)!!) \leq \frac{m}{p-1} + \lceil \log_p(m) \rceil \leq \frac{m}{p-1} + \log_p(m) + 1.$$

For a non-negative integer n, the notation $n!!$ stands for the product of all the integers from 1 to n that have the same parity as n.

Lemma 4.5.6. *Let* $\beta \in \mathbb{A}^1(K)$ *with* $\|\beta\|_p \leq 1$ *and* $\|g(\beta)\|_p = 1$. *Take* $\alpha \in \pi^{-1}(\beta)$ *and set* $z = (x-\beta)^{-1}$. *Let* η *be an odd meromorphic* 1*-form on* X *such that*

$$m := \deg_z(\mathrm{PP}_{\omega_0,\alpha}(\eta)) - 1 > 0.$$

Then there exists a unique polynomial $q(t) \in K[t]$ *of degree* m *such that*

$$\eta' := \eta - d(q((x-\beta)^{-1})y)$$

has at worst simple poles at points above β *and*

$$\|q(t)\|_p \leq p^{m/(p-1)} \|\mathrm{PP}_{\omega_0,\alpha}(\eta)\|_p.$$

Moreover, $\mathrm{PP}_{\omega_{-1},\infty}(\eta - \eta')$ *is a* K*-linear combination of* $\mathrm{PP}_{\omega_{-1},\infty}(\mu_{\beta,1}), \ldots, \mathrm{PP}_{\omega_{-1},\infty}(\mu_{\beta,d-1})$ *with coefficients with norm at most*

$$p^{m/(p-1)} \|\mathrm{PP}_{\omega_0,\alpha}(\eta)\|_p.$$

Proof. Let V be the m-dimensional K-vector space spanned by the meromorphic functions

$$\{(x-\beta)^{-1}y, (x-\beta)^{-2}y, \ldots, (x-\beta)^{-m}y\}$$

and let W be the K-vector space spanned by $\{z^2, z^3, \dots, z^{m+1}\}$. Define

$$\begin{array}{rcl} L\colon V & \to & W \\ h & \mapsto & T(\mathrm{PP}_{\omega_0,\alpha}(dh)) \end{array}$$

where T takes $z^1 \mapsto 0$ and $z^i \mapsto z^i$ for $i \geq 2$. By Remark 4.5.1, its matrix M in these bases is upper triangular. In fact, the diagonal entries of M are

$$M_{ii} = (-2i)g(\beta).$$

As $g(\beta) \neq 0$, the matrix M is invertible and we can find a polynomial $q(t)$ such that $q((x-\beta)^{-1})y \in V$ satisfies $L(q((x-\beta)^{-1})y) = T(\mathrm{PP}_{\omega_0,\alpha}(\eta))$.

To get control over the coefficients of $q(t)$, we use Cramer's rule. The coefficients of $q(t)$ are equal to $\det(M_j)/\det(M)$ where M_j is the matrix formed by replacing the jth column of M by the coefficients of $T(\mathrm{PP}_{\omega_0,\alpha}(\eta))$. By Legendre's formula, we have

$$\|\det(M)\|_p \geq p^{-m/(p-1)}.$$

By (4.4), the coefficients of M are bounded above in p-adic absolute value by 1, so

$$\|\det(M_j)\|_p \leq \|T(\mathrm{PP}_{\omega_0,\alpha}(\eta))\|_p.$$

Consequently, the coefficients of $q(t)$ are bounded above by

$$p^{m/(p-1)}\|\mathrm{PP}_{\omega_0,\alpha}(\eta)\|_p.$$

The bound on pole order at infinity and on the coefficients of the principal part at ∞ follow from Lemma 4.5.5. $\square$

Algorithm 2: Pole reduction at finite non-Weierstrass points

Input:

- $\alpha \in \pi^{-1}(\beta)$ where β is a non-root of $g(x)$.
- An odd meromorphic form η with pole at α of order m.

Output: A function F such that the form $\eta - dF$ has at worst simple poles at points above β.

1. For $j = 1, \dots, m-1$, compute the expansions

$$\mu_{\beta,j} = \big(w_{-j-1}(x-\beta)^{-j-1} + \text{higher order terms}\big)\,\frac{dx}{2y}.$$

2. Until η has at worst a simple pole at α, do the following:

   (a) Compute the expansion

$$\eta = \big(u_{-j}(x-\beta)^{-j} + \text{higher order terms}\big)\,\frac{dx}{2y}.$$

   (b) Define $a_{j-1} = u_{-j}/w_{-j}$ and set $\eta := \eta - a_{j-1}\mu_{\beta,j-1}$.

3. Return

$$F = \left(\frac{a_{m-1}}{(x-\beta)^{m-1}} + \frac{a_{m-2}}{(x-\beta)^{m-2}} + \cdots + \frac{a_1}{x-\beta}\right)y.$$

Now, we consider a root β of $g(x)$.

Lemma 4.5.7. *Let $\beta \in \mathbb{A}^1(K)$ be a root of $g(x)$ so that $\|\beta\|_p \le 1$. Suppose $\|g'(\beta)\|_p = 1$. Let $\alpha = \pi^{-1}(\beta)$ and set $z = (x-\beta)^{-1}$. Let η be an odd meromorphic 1-form on X such that*

$$m := \deg_z(\mathrm{PP}_{\omega_0,\alpha}(\eta)) > 0.$$

Then there exists a unique polynomial $q(t) \in K[t]$ of degree m such that

$$\eta' := \eta - d(q((x-\beta)^{-1})y)$$

is regular at α and

$$\|q(t)\|_p \leq mp^{1+m/(p-1)} \|\mathrm{PP}_{\omega_0,\alpha}(\eta)\|_p.$$

Moreover, $\mathrm{PP}_{\omega_{-1},\infty}(\eta - \eta')$ *is a K-linear combination of* $\mathrm{PP}_{\omega_{-1},\infty}(\mu_{\beta,1}), \ldots, \mathrm{PP}_{\omega_{-1},\infty}(\mu_{\beta,d-1})$ *with coefficients with norm at most*

$$mp^{1+m/(p-1)} \|\mathrm{PP}_{\omega_0,\alpha}(\eta)\|_p.$$

Proof. Let V be the m-dimensional K-vector space spanned by the meromorphic functions

$$\{(x-\beta)^{-1}y, (x-\beta)^{-2}y, \ldots, (x-\beta)^{-m}y\}$$

and let W be the K-vector space spanned by $\{z, z^2, z^3, \ldots, z^m\}$. Define the map

$$\begin{array}{rcl} L\colon V & \to & W \\ h & \mapsto & \mathrm{PP}_{\omega_0,\alpha}(dh). \end{array}$$

By Remark 4.5.1, its matrix M (in these bases) is an upper triangular matrix with diagonal entries

$$M_{ii} = (1-2i)g'(\beta).$$

Because $\|g'(\beta)\|_p = 1$, M is nonsingular and we can find $q((x-\beta)^{-1})y \in V$ with $L(q((x-\beta)^{-1})y) = \mathrm{PP}_{\omega_0,\alpha}(\eta)$.

Again, we use Cramer's rule to get control over the coefficients of h. The determinant of M has p-adic absolute value

$$\|\det(M)\|_p = \|(2m-1)!!\|_p \geq \frac{1}{m} p^{-1-m/(p-1)}$$

where the last inequality follows from Legendre's formula for odd factorials. Let M_j be the matrix formed by replacing the jth column of M by the coefficients of $\mathrm{PP}_{\omega_0,\alpha}(\eta)$. By (4.4), the coefficients of M are bounded above in p-adic absolute value by 1, so

$$\|\det(M_j)\|_p \leq \|\mathrm{PP}_{\omega_0,\alpha}(\eta)\|_p.$$

Consequently, the coefficients of $p(t)$ are bounded above by

$$mp^{1+m/(p-1)} \|\mathrm{PP}_{\omega_0,\alpha}(\eta)\|_p.$$

The bound on pole order at infinity and on the coefficients of the principal part at ∞ again follow from Lemma 4.5.5. $\square$

Algorithm 3: Pole reduction at finite Weierstrass points

Input:

- $\alpha = \pi^{-1}(\beta)$ where β is a root of $g(x)$.
- An odd meromorphic form η with pole at α of order $2m$.

Output: A function F such that the form $\eta - dF$ is regular at α.

1. For $j = 1, \ldots, m$, compute the expansions

$$\mu_{\beta,j} = \left(w_{-j}(x-\beta)^{-j} + \text{higher order terms}\right)\frac{dx}{2y}.$$

2. Until η is regular at α, do the following:

   (a) Compute the expansion

$$\eta = \left(u_{-j}(x-\beta)^{-j} + \text{higher order terms}\right)\frac{dx}{2y}.$$

   (b) Define $a_j = u_{-j}/w_{-j}$ and set $\eta \coloneqq \eta - a_j\mu_{\beta,j}$.

3. Return

$$F = \left(\frac{a_m}{(x-\beta)^m} + \frac{a_{m-1}}{(x-\beta)^{m-1}} + \cdots + \frac{a_1}{x-\beta}\right)y.$$

The main difference between Algorithms 2 and 3 is that, by subtracting off exact forms, poles at Weierstrass points can be removed completely but only non-simple poles can be removed at non-Weierstrass points.

We also need to lower the power of x in the numerator of a 1-form. This is the order reduction step in Kedlaya's algorithm.

Lemma 4.5.8. *Let η be an odd meromorphic 1-form on X such that*

$$m \coloneqq \deg_x(\mathrm{PP}_{\omega_{-1},\infty}(\eta)) - (d-1) > 0.$$

Let T be the truncation of a polynomial to degree $d-1$ and $U = \mathrm{Id} - T$. Then there exists a unique polynomial $q(t) \in K[t]$ of degree m such that

$$\eta' \coloneqq \eta - d(q(x)y)$$

has $\deg_x(\mathrm{PP}_{\omega_{-1},\infty}(\eta')) \leq d-1$ *and*

$$\|q(t)\|_p \leq d(d+m)p^{2+m/(p-1)}\|U(\mathrm{PP}_{\omega_{-1},\infty}(\eta))\|_p.$$

Moreover, $T(\mathrm{PP}_{\omega_{-1},\infty}(\eta-\eta'))$ *has coefficients with norm at most*

$$d(d+m)p^{2+m/(p-1)}\|U(\mathrm{PP}_{\omega_{-1},\infty}(\eta))\|_p.$$

Proof. Let V be the m-dimensional K-vector space spanned by the meromorphic functions

$$\{y, xy, \ldots, x^{m-1}y\}$$

and let W be the K-vector space spanned by $\{x^d, x^{d+1}, \ldots, x^{d+m-1}\}$. Define the map

$$\begin{array}{rcl} L\colon V & \to & W \\ h & \mapsto & U(\mathrm{PP}_{\omega_{-1},\infty}(dh)). \end{array}$$

By Remark 4.5.1, its matrix M in these bases is upper triangular with diagonal entries

$$M_{ii} = d + 2(i-1).$$

Hence we can find $q(x)y \in V$ with $L(q(x)y) = U(\mathrm{PP}_{\omega_{-1},\infty}(\eta))$. By arguments analogous to the above, considering the cases of d even and odd seperately, we have

$$\|\det(M)\|_p \geq \frac{1}{d(d+m)}p^{-2-m/(p-1)}.$$

Let M_j be the matrix M with the jth column replaced by the coefficients of $U(\mathrm{PP}_{\omega_{-1},\infty}(\eta))$. The coefficients of M are integral and so

$$\|\det(M_j)\|_p \leq \|U(\mathrm{PP}_{\omega_{-1},\infty}(\eta))\|_p.$$

Consequently, the coefficients of $q(x)$ are bounded above by

$$d(d+m)p^{2+m/(p-1)}\|U(\mathrm{PP}_{\omega_{-1},\infty}(\eta))\|_p.$$

The bound on the coefficients of $T(\mathrm{PP}_{\omega_{-1},\infty}(\eta-\eta'))$ follows from Lemma 4.5.5. □

Algorithm 4: Pole reduction at infinity

Input: An odd meromorphic form η such that $\frac{\eta}{\omega_0}$ has degree $d-2+m$ for some positive integer m.

Output: A function F such that the $\frac{\eta - dF}{\omega_0}$ has degree at most $d-2$.

1. For $j = 0, \ldots, m-1$, compute the expansions

$$d(x^j y) = \left(w_{d-1+j}x^{d-1+j} + \text{lower order terms}\right)\frac{dx}{2y}.$$

2. Until $\frac{\eta}{\omega_0}$ has degree at most $d-2$, do the following:

   (a) Compute the expansion

$$\eta = \left(u_{d-1+j}x^{d-1+j} + \text{lower order terms}\right)\frac{dx}{2y}.$$

   (b) Define $a_j = u_{d-1+j}/w_{d-1+j}$ and set $\eta \coloneqq \eta - a_j d(x^j y)$.

3. Return

$$F = (a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_0)y.$$

We now apply the algorithms described above to find a primitive of η. We first subtract exact forms from η to remove the non-simple poles over non-roots of $g(x)$ and to remove the poles over roots of $g(x)$. Then, we reduce the pole order at ∞. Because the exact forms only affect the principal parts of one finite point at a time, we have the following:

Proposition 4.5.9. *Let $f_{\omega_0}, \ldots, f_{\omega_{d-2}}$ and $f_{\nu_1}, \ldots, f_{\nu_k}$ be Coleman primitives of $\omega_1, \ldots, \omega_{d-2}$ and $\nu_1, \ldots, \nu_k$, respectively. Let η be an odd 1-form on X such that $\frac{\eta}{\omega_0}$ has poles at points $\{\beta_1, \ldots, \beta_\ell, \infty\} \subset \mathbb{P}^1(K)$ of order $n_1, \ldots, n_\ell, n_\infty$. Suppose $\beta_1, \ldots, \beta_k$ are not roots of $g(x)$ and $\beta_{k+1}, \ldots, \beta_\ell$ are roots of $g(x)$. Moreover, we will suppose $\|\beta_i\|_p \leq 1$ for all i, $\|g(\beta_i)\|_p = 1$ for $i = 1, \ldots, k$ and $\|g'(\beta_i)\|_p = 1$ for $i = k+1, \ldots, \ell$. Let $\alpha_i \in \pi^{-1}(\beta_i)$. Then η has a Coleman primitive that is a linear combination of the following:*

1. *$\frac{y}{(x-\beta_i)^j}$ where $1 \leq j \leq n_i - 1$ for $i = 1, \ldots, k$ with coefficient with norm at most*

$$p^{n_i/(p-1)}\|\mathrm{PP}_{\omega_0,\alpha_i}(\eta)\|_p,$$

2. $\frac{y}{(x-\beta_i)^j}$ *where* $1 \leq j \leq n_i$ *for* $i = k+1, \ldots, \ell$ *with coefficient with norm at most*

$$n_i p^{1+n_i/(p-1)} \|\mathrm{PP}_{\omega_0,\alpha_i}(\eta)\|_p,$$

3. $x^j y$ *where* $0 \leq j \leq \max(n_\infty - d + 2, 2)$ *with coefficient with norm at most the maximum of the following:*

   (a) $d(d+n_\infty)p^{2+n_\infty/(p-1)}\|U(\mathrm{PP}_{\omega_{-1},\infty}(\eta))\|_p$,

   (b) $\max_{i=1,\ldots,k}\left(p^{n_i/(p-1)}\|\mathrm{PP}_{\omega_0,\alpha_i}(\eta)\|_p\right)$, *and*

   (c) $\max_{i=k+1,\ldots,\ell}\left(n_i p^{1+n_i/(p-1)}\|\mathrm{PP}_{\omega_0,\alpha_i}(\eta)\|_p\right)$.

4. f_{ν_i} *for* $i = 1, \ldots, k$ *with coefficient equal to*

$$\frac{\mathrm{Res}_{\alpha_i}(\eta)}{\mathrm{Res}_{\alpha_i}(\nu_i)},$$

5. f_{ω_i} *for* $i = 0, \ldots, d-2$ *with coefficient with norm at most*

$$p^{2+\max(n_i/(p-1),n_\infty/(p-1))}\max(d(d+n_\infty), n_i)\max(\|\mathrm{PP}_{\omega_0,\alpha_i}(\eta)\|_p, \|\mathrm{PP}_{\omega_{-1},\infty}(\eta)\|_p)$$

   where the maximum is taken over $\alpha \in \pi^{-1}(\{\beta_1, \ldots, \beta_\ell\})$.

Proof. Let η be an odd 1-form on X such that $\frac{\eta}{\omega_0}$ has poles at points $\{\beta_1, \ldots, \beta_\ell\} \subset \mathbb{A}^1(K)$ of order $n_1, \ldots, n_\ell$. Then we apply Lemma 4.5.6 and Lemma 4.5.7 to reduce the pole orders at the β's. Because the exact forms that we subtract for one β_i does not affect the principal parts at other β_i's, the pole order reduction steps are independent, and we have the above bounds on coefficients. These operations do affect the principal parts above ∞ in degrees up to $d-2$ according to the bounds in the lemmas. This leads to the bounds for the coefficient of f_{ω_i}. $\square$

Algorithm 5: Cohomology class for a meromorphic 1-form

Input: A meromorphic form

$$\eta = x^{n_\infty} \prod_{j=1}^{\ell} \frac{1}{(x-\beta_j)^{n_j}} \frac{dx}{2y}$$

for nonnegative integers $n_1, \dots, n_\ell, n_\infty$.

Output: A function F and constants c_i, d_j such that

$$\eta = dF + \sum_{i=0}^{d-2} c_i \omega_i + \sum_{j=1}^{k} d_j \nu_j.$$

1. *Constants d_j:* For $j = 1, \dots, k$, pick $\alpha_j \in \pi^{-1}(\beta_j)$ and compute
$$d_j = \frac{\mathrm{Res}_{\alpha_j}(\eta)}{\mathrm{Res}_{\alpha_j}(\nu_j)}.$$
2. *Non-Weierstrass points:* Using Algorithm 2, find a function F_{nw} such that the form $\eta - dF_{nw}$ has at worst simple poles at points above $\beta_1, \dots, \beta_k$.
3. *Finite Weierstrass points:* Using Algorithm 3, find a function F_w such that the form $\eta - dF_{nw} - dF_w$ is regular at points above $\beta_{k+1}, \dots, \beta_\ell$.
4. *The point(s) at infinity:* Using Algorithm 4, find a function F_∞ such that $\frac{\eta - dF_{nw} - dF_w - dF_\infty}{\omega_0}$ has degree at most $d-2$.
5. *Constants c_i:* Compute
$$\frac{\eta - dF_{nw} - dF_w - dF_\infty - \sum d_j \nu_j}{\omega_0} = c_0 + c_1 x + \cdots + c_{d-2} x^{d-2}.$$
6. Return
$$F = F_{nw} + F_w + F_\infty, \ \{c_i\}_i, \ \{d_j\}_j.$$

4.6 Power Series Expansion

We will write a power series expansion of 1-forms $x^i \frac{dx}{2y}$ on basic wide opens in a semistable covering of a hyperelliptic curve $\pi\colon X^{\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$ defined by $y^2 = f(x)$ following the methods in Subsection 4.3.9. We will suppose that $f(x)$ is a monic polynomial with integral coefficients in some finite extension of $\mathbb{Q}_p$ and, moreover, that the roots of $f(x)$ lie in a field K of ramification degree e over $\mathbb{Q}_p$. By our assumptions, these roots have p-adic valuation at most 1. Let S_f be the set of roots of $f(x)$. Let U be an element of a good semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ with respect to S_f. We have an embedding $\iota\colon U \to \mathbb{P}^{1,\mathrm{an}}$ such that the points of $\iota(S_f \cap U(\mathbb{C}_p))$ lie in distinct residue discs. We use x to denote the coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$. Without loss of generality, we may suppose that U is the open disc $B(0,R)$ (for some $R > 1$) minus some closed discs and that $S_f \cap U(\mathbb{C}_p) \subset \overline{B}(0,1)$. Let I_∞ be the set of roots of $f(x)$ lying outside of $B(0,R)$. Because the roots of $f(x)$ are K-points, the elements of $I_\infty \setminus \{\infty\}$ have norm at least $p^{1/e}$. We partition the roots of $f(x)$ in $B(0,R)$ by residue disc: $S_f \cap B(0,R) = \cup_{j=1}^m I_j$. Notice that some of I_j's may have only one element. We relabel these sets such that

1. for $j = 1, \ldots, k$, $|I_j| \geq 2$ and $|I_j|$ is even;

2. for $j = k+1, \ldots, \ell$, $|I_j| \geq 2$ and $|I_j|$ is odd; and

3. for $j = \ell+1, \ldots, m$, $|I_j| = 1$.

