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Detailing the Connection Between a Family of Polar Graphs and Tremain Equiangular Tight Frames

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DETAILING THE CONNECTION BETWEEN A FAMILY OF POLAR GRAPHS AND TREMAIN EQUIANGULAR TIGHT FRAMES

BY

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DISSERTATION ACCEPTANCE PAGE

Nicholas Brown

This dissertation is approved as a creditable and independent investigation by a candidate for the Doctor of Philosophy degree and is acceptable for meeting the dissertation requirements for this degree. Acceptance of this does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

John Jasper
Advisor Date

Kurt Cogswell
Department Head Date

Nicole Lounsbery, PhD
Director, Graduate School Date

I dedicate this work to my family, friends, and my dog, Hope. Their support through this process has been invaluable and I wouldn't be here today without them!

"*Success comes from knowing that you did your best to become the best that you are capable of becoming.*"

John Wooden

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CONTENTS

ABSTRACT

DETAILING THE CONNECTION BETWEEN A FAMILY OF POLAR GRAPHS AND TREMAIN EQUIANGULAR TIGHT FRAMES

NICHOLAS BROWN

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The relationship between strongly regular graphs and equiangular tight frames has been known for several years, and this relationship has been used to construct many of the most recent examples of new strongly regular graphs. In this paper, we present an explicit construction of a family of equiangular tight frames using the geometry of a quadratic space over the field of four elements. We observe that these frames give rise to a strongly regular graph on a subset of points of a quadratic space over the field with 4 elements. We then demonstrate an isomorphism between this graph and a classical construction of polar graphs. While this family of graphs is known to exist, their construction using a Tremain ETF is much simpler, requiring the existence of Steiner triple systems and Hadamard matrices of the appropriate size, whereas the original constructions require computing intersections of hyperplanes.

1 INTRODUCTION

Among the most useful tools available in mathematics are orthonormal bases. While bases are useful for general vector spaces, they are not necessarily the ideal tool for inner product spaces. In addition to the normal linear properties of a basis, orthonormal bases allow for the calculation of both norms and inner products efficiently. Despite their usefulness, orthonormal bases may not be the optimal spanning set for a given application. For example, we may want to introduce redundancy for the purpose of guarding against data loss in signal processing applications. In this case, introducing an overcomplete spanning set with similar properties to an orthonormal basis would be preferable. A common way to describe these types of overcomplete spanning sets is to use frames [5], [15]. A (d, N)-frame is a finite sequence of N vectors which span either $\mathbb{R}^d$ or $\mathbb{C}^d$. We will use Φ to denote both the sequence $\{\varphi_i\}_{i=1}^N$ in $\mathbb{C}^d$ or $\mathbb{R}^d$ and the $d \times N$ matrix whose ith column is φ_i. Since Φ spans a d-dimensional space, $\Phi\Phi^*$ is always invertible and given $y = \Phi\Phi^* x$, recovering x from y might be computationally challenging. Hence, we are particularly interested in frames which have the property that $\Phi\Phi^* = AI$ for some positive constant A. Frames with this property are known as *tight frames.* Tight frames are well studied and are fairly well understood [3], [11], [14]. A tight frame helps replace an orthonormal basis by providing an efficient reconstruction formula for the vectors in space using the frame vectors, namely

$$x = \frac{1}{A}\sum_{i=1}^{N}\langle x, \varphi_i\rangle\varphi_i.$$

Another important property of orthonormal bases is their orthogonality. If $N > d$ then the set of vectors in a tight frame would necessarily form a linearly dependent set, and so it is not possible for all of the vectors in such a tight frame to be mutually orthogonal.

However, this suggests looking for unit norm tight frames in which the coherence,

$$\nu := \max_{\substack{i,j\in\{1,\ldots,N\}\\ i\neq j}} |\langle \varphi_i, \varphi_j\rangle|,$$

is as small as possible. In general, unit norm frames with minimal coherence are called *Grassmannian frames.* Originally introduced by Strohmer and Heath [15], and Holmes and Paulsen [12], the study of Grassmannian frames has since garnered substantial consideration from frame theorists. A notable predecessor to these Grassmannian frames however was a paper by Welch [19], which provides various lower bounds for the coherence in terms of N and d. For our purposes, the most important of these bounds is

$$\nu \geqslant \sqrt{\frac{N-d}{d(N-1)}}.$$

This inequality has been dubbed the *Welch bound*. An important application of the Welch bound is that a frame with unit norm vectors whose coherence is equal to the Welch bound is what is called an *equiangular tight frame* [15]. An equiangular tight frame gets its name by being both a tight frame and being equiangular, that is, $|\langle \varphi_i, \varphi_j\rangle|$ is constant for all $i, j \in \{1, \ldots, N\}$ with $i \neq j$. Since equiangular tight frames (ETFs) have coherence equal to the a lower bound for coherence, they are necessarily Grassmannian.

The introduction of ETFs to solve some erasure problems in coding theory led to the discovery of many other applications. As noted in [6], equiangular tight frames have been useful in solving problems involving compressed sensing, including medical MRI advancements, digital fingerprinting, which is useful to secure systems, and multiple description coding, which uses ETFs to quickly and reliably send and receive messages over a channel with potential corruption of data. Unfortunately, for most pairs $(d, N) \in \mathbb{N}^2$, an ETFs with N vectors in d-dimensional space either does not exist or is not known to exist [17]. Therefore, it is desirable to connect ETFs with other branches of mathematics in an attempt to discover and build new ETFs.

One application of an equiangular sequence of vectors was discovered prior to the introduction of ETFs. In 1966, van Lint and Seidel [13] introduced an identification between equiangular sequences of vectors and graphs, where vertices corresponded to vectors in the sequence and edges are drawn using the sign of the corresponding inner product. If we define the Gram matrix, G, of an equiangular sequence of vectors to be $G(i,j) = \langle \varphi_i, \varphi_j \rangle$ for all $i,j \in \{1,\ldots,N\}$ then we can express this matrix in terms of the adjacency matrix, A, of the graph. In the case of an equiangular tight frame that is real, that is, an ETF whose Gram matrix contains only real entries, then the tightness of the frame implies that G is a multiple of a projection, and so A satisfies a related quadratic equation. More recently, this result led to the establishment of a one-to-one correspondence between real ETFs and a family of *strongly regular graphs* [7], [13], [18]. Because of its importance to this paper, we note that a strongly regular graph (SRG) is a class of graphs that are regular and that satisfy two additional regularity conditions. The correspondence between ETFs and SRGs is the primary motivating result for this paper.

The discovery of the correspondence between ETFs and a family of SRGs has inspired a large effort to build ETFs from the previously constructed SRGs. One such construction of ETFs that give previously known SRGs is what are called Steiner ETFs [9]. Steiner ETFs are constructed using a tensor-like combination of the incidence matrix of a Steiner System, a particular type of block design, and a Hadamard matrix. Despite their novelty in the frame community, these ETFs were reformulations of previously known strongly regular graphs [10]. Considering that SRGs have been intensely studied for over 50 years, the existence of new graphs and corresponding ETFs will likely be difficult. However in [8], the authors discovered a new family of real ETFs and consequently a new infinite family strongly regular graphs. These new real ETFs, called Tremain ETFs, are not constructed in an intuitive way. Tremain ETFs are constructed by taking a particular family of Steiner ETFs built from Steiner Triple Systems and cleverly adding some rows and columns to obtain a larger ETF.

The purpose of this paper is to strengthen the connection between strongly regular graphs and equiangular tight frames. In particular, we will explore and deconstruct a family of strongly regular graphs denoted as $NO^+_{2n+1}(q)$ for some prime power q and demonstrate its connection to Tremain ETFs. Using the construction of Wilbrink [2], this graph has points which are the hyperplanes in an odd dimensional vector space over a finite field. While this construction works for any prime power, q, we restrict our focus to the case where $q = 2^h$ for some $h \in \mathbb{N}$. In this paper we will present two main results. First, we will define a new graph isomorphic to $NO^+_{2n+1}(q)$ which is defined on vectors instead of hyperplanes. Then using this isomorphic graph, we will show that there is a natural way to define a particular family of Tremain ETFs. And in this sense, $NO^+_{2n+1}(q)$ arises naturally as an instance of a Tremain ETF of this form. This new explicit connection between these strongly regular graphs and the Tremain ETF construction provides us potential new avenues to discover both new families of ETFs and strongly regular graphs. By deconstructing other constructions of SRGs over other finite fields, it may be possible to build new families of ETFs. Additionally, we may be able to generalize the Tremain construction further and potentially obtain new strongly regular graphs.

This dissertation will be laid out as follows. In Section 2 we provide some preliminary results about ETFs. Section 3 contains the basics on constructing both Steiner ETFs and Tremain ETFs. Section 4 will introduce and explore the finite geometry of regular quadratic spaces over a field of characteristic 2 while cataloging a number of useful results to prove the connection between $NO^+_{2n+1}(q)$ and Tremain ETFs. In Section 5 we develop a bijection that maps hyperbolic hyperplanes used in the construction of $NO^+_{2n+1}(q)$ to the points of a quadratic space. In section 6, we define two new related graphs and prove that one is isomorphic to $NO^+_{2n+1}(q)$. In Section 7 using these graphs, we present a natural method for constructing a Steiner ETF and a related Tremain ETF thus demonstrating a new connection between Tremain ETFs and strongly regular graphs. We wish to distinguish known results from new results, and we use the convention that theorems are either new results or previously known, named results for example, Theorem 4.32, the Witt Extension Theorem.

2 ETF BASICS

In order to work with equiangular tight frames, we collect a handful of results. We first start with the definition of a frame and develop the properties of ETFs.

Definition 2.1. Let $N, d \in \mathbb{N}$ such that $N \geqslant d$, and $\{\varphi_i\}_{i=1}^N$ be a sequence of vectors in $\mathbb{C}^d$, or $\mathbb{R}^d$. The sequence $\{\varphi_i\}_{i=1}^N$ is a called a **frame** if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\|x\|^2 \leqslant \sum_{i=1}^N |\langle x, \varphi_i\rangle|^2 \leqslant B\|x\|^2$$

for all $x \in \mathbb{C}^d$. We often consider the matrix whose columns consist of the frame vectors $\Phi = \begin{bmatrix} \varphi_1 & \cdots & \varphi_N \end{bmatrix}$. Indeed, we will often abuse notation and refer to Φ as a $d \times N$ frame.

Definition 2.2. A frame Φ is called a **tight frame** if $\Phi\Phi^* = AI$ for some $A > 0$.

If a frame is tight, then similarly to orthonormal bases, we can obtain a reconstruction formula for any vector in space.

Proposition 2.3. *If Φ is a tight frame such that $\Phi\Phi^* = AI$ for some $A > 0$, then for every $x \in \mathbb{C}^d$, $x = A^{-1}\sum_{i=1}^N \langle x, \varphi_i\rangle\varphi_i$.*

Proof. Since Φ is a tight frame then we know that $\Phi\Phi^* = AI$ for some $c > 0$. Therefore, we know that

$$Ax = \Phi\Phi^* x = \Phi(\Phi^* x) = \sum_{i=1}^N \langle x, \varphi_i\rangle\varphi_i.$$

Therefore, we have that $x = A^{-1}\sum_{1=i}^N \langle x, \varphi_i\rangle\varphi_i$. □

Note that orthonormal bases also have the property that any two distinct vectors are orthogonal. We now want to find a similar condition to impose on frames and we arrive at a definition about the maximum inner product for distinct frame vectors.

Definition 2.4. Suppose that Φ is a frame with $\|\varphi_i\| = 1$ for all $i \in \{1, \ldots, N\}$. The **coherence** of Φ is defined to be $\nu = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|$.

Definition 2.5. A sequence of vectors $\{\varphi_i\}_{i=1}^N$ is called **equiangular** if $|\langle \varphi_i, \varphi_j \rangle|$ is constant for all $i \neq j$ and $\|\varphi_i\|^2 = s > 0$ for all i.

We now want to prove the Welch bound as given by [19]. To accomplish this we need to define an appropriate matrix norm.

Definition 2.6. Let X be a complex matrix. The **Frobenius norm** of X is $\|X\|_F = \sqrt{\operatorname{tr}(XX^*)}$.

Proposition 2.7. *If $\Phi = \{\varphi_i\}_{i=1}^N$ is a frame where $\|\varphi_i\| = 1$ for all $i \in \{1, \ldots, N\}$ and has coherence ν, then $\nu \geqslant \sqrt{\frac{N-d}{d(N-1)}}$. Moreover, if equality holds, then Φ is a tight frame and $|\langle \varphi_i, \varphi_j \rangle| = \sqrt{\frac{N-d}{d(N-1)}}$ for all $i \neq j$.*

Proof. Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a frame with $\|\varphi_i\| = 1$ for all $i \in \{1, \ldots, N\}$. We begin by noting that

$$\|\Phi^*\Phi\|_F^2 = \operatorname{tr}\big((\Phi^*\Phi)(\Phi^*\Phi)^*\big) = \operatorname{tr}\big(\Phi^*(\Phi\Phi^*\Phi)\big) = \operatorname{tr}\big((\Phi\Phi^*\Phi)\Phi^*\big) = \|\Phi\Phi^*\|_F^2.$$

Also, denote the entries of $[f_{ij}] = [\Phi\Phi^*]_{ij}$. Now we know that

$$
\begin{aligned}
0 \leqslant \|\Phi\Phi^* - \frac{N}{d}I\|_F^2 &= \sum_{i\neq j}|f_{ij}|^2 + \sum_{i=1}^{d}|f_{ii} - \frac{N}{d}|^2 = \sum_{i\neq j}|f_{ij}|^2 + \sum_{i=1}^{d}(f_{ii} - \frac{N}{d})^2 \\
&= \sum_{i\neq j}|f_{ij}|^2 + \sum_{i=1}^{d}(f_{ii} - \frac{2N}{d}f_{ii} + \frac{N^2}{d^2}) = \|\Phi\Phi^*\|^2 + \sum_{i=1}^{d}(-\frac{2N}{d}f_{ii} + \frac{N^2}{d^2}) \\
&= \|\Phi\Phi^*\|^2 - \frac{2N}{d}\operatorname{tr}(\Phi\Phi^*) + \frac{N^2}{d} = \|\Phi\Phi^*\|^2 - \frac{2N}{d}\operatorname{tr}(\Phi^*\Phi) + \frac{N^2}{d} \\
&= \|\Phi\Phi^*\|^2 - \frac{2N}{d}(N) + \frac{N^2}{d} = \|\Phi\Phi^*\|^2 - \frac{N^2}{d} \\
&= \|\Phi^*\Phi\|^2 - \frac{N^2}{d} = N + \sum_{i\neq j}|\langle\varphi_i, \varphi_j\rangle|^2 - \frac{N^2}{d} \leqslant N + N(N-1)\nu^2 - \frac{N^2}{d}.
\end{aligned}
$$

This inequality implies that $\nu \geqslant \sqrt{\frac{N-d}{d(N-1)}}$ and so the coherence is bounded below.

Additionally, if there is equality in the above inequality, then we can see that $\|\Phi\Phi^* - \frac{N}{d}I\| = 0$ which implies that $\Phi\Phi^* = \frac{N}{d}I$ and so Φ is a tight frame. Additionally, equality also implies that $|\langle\varphi_i, \varphi_j\rangle| = \sqrt{\frac{N-d}{d(N-1)}}$ for all $i \neq j$. $\square$

The moreover of Proposition 2.7 leads us to the definition of an equiangular tight frame since all the inner products of distinct columns are equal, and the frame is tight.

Definition 2.8. A frame, Φ, is called an **equiangular tight frame** if the coherence of Φ is equal to the Welsh bound.

According to [17], equiangular tight frames are quite uncommon and generally an ETF of N vectors in d-dimensional space does not occur or is not known to occur for arbitrary pairs of $(d, N) \in \mathbb{N}^2$. The following propositions prove some results that give us more information about what possible pairs of (d, N) to look for.

Proposition 2.9. *If $\{\varphi_i\}_{i=1}^N \subseteq \mathbb{R}^d$ is a frame such that $\langle\varphi_i, \varphi_j\rangle = \alpha \in [0, 1)$ for all $i \neq j$ and $\|\varphi_i\| = 1$ for all $i \in \{1, \ldots, N\}$, then $N \leqslant d$.*

Proof. We know that $\Phi^*\Phi = (1-\alpha)I + \alpha J$. A straightforward calculation gives us that the eigenvalues of $\Phi^*\Phi$ are $1 + \alpha(N-1)$ and $1 - \alpha$. Note that because $\alpha \in [0, 1)$

then we have that both eigenvalues are nonzero. Hence $\operatorname{rank}(\Phi) = \operatorname{rank}(\Phi^*\Phi) = N$. But we know that since Φ is a frame, then $\operatorname{rank}(\Phi) \leqslant d$. Therefore we have $N \leqslant d$. $\square$

Proposition 2.10. *Let Φ be a $d \times N$ equiangular tight frame. If Φ is real, then $N \leqslant \binom{d+1}{2}$ and if Φ is complex, then $N \leqslant d^2$.*

Proof. Let Φ be an equiangular tight frame. Then we know that $|\langle \varphi_i, \varphi_j \rangle|$ is constant for all $i \neq j$ and $\|\varphi_i\| = 1$ for all i. Note that

$$\|\varphi_i \varphi_j^*\|_F^2 = \operatorname{tr}((\varphi_i \varphi_j^*)(\varphi_i \varphi_j^*)^*) = \operatorname{tr}(\varphi_i \varphi_j^* \varphi_j \varphi_i^*) = \operatorname{tr}((\varphi_j^* \varphi_j)(\varphi_i^* \varphi_i)) = 1.$$

Additionally, note that

$$\begin{aligned}\langle \varphi_i \varphi_i^*, \varphi_j \varphi_j^* \rangle = \operatorname{tr}(\varphi_i \varphi_i^* (\varphi_j \varphi_j^*)^*) = \operatorname{tr}(\varphi_i \varphi_i^* \varphi_j \varphi_j^*) = \operatorname{tr}((\varphi_i^* \varphi_j)(\varphi_j^* \varphi_i)) = \operatorname{tr}(|\langle \varphi_j, \varphi_i \rangle|^2) \\ = |\langle \varphi_j, \varphi_i \rangle|^2.\end{aligned}$$

Therefore the set of matrices $\{\varphi_i \varphi_i^*\}_{i=1}^N$ forms an equiangular sequence in the vector space

$V = \{A \in \mathbb{F}^{d \times d} : A = A^*\}$ since $\varphi_i \varphi_i^*$ is self adjoint for each $i \in \{1, \ldots, N\}$. If Φ is a real frame then V is a real vector space of symmetric matrices. If we denote E_{ij} to be the matrix with all zeros except for the i, j element equal to 1, then a basis for V is the set $\{E_{ij} + E_{ji} : 1 \leqslant i \leqslant j \leqslant d\}$. This implies that there are preciesly $d + (d-1) + \cdots + 2 + 1 = \binom{d+1}{2}$ basis elements and so we must have that $N \leqslant \binom{d+1}{2}$ by the previous proposition. If Φ is a complex frame, then V is a complex vector space of self-adjoint matrices. Therefore, a basis for V is $\{E_{ii} : 1 \leqslant i \leqslant N\} \cup \{E_{ij} + E_{ji} : 1 \leqslant i < j \leqslant N\} \cup \{iE_{ij} - iE_{ji} : 1 \leqslant i < j \leqslant N\}$. Hence $\dim V = 2\frac{d(d-1)}{2} + d = d^2$ so we have $N \leqslant d^2$. $\square$

3 STEINER AND TREMAIN ETF BASICS

Now that we have some basic ETF theory, we now want to define a couple of ETFs that will be the primary focus of this paper. In this section, we will define all of the components necessary to building both a Steiner ETF and a Tremain ETF. Along the way, we will also provide examples to better illustrate these constructions. To first build a Steiner ETF, we need a Steiner System which is a balanced incomplete block design (BIBD). And we start with that definition.

Definition 3.1. A **balanced incomplete block design** (BIBD) is a a pair $(V, \mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of b k-subsets of V such that each element of V is contained in exactly r bocks and any $2-$subset of V is contained in exactly λ blocks. The numbers, $v,\ b,\ r,\ k$, and λ are called the parameters of the BIBD. A **Steiner System** is a BIBD whith $\lambda = 1$.

To build a Steiner ETF, we will need both the incidence matrix of the Steiner System and a Hadamard matrix.

Definition 3.2. The **incidence matrix** of a BIBD is a $b \times v$ matrix A, with rows indexed by $\mathcal{B}$ and columns indexed by V given by

$$A(B,x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if else.} \end{cases}$$

Example 3.3. The incidence matrix A for a BIBD(7,3,1) is

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \end{bmatrix}$$

where $\cdot$ corresponds to entries of 0.

Note that the Steiner system from the previous example has a block size of 3. If a Steiner system has a parameter $k = 3$, we call this a **Steiner triple system.**

Definition 3.4. A **Hadamard matrix** is a square matrix whose entries are $+1$ or -1 and whose rows are mutually orthogonal.

Example 3.5. An example of a Hadamard matrix is

$$H_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix}.$$

Definition 3.6. Let $(V, \mathcal{B})$ be a BIBD with parameters v, b, r, k, and $\lambda = 1$. If Φ is an ETF with rows indexed by $\mathcal{B}$ and columns indexed by $V \times \mathcal{R}$ where $|\mathcal{R}| = r + 1$, and

$$|\Phi(B, (x, j))| = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B, \end{cases}$$

then we call Φ a **Steiner ETF.**

Example 3.7. We will combine the incidence matrix and the last three rows of the Hadamard matrix from the previous two examples to create a Steiner ETF by replacing the nonzero entries from each column of the incidence matrix and replacing it with a distinct row of the Hadamard matrix. We note that in the figure

+	−	+	−	+	−	+	−	+	−	+	−																
+	+	−	−									+	−	+	−	+	−	+	−								
+	−	−	+																	+	−	+	−	+	−	+	−
				+	+	−	−					+	+	−	−					+	+	−	−				
				+	−	−	+									+	+	−	−					+	+	−	−
								+	+	−	−	+	−	−	+									+	−	−	+
								+	−	−	+					+	−	−	+	+	−	−	+				

Figure 1: 7×28 Steiner ETF

above, the blank spaces are zeros and the symbols + and − correspond to 1 and -1 respectively. Consider that this matrix has rows indexed by the blocks of the BIBD in Example 3.3, and columns indexed by the $V \times \mathcal{R}$ where $\mathcal{R}$ is a set which orders the columns of the Hadamard matrix from Example 3.5. Therefore, Definition 3.6 applies to this matrix.

Note that in this construction that the dot product of any two rows of is equal to the dot product of any two rows of a Hadamard matrix, so the rows of this matrix are mutuallly orthogonal. Additionally, the rows all contain 12 entries of modulus 1, and so the row norms are equal as well. This implies the above matrix is tight. To show that this matrix is equiangular, we note that each column has norm squared equal to 3, and that because $\lambda = 1$, then clearly the dot product of any two distinct columns is modulus 1. Therefore, the above matrix is clearly a Steiner ETF.

