Transportation Research Part B

The cell transmission model with free-flow speeds varying over time or space

--Manuscript Draft--

Manuscript Number:	TRB_2018_842R2
Article Type:	Research Paper
Keywords:	cell transmission model; free flow; free flow speed less than 1; uncongested traffic; approximation error; corrected outflows; corrected travel times
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Abstract:	In the cell transmission model (CTM), time is discretised into time-steps and links discretised into cells. In the original CTM, and usually thereafter, the cell lengths are chosen so that, at free-flow speeds (ffs), traffic travels exactly one cell per time-step (1 cpts), so that the ffs, denoted by α, is exactly α = 1 cpts and, to avoid computational complications, the length of each cell is normally held constant over time. But the actual observed ffs's in a cell will often differ by time of day or traffic type or traffic lane, or due to speed limits that vary over time or space, or due to stochastic effects. By construction, the maximum ffs in each cell is 1 cpts (α = 1), hence when the ffs is varying within a cell, it will often be less than 1 (α < 1).
	   We show that when traffic in a cell has a ffs α < 1 cpts then the flows and occupancies obtained from the standard CTM can be very inaccurate. For example, consider a cell of length 1 that is in a free flow state with free-flow speed α < 1 and no further flow into the cell. Then all traffic in the cell will have exited by time 1/ α and the cell will then be empty. In contrast, for the same scenario, the CTM lets a fraction α < 1 of the remaining traffic in the cell exit in each time step, so that the cell outflow and occupancy decline geometrically toward zero, so that the cell never fully empties.
	  The problem is serious since in traffic networks there may be large numbers, or a large proportion, or a majority, of cells and links that are in a free-flow state for all or part of the time span being modelled. To overcome the above problem, we propose that the CTM not be applied to cells that are in a free-flow state with ffs α < 1 cpts. Instead, for those cells and links, we let traffic move forward at its ffs rather than as computed from the CTM. This is easily accomplished since, in the CTM the computations roll forward one time step at a time and, in each time step, the cell occupancy is updated from the previous time step, hence is known and the known occupancy immediately indicates whether the cell will be in a free-flow state.

Response to Reviewers:	To AE, Prof Michael Zhang. Dear Michael, Thank you very much for your decision letter of 24th April, which included a reviewer report. In your decision letter you advise that the paper would be accepted when I make a minor revision to tidy up the formatting of the paper. I have tidied the formatting in various places and did the following. 1. I have removed all of the yellow highlighting of text (phrases, sentences, paragraphs, and in some places a few successive paragraphs) in the paper, which had been inserted to make it easier for a reviewer to see how/ where the text had been revised in response to their comments, 2. I have removed lines that had been placed down left margins within parts of some paragraphs to indicate new or substantially revised text. Those lines had caused paragraphs to split up, and removing them brings the paragraphs together again, which improves the format and readability.

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Statement of contribution/ potential impact.

In the cell transmission model (CTM), time is discretised into time-steps and links discretised into cells. In the original CTM, and usually thereafter, the cell lengths are chosen so that, at free-flow speeds (ffs), traffic travels exactly one cell per time-step (1 cpts), so that the ffs, denoted α, is exactly $\alpha = 1$ cpts and, to avoid computational complications, the length of each cell is normally held constant over time. But the actual observed ffs's in a cell will often differ by time of day or traffic type or traffic lane, or due to speed limits that vary over time or space, or due to stochastic effects. By construction, the *maximum* ffs in each cell is 1 cpts ($\alpha = 1$), hence when the ffs is varying within a cell, it will often be less than 1 ($\alpha < 1$).

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Highlights for TRB_2018_842,
"The cell transmission model with free-flow speeds varying over time or space".

- The CTM assumes free-flow speeds (ffs's) are exactly one cell per time step (1 cpts) but in reality, ffs's vary over time and space – hence often less than 1 cpts
- Allowing ffs's less than 1 cpts, makes CTM solutions very inaccurate
- Proposed remedy: for cells with ffs less than 1 cpts, use actual ffs, not the CTM
- Problem and remedy (CTM and 'revised' CTM) are illustrated for simple examples

The cell transmission model with free-flow speeds varying over time or space

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29 April 2020.

Abstract

In the cell transmission model (CTM), time is discretised into time-steps and links discretised into cells. In the original CTM, and usually thereafter, the cell lengths are chosen so that, at free-flow speeds (ffs), traffic travels exactly one cell per time-step (1 cpts), so that the ffs, denoted α, is exactly $\alpha = 1$ cpts and, to avoid computational complications, the length of each cell is normally held constant over time. But the actual observed ffs's in a cell will often differ by time of day or traffic type or traffic lane, or due to speed limits that vary over time or space, or due to stochastic effects. By construction, the *maximum* ffs in each cell is 1 cpts ($\alpha = 1$), hence when the ffs is varying within a cell, it will often be less than 1 ($\alpha < 1$).

We show that when traffic in a cell has a ffs $\alpha < 1$ cpts then the flows and occupancies obtained from the standard CTM can be very inaccurate. For example, consider a cell of length 1 that is in a free flow state with free-flow speed $\alpha < 1$ and no further flow into the cell. Then all traffic in the cell will have exited by time $1/\alpha$ and the cell will then be empty. In contrast, for the same scenario, the CTM lets a fraction $\alpha < 1$ of the remaining traffic in the cell exit in each time step, so that the cell outflow and occupancy decline geometrically toward zero, so that the cell never fully empties.

The problem is serious since in traffic networks there may be large numbers, or a large proportion, or a majority, of cells and links that are in a free-flow state for all or part of the time span being modelled. To overcome the above problem, we propose that the CTM not be applied to cells that are in a free-flow state with ffs $\alpha < 1$ cpts. Instead, for those cells and links, we let traffic move forward at its ffs rather than as computed from the CTM. This is easily accomplished since, in the CTM the computations roll forward one time step at a time and, in each time step, the cell occupancy is updated from the previous time step, hence is known and the known occupancy immediately indicates whether the cell will be in a free-flow state.

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1. Introduction

In the cell transmission model (CTM), time and space are discretised so that traffic moves forward at most one cell per time-step. Typically, and in the original version of the CTM (Daganzo (1994, 1995a, 1995b)), the discretisation is chosen so that the highest speed (the free-flow speed) is exactly one cell per time-step, to give a closer approximation to the continuous LWR model (Lighthill and Whitham (1955), Richards (1956)), for given cell lengths and time-step sizes. Nevertheless, it is often useful or necessary to explicitly or implicitly assume free-flow speeds for some or all cells are less than one cell per time-step, as discussed and illustrated in Section 2 below.