For $j = 1, \ldots, \ell$, pick $\beta_j \in \mathbb{A}^1(K) \setminus U(K)$ in the same residue disc as the points in I_j (we may even take β_j to be an element of I_j); and for $j = \ell+1, \ldots, m$, let β_j denote the unique element of I_j. Notice that $\|\beta_j\|_p \leq 1$ for all j. Define

$$L_j = \begin{cases} |I_j|/2 & \text{for } j = 1, \ldots, k; \\ (|I_j|-1)/2 & \text{for } j = k+1, \ldots, \ell \end{cases}$$

and set

$$g(x) = \prod_{j=k+1}^{m} (x - \beta_j),$$

$$h(x) = \prod_{j=1}^{\ell} (x - \beta_j)^{L_j},$$

$$k(x) = \left(\prod_{j=1}^{\ell} \prod_{\beta \in I_j} \left(\frac{x - \beta}{x - \beta_j} \right) \right) \left(\prod_{\beta \in I_\infty \setminus \{\infty\}} (x - \beta) \right).$$

Observe that $f(x) = g(x)h(x)^2 k(x)$. Since $g(x)$ has at most one root in each residue disc and for $j = 1, \ldots, k$, the element β_j is not in the same residue disc as a root of $g(x)$, we have

$$\|g(\beta_j)\|_p = 1 \text{ for } j = 1, \ldots, k,$$
$$\|g'(\beta_j)\|_p = 1 \text{ for } j = k+1, \ldots, \ell.$$

Set

$$\tilde{y} = \frac{y}{h(x)k(x)^{1/2}}.$$

Note that $\frac{1}{h(x)k(x)^{1/2}}$ is an analytic function on U by construction; so $\tilde{y}^2 = g(x)$ for $x \in U$ defines a union of at most two basic wide opens in X^{an}. Write $\tilde{X}$ for the complete curve defined $\tilde{y}^2 = g(x)$. We may write $\tilde{X}_{g(x)}$ for $\tilde{X}$ when the polynomial $g(x)$ needs to be specified.

We have

$$\omega_i = x^i \frac{dx}{2y} = \frac{x^i}{h(x)k(x)^{1/2}} \frac{dx}{2\tilde{y}}.$$

We will expand ω_i in a power series on $\pi^{-1}(U)$. We may write

$$k_j(x) = \prod_{\beta \in I_j} \left(1 - \frac{\beta - \beta_j}{x - \beta_j} \right) \text{ for } j = 1, \ldots, \ell,$$

$$k_\infty(x) = \prod_{\beta \in I_\infty \setminus \{\infty\}} (-\beta)(1 - \beta^{-1} x),$$

so $k(x) = \big(\prod_j k_j(x)\big)k_\infty(x)$. Now,

$$\frac{1}{k_j(x)^{1/2}} = \prod_{\beta\in I_j}\Big(1-\frac{\beta-\beta_j}{x-\beta_j}\Big)^{-1/2} = \sum_{n=0}^{\infty}\frac{B_{jn}}{(x-\beta_j)^n}$$

$$\frac{1}{k_\infty(x)^{1/2}} = \prod_{\beta\in I_\infty\setminus\{\infty\}}(-\beta)^{-1/2}(1-\beta^{-1}x)^{-1/2} = \sum_{n=0}^{\infty}B_{\infty n}x^n$$

for some B_{jn}'s and $B_{\infty n}$'s. Then,

$$\begin{aligned}\frac{x^i}{h(x)k(x)^{1/2}} &= \left(\prod_{j=1}^{\ell}\sum_{n=0}^{\infty}\frac{B_{jn}}{(x-\beta_j)^{n+L_j}}\right)\left(\sum_{n=0}^{\infty}B_{\infty n}x^{n+i}\right)\\ &= \sum_{\substack{n_1\geq L_1,\ldots,n_\ell\geq L_\ell\\ n_\infty\geq i}}\left(B_{n_1,\ldots,n_\ell,n_\infty}x^{n_\infty}\prod_{j=1}^{\ell}\frac{1}{(x-\beta_j)^{n_j}}\right)\end{aligned}$$

for some $B_{n_1,\ldots,n_\ell,n_\infty}$'s. We may bound these coefficients as follows.

Proposition 4.6.1. *There is a constant C such that*

$$\|B_{n_1,\ldots,n_\ell,n_\infty}\|_p \leq Cp^{-\frac{(n_\infty-i)+\sum_{j=1}^{\ell}(n_j-L_j)}{e}}.$$

Proof. First observe that because $p\neq 2$, the coefficients of

$$(1-y)^{-1/2} = \sum_{n=0}^{\infty}\frac{1}{2^{2n}}\binom{2n}{n}y^n$$

are p-adic integers. Since $\|\beta-\beta_j\|_p \leq p^{-1/e}$ for each $\beta\in I_j$, by the ultrametric triangle inequality, we have $\|B_{jn}\|_p \leq C_jp^{-n/e}$ for some constant C_j. By an identical argument, we have

$$\|B_{\infty n}\|_p \leq C_\infty p^{-n/e}$$

for some constant C_∞. By multiplying together our inequalities, we get the desired conclusion. □

Consequently, the expression

$$\omega_i = \sum_{\substack{n_1\geq L_1,\ldots,n_\ell\geq L_\ell\\ n_\infty\geq i}}\left(B_{n_1,\ldots,n_\ell,n_\infty}x^{n_\infty}\prod_{j=1}^{\ell}\frac{1}{(x-\beta_j)^{n_j}}\frac{dx}{2\tilde{y}}\right) \tag{4.5}$$

makes sense.

Proposition 4.6.2. *Let $\tilde{\omega}_i = x^i \frac{dx}{2\tilde{y}}$. Then,*

$$\left\| \mathrm{PP}_{\tilde{\omega}_0,\alpha} \left(x^{n_\infty} \prod_{j=1}^{\ell} \frac{1}{(x-\beta_j)^{n_j}} \frac{dx}{2\tilde{y}} \right) \right\|_p \leq 1,$$

$$\left\| \mathrm{PP}_{\tilde{\omega}_{-1},\infty} \left(x^{n_\infty} \prod_{j=1}^{\ell} \frac{1}{(x-\beta_j)^{n_j}} \frac{dx}{2\tilde{y}} \right) \right\|_p \leq 1$$

where α is a point over any β_i.

Proof. If α is a point over some β_i, set $t = x - \beta_i$. Then, we have the following bounds on the coefficients of these power series, considered as Laurent series in t:

$$\begin{aligned} \left\| \frac{1}{(x-\beta_i)^{n_i}} \right\|_p &= 1, \\ \|x^{n_\infty}\|_p &= 1, \\ \left\| \frac{1}{(x-\beta_j)^{n_j}} \right\|_p &\leq 1, \quad j \neq i. \end{aligned}$$

Here, the last inequality follows from the observation that $\|\beta_i - \beta_j\|_p \geq 1$ for $i \neq j$. Because the Gauss norm $\|\cdot\|_p$ is multiplicative, for Laurent series f and g,

$$\|\mathrm{PP}(fg)\|_p \leq \|fg\|_p = \|f\|_p \|g\|_p$$

from which the conclusion follows. An analogous arguments holds for ∞ using $t = 1/x$. $\square$

Proposition 4.6.3. *Suppose $e < p-1$ and set $r = \frac{1}{e} - \frac{1}{p-1}$. Pick R with $1 < R < p^r$ and let*

$$D = \overline{B}(0,R) \setminus \bigcup_{i=1}^{\ell} B(\beta_i, 1/R).$$

The 1-form

$$\omega_i = \frac{x^i}{h(x)k(x)^{1/2}} \frac{dx}{2\tilde{y}}$$

has a Coleman primitive on $\pi^{-1}(D)$ given as the sum of terms of the following form:

1. *$a_{ij}\frac{y}{(x-\beta_i)^j}$ for $i = 1, \ldots, \ell$ and $j = 1, 2, \ldots$,*

2. *$b_j x^j y$ for $j = 0, 1, \ldots$,*

3. *$c_i f_{\omega_i}$ for $i = 0, \ldots, d-2$, and*

4. *$d_i f_{\nu_i}$ for $i = 1, \ldots, k$,*

where f_{ω_i} and f_{ν_i} are as in Proposition 4.5.9.

Proof. The coefficients for the power series expansion of ω in (4.5) decay at the rate of $p^{-N/e}$ where $N = n_\infty + \sum_{j=1}^{\ell} n_j$. By examining the summands, we see that they converge uniformly on D. On the other hand, by Proposition 4.5.9, the coefficients of the primitives of the summands (expressed in terms of the form (1)-(4)) grow slower than $Np^{N/(p-1)}$. We call the series produced by integrating the power series expansion, **the series of primitives**. For any given term of the type (1)-(4), its coefficient is convergent in the series of primitives.

We have to verify the hypotheses of Proposition 3.2.5 to show that the sum of the series of primitives is equal to the primitive of the sum of the series. By construction, the series of primitives is locally uniformly convergent. Moreover, for any lift of Frobenius ϕ and annihilating polynomial P, $P(\phi^*)$ applied to the series of primitives converges uniformly on a Frobenius neighborhood within D. Moreover, the restriction of the series of primitives to boundary annuli is uniformly convergent. $\square$

Remark 4.6.4. While we employ the algorithms from [BB12] to integrate the 1-forms ν_j on hyperelliptic basic wide opens, we can formulate a different integration algorithm similar to the work of Tuitman [Tui16, Tui17] using our techniques. Specifically, we can pick a lift of Frobenius ϕ on a hyperelliptic basic wide open. By replacing the lift of Frobenius by some power, we can ensure that it preserves the residue discs containing $\beta_1, \ldots, \beta_\ell$. Consequently, $\phi^*\nu_j$ can be written as a power series as in this section. By using the techniques of the previous section, $\phi^*\nu_j$ can be rewritten as a linear combination of 1-forms $\{\omega_0, \ldots, \omega_{d-2}, \nu_1, \ldots, \nu_k\}$ and an exact form dh_j. Hence, one obtains a matrix representing the action of Frobenius on odd cohomology and uses it to determine p-adic integrals. We will explore this in future work.

4.7 Integration on Curves

4.7.1 Berkovich–Coleman Integration on Paths

We explain how to perform Berkovich–Coleman integrals on a hyperelliptic curve X^{an}. Such an integral is to be done along a path γ in X^{an}. We will break up the path into smaller paths lying in hyperelliptic basic wide opens. Fix a holomorphic 1-form ω on X^{an}.

Let $\mathcal{C}$ be a semistable covering of X^{an} with dual graph Γ. For a vertex v of Γ, let U_v be the corresponding element of the covering. For $e = vw$, let U_e be the corresponding component of the intersection $U_v \cap U_w$. Pick a point P_v in each U_v and a point P_e in each U_e. These are called **reference points**. For each oriented edge e, write $i(e)$ and $t(e)$ for the initial and terminal point of e, respectively.

To a path $\gamma = e_1 e_2 \ldots e_\ell$ in Γ from v to w, we can consider the Berkovich–Coleman integral of ω from the reference point P_v to the reference point P_w along the path γ. In fact, because Γ is identified with the skeleton of X^{an}, there is a unique path $\tilde{\gamma}_{vw}$ in X^{an} from P_v to P_w (up to fixed endpoint homotopy) whose image under $\tau\colon X^{\mathrm{an}} \to \Gamma$ is γ. We have

$$^{\mathrm{BC}}\!\!\int_{\tilde{\gamma}_{vw}} \omega = \sum_{i=1}^{\ell} \left(^{\mathrm{BC}}\!\!\int_{P_{i(e_i)}}^{P_{e_i}} \omega + {}^{\mathrm{BC}}\!\!\int_{P_{e_i}}^{P_{t(e_i)}} \omega \right).$$

Here the integral from $P_{i(e_i)}$ to P_{e_i} is to be performed on $U_{i(e_i)}$ and the integral from P_{e_i} to $P_{t(e_i)}$ is to be performed on $U_{t(e_i)}$. Indeed, we can see the path $\tilde{\gamma}_{vw}$ as the concatenation (over i) of the path from $P_{i(e_i)}$ to P_{e_i} in $U_{i(e_i)}$ followed by the path from P_{e_i} to $P_{t(e_i)}$ in $U_{t(e_i)}$.

Now, given $x \in U_v$, $y \in U_w$ and a path γ from v to w in Γ, we may consider the Berkovich–Coleman integral of ω from x to y along γ. Indeed, it is the integral along any path $\tilde{\gamma}$ from x to y tropicalizing to γ:

$$^{\mathrm{BC}}\!\!\int_{\tilde{\gamma}} \omega = {}^{\mathrm{BC}}\!\!\int_{x}^{P_v} \omega + {}^{\mathrm{BC}}\!\!\int_{\tilde{\gamma}_{vw}} \omega + {}^{\mathrm{BC}}\!\!\int_{P_w}^{y} \omega$$

where the first and last integrals on the right side are performed on U_v and U_w, respectively. This integral is independent of the choices of reference points.

Finally, for a closed path γ in Γ at a vertex v, we may consider the **Berkovich–Coleman period**

$$^{\mathrm{BC}}\!\!\int_{\tilde{\gamma}} \omega = {}^{\mathrm{BC}}\!\!\int_{\tilde{\gamma}_{vv}} \omega.$$

Again, this is independent of the choice of reference points. Indeed, it depends only on the homology class of γ.

This gives the following algorithm for performing Berkovich–Coleman integration of ω. In particular, we can compute the periods of ω around closed loops.

Algorithm 6: Computing Berkovich–Coleman integrals

Input:

- A holomorphic 1-form ω on X^{an}.
- Points $x \in U_v, y \in U_w$.
- A path $\gamma = e_1 e_2 \dots e_\ell$ from v to w in Γ.

Output: Berkovich–Coleman integral of ω from x to y along $\tilde{\gamma}$.

1. For each i, compute the integrals
$$^{\mathrm{BC}}\!\!\int_{P_{i(e_i)}}^{P_{e_i}} \omega, \quad ^{\mathrm{BC}}\!\!\int_{P_{e_i}}^{P_{t(e_i)}} \omega$$
on the basic wide opens $U_{i(e_i)}$ and $U_{t(e_i)}$, respectively.

2. Compute the sum
$$^{\mathrm{BC}}\!\!\int_{\tilde{\gamma}_{vw}} \omega = \sum_{i=1}^{\ell} \left(^{\mathrm{BC}}\!\!\int_{P_{i(e_i)}}^{P_{e_i}} \omega + {}^{\mathrm{BC}}\!\!\int_{P_{e_i}}^{P_{t(e_i)}} \omega \right).$$

3. Compute the integrals
$$^{\mathrm{BC}}\!\!\int_{x}^{P_v} \omega, \quad ^{\mathrm{BC}}\!\!\int_{P_w}^{y} \omega$$
on the basic wide opens U_v and U_w, respectively.

4. Return
$$^{\mathrm{BC}}\!\!\int_{\tilde{\gamma}} \omega = {}^{\mathrm{BC}}\!\!\int_{x}^{P_v} \omega + {}^{\mathrm{BC}}\!\!\int_{\tilde{\gamma}_{vw}} \omega + {}^{\mathrm{BC}}\!\!\int_{P_w}^{y} \omega.$$

4.7.2 Abelian Integration

We have an algorithm for computing abelian integrals on a hyperelliptic curve X using Theorem 4.2.11 given a semistable cover $\mathcal{C}$ and its dual graph Γ.

Algorithm 7: Computing abelian integrals

Input:

- A holomorphic 1-form ω on X.
- Points $x, y \in X(K)$.

Output: Abelian integral of ω from x to y.

1. Pick a path γ in Γ from v to w for v, w such that $x \in U_v$ and $y \in U_w$.
2. Using Algorithm 6, compute the Berkovich–Coleman integral
$$\prescript{\mathrm{BC}}{}{\int_{\tilde{\gamma}}} \omega.$$
3. Pick a basis $C_1, \ldots, C_h$ for $H_1(\Gamma; \mathbb{Z})$ and a basis $\eta_1, \ldots, \eta_h$ of $\Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{R})$ dual to $C_1, \ldots, C_h$ (see Remark 4.7.3).
4. For each i, pick a loop γ_i in X^{an} whose homology class is C_i, and using Algorithm 6, compute the Berkovich–Coleman periods
$$\prescript{\mathrm{BC}}{}{\int_{\gamma_i}} \omega,\ i = 1, \ldots, h.$$
5. Compute the tropical integrals
$$\prescript{t}{}{\int_{\gamma}} \eta_i,\ i = 1, \ldots, h.$$
6. Return
$$\prescript{\mathrm{Ab}}{}{\int_x^y} \omega = \prescript{\mathrm{BC}}{}{\int_{\tilde{\gamma}}} \omega - \sum_i \left(\prescript{\mathrm{BC}}{}{\int_{\gamma_i}} \omega \right) \left(\prescript{t}{}{\int_{\gamma}} \eta_i \right).$$

In practice, computing tropical integrals is quite easy. It might happen that ${}^{t}\!\int_{\gamma} \eta_i = 0$ for some i, in which case there is no need to compute ${}^{\mathrm{BC}}\!\int_{\gamma_i} \omega$ of course.

Remark 4.7.3. A basis of $H_1(\Gamma; \mathbb{Z})$ and a dual tropical basis can be obtained from the tree T as in Proposition 4.3.16. Let $C'_1, \ldots, C'_h$ be a basis of $H_1(T_o, V_e; \mathbb{Z})$ and let $D'_1, \ldots, D'_h$ be a dual basis with respect to $\langle \cdot, \cdot \rangle$. Let $C_i = \iota(C'_i)$ and $D_i = \frac{1}{2}\iota(D'_i)$ where ι is given in Proposition 4.3.16. Then, by Proposition 4.3.17, $\{C_i\}$ and $\{\eta_i = \eta_{D_i}\}$ form a basis of $H_1(\Gamma; \mathbb{Z})$ and a dual tropical basis of 1-forms on Γ, respectively.