Definition 3.8. Let $(V, \mathcal{B})$ be a BIBD with parameters v, b, r, $\lambda = 1$, and $k = 3$. If Ψ is an ETF with rows indexed by $X = \mathcal{B} \cup V \cup \{0\}$ and columns indexed by

$Y = (V \times \mathcal{R}) \cup S$, where $|\mathcal{R}| = r + 1$ and $|S| = 2r + 2$, and

$$|\Psi(x,y)| = \begin{cases} 1 & \text{if } x = B \in \mathcal{B},\ y = (z,j) \in (V \times \mathcal{R}),\ z \in B \\ \sqrt{2} & \text{if } x = z \in V,\ y = (z,j) \in (V \times \mathcal{R}) \\ \sqrt{\frac{1}{2}} & \text{if } x = z \in V,\ y \in S \\ \sqrt{\frac{3}{2}} & \text{if } x = 0,\ y \in S \\ 0 & \text{else,} \end{cases}$$

then we say that Ψ is a **Tremain ETF.**

Example 3.9. We will combine the Steiner ETF from the previous example and add the rows and columns of two scaled Hadamard matrices appropriately and we obtain the figure below. We note that the blank spaces are zeros and the sybmols $+$

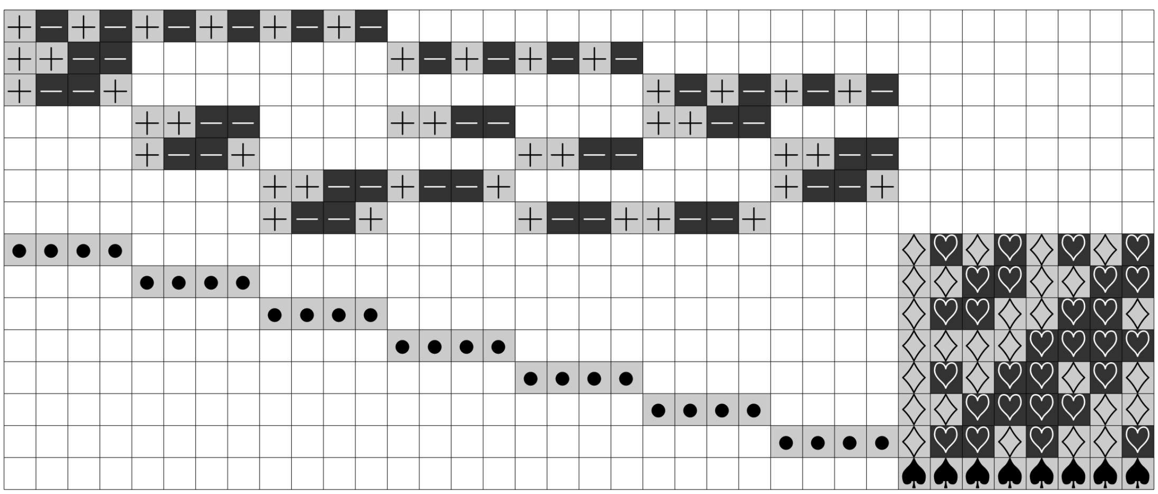

Figure 2: Tremain ETF

and $-$ correspond to 1 and -1 respectively. Meanwhile $\bullet = \sqrt{2}$, $\diamondsuit = \sqrt{\frac{1}{2}}$, $\heartsuit = -\sqrt{\frac{1}{2}}$ and $\spadesuit = \sqrt{\frac{3}{2}}$. The rows and columns of this matrix can be indexed as described in Definition 3.8 and routine calculations show that this matrix has orthogonal, equal norm rows, equal norm columns, and the modulus of the dot product between any two distinct columns is 1. Therefore, this matrix is a Tremain ETF.

We now present another small example of a Tremain ETF.

Example 3.10. Let the incidence matrix of a trivial Steiner Triple system be

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

We will build the the Steiner ETF,

$$\Phi = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

We will then add the rows and columns of two scaled Hadamard matrices appropriately and obtain the matrix below.

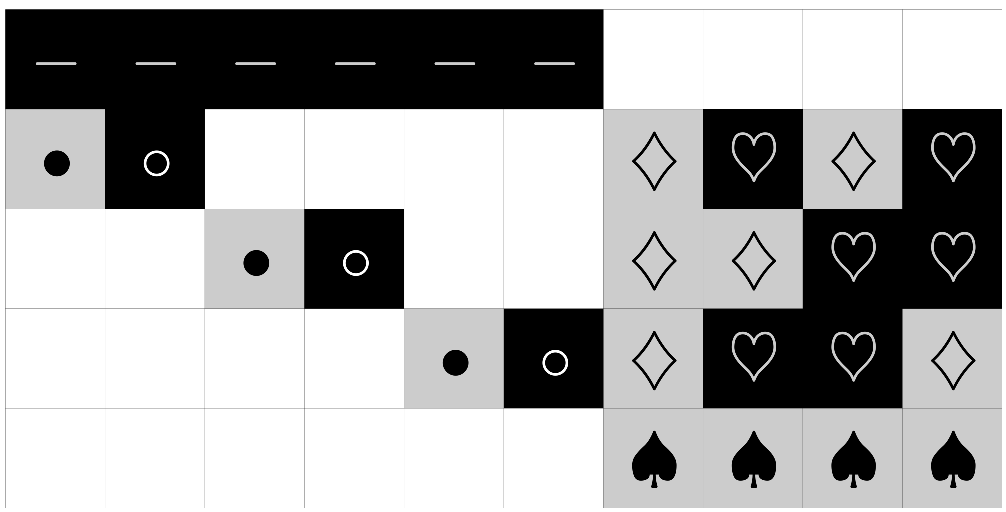

Figure 3: 5×10 Tremain ETF

4 GEOMETRY OF QUADRATIC SPACES OVER FIELDS OF CHARACTERISTIC 2

The purpose of this paper is to strengthen the connection between Tremain ETFs and the family of graphs known as, $NO^{+}_{2n+1}(2^h)$. In order to construct the graph, $NO^{+}_{2n+1}(2^h)$, given in [2], we require the construction of a collection of hyperplanes in an odd dimensional vector space over a field of characteristic 2 with a nondegenerate quadratic form. This section will contain a number of results which are necessary to describing the graph and better understanding its construction.

Assumption. Let V be a vector space over a field, $\mathbb{F}$, of characteristic 2.

Definition 4.1. A **quadratic form** on a vector space is a function $Q : V \to \mathbb{F}$ with the property

$$Q(ax) = a^2 Q(x)$$

for all $a \in \mathbb{F}$, all $x \in V$, and where the function $B : V \times V \to \mathbb{F}$ given by

$$B(x, y) = Q(x + y) + Q(x) + Q(y)$$

is a bilinear form on V.

While the common definition of a bilinear form on V associated with a given quadratic form is $B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y))$, our definition is specific to quadratic spaces over fields of characteristic 2.

Definition 4.2. If Q is a quadratic form on V and $x \in V \backslash \{0\}$ such that $Q(x) = 0$, then we say that x is a **singular vector**. Additionally, if V contains a singular vector then we call V a **singular space**. If V contains no singular vectors, then we call V a **nonsingular space**.

This new quadratic form and bilinear form are analogous to a norm and inner product defined on inner product spaces. We will now investigate the properties of the bilinear form on V.

Definition 4.3. Let $W \subseteq V$ be a subspace. Define the **orthogonal complement** of W in V to be

$$W^{\perp} = \{x \in V : B(x, w) = 0 \text{ for all } w \in W\}.$$

Definition 4.4. Given a vector space V and a quadratic form Q, define the **radical** of V to be

$$\operatorname{rad} V = \{x \in V : B(x, v) = 0, \text{ for all } v \in V\}.$$

If $\operatorname{rad} V \neq \{0\}$ then we say that V is **defective**, and if $\operatorname{rad} V = \{0\}$, then we say that V is **nondefective**.

In order to better illustrate these definitions, we present two basic examples in three and four dimensional space.

Example 4.5. Let $V = \mathbb{F}_4^3$ and let $\{e_1, e_2, e_3\}$ be the standard basis for V. Then for any

$x = a_1e_1 + a_2e_2 + a_3e_3 \in V$ a quadratic form on V is given by

$$Q(x) = a_1a_2 + a_3^2.$$

The induced bilinear form on V is given by

$$B(x, y) = (a_1 + b_1)(a_2 + b_2) + (a_3 + b_3)^2 + a_1a_2 + a_3^2 + b_1b_2 + b_3^2 = a_1b_2 + a_2b_1.$$

It is easy to see that V is a singular space since $Q(e_1) = 0$. Additionally, we can see that $B(e_3, x) = 0$ for any $x \in V$ and so $\operatorname{rad} V = \langle e_3 \rangle$ and so V is defective.

Note. Any quadratic space over a field of characteristic 2 has the property that for any $a \in \mathbb{F}$, $B(x, x) = B(x, ax) = 0$.

Example 4.6. Let $V = \mathbb{F}_4^4$ and let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for V. Then for any

$x = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 \in V$, a quadratic form on V is given by

$$Q(x) = a_1a_2 + a_3a_4.$$

The induced bilinear form on V is given by

$$\begin{aligned} B(x, y) &= (a_1 + b_1)(a_2 + b_2) + (a_3 + b_3)(a_4 + b_4) + a_1a_2 + a_3a_4 + b_1b_2 + b_3b_4 \\ &= a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3. \end{aligned}$$

Note again that V is a singular space since $Q(e_i) = 0$ for any $i \in \{1, 2, 3, 4\}$. For this example, if for any $x \in V$ we have $B(x, y) = 0$ for all $y \in V$, then x or y equals zero. This is an example of a nondefective space.

With the definition of a bilinear form on V, a natural question is to determine the dimension of the orthogonal complement of an arbitrary subspace of a quadratic space. We first need a couple definitions and provide an example.

Definition 4.7. Let $V, W,$ be subspaces of a vector space. Then

$$V + W = \{v + w : v \in V,\ w \in W\}.$$

If $V \cap W = \{0\}$, we denote the sum as $V \oplus W$ and call it the **direct sum**. If in addition, we have that $V \perp W$ then we denote the sum as $V \obot W$ and call it the **orthogonal direct sum**.

Example 4.8. Let V be the vector space as given in Example 4.5 and let $W = \langle e_1 \rangle$ be a subspace of V. Note that $B(e_1, e_2) = 1$ and $B(e_1, e_3) = 0$. This implies that $W^\perp = \langle e_1, e_3 \rangle$. Therefore, we can see that $\dim(W) + \dim(W^\perp) = \dim(V)$ but we do not have $W \obot W^\perp = V$ since $e_2 \notin W \obot W^\perp$.

Proposition 4.9. *If $W \subseteq V$ is a subspace of a quadratic space (V, Q), then*

$$\dim W^{\perp} = \begin{cases} \dim V - \dim W & \text{if } W \cap \operatorname{rad} V = \{0\} \\ \dim V - \dim W + \dim(\operatorname{rad} V) & \text{if } \operatorname{rad} V \subseteq W. \end{cases}$$

Proof. First, suppose that $W \cap \operatorname{rad} V = \{0\}$, that $\dim W = k$ and that $\{w_1, \ldots, w_k\}$ is a basis for W. Now, define the function $\varphi : V \to \mathbb{F}^k$ given by

$$\varphi(v) = (B(v, w_1), \ldots, B(v, w_k)).$$

Note that $x \in W^{\perp}$ if and only if $\varphi(x) = 0$ and so $\ker \varphi = W^{\perp}$. Therefore by Rank-Nullity, we have that $\dim W^{\perp} = \dim V - \dim \varphi(V) \geqslant \dim V - k = \dim V - \dim W$. Now suppose that $\dim \varphi(V) = p \leqslant k = \dim W$ and let $\{x_1, \ldots x_p\}$ be a basis for $\varphi(V)$. Let $v_1, v_2, \ldots, v_p \in V$ such that $\varphi(v_i) = x_i$ for all $i \in \{1, \ldots, p\}$. Note that if $0 = \sum_{i=1}^{p} \beta_i v_i$ for some $\beta_i \in \mathbb{F}$ for all i implies that $0 = \sum_{i=1}^{p} \beta \varphi(v_i) = \sum_{i=1}^{p} \beta_i x_i$. This implies that $\beta_1 = \cdots = \beta_p = 0$ and so $\{v_1, \ldots, v_p\}$ is an independent set. Let $X = \langle v_1, \ldots, v_p \rangle$. Suppose that if $x \in X \cap W^{\perp}$ then we have $x = \sum_{i=1}^{p} \beta_i v_i$ and that $0 = \varphi(x) = \sum_{i=1}^{p} \beta_i \varphi(v_i) = \sum_{i=1}^{p} \beta_i x_i$. This implies that $x = 0$, and so $X \cap W^{\perp} = \{0\}$. Now consider the set $X \oplus W^{\perp}$. We have that $\dim W^{\perp} = \dim V - \dim \varphi(V)$ and so $\dim V = \dim W^{\perp} + \dim \varphi(V) = \dim W^{\perp} + \dim X$. Hence we know that $X \oplus W^{\perp} = V$. Next, define the map $\psi : W \to \mathbb{F}^p$ given by $\psi(w) = (B(v_1, w), B(v_2, w), \ldots, B(v_p, w))$. We claim that $\ker \psi \subseteq \operatorname{rad} V$. Suppose that $w \in \ker \psi$. Then $w \in X^{\perp}$ and $w \in W$. Then for any $v \in V$ we have that $v = x + y$ for some $x \in X$ and $y \in W^{\perp}$. Hence we have that $B(v, w) = B(x + y, w) = B(x, w) + B(y, w) = 0$, and so we have $\ker \psi \subseteq \operatorname{rad} V$.

Therefore, we have that $p = k$ and so

$$\dim W^{\perp} = \dim V - \dim W.$$

Now suppose that $\operatorname{rad} V \subseteq W$, and $\dim W = k$. Then suppose that $\operatorname{rad} V$ has a basis $\{r_1, \ldots, r_p\}$ and let $\{r_1, \ldots, r_p, w_{k-p}, w_{k-p+1}, \ldots, w_k\}$ be a basis for W. Let $W' = \langle w_{k-p}, \ldots, w_k \rangle$. Note that $W' \cap \operatorname{rad} V = \{0\}$ and clearly $W^{\perp} = (W')^{\perp}$ and so we can apply the first case and hence

$$\dim W^{\perp} = \dim (W')^{\perp} = \dim V - \dim W' = \dim V - \dim W + \dim \operatorname{rad} V. \qquad \square$$

Now that we have the basic definitions of quadratic spaces, the construction of $NO^{+}_{2n+1}(2^h)$ utilizes a specific type of hyperplanes. Therefore, we will define and explore the properties of hyperplanes of a quadratic space.

Definition 4.10. Let Q be a quadratic form on V. We call the pair (V, Q) a **quadratic space**, and we say that V is **regular** with respect to Q if for all $x \in \operatorname{rad} V \backslash \{0\}$, we have $Q(x) \neq 0$.

We will begin by defining two different types of planes in a quadratic space, and we also want to discuss the idea of regular subspaces of a quadratic space. We say a subspace $H \subseteq V$ is regular if the induced quadratic space $(H, Q|_H)$ is regular.

Definition 4.11. A **hyperbolic pair** is an pair (x, y) of singular vectors with $B(x, y) = 1$. A two dimensional space which contains a hyhperbolic pair is a **hyperbolic plane**.

Definition 4.12. Let (V, Q) be a quadratic space, and $E \subseteq V$. If $\dim E = 2$ and E is nonsingular, then we say that E is an **elliptic plane**.

Now that we have defined two different types of planes, we want to ensure that any regular plane of a quadratic space is either hyperbolic or elliptic. This allows will allow us to be precise when we decompose hyperplanes into orthogonal planes later in this section.

Proposition 4.13. *If (V, Q) is a regular quadratic space of dimension 2, then V has zero or two singular subspaces of dimension 1.*

Proof. Let N denote the number of one dimensional singular subspaces of V, and suppose that V is regular. If $N = 3$, then let $\langle x\rangle, \langle y\rangle, \langle z\rangle$ be distinct one dimensional singular subspaces of V. Without loss of generality, suppose that $V = \langle x, y\rangle$. If $B(x, y) = 0$, then $\langle x\rangle \subseteq \operatorname{rad} V$, but this is a contradiction since V is regular. Hence $B(x, y) \neq 0$. Without loss of generality, suppose that $B(x, y) = 1$. Now if $B(x, z) = B(y, z) = 0$ then we have $\langle z\rangle \subseteq \operatorname{rad} V$ and so V is not regular, a contradiction. If $B(x, z) = 0$ but $B(y, z) \neq 0$, then consider that $B(z + B(z, y)x, x) = 0$, $B(z + B(y, z)x, y) = B(y, z) + B(y, z) = 0$ and that $Q(z + B(y, z)x) = Q(z) + (B(y, z))^2 Q(x) + B(z, B(y, z)x) = 0$ and so V is not regular which is a contradiction. Now suppose that $B(x, z) \neq 0$ and $B(y, z) \neq 0$. Then we claim that $\{x, y, z\}$ is an independent set. Suppose that $ax + by + cz = 0$ for some $a, b, c \in \mathbb{F}$. Without loss of generality, suppose that $B(x, y) = B(x, z) = 1$ and $B(y, z) = \beta$ for some $\beta \neq 0$. Then we have $0 = B(ax + by + cz, x) = b + c$, $0 = B(ax + by + cz, y) = a + \beta c$, and $0 = B(ax + by + cz, z) = a + \beta b$. This implies that $a = b = c = 0$ and so $\{x, y, z\}$ is an independent set which is a contradiction since $\dim V = 2$. Therefore, we know that $N \leqslant 2$.

Now suppose that $N = 1$ and that $\langle x\rangle \subseteq V$ for some $x \neq 0$ with $Q(x) = 0$. Since V is regular, $\langle x\rangle \cap \operatorname{rad} V = \{0\}$. Hence there exists $y \in V$ such that $B(x, y) \neq 0$. Without loss of generality, suppose that $B(x, y) = 1$. Note also that $Q(y) \neq 0$ since $N = 1$. Now consider the subspace $\langle y + Q(y)x\rangle$. Then we have $Q(y + Q(y)x) = Q(y) + (Q(y))^2 Q(x) + Q(y)B(x, y) = 0$. Hence if $N > 0$, then $N = 2$. Note that if $N = 0$, then V is nonsingular, and therefore is regular. Therefore, $N \in \{0, 2\}$ if V is regular. $\square$

Corollary 4.14. *Every regular two dimensional quadratic space is either hyperbolic or elliptic.*

The next theorem gives us a method of determining whether a regular quadratic plane is elliptic or hyperbolic.

Proposition 4.15. *Let (V, Q) be a quadradic space of dimension 2. Let $u, v \in V$ and $f \in \mathbb{F}[x]$ be defined by $f(x) = Q(xu + v) = Q(u)x^2 + B(u, v)x + Q(v)$. If V is elliptic, then f is irreducible for any basis $\{u, v\}$. Conversely, if there exists a basis $\{u, v\}$ for V with $Q(u) \neq 0$ such that f is irreducible, then V is elliptic.*

Proof. Let (V, Q) be a quadratic space of dimension 2 and that V is elliptic. Then we know that V is nonsingular. Let $\{u, v\}$ be a basis for V. Consider that $f(x) = Q(xu + v) \neq 0$ since $xu + v \in V$ for all $x \in \mathbb{F}$. Hence f does not have a root and must therefore be irreducible.

Conversely, let $\{u, v\}$ be a basis for V such that f is irreducible and $Q(u) \neq 0$. Then we know that the line $\langle xu + v\rangle$ is nonsingular for each $x \in \mathbb{F}$. Also, since $Q(u) \neq 0$ then we have no singular lines in V and so V must be an elliptic plane. □

In the light of Proposition 4.15 and Corollary 4.14, we also see that if V is hyperbolic, then f is has a root, and if there is a basis for which f has a root, and V is regular, then V is hyperbolic.

Assumption. We now assume that (V, Q) is a regular quadratic space in addition to the previous assumption that $\mathbb{F}$ is a field of characteristic 2.

A very useful tool in navigating quadratic spaces is to be able to identify a hyperbolic plane which contains a given vector. The next proposition gives us a method of constructing such a hyperbolic plane.

Proposition 4.16. *If $x \in V$ is singular, then there exists $y \in V$ such that (x, y) is a hyperbolic pair.*

Proof. Let $x \in V$ such that $Q(x) = 0$. Since $x \notin \operatorname{rad} V$ there exists $z \in V$ so that $B(x, z) \neq 0$. Without loss of generality, suppose that $B(x, z) = 1$. Consider the element $y = Q(z)x + z$. Then we know that
$Q(y) = Q(z)^2 Q(x) + Q(z)B(x, z) + Q(z) = 0$ and
$B(x, y) = B(x, Q(z)x + z) = Q(z)B(x, x) + B(x, z) = 1$. Hence (x, y) is a hyperbolic pair. □

One of our major results in this section will be that regular hyperplanes can be written as an orthogonal direct sum of regular planes. The following result tells us that taking the orthogonal complement of a hyperbolic plane is regular.

Proposition 4.17. *If H is a regular plane of a regular quadratic space (V, Q), then $V = H \oplus H^\perp$. Moreover, $H^\perp$ is a regular subspace of V.*

Proof. Suppose $H = \langle x, y \rangle$ is a regular plane in a regular quadratic space (V, Q) of dimension n. First note that if $z \in H \cap \operatorname{rad} V$ then we have $z = ax + by$ for some $a, b \in \mathbb{F}$ and that $B(x, y) \neq 0$. Also, we know that $0 = B(z, x) = b$ and $0 = B(z, y) = a$ and so $z = 0$. Hence, $H \cap \operatorname{rad} V = \{0\}$. Therefore, by Proposition 4.9 we know that $\dim H^\perp = \dim V - \dim H = n - 2$. Additionally by the same argument as above, we have that $H \cap H^\perp = \{0\}$ and by definition $H \perp H^\perp$. Therefore, $\dim(H \oplus H^\perp) = 2 + n - 2 = n = \dim V$ and clearly $(H \oplus H^\perp) \subseteq V$ and so we have $H \oplus H^\perp = V$.

Now by way of contradiction, suppose that $H^\perp$ is not regular. Then there exists a vector $r \in H^\perp$ such that $B(r, h) = 0$ for all $h \in H^\perp$ and $Q(r) = 0$. But then $B(r, k) = 0$ for all $k \in H$ since $r \in H^\perp$ and so r is a singular vector in the radical. This contradicts that V is a regular space, so it must be the case that $H^\perp$ is a regular subspace of V. □

Because $\mathbb{F}$ is a field of characteristic 2, we will want to leverage the properties of such fields. Hence the following definition.

Definition 4.18. A field $\mathbb{F}$ is called a **perfect field** if either $\mathbb{F}$ has characteristic 0, or, when $\mathbb{F}$ has characteristic $p > 0$, the Frobenius endomorphism $x \mapsto x^p$ is an automorphism of $\mathbb{F}$.

In our case, we have that $\mathbb{F}$ is a field of characteristic 2. Therefore, if $\mathbb{F}$ is perfect, then Definition 4.18 tells us that $x \mapsto x^2$ is an automorphism of $\mathbb{F}$ and so each element of $\mathbb{F}$ has a square root. Another useful characterization of planes is to determine their regularity based on their bases. A majority of our analysis of the geometry will require our spaces to be regular.

Proposition 4.19. *Let (V, Q) be a quadratic space with $\dim V = 2$. V is regular if and only if no basis for V is orthogonal.*

Proof. Let (V, Q) be a quadratic space with $\dim V = 2$. Suppose that $\{u, v\}$ is a basis for V such that $B(u, v) = 0$. Then if $Q(u) = 0$ or $Q(v) = 0$ then V is not regular. Then consider the polynomial
$f(x) = Q(xu + v) = Q(u)x^2 + B(u, v)x + Q(v) = Q(u)x^2 + Q(v)$. Note that since $\mathbb{F}$ is a perfect field, there exists $x_0 \in \mathbb{F}$ such that $x_0^2 = \frac{Q(v)}{Q(u)}$ and so f has a root and hence $Q(x_0u + v) = 0$ Also, note that $B(x_0u + v, u) = B(x_0u + v, v) = 0$ and so V is not regular.

Now let $\{u, v\}$ be a basis for V such that $B(u, v) \neq 0$. Let $x \in \operatorname{rad} V$. Then $x = au + bv$ for some $a, b \in \mathbb{F}$ and we have $0 = B(x, u) = B(au + bv, u) = bB(v, u)$ and $0 = B(x, v) = B(au + bv, v) = aB(u, v)$. But since $B(u, v) \neq 0$, then we must have $a = b = 0$ and so $x = 0$. Hence V is regular. $\square$

The property of perfection will allow us to determine the maximum dimension of the radical of a given quadratic space as well as determine that quadratic spaces of high enough dimension are singular.