This paper focuses on the CTM and there is not space here to consider related models, such as the link-transmission model (LTM) (Yperman *et al*. (2006), Yperman (2007)), the point queue (P-Q) model (e.g., Vickrey (1969), Zhang, Nie and Qian (2013)), and the spatial queue (S-Q) model (e.g., Nie and Zhang (2010), Qian, Shen and Zhang (2012), Zhang, Nie and Qian (2013)).[1]

If the discretisation in the CTM is chosen so that the free-flow speed α for some cells is less than one cell per time-step (i.e. $\alpha < 1$ cell per time-step) then, if the cell is in a free-flow state, the cell outflow computed by the CTM can be very inaccurate, which can be easily illustrated as follows. Consider a cell at time t containing uniformly distributed traffic travelling at free-flow speed $\alpha < 1$, and suppose no further traffic enters the cell after time t. As the traffic is travelling at free-flow speed $\alpha < 1$, in cells per time-step, it should take $1/\alpha > 1$ time-steps to traverse the cell, hence should exit from the cell at a constant rate from time t to time $t + 1/\alpha$. Thereafter the cell should be empty. However, if we instead apply the CTM we find that a fraction $\alpha < 1$ of the traffic in the cell will exit in time-step t+1, a fraction $\alpha < 1$ of the remaining traffic will exit in time-step t+2 and so on in all future time-steps, so that the traffic in the cell declines at a geometric rate over time and the cell never fully empties. This is of course unrealistic and is in contrast to the correct constant outflow rate noted above.

It is important to be clear that the example and comments above, and later in this paper, do not imply any flaw in the original CTM. In the original CTM the free-flow speed is assumed to be $\alpha = 1$ cell per time-step for all cells and time-steps, while here we are considering $\alpha < 1$ in the CTM for some cells that are in a free-flow state for some or all time-steps. Daganzo (1995b), top of page 264, using different notation than used here, noted that α must be set to $0 < \alpha \leq 1$ (to ensure convergence, see end of Section 3 below) and that the CTM is most accurate if the step length and time-step are chosen so that $\alpha = 1$ rather than $\alpha < 1$. He did not give any discussion of the inaccuracies that result from setting $\alpha < 1$, which is the topic of the present paper. We will sometimes contrast the CTM solution with the exact or correct solution and refer to the difference between them as an error in the CTM solution, but this is only when we are assuming traffic in a free-flow state and with a free-flow speed $\alpha < 1$. If $\alpha = 1$ is assumed for all cells and time-steps, as in the original CTM, then the problems considered in this paper disappear. In Section 2 we set out several reasons why it is important to let $\alpha < 1$ for some cells that are in a free-flow state in the CTM for some or all time-steps.

In this paper we propose that, when using the CTM, if a cell is in a free-flow state, with free-flow speed $\alpha < 1$, then the outflows from that cell should be computed using an exact or 'corrected' outflow rate (Section 5 below) instead of the CTM computed rate. This is easily accomplished since, in the CTM the computations roll forward one time step at a time and, in each time step, the cell occupancy is updated from the previous time step, hence is known and the known occupancy immediately indicates whether the cell will be in a free-flow state.

The behaviour of the CTM, described in the third paragraph of this introduction, has a simple explanation as follows. The CTM does not consider the location of traffic within each cell and implicitly assumes that the traffic in a cell is always uniformly distributed within the cell. If a cell contains uncongested free-flow traffic uniformly distributed within the cell, then a fraction $1/\alpha$ of this traffic exits in the first time-step. However, the CTM implicitly assumes that the remaining traffic immediately redistributes itself uniformly within the cell so that a fraction $1/\alpha$ of the remaining traffic exits in the second time-step, and so on in all future time-steps. It is this assumption that, in each time-step, traffic redistributes itself uniformly within the cell, that causes the unrealistic tail of outflows referred to in the second paragraph of this section. If inflow to a cell has stopped, or is declining, then redistributing traffic uniformly within the cell requires spreading some traffic

[1] The CTM is still the most widely cited of these. For example, the Daganzo (1994) paper has been cited more than 2,800 times and Daganzo (1995a) more than 2,000 times. Both of these papers continue to be cited at an increasing rate, e.g., according to Google Scholar (2020-03-20), the Daganzo (1994) paper has been cited 637 times in the five years 2010-2014 inclusive and 840 times in the next five years 2015-2019. The Daganzo (1995a) paper has been cited 809 times in the five years 2010-2014 and 1,020 times in the next five years 2015-2019.

backwards within the cell. That violates a "causality" property of traffic flow: causality refers to the property that vehicles are influenced only by traffic ahead and not by traffic behind.

The above phenomenon was remarked on in Carey (2004), Propositions 3(b) and 7(b), for an earlier class of exit-flow models namely those originating from the Merchant-Nemhauser (MN) model (Merchant and Nemhauser (1978a, 1978b)). For traffic in a free-flow state the CTM reduces to the MN model. (The cell outflow in the CTM is the minimum of the sending capacity of the current cell and the receiving capacity of the next downstream cell, while in the MN model the receiving capacity was omitted so that the outflow was given by only the sending capacity of the current cell. In free-flow conditions the outflow equation in the CTM reduces to the sending capacity of the current cell, which is the same as in the MN model.)

Improving the accuracy in modelling free-flow traffic is important when modelling network flows, since in most traffic networks many of the network links will be uncongested for some or all of the time span being modelled. The above inaccuracies can also cumulate and increase through knock-on effects into later downstream links. They can also cause knock-on inaccuracies in flows on upstream links if the model includes route choice, since upstream traffic may then choose routes based on inaccurate information about downstream flows and travel times.

Since in this paper we are concerned with the behaviour of the CTM, and a modified CTM, when traffic is in a free-flow state with free-flow speed $\alpha < 1$, to motivate the paper it is important to first illustrate how and why this $\alpha < 1$ scenario arises in practice. This is set out and discussed in Section 2.
Section 3 sets out some standard concepts and terms used in this paper. Section 4 sets out the CTM solution, and also a modified corrected solution, for cells that are in a free-flow state with free-flow speed $\alpha < 1$. Solutions are obtained for some simple inflow profiles (in Sections 4.1 and 4.2) and also when refining the discretisation to its continuous limit. (in Sections 4.3). The modified corrected solutions take the cell free-flow travel times (cfftt) as cfftt = 1/ffs = $1/\alpha$, instead of using the CTM. Section 5 applies the modified/ corrected CTM solution. Section 5.1 set this out for a single cell, when the cfftt (i.e. $1/\alpha$) may be integer or non-integer. Section 5.2 extends the modified CTM from a single cell to networks. This is relatively simple when the (modified) CTM is used for one-pass network loading as in Section 5.2.1, and a bit more complex when it is used with iterative adjustment of spatial path allocations to obtain a user equilibrium. Section 6 proposes some extensions and concluding remarks.