4.8 Numerical Examples

Here, we illustrate our methods with numerical examples computed in Sage. But first, we make the following remarks:

- *Sage restriction.* Let X be a curve defined over $\mathbb{Q}_p$. An abelian integral on X between $\mathbb{Q}_p$-rational points is an element of $\mathbb{Q}_p$. In our approach, such an integral is expressed as a sum of other integrals, each of which is an element of a possibly different finite extension of $\mathbb{Q}_p$. More precisely, reference points corresponding to edges might lie in highly ramified extensions and taking square roots might force us to work with unramified extensions. In Sage, one can define these extensions individually, however, conversion between p-adic extensions has not been implemented yet. In order to deal with this restriction, in each of our examples, all computations will take place in a single extension.

- *Weierstrass endpoints.* Let X be an odd degree hyperelliptic curve with the Abel–Jacobi map $\iota\colon X \to J$ with base-point ∞. For Weierstrass points $R, S \in X(\mathbb{C}_p)$, the class $[S] - [R]$ represents a 2-torsion point of $J(\mathbb{C}_p)$ since
$$\mathrm{div}(x - \alpha) = 2(\alpha, 0) - 2\infty$$
for any root α of the polynomial defining X. This implies by Remark 4.1.1 that the abelian integrals with Weierstrass endpoints must vanish. We will observe this vanishing numerically to test the correctness of our algorithm.

- *Branch of logarithm.* As we discussed before, the Berkovich–Coleman integration requires a branch of the p-adic logarithm. We pick the Iwasawa branch, i.e., the one characterized by $\operatorname{Log}(p) = 0$. Abelian integration does not depend on this choice.

In the examples below, as usual, ω_i will denote the holomorphic 1-form $x^i \frac{dx}{2y}$ on the corresponding curve.

Example 4.8.1. Consider the elliptic curve $X/\mathbb{Q}$ [LMF20, 272.b2] given by

$$y^2 = f(x) = (x-6)(x-5)(x+11).$$

Its Mordell–Weil group is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the point $P = (-3, 24)$ is a generator of the free part.

Hereafter, we consider X over the field $\mathbb{Q}_{17}$; clearly this curve has split multiplicative reduction. Set $R = (23, 102), S = (7, 6)$. Using the formal logarithm implementation in Sage, one can easily check

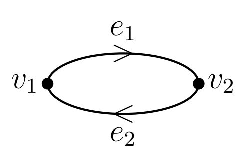
$$\int_S^{\mathrm{Ab}\,R} \omega_0 = 12 \cdot 17 + 8 \cdot 17^2 + 15 \cdot 17^3 + 9 \cdot 17^4 + 16 \cdot 17^5 + 8 \cdot 17^6 + O(17^7). \quad (4.6)$$

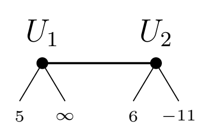
We will compute this integral using our techniques and compare the results.

The set $\{U_1, U_2\}$ is a good semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ with respect to $S_f = \{6, 5, -11, \infty\}$ where

$$U_1 = \mathbb{P}^{1,\mathrm{an}} \setminus \overline{B}(6, 1/17),\ U_2 = B(6, 1)$$

and we have the dual graphs Γ and T





respectively. Note that $R \in \pi^{-1}(U_2),\ S \in \pi^{-1}(U_1)$.

The cycle $C = e_1 + e_2$ and the tropical 1-form $\eta = \frac{1}{2}\eta_C$ are as in Corollary 4.2.8. Now, we pick reference points. Let P_{v_1} and P_{v_2} be points whose x-coordinates are 1 and -28, respectively; hence $P_{v_i} \in \pi^{-1}(U_i)$. Let P_{e_1} and P_{e_2} denote the two different points whose x-coordinates are both $a + 6$ where $a^2 = 17$. Notice that these points lie in the intersection $\pi^{-1}(U_1) \cap \pi^{-1}(U_2)$. We assume that the point P_{e_1} lies in the component corresponding to the edge e_1.

We have

$$\begin{aligned}\pi^{-1}(U_1) &\simeq \{(x,\tilde{y}) \mid \tilde{y}^2 = x-5, x \in U_1\}\\ &= \tilde{X}^{\mathrm{an}}_{x-5} \setminus D_1,\ D_1 = \{(x,\tilde{y}) \mid \tilde{y}^2 = x-5, x \in \overline{B}(6,1/17)\}\end{aligned}$$

where

$$\tilde{y} = \frac{y}{\ell(x)},\ \ell(x) = (x-6)\Big(1+\frac{17}{x-6}\Big)^{1/2}.$$

Define

$$\begin{array}{ccc}\mathbb{P}^1 & \to & \tilde{X}_{x-5}\\ T & \mapsto & \begin{cases}\infty & \text{if } T=\infty,\\ (T^2+2T+6, T+1) & \text{otherwise.}\end{cases}\end{array}$$

This is a parametrization and induces an isomorphism

$$\mathbb{P}^{1,\mathrm{an}} \setminus \big(\overline{B}(0,1/17) \cup \overline{B}(-2,1/17)\big) \simeq \tilde{X}^{\mathrm{an}}_{x-5} \setminus D_1.$$

This annulus is isomorphic to a standard annulus by

$$\begin{array}{ccc}A(1/17^2,1) & \xrightarrow{\sim} & \mathbb{P}^{1,\mathrm{an}} \setminus \big(\overline{B}(0,1/17) \cup \overline{B}(-2,1/17)\big)\\ t & \mapsto & \dfrac{-2t}{t-17}\end{array}$$

and, under these isomorphisms, the 1-form $\omega_{0|\pi^{-1}(U_1)}$ is represented on $A(1/17^2,1)$ by

$$\Big(1+\frac{(t-17)^2}{4t}\Big)^{-1/2}\frac{dt}{2t}.$$

Similarly, we have

$$\begin{aligned}\pi^{-1}(U_2) &\simeq \{(\tilde{x},\tilde{y}) \mid \tilde{y}^2 = \tilde{x}(\tilde{x}+1), \tilde{x} \in B(0,17)\}\\ &= \tilde{X}^{\mathrm{an}}_{\tilde{x}(\tilde{x}+1)} \setminus D_2,\ D_2 = \{(\tilde{x},\tilde{y}) \mid \tilde{y}^2 = \tilde{x}(\tilde{x}+1), \tilde{x} \in \overline{B}(\infty,1/17)\}\end{aligned}$$

where

$$\tilde{x} = \frac{x-6}{17},\ \tilde{y} = \frac{y}{17\ell(\tilde{x})},\ \ell(\tilde{x}) = (1+17\tilde{x})^{1/2}.$$

Define

$$\begin{array}{rcl} \mathbb{P}^1 & \to & \tilde{X}_{\tilde{x}(\tilde{x}+1)} \\ T & \mapsto & \begin{cases} \infty^+ & \text{if } T = 0, \\ \infty^- & \text{if } T = \infty, \\ \left(\frac{1}{2}\left(T + \frac{1}{4T}\right) - \frac{1}{2}, \frac{1}{2}\left(T - \frac{1}{4T}\right)\right) & \text{otherwise.} \end{cases} \end{array}$$

This is a parametrization and induces an isomorphism

$$\mathbb{P}^{1,\mathrm{an}} \setminus \left(\overline{B}(0, 1/17) \cup \overline{B}(\infty, 1/17)\right) \simeq \tilde{X}^{\mathrm{an}}_{\tilde{x}(\tilde{x}+1)} \setminus D_2.$$

The annulus on the left is isomorphic to a standard annulus by

$$\begin{array}{rcl} A(1/17^2, 1) & \xrightarrow{\sim} & \mathbb{P}^{1,\mathrm{an}} \setminus \left(\overline{B}(0, 1/17) \cup \overline{B}(\infty, 1/17)\right) \\ t & \mapsto & t/17 \end{array}$$

and, under these isomorphisms, the 1-form $\omega_{0|\pi^{-1}(U_2)}$ is represented on $A(1/17^2, 1)$ by

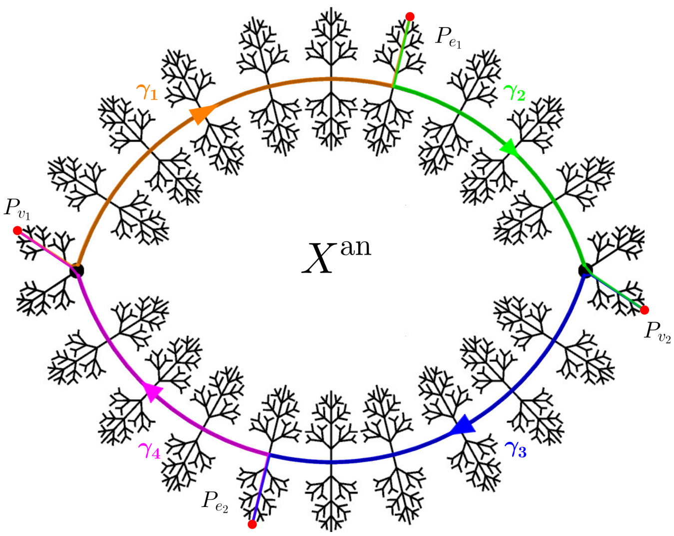
$$\left(1 + \frac{(t - 17/2)^2}{2t}\right)^{-1/2} \frac{dt}{2t}.$$

Let γ be the concatenation of a path from S to P_{e_1} in $\pi^{-1}(U_1)$ and a path from P_{e_1} to R in $\pi^{-1}(U_2)$. Then $\tau(\gamma) = e_1$ and we have

$${}^{\mathrm{BC}}\!\!\int_\gamma \omega_0 = 15 \cdot a^4 + 11 \cdot a^6 + 12 \cdot a^8 + a^{10} + 11 \cdot a^{12} + O(a^{14}),$$

$${}^{t}\!\!\int_{e_1} \eta = \frac{1}{2}.$$

Consider the loop $\gamma_C = \gamma_1\gamma_2\gamma_3\gamma_4$ in X^{an} where γ_1 is a path from P_{v_1} to P_{e_1} in $\pi^{-1}(U_1)$, γ_2 is a path from P_{e_1} to P_{v_2} in $\pi^{-1}(U_2)$, γ_3 is a path from P_{v_2} to P_{e_2} in $\pi^{-1}(U_2)$ and γ_4 is a path from P_{e_2} to P_{v_1} in $\pi^{-1}(U_1)$. The paths are shown in a figure modified from one in [KRZB18].



The homology class of γ_C is C and the period integral is

$$\prescript{\mathrm{BC}}{}{\int_{\gamma_C}} \omega_0 = 10 \cdot a^2 + 12 \cdot a^4 + 9 \cdot a^6 + 5 \cdot a^8 + 4 \cdot a^{10} + 4 \cdot a^{12} + O(a^{14}).$$

Finally,

$$\begin{aligned} \prescript{\mathrm{Ab}}{}{\int_S^R} \omega_0 &= \prescript{\mathrm{BC}}{}{\int_{\gamma}} \omega_0 - \left(\prescript{\mathrm{BC}}{}{\int_{\gamma_C}} \omega_0\right)\left(\int_{e_1}^{t} \eta\right) \\ &= 12 \cdot a^2 + 8 \cdot a^4 + 15 \cdot a^6 + 9 \cdot a^8 + 16 \cdot a^{10} + 8 \cdot a^{12} + O(a^{14}) \end{aligned}$$

which is the same result as in (4.6) since $a^2 = 17$.

We also note that, using the addition law on elliptic curves, for each $i \in \{0, \ldots, 100\}$ our methods give

$$\prescript{\mathrm{Ab}}{}{\int_{(5,0)+iP}^{(6,0)+iP}} \omega_0 = \prescript{\mathrm{Ab}}{}{\int_{(5,0)+iP}^{(-11,0)+iP}} \omega_0 = O(a^{14})$$

demonstrating the vanishing of integrals between points whose difference is torsion.

Example 4.8.2. Let $X/\mathbb{Q}_7$ be the genus 2 curve defined by

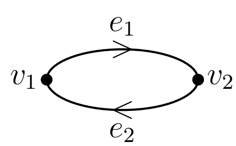
$$y^2 = f(x) = x(x-1)(x-2)(x-3)(x-7).$$

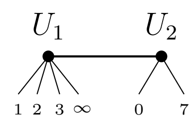
Set $R = (0,0), S = (1,0)$; we already know that the abelian integral of ω from S to R vanishes for every holomorphic form ω on X. Using our techniques, we will verify this up to a certain precision.

The set $\{U_1, U_2\}$ is a good semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ with respect to $S_f = \{0,1,2,3,7,\infty\}$ where

$$U_1 = \mathbb{P}^{1,\mathrm{an}} \setminus \overline{B}(0,1/7),\ U_2 = B(0,1)$$

and we have the dual graphs Γ and T





respectively. Notice that $R \in \pi^{-1}(U_2),\ S \in \pi^{-1}(U_1)$.

The cycle $C = e_1 + e_2$ and the tropical 1-form $\eta = \frac{1}{2}\eta_C$ are as in Corollary 4.2.8. Again, we pick reference points. Let P_{v_1} and P_{v_2} be points whose x-coordinates are -1 and 14, respectively; hence $P_{v_i} \in \pi^{-1}(U_i)$. Let P_{e_1} and P_{e_2} denote the two different points whose x-coordinates are both a where $a^2 = 7$. Notice that these points lie in the intersection $\pi^{-1}(U_1) \cap \pi^{-1}(U_2)$. We assume that the point P_{e_1} lies in the component corresponding to the edge e_1.

The analytic open $\pi^{-1}(U_1)$ is embedded into a good reduction elliptic curve. In fact, we have

$$\pi^{-1}(U_1) \simeq \{(x, \tilde{y}) \mid \tilde{y}^2 = (x-1)(x-2)(x-3), x \in U_1\}$$

where

$$\tilde{y} = \frac{y}{\ell(x)},\ \ell(x) = x\Big(1 - \frac{7}{x}\Big)^{1/2}.$$

In the new coordinates, the 1-form $\omega_{i|\pi^{-1}(U_1)}$ is given by

$$\Big(1 - \frac{7}{x}\Big)^{-1/2} x^{i-1}\frac{dx}{2\tilde{y}}.$$

For the other component, we have

$$\begin{aligned}\pi^{-1}(U_2) &\simeq \{(\tilde{x}, \tilde{y}) \mid \tilde{y}^2 = \tilde{x}(\tilde{x}-1), \tilde{x} \in B(0,7)\}\\ &= \tilde{X}^{\mathrm{an}}_{\tilde{x}(\tilde{x}-1)} \setminus D,\ D = \{(\tilde{x}, \tilde{y}) \mid \tilde{y}^2 = \tilde{x}(\tilde{x}-1), \tilde{x} \in \overline{B}(\infty, 1/7)\}\end{aligned}$$

where

$$\tilde{x} = \frac{x}{7},\ \tilde{y} = \frac{y}{7\ell(\tilde{x})},\ \ell(\tilde{x}) = \big((7\tilde{x}-1)(7\tilde{x}-2)(7\tilde{x}-3)\big)^{1/2}.$$

Define

$$\begin{array}{rcl} \mathbb{P}^1 & \to & \tilde{X}_{\tilde{x}(\tilde{x}-1)} \\ T & \mapsto & \begin{cases} \infty^+ & \text{if } T = 0, \\ \infty^- & \text{if } T = \infty, \\ \Big(\frac{1}{2}\big(T + \frac{1}{4T}\big) + \frac{1}{2}, \frac{1}{2}\big(T - \frac{1}{4T}\big)\Big) & \text{otherwise.} \end{cases} \end{array}$$

This is a parametrization and induces an isomorphism

$$\mathbb{P}^{1,\mathrm{an}} \setminus \big(\overline{B}(0,1/7) \cup \overline{B}(\infty,1/7)\big) \simeq \tilde{X}^{\mathrm{an}}_{\tilde{x}(\tilde{x}-1)} \setminus D.$$

The annulus on the left is isomorphic to a standard annulus by

$$\begin{array}{rcl} A(1/7^2,1) & \xrightarrow{\sim} & \mathbb{P}^{1,\mathrm{an}} \setminus \big(\overline{B}(0,1/7) \cup \overline{B}(\infty,1/7)\big) \\ t & \mapsto & t/7 \end{array}$$

and, under these isomorphisms, the 1-form $\omega_{i|\pi^{-1}(U_2)}$ is expressed on $A(1/7^2,1)$ as

$$\Big(\frac{(t+7/2)^2}{2t}\Big)^i \bigg(\Big(\frac{(t+7/2)^2}{2t} - 1\Big)\Big(\frac{(t+7/2)^2}{2t} - 2\Big)\Big(\frac{(t+7/2)^2}{2t} - 3\Big)\bigg)^{-1/2} \frac{dt}{2t}.$$

As in the previous example, take a path γ from S to R such that $\tau(\gamma) = e_1$ and take a loop γ_C whose homology class is C. Then our computations give

$$\begin{aligned} {}^{\mathrm{BC}}\!\!\int_\gamma \omega_0 &= 4\cdot a^6 + 2\cdot a^8 + 2\cdot a^{10} + 5\cdot a^{12} + O(a^{14}), \\ {}^{\mathrm{BC}}\!\!\int_\gamma \omega_1 &= 6\cdot a^2 + 6\cdot a^6 + 4\cdot a^{10} + 6\cdot a^{12} + O(a^{14}), \\ {}^{\mathrm{BC}}\!\!\int_{\gamma_C} \omega_0 &= a^6 + 5\cdot a^8 + 4\cdot a^{10} + 3\cdot a^{12} + O(a^{14}), \\ {}^{\mathrm{BC}}\!\!\int_{\gamma_C} \omega_1 &= 5\cdot a^2 + a^4 + 5\cdot a^6 + a^8 + a^{10} + 6\cdot a^{12} + O(a^{14}), \end{aligned}$$

$$\int_{e_1}^{t} \eta = \frac{1}{2}.$$

Combining these, we get

$$ {}^{\mathrm{Ab}}\!\!\int_S^R \omega_i = {}^{\mathrm{BC}}\!\!\int_\gamma \omega_i - \left({}^{\mathrm{BC}}\!\!\int_{\gamma_C} \omega_i \right) \left(\int_{e_1}^{t} \eta \right) = O(a^{14}),\ i = 0, 1$$

from which our aim follows as every holomorphic 1-form is a linear combination of ω_0 and ω_1.

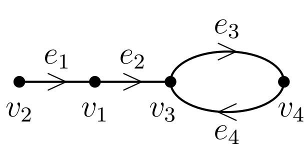
Example 4.8.3. Let $X/\mathbb{Q}_{13}$ be the genus 3 curve given by

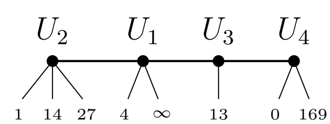
$$y^2 = f(x) = x(x-13)(x-169)(x-1)(x-14)(x-27)(x-4).$$

The set $\{U_1, U_2, U_3, U_4\}$ is a good semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ with respect to $S_f = \{0, 13, 169, 1, 14, 27, 4, \infty\}$ where

$$\begin{aligned} &U_1 = \mathbb{P}^{1,\mathrm{an}} \setminus \big(\overline{B}(1, 1/13) \cup \overline{B}(0, 1/13)\big),\\ &U_2 = B(1, 1),\\ &U_3 = A(1/169, 1),\ U_4 = B(0, 1/13), \end{aligned}$$

and we have the dual graphs Γ and T





respectively.