Proposition 4.20. *Let V be a vector space over a perfect field. If V is a regular quadratic space with respect to Q, then the radical of V has either dimension 0 or 1.*

Proof. Suppose that $\dim(\operatorname{rad} V) \geqslant 2$. Then we know that there exists an independent set, $\{x, y\} \subseteq \operatorname{rad} V$. Since V is a regular space, we know that $Q(x) \neq 0$ and $Q(y) \neq 0$. Since $\mathbb{F}$ is perfect, we may assume that $Q(x) = Q(y) = 1$. Consider that since $x, y \in \operatorname{rad} V$, we have $0 = B(x, y) = Q(x+y) + Q(x) + Q(y) = Q(x+y)$. Since $\{x, y\}$ is independent, then we know that $x + y \neq 0$ and so there exists a nonzero singular vector in the radical of V and so V is not regular. Hence, $\dim(\operatorname{rad} V) < 2$ and so we have $\dim(\operatorname{rad} V) \in \{0, 1\}$. $\square$

Proposition 4.21. *If (V, Q) is a quadratic space over a perfect field with $\dim V \geqslant 3$, then V is singular.*

Proof. Let $x \in V \backslash \{0\}$ and let $y \in \langle x \rangle^{\perp} \backslash \langle x \rangle$. Note that since $x \neq 0$, we have by Proposition 4.9 that $\dim \langle x \rangle^{\perp} \geqslant 2$. If $Q(x) = 0$ then V is singular. If $Q(x) \neq 0$, then $\frac{Q(y)}{Q(x)} \in \mathbb{F}$. Since $\mathbb{F}$ is perfect, there is an element $a \in \mathbb{F}$ such that $a^2 = \frac{Q(y)}{Q(x)}$. Consider $z = ax + y$. Then $Q(z) = a^2 Q(x) + aB(x, y) + Q(y) = Q(y) + Q(y) = 0$. Hence V is singular. $\square$

Corollary 4.22. *If (V, Q) is a quadratic space over a perfect field with $\dim V \geqslant 3$, then there is a hyperbolic plane in V.*

The previous proposition allows us to factor regular, odd dimensional quadratic spaces into an orthogonal direct sum of hyperbolic planes and the radical.

Proposition 4.23. *If (V, Q) is an odd dimensional regular quadratic space with with $\dim V = 2n + 1 \geqslant 3$, then there exists a set of hyperbolic planes $\{H_i\}_{i=1}^{n}$ such that $V = \bigoplus_{i=1}^{n} H_i \oplus \operatorname{rad} V$.*

Proof. We will induct on n where $\dim V = 2n + 1$. Let $n = 1$. Then $\dim V = 2n + 1 = 3$. By Corollary 4.22 and Proposition 4.17, we see that there exists a hyperbolic plane, H, in V. Therefore, $V = H \oplus H^{\perp}$ with $\dim H^{\perp} = 1$. Since V is regular and clearly $B(h, v) = 0$ for all $h \in H^{\perp}$, and $v \in V$, then $H^{\perp}$ is nonsingular and hence $H^{\perp} = \operatorname{rad} V$ so $V = H \oplus \operatorname{rad} V$.

Suppose that the proposition is true for when $n = k$. Let $n = k + 1$ and $\dim V = 2k + 3$. Since $2k + 3 > 3$, we have that there is a hyperbolic plane, $H_1 \subseteq V$, by Corollary 4.22 and that $V = H_1 \oplus H_1^{\perp}$. Note that $H_1^{\perp}$ is a regular and $\dim H_1^{\perp} = 2k + 1$. By the induction hypothesis, there exists a set of hyperbolic planes, $\{H_i\}_{i=1}^{k}$ such that $H_1^{\perp} = \bigoplus_{i=2}^{k} H_i \oplus \operatorname{rad} H_1^{\perp}$. Hence
$V = \bigoplus_{i=1}^{k} H_i \oplus \operatorname{rad} H_1^{\perp} = \bigoplus_{i=1}^{k+1} H_i \oplus \operatorname{rad} V$. □

The previous proposition covers all of the odd dimensional regular quadratic spaces, so now we want to determine a decomposition for the even dimensional regular hyperplanes of these odd dimensional spaces.

Definition 4.24. Let (V, Q) be a regular quadratic space. We say that V is a **hyperbolic space** if there exists a collection of hyperbolic planes $\{H_i\}_{i=1}^{n}$ such that $V = \bigoplus_{i=1}^{n} H_i$.

Definition 4.25. Let (V, Q) be a regular quadratic space of dimension $2n + 1$ and let $H \subseteq V$ be a hyperplane. We say that H is a **hyperbolic hyperplane** if the induced quadratic space $(H, Q|_H)$ is a hyperbolic space.

Definition 4.26. Let (V, Q) be a regular quadratic space of dimension $2n + 1$ and let $H \subseteq V$ be a hyperplane. We say that H is a **elliptic hyperplane** if the induced quadratic space $(H, Q|_H)$ is regular and not hyperbolic.

To ensure that our definitions of hyperbolic hyperplanes and elliptic hyperplanes are consistent, we prove that they are regular.

Proposition 4.27. *Let V be a regular quadratic space of dimension $2n + 1$. If H is a hyperbolic hyperplane then H is a regular hyperplane.*

Proof. Note that if V is odd dimensional, then H is even dimensional. Let $\{H_i\}_{i=1}^{n}$ be a collection of hyperbolic planes such that $H = \bigoplus_{i=1}^{n} H_i$. Then if we let (h_{i1}, h_{i2}) be a hyperbolic pair for H_i then we have $\{h_{11}, h_{12}, h_{21}, h_{22}, \ldots, h_{n1}, h_{n2}\}$ as a basis for H. Let $x \in \operatorname{rad} H$. Then we know that $x = \sum_{i=1}^{n}(a_{i1}h_{i1} + a_{i2}h_{i2})$ for some scalars $a_{ij} \in \mathbb{F}$. Note that for all $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$ we have $0 = B(x, h_{ij}) = a_{ik}$ for $k \in \{1, 2\} \backslash \{j\}$. Hence we have that $x = 0$ and so H is regular. $\square$

Note that Proposition 4.27 along with the definition of elliptic hyperplanes tell us that every regular hyperplane of an odd dimensional quadratic space is either hyperbolic or elliptic. This now allows us to decompose any elliptic hyperplane of an odd dimensional regular quadratic space.

Proposition 4.28. *Let (V, Q) be a regular quadratic space of dimension $2n + 1$. If H is an elliptic hyperplane, then there exists a collection of hyperbolic planes $\{H_i\}_{i=1}^{n-1}$ and an elliptic plane W such that*

$$H = \bigoplus_{i=1}^{n-1} H_i \oplus W.$$

Proof. If H is an elliptic hyperplane, then we know that H is regular. If $n = 1$ then $\dim H = 2$ and so H is an elliptic plane. Now suppose $n > 1$. Then $\dim H = 2n > 3$. By Corollary 4.22, there exists a hyperbolic plane $H_1 \subseteq H$. By Proposition 4.17, we can decompose $H = H_1 \oplus H_1^{\perp}$. By successively applying Proposition 4.17, we have $H = \bigoplus_{i=1}^{n-1} H_i \oplus W$ where W is regular and $\dim W = 2$. If W contains a singular vector, then W would be a hyperbolic plane and hence H would then be a hyperbolic hyperplane. Thus it must be the case that W does not contain any singular vectors. Hence $H = \bigoplus_{i=1}^{n-1} H_i \oplus W$. $\square$

We now want to leverage our decomposition of regular hyperplanes to help us quickly determine whether a hyperplane is elliptic or hyperbolic without having to decompose it. This leads us to the following definitions.

Definition 4.29. Let (V, Q) be a quadratic space and $T \subseteq V$ be a subspace. We say that T is a **totally singular subspace** of V if for all $x \in T$ we have $Q(x) = 0$.

Definition 4.30. Let T be a totally singular subspace of V. We say that T is a **maximal totally singular subspace** if for any totally singular subspace S such that $T \subseteq S$ then $T = S$.

Using these totally singular subspaces will allow us to determine which type of regular hyperplane it is. However, we still must prove that the cardinality of any two maximal totally singular subspaces are the same. The next definition and theorem will allow us to do just that.

Definition 4.31. Suppose that Q_1 and Q_2 are quadratic forms on quadratic spaces V_1 and V_2 respectively. An **isometry** relative to B_1 and B_2 is an $\mathbb{F}$-linear injection $f : V_1 \to V_2$ satisfying $Q_2(f(v)) = Q_1(v)$ for all $v \in V_1$.

Note that in light of the previous definition, we also have that

$$\begin{aligned} B_2(f(x), f(y)) &= Q_2(f(x) + f(y)) + Q_2(f(x)) + Q_2(f(y)) \\ &= Q_1(x + y) + Q_1(x) + Q_1(y) = B_1(x, y). \end{aligned}$$

Theorem 4.32 (Taylor 7.4 Witt's Extension Theorem)**.** *Suppose that U is a subspace of V and that the map $f : U \to V$ is a linear isometry. Then there is a linear isometry $g : V \to V$ such that $g(u) = f(u)$ for all $u \in U$ if and only if $f(U \cap \operatorname{rad} V) = f(U) \cap \operatorname{rad} V$.*

We omit the proof, but it can be found in [16].

Corollary 4.33. *Any two maximal totally singular subspaces of V are isometric and have the same dimension.*

Proof. Let W_1 be a maximal totally singular subspace of V with $\dim W_1 = n$ and let W_2 be a totally singular subspace with $\dim W_2 \leqslant n$. Let $\{u_1, \ldots, u_n\}$ be a basis for W_1 and $\{v_1, \ldots, v_m\}$ be a basis for W_2. Since V is regular, we have that $W_1 \cap \operatorname{rad} V = \{0\}$ and $W_2 \cap \operatorname{rad} V = \{0\}$. Let $f : W_2 \to V$ be the linear map given by $v_i \mapsto u_i$. Note that for each basis element of W_2 we have that $Q(v_i) = Q(f(v_i)) = Q(u_i) = 0$. Hence f is an isometry. Additionally, $f(W_2 \cap \operatorname{rad} V) = f(\{0\}) = \{0\} = f(W_2) \cap \operatorname{rad} V$ since $f(W_2)$ is a totally singular subspace of V. Then there exists a linear isometry $g : V \to V$ such that $f(w) = g(w)$ for all $w \in W_2$. Since g is an isometry, $g^{-1}(W_1)$ is a totally singular subspace and since g is a bijection, $n = \dim W_2 = \dim g(W) = \dim W_1$. □

Definition 4.34. Let (V, Q) be a quadratic space. The dimension of a maximal totally singular subspace is called the **Witt index** and is denoted $m(V)$.

Now that we have proven that the dimension of maximal totally singular subspaces are invariant, we want to catalog the Witt indices of odd dimensional regular quadratic spaces and their regular hyperplanes.

Proposition 4.35. *If* (V, Q) *is regular quadratic space where* $\dim V = n$ *is odd, then* $m(V) = \frac{n-1}{2}$.

Proof. Let (V, Q) be a regular quadratic space with $\dim V = n$ where n is odd. Then by Proposition 4.23 we have $V = \bigoplus_{i=1}^{\frac{n-1}{2}} H_i \oplus \operatorname{rad} V$ where H_i is a hyperbolic plane for each $i \in \{1, \dots, \frac{n-1}{2}\}$. Let (h_{i1}, h_{i2}) be a hyperbolic pair for each H_i. Then the subspace $T = \langle h_{11}, h_{21}, \dots, h_{\frac{n-1}{2}1} \rangle$ is a totally singular subspace of dimension $\frac{n-1}{2}$.

Now by way of contradiction, suppose that T is not maximal. That is, there exists $S \subseteq V$ such that $T \subseteq S$ but $T \neq S$. This implies that $\dim S > \dim T$. Suppose that $\dim S = \dim T + 1$. Let $\langle h_{11}, h_{21}, \dots, h_{\frac{n-1}{2}1}, s \rangle$ be a basis for S. Since $B(s, h_{i1}) = 0$ for all $i \in \{1, \dots, \frac{n-1}{2}\}$, Proposition 4.19 implies that $s \notin H_i$ for every i since each H_i is regular. Therefore, $\langle s \rangle \subseteq \operatorname{rad} V$ which implies that V is not regular. This is a contradiction, and so T must be a maximal totally singular subspace, and $\dim T = \frac{n-1}{2}$. □

Proposition 4.36. *Suppose* (V, Q) *is a regular quadratic space of dimension* n. *The space* V *is hyperbolic if and only if* $m(V) = \frac{n}{2}$.

Proof. Suppose that (V, Q) is a quadratic space of dimension n. If V is hyperbolic, then by definition, we have $n = 2m$ for some $m \in \mathbb{N}$ and $V = \bigoplus_{i=1}^{m} H_i$ where $\{H_i\}_{i=1}^{m}$ is a collection of hyperbolic planes. Note that each H_i contains a hyperbolic pair (h_{i1}, h_{i2}). Then subspace $T = \langle h_{11}, h_{21}, \ldots, h_{m1} \rangle$ is a totally singular subspace and $\dim T = m = \frac{n}{2}$.

Now suppose by way of contradiction, that T is not maximal. Then there exists $S \subseteq V$ such that $T \subseteq S$ but $T \neq S$. Let $\dim S = \dim T + 1$ and let $\{h_{11}, \ldots, h_{m1}, s\}$ be a basis for S. Since $B(s, h_{i1}) = 0$ for all $i \in \{1, \ldots, m\}$, Proposition 4.19 implies that $s \notin H_i$ for every i since each H_i is regular. Therefore, $V = V = \bigoplus_{i=1}^{m} H_i \oplus \langle s \rangle$ and so $\dim V = 2n + 1$ which is a contradiction since V is a hyperbolic space. Therefore, T must be maximal and $\dim T = \frac{n}{2}$.

Now suppose that V contains a maximal totally singular subspace, T, of of dimension $m = \frac{n}{2}$. Note that since V is regular we have $T \cap \operatorname{rad} V = \{0\}$. Now induct on the dimension of T. If $m = 1$ then $T = \langle t_1 \rangle$. By Proposition 4.16, we have that there exists a hyperbolic plane $H = \langle t_1, s_1 \rangle$. Since we know that if $m = 1$, then $\dim V = 2$, we must have $V = H$ and so V is hyperbolic. Now suppose that if $\dim V = 2m - 2$ and V contains an $m - 1$ dimensional maximal totally singular subspace, then V is hyperbolic. Let $\{t_1, \ldots, t_m\}$ be a basis for T. Let $C = \langle t_2, \ldots, t_m \rangle^{\perp}$. Note that $\dim C = \frac{n}{2} + 1$ by Proposition 4.9. Additionally, $T \subseteq C$. Note that since $\dim C > \dim T$, there exists $s \in C$ such that $s \notin T$ but $B(t_1, s) \neq 0$. Without loss of generality, suppose that $B(s, t_1) = 1$. If s is singular, then we have the hyperbolic pair (t_1, s). If s is not singular, then $(t_1, Q(s)t_1 + s)$ is a hyperbolic pair. Let H_1 be the hyperbolic plane that contains t_1 and is orthogonal to C. Then $V = H_1 \oplus H_1^{\perp}$ and $\langle t_2, \ldots, t_m \rangle \subseteq H_1^{\perp}$. Note that $\dim H_1^{\perp} = 2m - 2$ and it contains the maximal totally singular subspace $T' = \langle t_2, \ldots, t_m \rangle$. Since $H_1^{\perp}$ is regular $\dim T' = m - 1$, the induction hypothesis gives us that $H_1^{\perp}$ is a hyperbolic space. Hence there exist hyperbolic planes $\{H_i\}_{i=2}^{m}$ such that $H_1^{\perp} = \bigoplus_{i=2}^{m} H_i$ and so

$V = \bigoplus_{i=1}^{m} H_i$ and so V is a hyperbolic space. $\square$

Note a consequence of Proposition 4.36, we have that if (V, Q) is a regular quadratic space with $\dim V = 2n$, and that V is elliptic, then $m(V) = \frac{n}{2} - 1$.

It is at this point that we focus our attention on quadratic spaces over finite fields. We will utilize the trace function given by finite fields to aid in our understanding of regular hyperplanes.

Definition 4.37. The field $\mathbb{F}_q$ is the field containing q elements.

Assumption. Suppose that $|\mathbb{F}| = (2^h)$ for some $h \in \mathbb{N}$.

Definition 4.38. Let $\mathbb{F}$ be a finite field of characteristic 2 such that $|\mathbb{F}| = 2^h$ for some $h \in \mathbb{N}$. Define $\mathrm{tr} : \mathbb{F} \to \mathbb{F}_2$ given by $\mathrm{tr}(\alpha) = \alpha + \alpha^2 + \cdots + \alpha^{2^{h-1}}$.

Proposition 4.39. *If $\mathbb{F}$ is a finite field of characteristic 2 such that $|\mathbb{F}| = 2^h$ for some $h \in \mathbb{N}$, then $\mathrm{tr}(\alpha + \beta) = \mathrm{tr}(\alpha) + \mathrm{tr}(\beta)$, $\mathrm{tr}(\alpha^2) = \mathrm{tr}(\alpha)$ for all $\alpha, \beta \in \mathbb{F}$.*

Proof. Let $\mathbb{F}$ be a finite field of characteristic 2 such that $|\mathbb{F}| = 2^h$ for some $h \in \mathbb{N}$. First let $\alpha, \beta \in \mathbb{F}$. Then $\mathrm{tr}(\alpha + \beta) = (\alpha + \beta) + (\alpha + \beta)^2 + \cdots + (\alpha + \beta)^{2^{h-1}} =$ $\alpha + \alpha^2 + \cdots + \alpha^{2^{h-1}} + \beta + \beta^2 + \cdots + \beta^{2^{h-1}} = \mathrm{tr}(\alpha) + \mathrm{tr}(\beta)$.

Next, consider that since $\mathbb{F}$ is a finite field, we have that $\alpha \mapsto \alpha^{2^h}$ is the identity map since the nonzero elements of $\mathbb{F}$ form a cyclic group under multiplication. Therefore,

$$\begin{aligned}
\mathrm{tr}(\alpha^2) + \mathrm{tr}(\alpha) &= \alpha^2 + \alpha^4 + \cdots + \alpha^{2^{h-1}} + \alpha^{2^h} + \alpha + \alpha^2 + \cdots + \alpha^{2^{h-1}} \\
&= \alpha^2 + \alpha^4 + \cdots + \alpha^{2^{h-1}} + \alpha + \alpha + \alpha^2 + \cdots + \alpha^{2^{h-1}} = 0.
\end{aligned}$$

Hence, $\mathrm{tr}(\alpha^2) = \mathrm{tr}(\alpha)$ for all $\alpha \in \mathbb{F}$. $\square$

Proposition 4.40. *Let $\mathbb{F}$ be a finite field. The polynomial $x^2 + x + \beta \in \mathbb{F}[x]$ is irreducible if and only if $\mathrm{tr}(\beta) = 1$.*

Proof. Suppose that $x^2 + x + \beta \in \mathbb{F}[x]$ is reducible. Then for some $\gamma \in \mathbb{F}$ we have $\gamma^2 + \gamma + \beta = 0$, or equivalently, $\gamma^2 + \gamma = \beta$. But then $\mathrm{tr}(\beta) = \mathrm{tr}(\gamma^2 + \gamma) = \mathrm{tr}(\gamma^2) + \mathrm{tr}(\gamma) = 2\,\mathrm{tr}(\gamma) = 0$.

Now suppose that $\mathbb{F} = \mathbb{F}_{2^n}$ for some $n \in \mathbb{N}$ and that $0 = \mathrm{tr}(\beta) = \beta + \beta^2 + \cdots + \beta^{2^{n-1}}$. Note that for any $\alpha \in \mathbb{F}\backslash\{0\}$ that we have $\alpha^{2^n-1} = 1$ and so $\alpha^{2^n} = \alpha$. If n is odd, then $n = 2m+1$ for some $m \in \mathbb{Z}$. Let $x_0 = \beta + \beta^4 + \beta^{16} + \cdots + \beta^{2^{2m-2}} + \beta^{2^{2m}}$ and consider

$$\begin{aligned} x_0^2 + x_0 + \beta &= (\beta + \beta^4 + \beta^{16} + \cdots + \beta^{2^{2m-2}} + \beta^{2^{2m}})^2 \\ &\quad + (\beta + \beta^4 + \beta^{16} + \cdots + \beta^{2^{2m-2}} + \beta^{2^{2m}}) + \beta \\ &= \beta^2 + \beta^8 + \beta^{32} + \cdots + \beta^{2^{2m-1}} + \beta^{2^{2m+1}} + \beta + \beta^4 + \cdots + \beta^{2^{2m-2}} \\ &\quad + \beta^{2^{2m}} + \beta \\ &= \beta + \beta + \beta + \beta^2 + \beta^4 + \beta^8 + \cdots + \beta^{2^{2m-1}} + \beta^{2^{2m}} = \mathrm{tr}(\beta) = 0. \end{aligned}$$

Hence, we know that $x^2 + x + \beta$ has a root and so it is reducible.

Now suppose that n is even, and let $\gamma \in \mathbb{F}$ such that $\mathrm{tr}(\gamma) = 1$. Then define $x_0 = \gamma\beta^2 + (\gamma + \gamma^2)\beta^4 + \cdots + (\gamma + \gamma^2 + \cdots + \gamma^{2^{n-2}})\beta^{2^{n-1}}$ and consider

$$\begin{aligned} x_0^2 + x_0 + \beta &= (\gamma\beta^2 + (\gamma + \gamma^2)\beta^4 + \cdots + (\gamma + \gamma^2 + \cdots + \gamma^{2^{n-2}})\beta^{2^{n-1}})^2 \\ &\quad + (\gamma\beta^2 + \cdots + (\gamma + \gamma^2 + \cdots + \gamma^{2^{n-2}})\beta^{2^{n-1}}) + \beta \\ &= \gamma\beta^2 + \gamma\beta^4 + \cdots + \gamma\beta^{2^{n-1}} + (\gamma + 1)\beta^{2^n} + \beta = \gamma\,\mathrm{tr}(\beta) = 0. \end{aligned}$$

Hence we know that $x^2 + x + \beta$ has a root and so it is reducible. □

Proposition 4.41. *Let $\mathbb{F}$ be a finite field and $a, b \in \mathbb{F}\backslash\{0\}$. The polynomial $ax^2 + bx + c \in \mathbb{F}[x]$ has a root if and only if* $\mathrm{tr}(\frac{ac}{b^2}) = 0$.

Proof. Let $\mathbb{F}$ be a finite field, and let $f(x) = ax^2 + bx + c \in \mathbb{F}[x]$. Note that since $\mathbb{F}$ is finite and has characteristic 2, then $\mathbb{F}$ is perfect. Suppose that f is irreducible. Now consider the polynomial $g(x) = f(\frac{b}{a}x) = \frac{b^2}{a^2}ax^2 + \frac{b^2}{a}x + c = \frac{b^2}{a}(x^2 + x + \frac{ac}{b^2})$.

Note that by Proposition 4.40 $\operatorname{tr}(\frac{ac}{b^2}) = 1$ if and only if g is irreducible if and only if f is irreducible. $\square$

The previous propositions provide us another way to test if a plane is hyperbolic or elliptic when using Proposition 4.15.

Proposition 4.42. *If $\mathbb{F}$ is a finite field of characteristic 2 such that $|\mathbb{F}| = 2^h$ for some $h \in \mathbb{N}$, then for each $c \in \mathbb{F}_2$, there are exactly 2^{h-1} elements $\alpha \in \mathbb{F}$ such that* $\operatorname{tr}(\alpha) = c$.

Proof. Let $\mathbb{F}$ be a finite field of characteristic 2 such that $|\mathbb{F}| = 2^h$ for some $h \in \mathbb{N}$. First, suppose that $\operatorname{tr}(\mathbb{F}) = 0$. Then by Proposition 4.41, we know that every monic quadratic polynomial, $x^2 + bx + c \in \mathbb{F}[x]$ is reducible. Therefore, we know that every monic quadratic polynomial is of the form $(x - \beta_1)(x - \beta_2)$ for some $\beta_1, \beta_2 \in \mathbb{F}$. Consider the set of monic, quadratic polynomials over this field, that is $\mathbb{F}[x] = \{x^2 + \alpha_1 x + \alpha_2 : \alpha_1, \alpha_2 \in \mathbb{F}\}$. Let $h : \mathbb{F}^2 \to \mathbb{F}[x]$ be given by $h(\alpha_1, \alpha_2) = x^2 + \alpha_1 x + \alpha_2$. This function is clearly a bijection and therefore, $|\mathbb{F}[x]| = |\mathbb{F}|^2$. However, since each monic polynomial is reducible, we have that $\mathbb{F}[x] = \{(x - \beta_1)(x - \beta_2) : \beta_1, \beta_2 \in \mathbb{F}\}$. Let $g : \mathbb{F}^2 \to \mathbb{F}[x]$ be given by $g(\beta_1, \beta_2) = (x - \beta_1)(x - \beta_2)$. Note that $g(\beta_1, \beta_2) = g(\beta_2, \beta_1)$ and so g is not injective. This implies that $|\{(x - \beta_1)(x - \beta_2) : \beta_1, \beta_2 \in \mathbb{F}\}| < |\mathbb{F}|^2$ which is a contradiction. Hence there exists an element $\beta \in \mathbb{F}$ such that $\operatorname{tr}(\beta) = 1$. Now note that for any $\alpha \in \ker(\operatorname{tr})$, we have that $\operatorname{tr}(\alpha + \beta) = 1$. Now we can build the coset of $\beta + \ker(\operatorname{tr})$ in the additive structure of $\mathbb{F}$, that this is a well-defined coset. Note that since $\ker(\operatorname{tr})$ is a subgroup of $\mathbb{F}$, we know that by Lagrange's Theorem, $[\mathbb{F} : \ker(\operatorname{tr})] = 2$ and so $|\ker(\operatorname{tr})| = |\beta + \ker(\operatorname{tr})| = \frac{|\mathbb{F}|}{2} = 2^{h-1}$. $\square$

The final result of this section is to show that if a quadratic space is written as the orthogonal direct sum of two elliptic planes, then it is actually a hyperbolic space. This result allows us to obtain ideal bases for hyperbolic spaces.