2. Reasons for considering free-flow speeds $\alpha < 1$ for some cells in the CTM

The free-flow speed α is measured in cells traversed per time-step, or rather, it is the fraction of a cell that is traversed per time-step since, in the CTM, the discretisation into time-steps and cell lengths is chosen so that no more than one cell can be traversed per time-step. The free-flow speed α, in cells traversed per time-step, is related to the free-flow speed s measured in 'natural units' (e.g., metres per sec), as follows

$$\alpha = (\varepsilon/d)s \tag{1}$$

where d is the cell length (e.g. in metres), ε is the length (duration) of the time-step (e.g. in seconds), so that ε/d is the time-space discretisation ratio, in seconds per metre.

Several reasons why we wish to consider free-flow speeds $\alpha < 1$ are set out in cases 1 to 6 later below. In general, the reason is that the free-flow speed α may vary across the network or over time hence it can not have the same value ($\alpha = 1$) everywhere. As the maximum value of α is 1, it follows that for some cells in some time steps it must be that $\alpha < 1$.

In the original Daganzo papers on the CTM, d, ε and s are held constant for all time steps and cells and this practice has been followed by later authors. When these are held constant the discretisation ratio ε/d and hence $\alpha = (\varepsilon/d)s$ are also constant over time steps and cells. Daganzo (1995b) shows that $1 \geq (\varepsilon/d)s$ "is needed for convergence and that the algorithm is most accurate if $d = s\varepsilon$ (hence

$1 = (\varepsilon/d)s$) as recommended in Daganzo (1994)." The "algorithm" refers to an algorithm in Daganzo (1995b) to obtain a finite difference approximation of the kinematic wave model of traffic flow.

In cases 1 to 6 later below, we set out a range of scenarios in which the free-flow speed *s* is likely to vary over time and/ or over cells. In that case, if d and ε continue to be held constant, then it follows from (1) that α varies over time steps and/or cells in the same way as *s*. In that case, we no longer have $d = s\varepsilon$ and hence $\alpha = (\varepsilon/d)s = 1$ as recommended by Daganzo. This suggests the following two interesting questions.

Q1. If the free-flow speed *s* (i.e., in 'natural' units, e.g. metres per second) varies from cell to cell, could the discretisation (the cell length d) also be varied from cell to cell to exactly compensate for the variation in the free-flow speed *s* so that the free-flow speed α in cell-based units remains constant at $\alpha = 1$?

Q2. Even if the answer to Q1 is yes, can we instead simply move traffic forward at free-flow speed $\alpha \leq 1$, which may be different in each cell or link (and hence avoid using the CTM flow equation (2) and avoid the complications involve in varying d and/or ε as in Q1)?

As discussed below, the answer to Q1 may be a qualified 'yes' but it can become very complicated, and involves deviating from the simple CTM model. The answer to Q2 is also yes. To consider question Q1, we consider three scenarios Q1(i), Q1(ii) and Q1(iii) that arise in Q1.

Q1(i). Consider varying the step length *d* in (1) over time for a cell *i*.

Suppose we have $\alpha = (\varepsilon/d_t)s_t = 1$ for free-flow traffic in time step *t*, then to ensure $\alpha = (\varepsilon/d_{t+1})s_{t+1} = 1$ in the next time step *t*+1, we must set $s_{t+1}/d_{t+1} = s_t/d_t$, hence set $d_{t+1} = (s_{t+1}/s_t)\ d_t$. That is, the cell length in time step *t*+1 is obtained by multiplying the cell length in time step *t* by a factor (s_{t+1}/s_t), so that the cell lengths are different in time steps *t* and *t*+1.

Having different cell lengths in different in time steps would cause substantial complications in the. Suppose, for example, that $d_{t+1} < d_t$. In that case, cell *i* in time step *t*+1 may lie entirely within the length of a cell from time step *t*, or it may overlap part of two adjacent cells from time step *t*. The free-flow speeds (s_i and s_{i+1}) may be different for those two cells in time step *t*, hence cells in time step *t*+1 will inherit two cohorts with different free-flow speeds.

An additional complication or difficulty is that a cell may be in a free-flow state for some time steps and may not be in a free-flow state for some earlier or later time steps. In that case, if the step length *d* is changed or varied as above to suit free-flow cells, then the resulting new step lengths may not be lengths that we would have chosen for those earlier or later time steps.

Q1(ii). Consider the time-step length (duration) ε in (1) varying between cells along a link or route.

In (1), let *d* remain constant over time steps *t* and let ε vary from ε_t in time step *t* to ε_{t+1} in time step *t*+1, so that $\alpha = (\varepsilon_t/d)s_t$ and $\alpha = (\varepsilon_{t+1}/d)s_{t+1}$ for free-flow traffic in time steps *t* and *t*+1 respectively. Assuming $\alpha = 1$ in time step *t* then, to ensure that it is still $\alpha = 1$ in time step *t*+1, we must have $(\varepsilon_t/d)s_t = (\varepsilon_{t+1}/d)s_{t+1}$, hence $s_{t+1}\varepsilon_{t+1} = s_t\varepsilon_t$, hence $\varepsilon_{t+1} = (s_t/s_{t+1})\ \varepsilon_t$. That is, the time step length in time step *t*+1 is obtained by multiplying the time step length in time step *t* by a factor (s_t/s_{t+1}). The free-flow speeds s_t and s_{t+1}, and hence their ratio (s_t/s_{t+1}), and hence the length of the time step $\varepsilon_{t+1} = (s_t/s_{t+1})\ \varepsilon_t$, may be different in adjacent cells within a link or route.

The above (having time steps of different durations in adjacent cells within a link or route) would cause substantial complications in the CTM. The duration of time step *t*+1 in cell *i* may lie entirely within the duration of a time step in the next downstream cell, or it may overlap with the durations of two adjacent time steps in the next downstream cell. That means that two traffic cohorts, which may be travelling at two different speeds, may enter the next downstream cell and have to be somehow merged together at a new composite speed.

Q1(iii). Consider varying the step length *d* in (1) as in Q1(i) and *simultaneously* varying the time-step length ε in (1) along a link or route as in Q1(ii).