The cycle $C = e_3 + e_4$ and the tropical 1-form $\eta = \frac{1}{2}\eta_C$ are as in Corollary 4.2.8. Let $P_{v_1}, P_{v_2}, P_{v_3}, P_{v_4}$ be points whose x-coordinates are $2, 20/7, -13/12, 169/14$, respectively; hence $P_{v_i} \in \pi^{-1}(U_i)$. For an a such that $a^4 = 13$, let P_{e_1} and P_{e_2} denote points whose x-coordinates are $a^2 + 1$ and a^2, respectively; and let P_{e_3} and P_{e_4} be the two different points whose x-coordinates are both $13a^2$. Notice that

$$P_{e_1} \in \pi^{-1}(U_2) \cap \pi^{-1}(U_1),\ P_{e_2} \in \pi^{-1}(U_1) \cap \pi^{-1}(U_3)$$

and that

$$P_{e_3}, P_{e_4} \in \pi^{-1}(U_3) \cap \pi^{-1}(U_4).$$

We assume that the point P_{e_3} lies in the component corresponding to the edge e_3.

The preimage $\pi^{-1}(U_1)$ is embedded into a good reduction elliptic curve:

$$\pi^{-1}(U_1) \simeq \{(x, \tilde{y}) \mid \tilde{y}^2 = x(x-1)(x-4), x \in U_1\}$$

where

$$\tilde{y} = \frac{y}{\ell(x)},\ \ell(x) = x(x-1)\Big(\Big(1-\frac{13}{x}\Big)\Big(1-\frac{169}{x}\Big)\Big(1-\frac{13}{x-1}\Big)\Big(1-\frac{26}{x-1}\Big)\Big)^{1/2}.$$

The 1-form $\omega_{i|\pi^{-1}(U_1)}$ is represented by

$$\Big(\Big(1-\frac{13}{x}\Big)\Big(1-\frac{169}{x}\Big)\Big(1-\frac{13}{x-1}\Big)\Big(1-\frac{26}{x-1}\Big)\Big)^{-1/2}\frac{x^{i-1}}{x-1}\frac{dx}{2\tilde{y}}.$$

Similarly, $\pi^{-1}(U_2)$ is also embedded into an elliptic curve:

$$\pi^{-1}(U_2) \simeq \{(\tilde{x}, \tilde{y}) \mid \tilde{y}^2 = \tilde{x}(\tilde{x}-1)(\tilde{x}-2), \tilde{x} \in B(0,13)\}$$

where

$$\tilde{x} = \tfrac{x-1}{13},\ \tilde{y} = \tfrac{y}{13\sqrt{13}\cdot\ell(\tilde{x})},\ \ell(\tilde{x}) = \big((13\tilde{x}+1)(13\tilde{x}-12)(13\tilde{x}-168)(13\tilde{x}-3)\big)^{1/2}.$$

The 1-form $\omega_{i|\pi^{-1}(U_2)}$ becomes

$$\frac{1}{\sqrt{13}}(13\tilde{x}+1)^i\big((13\tilde{x}+1)(13\tilde{x}-12)(13\tilde{x}-168)(13\tilde{x}-3)\big)^{-1/2}\frac{d\tilde{x}}{2\tilde{y}}.$$

Now, $\pi^{-1}(U_3)$ is embedded into a rational curve:

$$\pi^{-1}(U_3) \simeq \{(x, \tilde{y}) \mid \tilde{y}^2 = x-13, x \in U_3\}$$

where

$$\tilde{y} = \frac{y}{\ell(x)},\ \ell(x) = x\Big(1-\frac{169}{x}\Big)^{1/2}\big((x-1)(x-14)(x-27)(x-4)\big)^{1/2}.$$

Under this isomorphism, the 1-form $\omega_{i|\pi^{-1}(U_3)}$ is represented by

$$\Big(1-\frac{169}{x}\Big)^{-1/2}\big((x-1)(x-14)(x-27)(x-4)\big)^{-1/2}x^{i-1}\frac{dx}{2\tilde{y}}.$$

The analytic open $\pi^{-1}(U_4)$ is also embedded into a rational curve:

$$\pi^{-1}(U_4) \simeq \{(\tilde{x},\tilde{y}) \mid \tilde{y}^2 = \tilde{x}(\tilde{x}-1), \tilde{x} \in B(0,13)\}$$

where

$$\tilde{x} = \tfrac{x}{169},\ \tilde{y} = \tfrac{y}{169\cdot \ell(\tilde{x})},\ \ell(\tilde{x}) = \big((169\tilde{x}-13)(169\tilde{x}-1)(169\tilde{x}-14)(169\tilde{x}-27)(169\tilde{x}-4)\big)^{1/2}.$$

In the new coordinates, the 1-form $\omega_{i|\pi^{-1}(U_4)}$ becomes

$$\big((169\tilde{x}-13)(169\tilde{x}-1)(169\tilde{x}-14)(169\tilde{x}-27)(169\tilde{x}-4)\big)^{-1/2}(169\tilde{x})^i \frac{d\tilde{x}}{2\tilde{y}}.$$

We start by verifying (up to a certain precision) that the abelian integral of ω_0 vanishes between the Weierstrass points $R = (13,0), S = (1,0)$. Note that $R \in \pi^{-1}(U_3)$ and that $S \in \pi^{-1}(U_2)$. Consider the concatenation $\gamma = \gamma_1\gamma_2\gamma_3$, where γ_1 is a path from S to P_{e_1} in $\pi^{-1}(U_2)$, γ_2 is a path from P_{e_1} to P_{e_2} in $\pi^{-1}(U_1)$ and γ_3 is a path from P_{e_2} to R in $\pi^{-1}(U_3)$; hence $\tau(\gamma) = e_1e_2$. Since the tropical integral of η along e_1e_2 is 0, we have the equality

$$\prescript{\mathrm{Ab}}{}{\int_S^R} \omega_0 = \prescript{\mathrm{BC}}{}{\int_\gamma} \omega_0 = \prescript{\mathrm{BC}}{}{\int_S^{P_{e_1}}} \omega_0 + \prescript{\mathrm{BC}}{}{\int_{P_{e_1}}^{P_{e_2}}} \omega_0 + \prescript{\mathrm{BC}}{}{\int_{P_{e_2}}^{R}} \omega_0.$$

Our methods yield

$$\prescript{\mathrm{BC}}{}{\int_S^{P_{e_1}}} \omega_0 = 2\cdot a^{-1} + 8\cdot a + 6\cdot a^3 + 9\cdot a^5 + 8\cdot a^7 + 3\cdot a^9 + 5\cdot a^{11} + O(a^{13}),$$
$$\prescript{\mathrm{BC}}{}{\int_{P_{e_1}}^{P_{e_2}}} \omega_0 = 4\cdot a^{-1} + 6\cdot a + 3\cdot a^3 + 10\cdot a^5 + 8\cdot a^7 + 9\cdot a^9 + 11\cdot a^{11} + O(a^{13}),$$
$$\prescript{\mathrm{BC}}{}{\int_{P_{e_2}}^{R}} \omega_0 = 7\cdot a^{-1} + 12\cdot a + 3\cdot a^3 + 5\cdot a^5 + 9\cdot a^7 + 12\cdot a^9 + 8\cdot a^{11} + O(a^{13}),$$

from which we get

$$\prescript{\mathrm{Ab}}{}{\int_S^R} \omega_0 = O(a^{13})$$

as required.

To demonstrate our methods, we compute the abelian integral of $\omega = \omega_1 + \omega_2$ between the following two points lying in different basic wide opens:

$$R = (13^3, 2\cdot 13^3 + 2\cdot 13^4 + 10\cdot 13^5 + 11\cdot 13^6 + O(13^7)) \in \pi^{-1}(U_4),$$
$$S = (7, 4 + 7\cdot 13^2 + 12\cdot 13^4 + 6\cdot 13^5 + O(13^7)) \in \pi^{-1}(U_1).$$

Set $\gamma = \gamma_1\gamma_2\gamma_3$ where γ_1 is a path from S to P_{e_2} in $\pi^{-1}(U_1)$, γ_2 is a path from P_{e_2} to P_{e_3} in $\pi^{-1}(U_3)$ and γ_3 is a path from P_{e_3} to R in $\pi^{-1}(U_4)$; thus $\tau(\gamma) = e_2e_3$. For this path, we have

$$\begin{aligned}{}^{\mathrm{BC}}\!\!\int_{\gamma} \omega = 11 + 4 \cdot a^2 + 2 \cdot a^4 + 10 \cdot a^6 + 6 \cdot a^8 + 7 \cdot a^{10} + 8 \cdot a^{12} + 9 \cdot a^{14} \\ + 9 \cdot a^{16} + 11 \cdot a^{18} + 9 \cdot a^{20} + 6 \cdot a^{22} + 4 \cdot a^{24} + 10 \cdot a^{26} + O(a^{28}),\end{aligned}$$

$${}^{t}\!\!\int_{\tau(\gamma)} \eta = \frac{1}{2}.$$

Consider the loop $\gamma_C = \gamma_1\gamma_2\gamma_3\gamma_4$ in X^{an} where γ_1 is a path from P_{v_3} to P_{e_3} in $\pi^{-1}(U_3)$, γ_2 is a path from P_{e_3} to P_{v_4} in $\pi^{-1}(U_4)$, γ_3 is a path from P_{v_4} to P_{e_4} in $\pi^{-1}(U_4)$ and γ_4 is a path from P_{e_4} to P_{v_3} in $\pi^{-1}(U_3)$. The homology class of this loop is C and the period integral is

$${}^{\mathrm{BC}}\!\!\int_{\gamma_C} \omega = 8 \cdot a^2 + 7 \cdot a^6 + 2 \cdot a^{10} + 6 \cdot a^{14} + 10 \cdot a^{18} + 8 \cdot a^{26} + O(a^{28}).$$

Consequently, we have

$$\begin{aligned}{}^{\mathrm{Ab}}\!\!\int_{S}^{R} \omega &= {}^{\mathrm{BC}}\!\!\int_{\gamma} \omega - \left({}^{\mathrm{BC}}\!\!\int_{\gamma_C} \omega\right)\left({}^{t}\!\!\int_{\tau(\gamma)} \eta\right) \\ &= 11 + 2 \cdot a^4 + 6 \cdot a^8 + 8 \cdot a^{12} + 9 \cdot a^{16} + 9 \cdot a^{20} + 4 \cdot a^{24} + O(a^{28}) \\ &= 11 + 2 \cdot 13 + 6 \cdot 13^2 + 8 \cdot 13^3 + 9 \cdot 13^4 + 9 \cdot 13^5 + 4 \cdot 13^6 + O(13^7).\end{aligned}$$

Example 4.8.4. Consider the even degree hyperelliptic curve $X/\mathbb{Q}$ [LMF20, 3200.f.819200.1] defined by the equation

$$y^2 = f(x) = (x^2 - 2)(x^2 - x - 1)(x^2 + x - 1).$$

According to the database, this curve has exactly six rational points. In this final example, we will identify the annihilating differential to be used in the Chabauty–Coleman method at a prime of bad reduction. See the survey [MP12] (especially Appendix A) for a detailed account of the method with many references.

The curve X has bad reduction at the prime 5 and its minimal regular model $\mathfrak{X}$ over $\mathbb{Z}_5$ is given by the same equation as the above Weierstrass model. The Chabauty–Coleman bound [LT02, Corollary 1.11] (see also [KZB13, Theorem 1.4] for a refinement) gives

$$\#X(\mathbb{Q}) \leq \#\mathfrak{X}^{\mathrm{sm}}_{\mathbb{F}_5}(\mathbb{F}_5) + 2 = 8$$

where $\mathfrak{X}_{\mathbb{F}_5}^{\mathrm{sm}}$ denotes the smooth locus of the special fiber of $\mathfrak{X}$. A point count in Magma reveals the set of all rational points of naive height bounded by 10^5:

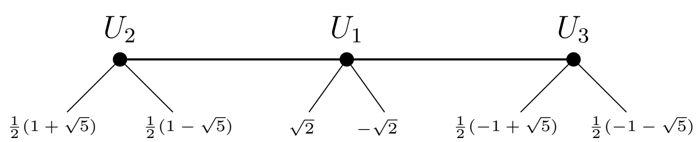
$$\{\infty^+, \infty^-, (1, \pm 1), (-1, \pm 1)\} \subseteq X(\mathbb{Q}). \tag{4.7}$$

Another computation in Magma shows that the Mordell-Weil rank of the Jacobian of X is equal to 1. Therefore, in order to check whether or not the curve X has more rational points, one can use [MP12, Theorem A.5.(1)]. The crucial step is to construct the unique, up to a scalar multiple, annihilating differential on X. The fact that the known rational points are all in different residue discs makes it necessary to compute non-tiny integrals; this can be achieved by using our techniques.

The set $\{U_1, U_2, U_3\}$ is a good semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ with respect to the set $S_f = \{\pm\sqrt{2}, \frac{1}{2}(1 \pm \sqrt{5}), \frac{1}{2}(-1 \pm \sqrt{5})\}$, where

$$\begin{aligned} U_1 &= \mathbb{P}^{1,\mathrm{an}} \setminus \left(\overline{B}(1/2, 1/\sqrt{5}) \cup \overline{B}(-1/2, 1/\sqrt{5})\right), \\ U_2 &= B(1/2, 1), \\ U_3 &= B(-1/2, 1), \end{aligned}$$

with dual graph T



Consider the points $R = (1, -1)$ and $S = (1, 1)$, both belong to the space $\pi^{-1}(U_1)$. Therefore, for every holomorphic 1-form ω on X, we have the equality

$$\prescript{\mathrm{Ab}}{}{\int_S^R} \omega = \prescript{\mathrm{BC}}{}{\int_S^R} \omega.$$

The basic wide open $\pi^{-1}(U_1)$ is embedded into a rational curve:

$$\pi^{-1}(U_1) \simeq \{(x, \tilde{y}) \mid \tilde{y}^2 = x^2 - 2, x \in U_1\}$$

where

$$\tilde{y} = \frac{y}{\ell(x)}, \ \ell(x) = \left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right)\left(\left(1 - \frac{5/4}{(x-1/2)^2}\right)\left(1 - \frac{5/4}{(x+1/2)^2}\right)\right)^{1/2}.$$

Under this isomorphism, the 1-form $\omega_{i|\pi^{-1}(U_1)}$ is represented by

$$\left(\left(1-\frac{5/4}{(x-1/2)^2}\right)\left(1-\frac{5/4}{(x+1/2)^2}\right)\right)^{-1/2}\frac{x^i}{x^2-1/4}\frac{dx}{2\tilde{y}}.$$

Our methods yield

$$\begin{aligned}
a &:= {}^{\mathrm{BC}}\!\!\int_S^R \omega_0 = 2\cdot 5+5^4+3\cdot 5^6+2\cdot 5^7+2\cdot 5^8+4\cdot 5^9+O(5^{10}),\\
b &:= {}^{\mathrm{BC}}\!\!\int_S^R \omega_1 = O(5^{10}),
\end{aligned}$$

which give the annihilating differential as

$$\omega := b\omega_0 - a\omega_1.$$

It can be shown that the inclusion in (4.7) is actually an equality applying [MP12, Theorem A.5.(1)] to the annihilating differential ω. In particular, we have

$$X(\mathbb{Q}) = \{\infty^+, \infty^-, (1,\pm 1), (-1,\pm 1)\}.$$

Remark 4.8.5. We end this example with a remark about the importance of the Chabauty–Coleman method at a prime of bad reduction. An illustration was provided in Katz–Zureick-Brown [KZB13, Example 5.1]. In this example, for a certain curve $X/\mathbb{Q}$ that has bad reduction at 5, it is shown that 5 is the only prime at which the refined Chabauty–Coleman bound [KZB13, Theorem 1.4] is sharp. Hence, one cannot determine the set $X(\mathbb{Q})$ by using the Chabauty–Coleman bound at primes of good reduction alone; it is necessary to work with a prime of bad reduction or to make use of other techniques.

Chapter 5

Explicit Vologodsky Integration for Hyperelliptic Curves

In the present chapter, which is based on [Kay20], we extend the techniques of Chapter 4 in a natural way to also cover meromorphic forms, building on the recent work of Besser and Zerbes [BZ17] which *relates* Vologodsky integration on bad reduction curves to Coleman primitives. In particular, we present an algorithm for computing Vologodsky integrals of general meromorphic 1-forms on bad reduction hyperelliptic curves for $p \neq 2$.

The base field in [BZ17] is a finite extension of $\mathbb{Q}_p$, hence, in this chapter, we assume that our field K is finite over $\mathbb{Q}_p$.

5.1 Comparison of the Integrals

We fix a smooth, proper and geometrically connected (algebraic) curve X over K. In this section, we give a formula for passing between Vologodsky and Berkovich–Coleman integration. As in Section 4.2, the relation between the two will be provided by tropical integration. We first modify its definition slightly: we replace "$\mathbb{R}$" by "$\mathbb{C}_p$" in Subsection 4.2.4; this will allow us to incorporate the main result of Besser–Zerbes [BZ17] into our framework.

5.1.1 Tropical Integration Revisited

Let Γ be a finite and connected graph.

Definition 5.1.2. A **tropical** 1**-form** on Γ is a function $\eta\colon E(\Gamma) \to \mathbb{C}_p$ such that

1. $\eta(-e) = -\eta(e)$, and
2. for each $v \in V(\Gamma)$, we have $\sum_e \eta(e) = 0$ where the sum is taken over all edges that are directed away from v.

Denote the space of such functions by $\Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{C}_p)$.

Note that $\Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{C}_p) \subset C_1(\Gamma, \mathbb{C}_p)^*$. For later use, we provide the following well known proposition which describes the difference.

Proposition 5.1.3. *We have*

$$C_1(\Gamma, \mathbb{C}_p)^* = \Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{C}_p) \oplus \mathrm{Im}(d^*)$$

where $d^*\colon C_0(\Gamma, \mathbb{C}_p)^* \to C_1(\Gamma, \mathbb{C}_p)^*$ is the coboundary map.

Now let η be a tropical 1-form and let γ be a path in Γ. We then define the tropical integral ${}^t\!\int_\gamma \eta$ as in Subsection 4.2.4. We remark that the modified tropical integration takes values in $\mathbb{C}_p$, that is, ${}^t\!\int_\gamma \eta \in \mathbb{C}_p$.

5.1.4 The Comparison Formula

Recall that we have the curve X over K. By extending the field of definition if necessary, we assume that the curve X admits a semistable $\mathcal{O}_K$-model $\mathfrak{X}$. Let $\Gamma \subset X^{\mathrm{an}}$ and $\tau\colon X^{\mathrm{an}} \to \Gamma$ be the corresponding skeleton and retraction, respectively. After identifying Γ with the dual graph of the special fiber $\mathfrak{X}_{\Bbbk}$, we have the following result whose proof is almost identical to that of Corollary 4.2.8.