Proposition 4.43. *Let $\mathbb{F}$ be a finite field. If E_1, E_2 are elliptic planes in a quadratic space (V, Q) and $E_1 \subseteq E_2^{\perp}$, then $E_1 \oplus E_2$ is a hyperbolic space.*

Proof. Let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ be bases for E_1 and E_2 respectively. Without loss of generality we can suppose that $Q(u_1) = B(u_1, v_1) = 1$ and $Q(u_2) = B(u_2, v_2) = 1$. Now let $x \in \operatorname{rad}(E_1 \oplus E_2)$. Then $x = au_1 + bv_1 + cu_2 + dv_2$. Note that $B(x, u_1) = B(x, v_1) = B(x, u_2), B(x, v_2) = 0$ implies that $a = b = c = d = 0$ and so $x = 0$. Therefore, $\operatorname{rad}(E_1 \oplus E_2) = \{0\}$ so $E_1 \oplus E_2$ is regular.

We will show that $E_1 \oplus E_2$ contains a two dimensional totally singular subspace and we consider two cases.

Case 1. Suppose that $Q(v_1) = Q(v_2)$. Then the subspace $T = \langle u_1 + u_2, v_1 + v_2 \rangle$ has the properties that

$$Q(u_1 + u_2) = Q(u_1) + Q(u_2) + B(u_1, u_2) = 1 + 1 + 0 = 0,$$

that

$$Q(v_1 + v_2) = Q(v_1) + Q(v_2) + B(v_1, v_2) = 0,$$

and that

$$B(u_1 + u_2, v_1 + v_2) = B(u_1, v_1) + B(u_2, v_2) = 1 + 1 = 0.$$

Hence T is a two dimensional totally singular subspace and by Proposition 4.36, $E_1 \oplus E_2$ is a hyperbolic space.

Case 2. Suppose that $Q(v_1) \neq Q(v_2)$ and define the polynomials $f(x) = Q(xu_1 + v_1) = x^2 + x + Q(v_1)$ and $g(x) = Q(xu_2 + v_2)$. Clearly $f \neq g$. Since E_1, E_2 are elliptic planes, we know that $\operatorname{tr}(Q(v_1)) = \operatorname{tr}(Q(v_2)) = 1$. Therefore, we know that $\operatorname{tr}(Q(v_1) + Q(v_1)) = 0$. Hence the polynomial $h(x) = x^2 + x + (Q(v_1 + Q(v_2)))$ is reducible. Since h is reducible, then there exists $s \in \mathbb{F}$ such that $h(s) = 0$. Consider the basis $\{u_2, su_2 + v_2\}$ for E_2. This implies that

$Q(xu_2 + su_2 + v_2) = x^2 + x + (s^2 + s + Q(v_2)) = x^2 + x + Q(v_1) = f(x)$. Then the subspace $T = \langle u_1 + u_2, v_1 + su_2 + v_2 \rangle$ has the properties that

$$Q(u_1 + u_2) = Q(u_1) + Q(u_2) + B(u_1, u_2) = 1 + 1 + 0 = 0,$$

that

$$Q(v_1 + su_2 + v_2) = Q(v_1) + Q(su_2 + v_2) = Q(v_1) + Q(v_1) = 0,$$

and that

$$B(u_1+u_2, v_1+su_2+v_2) = B(u_1+u_2, v_1)+sB(u_1+u_2, u_2)+B(u_1+u_2, v_2) = 1+0+1 = 0.$$

Therefore, T is a two dimensional totally singular subspace and by Proposition 4.36 $E_1 \oplus E_2$ is a hyperbolic space. $\square$

5 A CORRESPONDENCE BETWEEN POINTS AND HYPERPLANES

The graph $NO^+_{2n+1}(2^h)$ is defined on hyperplanes of an odd dimensional regular quadratic space. Usually, it is useful to consider a set of vectors instead of hyperplanes. Since $\mathbb{F}$ is characteristic 2, the usual convention of identifying hyperplanes with the vector in its orthogonal complement doesn't work here. So instead, we will develop a correspondence between the regular hyperplanes of an odd dimensional regular quadratic space and vectors in a regular quadratic space of one fewer dimension.

Definition 5.1. Two maximal totally singular subspaces are said to be **complementary** if they have trivial intersection.

This definition allows us to develop bases for two complementary totally singular subspaces where the vectors are almost all orthogonal.

Proposition 5.2. *Let (V, Q) be a regular quadratic space with Witt index m. The* *subspaces T and S are complementary if and only if there exist bases $\{t_1, \ldots, t_m\}$ and $\{s_1, \ldots, s_m\}$ for T and S, resepectively, such that $B(t_i, s_j) = \delta_{ij}$.*

Proof. Suppose that there exist bases for $T = \langle t_1, \ldots, t_n \rangle$ and $S = \langle s_1, \ldots, s_n \rangle$ such that $B(t_i, s_j) = \delta_{ij}$. Let $x \in T \cap S$. Then $x = \sum_{i=1}^{n} a_i t_i$ and $x = \sum_{j=1}^{n} b_j s_j$. Note that since $x \in S$, we have $B(x, s_i) = 0$ for all $i \in \{1, \ldots, n\}$. Hence, we have $0 = B(\sum_{i=1}^{n} a_i t_i, s_j) = a_j$ for all $j \in \{1, \ldots, n\}$. Hence we must have $x = 0$ and so the intersection is trivial.

Now suppose that T and S are maximal and that $T \cap S = \{0\}$. We will induct on the Witt Index of V. First suppose that $\dim T = \dim S = m = 1$. Let $T = \langle t_1 \rangle$ and $S = \langle s_1 \rangle$. Since $m = 1$, then $\dim V = 2$ or $\dim V = 3$. Suppose that $\dim V = 2$ and that $B(t_1, s_1) = 0$. Since $\{t_1, s_1\}$ is an independent set, then $\{t_1, s_1\}$ is an orthogonal basis for V. This implies that V is not regular by Proposition 4.19, which is a contradiction. Hence $B(t_1, s_1) \neq 0$ and therefore, there is a basis $\{t_1, s_1'\}$ such that $B(t_1, s_1') = 1$.

Now suppose that $\dim V = 3$ and $B(t_1, s_1) = 0$. Since V is regular, then V contains a nontrivial and nonsingular radical. Let $\operatorname{rad} V = \langle r \rangle$. But then we know that $\{t_1, s_1, r\}$ is an independent set and a basis for V. However, this implies that $t_1 \in \operatorname{rad} V$ which implies that V is not regular, a contradiction. Therefore, $B(t_1, s_1) \neq 0$ and thus there is a basis $\{t_1, s_1'\}$ such that $B(t_1, s_1') = 1$.

Now suppose that when $m = k$ then there exist bases $\{t_1, \ldots, t_k\}$ and $\{s_1, \ldots, s_k\}$ for T and S, respectively, such that $B(t_i, s_j) = \delta_{ij}$. Let $m = k + 1$. This implies that $\dim V = 2k + 2$ or $\dim V = 2k + 3$. Also, $\dim T = \dim S = k + 1$ and that $T \cap S = \{0\}$. Let $t_1 \in T \backslash \{0\}$. If $B(t_1, s) = 0$ for all $s \in S$ then S is not maximal which is a contradiction. Therefore, since S is maximal, then without loss of generality, there exists $s_1 \in S \backslash \{0\}$ such that $B(t_1, s_1) = 1$. This implies that (t_1, s_1) is a hyperbolic pair and that $V = \langle t_1, s_1 \rangle \ominus \langle t_1, s_1 \rangle^{\perp}$ where $\langle t_1, s_1 \rangle^{\perp}$ is a regular space with Witt index $m = k$. Consider that the subspaces $T' = T \cap \langle t_1, s_1 \rangle^{\perp}$ and

$S' = S \cap \langle t_1, s_1 \rangle^{\perp}$ have trivial intersection since $T \cap S = \{0\}$ and that $\dim T' = \dim S' = k$. Therefore by the induction hypothesis, there exist bases $\{t_2, \ldots, t_{k+1}\}$ and $\{s_2, \ldots, s_{k+1}\}$ for T' and S', respectively, such that $B(t_i, s_j) = \delta_{ij}$. Therefore, the bases $\{t_1, \ldots, t_{k+1}\}$ and $\{s_1, \ldots, s_{k+1}\}$ for T and S, respectively, have the property that $B(t_i, s_j) = \delta_{ij}$. □

The previous proposition will allow us to write down a specific basis for any odd dimensional regular quadratic space. This basis lets us give an formula for all of the regular hyperplanes that space.

Proposition 5.3. *If (V, Q) is a regular quadratic space with $\dim V = n$ for some odd $n \in \mathbb{N}$ and if $T, S \subseteq V$ are maximal, complementary, totally singular subspaces, then $V = (T \oplus S) \ominus \operatorname{rad} V$.*

Proof. If n is odd then we have $m(V) = \frac{n-1}{2}$ by Proposition 4.35. Hence, T, S are complementary totally singular subspaces of V implies that $\dim(T \oplus S) = n - 1$ and so $\dim((T \oplus S) \ominus \operatorname{rad} V) = 2n + 1$ and so $V = (T \oplus S) \ominus \operatorname{rad} V$. □

Corollary 5.4. *If $\dim V = 2n + 1$ and (V, Q) is regular, then there is a basis for*

$$V = \langle b_1, b_2, \ldots, b_{2n-1}, b_{2n}, r \rangle$$

where $r \in \operatorname{rad} V \backslash \{0\}$, $Q(r) = 1$, and $B(b_{2i-1}, b_{2j}) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$.

Proposition 5.5. *If (V, Q) is a hyperbolic quadratic space with $\dim V = n$ for some even $n \in \mathbb{N}$ and if $T, S \subseteq V$ are maximal, complementary totally singular subspaces, then $V = T \oplus S$.*

Proof. Since n is even, then we have $m(V) = \frac{n}{2}$ by Proposition 4.36, and so we have $\dim T = \dim S = \frac{n}{2}$ and so $\dim(T \oplus S) = n$ and hence $V = T \oplus S$. □

Corollary 5.6. *If $\dim V = 2n$ and (V, Q) is hyperbolic, then there is a basis for $V = \langle b_1, b_2, \ldots, b_{2n-1}, b_{2n} \rangle$ and $B(b_{2i-1}, b_{2j}) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$.*

Since the direct sum of complementary totally singular subspaces will always be a subspace of their quadratic spaces, we will find it useful to identify vectors with their components from the two complementary totally singular subspaces.

Definition 5.7. Let T, S be complementary totally singular subspaces of a regular quadratic space, (V, Q), and let $x = t + s + r \in V$ for some $t \in T$, $s \in S$, and $r \in \operatorname{rad} V$. Define the $\operatorname{proj}_T(x) = t$ to be the **projection** of x onto T.

The next proposition allows us to find a hyperbolic plane containing any given vector in space. This allows us to determine the intersection of our hyperplanes.

Proposition 5.8. *If (V, Q) be a singular quadratic space with $\dim V = 2n$ for some $n \in \mathbb{N}$, then any vector $x \in V \backslash \{0\}$ is contained in a hyperbolic plane.*

Proof. Since V is regular, if $Q(x) = 0$ then by Proposition 4.16, x is in a hyperbolic plane. Now suppose that $Q(x) \neq 0$. Since $\dim V = 2n$, then we know that $x \notin \operatorname{rad} V$. Therefore, there exists $y \in V$ such that $B(x, y) \neq 0$. Let $f(\gamma) = Q(x + \gamma y) = 1 + \gamma B(x, y) + \gamma^2 Q(y)$. If f is reducible, then there exists $z \in \langle x, y \rangle$ such that $Q(z) = 0$ and $B(x, z) = 1$. Therefore, if $w = Q(x)z + x$ then $Q(w) = Q(x)^2 Q(z) + Q(x) + Q(x)B(x, z) = 0$ and so $\langle z, w \rangle = \langle x, y \rangle$ is a hyperbolic plane that contains x.

If f is irreducible, then $\langle x, y \rangle$ contains no singular vectors and must be an elliptic plane. Since V is singular, then we know that $\langle x, y \rangle^\perp$ is singular and hence there exists a hyperbolic plane, $H \subseteq \langle x, y \rangle^\perp$. Therefore, there exists $w \in H$ such that $Q(w) = Q(y)$. Then $B(w + y, x) = B(x, y) \neq 0$ and $Q(w + y) = Q(w) + Q(y) = 0$. Therefore by a similar argument to above, $\langle x, w + y \rangle$ spans a hyperbolic plane that contains x. ∎

The next proposition begins our work towards developing the correspondence between regular hyperplanes and the points in a vector space that is the same dimension as the hyperplanes. Here we identify a sequence in $\mathbb{F}$ which will become our point that we associate with the hyperplane. It is at this point which we utilize the previously known results to identify the regular hyperplanes of an odd dimensional quadratic space and vectors of a regular quadratic space of one dimension fewer which is a new identification.

Theorem 5.9. *Let (V, Q) be a regular quadratic space with $\dim V = 2n+1$ for some $n \in \mathbb{N}$. If a hyperplane $H \subseteq V$ is regular, then there exists a unique sequence of elements, $\{\beta_i\}_{i=1}^{2n}$, in $\mathbb{F}$, such that*

$H = \bigoplus_{i=1}^{n} \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle$. Conversely, if $\{\beta_i\}_{i=1}^{2n} \in \mathbb{F}^{2n}$ is a sequence, then the hyperplane $H = \bigoplus_{i=1}^{n} \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle$ is a regular hyperplane.

Proof. Let $H \subseteq V$ be a regular hyperplane. And let $\{b_1, \ldots, b_{2n}, r\}$ be a basis for V given by Corollary 5.4. Then we know that $r \notin H$, and so $\dim(H \cap \langle b_{2i-1}, b_{2i}, r \rangle) = 2$ for each $i \in \{1, \ldots, n\}$. Now let $\{x, y\}$ be a basis for $H \cap \langle b_{2i-1}, b_{2i}, r \rangle$. Then $x = ab_{2i-1} + cb_{2i} + dr$ and $y = eb_{2i-1} + fb_{2i} + gr$ for some $a, c, d, e, f, g \in \mathbb{F}$. Since x and y are independent and H is regular, then we know that either $a \neq 0$ or $c \neq 0$ and that $e \neq 0$ or $f \neq 0$. If $a \neq 0$ then let $p \in \mathbb{F}$ such that $y' = y + px = f' b_{2i} + g' r$. And then let $q \in \mathbb{F}$ such that $x' = x + qy' = a' b_{2i-1} + d' r$. A similar construction can be made for x' and y' if $c \neq 0$. In either case, without loss of generality, there exists a basis for $H \cap \langle b_{2i-1}, b_{2i}, r \rangle$ of the form $\{b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r\}$ for some $\beta_{2i-1}, \beta_{2i} \in \mathbb{F}$. Therefore, we know that for each $i \in \{1, \ldots, n\}$ we have $H \cap \langle b_{2i-1}, b_{2i}, r \rangle = \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle$. By construction and the basis for V from Corollary 5.4, the planes are orthogonal for distinct $i, j \in \{1, \ldots, n\}$. Additionally, if $\langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle \cap \langle b_{2j-1} + \beta_{2j-1} r, b_{2j} + \beta_{2j} r \rangle \neq \{0\}$ then $\{b_1, \ldots, b_{2n-1}, r\}$ is not an independent set which is a contradiction. Hence we have

$$H = \bigoplus_{i=1}^{n} \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle.$$

Now consider $H = \bigoplus_{i=1}^{n} \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle$ and

$H = \bigoplus_{i=1}^{n} \langle b_{2i-1} + \gamma_{2i-1} r, b_{2i} + \gamma_{2i} r \rangle$. Since H is written as a direct sum, the intersection of any two planes must be trivial. Hence, for all $i \in \{1, \ldots, n\}$, $H \cap \{b_{2i-1}, b_{2i}, r\} = \langle b_{2i-1} + \beta_{2i} r, b_{2i} + \beta_{2i} r \rangle = \langle b_{2i-1} + \gamma_{2i-1} r, b_{2i} + \gamma_{2i} r \rangle$. Note that if $\beta_{2i-1} \neq \gamma_{2i-1}$ then we have $(\beta_{2i-1} + \gamma_{2i-1}) r \in H \cap \langle b_{2i-1}, b_{2i}, r \rangle$ which is a contradiction since H is a regular hyperplane. Hence $\beta_{2i-1} = \gamma_{2i-1}$. A similar argument shows that $\beta_{2i} = \gamma_{2i}$ and so the sequence of $\{\beta_i\}_{i=1}^{2n}$ is unique.

Now suppose that for some sequence $\{\beta_i\}_{i=1}^{n}$ that the hyperplane $H = \bigoplus_{i=1}^{n} \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle$ but also that H is not regular. Let $x \in \operatorname{rad} H$. Then we know that $x = \sum_{i=1}^{n} (a_{2i-1}(b_{2i-1} + \beta_{2i-1} r) + a_{2i}(b_{2i} + \beta_{2i} r))$ where $a_i \in \mathbb{F}$ for all $i \in \{1, \ldots, 2n\}$. Additionally, we have that

$$B(x, b_{2i-1} + \beta_{2i-1} r) = B(a_{2i}(b_{2i} + \beta_{2i} r), b_{2i-1} + \beta_{2i-1} r) = a_{2i} = 0$$

for all $i \in \{1, \ldots, n\}$. A similar argument shows that $a_{2i-1} = 0$ for all $i \in \{1, \ldots, n\}$. Therefore, $x = 0$ and so $\operatorname{rad} H = \{0\}$. Therefore, H is a regular hyperplane. $\square$

Now that we can identify a given sequence with regular hyperplanes, we now want to determine which hyperplanes are hyperbolic and elliptic given the sequence.

Theorem 5.10. *Let $H \subseteq V$ be a regular hyperplane and let $H = \bigoplus_{i=1}^{n} \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle$. The hyperplane H is hyperbolic if and only if $\operatorname{tr}(\sum_{i=1}^{n} \beta_{2i-1}\beta_{2i}) = 0$.*

Proof. Let $P_i = \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r \rangle$. Since P_i must be regular for all $i \in \{1, \ldots, n\}$ by Proposition 5.9, then by Corollary 4.14, P_i is either a hyperbolic plane or an elliptic plane.

Now consider the polynomial $f(x) = Q(x(b_{2i-1} + \beta_{2i-1} r) + b_{2i} + \beta_{2i} r) = x^2 \beta_{2i-1}^2 + x + \beta_{2i}^2$. By Proposition 4.41, P_i is hyperbolic if and only if $\operatorname{tr}(\beta_{2i-1}^2 \beta_{2i}^2) = \operatorname{tr}((\beta_{2i-1}\beta_{2i})^2) = \operatorname{tr}(\beta_{2i-1}\beta_{2i}) = 0$ and P_i is elliptic if and only if $\operatorname{tr}(\beta_{2i-1}\beta_{2i}) = 1$.

Next, suppose that H is a hyperbolic hyperplane. If P_i is hyperbolic for all i, then we have

$$\sum_{i=1}^{n} \operatorname{tr}(\beta_{2i-1}\beta_{2i}) = \operatorname{tr}(\sum_{i=1}^{n} \beta_{2i-1}\beta_{2i}) = 0.$$

If P_i is not hyperbolic for all i, then consider the set

$$E = \{j : \operatorname{tr}(\beta_{2j-1}\beta_{2j}) = 1\}.$$

By Proposition 4.43, we know that if $j, k \in E$ then

$$\operatorname{tr}(\beta_{2j-1}\beta_{2j}) + \operatorname{tr}(\beta_{2k-1}\beta_{2k}) = 0$$

and therefore $P_j \oplus P_k$ is a hyperbolic space.

Since H is hyperbolic, the Witt index is n. Suppose that $|E| \bmod 2 = 1$. Then there exists an elliptic plane P_i so that $H = \oplus_{i=1}^{n-1} H_i \oplus P_i$ where H_i is a hyperbolic plane for all $i \in \{1, \ldots, n-1\}$. But then since P_i is elliptic, it contains no singular vectors and thus $m(H) = n - 1$ which is a contradiction. Thus we have $|E| \bmod 2 = 0$ and hence

$$\operatorname{tr}\left(\sum_{i=1}^{n} \beta_{2i-1}\beta_{2i}\right) = 0.$$

Now suppose that $0 = \operatorname{tr}(\sum_{i=1}^{n} \beta_{2i-1}\beta_{2i}) = \sum_{i=1}^{n} \operatorname{tr}(\beta_{2i-1}\beta_{2i})$. Now if $\operatorname{tr}(\beta_{2i-1}\beta_{2i}) = 0$ for all i, then each P_i is a hyperbolic plane and so H is a hyperbolic hyperplane. If $\operatorname{tr}(\beta_{2i-1}\beta_{2i}) \neq 0$ for all i then let $E = \{j : \operatorname{tr}(\beta_{2j-1}\beta_{2j}) = 1\}$. Clearly we must have $|E| \mod 2 = 0$ and so by Proposition 4.43, there exist hyperbolic planes $\{H_i\}_{i=1}^{n}$ such that $H = \bigoplus_{i=1}^{n} H_i$ and so H is a hyperbolic hyperplane. □

Since the construction of $NO_{2n+1}^{+}(2^h)$ involves computing the intersection of hyperbolic hyperplanes, we now compute these intersections in general.

Proposition 5.11. *Let H, K be regular hyperplanes of V. If*

$H = \bigoplus_{i=1}^{n}\langle b_{2i-1} + \beta_{2i-1}r, b_{2i} + \beta_{2i}r\rangle$ *and* $K = \bigoplus_{i=1}^{n}\langle b_{2i-1} + \gamma_{2i-1}r, b_{2i} + \gamma_{2i}r\rangle$, *then*

$$H \cap K = \left\{ \sum_{i=1}^{n} a_i(b_i + \beta_i r) : \sum_{i=1}^{n} a_i(\beta_i + \gamma_i) = 0 \right\}.$$

Proof. Let $x \in H \cap K$. Then for some scalars $a_i, c_i \in \mathbb{F}$ for $i \in \{1, \dots, 2n\}$ we have

$$x = \sum_{i=1}^{2n} a_i(b_i + \beta_i r) = \sum_{i=1}^{2n} c_i(b_i + \gamma_i r).$$

Thus,

$$0 = \sum_{i=1}^{2n} ((a_i + c_i)b_i + (a_i\beta_i + c_i\gamma_i)r) = \sum_{i=1}^{2n} ((a_i + c_i)b_i) + \left(\sum_{i=1}^{2n}(a_i\beta_i + c_i\gamma_i)\right) r.$$

Since $\{b_1, \dots, b_{2n}, r\}$ is a basis for V, then $a_i = c_i$ for all $i \in \{1, \dots, 2n\}$ and $\sum_{i=1}^{2n} a_i(\beta_i + \gamma_i) = 0$. □

In our final result, we establish a bijection between the hyperbolic hyperplanes of an odd dimensional quadratic space and the points of a quadratic space of one dimension fewer. We will use this bijection to prove a graph isomorphism in Section 5.

Proposition 5.12. *Let* (V_1, Q_1) *be a regular quadratic space of dimension* $2n + 1$ *for some* $n \in \mathbb{N}$ *and* (V_2, Q_2) *be a hyperbolic quadratic spaces of dimension* $2n$. *If* $\mathcal{H} = \{H \subseteq V_1 : H$ *is a hyperbolic hyperplane*$\}$ *and* $\mathcal{P}_H = \{v \in V_2 : \mathrm{tr}(Q(v)) = 0\}$, *then* $|\mathcal{H}| = |\mathcal{P}_H|$.