We will not discuss this case here but suffice to say that it is at least a bit more complicated than either Q1(i) or Q1(ii) above.

The approach proposed in this paper, for handling free-flow traffic when $\alpha_i < 1$, as in cases 1 to 6 below, is very much simpler than the approach in Q1(i)-(iii) above. It is outlined in the first paragraph of Section 3 and in the third paragraph of Section 1 and, in Propositions 1A and 2A, is contrasted with using the CTM when $\alpha_i < 1$. It does not involve adjusting the cell lengths as in Q1(i)-(iii), and does not involve trying to force the problem into a CTM framework, which turns out to be not necessary or appropriate when $\alpha_i < 1$. Thus, the answer to question Q2 is, yes.

The following are scenarios in which free-flow speed typically varies along a route at a given time or time step (case 1 below) or varies over time in a cell or link or route (cases 2 to 6 below).

1. *Free-flow speeds varying along a link or route in a given time or time step*. Free-flow speeds are likely to vary along a link or route. For example, free-flow speed may vary along a road because of varying posted speed limits, the quality of the road surface, the width of the road surface, whether it is straight or curved or winding, uphill or downhill, is through a shopping area with pedestrians crossing the road, street lighting, etc.

2. *Posted speed limits varying over time*. Even if the maximum free-flow speed for a section of roadway is $\alpha = 1$, the posted speed limit may vary throughout the day or at certain times of day (e.g. as in the case of intelligent speed control systems) and at those times the free-flow speed will be less than the $\alpha = 1$ maximum. In practice, traffic may exceed the posted speed limit, so that the actual average traffic speed or prevailing speed sometimes exceeds the posted speed limit. But, nevertheless, when the posted speed limit is lower than the maximum (free flow) speed this normally causes a reduction in the average or prevailing speed. Hence, if we let $\alpha = 1$ denote the free-flow speed in the absence of a posted speed limit then α will normally be less than 1 when the posted speed limit is less than the unrestricted free-flow speed.

3. *Free-flow speeds different in different lanes*. An extension of the basic CTM considered by a number of authors is in introducing lanes and lane-changing. Typically, we may not expect to observe the same free-flow speed in all lanes. For example, in some countries faster traffic is expected to use outer/ inner lanes. In some countries there are restrictions on the types of vehicles permitted to use various lanes, so that vehicle-type composition will systematically vary by lane, resulting in differing free-flow speeds in different lanes.

4. *Free-flow speeds different for different vehicle types*. Different vehicle types, such as heavy goods vehicles and other vehicles, may have different free-flow speeds. At low free-flow speeds, the cell occupancy or density will be low hence, even if travelling at different speeds, they may overtake each other without significantly affecting their speeds. In that case, if the fastest vehicles travel at free-flow speed $\alpha = 1$ then other vehicle types will travel at free-flow speeds $\alpha < 1$.

5. *Other factors causing the free-flow speed to vary over time*. Free-flow speeds at a location may also vary during the day for reasons other than those noted in items 2, 3 and 4 above, that is, even if the posted speed limit does not change and even if there are no significant differences between lane speeds and vehicle types. For example, free-flow speeds may be higher at times when parking is not allowed along a roadway. Free-flow speeds may be lower at times of day when there are more pedestrians around, causing drivers to be more cautious, for example, at opening or closing times of schools, places of employment, shops or public events. Free-flow speeds may also vary with changing lighting conditions, being lower as darkness falls in the evening or clears in the morning.

6. *Free-flow speeds varying due to other stochastic effects*. Free-flow speeds may vary due to stochastic effects, such as weather conditions including fog, rain, frost or snow. The timing of changes in weather conditions during the day are usually difficult to predict accurately in advance, or at least have a large random component, though they are to some extent predicable in some climates, for example, the timing of daily monsoon rains.

3. Preliminaries

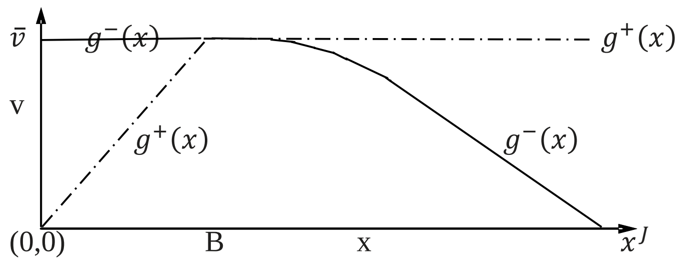
Free-flow traffic. When traffic is light, so that there is little interaction between vehicles, then the traffic speed and flow rate are not significantly affected by the traffic density, and traffic is said to be in a free-flow state. In that case, traffic that enters a road segment or cell at time *t* exits from it a time *t*+cfftt where cfftt is the cell free-flow travel time. The time profile of traffic exiting from the road segment is the same as the time profile of traffic entering it but with a time lag equal to the cfftt. For modelling purposes, let the road length be divided into cells and time be divided into time-steps, as in the CTM. Then, if the free-flow traffic speed in cell *i* is $\alpha_i \leq 1$ cells per time-step (cpts), the time taken to traverse cell *i* (the cell free-flow travel time cfftt) is 1/(free-flow speed in cells per time-step) = $1/\alpha$ time-steps.
The subscript *i* indicates that the free-flow speed α_i in a cell, and hence free-flow travel time $1/\alpha_i$ for a cell, may vary from cell to cell along a link and along a route.

The cell transmission model

The cell transmission model was introduced in Daganzo (1994, 1995a and 1995b) as a discrete approximation to the LWR model (Lighthill and Whitham (1955), Richards (1956)), which is also referred to as the hydrodynamic model or kinematic wave model of traffic flow. The CTM is a finite difference approximation to the partial differential equations of the LWR model. In the CTM each link is divided into cells $i = 1, \dots, I$, and time is divided into time-steps or 'clock ticks' $t = 1, \dots, T$. The flow-occupancy function for cell *i* in time-step *t* can be written as $g_i(x_i^t)$ where x_i^t is the occupancy of the cell in time-step *t*, that is, the number of vehicles in the cell, measured in standardised units, and $g_i(.)$ is a nonnegative unimodal function. Commonly used forms of the flow-occupancy function $g_i(x_i^t)$ include triangular, trapezoidal, general piecewise-linear, general nonlinear, and general piecewise-nonlinear. Then the CTM traffic flow function can be written in a very general form as follows, as in Daganzo (1995b),

$$v_i^t = \min\{g_i^+(x_i^t),\ g_{i+1}^-(x_{i+1}^t)\} \tag{2}$$
= min{(sending capacity of cell *i* in time-step *t*),
(receiving capacity of the next downstream cell *i*+1 in time-step *t*)}.

where v_i^t is the outflow from cell *i* in time-step *t*. Equation (2) and general nonlinear forms of the functions $g_i^+(x_i^t)$ and $g_{i+1}^-(x_{i+1}^t)$ are illustrated in Fig. 2 on page 266 of Daganzo (1995b). Since we are using different notations in this paper we have reproduced a similar figure (Fig. 1) to illustrate (2) and general nonlinear forms of $g_i^+(x_i^t)$ and $g_{i+1}^-(x_{i+1}^t)$.