Proposition 5.1.5. *There exists a basis $C_1, \ldots, C_h$ of $H_1(\Gamma; \mathbb{C}_p)$ and a basis $\eta_1, \ldots, \eta_h$ of $\Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{C}_p)$ with the property that*

$${}^t\!\!\int_{C_i} \eta_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Now, we give a comparison theorem for Vologodsky and Berkovich–Coleman integrals.

Theorem 5.1.6. *With the above notation, pick a loop γ_i in X^{an} satisfying $\tau(\gamma_i) = C_i$ for each $i = 1, \dots, h$. Let ω be a meromorphic 1-form on X, let $x, y \in X(K)$ and pick a path γ in X^{an} with $\gamma(0) = x$, $\gamma(1) = y$. Then*

$$ {}^{\mathrm{BC}}\!\!\int_{\gamma} \omega - {}^{\mathrm{Vol}}\!\!\int_{x}^{y} \omega = \sum_{i=1}^{h} \left({}^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega \right) \left(\int_{\tau(\gamma)}^{t} \eta_i \right). \tag{5.1} $$

The proof of this will occupy the rest of this section. Before giving the proof, we make a few preliminary remarks.

Remark 5.1.7. The path γ exists because the assumption that X is geometrically connected implies that X^{an} is path-connected; and the integral ${}^{\mathrm{BC}}\!\int_{\gamma_i} \omega$ is well-defined as γ_i is unique up to fixed endpoint homotopy.

Remark 5.1.8. The Vologodsky and Berkovich–Coleman integrals coincide when Γ is a tree, or equivalently, when the Jacobian of X has (potentially) good reduction; and, in particular, when X has (potentially) good reduction.

Remark 5.1.9. Theorem 5.1.6 generalizes Theorem 4.2.11 since, when the integrand is holomorphic, the Vologodsky integral is the same as the abelian integral.

Theorem 5.1.6 follows from the main result of Besser and Zerbes [BZ17], which we briefly recall here.

Write $\mathfrak{X}_{\Bbbk} = \cup_i T_i$. By blowing up if necessary, we may assume that every irreducible component is smooth and that two different components intersect at at most one point.

The reduction map $\mathrm{red}\colon X \to \mathfrak{X}_{\Bbbk}$ allows us to cover X by basic wide open spaces: define $U_v = \mathrm{red}^{-1} T_v$ for each $v \in V(\Gamma)$. These spaces intersect along annuli corresponding bijectively to the quotient set of unoriented edges $E(\Gamma)/\pm$. An orientation of an annulus fixes a sign for the residue along this annulus. We make a bijection between oriented edges and oriented annuli by choosing the following convention:

> For an edge e, the orientation of the corresponding annulus is the one corresponding to it being the annulus end of $T_{i(e)}$.

We will use the same notation for an edge and for the corresponding annulus.

Choose Coleman primitives F_v for ω on U_v for each $v \in V(\Gamma)$. Notice that, for an oriented edge e, the function $F_{t(e)}|_e - F_{i(e)}|_e$ is constant because both Coleman primitives differentiate to ω. This observation gives a map

$$ \eta_\omega \colon E(\Gamma) \to \mathbb{C}_p, \quad e \mapsto F_{t(e)}|_e - F_{i(e)}|_e $$

which obviously satisfies $\eta_\omega(-e) = -\eta_\omega(e)$. By Proposition 5.1.3, there is a unique way (up to a global constant) of choosing the F_v's such that the map η_ω is a tropical 1-form. With these choices, thanks to [BZ17, Theorem 1.1], the following holds.

Theorem 5.1.10. *If v, w are vertices such that $x \in U_v$ and $y \in U_w$, then*

$$\,^{\mathrm{Vol}}\!\!\int_x^y \omega = F_w(y) - F_v(x).$$

In other words, Vologodsky integration is *locally* given by Coleman primitives.

In the following proof, we use the fact that Berkovich–Coleman integrals on basic wide open spaces are ordinary Coleman integrals (under the identification of a Berkovich space and its corresponding rigid space).

Proof of Theorem 5.1.6. We continue with the above notation. In the case that $v = w$, the formula in Theorem 5.1.6 is the same as the formula in Theorem 5.1.10. Otherwise, if $\tau(\gamma) = e_1 e_2 \dots e_\ell$, then we can write $\gamma = \gamma^1\gamma^2 \dots \gamma^{\ell+1}$ as a concatenation of paths, each staying in a basic wide open space. Set

$$P_i = \gamma^i(1) = \gamma^{i+1}(0), \quad i = 1, 2, \dots, \ell.$$

Then we get

$$F_w(y) - F_v(x) = \,^{\mathrm{BC}}\!\!\int_x^{P_1} \omega + \,^{\mathrm{BC}}\!\!\int_{P_1}^{P_2} \omega + \dots + \,^{\mathrm{BC}}\!\!\int_{P_\ell}^{y} \omega + \sum_{i=1}^{\ell} \eta_\omega(e_i) = \,^{\mathrm{BC}}\!\!\int_\gamma \omega + \eta_\omega(\tau(\gamma)).$$

Therefore, we need to show that

$$\eta_\omega(\tau(\gamma)) = -\sum_{i=1}^{h} \left(\,^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega\right) \left(\,^{t}\!\!\int_{\tau(\gamma)} \eta_i\right).$$

Since η_ω is a tropical 1-form, $\eta_\omega = \sum_{i=1}^{h} c_i \eta_i$ for some constants c_i. The computation

$$\eta_\omega(C_j) = \sum_{i=1}^{h} c_i \eta_i(C_j) = \sum_{i=1}^{h} c_i \,^{t}\!\!\int_{C_j} \eta_i = c_j$$

gives

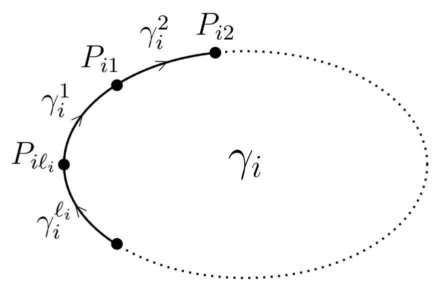
$$\eta_\omega(\tau(\gamma)) = \sum_{i=1}^{h} \eta_\omega(C_i) \eta_i(\tau(\gamma)) = \sum_{i=1}^{h} \eta_\omega(C_i) \left(\,^{t}\!\!\int_{\tau(\gamma)} \eta_i\right).$$

It remains to verify that

$$\eta_\omega(C_i) = -{}^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega, \quad i = 1, \dots, h.$$

If $C_i = e_{i1}e_{i2}\dots e_{i\ell_i}$, then we can write $\gamma_i = \gamma_i^1\gamma_i^2\dots\gamma_i^{\ell_i}$ as a concatenation of paths, each staying in a basic wide open space. Set

$$P_{ij} = \begin{cases} \gamma_i^j(1) = \gamma_i^{j+1}(0) & \text{if } j = 1, 2, \dots, \ell_i - 1, \\ \gamma_i^{\ell_i}(1) = \gamma_i^1(0) & \text{if } j = \ell_i. \end{cases}$$



By setting $F_{ij} = F_{i(e_{ij})}$ for $j = 1, 2, \dots, \ell_i$, we see that

$$\begin{aligned} {}^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega &= {}^{\mathrm{BC}}\!\!\int_{\gamma_i^1} \omega + {}^{\mathrm{BC}}\!\!\int_{\gamma_i^2} \omega + \cdots + {}^{\mathrm{BC}}\!\!\int_{\gamma_i^{\ell_i}} \omega \\ &= \big(F_{i1}(P_{i1}) - F_{i2}(P_{i1})\big) + \big(F_{i2}(P_{i2}) - F_{i3}(P_{i2})\big) + \cdots + \big(F_{i\ell_i}(P_{i\ell_i}) - F_{i1}(P_{i\ell_i})\big) \\ &= -\eta_\omega(e_{i1}) - \eta_\omega(e_{i2}) - \cdots - \eta_\omega(e_{i\ell_i}) = -\eta_\omega(C_i) \end{aligned}$$

as required. □

5.2 Computation of the Integrals

Let X be a hyperelliptic curve given by $y^2 = f(x)$ for some monic polynomial $f(x)$ with coefficients in $\mathcal{O}_K$, and, as constructed in Subsection 4.3.9, let $\mathcal{D}$ be the semistable covering of X^{an} that is good with respect to the set of Weierstrass points. Recall that, in the case when $f(x)$ is of even degree, ∞^+ and ∞^- denote the points $(1 : 1 : 0)$ and $(1 : -1 : 0)$ at infinity, respectively.

5.2.1 Berkovich–Coleman Integrals

Chapter 4 describes an effective method for numerically computing Berkovich–Coleman integrals of regular 1-forms on X^{an}. Here is a rough outline of the method:

1. Reduce the problem of computing ${}^{\mathrm{BC}}\!\int_\gamma \omega$ on X^{an} to computing Berkovich–Coleman integrals on certain elements of $\mathcal{D}$.
2. For each element Y of $\mathcal{D}$ of interest, go through the following steps:
   (a) Identify Y with a basic wide open Z inside the analytification of a curve $\tilde{X}$ of good reduction.
   (b) Expand the pull back of the form $\omega|_Y$ to Z as a power series in certain meromorphic forms on $\tilde{X}$.
   (c) By a pole reduction argument, rewrite the terms in the power series expansion in terms of basis elements.
   (d) Employ the known integration algorithms on $\tilde{X}$.

In this subsection, we extend the method in a natural way to also cover meromorphic forms. This will make it possible to compute Vologodsky integrals by Theorem 5.1.6.

We can proceed as before up to 2c, where things become more interesting. We begin by breaking up the integrand into smaller, more manageable pieces. For an integer i and an element β in $\mathbb{A}^1(\bar{K})$, define the differentials

$$\omega_i = x^i \frac{dx}{2y}, \quad \nu_\beta = \frac{1}{x-\beta}\frac{dx}{2y}.$$

Proposition 5.2.2. *Let ω be a meromorphic 1-form on X. If $f(x)$ is of degree d, then the problem of computing ${}^{\mathrm{BC}}\!\int \omega$ reduces to computing*

1. ${}^{\mathrm{BC}}\!\int \omega_i,\quad i = 0, 1, \ldots, d-2$*; and*

2. ${}^{\mathrm{BC}}\!\int \nu_\beta,\quad \beta$ *is a non-root of* $f(x)$.

Proof. We can write ω as a linear combination, $\omega = \rho + \sum_j d_j \nu_j$, where ρ is of the second kind, $d_j \in \bar{K}$ and ν_j is of the third kind. This gives

$${}^{\mathrm{BC}}\!\!\int \omega = {}^{\mathrm{BC}}\!\!\int \rho + \sum_j d_j \,{}^{\mathrm{BC}}\!\!\int \nu_j.$$

In both the odd and even degree cases, the set $\{[\omega_0], [\omega_1], \ldots, [\omega_{d-2}]\}$ forms a spanning set for $H^1_{\mathrm{dR}}(X)$. Therefore, we may write ρ as a linear combination of $\omega_0, \omega_1, \ldots, \omega_{d-2}$ together with an exact form. On the other hand, recalling the exact sequence (2.1) induced by the residual divisor homomorphism, we may assume that

$$\mathrm{Res}(\nu_j) = (P_j) - (Q_j).$$

Then, up to a holomorphic differential, we have

$$\nu_j = \begin{cases} \left(\dfrac{y + y(P_j)}{x - x(P_j)} - \dfrac{y + y(Q_j)}{x - x(Q_j)}\right) \dfrac{dx}{2y} & \text{if } P_j \text{ and } Q_j \text{ are finite;} \\ \dfrac{y + y(P_j)}{x - x(P_j)} \dfrac{dx}{2y} & \text{if } d \text{ is odd, } P_j \text{ is finite, } Q_j \text{ is infinite;} \\ 2\omega_g & \text{if } d \text{ is even, } P_j = \infty^-,\ Q_j = \infty^+; \\ \dfrac{y + y(P_j)}{x - x(P_j)} \dfrac{dx}{2y} - \omega_g & \text{if } d \text{ is even, } P_j \text{ is finite, } Q_j = \infty^-; \\ \dfrac{y + y(P_j)}{x - x(P_j)} \dfrac{dx}{2y} + \omega_g & \text{if } d \text{ is even, } P_j \text{ is finite, } Q_j = \infty^+. \end{cases}$$

In any case, the form ν_j is a linear combination of $\omega_0, \omega_1, \ldots, \omega_{g-1}, \omega_g$, differentials of the form $\frac{y(P)}{x-x(P)} \frac{dx}{2y}$ and logarithmic differentials. As $\frac{y(P)}{x-x(P)} \frac{dx}{2y} = 0$ when P is a Weierstrass point, the claim follows. $\square$

The forms ω_i are already dealt with in Sections 4.4 through 4.7 (arguments are for $i = 0, 1, \ldots, g-1$ but work in general). We can proceed in the same way for the forms ν_β since, analogously to (4.5), the power series expansion of a ν_β is nothing but

$$\nu_\beta = \sum_{\substack{n_1 \geq L_1, \ldots, n_\ell \geq L_\ell \\ n_\infty \geq 0}} \left(B_{n_1, \ldots, n_\ell, n_\infty} \frac{x^{n_\infty}}{x - \beta} \prod_{j=1}^{\ell} \frac{1}{(x - \beta_j)^{n_j}} \frac{dx}{2\tilde{y}} \right).$$

5.2.3 Vologodsky Integrals

Here, we use the material from the previous sections to give an algorithm for computing Vologodsky integrals on X. Let Γ denote the dual graph of $\mathcal{D}$ as defined in Definition 4.3.2. As before, we write $\tau\colon X^{\mathrm{an}} \to \Gamma$ for the retraction map by identifying Γ with the skeleton of X^{an}.

Algorithm 8: Computing Vologodsky integrals

Input:

- A meromorphic 1-form ω on X.
- Points $x, y \in X(K)$.

Output: Vologodsky integral of ω from x to y.

1. Pick a path γ in X^{an} from x to y and compute the Berkovich–Coleman integral
$$ {}^{\mathrm{BC}}\!\!\int_{\gamma} \omega $$
as in Subsection 5.2.1.

2. Determine a basis $C_1, \ldots, C_h$ of $H_1(\Gamma; \mathbb{C}_p)$ and a basis $\eta_1, \ldots, \eta_h$ of $\Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{C}_p)$ such that
$$ {}^{t}\!\!\int_{C_i} \eta_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} $$

3. For each i, pick a loop γ_i in X^{an} with the property that $\tau(\gamma_i) = C_i$; and compute the Berkovich–Coleman periods
$$ {}^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega, \quad i = 1, \ldots, h $$
as in Subsection 5.2.1.

4. Compute the tropical integrals
$$ {}^{t}\!\!\int_{\tau(\gamma)} \eta_i, \quad i = 1, \ldots, h. $$

5. Return
$$ {}^{\mathrm{Vol}}\!\!\int_x^y \omega = {}^{\mathrm{BC}}\!\!\int_{\gamma} \omega - \sum_{i=1}^{h} \left({}^{\mathrm{BC}}\!\!\int_{\gamma_i} \omega \right) \left({}^{t}\!\!\int_{\tau(\gamma)} \eta_i \right). $$

5.3 Numerical Examples

In this final section, we provide two examples computed in Sage, with assistance from Magma.

As discussed before, the Vologodsky and Berkovich–Coleman integrals require a branch of the p-adic logarithm. We choose the branch that takes the value 0 at p.

In the examples below, ω_i will denote the differential $x^i \frac{dx}{2y}$ on the corresponding hyperelliptic curve.

Example 5.3.1. Consider the elliptic curve $X/\mathbb{Q}$ [LMF20, 6622.i3] given by

$$y^2 = f(x) = x^3 - 1351755x + 555015942$$

which has split multiplicative reduction at the prime $p = 43$. In this example, we will compute several Vologodsky integrals involving the points

$$Q = (2523, 114912), \quad R = (219, 16416) \in X(\mathbb{Q}).$$

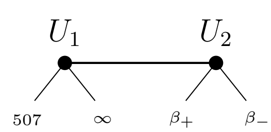
The polynomial $f(x)$ factors as

$$f(x) = (x - 507)(x - \beta_+)(x - \beta_-), \quad \beta_\pm = -\frac{3}{2}(169 \pm 33\sqrt{473})$$

and the set $\{U_1, U_2\}$ is a semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ that is good with respect to $S_f = \{507, \beta_\pm, \infty\}$ where

$$U_1 = \mathbb{P}^{1,\mathrm{an}} \setminus \overline{B}(26, 1/\sqrt{43}), \quad U_2 = B(26, 1).$$

The corresponding dual graph is as follows:



The points $Q, -Q, R, -R$ lie in the component $\pi^{-1}(U_1)$, which is embedded into a rational curve:

$$\pi^{-1}(U_1) \simeq \{(x, \tilde{y}) \mid \tilde{y}^2 = x - 507,\ x \in U_1\}$$

where

$$\tilde{y} = \frac{y}{\ell(x)}, \quad \ell(x) = \Big(x + \frac{507}{2}\Big)\Big(1 - \frac{4635873/4}{(x + 507/2)^2}\Big)^{1/2}.$$

Our computations give

$$\begin{aligned}
{}^{\mathrm{Vol}}\!\!\int_{-Q}^{Q} \omega_0 &= 12 \cdot 43^2 + 43^3 + 18 \cdot 43^4 + 40 \cdot 43^5 + O(43^6), \\
{}^{\mathrm{Vol}}\!\!\int_{-Q}^{Q} \omega_1 &= 25 + 11 \cdot 43 + 34 \cdot 43^2 + 26 \cdot 43^3 + 25 \cdot 43^4 + 34 \cdot 43^5 + O(43^6), \\
{}^{\mathrm{Vol}}\!\!\int_{-Q}^{Q} \frac{y(R)}{x - x(R)} \frac{dx}{y} &= 29 \cdot 43 + 21 \cdot 43^2 + 35 \cdot 43^3 + 20 \cdot 43^4 + 10 \cdot 43^5 + O(43^6);
\end{aligned}$$

and

$$\begin{aligned}
{}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \omega_0 &= 12 \cdot 43^2 + 43^3 + 18 \cdot 43^4 + 40 \cdot 43^5 + O(43^6), \\
{}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \omega_1 &= 40 + 8 \cdot 43 + 34 \cdot 43^2 + 26 \cdot 43^3 + 25 \cdot 43^4 + 34 \cdot 43^5 + O(43^6), \\
{}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \frac{y(Q)}{x - x(Q)} \frac{dx}{y} &= 29 \cdot 43 + 29 \cdot 43^2 + 18 \cdot 43^3 + 29 \cdot 43^4 + 3 \cdot 43^5 + O(43^6).
\end{aligned}$$

We notice that ${}^{\mathrm{Vol}}\!\int_{-Q}^{Q} \omega_0 = {}^{\mathrm{Vol}}\!\int_{-R}^{R} \omega_0$; this is consistent with the fact that $(Q) - (-Q) = (R) - (-R)$ and ω_0 is regular. In fact, there is good reason to believe these numbers to be correct: in Example 6.4.1, we will do a p-adic height computation on X using these numbers, and the result will have the *desired* from; see Example 6.4.1 for details.