Proof. Let $n \in \mathbb{N}$ be given, let (V_1, Q_1) be a regular quadratic space with $\dim V_1 = 2n + 1$ and let $\{b_1, \dots, b_{2n}, r\}$ be the basis for V_1 given by Corollary 5.4. Let (V_2, Q_2) be a hyperbolic quadratic space with $\dim V_2 = 2n$, and let $\{e_1, \dots, e_{2n}\}$ be the hyperbolic basis for V_2 given by Corollary 5.6. This implies that if $v \in \mathcal{P}_H$, then

$$v = \sum_{i=1}^{2n} \beta_i e_i$$

such that

$$\operatorname{tr}\left(\sum_{i=1}^{n} \beta_{2i-1}\beta_{2i}\right) = 0.$$

Define the function $f : \mathcal{P}_H \to \mathcal{H}$ given by $f(v) = \bigoplus_{i=1}^{n}\langle b_{2i-1} + \beta_{2i-1}r, b_{2i} + \beta_{2i}r\rangle$. Note that since $\{e_i\}_{i=1}^{2n}$ is a basis for V_2 that f is well-defined. Also, by Theorem 5.10 we have that

$$f(v) = \bigoplus_{i=1}^{n}\langle b_{2i-1} + \beta_{2i-1}r, b_{2i} + \beta_{2i}r\rangle$$

is a hyperbolic hyperplane of V_1. We claim that f is a graph isomorphism.

Note that if $H \in \mathcal{H}$ then by Theorem 5.9 there is a sequence $\{\beta_i\}_{i=1}^{2n}$ such that

$$H = \bigoplus_{i=1}^{n}\langle b_{2i-1} + \beta_{2i-1}r, b_{2i} + \beta_{2i}r\rangle.$$

Clearly, if we let $v = \sum_{i=1}^{2n} \beta_i e_i$ then $f(v) = H$, and $v \in \mathcal{P}_H$ by Theorem 5.10 so f is surjective. Let $u, v \in \mathcal{P}_H$ such that

$$u = \sum_{i=1}^{2n} \gamma_i e_i.$$

Now suppose that $f(v) = f(u)$. Then

$$\bigoplus_{i=1}^{n}\langle b_{2i-1} + \beta_{2i-1}r, b_{2i} + \beta_{2i}r\rangle = \bigoplus\langle b_{2i-1} + \gamma_{2i-1}r, b_{2i} + \gamma_{2i}r\rangle$$

and by Theorem 5.9, $u = v$ and so f is injective. Hence f is a bijection and $|\mathcal{H}| = |\mathcal{P}_H|$. □

6 ISOMORPHIC STRONGLY REGULAR GRAPHS

In this section, we formally define the graph $NO^{+}_{2n+1}(2^h)$. In addition, we define two new graphs on the points of an even quadratic space. We will then show that one of our new graphs is isomorphic to $NO^+_{2n+1}(2^h)$ and prove that both new graphs are strongly regular.

Definition 6.1. Let (V,Q) be a regular quadratic space with $\dim V = 2n+1$ for some $n \in \mathbb{N}$, and let $\mathcal{H}$ be the set of hyperbolic hyperplanes of V and let $\mathcal{E}$ be the set of elliptic hyperplanes of V. Define the graph on $\mathcal{H}$ or $\mathcal{E}$ with $H \sim K$ if and only if $H \cap K$ is not regular. The graph on $\mathcal{H}$ is denoted by $\mathrm{NO}^+_{2n+1}(2^h)$ and the graph on $\mathcal{E}$ is denoted by $\mathrm{NO}^-_{2n+1}(2^h)$.

Definition 6.2. Let (V,Q) be a hyperbolic quadratic space with $\dim V = 2n$ for some $n \in \mathbb{N}$, and where $|\mathbb{F}| = 2^h$ for some $h \geqslant 2$. Let $\mathcal{P}_H$ be the vectors in V such that $\mathrm{tr}(Q(v)) = 0$ and let $\mathcal{P}_E$ be the vectors in V such that $\mathrm{tr}(Q(v)) = 1$. We will call the set $\mathcal{P}_H$ the **hyperbolic points**. Define the graphs on $\mathcal{P}_H$ and $\mathcal{P}_E$ with $v \sim u$ if and only if $v \neq u$ and $Q(u+v) + B(v,u)^2 = 0$. We denote this graph on $\mathcal{P}_H$ by $\mathrm{NO}^+_{2n}(2^h)$ and we denote the graph on $\mathcal{P}_E$ by $\mathrm{NO}^-_{2n}(2^h)$. We will call $NO^+_{2n}(2^h)$ the **hyperbolic points graph**.

In order to prove that the hyperbolic points graph is isomorphic to $NO^+_{2n+1}(2^h)$, we will need to show that the number of vectors with trace equal to zero is the same number as hyperbolic hyperplanes used in $NO^+_{2n+1}(2^h)$. Since every singular vector has trace equal to zero, we begin by counting the number of singular vectors in an even dimensional quadratic space.

Lemma 6.3. *Let (V,Q) be a regular quadratic space of dimension $2n$ where $n \in \mathbb{N}$ and assume $|\mathbb{F}| = 2^h$ for some $h \in \mathbb{N}$. If (V,Q) is a hyperbolic quadratic space, then $|\ker Q| = (2^h - 1)(2^h)^{n-1} + (2^h)^{2n-1}$.*

Proof. We first note that since (V,Q) is a hyperbolic quadratic space, then there exists a basis $\{b_i\}_{i=1}^{2n}$ such that $V = \bigoplus_{i=1}^{n} \langle b_{2i-1}, b_{2i} \rangle$ where (b_{2i-1}, b_{2i}) is a hyperbolic pair. Hence for any $x \in V$ we have

$$Q(x) = Q\left(\sum_{i=1}^{2n} a_i b_i\right) = \sum_{i=1}^{n} a_{2i-1} a_{2i}.$$

Suppose $n = 1$. Then $\dim V = 2$ and so $Q(a_1b_1 + a_2b_2) = a_1a_2 = 0$ if and only if $a_1 = 0$ or $a_2 = 0$. Thus
$|\ker Q| = 2^{h+1} - 1 = (2^h - 1)(1) + 2^h = (2^h - 1)(2^h)^0 + (2^h)^1$.

Now for ease of notation, let $q = 2^h$. Also suppose that for $n = m$ for some $m \geqslant 1$ we have

$$|\ker Q| = (q-1)q^{m-1} + q^{2m-1}.$$

Now let $n = m + 1$. Then we have $\dim V = 2m + 2$ and

$$Q(x) = Q\left(\sum_{i=1}^{2m+2} a_ib_i\right) = \sum_{i=1}^{m+1} a_{2i-1}a_{2i} = \sum_{i=1}^{m} a_{2i-1}a_{2i} + a_{2m+1}a_{2m+2}.$$

If $\sum_{i=1}^{m} a_{2i-1}a_{2i} = 0$ then $Q(x) = 0$ if and only if $a_{2m+1} = 0$ or $a_{2m+2} = 0$. There are $2q - 1$ choices for such coefficents. If $\sum_{i=1}^{m} a_{2i-1}a_{2i} \neq 0$ then $a_{2m+1} \neq 0$ and $a_{2m+2} \neq 0$. Additionally, we know that without loss of generality, we can choose a_{2m+1} to be any nonzero element, and then a_{2m+2} is determined. Hence

$$\begin{aligned}|\ker Q| &= (2q-1)[(q-1)q^{m-1} + q^{2m-1}] + (q-1)[q^{2m} - [(q-1)q^{m-1} + q^{2m-1}]] \\ &= q^{m+1} - q^m + q^{2m+1} \\ &= (q-1)q^m + q^{2m+1}\end{aligned}$$

Hence by induction, we have $|\ker Q| = (2^h - 1)(2^h)^{n-1} + (2^h)^{2n-1}$. $\square$

Using Lemma 6.3, we now will be able to find the cardinality of the set of hyperbolic points and set of elliptic points.

Theorem 6.4. *Let (V, Q) be a hyperbolic quadratic space with $\dim V = 2n$ for some $n \in \mathbb{N}$. Also let $|\mathbb{F}| = q$. If $\mathcal{P}_H = \{v \in V : \operatorname{tr}(Q(v)) = 0\}$, then $|\mathcal{P}_H| = \frac{1}{2}(q^{2n} + q^n)$.*

Proof. First we note that $\operatorname{tr}(Q(v)) = 0$ if and only if $Q(v) = 0$ or if $Q(v) = \alpha$ where $\operatorname{tr}(\alpha) = 0$. Note that by Proposition 4.42, we know that there are $q^{n-1} - 1 = \frac{q}{2} - 1$ nonzero elements with trace equal to 0. Next we note that by Lemma 6.3, we have

$|\ker Q| = (q-1)q^{n-1} + q^{2n-1}$. This implies that there are $\frac{|\ker Q|-1}{q-1}$ singular lines in V and there are $\frac{q^{2n}-1}{q-1} - \frac{|\ker Q|-1}{q-1} = q^{2n-1} - q^{n-1}$ nonsingular lines in V. Note that since $\mathbb{F}$ is perfect, for each element $\alpha \in \mathbb{F}$ there exists a unique element, x, of a nonsingular line such that $Q(x) = \alpha$. Therefore

$|\mathcal{P}_H| = (q-1)q^{n-1} + q^{2n-1} + \left(\frac{q}{2} - 1\right)\left(q^{2n-1} - q^{n-1}\right) = \frac{1}{2}(q^{2n} + q^n)$. □

Corollary 6.5. *Let* (V, Q) *be a hyperbolic quadratic space with* $\dim V = 2n$ *for some* $n \in \mathbb{N}$. *Also let* $|\mathbb{F}| = q$. *If* $\mathcal{P}_E = \{v \in V : \operatorname{tr}(Q(v)) = 1\}$, *then* $|\mathcal{P}_E| = \frac{1}{2}(q^{2n} - q^n)$.

Proof. Since $\operatorname{tr} : \mathbb{F} \to \mathbb{F}_2$, we know that any vector in $\mathbb{F}$ that does not map to zero, must map to 1. Hence $|\mathcal{P}_E| = |V| - |\mathcal{P}_H| = q^{2n} - \frac{1}{2}(q^{2n} - q^n) = \frac{1}{2}(q^{2n} - q^n)$. □

With the cardinality of the hyperbolic points set determined, we can now use our bijection from Proposition 5.12, to show that the hyperbolic points graph and $NO^{+}_{2n+1}(2^h)$ are isomorphic.

Theorem 6.6. $NO^{+}_{2n+1}(2^h)$ *is isomorphic to* $NO^{+}_{2n}(2^h)$.

Proof. Let $n \in \mathbb{N}$ be given, let (V_1, Q_1) be a regular quadratic space with $\dim V_1 = 2n + 1$ and let $\{b_1, \ldots, b_{2n}, r\}$ be the basis for V_1 given by Corollary 5.4. Let (V_2, Q_2) be a hyperbolic quadratic space with $\dim V_2 = 2n$. Let $\{e_1, \ldots, e_{2n}\}$ be the hyperbolic basis for V_2 given by Corollary 5.6. Recall that

$\mathcal{P}_H = \{v \in V_2 : \operatorname{tr}(Q(v)) = 0\}$ and $\mathcal{H} = \{H \subseteq V_1 :\ H$ is a hyperbolic hyperplane$\}$.

Let $v \in \mathcal{P}_H$. That is

$$v = \sum_{i=1}^{2n} \beta_i e_i$$

such that

$$\operatorname{tr}\left(\sum_{i=1}^{n} \beta_{2i-1}\beta_{2i}\right) = 0.$$

Let $f : \mathcal{P}_H \to \mathcal{H}$ given by $f(v) = \bigoplus_{i=1}^{n} \langle b_{2i-1} + \beta_{2i-1} r, b_{2i} + \beta_{2i} r\rangle$ be the bijection from Proposition 5.12. Additionally, this gives us that $|\mathcal{H}| = |\mathcal{P}_H|$. Now let

$v, u \in \mathcal{P}_H$. By Proposition 5.11

$$f(v) \cap f(u) = \left\{ \sum_{i=1}^{n} a_i(b_i + \beta_i r) : \sum_{i=1}^{n} a_i(\beta_i + \gamma_i) = 0 \right\}.$$

Let $\{c_i\}_{i=1}^{2n}$ be a sequence in $\mathbb{F}$ such that

$$c_i = \begin{cases} \beta_{i-1} + \gamma_{i-1} & \text{if } i = 0 \bmod 2 \\ \beta_{i+1} + \gamma_{i+1} & \text{if } i = 1 \bmod 2. \end{cases}$$

We claim that $x = \sum_{i=1}^{2n} c_i(b_i + \beta_i r) \in \operatorname{rad}(f(v) \cap f(u)) \setminus \{0\}$. Clearly, $x \neq 0$, and we have that

$$\sum_{i=1}^{2n} c_i(\beta_i + \gamma_i) = (\beta_2 + \gamma_2)(\beta_1 + \gamma_1)$$
$$+ (\beta_1 + \gamma_1)(\beta_2 + \gamma_2) + \cdots + (\beta_{2n-1} + \gamma_{2n-1})(\beta_{2n} + \gamma_{2n}) = 0.$$

Hence $x \in f(v) \cap f(u)$. Note also that

$$\begin{aligned} B\left(x, \sum_{i=1}^{2n} a_i(b_i + \beta_i r)\right) &= \sum_{i=1}^{2n} \sum_{j=1}^{2n} a_i c_j B(b_j, b_i) \\ &= a_1c_2 + a_2c_1 + a_3c_4 + a_4c_3 + \cdots + a_{2n-1}c_{2n} + a_{2n}c_{2n-1} \\ &= \sum_{i=1}^{2n} a_i(\beta_i + \gamma_i) = 0. \end{aligned}$$

Hence $x \in \operatorname{rad}(f(v) \cap f(u)) \setminus \{0\}$.

Now let $y \in \operatorname{rad}(f(v) \cap f(u))$. Then

$$y = \sum_{i=1}^{2n} d_i(b_i + \beta_i r)$$

where $d_i \in \mathbb{F}$ for all $i \in \{1, \ldots, n\}$ and

$$\sum_{i=1}^{2n} d_i(\beta_i + \gamma_i) = 0.$$

Now let $z \in f(v) \cap f(u)$. Then

$$z = \sum_{i=1}^{2n} f_i(b_i + \beta_i r)$$

where $f_i \in \mathbb{F}$ for all $i \in \{1, \ldots, n\}$, and

$$\sum_{i=1}^{2n} f_i(\beta_i + \gamma_i) = 0.$$

Then we have that

$$0 = B(y, z) = \sum_{i=1}^{2n} \sum_{j=1}^{2n} d_i f_j B(b_i, b_j) = d_1 f_2 + d_2 f_1 + \cdots + d_{2n-1} f_{2n} + d_{2n} f_{2n-1}$$

This implies that that

$$d_i = \begin{cases} c(\beta_{i-1} + \gamma_{i-1}) & \text{if } i = 0 \bmod 2 \\ c(\beta_{i+1} + \gamma_{i+1})) & \text{if } i = 1 \bmod 2. \end{cases}$$

Hence, $y \in \langle x \rangle$ and so $\mathrm{rad}(f(v) \cap f(u)) = \langle x \rangle$.

Additionally, we have that

$$\begin{aligned}
Q(x) &= Q\left(\sum_{i=1}^{2n} c_i(b_i + \beta_i r)\right) = Q\left(\sum_{i=1}^{2n} c_i b_i + \left(\sum_{i=1}^{2n} c_i\beta_i\right) r\right) \\
&= Q\Bigg(\Big(\sum_{i=1}^{2n} c_i b_i\Big)\Bigg) + \left(\sum_{i=1}^{2n} c_i\beta_i\right)^2 \\
&= (\beta_1+\gamma_1)(\beta_2+\gamma_2) + \cdots + (\beta_{2n-1}+\gamma_{2n-1})(\beta_{2n}+\gamma_{2n}) + \left(\sum_{i=1}^{2n} c_i\beta_i\right)^2 \\
&= Q(\beta+\gamma) \\
&\quad + ((\beta_2+\gamma_2)\beta_1 + (\beta_1+\gamma_1)\beta_2) + \cdots + (\beta_{2n}+\gamma_{2n})\beta_{2n-1} + (\beta_{2n-1}+\gamma_{2n-1})\beta_{2n})^2 \\
&= Q(\beta+\gamma) + B(\beta,\gamma)^2.
\end{aligned}$$

Then we have $\beta \sim \gamma$ if and only if $Q(\beta+\gamma) + B(\beta,\gamma)^2 = 0$ if and only if $Q(x) = 0$ if and only if $f(\beta) \cap f(\gamma)$ is not regular.

Therefore, f is a graph isomorphism. Hence $\mathrm{NO}^+_{2n+1}(2^h)$ is isomorphic to $\mathrm{NO}^+_{2n}(2^h)$. □

Theorem 6.7. *$NO^-_{2n+1}(2^h)$ is isomorphic to $NO^-_{2n}(2^h)$.*

Proof. The function $f : \mathcal{P}_E \to \mathcal{E}$ given by $f(v) = \bigoplus_{i=1}^{n}\langle b_{2i-1} + \beta_{2i-1}r, b_{2i} + \beta_{2i}r\rangle$ is clearly a graph isomorphism and the proof is identical to that of Theorem 6.6. □

We note that it is known [4] that $\mathrm{NO}^+_{2n+1}(2^h)$ and that $\mathrm{NO}^-_{2n+1}(2^h)$ are strongly regular, but we prove that $\mathrm{NO}^+_{2n}(2^h)$ is strongly regular for completeness. But first, we need to catalog some properties of a useful function to prove that the hyperbolic points graph is strongly regular.

Definition 6.8. A graph is called a **strongly regular graph** if each vertex has the same number of neighbors, k, that adjacent vertices have the same number of common neighbors, λ, and that non-adjacent vertices have the same number of common neighbors, μ. We will often say a graph is an $\mathrm{SRG}(v, k, \lambda, \mu)$.

Lemma 6.9. *For all $x \in V$ there exists a linear bijection $\tau_x : V \to V$ with the following properties:*

1. *For $y \in V$,*
$$\tau_x(y) = y. \tag{6.1}$$
if and only if $B(x, y) = 0$. In particular, $\tau_x(x) = x$.

2. $$\tau_x(\tau_x(y)) = y \tag{6.2}$$
for all $y \in V$.

3. *For all $y, z \in V$*
$$B(\tau_x(y), z) = B(y, \tau_x(z)). \tag{6.3}$$

4. *If $x \in \mathcal{P}_H$, then*
$$Q(\tau_x(y)) = Q(y) + B(x, y)^2. \tag{6.4}$$

Proof. Let $x \in V$. Define the following function $\tau_x : V \to V$ given by $\tau_x(y) = y + \gamma B(x, y)x$ where $\gamma \in \mathbb{F} \backslash \{0\}$. First, let $a \in \mathbb{F}$ and $y, z \in V$. Note that

$$\begin{aligned} \tau_x(ay + z) &= ay + z + \gamma B(ay + z, x)x = a(y + \gamma B(y, x)x) + (z + \gamma B(z, y)x) \\ &= a\tau_x(y) + \tau_x(z). \end{aligned}$$

Therefore, τ_x is a linear function.

Now, suppose $\tau_x(y) = \tau_x(z)$. Then $y + \gamma B(y, x)x = z + \gamma B(z, x)x$. Hence we have that $y + z = \gamma B(y + z, x)x$. Hence $y + z \in \langle x \rangle$. This implies that $B(y + z, x) = 0$ and so $y = z$. Hence τ_x is injective.

Next, we know that $B(x, y) = 0$ if and only if $\tau_x(y) = y + \gamma B(x, y)x = y$.

Additionally, we know that

$$\tau_x(\tau_x(y)) = \tau_x(y + \gamma B(y,x)x) = y + \gamma B(y,x)x + \gamma B(y + \gamma B(y,x)x, x)x = y.$$

Next, note that

$$\begin{aligned} B(\tau_x(y), z) &= B(y + \gamma B(x,y)x, z) = B(y,z) + \gamma B(x,y)B(x,z) \\ &= B(y, z + \gamma B(x,z)x) = B(y, \tau_x(z)). \end{aligned}$$

Finally, suppose that $x \in \mathcal{P}_H$. Then,

$$\begin{aligned} Q(\tau_x(y)) &= Q(y + \gamma B(x,y)x) \\ &= Q(y) + Q(\gamma B(y,x)x) + B(y, \gamma B(y,x)x) = Q(y) + B(x,y)^2(\gamma^2 Q(x) + \gamma) \end{aligned}$$

Note that $\gamma^2 Q(x) + \gamma + 1 = 0$ has a solution if and only if $\operatorname{tr}(Q(x)) = 0$. Since $x \in \mathcal{P}_H$, then there exists $\gamma_0 \in \mathbb{F}$ such that $\gamma_0^2 Q(x) + \gamma_0 + 1 = 0$ and hence τ_x given by $\tau_x(y) = y + \gamma_0 B(y,x)x$ has the property that $Q(\tau_x(y)) = Q(y) + B(x,y)^2$. $\square$

The next Lemma will help us determine the set of hyperbolic points connected to a given point in the hyperbolic points graph, called neighbors. This characterization of this set will help us compute the intersection between two sets of neighbors. In other words, we will be finding the set of common neighbors to two hyperbolic points.

Lemma 6.10. *If $x \in \mathcal{P}_H$, and $V_x = \{z \in \mathcal{P}_H : z \sim x\}$, then*

$$V_x = x + \{\tau_x(y) : y \in \ker Q \backslash \{0\}\}.$$

Proof. Let $x \in \mathcal{P}_H$ be given let $\tau_x : V \to V$ be the map from Lemma 6.9. Note that for all $z \in V$, $z = x + \tau_x(x + \tau_x(z))$ and that if $z = x + \tau_x(y)$ for some $y \in V$ then $y = x + \tau_x(z)$ by the injectivity of τ_x. Therefore, $z \in \{y : Q(x + \tau_x(y)) = 0\}$ if and only if $x + \tau_x(z) \in \ker Q$ if and only if

$$x + \tau_x(x + \tau_x(z)) = z \in x + \{\tau_x(y) : y \in \ker Q\}.$$

Hence, $\{y : Q(x + \tau_x(y)) = 0\} = x + \{\tau_x(y) : y \in \ker Q\}$. Since $y \in V_x$ if and only if $Q(x + y) + B(x, y)^2 = 0$ and $x \neq y$ if and only if $Q(\tau_x(x + y)) = 0$. Therefore, $V_x = x + \{\tau_x(y) : y \in \ker Q \backslash \{0\}\}$. $\square$

Lemma 6.11. *If $x, y \in \mathcal{P}_H$, $V_x = \{z \in \mathcal{P}_H : z \sim x\}$, $V_y = \{p \in \mathcal{P}_H : p \sim y\}$, and τ_x be the linear map from Lemma 6.9, then*

$$V_x \cap V_y = \{x + \tau_x(z) : z \in \ker Q \backslash \{0\},\ Q(x + \tau_x(y)) + B(x + \tau_x(y), z) + B(x + \tau_x(y), z)^2 = 0\}.$$

Proof. Let $w \in V_x$. Then by Lemma 6.10, we know that $w = x + \tau_x(z)$ for some singular z. Therefore using properties of τ_x from Lemma 6.9, $w \in V_y$ if and only if

$$\begin{aligned} 0 = Q(y + w) + B(y, w)^2 &= Q(y + x + \tau_x(z)) + B(y, x + \tau_x(z)) \\ &= Q(x + y) + Q(\tau_x(z)) + B(x + y, \tau_x(z)) + B(x, y)^2 + B(y, \tau_x(z))^2 \\ &= Q(x + y) + Q(z) + B(x, z)^2 + B(\tau_x(x + y), z) + B(x, x + y)^2 + B(\tau_x(y), z)^2 \\ &= Q(x + \tau_x(y)) + B(x + \tau_x(y), z) + B(x + \tau_x(y), z)^2. \end{aligned}$$

Thus,

$$V_x \cap V_y = \{x + \tau_x(z) : z \in \ker Q \backslash \{0\},\ Q(x + \tau_x(y)) + B(x + \tau_x(y), z) + B(x + \tau_x(y), z)^2 = 0\}.$$

$\square$

This next lemma will give us the cardinality of the set of singular vectors that are orthogonal to a given vector. We will use this to help determine the number of common neighbors between two hyperbolic points.