Dashed line denotes $g^+(x)$.
Solid line denotes $g^-(x)$.

Fig. 1. Functions $g^+(x)$, $g^-(x)$ and (2) yield mound shaped flow-occupancy function $v = g(x)$.

These functions can be described as follows. $g_i^+(x_i^t)$ is obtained by taking the upward-sloping part of $g_i(x_i^t)$ and extending it to the right in a horizontal straight line from its peak. $g_{i+1}^-(x_{i+1}^t)$ is obtained by taking the downward-sloping part of $g_{i+1}(x_{i+1}^t)$ and extending it back to the vertical axis in a horizontal straight line from its peak. The outflow equation (2) allows all of, the piecewise linear and nonlinear forms referred to above, by suitably defining the forms of $g_i^+(x_i^t)$ and $g_{i+1}^-(x_{i+1}^t)$. Equation (2) together with a conservation equation

$$x_i^{t+1} = x_i^t + u_i^t - v_i^t \tag{3}$$

yields the cell transmission model ((2),(3)), where u_i^t and v_i^t denote the inflow and outflow for cell *i* in time-step *t*.

In this paper we are particularly concerned with traffic in a free-flow state, with a free-flow speed $\alpha \leq 1$ in cells per time-step, in other words α is the fraction of the cell length that is traversed in one time step. If we assume that the traffic is uniformly distributed along the cell, then a fraction α of the cell length contains a fraction α of the traffic in the cell. Hence the amount of traffic that exits from the cell in one time step (i.e., v_i^t) is a fraction α of the total amount of traffic in the cell (the cell occupancy x_i^t) or, more formally,

$$v_i^t = \alpha x_i^t. \tag{4}$$

That is, when traffic is in a free-flow state, we can replace the exit flow function (2) with (4). [Another way to state this is that, in free-flow conditions, (2) reduces to $v_i^t = g_i^+(x_i^t)$ and $g_i^+(x_i^t)$ reduces to $g_i^+\left(x_i^t\right) = \alpha x_i^t$.] This reduces the CTM (2)-(3) to (4)-(3).

The CTM in 'cell-based' units and in 'natural' units

The CTM equations (2) and (4) are stated in cell-based units, that is, with speed α in cells per time-step rather than in for example metres per second and using cell occupancy x_i^t rather than cell density. This is the form of the CTM that is used throughout this paper, but for reference it is perhaps worth noting that in applications the CTM is also often stated in "natural" units, e.g. with speed *s* in metres per second rather than cells per time-step and using cell density rather than cell occupancy. To convert (2) and (4) to natural units, as in (2') and (4') below, let *d* denote the cell length (e.g. in metres) and ε denote the time-step length (e.g. in seconds). Then $v_i^t = q_i^t \varepsilon$ where q_i^t is the cell outflow in vehicles per second, $x_i^t = k_i^t d$ where k_i^t is the cell density in vehicles per metre. Substituting these into (4) gives $q_i^t \varepsilon = \alpha k_i^t d$ and using (1) to substitute for α in this gives $q_i^t \varepsilon = (\varepsilon/d) s\, k_i^t d$ which reduces to

$$q_i^t = s\, k_i^t \tag{4'}$$

It is noticeable that (4') has the same form as (4) though it is in natural units while (4) is in cell-based units.

Similarly, substituting $v_i^t = q_i^t \varepsilon$ and $x_i^t = k_i^t d$ from above into (2) gives

$$q_i^t \varepsilon = \min\{ g_i^+(k_i^t d), g_{i+1}^-(k_{i+1}^t d)\}$$

hence

$$q_i^t = \min\{ g_i^+(k_i^t d)/\varepsilon, g_{i+1}^-(k_{i+1}^t d)/\varepsilon\}.$$

Then rewriting $g_i^+(k_i^t d)/\varepsilon$ as $h_i^+\left(k_i^t\right)$ and $g_i^-(k_i^t d)/\varepsilon$ as $h_i^-\left(k_i^t\right)$ gives

$$q_i^t = \min\{ h_i^+(k_i^t), h_{i+1}^-\left(k_{i+1}^t\right)\} \tag{2'}$$

Again, (2') is of the same form as (2) but in natural units while (2) is in cell-based units.

The CFL condition. The Courant-Friedrichs-Lewy (CFL) condition (Courant *et al*. (1967)) is a necessary condition for a finite difference scheme to converge to the solution of the corresponding partial differential equation as the discretisation step sizes go to zero. In the present case the CFL condition is that $0 < \alpha \leq 1$. Daganzo (1994) assumed $\alpha = 1$. Without explicitly referring to CFL, he showed that the solution of the CTM converges to the solution of the LWR model as the discretisation (d, ε) is refined to the continuous limit, while retaining the ratio d/ε unchanged. Daganzo (1995b), at top of page 264 using notation different than used here, stated that $0 < \alpha \leq 1$ is needed for convergence and that the solution algorithm is most accurate if $\alpha = 1$. When $\alpha = 1$, then (4) also has the nice property that it reduces to $v_i^t = x_i^t$, which means that traffic advances exactly one cell per time-step when travelling at free-flow speed.

4. Error associated with the CTM when a cell is in a free-flow state with free-flow speed $\alpha < 1$

If traffic in a road segment or cell is in a free-flow state, then the outflows and occupancy for the segment will be as outlined in the first paragraph of Section 3 and in the third paragraph of Section 1. However, if we use the CTM to obtain the outflows for this free-flow scenario then the outflows are governed by (2) which reduces to (4). If we also let $\alpha < 1$ then the CTM yields outflows and occupancies for the segment or cell that exhibit a geometric decay property, as set out in Propositions 1A, 2A and 3 below. These geometric decay properties and profiles are substantially different than those set out in the first paragraph of Section 3 and hence are an unrealistic, poor approximation to the latter.