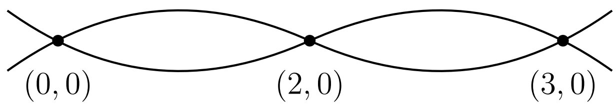
Remark 5.3.2. The curve in this example may appear very special since the coefficients of its defining polynomial are quite large, but it is not. There is only one additional constraint: the existence of two distinct points $Q, R \in X(\mathbb{Q})$ such that $(Q) - (-Q) = (R) - (-R)$ (this will be important in Example 6.4.1). Because this restriction is fairly mild, similar computations can be performed for curves given by polynomials with reasonably small coefficients.

Example 5.3.3. Consider the hyperelliptic curve $X/\mathbb{Q}$ [LMF20, 3950.b.39500.1] given by

$$y^2 = f(x) = (x^2 - x - 1)(x^4 + x^3 - 6x^2 + 5x - 5).$$

According to the database, the Mordell-Weil group of the Jacobian of X over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}/12\mathbb{Z}$. Therefore, for any two points $R, S \in X(\mathbb{Q})$, the Vologodsky integrals of holomorphic forms against the divisor $(R) - (S)$ must vanish. In this example, we will compute the integrals of $\omega_0, \omega_1, \omega_2, \omega_3$ and ω_4 between two rational points and observe this vanishing numerically.

Note that $p = 5$ is a prime of bad reduction for X. Moreover, the corresponding (stable) reduction is a genus 2 *banana* curve, i.e., the union of two projective lines meeting transversally at three points, as represented in the following figure:



Hereafter, we consider X over the field $\mathbb{Q}_5$. Then the polynomial $f(x)$ factors as the product of three quadratic monic polynomials:

$$f(x) = f_1(x)f_2(x)f_3(x), \quad f_j(x) = x^2 + A_jx + B_j \in \mathbb{Q}_5[x].$$

Relabeling if necessary, we may assume that

$$f_1(x) \equiv x^2, \quad f_2(x) \equiv (x-2)^2, \quad f_3(x) \equiv (x-3)^2 \pmod 5.$$

We begin by constructing coverings. The set $\mathcal{C} = \{U, U_1, U_2, U_3\}$ is a semistable covering of $\mathbb{P}^{1,\mathrm{an}}$ that is good with respect to the roots of $f(x)$ where

$$\begin{aligned}
U\ &= \mathbb{P}^{1,\mathrm{an}} \setminus \big(\overline{B}(0, 1/\sqrt{5}) \cup \overline{B}(2, 1/\sqrt{5}) \cup \overline{B}(3, 1/\sqrt{5})\big),\\
U_1 &= B(0, 1),\\
U_2 &= B(2, 1),\\
U_3 &= B(3, 1).
\end{aligned}$$

Set

$$\ell_j(x) = \left(x + \frac{A_j}{2}\right)\left(1 + \frac{B_j - A_j^2/4}{(x + A_j/2)^2}\right)^{1/2}, \quad j = 1, 2, 3.$$

Then $\ell_j(x)^2 = f_j(x)$ for x in the domain of convergence. For the first component, we have

$$\pi^{-1}(U) \simeq \{(x, \tilde{y}) \mid \tilde{y}^2 = 1,\ x \in U\} = \{(x, \pm 1) \mid x \in U\}$$

where

$$\tilde{y} = \frac{y}{\ell(x)}, \quad \ell(x) = \ell_1(x)\ell_2(x)\ell_3(x).$$

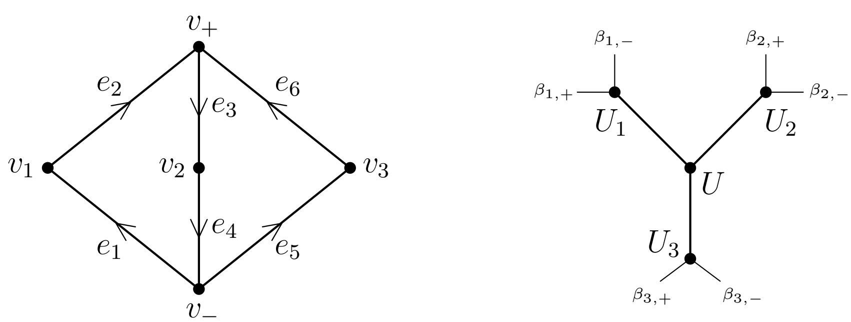
Let v_+ (resp. v_-) denote the component of $\pi^{-1}(U)$ corresponding to $\{(x,1) \mid x \in U\}$ (resp. $\{(x,-1) \mid x \in U\}$). For the other components, we have

$$v_j \coloneqq \pi^{-1}(U_j) \simeq \{(x,\tilde{y}) \mid \tilde{y}^2 = f_j(x),\ x \in U_j\}, \quad j = 1,2,3$$

where

$$\tilde{y} = \frac{y}{\ell(x)}, \quad \ell(x) = \begin{cases} \ell_2(x)\ell_3(x) & \text{if } j = 1, \\ \ell_1(x)\ell_3(x) & \text{if } j = 2, \\ \ell_1(x)\ell_2(x) & \text{if } j = 3. \end{cases}$$

Consequently, $\mathcal{D} = \{v_\pm, v_1, v_2, v_3\}$ forms a semistable covering of X^{an} that is good with respect to the set of Weierstrass points. Here are the dual graphs Γ and T:



where $\beta_{j,\pm}$ denote the roots of f_j. Let $\tau\colon X^{\mathrm{an}} \to \Gamma$ be the retraction map.

Returning now to integrals, let $R = (1,2)$, $S = (1,-2)$ be points on X. We first compute integrals along a path from S to R. Clearly, the points R and S belong, respectively, to the components v_+ and v_-. In order to pass from v_- to v_+ (via v_1), we pick the following reference points:

$$P_{e_1} = (a, 4\cdot a + a^3 + 2\cdot a^5 + 4\cdot a^6 + a^7 + O(a^8)),$$
$$P_{e_2} = (a, a + 4\cdot a^3 + 2\cdot a^5 + a^6 + 3\cdot a^7 + O(a^8)),$$

where $a^4 = 5$. Now let $\gamma = \gamma^1\gamma^2\gamma^3$ where

- γ^1 is a path from S to P_{e_1} in v_-,
- γ^2 is a path from P_{e_1} to P_{e_2} in v_1, and
- γ^3 is a path from P_{e_2} to R in v_+,

so that $\tau(\gamma) = e_1 e_2$. Our computations give

$$\int_{\gamma}^{\mathrm{BC}} \omega_i = \begin{cases} 2\cdot a^4 + 3\cdot a^8 + 4\cdot a^{12} + 2\cdot a^{16} + a^{20} + 2\cdot a^{24} + O(a^{32}) & \text{if } i = 0, \\ a^4 + a^8 + a^{12} + a^{24} + a^{28} + O(a^{32}) & \text{if } i = 1, \\ a^4 + 2\cdot a^{24} + O(a^{32}) & \text{if } i = 2, \\ 1 + 3\cdot a^4 + 3\cdot a^8 + 2\cdot a^{12} + 4\cdot a^{16} + a^{20} + O(a^{32}) & \text{if } i = 3, \\ 3 + 4\cdot a^4 + 2\cdot a^8 + 4\cdot a^{12} + 2\cdot a^{16} + 2\cdot a^{20} + a^{24} + 3\cdot a^{28} + O(a^{32}) & \text{if } i = 4. \end{cases}$$

Now, we compute the period integrals. The 1-cycles

$$C_1 = e_1 + e_2 + e_3 + e_4, \quad C_2 = e_3 + e_4 + e_5 + e_6$$

are a basis for $H_1(\Gamma; \mathbb{C}_p)$, and the tropical 1-forms

$$\eta_1(e_i) = \begin{cases} 1/3 & \text{if } i = 1 \text{ or } 2, \\ 1/6 & \text{if } i = 3 \text{ or } 4, \\ -1/6 & \text{if } i = 5 \text{ or } 6; \end{cases} \quad \eta_2(e_i) = \begin{cases} -1/6 & \text{if } i = 1 \text{ or } 2, \\ 1/6 & \text{if } i = 3 \text{ or } 4, \\ 1/3 & \text{if } i = 5 \text{ or } 6; \end{cases}$$

are a basis for $\Omega^1_{\mathrm{trop}}(\Gamma, \mathbb{C}_p)$ so that

$$\int_{C_i}^{t} \eta_j = \begin{cases} 1 & \text{if } \ i = j, \\ 0 & \text{if } \ i \neq j. \end{cases}$$

For each of the remaining edges, we pick the following reference points:

$$\begin{aligned} P_{e_3} &= (a+2, 3\cdot a + a^2 + 2\cdot a^3 + a^4 + 4\cdot a^5 + O(a^7)), \\ P_{e_4} &= (a+2, 2\cdot a + 4\cdot a^2 + 3\cdot a^3 + 4\cdot a^4 + a^6 + O(a^7)), \\ P_{e_5} &= (a+3, 2\cdot a + a^2 + 4\cdot a^4 + 4\cdot a^5 + 3\cdot a^6 + O(a^7)), \\ P_{e_6} &= (a+3, 3\cdot a + 4\cdot a^2 + a^4 + a^6 + a^7 + O(a^8)), \end{aligned}$$

recalling $a^4 = 5$. Let $\gamma_1 = \gamma_1^1 \gamma_1^2 \gamma_1^3 \gamma_1^4$ where

- γ_1^1 is a path from P_{e_1} to P_{e_2} in v_1,
- γ_1^2 is a path from P_{e_2} to P_{e_3} in v_+,
- γ_1^3 is a path from P_{e_3} to P_{e_4} in v_2, and
- γ_1^4 is a path from P_{e_4} to P_{e_1} in v_-.

Then, by construction, γ_1 is a loop in X^{an} such that $\tau(\gamma_1) = C_1$ and the corresponding period integrals are

$$
{}^{\mathrm{BC}}\!\!\int_{\gamma_1} \omega_i = \begin{cases}
a^8 + 3\cdot a^{16} + a^{20} + O(a^{32}) & \text{if } i = 0,\\
2\cdot a^4 + a^{12} + 3\cdot a^{24} + 4\cdot a^{28} + O(a^{32}) & \text{if } i = 1,\\
a^{12} + 4\cdot a^{16} + 3\cdot a^{28} + O(a^{32}) & \text{if } i = 2,\\
2 + 3\cdot a^4 + 2\cdot a^8 + 4\cdot a^{16} + 2\cdot a^{20} + a^{24} + a^{28} + O(a^{32}) & \text{if } i = 3,\\
2 + 3\cdot a^4 + a^8 + 2\cdot a^{12} + 2\cdot a^{16} + 4\cdot a^{20} + 4\cdot a^{24} + 4\cdot a^{28} + O(a^{32}) & \text{if } i = 4.
\end{cases}
$$

By constructing γ_2 analogously, we have

$$
{}^{\mathrm{BC}}\!\!\int_{\gamma_2} \omega_i = \begin{cases}
4\cdot a^4 + a^8 + a^{12} + 2\cdot a^{16} + 3\cdot a^{20} + 3\cdot a^{24} + 3\cdot a^{28} + O(a^{32}) & \text{if } i = 0,\\
a^4 + 2\cdot a^8 + 3\cdot a^{12} + 4\cdot a^{16} + 4\cdot a^{20} + 2\cdot a^{24} + O(a^{32}) & \text{if } i = 1,\\
2\cdot a^4 + 4\cdot a^8 + a^{12} + 3\cdot a^{16} + a^{20} + 4\cdot a^{24} + 4\cdot a^{28} + O(a^{32}) & \text{if } i = 2,\\
4 + a^4 + 4\cdot a^8 + 2\cdot a^{12} + 4\cdot a^{16} + 4\cdot a^{20} + a^{24} + 2\cdot a^{28} + O(a^{32}) & \text{if } i = 3,\\
3\cdot a^4 + 4\cdot a^8 + a^{16} + a^{20} + O(a^{32}) & \text{if } i = 4.
\end{cases}
$$

Finally, using the comparison formula in Theorem 5.1.6, we compute

$$
\begin{aligned}
{}^{\mathrm{Vol}}\!\!\int_S^R \omega_i &= {}^{\mathrm{BC}}\!\!\int_{\gamma} \omega_i - \left({}^{\mathrm{BC}}\!\!\int_{\gamma_1} \omega_i\right)\left({}^{t}\!\!\int_{\tau(\gamma)} \eta_1\right) - \left({}^{\mathrm{BC}}\!\!\int_{\gamma_2} \omega_i\right)\left({}^{t}\!\!\int_{\tau(\gamma)} \eta_2\right)\\
&= O(a^{32}) = O(5^8), \quad i = 0,1,2,4;\\
{}^{\mathrm{Vol}}\!\!\int_S^R \omega_3 &= {}^{\mathrm{BC}}\!\!\int_{\gamma} \omega_3 - \left({}^{\mathrm{BC}}\!\!\int_{\gamma_1} \omega_3\right)\left({}^{t}\!\!\int_{\tau(\gamma)} \eta_1\right) - \left({}^{\mathrm{BC}}\!\!\int_{\gamma_2} \omega_3\right)\left({}^{t}\!\!\int_{\tau(\gamma)} \eta_2\right)\\
&= 1 + 3\cdot a^4 + a^8 + 3\cdot a^{12} + a^{16} + 3\cdot a^{20} + a^{24} + 3\cdot a^{28} + O(a^{32})\\
&= 1 + 3\cdot 5 + 5^2 + 3\cdot 5^3 + 5^4 + 3\cdot 5^5 + 5^6 + 3\cdot 5^7 + O(5^8)
\end{aligned}
$$

consistent with the fact that ω_0, ω_1 are regular but ω_3 is not.[6]

Let γ' (resp. γ'') be a path from S to R such that $\tau(\gamma') = (-e_4)(-e_3)$ (resp. $\tau(\gamma'') = e_5e_6$). As a consistency check, we replaced the path γ in the computations above by γ' and γ'', respectively. As expected, these changes did not affect the (final) results.

---

[6]Incidentally, the integrals ${}^{\mathrm{Vol}}\!\!\int_S^R \omega_2$ and ${}^{\mathrm{Vol}}\!\!\int_S^R \omega_4$ vanish; we do not know the reason behind this.

5.4 Future Work

5.4.1 Implementation

We would ultimately like to provide a general implementation of our algorithms in Sage. Since we work with infinite series for computing Coleman–Berkovich integrals in our method, a careful precision analysis is required to get provably correct results.

5.4.2 Double Vologodsky Integration for Hyperelliptic Curves

Balakrishnan [Bal13, Bal15] gave algorithms to compute double Coleman integrals on good reduction hyperelliptic curves. In another direction, we would like to devise an algorithm for computing double Vologodsky integrals on semistable hyperelliptic curves. To do so, we will combine the techniques of Chapters 4-5 with the forthcoming work of Katz and Litt [KL].

We note that p-adic double integrals appear in the quadratic Chabauty method, see Subsection 6.5.1.

5.4.3 Beyond Hyperelliptic Curves

Together with Katz, we intend to increase the class of curves for which we can compute Berkovich–Coleman integrals effectively; at present, we are limited to hyperelliptic curves. Since computing tropical integrals is quite easy in practice, this will allow us to compute Vologodsky integrals.

It is our hope that we can integrate on any *schön* planar curve, a natural tropical condition implying smoothness; see [HK12]. The main technical result we need is a certain theorem on parameterizing basic wide open spaces in smooth planar curves, based on the work of Tuitman [Tui16, Tui17].

Chapter 6

Explicit p-adic Heights for Hyperelliptic Curves

In the literature, there are several definitions of p-adic height pairings on abelian varieties defined over number fields. Some of these definitions were given by Schneider [Sch82], Mazur–Tate [MT83] and Nekovář [Nek93]; for Jacobians of curves, there is a fourth definition due to Coleman–Gross [CG89] and Besser [Bes17]. Most of these constructions are quite similar to constructions of the Néron–Tate height pairing, which is a symmetric bilinear map

$$A(F) \times A(F) \to \mathbb{R},$$

where A is an abelian variety over a number field F.

Algorithms for computing p-adic heights

- allow one to compute p-adic regulators, some of which fit into p-adic versions of Birch and Swinnerton–Dyer conjecture ([MST06, Har08, SW13, BMS16]), and
- play a crucial role in carrying out the quadratic Chabauty method ([BBM16, BBM17, BD18, BD21, BDM$^+$19, Bia20, BBBM21]).

The p-adic height pairing of Coleman–Gross is particularly relevant to us for two reasons:

- It can be described solely in terms of the curve.

- It is, by definition, a sum of local height pairings at each finite place and the local components at the places above p are given in terms of Vologodsky integrals of non-holomorphic differential forms.

In this chapter, which constitutes some parts of joint work with Bianchi and Müller [BKM], we give an algorithm to compute the (p-part of the) Coleman–Gross p-adic height pairing on Jacobians of hyperelliptic curves of arbitrary reduction type.

We will use the following notation. Let F denote a number field with ring of integers $\mathcal{O}_F$. For a place v of F, we write F_v and $\mathcal{O}_{F_v}$ for the completions at v. Fix a uniformizer π_v of $\mathcal{O}_{F_v}$.

6.1 The (Extended) Coleman–Gross Height Pairing

Suppose C is a smooth complete curve defined over F. In this section, we review the definition of the extended Coleman–Gross height pairing on C following [Bes17, Section 2]. When C has good reduction at places dividing p, the construction is due to Coleman–Gross [CG89], hence the name "Coleman–Gross"; Besser [Bes17, Definition 2.1] later gave an extended definition of the Coleman–Gross pairing without any assumptions on the reduction type, hence the adjective "extended". For simplicity, we might drop the word "extended".

We assume that C has a F-rational point. The pairing, which we denote by h, is a function from $\mathrm{Div}^0(C) \times \mathrm{Div}^0(C)$ to $\mathbb{Q}_p$ and depends on some choices, where $\mathrm{Div}^0(C)$ denotes the group of degree 0 divisors on C over F.

First, let

$$\ell = (\ell_v)_v \colon \mathbb{A}_F^\times / F^\times \to \mathbb{Q}_p$$

be a continuous idèle class character, from which we obtain the following for a fixed finite place v of F. If $v \nmid p$ or if ℓ_v is unramified, then ℓ_v is determined by $\ell_v(\pi_v)$ since $\ell_v(\mathcal{O}_{F_v}^\times) = 0$. On the other hand, if $v \mid p$ and ℓ_v is ramified, we decompose ℓ_v on $\mathcal{O}_{F_v}^\times$ as

$$\ell_v = t_v \circ \log_v$$

where t_v is a $\mathbb{Q}_p$-linear map from F_v to $\mathbb{Q}_p$. Moreover, we extend $\log_v$ to

$$\log_v \colon F_v^\times \to F_v \tag{6.1}$$

for which the diagram

$$\begin{tikzcd} F_v^\times \arrow[rr, "\ell_v"] \arrow[dr, "\log_v"'] & & \mathbb{Q}_p \\ & F_v \arrow[ur, "t_v"'] & \end{tikzcd}$$

is commutative.