Lemma 6.12. *Let (V,Q) be a regular quadratic space of dimension $2n$ for some $n \in \mathbb{N}$. If $v \in \mathcal{P}_H$, then*

$$|\ker Q \cap \langle v\rangle^{\perp}| = \begin{cases} q^{2n-2} + q^n - q^{n-1} & \text{if } v \in \ker Q \\ q^{2n-2} & \text{if } v \notin \ker Q. \end{cases}$$

Proof. Suppose $v \in \mathcal{P}_H$. First, suppose $v \in \ker Q$. Then there exists $u \in V$ such that $\langle u, v\rangle$ is a hyperbolic plane. Therefore, $V = \langle u, v\rangle \oplus \langle u, v\rangle^{\perp}$. We will show that $\ker Q \cap \langle v\rangle^{\perp} = \{bv + p : b \in \mathbb{F},\ p \in \langle u, v\rangle^{\perp},\ Q(p) = 0\}$. Note that since $\langle u, v\rangle^{\perp}$ is regular, then by Lemma 6.3,

$$|\ker Q|_{\langle u,v\rangle^{\perp}} = q^{2n-3} + q^{n-1} - q^{n-2}.$$

Let $z \in \ker Q \cap \langle v\rangle^{\perp}$.

Then $z = au + bv + p$ for some $a, b \in \mathbb{F}$ and $p \in \langle u, v\rangle^{\perp}$. Since $z \in \langle v\rangle^{\perp}$,

$$B(z, v) = B(au + bv + p, v) = aB(u, v) = 0$$

implies that $a = 0$. Hence $z = bv + p$. Additionally, $Q(z) = 0$ if and only if $Q(p) = 0$. Thus

$$z \in \{bv + p : b \in \mathbb{F},\ p \in \langle u, v\rangle^{\perp},\ Q(p) = 0\}.$$

Now suppose that $z \in \{bv + p : b \in \mathbb{F},\ p \in \langle u, v\rangle^{\perp},\ Q(p) = 0\}$. Then

$$B(z, v) = B(bv + p, v) = bB(v, v) + B(p, v) = 0$$

and so $z \in \langle v\rangle^{\perp}$. Also, $Q(z) = Q(bv + p) = b^2 Q(v) + Q(p) + B(bv, p) = 0$. Hence $z \in \ker Q$. Therefore, $\ker Q \cap \langle v\rangle^{\perp} = \{bv + p : b \in \mathbb{F},\ p \in \langle u, v\rangle^{\perp},\ Q(p) = 0\}$.

Hence if $v \in \ker Q$, then

$$|\ker Q \cap \langle v\rangle^{\perp}| = |\{bv + p : b \in \mathbb{F},\ p \in \langle u, v\rangle^{\perp},\ Q(p) = 0\}| = |\mathbb{F} \times \ker Q|_{\langle u,v\rangle^{\perp}}|$$
$$= q(q^{2n-3} + q^{n-1} - q^{n-2}) = q^{2n-2} + q^{n} - q^{n-1}.$$

Now suppose $v \notin \ker Q$. Then by Proposition 5.8 there exists $H \subseteq V$ such that H is a hyperbolic plane, and $v \in H$. Therefore, $V = H \oplus H^{\perp}$. We will show $\ker Q \cap \langle v\rangle^{\perp} = \{av + p : a \in \mathbb{F},\ p \in H^{\perp}, Q(p) = Q(av)\}$. Suppose $z \in \ker Q \cap \langle v\rangle^{\perp}$. Then $z = h + p$ for some $h \in H$ and $p \in H^{\perp}$. Since $B(z, v) = 0$, we have that $B(h, v) = 0$ which implies that $h = av$ for some $a \in \mathbb{F}$. Additionally, $Q(z) = 0$ if and only if

$$Q(z) = Q(av + p) = Q(av) + Q(p) + B(av, p) = Q(av) + Q(p) = 0.$$

Hence $z \in \{av + p : a \in \mathbb{F},\ p \in H^{\perp},\ Q(p) = Q(av)\}$.

Now let $z \in \{av + p : a \in \mathbb{F},\ p \in H^{\perp},\ Q(p) = Q(av)\}$. Then $B(z, v) = B(av + p, v) = 0$ and so $z \in \langle v\rangle^{\perp}$. Additionally, $Q(z) = Q(av) + Q(p) = 0$ and so $z \in \ker Q$. Hence $z \in \ker Q \cap \langle v\rangle^{\perp}$. Therefore, $\ker Q \cap \langle v\rangle^{\perp} = \{av + p : a \in \mathbb{F},\ p \in H^{\perp},\ Q(p) = Q(av)\}$.

Now we wish to compute $|\{av + p : a \in \mathbb{F},\ p \in H^{\perp},\ Q(p) = Q(av)\}|$. First, for $a \in \mathbb{F}$, set

$$c_a = |\{p \in H^{\perp} : Q(p) = Q(av)\}|.$$

We claim that

$$c_a = \begin{cases} q^{2n-3} + q^{n-1} - q^{n-2} & \text{if } a = 0 \\ q^{2n-2} - q^{n-2} & \text{if } a \neq 0. \end{cases}$$

Suppose that $a = 0$. Then $av + p = p \in \ker Q|_{H^{\perp}}$ and so $c_0 = |\ker Q|_{H^{\perp}}| = q^{2n-3} + q^{n-1} - q^{n-2}$.

Now suppose $a \neq 0$. Then for each nonsingular line of $H^{\perp}$ there exists exactly one vector such that $Q(av) = Q(p)$. Therefore,

$$c_a = \frac{q^{2n-2} - q^{2n-3} - q^{n-1} + q^{n-2}}{q-1} = q^{2n-3} - q^{n-2}.$$

Hence if $v \notin \ker Q$, we have that

$$\begin{aligned} |\ker Q \cap \langle v \rangle^{\perp}| &= |\{av + p : a \in \mathbb{F},\ p \in H^{\perp},\ Q(p) = Q(av)\}| = \sum_{a \in \mathbb{F}} c_a \\ &= q^{2n-3} + q^{n-1} - q^{n-2} + (q-1)(q^{2n-3} - q^{n-2}) = q^{2n-2}. \quad \square \end{aligned}$$

Theorem 6.13. *$NO^{+}_{2n}(2^h)$ is an*

$$SRG\left(\frac{q^{2n}+q^n}{2},\ q^{2n-1} + q^n - q^{n-1} - 1,\ 2(q^{2n-2} - 1)q^{n-1}(q-1),\ 2(q^{2n-2} + q^{n-1})\right).$$

Proof. Note that if $q = 2^h$ then by Theorem 6.4 we have the number of vertices of $\mathrm{NO}^{+}_{2n}(2^h)$ is

$|\mathcal{P}_H| = \frac{1}{2}(q^{2n} + q^n)$.

Let $\sim$ denote adjacency the graph and let $V_x = \{y : y \sim x\}$. By Lemma 6.10,

$$V_x = x + \{\tau_x(y) : y \in \ker Q \backslash \{0\}\}.$$

Hence by Lemma 6.3, the degree of $\mathrm{NO}^{+}_{2n}(2^h)$ is

$$k = |V_x| = |\ker Q| - 1 = (q-1)(q)^{n-1} + (q)^{2n-1} - 1.$$

Now suppose that $x \sim y$. This implies that $Q(x + \tau_x(y)) = 0$. Hence if we let

$v = x + \tau_x(y)$ then by Lemma 6.11,

$$\begin{aligned}
V_x \cap V_y &= \{x + \tau_x(z) : z \in \ker Q \backslash \{0, v\}, B(v,z)^2 + B(v,z) = 0\} \\
&= \{x + \tau_x(z) : z \in \ker Q \backslash \{0, v\},\ B(v,z) \in \{0,1\}\} \\
&= \{x + \tau_x(z) : z \in \ker Q \backslash \{0, v\},\ B(v,z) = 0\} \\
&\quad \cup \{x + \tau_x(z) : z \in \ker Q \backslash \{0, v\},\ B(v,z) = 1\}.
\end{aligned}$$

By Lemma 6.9, the map $z \mapsto x + \tau_x(z)$ is a bijection, and so

$$\begin{aligned}
|\{x + \tau_x(z) : z \in \ker Q \backslash \{0, v\},\ B(v,z) = 0\}| &= |\{z \in \ker Q \backslash \{0, v\},\ B(v,z) = 0\}| \\
&= |(\ker Q \cap \langle v \rangle^\perp) \backslash \{0, v\}| \\
&= q^{2n-2} + q^n - q^{n-1} - 2.
\end{aligned}$$

Now to compute $|\{x + \tau_x(z) : z \in \ker Q \backslash \{0, v\},\ B(v,z) = 1\}|$ we first note that $|\ker Q \backslash \langle v \rangle^\perp| = |\ker Q| - |\ker Q \cap \langle v \rangle^\perp| = q^{2n-1} - q^{2n-2}$. This implies there are $\frac{q^{2n-1} - q^{2n-2}}{q-1} = q^{2n-2}$ lines in $\ker Q$ that are not orthogonal to v. From each of these lines, there is exactly one vector, z_l, such that $B(v, z_l) = 1$. Therefore,

$$\lambda = |V_x \cap V_y| = q^{2n-2} + q^n - q^{n-1} - 2 + q^{2n-2} = 2(q^{2n-2} - 1) + q^{n-1}(q-1).$$

Now suppose that $x \nsim y$. Note that since $x, y \in \mathcal{P}_H$ we have $\operatorname{tr}(Q(x)) = \operatorname{tr}(Q(y)) = 0$, and

$$\begin{aligned}
\operatorname{tr}(Q(x + \tau_x(y))) &= \operatorname{tr}(Q(x) + Q(y) + B(x,y) + B(x,y)^2) \\
&= \operatorname{tr}(Q(x)) + \operatorname{tr}(Q(y)) + \operatorname{tr}(B(x,y)) + \operatorname{tr}(B(x,y)^2) \\
&= 0.
\end{aligned}$$

Also,

$$V_x \cap V_y = \{x + \tau_x(z) : z \in \ker Q \backslash \{0\},\ Q(x + \tau_x(y)) + B(x + \tau_x(y), z) + B(x + \tau_x(y), z)^2 = 0\}.$$

Additionally, if $v = x + \tau_x(y)$ then by Lemma 6.9,
$|V_x \cap V_y| = |\{z \in \ker Q : Q(v) + B(v, z) + B(v, z)^2 = 0\}|$. Note that by Lemma 6.9 6.4, we have that

$$Q(v) = Q(\tau_x(x+y)) = Q(x+y) + B(x, x+y)^2 = Q(x) + Q(y) + B(x, y) + B(x, y)^2 \neq 0.$$

We wish to find $|\{z \in \ker Q : B(v, z) = 1\}|$. By Lemma 6.12

$$|\ker Q \backslash \langle v\rangle^{\perp}| = |\ker Q| - |\ker Q \cap \langle v\rangle^{\perp}| = q^{2n-1} + q^n - q^{n-1} - q^{2n-2}.$$

This implies there are $\frac{q^{2n-1} - q^{2n-2} + q^n - q^{n-1}}{q-1} = q^{2n-2} + q^{n-1}$ lines in $\ker Q \backslash \langle v\rangle^{\perp}$. Therefore, for each line in $\ker Q \backslash \langle v\rangle^{\perp}$, there exists z_l such that $B(v, z_l) = 1$ and so $|\{z \in \ker Q : B(v, z) = 1\}| = q^{2n-2} + q^{n-1}$. Now let $z' \in \{z \in \ker Q : B(v, z) = 1\}$ and let $\beta \in \mathbb{F}$. Then we have that $\beta z' \in \{z \in \ker Q : Q(v) + B(v, z) + B(v, z)^2 = 0\}$ if and only if

$$0 = Q(v) + B(v, \beta z') + B(v, \beta z')^2 = Q(v) + \beta + \beta^2.$$

Since $\mathrm{tr}(Q(v)) = 0$, then there exist $\beta_1, \beta_2 \in \mathbb{F}$ for each $z' \in \{z \in \ker Q : B(v, z) = 1\}$ such that $\beta_1 z', \beta_2 z' \in \{z \in \ker Q : Q(v) + B(v, z) + B(v, z)^2 = 0\}$. Hence,

$$\mu = |V_x \cap V_y| = |\{z \in \ker Q : Q(v) + B(v, z) + B(v, z)^2 = 0\}| = 2(q^{2n-2} + q^{n-1}).$$

Therefore $\mathrm{NO}^+_{2n}(2^h)$ is a
$\mathrm{SRG}(\frac{1}{2}(q^{2n} + q^n),\ q^{2n-1} + q^n - q^{n-1} - 1,\ 2(q^{2n-2} - 1) + q^{n-1}(q-1),\ 2(q^{2n-2} + q^{n-1}))$. □

Now that we have proven that the hyperbolic points graph is strongly regular, we now want to define a graph that is related to the hyperbolic points graph. We will define this graph on a subset of the hyperbolic points, namely all of the points which do not lie in a given totally singular subspace. We define adjacency similarly to that of the hyperbolic points graph, but alter adjacency in a key way. Additionally, we need to note that we will also restrict ourselves to building this graph only if $\mathbb{F} = \mathbb{F}_4$. This is because in the proof that it is strongly regular, we require that the trace function have the desired properties.

Definition 6.14. Let $\mathbb{F} = \mathbb{F}_4$ and (V, Q) be a hyperbolic quadratic space of dimension $2n$. Also let T, S be complementary, maximal totally singular subspaces of V. Define the graph on $\mathcal{P}_H \backslash S$ in the following way: The vertices $x \sim_s y$ if either

1. $\text{proj}_T(x) = \text{proj}_T(y)$ and $Q(x+y) + B(x,y)^2 = 1$, or
2. $\text{proj}_T(x) \neq \text{proj}_T(y)$ and $Q(x+y) + B(x,y)^2 = 0$.

We call this graph the **hyperbolic subpoints graph**.

Note that because of the clear relationship between $NO^+_{2n}(4)$ and the hyperbolic subpoints graph, we distinguish between adjacency in $NO^+_{2n}(4)$ as $\sim$ and adjacency in the subpoints graph as $\sim_s$. We now present a handful of lemmas that determine the number of common neighbors that are in certain sets.

Lemma 6.15. *Let (V, Q) be a regular quadratic space of dimension $2n$ for some $n \in \mathbb{N}$ over $\mathbb{F}_4$ and let T, S be maximal, complementary, totally singular subspaces of V. If $x, y \in \mathcal{P}_H \backslash S$ are distinct,*

$$A_x := \{z \in \mathcal{P}_H \backslash S : \text{proj}_T(x) = \text{proj}_T(z),\ Q(x+z) + B(x,z)^2 = 0\},$$

$$B_x := \{z \in \mathcal{P}_H \backslash S : \text{proj}_T(x) = \text{proj}_T(z),\ Q(x+z) + B(x,z)^2 = 1\},$$

and

$$C_x := \{z \in \mathcal{P}_H \backslash S : \text{proj}_T(x) \neq \text{proj}_T(z),\ Q(x+z) + B(x,z)^2 = 0\},$$

then

1.

$$|A_x| = |B_x|, \tag{6.5}$$

2.

$$|A_x \cap A_y| = |B_x \cap B_y|, \tag{6.6}$$

and

3.

$$|A_x \cap C_y| = |B_x \cap C_y|. \tag{6.7}$$

Proof. Suppose that $x, y, z \in \mathcal{P}_H \backslash S = U$ such that $\mathrm{proj}_T(x) = \mathrm{proj}_T(y) = \mathrm{proj}_T(z) = t$ for some $t \in T$. Then for some $s_1, s_2, s_3 \in S$ we have $x = t + s_1$, $y = t + s_2$, and $z = t + s_3$. Therefore

$$Q(x+z) + B(x,z)^2 = Q(s_1 + s_2) + (B(t, s_1) + B(t, s_2))^2 = (Q(x) + Q(z))^2 .$$

Hence, $Q(x+z) + B(x,z)^2 = 1$ if and only if $Q(x) = Q(z) + 1$. By Proposition 5.2 there exists $s \in S$ such that $B(t,s) = 1$. Define the map χ_1 from B_x given by $z \mapsto z + s$. Then if $z \in B_x$ then $\mathrm{proj}_T(z+s) = t$ and $Q(z+s) = Q(z) + Q(s) + B(z,s) = Q(z) + 1$. This implies that $Q(x+z+s) + B(x, z+s)^2 = 0$ and so $z + s \in A_x$. Additionally, if $z \in A_x \cap A_y \subseteq A_x$ then $\chi_1(z) \in B_x \cap B_y \subseteq B_x$. Note that $\chi_1(\chi_1(z)) = z$ and so χ_1 is its own inverse and therefore a bijection. Hence $|A_x| = |B_x|$ and $|A_x \cap A_y| = |B_x \cap B_y|$.

Next we will show that $|A_x \cap C_y| = |B_x \cap C_y|$. Let $x, y, z \in U$ such that $\mathrm{proj}_T(x) = \mathrm{proj}_T(z) = t_1$ and that $\mathrm{proj}_T(y) = t_2 \neq t_1$. Suppose $z \in A_x \cap C_y$. We will now proceed in two cases.

Case 1. Suppose that $t_2 = \beta t_1$ for some $\beta \in \mathbb{F}_4 \backslash \{0, 1\}$. Then by Proposition 4.42,

$\text{tr}(\beta) = 1$, and by Proposition 5.2, there exists $s \in S$ such that $B(t_1, s) = 1$. This implies that $B(t_2, s) = \beta$. Now set $z' = z + s$. Note that
$Q(z') = Q(z) + B(z, s) = Q(z) + 1$ and so $z' \in U$. Also,
$Q(x + z') + B(x, z') = B(x, z + s) = 1$, and

$$\begin{aligned} Q(y + z') + B(y, z')^2 &= Q(y + z) + Q(s) + B(y + z, s) + B(y, z + s)^2 \\ &= B(s, z) + B(s, y) + B(s, y)^2 = 1 + \beta + \beta^2 = 1 + \text{tr}(\beta) = 0. \end{aligned}$$

Therefore, $z' \in B_x \cap C_y$ and so the map χ_2 given by $z \mapsto z + s$ is an invertible map which shows that $|A_x \cap C_y| = |B_x \cap C_y|$.

Case 2. Now suppose that $\{t_1, t_2\}$ is an independent set. Then by Proposition 5.2 there exists $s_1, s_2 \in S$ such that $B(t_i, s_j) = \delta_{ij}$ for $i, j \in \{1, 2\}$. Set $s = s_1 + \beta s_2$ such that $\beta \in \mathbb{F}_4 \backslash \{0, 1\}$. By Proposition 4.42, $\text{tr}(\beta) = 1$. Set $z' = z + s$. Then we have that $Q(z') = Q(z) + B(z, s) = Q(z) + 1$ and so $z' \in U$. Also,
$Q(x + z') + B(x, z') = B(x, z + s) = 1$, and

$$\begin{aligned} Q(y + z') + B(y, z')^2 &= Q(y + z) + Q(s) + B(y + z, s) + B(y, z + s)^2 \\ &= B(s, z) + B(s, y) + B(s, y)^2 = 1 + \beta + \beta^2 = 1 + \text{tr}(\beta) = 0. \end{aligned}$$

Therefore, $z' \in B_x \cap C_y$ and so the map χ_3 given by $z \mapsto z + s$ is an invertible map which shows that $|A_x \cap C_y| = |B_x \cap C_y|$.

Therefore, $|A_x \cap C_y| = |B_x \cap C_y|$. □

The next lemma will count the number of common neighbors inside of a given totally singular subspace.

Lemma 6.16. *Let (V, Q) be a regular quadratic space of dimension $2n$ for some $n \in \mathbb{N}$ over $\mathbb{F}_4$ and let T, S be maximal, complementary, totally singular subspaces of V. If $x \in \mathcal{P}_H \backslash S$ and $S_x = \{s \in S : s \sim x\}$, then $|S_x| = \frac{|S|}{2}$. Moreover, if $x, y \in \mathcal{P}_H \backslash S$,* *then*

$$|S_x \cap S_y| = \begin{cases} 0 & \text{if } \operatorname{proj}_T(x) = \operatorname{proj}_T(y) \text{ and } x \sim_s y \\ \frac{|S|}{2} & \text{if } \operatorname{proj}_T(x) = \operatorname{proj}_T(y) \text{ and } x \nsim_s y \\ \frac{|S|}{4} & \text{if } \operatorname{proj}_T(x) \neq \operatorname{proj}_T(y). \end{cases}$$

Proof. Let $U = \mathcal{P}_H \backslash S$, $x \in U$, and let $y \in S$. Then $\operatorname{proj}_T(x) = t \neq 0$ and $\operatorname{proj}_T(y) = 0$ then $x = t + s_1$ and $y = s_2$ for some $s_1, s_2 \in S$. Then
$0 = Q(x+y) + B(x,y)^2 = Q(x) + B(t, s_2) + B(t, s_2)^2 = Q(x) + \operatorname{tr}(B(t, s_2))$.
Therefore, note the set

$$\{s \in S : x \sim s\} = \{s \in S : \operatorname{tr}(B(t,s)) = 0\} = \{s \in S : B(t,s) \in \{0,1\}\}.$$

Note that by Proposition 5.2, there is a basis for S so that $S = \langle w_1, \ldots, w_n \rangle$ where $B(t, w_i) = \delta_{1i}$. Then we know that $B(t,s) = 0$ if and only if $s \in \langle w_2, \ldots, w_n \rangle$. Additionally, we know that $B(t,s) = 1$ if and only if $s \in w + \langle w_2, \ldots, w_n \rangle$. Therefore,

$$\begin{aligned} |\{y \in S : Q(x+y) + B(x,y)^2 = 0\}| &= |\langle w_2, \ldots, w_n \rangle| + |w_1 + \langle w_2, \ldots, w_n \rangle| \\ &= \frac{|S|}{4} + \frac{|S|}{4} = \frac{|S|}{2}. \end{aligned}$$

Now let $x, y \in U$. Suppose that $\operatorname{proj}_T(x) = \operatorname{proj}_T(y) = t$, $x = t + s_1$, $y = t + s_2$, and $x \sim_s y$. This implies that $1 = Q(x+y) + B(x,y)^2 = B(x,y)^2 = B(x,y)$ and $Q(x) = Q(y) + 1$. Now let $s \in S$ and $x \sim s$. Then

$$\begin{aligned} Q(x) &= B(t + s_1, s) + B(t + s_1, s)^2 \\ Q(y) + 1 &= B(t,s) + B(t,s)^2 \\ 1 &= Q(y) + B(t + s_2, s) + B(t + s_2, s)^2 = Q(y+s) + B(y,s)^2. \end{aligned}$$

Therefore, $s \nsim y$ and so $|S_x \cap S_y| = |\varnothing| = 0$.

Now suppose that $x \nsim_s y$. This implies that $B(x,y) = 0$ and that

$Q(x) = Q(y)$. If $s \in S_x$, then

$$\begin{aligned} 0 &= Q(x) + Q(s) + B(t + s_1, s) + B(t + s_1, s)^2 \\ &= Q(y) + Q(s) + B(t + s_2, s) + B(t + s_2, s)^2 = Q(y + s) + B(y, s)^2 \end{aligned}$$

Hence, we have $s \in S_y$ and so $|S_x \cap S_y| = \frac{|S|}{2}$.

Finally, suppose that $\text{proj}_T(x) = t_1$ and $\text{proj}_T(y) = t_2 \neq t_1$. We will proceed in two cases.

Case 1. Suppose $t_2 = \beta t_1$ for some $\beta \in \mathbb{F}_4 \backslash \{0, 1\}$. Let $S = \langle w_1, \ldots, w_n \rangle$ such that $B(t, w_j) = \delta_{1j}$. Also let $w \in S$ and suppose that $w \in S_x \cap S_y$. Then $Q(x) + B(x, w) + B(x, w)^2 = 0$ and $Q(y) + B(y, w) + B(y, w)^2 = 0$. Since $B(x, w) = B(t, w)$ and $B(y, w) = B(\beta t, w)$ then $B(t, w), B(\beta t, w) \in \{0, 1\}$ since $x, y \in U$. Suppose by way of contradiction that $w = w_1 + s$ for some $s \in \langle w_2, \ldots, w_n \rangle$. Then $B(t, w) = 1$ and $B(\beta t, w) = \beta \notin \{0, 1\}$ and so $w \notin S_y$, a contradiction. Hence, it must be the case that $w \in \langle w_2, \ldots, w_n \rangle$ and therefore $|S_x \cap S_y| = |\langle w_2, \ldots, w_n \rangle| = \frac{|S|}{4}$.