The above error is illustrated in the third paragraph in Section 1. A numerical illustration is as follows. Suppose $\alpha = 0.4$, the initial occupancy of the cell is *x* and there are no further inflows to the cell. A free-flow speed $\alpha = 0.4$ implies a cell traversal time (in cell lengths per time-step) of 1/0.4 = 2.5 time-steps, which means that traffic that entered the cell at the beginning of time-step *t* should all exit from the cell after 2.5 time-steps. However, if the outflows from the cell are instead computed using the CTM ((4),(3)), then the cell exit flow in the first time-step will be 0.4*x*, leaving 0.6*x* remaining, the exit flow in the second time-step will be 0.4(0.6*x*) = 0.24*x* leaving (0.6 – 0.24)*x* = 0.36*x* remaining, and so on. Thus, when using the CTM, instead of the outflow ending after 2.5 time-steps it continues exiting at a geometrically declining rate.

The above property is set out more fully and formally in the propositions and discussion below. In Propositions 1A and 1B we assume that there is no further inflow to the cell after time *t* = 0 while in Propositions 2A and 2B we assume that inflows continue after time *t* = 0. In propositions 1A and 2A we assume that the cell is in a free-flow state throughout.

4.1. *Comparing the exact (correct) solution and the CTM solution for a single cell, with $\alpha \leq 1$, while assuming no further inflows to the cell.*

Proposition 1A. Consider a cell *i*, of length 1 unit, that is in a free-flow state with free-flow traffic speed α, $0 < \alpha \leq 1$. For simplicity, let there be no further inflows to the cell after time $t = 0$, no restrictions on exiting from the cell. and let the cell occupancy be x_i^0 at time $t = 0$.

The exact (correct) solution.

(i) Up to time $t = 1/\alpha$.

The cell outflow rate.

The constant free-flow speed α implies a constant cell traversal time $1/\alpha$. If traffic enters the cell at a constant free-flow rate $u_i^t = u$ then it will exit from it at the same constant rate $v_i^t = u$ and the cell occupancy declines at a constant rate u, from $x_i^t = x_i^0$ at time $t = 0$ until the cell is empty at time $t = 1/\alpha$.
We assume that traffic that exits from the cell from time $t = 0$ onwards has been in a free-flow state since it entered it from time $t = -1/\alpha$ onwards. Note that we do not assume, and do not need to assume, that this traffic enters or exits at a constant flow rate. The flow rate can vary over time, but the free-flow assumption implies that the exit flow rate at time t (i.e., v_i^t) is exactly equal to the entry flow rate at time $t - 1/\alpha$ (i.e., $u_i^{t-1/\alpha}$), that is, $v_i^t = u_i^{t-1/\alpha}$, from time $t = 0$ until the cell is empty at time $t = 1/\alpha$.

The cell occupancy.

The cell occupancy is x_i^0 at time $t = 0$ and, since there are no further inflows to the cell, hence the cell occupancy at time t is $x_i^t = x_i^0 - \int_{t'=0}^{t} v_i^{t'}$ and, since the cell outflow rate (see above) is $v_i^t = u_i^{t-1/\alpha}$ we also have $x_i^t = x_i^0 - \int_{t'=0}^{t} u_i^{t'-1/\alpha}$, from time $t = 0$ until the cell is empty at time $t = 1/\alpha$.

(ii) From time $t = 1/\alpha$ onwards, the cell outflow rate and cell occupancy are both 0.

The CTM solution.

If we now instead apply the CTM ((4),(3)) to the same scenario as stated just before the exact solution (i)-(ii) above, and let $0 < \alpha < 1$, then the results are quite different than in the exact solution. [Note that in the "exact" solution above we assumed that the cell was in a free-flow state for some time before time $t = 0$. However, we do not need to assume that for the CTM, since the CTM solution will the same regardless of whether the cell was, or was not, in a free-flow state before time-step $t = 0$.]

If $0 < \alpha < 1$, then the CTM yields cell occupancies and cell outflow rates that continue for all time-steps $t > 0$ in a geometrically decreasing tail, as follows.

(a) Using (4) to substitute for v_i^t in (3) reduces the cell occupancy (3) to $x_i^t = x_i^{t-1} - \alpha x_i^{t-1} = (1-\alpha)x_i^{t-1}$ for time-steps $t > 0$, which, by recursive substitution, reduces to $x_i^t = (1-\alpha)^t x_i^0$ for $t > 0$. Since $0 < (1-\alpha) < 1$ this implies $x_i^t \to +0$ from above as $t \to +\infty$.

(b) The cell outflow rate $v_i^t = \alpha x_i^t$ starts from $v_i^0 = \alpha x_i^0 > 0$ in time-step $t = 0$, thereafter decreases, and $v_i^t \to +0$ from above as $t \to +\infty$.

On the other hand, if $\alpha = 1$, then

(c) the above CTM solution reduces to the correct solution (i)-(ii).

Proof:

(a) The derivation is included in the proposition above, and assumes $x_i^0 > 0$ and, for all $t \geq 0$, $u_i^t = 0$ and $v_i^t = \alpha x_i^t$ from (4).

(b) This follows immediately from (a) and $v_i^t = \alpha x_i^t$ from (4).

(c) Substituting $\alpha = 1$ into the equations in (a) reduces them to $x_i^t = 0$, that is, the traffic in the cell has all exited by the end of each time-step, which, in the assumed free-flow conditions, is also the solution in (i) and (ii). ■

Corollary. (The error of approximation in a CTM solution for free-flow traffic with $\alpha < 1$.)

Comparing the CTM solution ((4),(3)), set out in (a)-(b) above, with the correct solution set out in (i)-(ii) above we see that:

(a) After time $t = 1/\alpha$, the correct values of the cell occupancy and the cell outflow rate are zero, while in the CTM solution both of these are positive (decreasing geometrically over time) hence overestimate the outflows for all time $t > 1/\alpha$.