Second, for each $v \mid p$ such that ℓ_v is ramified, let W_v be a subspace of $H^1_{\mathrm{dR}}(C \otimes F_v/F_v)$ that is complementary to the space of holomorphic forms; that is,

$$H^1_{\mathrm{dR}}(C \otimes F_v/F_v) = H^{1,0}_{\mathrm{dR}}(C \otimes F_v/F_v) \oplus W_v. \tag{6.2}$$

We remark that, when $C \otimes F_v$ has semistable ordinary reduction in the sense of [MT83, Section 1.1], there is a canonical choice of such a complementary space: the unit root subspace for the action of the Frobenius endomorphism. See, for example, the paragraph after [Iov00, Lemma 2.2] for some generalities on unit root subspaces. We also remark that, in the special case where $C \otimes F_v$ is an odd degree hyperelliptic curve that has good ordinary reduction, one can use [BB12, Proposition 6.1] to compute a basis for the unit root subspace. In general, this is highly non-trivial, but the explicit description of the Frobenius endomorphism given in Coleman–Iovita [CI99] might help; see also Remark 6.3.2.

We now describe the height pairing $h = h_{\ell,\{W_v\}_v}$. It is, by definition, a sum of local height pairings over all finite places of F. In more precise terms, let D_1 and D_2 be two elements of $\mathrm{Div}^0(C)$, and for a finite place v of F, let $h_v(D_1, D_2)$ denote the local height pairing at v, which will be defined below. Then

$$h(D_1, D_2) = \sum_v h_v(D_1, D_2).$$

Let us define the local components. We first consider the case where D_1 and D_2 have disjoint support.

When $v \nmid p$ or when ℓ_v is unramified, the local term is described using arithmetic intersection theory; more precisely,

$$h_v(D_1, D_2) = \ell_v(\pi_v) \cdot (D_1, D_2),$$

where (D_1, D_2) denotes the intersection multiplicity of certain extensions of D_1 and D_2 to a regular model of C over $\mathcal{O}_{F_v}$. We note that the pairing $h_v(D_1, D_2)$ is bi-additive and symmetric. See [CG89, Proposition 1.2] and its proof.

The local term at a place $v \mid p$ such that ℓ_v is ramified is given in terms of a Vologodsky integral, for which we need to choose a branch of the p-adic logarithm. But we have already deduced a branch from our idèle class character ℓ, namely (6.1); we will use this one. Set

$$K = F_v, \quad X = C \otimes K$$

and consider D_1 and D_2 as divisors on X. By the exact sequence (2.1), there exists a form of the third kind defined over K whose residue divisor is D_1 and this form is unique only up to a holomorphic differential. Our complementary subspace W_v gives a unique choice ω_{D_1} satisfying

$$\mathrm{Res}(\omega_{D_1}) = D_1, \quad \Psi(\omega_{D_1}) \in W_v,$$

with Ψ as in (2.2). The uniqueness follows from the decomposition (6.2) and the fact that Ψ is the identity on differentials of the first kind. The local term is then defined via

$$h_v(D_1, D_2) = t_v \left({}^{\mathrm{Vol}}\!\!\int_{D_2} \omega_{D_1} \right).$$

The pairing $h_v(D_1, D_2)$ is bi-additive, but not symmetric in general. It is symmetric precisely when the subspace W_v is isotropic with respect to the cup product pairing; see [Bes05, Section 3].

If D_1 and D_2 have common support, we can still define local height pairings $h_v(D_1, D_2)$ thanks to the work of Balakrishnan–Besser [BB15]. This involves choosing a tangent vector at each point in $\mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2)$. Although the local terms depend on the tangent vectors, the global height pairing does not (provided that we make these choices consistently at all places). We finally note that in order to define the local term $h_v(D_1, D_2)$ at a place v above p such that ℓ_v is ramified, we need to impose the following additional assumption (which was not necessary, but helpful, for the case of disjoint support):

> The subspace W_v is isotropic with respect to the cup product pairing.

See [BB15, Sections 2-3] for details.

We finish this section by noting that the Coleman–Gross height pairing factors through the Jacobian. Let J/F denote the Jacobian variety of C and let P_1, P_2 be two elements of $J(F)$. By our assumption that $C(F)$ is non-empty, we can find elements D_1 and D_2 in $\mathrm{Div}^0(C)$ that represent P_1 and P_2, respectively. The value $h(D_1, D_2)$ is independent of the choice of D_1 and D_2. This follows from the fact that for every finite place v of F and every non-zero rational function on $C \otimes F_v$ we have

$$h_v((f), D_2) = \ell_v(f(D_2)),$$

and that ℓ is an idèle class character; see [Bes17, Proposition 2.2]. In summary, we obtain a bilinear pairing

$$h\colon J(F) \times J(F) \to \mathbb{Q}_p$$

which is symmetric when W_v's are isotropic with respect to the cup product.

6.1.1 Local and Global Symbols

We keep the notation introduced above. A crucial step in computing $h_v(D_1, D_2)$, for a place v dividing p with ℓ_v ramified, is the construction of the form ω_{D_1}. This requires the explicit computation of the map Ψ, but its original definition is not suitable for computations. As in [BB12], we will express this map in terms of *local* and *global symbols* in order to make it more explicit.

Definition 6.1.2. Let ω be a meromorphic form. Let ρ be a form of the second kind and fix a point Z at which ρ is regular. For a point A, the **local symbol** is defined as

$$\langle \omega, \rho \rangle_A = -\mathrm{Res}_A \left(\omega \,{}^{\mathrm{Vol}}\!\!\int \rho \right)$$

where by ${}^{\mathrm{Vol}}\!\!\int \rho$ we mean the formal integral of a local expansion of ρ around A whose constant term is the Vologodsky integral ${}^{\mathrm{Vol}}\!\!\int_Z^A \rho$ (resp. ${}^{\mathrm{Vol}}\!\!\int_Z^{A'} \rho$ for a nearby point A') if ρ is regular (resp. singular) at A[7]. Then the **global symbol** is defined as

$$\langle \omega, \rho \rangle = \sum_A \langle \omega, \rho \rangle_A$$

where A runs over all points where either ω or ρ has a singularity.

The following result is due to Besser and allows us to compute Ψ by computing global symbols.

Proposition 6.1.3. ([Bes05, Corollary 3.14]) *Let ω and ρ be as in Definition 6.1.2. Then the global symbol $\langle \omega, \rho \rangle$ is nothing but the cup product $\Psi(\omega) \cup [\rho]$.*

[7]When $\mathrm{Res}_A(\omega) = 0$ (in particular, when ω is regular at A), the choice of constant of integration does not matter.

6.2 Computing Coleman–Gross Heights

In this section, we present an algorithm for computing Coleman–Gross p-adic height pairings on hyperelliptic curves. Let C be a genus-g hyperelliptic curve over F with affine model

$$y^2 = f(x)$$

for some monic polynomial $f(x)$ of degree d with integral coefficients, and choose $(\ell, \{W_v\}_v)$ as in Section 6.1.

When $v \nmid p$ or when ℓ_v is unramified, the local height pairing h_v behaves in much the same way as local components at v of real-valued heights and can be computed as explained in [BMS16, Section 3.1]; see [Hol12, Mül14, VBHM20] for a detailed account. Our interest lies in the local height pairings at places $v \mid p$ such that ℓ_v is ramified, where p-adic integration is used.

Fix a finite place v of F with the property that $v \mid p$ and ℓ_v is ramified. Recall that from ℓ_v we deduce a branch $\log_v$ of the p-adic logarithm and a linear map t_v. Set $K = F_v$ and $X = C \otimes K$.

Algorithm 9: Computing local heights above p

Input:

- The subspace W_v, branch $\log_v$ of logarithm and linear map t_v.
- Divisors $D_1, D_2 \in \mathrm{Div}^0(X)$ with disjoint support.

Output: The local height pairing $h_v(D_1, D_2)$.

1. Choose a differential ω of the third kind defined over K with residue divisor D_1 (see 6.2.1).
2. Determine the holomorphic form η such that $\Psi(\omega - \eta)$ lies in the complementary subspace W_v (see 6.2.4); then $\omega_{D_1} = \omega - \eta$.
3. Compute the Vologodsky integral ${}^{\mathrm{Vol}}\!\int_{D_2} \omega_{D_1}$ as described in Chapters 4-5 with respect to $\log_v$.
4. Return $t_v\left({}^{\mathrm{Vol}}\!\int_{D_2} \omega_{D_1}\right)$.

Below, ∞^+ and ∞^- stand for $(1:1:0)$ and $(1:-1:0)$ when d is even, and

ω_i denotes the 1-form $x^i \frac{dx}{2y}$.

6.2.1 Constructing a Form with Given Residue Divisor

Let D be a divisor of degree 0 on X over K and write

$$D = \sum_j ((P_j) - (Q_j))$$

with points P_j, Q_j on X. Then $\omega = \sum_j \nu_j$ is a form of the third kind such that $\mathrm{Res}(\omega) = D$, where ν_j is defined by

$$\nu_j = \begin{cases} \left(\dfrac{y + y(P_j)}{x - x(P_j)} - \dfrac{y + y(Q_j)}{x - x(Q_j)}\right)\dfrac{dx}{2y} & \text{if } P_j \text{ and } Q_j \text{ are finite;} \\ \dfrac{y + y(P_j)}{x - x(P_j)}\dfrac{dx}{2y} & \text{if } d \text{ is odd, } P_j \text{ is finite, } Q_j \text{ is infinite;} \\ 2\omega_g & \text{if } d \text{ is even, } P_j = \infty^-,\ Q_j = \infty^+; \\ \dfrac{y + y(P_j)}{x - x(P_j)}\dfrac{dx}{2y} - \omega_g & \text{if } d \text{ is even, } P_j \text{ is finite, } Q_j = \infty^-; \\ \dfrac{y + y(P_j)}{x - x(P_j)}\dfrac{dx}{2y} + \omega_g & \text{if } d \text{ is even, } P_j \text{ is finite, } Q_j = \infty^+. \end{cases}$$

Note that the points P_j, Q_j, and hence the form ν_j, are not necessarily defined over K, but since D is defined over K, so is ω.

6.2.2 Computing Ψ

For $j = 0, 1, \ldots, 2g - 1$, define

$$\rho_j = \begin{cases} \omega_j & \text{if } j = 0, \ldots, g - 1; \\ \omega_j & \text{if } j = g, \ldots, 2g - 1 \text{ and } d \text{ is odd;} \\ \omega_{j+1} + 2\mathrm{Res}_{\infty^+}(\omega_{j+1})\omega_g & \text{if } j = g, \ldots, 2g - 1 \text{ and } d \text{ is even.} \end{cases}$$

By construction, each ρ_j is of the second kind, so that the class $[\rho_j]$ is an element of $H^1_{\mathrm{dR}}(X/K)$. Moreover, the elements $[\rho_0], \ldots, [\rho_{2g-1}]$ are linearly independent, which implies that the set $\mathcal{B} = \{[\rho_0], \ldots, [\rho_{2g-1}]\}$ forms a basis for $H^1_{\mathrm{dR}}(X/K)$ since this space is $2g$-dimensional.

Now let ω be a form of the third kind defined over K. Then

$$\Psi(\omega) = c_0[\rho_0] + \cdots + c_{2g-1}[\rho_{2g-1}]$$

for some constants c_i. By Proposition 6.1.3,

$$\langle \omega, \rho_j \rangle = \Psi(\omega) \cup [\rho_j] = \sum_{i=0}^{2g-1} c_i([\rho_i] \cup [\rho_j]), \quad j = 0, 1, \ldots, 2g-1. \tag{6.3}$$

As discussed in Section 2.2, the pairing $\cup$ is non-degenerate. Let N denote the cup product matrix with respect to the basis $\mathcal{B}$. From (6.3), we get

$$\begin{pmatrix} c_0 \\ \vdots \\ c_{2g-1} \end{pmatrix} = -N^{-1} \begin{pmatrix} \langle \omega, \rho_0 \rangle \\ \vdots \\ \langle \omega, \rho_{2g-1} \rangle \end{pmatrix}.$$

Therefore, in order to compute $\Psi(\omega)$, it is enough the compute the matrix N and the global symbols $\langle \omega, \rho_j \rangle$. The former is an easy task. The latter can be done using the techniques in Chapters 4-5; the situation is even better if the residue divisor of ω contains only affine points:

Proposition 6.2.3. *If the residue divisor D of ω does not contain the point(s) at infinity, we have*

$$\langle \omega, \rho_j \rangle = \begin{cases} -\,{}^{\mathrm{Vol}}\!\!\int_D \rho_j - \operatorname{Res}_\infty\left(\omega\, {}^{\mathrm{Vol}}\!\!\int \rho_j\right) & \text{if } d \text{ is odd}, \\ -\,{}^{\mathrm{Vol}}\!\!\int_D \rho_j - \operatorname{Res}_{\infty^+}\left(\omega\, {}^{\mathrm{Vol}}\!\!\int \rho_j\right) - \operatorname{Res}_{\infty^-}\left(\omega\, {}^{\mathrm{Vol}}\!\!\int \rho_j\right) & \text{if } d \text{ is even}. \end{cases}$$

Proof. This is a straightforward generalization of [BB12, Proposition 5.12] (or rather, its corrected version in the errata [BB19]). The key observation is that the local symbol at a point A in the support of D equals

$$-(\text{the multiplicity of } D \text{ at } A) \cdot {}^{\mathrm{Vol}}\!\!\int_Z^A \rho_j$$

since ω has a simple pole at A. Here, Z is a fixed point throughout the global symbol computation. Therefore, summing over all points gives $-\,{}^{\mathrm{Vol}}\!\!\int_D \rho_j$. $\square$

6.2.4 From D to ω_D

Let ω be a form of the third kind defined over K with residue divisor D. Let $\eta_0, \ldots, \eta_{g-1}$ be differentials of the second kind whose classes generate W_v. Since $H^1_{\mathrm{dR}}(X/K) = H^{1,0}_{\mathrm{dR}}(X/K) \oplus W_v$, there are constants d_i, e_i such that

$$\Psi(\omega) = d_0[\rho_0] + \cdots + d_{g-1}[\rho_{g-1}] + e_0[\eta_0] + \cdots + e_{g-1}[\eta_{g-1}]. \tag{6.4}$$

Set

$$\eta = d_0\rho_0 + \cdots + d_{g-1}\rho_{g-1} \in H^0(X, \Omega^1_{X/K}).$$

Then $\Psi(\omega - \eta)$ lies in W_v.

Let us explain how to determine the constants d_i. First, notice that η_j/ω_0 can be represented by a polynomial in x. Without loss of generality, we may assume that

$$\deg\left(\frac{\eta_j}{\omega_0}\right) = \begin{cases} j+g & \text{if } d \text{ is odd,} \\ j+g+1 & \text{if } d \text{ is even.} \end{cases} \tag{6.5}$$

As explained in 6.2.2, we can explicitly compute the constants c_i such that

$$\Psi(\omega) = c_0[\rho_0] + \cdots + c_{2g-1}[\rho_{2g-1}]. \tag{6.6}$$

Finally, the assumption (6.5) allows us to determine the constants d_i easily by comparing (6.4) and (6.6).

6.3 Other Height Pairings

Here, we briefly discuss the relations among the p-adic height pairings we touched upon in the beginning of this chapter.

Recall that in order to define the Coleman–Gross height pairing, we need to make some choices. The Nekovář height pairing requires the same data. Besser [Bes04, Bes17] showed, for a smooth complete curve defined over a number field with semistable reduction at each prime above p, that these two pairings are the same when they are defined with respect to the same auxiliary data.

As explained in [Nek93, Section 8.1], Nekovář's canonical height pairing coincides with the canonical height pairing of Mazur–Tate for abelian varieties with good ordinary reduction. It is widely believed that this relation continues to hold for abelian varieties with semistable ordinary reduction; see, for example, [Nek93, Section 8.2] and [Wer98, Section 7].

By [MT83, Section 4.4], the canonical Mazur–Tate height pairing is equal to the Schneider height pairing for abelian varieties with good ordinary reduction. These two pairings, however, differ in general: a formula for the difference in the case of semistable ordinary reduction was given by Werner [Wer98, Theorem 7.2].

Three remarks are in order.

Remark 6.3.1. By the discussion above, one can expect that the canonical height pairing of Coleman–Gross, the one whose components at ramified places above p are given with respect to the unit root subspace, and the canonical height pairing of Mazur–Tate coincide for Jacobian varieties with semistable ordinary reduction.

Remark 6.3.2. By the work of Papanikolas [Pap02, Section 5], the canonical Mazur–Tate height pairing can be expressed in terms of Norman's p-adic theta functions [Nor85, Nor86] for abelian varieties with semistable ordinary reduction. In joint work in progress with Bianchi and Müller, we give an algorithm to compute Norman's p-adic theta functions (hence canonical Mazur–Tate heights) on Jacobian surfaces with semistable ordinary reduction, adapting ideas of Blakestad [Bla18] to this setting. We prove that the canonical Mazur–Tate height is the same as the Coleman–Gross height with respect to a 2-dimensional canonical subspace of the first de Rham cohomology that comes from [Bla18, Section 3.2]. In the light of the previous remark, we suspect that this subspace is nothing but the unit root subspace. We also applied quadratic Chabauty to the curve

$$y^2 = x^5 + x^3 - 2x + 1$$

for $p = 5$, a prime of bad reduction, and determined its integral points.

Remark 6.3.3. In work in progress, Besser, Müller and Srinivasan develop a new theory of p-adic heights based on p-adic Arakelov theory [Bes05]. The local heights are defined using single and double Vologodsky integration. Our integration algorithms, as well as their additional refinements proposed in Section 5.4.2, will therefore be useful to compute this height and to apply it, for instance, in the context of quadratic Chabauty.

6.4 Numerical Examples

In the examples below, the base field will be $\mathbb{Q}$. Moreover, we will always work with the idèle class character ℓ given as follows: the component ℓ_p is the branch $\log_p$ of the p-adic logarithm vanishing on p; for $q \neq p$, the component ℓ_q is unramified with $\ell_q(q) = -\log_p(q)$.

As usual, ω_i will denote the differential $x^i \frac{dx}{2y}$ on the corresponding hyperelliptic curve.

Example 6.4.1. Let $C/\mathbb{Q}$ be an elliptic curve that has split multiplicative reduction at a prime number p. The canonical Mazur–Tate height of a point $P \in C(\mathbb{Q})$ is given in terms of the canonical p-adic sigma function σ_p of [MT91][8] by a simple formula: it is

$$2 \log_p \left(\frac{e(P)}{\sigma_p(t(P))} \right)$$

with the notation in [SW13, Section 4.2][9]; therefore it can be easily computed using Sage.