Case 2. Suppose that $\{t_1, t_2\}$ is an independent set and $x = t_1 + s_1$ and $y = t_2 + s_2$. Also let $S = \langle w_1, \ldots, w_n \rangle$ such that $B(t_1, w_j) = \delta_{1j}$ and $B(t_2, w_j) = \delta_{2j}$. Then we have $w \in S_x \cap S_y$ if and only if

$$Q(x) + B(t_1, s) + B(t_1, s)^2 = 0$$

and

$$Q(y) + B(t_2, s) + B(t_2, s) = 0$$

if and only if

$$Q(x) + Q(y) = B(t_1 + t_2, s) + B(t_1 + t_2, s)^2.$$

Note that since $x, y \in U$ then we know $Q(x) + Q(y) \in \{0, 1\}$ and so

If $Q(x) + Q(y) = 1$ then $\mathrm{tr}(B(t_1 + t_2, s)) = 1$. This implies that $B(t_1, s) + B(t_2, s) \in \{\beta_1, \beta_1 + 1\}$ where $\beta_1 \in \mathbb{F}_4 \backslash \{0, 1\}$. If $B(t_1, s) = \beta_1$ and $B(t_2, s) = 0$ then $s \in \beta_1 w_1 + \langle w_3, \ldots, w_n \rangle$. If $B(t_1, s) = \beta_1$ and $B(t_2, s) = 1$ then $s \in (\beta_1 w_1 + w_2) + \langle w_3, \ldots, w_n \rangle$. If $B(t_1, s) = 0$ and $B(t_2, s) = \beta_1$ then $s \in \beta_1 w_2 + \langle w_3, \ldots, w_n \rangle$. If $B(t_1, s) = 1$ and $B(t_2, s) = \beta_1$ then $s \in (w_1 + \beta_1 w_2) + \langle w_3, \ldots, w_n \rangle$. Therefore, $|S_x \cap S_y| = 4(4^{n-2}) = 4^{n-1} = \frac{|S|}{4}$.

If $Q(x) + Q(y) = 0$ then $\mathrm{tr}(B(t_1 + t_2, s)) = 0$. This implies that $B(t_1, s) + B(t_2, s) \in \{0, 1\}$. If $B(t_1, s) = 1$ and $B(t_2, s) = 0$ then $s \in w_1 + \langle w_3, \ldots, w_n \rangle$. If $B(t_1, s) = 1$ and $B(t_2, s) = 1$ then $s \in (w_1 + w_2) + \langle w_3, \ldots, w_n \rangle$. If $B(t_1, s) = 0$ and $B(t_2, s) = 1$ then $s \in w_2 + \langle w_3, \ldots, w_n \rangle$. If $B(t_1, s) = B(t_2, s) = 0$ then $s \in \langle w_3, \ldots, w_n \rangle$. And so, $|S_x \cap S_y| = 4(4^{n-2}) = 4^{n-1} = \frac{|S|}{4}$. Therefore, if $\mathrm{proj}_T(x) \neq \mathrm{proj}_T(y)$, then $|S_x \cap S_y| = \frac{|S|}{4}$. $\square$

We now want to prove that our new hyperbolic subpoints graph is strongly regular.

Proposition 6.17. *The hyperbolic subpoints graph is a*

$$SRG(2^{4n-1} - 2^{2n-1}, 4^{2n-1} + 4^{n-1}, 2(4^{2n-2} + 4^{n-1}), 2(4)^{2n-2} + 4^{n-1}).$$

Proof. Let $U = \mathcal{P}_H \backslash S$. First we know that from Theorem 6.4 that the size of the hyperbolic subpoints graph is

$$|U| = |\mathcal{P}_H| - 4^n = \frac{1}{2}(4^{2n} - 4^n) = 2^{4n-1} - 2^{2n-1}.$$

Now to show that this graph is regular, we first note that the degree of the hyperbolic points graph and the hyperbolic subpoints graph are related. Consider the disjoint sets

$$\begin{aligned}
A_x &= \{y \in U : \mathrm{proj}_T(x) = \mathrm{proj}_T(y),\ Q(x+y) + B(x,y)^2 = 0\},\\
B_x &= \{y \in U : \mathrm{proj}_T(x) = \mathrm{proj}_T(y),\ Q(x+y) + B(x,y)^2 = 1\},\\
C_x &= \{y \in U : \mathrm{proj}_T(x) \neq \mathrm{proj}_T(y), Q(x+y) + B(x,y)^2 = 0\},\ \text{and}\\
S_x &= \{z \in S : Q(x+z) + B(x,z)^2 = 0\}.
\end{aligned}$$

From Theorem 6.13 that for any $x \in \mathcal{P}_H$, set of adjacent points to x in $NO^+_{2n}(4)$ is

$$V_x = (A_x \backslash \{x\}) \cup C_x \cup S_x$$

and has degree $3(4)^{n-1} + 4^{2n-1} - 1$. Note that in the subpoints graph for any $x \in U$, we have

$$U_x = \{y \in U : y \sim_s x\} = B_x \cup C_x.$$

We can now compute the degree of the hyperbolic subpoints graph since we know that $x \not\sim_s x$ for all $x \in \mathcal{P}_H$, Lemmas 6.15 and 6.16 give that

$$|U_x| = |V_x| - |S_x| + 1 = 3(4)^{n-1} + 4^{2n-1} - 1 - 2(4)^{n-1} + 1. = 4^{n-1} + 4^{2n-1}.$$

Next, we want to determine $|U_x \cap U_y|$ for distinct $x, y \in U$. We begin by noting that

$$U_x \cap U_y = (B_x \cup C_x) \cap (B_y \cup C_y) = (B_x \cap B_y) \cup (B_x \cap C_y) \cup (B_y \cap C_x) \cup (C_x \cap C_y).$$

Additionally, we have

$$\begin{aligned}
V_x \cap V_y &= ((A_x \cup C_x \cup S_x)\backslash\{x\}) \cap ((A_y \cup C_y \cup S_y)\backslash\{y\})\\
&= ((A_x \cap A_y) \cup (A_x \cap C_y) \cup (A_y \cap C_x) \cup (S_x \cap S_y))\backslash\{x, y\}.
\end{aligned}$$

Suppose $x \sim_s y$. If $\mathrm{proj}_T(x) = \mathrm{proj}_T(y)$ then $x \not\sim y$ in $NO^+_{2n}(4)$ and by Lemmas 6.15 and 6.16, we have $|U_x \cap U_y| = |V_x \cap V_y| = 2(4^{2n-2} + 4^{n-1})$. If $\mathrm{proj}_T(x) \neq \mathrm{proj}_T(y)$ then $x \sim y$ in $NO^+_{2n}(4)$ and so Lemmas 6.15 and 6.16 give us that

$$|U_x \cap U_y| = |V_x \cap V_y| - |S_x \cap S_y| + 2 = 2(4)^{2n-2} - 2 + 3(4)^{n-1} - (4)^{n-1} + 2 = 2(4^{2n-2} + 4^{n-1}).$$

Now suppose that $x \not\sim_s y$. If $\mathrm{proj}_T(x) = \mathrm{proj}_T(y)$ then $x \sim y$ in $NO^+_{2n}(4)$ and by Lemmas 6.15 and 6.16, we have

$$|U_x \cap U_y| = |V_x \cap V_y| - |S_x \cap S_y| + 2 = 2(4)^{2n-2} - 2 + 3(4)^{n-1} - (4)^{n-1} + 2 = 2(4^{2n-2} + 4^{n-1}).$$

If $\mathrm{proj}_T(x) \neq \mathrm{proj}_T(y)$, then $x \not\sim y$ in $NO^+_{2n}(4)$ and so Lemmas 6.15 and 6.16, we have that

$$|U_x \cap U_y| = |V_x \cap V_y| - |S_x \cap S_y| = 2(4^{2n-2} + 4^{n-1}) - 4^{n-1} = 2(4)^{2n-2} + 4^{n-1}.$$

Therefore, the hyperbolic subpoints graph is a SRG$(\frac{4^{2n}-4^n}{2}, 4^{2n-1} + 4^{n-1}, 2(4^{2n-2} + 4^{n-1}), 2(4)^{2n-2} + 4^{n-1})$. $\square$

It is key that this graph and $NO^+_{2n}(2^h)$ are strongly regular. The identification of the points of the subpoints and hyperbolic points graphs will be key in building a Steiner ETF for the subpoints graph, and a Tremain ETF for the hyperbolic points graph.

7 CORRESPONDING ETFS FROM $NO^+_{2N}(4)$ AND THE HYPERBOLIC SUBPOINTS GRAPH

In our final section, we will establish the connection between $NO^+_{2n}(2^h)$ and the hyperbolic subpoints graph with a Tremain ETF and a Steiner ETF respectively. To accomplish this we first establish that the Gram matrix of an ETF with the properties our

Steiner and Tremain ETFs have give rise to an adjacency matrix for a strongly regular graph. Then we will show that if we have the adjacency matrix for a strongly regular graph, then it satisfies a quadratic relation.

Proposition 7.1. *If A is the adjacency matrix for a graph, G, which is a $SRG(n,k,\lambda,\mu)$, then*

$$A^2 = \lambda A + kI_n + \mu(J_n - I_n - A)$$

where I_n is an $n \times n$ identity matrix and J_n is an $n \times n$ all ones matrix.

Proof. We begin by noting that the A^2_{xy} entry contains the number of vertices adjacent to both x and y. If $x = y$ then this would be a diagonal entry and since G is regular, then $A^2_{xx} = k$ for all x. Note that if $x \sim y$ in G then A^2_{xy} corresponds to the number of vertices adjacent to two connected vertices. Since G is a strongly regular graph, then this number is λ for every nonzero entry of A. Lastly if $x \neq y$ and $x \nsim y$ then A^2_{xy} corresponds to the number of vertices adjacent to two disconnected vertices. This number is μ and is in every nonzero, non-diagonal entry of A. These cases are exhaustive, and so $A^2 = \lambda A + KI_n + \mu(J_n - A - I_n)$. $\square$

This lemma allows us to build an object which will act as our balanced incomplete block design for our Steiner ETF.

Lemma 7.2. *Let T, S be a maximal totally singular subspace of a regular quadratic space (V,Q) of dimension $2n$ so that $V = T \oplus S$ and let $\mathbb{F} = \mathbb{F}_4$. If $b = \{t_1, t_2, t_3\} \subseteq T\backslash\{0\}$ such that $t_1 + t_2 + t_3 = 0$ and $x = t_1 + s \in \mathcal{P}_H\backslash S$ for some $s \in S$, then the quantity $Q(x) + B(x,t) + B(x,t)^2$ is constant for all $t \in b\backslash\{\mathrm{proj}_T(x)\}$.*

Proof. Define the function $g : b \to \mathbb{F}$ given by $g(t) = Q(x) + B(x,t) + B(x,t)^2$. Note that $g(t_1) = Q(x) + B(t_1+s, t_1) + B(t_1+s, t_1)^2 = Q(x) + Q(x) + Q(x)^2 = Q(x)$. We now proceed by cases. Suppose that $Q(x) = 0$ and $g(t_2) = 0$. Then $g(t_3) = g(t_1+t_2) = Q(x) + B(x, t_1+t_2) + B(x, t_1+t_2)^2 =$

$Q(x) + B(x, t_2) + B(x, t_2) = g(t_2) = 0$. If $g(t_2) = 1$ then a similar calculation yields $g(t_3) = 1$.

Now if $Q(x) = 1$ and $g(t_2) = 0$ then a similar calculation yields $g(t_3) = 0$. And simliarly if $g(t_2) = 1$, then $g(t_3) = 1$. Hence g is constant for all $t \in B \backslash \{\text{proj}_T(x)\}$. $\square$

Lemma 7.3. *Let T, S be maximal, complementary totally singular subspaces of a regular quadratic space (V, Q) of dimension $2n$ and let $\mathbb{F} = \mathbb{F}_4$. If $s \in S \backslash \{0\}$, then*

$$|\{t \in T : \text{tr}(B(t, s)) = 0\}| = |\{t \in T : \text{tr}(B(t, s)) = 1\}|.$$

Proof. Let $s \in S \backslash \{0\}$. By Proposition 5.2, there exists $t' \in T$ such that $B(t', s) = 1$. Also, by Proposition 4.42, there exists $\beta \in \mathbb{F}_4$ such that $\text{tr}(\beta) = 1$. Let γ be a map from $\{t \in T : \text{tr}(B(t, s)) = 0\}$ to $\{t \in T : \text{tr}(B(t, s)) = 1\}$ given by $\gamma(t) = t + \beta t'$. Then we see that

$$\text{tr}(B(t + \beta t', s)) = \text{tr}(B(t, s)) + \text{tr}(B(\beta t', s)) = 0 + \text{tr}(\beta) = 1.$$

Therefore, $\gamma(t) \in \{t \in T : \text{tr}(B(t, s)) = 1\}$. Additionally, $\gamma(\gamma(t)) = t$ and so this is an invertible map and is therefore a bijection. Hence, it must be the case that

$$|\{t \in T : \text{tr}(B(t, s)) = 0\}| = |\{t \in T : \text{tr}(B(t, s)) = 1\}|. \qquad \square$$

Now we will build our Steiner ETF from the hyperbolic subpoints graph.

Definition 7.4. Let T, S be maximal complementary, totally singular subspaces of a regular quadratic space of dimension $2n$ so that $V = T \oplus S$ and let $\mathbb{F} = \mathbb{F}_4$. Also let $\mathcal{B} = \{\{t_1, t_2, t_3\} \subseteq T \backslash \{0\} : t_1 + t_2 + t_3 = 0\}$. We define the matrix Φ, with rows indexed by elements of $\mathcal{B}$ and columns indexed by elements of $\mathcal{P}_H \backslash S$ given by

$$\Phi(B,x) = \begin{cases} -1 & \text{if } \operatorname{proj}_T(x) \in B \text{ and } Q(x) + B(x,t_i) + B(x,t_i)^2 = 0 \text{ for any } t_i \in B \backslash \{\operatorname{proj}_T(x)\} \\ 1 & \text{if } \operatorname{proj}_T(x) \in B \text{ and } Q(x) + B(x,t_i) + B(x,t_i)^2 = 1 \text{ for any } t_i \in B \backslash \{\operatorname{proj}_T(x)\} \\ 0 & \text{otherwise.} \end{cases}$$

Example 7.5. In this example, we will present the smallest example of the Steiner ETF from the hyperbolic subpoints graph. In this case $V = \mathbb{F}_4^4$ and we let $V = \langle e_1, e_2, e_3, e_4 \rangle$ and the quadratic form is $Q(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) = a_1a_2 + a_3a_4$. We then note that we can let $T = \langle e_1, e_3 \rangle$ and $S = \langle e_2, e_4 \rangle$. Using this setup and Definition 7.4, we obtain the following matrix on the next page. We note that the rows are indexed by triples of elements that sum to zero but we only write the first two elements to save space. The third element of the set can be found by taking the sum of the two elements shown. We will prove that a matrix constructed like this is indeed an equiangular tight frame.

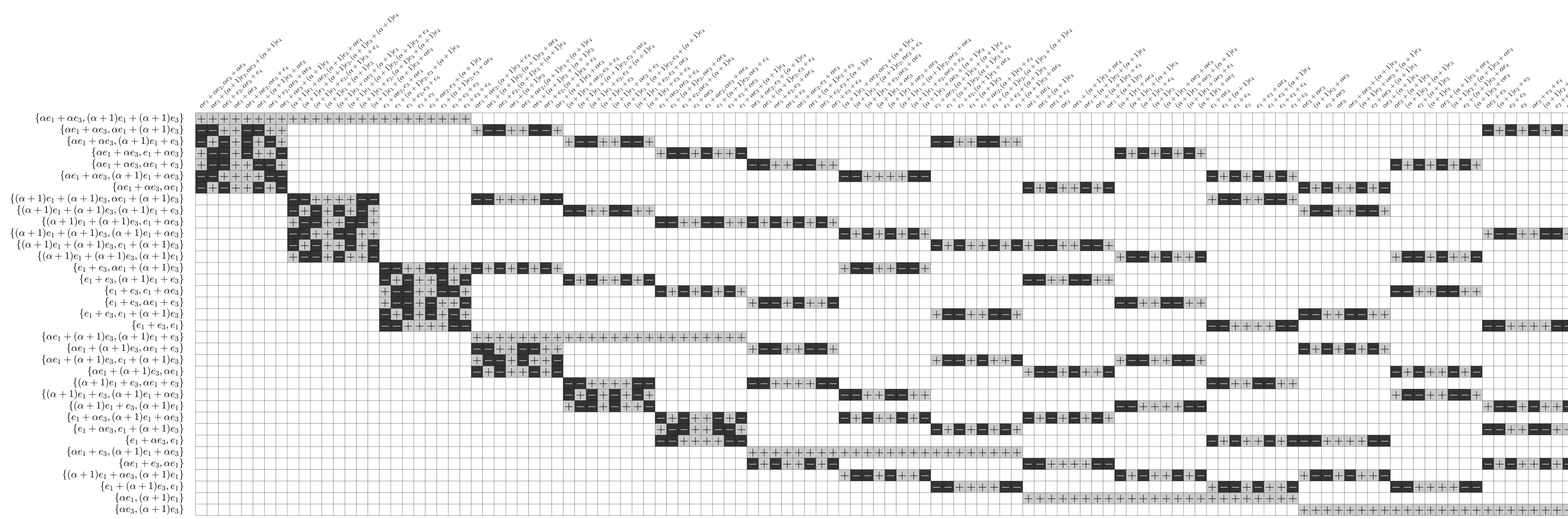

Figure 4: Steiner ETF Associated with Hyperbolic Subpoints Graph

Theorem 7.6. *The matrix Φ given by Definition 7.4 is a $\frac{4^{2n}-3(4)^n+2}{6} \times \frac{1}{2}(4^{2n}-4^n)$ Steiner equiangular tight frame.*

Proof. For ease of notation, denote the columns of Φ by φ_x where $x \in \mathcal{P}_H \backslash S = U$. Now consider

$$\|\varphi_x\|^2 = \langle \varphi_x, \varphi_x \rangle = \sum_{b \in B} \Phi(b,x)\Phi(b,x) = \sum_{\substack{b \in B \\ \mathrm{proj}_T(x) \in b}} 1 = |\{b \in B : \mathrm{proj}_T(x) \in b\}|$$

$$= |\{\{t_1, t_2\} \subseteq T \backslash \{0\} : \mathrm{proj}_T(x) = t_1 + t_2\}| = \frac{1}{2} |\{t_1 \in T \backslash \{0, \mathrm{proj}_T(x)\}\}|.$$

If we consider T as a $2n$ dimensional vector space over the subfield of $\mathbb{F}_4$ consisting of $\{0, 1\}$ then clearly $|\{t_1 \in T \backslash \{0, \mathrm{proj}_T(x)\}\}| = 2^{2n} - 2$. Therefore, $\|\varphi_x\|^2 = \frac{2^{2n}-2}{2} = 2(4^{n-1}) - 1$ for all $x \in U$.

Now we will show that Φ is equiangular. First, let $x, y \in U$ and let $\mathrm{proj}_T(x) = t_1$ and $\mathrm{proj}_T(y) = t_2$ where $t_1 \neq t_2$. Since t_1, t_2 are distinct, then there is exactly one block, $b_0 \in B$ such that$\{t_1, t_2\} \subseteq b_0$. Therefore,

$$|\langle \varphi_x, \varphi_y \rangle| = |\sum_{b \in B} \Phi(b,x)\Phi(b,y)| = |\Phi(b_0,x)\Phi(b_0,y)| = 1.$$

Now suppose that $\mathrm{proj}_T(x) = \mathrm{proj}_T(y) = t_1$ and that for some $s_1, s_2 \in S$ we have $x = t_1 + s_1$, and $y = t_1 + s_2$. In this case we have

$$|\langle \varphi_x, \varphi_y \rangle| = \left| \sum_{b \in B} \Phi(b,x)\Phi(b,y) \right| = \left| \sum_{\substack{b \in B \\ t_1 \in b}} \Phi(b,x)\Phi(b,y) \right|$$

$$= \left| \frac{1}{2} \sum_{t_2 \in T \backslash \{t_1, 0\}} \Phi(\{t_1, t_2, t_1 + t_2\}, x)\Phi(\{t_1, t_2, t_1 + t_2\}, y) \right|.$$

Let $t_2 \in T \backslash \{0, t_1\}$ and let $\varepsilon = Q(x) + Q(y) + \mathrm{tr}(B(t_2, s_1 + s_2))$. Then

$$\Phi(\{t_1, t_2, t_1 + t_2\}, x)\Phi(\{t_1, t_2, t_1 + t_2\}, y) = 1$$

if and only if $\varepsilon = 0$. Additionally, $\varepsilon = 1$ if and only if $\Phi(\{t_1, t_2, t_1 + t_2\}, x)\Phi(\{t_1, t_2, t_1 + t_2\}, y) = -1$. Therefore, we have

$$|\langle \varphi_x, \varphi_y \rangle| = \frac{1}{2} \left| \sum_{\substack{t_2 \in T \setminus \{t_1, 0\} \\ \mathrm{tr}(B(t_2, s_1 + s_2) = \varepsilon)}} (-1)^{\varepsilon} \right|.$$

By Lemma 7.3, we have that $|\{t_2 \in \mathbb{T} : \mathrm{tr}(B(t_2, s_1 + s_2)) = 0\}| = |\{t_2 \in T : \mathrm{tr}(B(t_2, s_1 + s_2)) = 1\}|$. However, we note that $0 \in \{t_2 \in \mathbb{T} : \mathrm{tr}(B(t_2, s_1 + s_2)) = 0\}$ and that $Q(x) + Q(y) = B(t_1, s_1 + s_2) \in \{0, 1\}$ and so $t_1 \in \{t_2 \in \mathbb{T} : \mathrm{tr}(B(t_2, s_1 + s_2)) = 0\}$. Therefore,

$$\sum_{\substack{t_2 \in T \setminus \{t_1, 0\} \\ \mathrm{tr}(B(t_2, s_1 + s_2) = \varepsilon)}} (-1)^{\varepsilon} = \begin{cases} 2 & \text{if } Q(x) = Q(y) \\ -2 & \text{if } Q(x) = Q(y) + 1. \end{cases}$$

Hence, $|\langle \varphi_x, \varphi_y \rangle| = 1$ and so Φ is equiangular.

Now we wish to show that $\langle \varphi_x, \varphi_y \rangle = 1$ if and only if $x \sim_s y$ in the hyperbolic subpoints graph. Suppose that $\mathrm{proj}_T(x) \neq \mathrm{proj}_T(y)$ and for some $s_1, s_2 \in S$ we have $x = t_1 + s_1$ and $y = t_2 + s_2$. Then we have $\langle \varphi_x, \varphi_y \rangle = 1$ if and only if $\Phi(b_0, x) = \Phi(b_0, y)$ if and only if $Q(x) + B(t_2, x) + B(t_2, x)^2 = Q(y) + B(t_1, y) + B(t_1, y)^2$ if and only if $Q(x) + Q(y) + B(x, y) + B(x, y)^2 = 0$. Thus in the hyperbolic subpoints graph we have $x \sim_s y$. If $\mathrm{proj}_T(x) = \mathrm{proj}_T(y)$ then we can see that $\langle \varphi_x, \varphi_y \rangle = 1$ if and only if $Q(x) + Q(y) = 1$ from the equiangularity proof. Therefore we know that in the hyperbolic subpoints graph, we have $x \sim_s y$. Therefore $\Phi^*\Phi(x, y) = 1$ if and only if $x \sim_s y$ in the hyperbolic subpoints graph.

Let A be the adjacency matrix for the hyperbolic subpoints graph and let I, J be the identity and all ones matrix of approriate dimension, then we can see that

$$\Phi^*\Phi = 2A + pI - J$$

where $p = 2(4)^{n-1}$. Since we know that A is the adjacency matrix for a strongly regular graph on n vertices, we know that $A^2 = \lambda A + kI + \mu(J - I - A)$. Hence,

$$\begin{aligned}(\Phi^*\Phi)^2 &= (2A + pI - J)(2A + pI - J)\\ &= (4\lambda - 4\mu + 4p)A + (4k - 4\mu + p^2)I + (4\mu + n - 4k - 2p)J\\ &= (2(4)^n - 2(4)^{n-1})(2A + pI - J).\end{aligned}$$

Hence we see that the gram matrix of Φ is a multiple of a projection, and so Φ is tight. Therefore, Φ is an equiangular tight frame.