(b) It follows from (a) that, up to time $t = 1/\alpha$, the CTM must have underestimated the total outflow from the cell. Also, up to time $t = 1/\alpha$, the correct solution depends on when (in which time-step) the traffic entered the cell, while the CTM solution does not. ■

Time-step by which cell occupancy in the CTM is reduced by 95%. In part (a) of Proposition 1A above we saw that the cell occupancy at time-step t is $x_i^t = (1-\alpha)^t x_i^0$, so the cell occupancy in the CTM declines asymptotically to zero as $t \to +\infty$. Another way to illustrate how the free-flow speed α, $0 < \alpha \leq 1$, affects the decline of the cell occupancy is by computing, for various values of α, the number of time steps needed for the cell occupancy to decline to say 0.05 (i.e. 5%) of its initial value x_i^0 at time $t = 0$. That is, we wish to find the smallest value of t such that $x_i^t \leq 0.05 x_i^0$ or, recalling

that $x_i^t = (1-\alpha)^t x_i^0$, we wish to find the smallest value of t such that $(1-\alpha)^t \leq 0.05$. Let $t_{0.05}$ denote this value of t.
Some examples: Suppose $\alpha = 0.3$. If $t = 8$ then $(1-\alpha)^t = 0.7^8 = 0.058$ and if $t = 9$ then $(1-\alpha)^t = 0.7^9 = 0.0404$, hence $t = t_{0.05} = 9$ is the smallest value of t such that $(1-\alpha)^t \leq 0.05$. That is, it takes $t_{0.05} = 9$ time steps in the CTM for the cell occupancy to decline to 5% of its initial value x_i^0 at time $t = 0$. This contrasts with the exact solution given by (i) in the proposition, namely that it takes $1/\alpha = 1/0.3 = 3.3\dot{3}$ time steps for the cell occupancy to decline to 0% of its initial value x_i^0.
Now, suppose $\alpha = 0.7$. By similar calculations as above we see that in the CTM $\alpha = 0.7 \Rightarrow t_{0.05} = 3$. This again contrasts with the exact solution given by (i) in the proposition, namely that it takes $1/\alpha = 1/0.7 = 1.43$ time steps for the cell occupancy to decline to 0% of its initial value x_i^0.

Throughout this section, including Proposition 1A and its Corollary, we assumed that the cell is in a free-flow state, i.e., the cell occupancy is to the left of point B in Fig. 1. For completeness, in Proposition 1B below we assume instead that the cell is not in a free-flow state, i.e., the cell occupancy is to the right of point B in Fig. 1.

The flow-occupancy curve assumed in Propositions 1B and 2B below.
Assume, as usual, that the flow-occupancy curve is unimodal. Also, assume that the initial upward sloping part is a straight line from the origin, as in Fig. 1. Beyond point B in Fig 1, the flow-occupancy curve has a flat peak followed by a downward sloping part which may be a straight line or a curve. If it is a straight line then the flow-occupancy curve is said to be trapezoidal. If the flat peak in Fig. 1 is omitted and the part beyond B is a downward sloping straight line then the flow-occupancy curve is said to be triangular.
Proposition 1B and 2B are concerned with the part of the flow-occupancy curve to the right of point B and Proposition 1A and 2A are concerned with the part of the flow-occupancy to the left of point B.

Proposition 1B. Consider a scenario in which a cell is not initially in a free-flow state, i.e., the cell occupancy is to the right of point B in Fig. 1. Assume, as in Proposition 1A, that inflows to the cell have stopped but outflows continue, so that the cell occupancy is declining over time, but has not yet fallen to the upward sloping free-flow part of the flow-occupancy curve in Fig. 1. Also assume that the free-flow speed (ffs) is $\alpha < 1$.
Then the outflows from the cell are as described by the usual CTM equations (2) and (3), until the cell occupancy has fallen to point B in Fig. 1.

Proof. As the cell occupancy is beyond point B in Fig. 1 and inflows to the cell have stopped (are zero) then the cell occupancy must decline. If the time-steps are small, or arbitrarily small, then the flow-occupancy point will move back along the flow-occupancy curve, back to point B, as described by the usual CTM equations (2) and (3).
The assumption that the free-flow speed (ffs) is $\alpha < 1$ does not affect the result, since that applies only when the flows are on the free-flow part of the curve, to the left of point B, and in this proposition it is assumed that they are not on that part of the curve. ■

4.2. *Extending Section 4.1 to allowing for further inflows to the cell.*

So far in this section, including Propositions 1A and 1B, we assumed that there are no further inflows to the cell from upstream from time-step $t = 0$ onwards. In Proposition 1A we then showed that, if α is set to $0 < \alpha < 1$ for a cell and the cell is in a free-flow state with no further inflows to it, then applying the CTM yields a solution with a substantial systematic approximation error. In case it might be thought that this error may be due to the absence of further inflows to the cell, we will now consider the same scenario but let the inflows continue after time $t = 0$, with an inflow profile that varies over time-steps and may or may not taper off to zero. We see, in Proposition 2A, below that in this revised scenario the CTM continues to yield a solution with a similar substantial systematic approximation error.

Proposition 2A. Consider the same scenario as in Proposition 1A but, unlike Proposition 1A, let inflows u_i^t continue to enter the cell from time $t = 0$ to time $t = n$ inclusive.

The exact (*correct*) *solution.*

(i) Up to time to $t = n + 1/\alpha$.

The cell outflow rate. The solution is the same as in Proposition 1A, except that traffic inflows continue up until time $t = n$ hence free-flow outflow continues up until time $t = n + 1/\alpha$, rather than $t = 1/\alpha$, and the cell is empty thereafter.
The cell occupancy. The occupancy of the cell at time *t* is $x_i^t = \int_{\tau=-\infty}^{t} u_i^\tau - v_i^\tau$ and when the cell is in free-flow state, from time $t = 0$ to $t = n + 1/\alpha$, then $v_i^\tau = u_i^{\tau-1/\alpha}$, hence $x_i^t = \int_{\tau=-\infty}^{t} u_i^\tau - \int_{\tau=-\infty}^{t} u_i^{\tau-1/\alpha}$.
[E.g. if inflow is constant over time, i.e. $u_i^\tau = u_i$ from time $t = 0$ to $t = n$, then this flow will exit from the cell at the same constant rate $v_i^t = u$ and the expression for x_i^t reduces to $x_i^t = (t - (t - 1/\alpha))u_i = u_i/\alpha$, so that occupancy is constant from time $t = 0$ to $t = n$.
At time *n*, inflow ceases and occupancy then declines at a constant rate *u* from $x_i^t = u/\alpha$ at time $t = n$ to $x_i^t = 0$ at time $t = n + 1/\alpha$.]

(ii) After time $t = n + 1/\alpha$, the cell outflow rate and cell occupancy will both be 0.

The CTM solution.

If we now instead apply the CTM ((4),(3)) to the same scenario as stated just before the exact solution (i)-(ii) above, and let $0 < \alpha < 1$, then the results are quite different than in the exact solution above. The CTM yields cell occupancies and cell outflow rates that continue for all time-steps $t > 0$, as follows.
Using (4) to substitute for v_i^t in (3) reduces the cell occupancy (3) to $x_i^t = x_i^{t-1} - \alpha x_i^{t-1} + u_i^t = (1-\alpha)x_i^{t-1} + u_i^t$ up to time-step $t = n$ and $x_i^t = (1-\alpha)x_i^{t-1}$ after time-step $t = n$. Then, by recursive substitution we obtain the following.