As a concrete example, consider the elliptic curve $C/\mathbb{Q}$ given by

$$y^2 = x^3 - 1351755x + 555015942$$

with

$$P := \left(\frac{330483}{361}, \frac{63148032}{6859} \right) \in C(\mathbb{Q}).$$

Notice that we already encountered this curve in Example 5.3.1, it has split multiplicative reduction at $p = 43$. The canonical Mazur–Tate height of the point P is

$$19 \cdot 43 + 7 \cdot 43^2 + 8 \cdot 43^3 + 2 \cdot 43^4 + 28 \cdot 43^5 + O(43^6). \tag{6.7}$$

Note that

$$P = (Q) - (-Q) = (R) - (-R),$$

where

$$Q = (2523, 114912), \quad R = (219, 16416) \in C(\mathbb{Q}).$$

In this example, we will compute the Coleman–Gross height

$$h((Q) - (-Q), (R) - (-R))$$

with respect to the unit root subspace $W = W_{43}$ and compare the results. For ease of notation, we write

$$D_Q = (Q) - (-Q), \quad D_R = (R) - (-R).$$

[8] This is consistent with Remark 6.3.2 in the following sense: Norman's p-adic theta functions are defined only up to scalar multiple, and Papanikolas showed that these functions are essentially the canonical p-adic sigma function of Mazur–Tate in the case of elliptic curves; see [Pap02, Proposition 5.5] for details.

[9] There, $\hat{h}_p$ is the Schneider height, which is not of interest for this example.

Away from 43, using the Magma implementation of the algorithm developed in van Bommel–Holmes–Müller [VBHM20], we have

$$\begin{aligned}\sum_{v\neq 43} h_v(D_Q, D_R) &= 9\cdot \mathrm{Log}(2)\\ &= 33\cdot 43 + 21\cdot 43^2 + 40\cdot 43^3 + 5\cdot 43^4 + 20\cdot 43^5 + O(43^6).\end{aligned}$$

For the component at 43, we first note that the space W is generated by $\alpha[\omega_0] + [\omega_1]$ where

$$\alpha = -\frac{e_2}{12\cdot C^2}$$

with the notation in [SW13, Section 4.2]; see [Kat73, Appendix 2] for details. This quantity can be easily computed using Sage:

$$\alpha = 17 + 37\cdot 43 + 20\cdot 43^2 + 11\cdot 43^3 + 38\cdot 43^4 + 6\cdot 43^5 + O(43^6).$$

We then have

$$h_{43}(D_Q, D_R) = {}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \omega_{D_Q} = {}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \frac{y(Q)}{x - x(Q)}\frac{dx}{y} + (c_1\alpha - c_0)\,{}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \omega_0$$

where

$$c_0 = {}^{\mathrm{Vol}}\!\!\int_{Q}^{-Q} \omega_1, \quad c_1 = {}^{\mathrm{Vol}}\!\!\int_{-Q}^{Q} \omega_0.$$

But we already computed these integrals in Example 5.3.1; this gives

$$h_{43}(D_Q, D_R) = 29\cdot 43 + 28\cdot 43^2 + 10\cdot 43^3 + 39\cdot 43^4 + 7\cdot 43^5 + O(43^6).$$

Putting all of this together, we get

$$\begin{aligned}h(D_Q, D_R) &= h_{43}(D_Q, D_R) + \sum_{v\neq 43} h_v(D_Q, D_R)\\ &= 19\cdot 43 + 7\cdot 43^2 + 8\cdot 43^3 + 2\cdot 43^4 + 28\cdot 43^5 + O(43^6)\end{aligned}$$

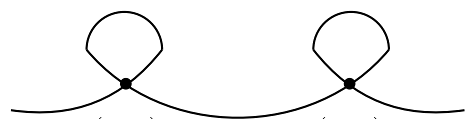
which is the same result as in (6.7). This is consistent with Remark 6.3.1.

We end this example by saying that the choice of a p-adic sigma function in the Mazur–Tate construction corresponds to the choice of a complementary subspace in the Coleman–Gross construction.

Example 6.4.2. Consider the hyperelliptic curve $C/\mathbb{Q}$ given by

$$y^2 = x^5 + 5x^4 - 168x^3 + 1584x^2 - 10368x + 20736.$$

Note that $p = 5$ is a prime of bad reduction for C. Moreover, the corresponding (stable) reduction is a projective line with two ordinary double points, as represented in the following figure:



Let $P = (-12, 720),\ Q = (-8, 528),\ R = (0, -144),\ S = (12, 432),\ T = (36, 7920)$ be points on C and set

$$D_1 = (Q) - (w(Q)), \qquad D_2 = (R) - (P),$$
$$D_3 = (S) - (w(P)), \qquad D_4 = (T) - (w(T)).$$

Using Cantor's algorithm, one can easily check that we have

$$D_1 = 4D_2, \quad D_4 = 6D_3$$

up to linear equivalence, which implies

$$6h(D_1, D_3) = h(D_1, D_4) = h(D_4, D_1) = 4h(D_4, D_2) \tag{6.8}$$

provided that the chosen complementary subspace for the prime 5 is isotropic with respect to the cup product pairing. In this example, we will verify this up to a certain precision.

Away from 5, using again the Magma implementation of the algorithm developed in van Bommel–Holmes–Müller [VBHM20], we have

$$\sum_{v \neq 5} h_v(D_1, D_3) = 0,$$
$$\sum_{v \neq 5} h_v(D_4, D_2) = -2 \cdot \mathrm{Log}(2) + \mathrm{Log}(3)$$
$$= 2 \cdot 5^2 + 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + 4 \cdot 5^6 + O(5^7).$$

For the local contributions at 5, let $W = W_5$ be the subspace generated by the following differentials

$$[\eta_0] = \alpha_0[\omega_0] + \alpha_1[\omega_1] + \alpha_2[\omega_2],$$
$$[\eta_1] = \beta_0[\omega_0] + \beta_1[\omega_1] + \beta_2[\omega_2] + \beta_3[\omega_3],$$

where

$$\begin{aligned}
\alpha_0 &= 2 + 3\cdot 5 + 4\cdot 5^2 + 2\cdot 5^3 + 5^4 + 3\cdot 5^6 + O(5^7),\\
\alpha_1 &= 3 + 4\cdot 5 + 5^3 + 3\cdot 5^4 + 2\cdot 5^5 + 4\cdot 5^6 + O(5^7),\\
\alpha_2 &= 1,\\
\beta_0 &= 2 + 3\cdot 5 + 2\cdot 5^2 + 4\cdot 5^3 + 5^4 + 2\cdot 5^5 + 5^6 + O(5^7),\\
\beta_1 &= 4 + 2\cdot 5 + 2\cdot 5^2 + 2\cdot 5^3 + 2\cdot 5^4 + 3\cdot 5^5 + O(5^7),\\
\beta_2 &= 15,\\
\beta_3 &= 3.
\end{aligned}$$

This space, constructed using the techniques in [Bla18, Section 3.2] (see Remark 6.3.2), is isotropic with respect to the cup product pairing.

Let us compute $h_5(D_1, D_3)$. The form

$$\omega := \frac{y(Q)}{x - x(Q)}\frac{dx}{y}$$

is of the third kind with residue divisor D_1 and

$$\Psi(\omega) = d_0[\omega_0] + d_1[\omega_1] + e_0[\eta_0] + e_1[\eta_1] \tag{6.9}$$

for some d_0, d_1, e_0, e_1. We then have

$$h_5(D_1, D_3) = {}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_{D_1} = {}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \frac{y(Q)}{x - x(Q)}\frac{dx}{y} - d_0\,{}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_0 - d_1\,{}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_1.$$

Using the techniques described in Chapters 4-5, we compute

$$\begin{aligned}
{}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \frac{y(Q)}{x - x(Q)}\frac{dx}{y} &= 2\cdot 5 + 5^2 + 4\cdot 5^3 + 3\cdot 5^4 + 4\cdot 5^5 + 2\cdot 5^6 + O(5^7),\\
{}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_0 &= 3\cdot 5 + 3\cdot 5^2 + 5^4 + 4\cdot 5^5 + O(5^7),\\
{}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_1 &= 3\cdot 5 + 4\cdot 5^2 + 3\cdot 5^3 + 4\cdot 5^4 + 2\cdot 5^6 + O(5^7).
\end{aligned}$$

In order to determine d_0 and d_1, write

$$\Psi(\omega) = c_0[\omega_0] + c_1[\omega_1] + c_2[\omega_2] + c_3[\omega_3]. \tag{6.10}$$

Comparing (6.9) and (6.10), we see that

$$\begin{aligned} e_1 &= c_3/\beta_3, & e_0 &= (c_2 - e_1 \cdot \beta_2)/\alpha_2, \\ d_1 &= c_1 - e_0 \cdot \alpha_1 - e_1 \cdot \beta_1, & d_0 &= c_0 - e_0 \cdot \alpha_0 - e_1 \cdot \beta_0. \end{aligned}$$

The cup product matrix N with respect to the basis $\mathcal{B} = \{[\omega_0], [\omega_1], [\omega_2], [\omega_3]\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & -10/3 \\ 0 & -1 & 0 & -56 \\ -1/3 & 10/3 & 56 & 0 \end{pmatrix}$$

and the global symbols are

$$\begin{pmatrix} \langle \omega, \omega_0 \rangle \\ \langle \omega, \omega_1 \rangle \\ \langle \omega, \omega_2 \rangle \\ \langle \omega, \omega_3 \rangle \end{pmatrix} = \begin{pmatrix} 2 \cdot 5 + 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + 3 \cdot 5^5 + 4 \cdot 5^6 + O(5^7) \\ 4 \cdot 5 + 2 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + O(5^7) \\ 2 + 3 \cdot 5 + 2 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^5 + O(5^7) \\ 2 \cdot 5^2 + 2 \cdot 5^3 + 3 \cdot 5^4 + 2 \cdot 5^5 + 4 \cdot 5^6 + O(5^7) \end{pmatrix}.$$

We then get that

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + 4 \cdot 5^4 + 3 \cdot 5^6 + O(5^7) \\ 2 + 4 \cdot 5 + 5^3 + 2 \cdot 5^5 + 5^6 + O(5^7) \\ 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 5^4 + 2 \cdot 5^5 + 5^6 + O(5^7) \\ 4 \cdot 5 + 2 \cdot 5^3 + 3 \cdot 5^5 + O(5^7) \end{pmatrix}$$

which implies that

$$\begin{pmatrix} e_1 \\ e_0 \\ d_1 \\ d_0 \end{pmatrix} = \begin{pmatrix} 3 \cdot 5 + 3 \cdot 5^2 + 5^5 + O(5^7) \\ 5 + 4 \cdot 5^2 + 5^3 + 4 \cdot 5^4 + 5^5 + 3 \cdot 5^6 + O(5^7) \\ 2 + 4 \cdot 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 3 \cdot 5^4 + 2 \cdot 5^5 + 5^6 + O(5^7) \\ 4 \cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^6 + O(5^7) \end{pmatrix}.$$

Combining all of this, we obtain

$$h_5(D_1, D_3) = 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 2 \cdot 5^4 + 3 \cdot 5^5 + 4 \cdot 5^6 + O(5^7).$$

A similar computation, which we omit, gives that

$$h_5(D_4, D_2) = 4 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^5 + 4 \cdot 5^6 + O(5^7).$$

Finally, we see

$$\begin{aligned} h(D_1, D_3) &= h_5(D_1, D_3) + \sum_{v \neq 5} h_v(D_1, D_3) \\ &= 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 2 \cdot 5^4 + 3 \cdot 5^5 + 4 \cdot 5^6 + O(5^7), \\ h(D_4, D_2) &= h_5(D_4, D_2) + \sum_{v \neq 5} h_v(D_4, D_2) \\ &= 4 \cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3 + 3 \cdot 5^4 + 2 \cdot 5^5 + 4 \cdot 5^6 + O(5^7), \end{aligned}$$

which give

$$\begin{aligned} 6h(D_1, D_3) &= 5 + 4 \cdot 5^2 + 5^5 + 3 \cdot 5^6 + O(5^7), \\ 4h(D_4, D_2) &= 5 + 4 \cdot 5^2 + 5^5 + 3 \cdot 5^6 + O(5^7). \end{aligned}$$

This is consistent with (6.8).

6.5 Future Work

6.5.1 Quadratic Chabauty at Bad Primes

As discussed in Section 1.3, working with primes of bad reduction might have some practical advantages for the Chabauty–Coleman method. It is a natural question whether this is also the case for quadratic Chabauty. If so, this could be used in previously inaccessible examples, for instance. It would be interesting to investigate this question. In order to do this, we need algorithms for

- computing p-adic heights, and
- computing p-adic double integrals.

The first bullet has already been discussed in Section 6.2. For the second, the proposed algorithm in Subsection 5.4.2 can be used.

6.5.2 p-adic BSD for Abelian Varieties

A p-adic analogue of the BSD conjecture for an elliptic curve over $\mathbb{Q}$ was given in Mazur–Tate–Teitelbaum (MTT) [MTT86] when p is a prime of good ordinary or multiplicative reduction, with the canonical regulator defined in terms of the Néron–Tate height replaced by the p-adic regulator associated to the Schneider height. Balakrishnan–Müller–Stein [BMS16] formulated a generalization of the MTT conjecture in the good ordinary case to higher-dimensional modular abelian varieties of GL_2-type over $\mathbb{Q}$. They also provided numerical evidence supporting their conjecture for Jacobians of hyperelliptic curves.

On the other hand, the MTT conjecture in the case of split multiplicative reduction, the *exceptional* case, is of special interest. One might expect that a generalization of this conjecture to higher-dimensional modular abelian varieties of GL_2-type over $\mathbb{Q}$ in the case of split purely toric reduction can be formulated. Formulating such a conjecture, as well as gathering numerical evidence for it, requires the computation Schneider heights. To that end, the findings of the proposed project in the next subsection can be used.

6.5.3 The Schneider Height Pairing

By the discussion of the previous subsection, it would be of great interest to develop an algorithm to compute Schneider heights on Jacobians of hyperelliptic curves.

One possibility is to compute the Coleman–Gross height first, using the techniques of Section 6.2, and afterwards convert it into the Schneider height, based on the discussion in Section 6.3. An important step is making Werner's formula [Wer98, Theorem 7.2], which gives the difference between the canonical Mazur–Tate height and the Schneider height, explicit and algorithmically computable. This will require two things. First, Iovita–Werner [IW03] proved that for abelian varieties with semistable ordinary reduction, the Mazur–Tate height pairing is induced by the unit root splitting of the Hodge filtration on the first de Rham cohomology group. In order to compute the unit root subspace, one can use the work of Coleman–Iovita [CI99], which gives an explicit description of Hyodo and Kato's Frobenius operator. Second, Werner's formula uses p-adic Abel–Jacobi maps and p-adic theta functions. It is neccessary to make these maps explicit; one can first try this for hyperelliptic curves whose Jacobians are modular, since more is known in these special cases and these are the ones we need for Subsection 6.5.2

Another possibility is to make use of non-Archimedean uniformization theory. Let $A/\mathbb{Q}$ be an abelian variety with split purely toric reduction at p. Werner [Wer97] expressed the Schneider height pairing on A in terms of rigid analytic uniformization of $A(\mathbb{Q}_p)$. This description might lead a practical algorithm for computing the Schneider height in favorable situations, as follows. To compute the rigid uniformization, one can use the work of Guitart–Masdeu [GM18], which gives a conjectural practical algorithm when A is modular of GL_2-type. In the case when A is the Jacobian of a curve X, one can also use work of Kadziela [Kad07a, Kad07b]; for this, one first computes the structure of X as a Mumford curve, encoded by a Schottky group Γ, and then computes the rigid uniformization from this using p-adic theta functions. See also the work of Werner [Wer96] and Morrison–Ren [MR15].

Summary

Theories of p-adic integration have numerous applications in arithmetic geometry and related areas, such as determining rational points on curves, deriving uniform bounds for the number of rational points on curves, computing p-adic heights on Jacobian varieties, and computing p-adic regulators in K-theory. Some of these applications rely heavily on explicit computation of integrals, which has been, thus far, restricted to curves of good reduction. Working with primes of bad reduction, on the other hand, might have some practical advantages. This thesis takes the first steps in considering the case of bad reduction.

Chapter 2 is devoted to preliminaries. Chapter 3 provides an overview of Vologodsky and Berkovich–Coleman integration theories, as well as a summary of the known p-adic integration algorithms.

In Chapters 4 and 5, we study theoretical and algorithmic aspects of p-adic integration theories on curves of arbitrary reduction type. More precisely, we prove a comparison theorem for Vologodsky and Berkovich–Coleman integrals. This theorem reduces the computation of Vologodsky integrals to the computation of Berkovich–Coleman integrals. We also develop algorithms for computing Berkovich–Coleman (and hence Vologodsky) integrals on hyperelliptic curves.

In Chapter 6, as an application of our integration algorithms, we give an algorithm for computing Coleman–Gross p-adic height pairings on hyperelliptic Jacobians; it is defined in terms of Vologodsky integration. This algorithm can be used in the quadratic Chabauty method in order to determine rational points on hyperelliptic curves of genus at least two.

Throughout, several numerical examples are given and several natural questions arising from our work are discussed. We believe that our findings can act as a starting point for many more interesting results.

Samenvatting

Theorieën voor p-adische integratie hebben talloze toepassingen in aritmetische meetkunde en gerelateerde gebieden, zoals het bepalen van rationale punten op krommen, het afleiden van uniformen grenzen voor het aantal rationale punten op krommen, het berekenen van p-adische hoogtes op Jacobiaanse variëteiten, en het berekenen van p-adische regulatoren in K-theorie. Enkele van deze toepassingen berusten sterk op expliciete berekening van integralen, wat tot zo ver beperkt is gebleven tot krommen met goede reductie. Werken met priemen van slechte reductie heeft daarentegen mogelijk enkele praktische voordelen. Deze thesis zet de eerste stappen voor het kijken naar priemen van slechte reductie.

Hoofdstuk 2 behandelt achtergrondkennis. Hoofdstuk 3 geeft een overzicht van Vologodsky- en Berkovich–Colemanintegratietheorieën, en ook een samenvatting van de bekende p-adische integratiealgoritmen.

In Hoofdstukken 4 en 5 bestuderen we theoretische en algoritmische aspecten van p-adische integratietheorieën op krommen van willekeurig reductietype. We bewijzen een vergelijkingsstelling voor Vodogodsky- en Berkovich–Colemanintegralen. Deze stelling reduceert de berekening van Vologodsky-integralen tot de berekening van Berkovich–Colemanintegralen. We ontwikkelen algoritmen voor het berekenen van Berkovich–Coleman- (en dus ook Vologodsky-) integralen.

In Hoofdstuk 6, als toepassing van onze integratiealgoritmen, geven we een algoritme voor het berekenen van Coleman-Gross p-adische hoogteparingen op hyperelliptische Jacobianen; deze zijn gedefinieerd in termen van Vologodsky-integratie. Dit algoritme kan worden gebruikt in de kwadratische Chabautymethode om de rationale punten op een hyperelliptische kromme van geslacht minstens twee te bepalen.

Door de hoofdstukken heen worden enkele voorbeelden gegeven en enkele natuurlijke vragen benoemd die uit ons werk voorkomen. We zijn van mening dat onze resultaten als startpunt kunnen dienen voor vele verdere interessante resultaten.

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