Lastly, we note that if we consider the nonzero elements of T to be the points and the set $\mathcal{B}$ as a collection of subsets of T, then clearly $(T\backslash\{0\}, \mathcal{B})$ is a BIBD with $\lambda = 1$, and $k = 3$. Additionally, since $\{s \in S : t + s \in \mathcal{P}_h\}$ for any $t \in T\backslash\{0\}$ is has cardinality 8, then Φ satisfies Definition 3.6 and so Φ is a Steiner ETF. $\square$

Now using this frame Φ we will build another frame of Tremain style. Let (V, Q) be a regular, hyperbolic quadratic space over $\mathbb{F}_4$ with complementary totally singular subspaces T, S such that $V = T \oplus S$. Let the matrix A indexed by elements of T and elements of $\mathcal{P}_H\backslash S$ be defined by

$$A(t,x) = \begin{cases} \sqrt{2} & \text{if } t \in T\backslash\{0\},\ \mathrm{proj}_T(x) = t,\ Q(x) = 0 \\ -\sqrt{2} & \text{if } t \in T\backslash\{0\},\ \mathrm{proj}_T(x) = t,\ Q(x) = 1 \\ 0 & \text{else} \end{cases}.$$

Note that A is a $4^n \times \frac{1}{2}(4^{2n} - 4^n)$ matrix.

Next we define the matrix C indexed by elements of T and elements of S defined by

$$C(t,s) = \begin{cases} \sqrt{\frac{1}{2}} & \text{if } \mathrm{tr}(Q(t+s)) = 0,\ t \neq 0 \\ -\sqrt{\frac{1}{2}} & \text{if } \mathrm{tr}(Q(t+s)) = 1 \\ \sqrt{\frac{3}{2}} & \text{if } t = 0 \end{cases}.$$

Note that C is a $4^n \times 4^n$ matrix.

Using the construction of Φ from Theorem 7.6, now lets define the following matrix,

$$\Psi = \begin{bmatrix} \Phi & 0 \\ A & C \end{bmatrix}.$$

Notably, this matrix has $\frac{1}{2}(4^{2n} + 4^n)$ columns. We will now give an equivalent definition of the entire matrix Ψ.

Definition 7.7. Let T, S be maximal complementary, totally singular subspaces of a regular quadratic space of dimension $2n$ so that $V = T \oplus S$ and let $\mathbb{F} = \mathbb{F}_4$. We define the matrix Ψ, with rows indexed by elements of $R = \mathcal{B} \cup T$ and columns indexed by elements of $\mathcal{P}_H$ given by

$$\Psi(r,x) = \begin{cases} -1 & \text{if } r \in \mathcal{B},\ \operatorname{proj}_T(x) \in r,\ Q(x) + B(x,t_i) + B(x,t_i)^2 = 0 \text{ for any } t_i \in r \backslash \{\operatorname{proj}_T(x)\} \\ 1 & \text{if } r \in \mathcal{B},\ \operatorname{proj}_T(x) \in r,\ Q(x) + B(x,t_i) + B(x,t_i)^2 = 1 \text{ for any } t_i \in r \backslash \{\operatorname{proj}_T(x)\} \\ \sqrt{2} & \text{if } r \in T \backslash \{0\},\ \operatorname{proj}_T(x) = r,\ Q(x) = 0 \\ -\sqrt{2} & \text{if } r \in T \backslash \{0\},\ \operatorname{proj}_T(x) = r,\ Q(x) = 1 \\ \sqrt{\frac{1}{2}} & \text{if } r \in T \backslash \{0\},\ x \in S,\ \operatorname{tr}(Q(r+x)) = 0 \\ -\sqrt{\frac{1}{2}} & \text{if } r \in T \backslash \{0\},\ x \in S,\ \operatorname{tr}(Q(r+x)) = 1 \\ \sqrt{\frac{3}{2}} & \text{if } r = 0, x \in S \\ 0 & \text{otherwise.} \end{cases}$$

Example 7.8. We continue with the same set up from Example 7.5. We will take the matrix from that example and add the appropriate rows and columns as described by Definition 7.7. In the figure below, we note that the blank spaces are zeros and the sybmols $+$ and $-$ correspond to 1 and -1 respectively.

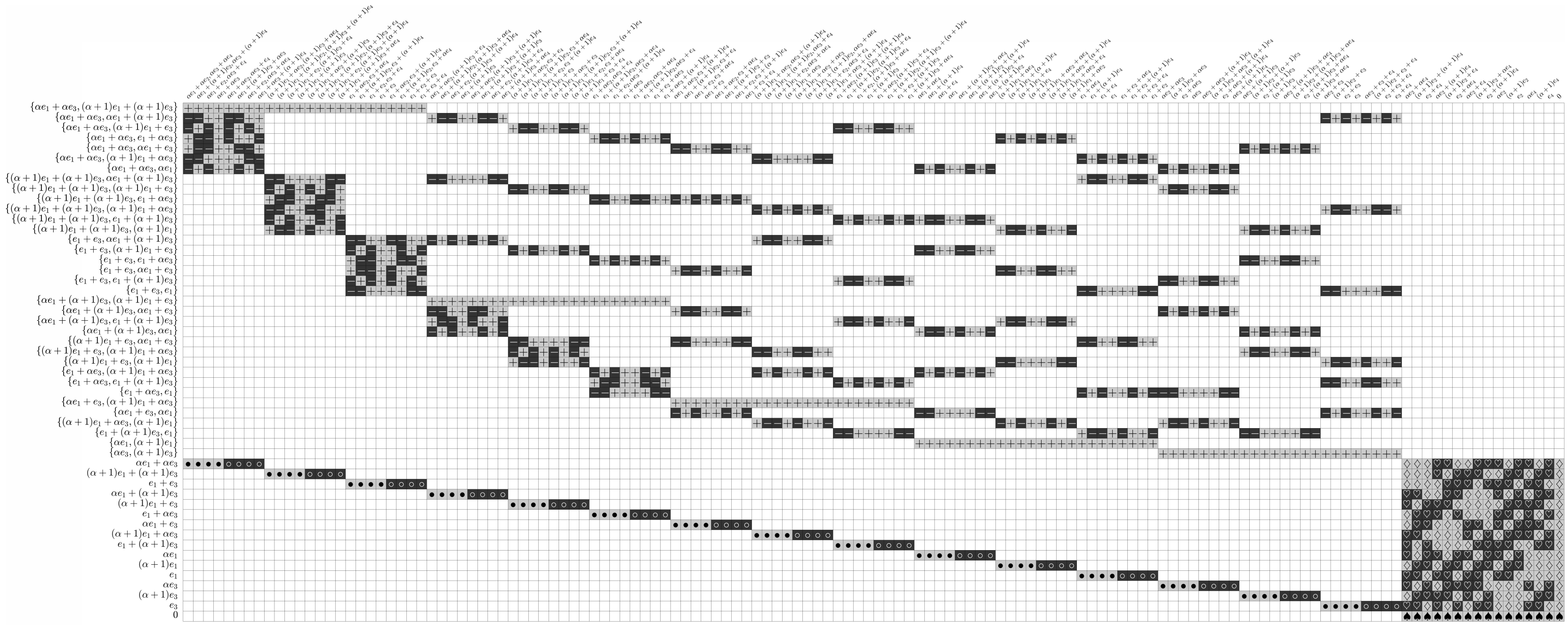

Figure 5: Tremain ETF Associated with $NO_4^+(4)$

$\bullet = \sqrt{2},$, $\circ = -\sqrt{2}$, $\diamondsuit = \sqrt{\frac{1}{2}}$, $\heartsuit = -\sqrt{\frac{1}{2}}$, $\spadesuit = \sqrt{\frac{3}{2}}$

Finally, we prove that the graph of $NO^+_{2n+1}(2^h)$ arises naturally from a Tremain ETF.

Theorem 7.9. *The matrix* Ψ *is a* $\frac{4^{2n}+3\cdot 4^n+2}{6} \times \frac{1}{2}(4^{2n}+4^n)$ *Tremain equiangular tight frame. Moreover, if* $G = \Psi^*\Psi$, D *is the adjacency matrix of* $NO^+_{2n}(4)$, *and* $p = 2(4)^{n-1}+2$, *then* $G = 2D + pI - J$ *where* I, J *are identity matrix and all ones matrix.*

Proof. We first begin by noting the relationship between Ψ and Φ from Theorem 7.6. Let ψ_z denote the column of Ψ associated with $z \in \mathcal{P}_H$ and φ_z be the column of Φ associated with $z \in \mathcal{P}_H \backslash S$. Now let $x \in \mathcal{P}_H \backslash S$. Then we have $\|\psi_x\|^2 = \langle \psi_x, \psi_x\rangle = \langle \varphi_x, \varphi_x \rangle + 2 = 2(4)^{n-1}+1$. Now if we consider $s \in S$ then

$$\begin{aligned}\|\psi_s\|^2 = \langle \psi_s, \psi_s\rangle &= \sum_{b\in B}\Psi(b,s)^2 + \sum_{t\in T\backslash\{0\}}\Psi(t,s)^2 + \Psi(0,s)^2 = 0 + \frac{1}{2}(4^n-1) + \frac{3}{2}\\ &= 2(4)^{n-1}+1.\end{aligned}$$

Hence each column of Ψ has equal norm.

Now we will show that Ψ is equiangular. And we will proceed in cases.

Case 1. First let $x, y \in \mathcal{P}_H\backslash S$ such that $\mathrm{proj}_T(x) = t_1$ and $\mathrm{proj}_T(y) = t_2$ where $t_1 \neq t_2$. This implies that $A(t_2, x) = A(t_1, y) = 0$ and so $|\langle \psi_x, \psi_y\rangle| = |\langle \varphi_x, \varphi_y\rangle| = 1$.

Case 2. Now consider $x, y \in \mathcal{P}_H\backslash S$ where $\mathrm{proj}_T(x) = \mathrm{proj}_T(y) = t$. Note that from the proof of Theorem 7.6 if $Q(x) = Q(y)$ then $\langle \varphi_x, \varphi_y\rangle = -1$. Additionally, we see that $A(t,x) = A(t,y)$ and so $|\langle \psi_x, \psi_y\rangle| = |\langle \varphi_x, \varphi_y\rangle + A(t,x)A(t,y)| = |-1+2| = 1$. Now consider that if $Q(x)+Q(y) = 1$ then $\langle \varphi_x, \varphi_y\rangle = 1$. Since in this case we have $A(t,x) = -A(t,y)$ then we have $|\langle \psi_x, \psi_y\rangle| = |\langle \varphi_x, \varphi_y\rangle + A(t,x)A(t,y)| = |1-2| = 1$.

Case 3. Now suppose $t \in T$ and $s_1 \in S$ such that $x = t + s_1 \in \mathcal{P}_H\backslash S$ and $y \in S$.

Then we can see that

$$|\langle\psi_x, \psi_y\rangle| = |\langle\varphi_x, 0\rangle + A(t, x)C(t, y)| = \left|(\pm\sqrt{2})\left(\pm\sqrt{\frac{1}{2}}\right)\right| = 1.$$

Case 4. Lastly, suppose that $x, y \in S$. Consider that when $t \neq 0$, $C(t, x)C(t, y) = \frac{1}{2}$ if and only if $\text{tr}(Q(t + x)) + \text{tr}(Q(t + y)) = \text{tr}(B(t, x + y)) = 0$. This also implies that $C(t, x)C(t, y) = -\frac{1}{2}$ if and only if $\text{tr}(B(t, x + y)) = 1$. Therefore, by a similar argument in the proof of Theorem 7.6, there are $2(4)^{n-1} - 1$ vectors in $T\backslash\{0\}$ such that $\text{tr}(B(t, x + y)) = 0$ and $2(4)^{n-1}$ vectors in $T\backslash\{0\}$ such that $\text{tr}(B(t, x + y)) = 1$. Then

$$\begin{aligned}|\langle\psi_x, \psi_y\rangle| &= |\sum_{t\in T\backslash\{0\}} C(t, x)C(t, y) + C(0, x)C(0, y)| \\ &= |\frac{1}{2}(2(4)^{n-1} - 1 - 2(4)^{n-1}) + \frac{3}{2}| \\ &= 1.\end{aligned}$$

Therefore, Ψ is equiangular.

Since the columns of Ψ are indexed by elements of $\mathcal{P}_H$, then we will show that for any $x, y \in \mathcal{P}_H$, $(\Psi^*\Psi)(x, y) = 1$ if and only if $x \sim y$ in $NO^+_{2n}(4)$.

First suppose that $x, y \in \mathcal{P}_H\backslash S$ and that $\text{proj}_T(x) \neq \text{proj}_T(y)$. Then from Theorem 7.6 we know that this implies that $x \sim_s y$ in the hyperbolic subpoints graph which implies that $x \sim y$ in $NO^+_{2n}(4)$. Next, if we suppose that $x, y \in \mathcal{P}_H\backslash S$ and that $\text{proj}_T(x) = \text{proj}_T(y)$, then $Q(x + y) + B(x, y)^2 = 0$ if and only if $Q(x) = Q(y)$ if and only if $\langle\psi_x, \psi_y\rangle = 1$. Therefore in this case we have $x \sim y$ if and only if $\Psi^*\Psi(x, y) = 1$.

Now suppose that $t \in T$ and $s_1 \in S$ such that $x = t + s_1 \in \mathcal{P}_H\backslash S$ and that $y \in S$. Then $\langle\psi_x, \psi_y\rangle = A(t, x)C(t, y) = 1$ if and only if $Q(x) + \text{tr}(Q(t + y)) = Q(x + y) + B(x, y)^2 = 0$ if and only if $x \sim y$.

Finally, note that if $x, y \in S$ then $\langle\psi_x, \psi_y\rangle = 1$ and we have $Q(x + y) + B(x, y)^2 = 0$ so $x \sim y$. Hence if we consider the gram matrix of Ψ, we have $\Psi^*\Psi(x,y) = 1$, if and only if $x \sim y$ in NO^4_{2n+1}.

Therefore, if D is the adjacency matrix for $NO^+_{2n}(4)$, then we have $\Psi^*\Psi = 2D + pI - J$ where $p = (2(4)^{n-1} + 2)$. Since we know that D is the adjacency matrix for a strongly regular graph we know that $D^2 = \lambda D + kI + \mu(J - I - D)$ where λ, μ, and k are the graph parameters of the hyperbolic points graph. Therefore we can see that

$$\begin{aligned}
(\Psi^*\Psi)^2 &= (2D + pI - J)(2D + pI - J) \\
&= (4\lambda - 4\mu + 4p)D + (4k - 4\mu + p^2)I + (4\mu + n - 4k - 2p)J \\
&= (6(4)^{n-1})(2D + pI - J).
\end{aligned}$$

Hence we can see that $\Psi^*\Psi$ is a multiple of a projection and is therefore tight. Thus, Ψ is an equiangular tight frame.

Clearly, the matrix Ψ is has rows indexed by $\mathcal{B} \cup T\backslash\{0\} \cup \{0\}$ and columns indexed by $(T \times \{s \in S : t + s \in \mathcal{P}_H\backslash S\}) \cup S$. We know that from the proof of Theorem 7.6, that $(T\backslash\{0\}, \mathcal{B})$ is a Steiner triple system, and that this matrix satisfies Definition 3.8. Hence Ψ is a Tremain ETF. $\square$

Example 7.10. We recall Example 3.10. This Tremain ETF is actually an instance of this construction. In this case we have $V = \mathbb{F}_4^2$ and let $V = \langle e_1, e_2 \rangle$ with a quadratic form $Q(ae_1 + be_2) = ab$. We then note that $T = \langle e_1 \rangle$ and $S = \langle e_2 \rangle$. Using this setup and Definition 7.4, we can construct a Steiner ETF with a BIBD consisting of a pointset of $\langle e_1 \rangle \backslash \{0\}$ and the set of blocks consists of a single set $\{e_1, \alpha e_1, (\alpha + 1)e_1\}$. This gives us a trivial Steiner ETF of

$$\Phi = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

Using this trivial Steiner ETF, we can build a nontrivial Tremain ETF according to Definition 7.7.

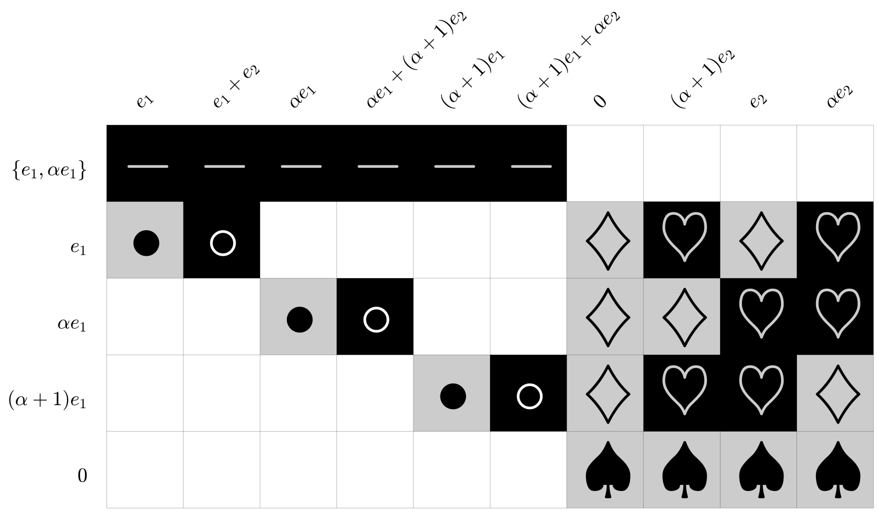

Figure 6: Tremain ETF Associated with $NO_2^+(4)$
$\bullet = \sqrt{2}$, $\circ = -\sqrt{2}$, $\diamondsuit = \sqrt{\frac{1}{2}}$, $\heartsuit = -\sqrt{\frac{1}{2}}$, $\spadesuit = \sqrt{\frac{3}{2}}$

8 CONCLUSION AND FUTURE WORK

In this paper we have analyzed the construction of $NO^+_{2n+1}(2^h)$ and developed a construction of an isomorphic strongly regular graph which allowed us to identify the vertices of the graph with vectors in a particular vector space instead of hyperplanes. This identification allowed us to observe a natural correspondence between $NO^+_{2n+1}(4)$ and a family of Tremain ETFs.

By strengthening the connection between a family of Tremain ETFs and $NO^+_{2n+1}(4)$ we hope to extend this correspondence to other contexts. It was critical to building this particular family of Tremain ETFs that we restricted to vector spaces over $\mathbb{F}_4$. We hope to duplicate this analysis over other fields of characteristic 2 as well as other finite fields. In [8], the discovery of the SRG$(820, 429, 228, 220)$ arose as a generalization of a particular family of Tremain ETFs. This particular SRG was not an instance of $NO^+_{2n+1}(2^h)$. By analyzing the construction of other polar graphs in a similar manner, we hope to find new extensions of ETFs similar to Tremain ETFs which may give new graphs that aren't polar graphs.

We may search for a frame representation that does not rely solely on ETFs since every strongly regular graph corresponds to a generalization of equiangular tight frames, called two-distance tight frames [1]. By developing an explicit connection between other polar graphs over fields that are not $\mathbb{F}_4$, then we can potentially discover generalizations of two-distance tight frames that may result in new strongly regular graphs.

REFERENCES

[1] A. Barg, A. Glazyrin, K. A. Okoudjou, and W.-H. Yu, "Finite two-distance tight frames," *Linear Algebra Appl.*, vol. 475, pp. 163–175, 2015, ISSN: 0024-3795. DOI: 10.1016/j.laa.2015.02.020. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1016/j.laa.2015.02.020.

[2] A. E. Brouwer and J. H. van Lint, "Strongly regular graphs and partial geometries," in *Enumeration and design (Waterloo, Ont., 1982)*, Academic Press, Toronto, ON, 1984, pp. 85–122.

[3] P. Casazza and J. Kovacevic, "Equal-norm tight frames with erasures," *Advances in Computational Mathematics*, vol. 18, pp. 387–430, Feb. 2003. DOI: 10.1023/A:1021349819855.

[4] C. J. Colbourn and J. H. Dinitz, Eds., *Handbook of combinatorial designs*, Second, ser. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2007, pp. xxii+984, ISBN: 978-1-58488-506-1; 1-58488-506-8.

[5] I. Daubechies, *Ten lectures on wavelets*, ser. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992, vol. 61, pp. xx+357, ISBN: 0-89871-274-2. DOI: 10.1137/1.9781611970104. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1137/1.9781611970104.

[6] M. Fickus, J. Jasper, and D. G. Mixon, "Packings in real projective spaces," *SIAM J. Appl. Algebra Geom.*, vol. 2, no. 3, pp. 377–409, 2018. DOI: 10.1137/17M1137528. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1137/17M1137528.

[7] M. Fickus, J. Jasper, D. G. Mixon, J. D. Peterson, and C. E. Watson, "Equiangular tight frames with centroidal symmetry," *Appl. Comput. Harmon. Anal.*, vol. 44, no. 2, pp. 476–496, 2018, ISSN: 1063-5203. DOI: 10.1016/j.acha.2016.06.004. [Online]. Available: https://doi.org/10.1016/j.acha.2016.06.004.

[8] M. Fickus, J. Jasper, D. G. Mixon, and J. Peterson, "Tremain equiangular tight frames," *J. Combin. Theory Ser. A*, vol. 153, pp. 54–66, 2018, ISSN: 0097-3165. DOI: 10.1016/j.jcta.2017.08.005. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1016/j.jcta.2017.08.005.

[9] M. Fickus, D. G. Mixon, and J. C. Tremain, "Steiner equiangular tight frames," *Linear Algebra Appl.*, vol. 436, no. 5, pp. 1014–1027, 2012, ISSN: 0024-3795. DOI: 10.1016/j.laa.2011.06.027. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1016/j.laa.2011.06.027.

[10] J.-M. Goethals and J. J. Seidel, "Strongly regular graphs derived from combinatorial designs," *Canadian J. Math.*, vol. 22, pp. 597–614, 1970, ISSN: 0008-414X. DOI: 10.4153/CJM-1970-067-9. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.4153/CJM-1970-067-9.

[11] V. K. Goyal, J. Kovačević, and J. A. Kelner, "Quantized frame expansions with erasures," *Appl. Comput. Harmon. Anal.*, vol. 10, no. 3, pp. 203–233, 2001, ISSN: 1063-5203. DOI: 10.1006/acha.2000.0340. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1006/acha.2000.0340.

[12] R. B. Holmes and V. I. Paulsen, "Optimal frames for erasures," *Linear Algebra Appl.*, vol. 377, pp. 31–51, 2004, ISSN: 0024-3795. DOI: 10.1016/j.laa.2003.07.012. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1016/j.laa.2003.07.012.

[13] J. H. van Lint and J. J. Seidel, "Equilateral point sets in elliptic geometry," *Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math.*, vol. 28, pp. 335–348, 1966.

[14] T. Strohmer, "Approximation of dual Gabor frames, window decay, and wireless communications," *Appl. Comput. Harmon. Anal.*, vol. 11, no. 2, pp. 243–262, 2001, ISSN: 1063-5203. DOI: 10.1006/acha.2001.0357. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1006/acha.2001.0357.

[15] T. Strohmer and R. W. Heath Jr., "Grassmannian frames with applications to coding and communication," *Appl. Comput. Harmon. Anal.*, vol. 14, no. 3, pp. 257–275, 2003, ISSN: 1063-5203. DOI: 10.1016/S1063-5203(03)00023-X. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1016/S1063-5203(03)00023-X.

[16] D. E. Taylor, *The geometry of the classical groups*, ser. Sigma Series in Pure Mathematics. Heldermann Verlag, Berlin, 1992, vol. 9, pp. xii+229, ISBN: 3-88538-009-9.

[17] J. A. Tropp, "Complex equiangular tight frames," in *Wavelets XI*, M. Papadakis, A. F. Laine, and M. A. Unser, Eds., International Society for Optics and Photonics, vol. 5914, SPIE, 2005, pp. 1 –11. DOI: 10.1117/12.618821. [Online]. Available: https://doi.org/10.1117/12.618821.

[18] S. Waldron, "On the construction of equiangular frames from graphs," *Linear Algebra Appl.*, vol. 431, no. 11, pp. 2228–2242, 2009, ISSN: 0024-3795. DOI: 10.1016/j.laa.2009.07.016. [Online]. Available: https://doi-org.ezproxy.lib.ou.edu/10.1016/j.laa.2009.07.016.

[19] L. Welch, "Lower bounds on the maximum cross correlation of signals (corresp.)," *IEEE Transactions on Information Theory*, vol. 20, no. 3, pp. 397–399, 1974. DOI: 10.1109/TIT.1974.1055219.