(a) If $t \le n$ then
.
$$x_i^t = (1-\alpha)^t x_i^0 + (1-\alpha)^{t-1}u_i^1 + (1-\alpha)^{t-2}u_i^2 + (1-\alpha)^{t-3}u_i^3 + \dots\dots$$
$$\dots\dots + (1-\alpha)^2 u_i^{t-2} + (1-\alpha)^1 u_i^{t-1} + (1-\alpha)^0 u_i^t$$
$$= (1-\alpha)^t x_i^0 + \sum_{\tau=0}^{t-1}(1-\alpha)^\tau u_i^{t-\tau}. \qquad (5.1)$$

If $t > n$ then (5.1) continues to hold except that the summation in (5.1) then starts from $\tau = n$ rather than from $\tau = 0$, thus
$$x_i^t = (1-\alpha)^t x_i^0 + (1-\alpha)^{t-1}u_i^1 + (1-\alpha)^{t-2}u_i^2 + \cdots + (1-\alpha)^n u_i^{t-n}$$
$$= (1-\alpha)^t x_i^0 + \sum_{\tau=n}^{t-1}(1-\alpha)^\tau u_i^{t-\tau}. \qquad (5.2)$$

(b) From (4), the cell outflow rate is $v_i^t = \alpha x_i^t$ and the occupancy x_i^t is given by (5.1) and (5.2) above, hence the cell outflow rate is $v_i^t = \alpha x_i^t$ is obtained by simply multiplying through (5.1) and (5.2) by $\alpha,\ 0 < \alpha < 1$.

On the other hand, if $\alpha = 1$, then
(c) the above CTM solution reduces to the exact (correct) solution (i)-(ii) above.

Proof:
(a) The derivation is included in the statement of the proposition above.
(b) The derivation is included in the statement of the proposition above.
(c) When $\alpha = 1$, then (5.1) and (5.2) reduce to $x_i^t = 0$, that is, the traffic in the cell has all exited at the end of each time-step. In the assumed free-flow conditions that is also the correct solution as in (i)-(ii). ■

Equations (5.1) and (5.2) in part (a) of the above proposition show that if a cell is in a free-flow state and $0 < \alpha < 1$ then inflows to the cell (i.e. u_i^0 u_i^1, u_i^2, ….) will continue to affect (increase) the cell occupancy x_i^t in all future time-steps. The effects of these earlier inflows from upstream decrease over time because the inflows are weighted by the factors $(1-\alpha)^{t-1}$, $(1-\alpha)^{t-2}$, $(1-\alpha)^{t-3}$, etc., which are decreasing over time because $0 < \alpha < 1$. Note that this is in contrast to the situation in Proposition 1A, where the cell occupancy in any time-step *t* does not depend on inflows from any upstream cells, since in Proposition 1A those inflows were assumed to be zero from time-step $t = 0$ onwards.

The long tail of cell occupancies in (5.1)-(5.2) and in the above paragraph are an approximation error and are not present in the correct solution (i)-(ii) above. Though the CTM with free-flow traffic and $\alpha < 1$ does not yield the correct solution, there is one free-flow scenario in which it at least converges to the correct solution as $t \to +\infty$. The scenario is when inflows u_i^t are constant over time as $t \to +\infty$. In that special case $u_i^t = u_i$, which reduces (5.1) to

$$x_i^t = (1-\alpha)^t x_i^0 + u_i \sum_{\tau=0}^{t-1}(1-\alpha)^\tau. \tag{5.3}$$

If $0 < \alpha < 1$ then, as $t \to +\infty$, $(1-\alpha)^t \to 0$ and $\sum_{\tau=0}^{t-1}(1-\alpha)^\tau \to 1/\alpha$. Hence (5.3) reduces to

$$x_i^t \to u_i/\alpha \text{ as } t \to +\infty, \text{ when } 0 < \alpha < 1 \text{ and inflows are constant at } u_i \text{ over time} \tag{5.4}$$

The asymptote $x_i = u_i/\alpha$ in (5.4) is identical to the exact solution for the cell occupancy, that was obtained in part (i) of Proposition 2A for the case when inflow is constant over time as is also assumed in (5.4). That is, the CTM solution (5.4) is not the exact (correct) solution but is converging asymptotically towards it as $t \to +\infty$.

Recall that in Proposition 2A the inflows u_i^t are let continue from time-step $t = 0$ to time-step $t = n$. This includes a wide range of possible inflow profiles, since the only restriction on inflows u_i^t that we are assuming here is that these do not exceed the maximum free-flow level. The number of time-steps *n* can be made as large or small as we wish. For example, inflows may taper off gradually to zero over say *n* time-steps (from time $t = 0$ to $t = n$) or they may increase and then decrease.

Throughout Proposition 2A above we assumed that the cell is in a free-flow state, i.e., the cell occupancy is to the left of point B in Fig.1. For completeness, in Proposition 2B below we assume instead that the cell is not in a free-flow state, i.e., the cell occupancy is initially to the right of point B in Fig.1.

Proposition 2B. This proposition is the same as Proposition 1B except that, instead of assuming that inflows to the cell stop at time $t = 0$ as in Propositions 1A and 1B, assume here that they continue until time $t = n$ as in Proposition 2A.

Proof. The proof is the same as for Proposition 1B since, letting the inflows continue for a time, from time $t = 0$ up to $t = n$ does not affect the proposition or proof. ■

4.3. *Refining the discretisation of time and space, in Sections 4.1 and 4.2 above, towards their continuous limits*.

An obvious question that arises is: can the approximation errors that arise when using the CTM with free-flow traffic and $\alpha < 1$, as set out in Propositions 1A and 2A, be reduced or eliminated by refining the discretisation of time and space towards their continuous limit? Unfortunately, the answer is no, as we see in the following proposition which considers the same scenario as in Proposition 1A and its corollary above, while refining the discretisation in the CTM to its continuous limit, holding the discretisation ratio ε/d and the traffic speed *s* fixed. The exact (correct) solution remains the same as in (i) and (ii) in Proposition 1A.

Proposition 3. Consider a cell with a positive initial occupancy at the start of time-step $t = 0$, in which traffic is travelling at free-flow speed α, $0 < \alpha < 1$, measured in cells per time-step, or equivalently, travelling at free-flow speed s measured in natural units (e.g. metres per second) where α and s are related by a fixed discretisation ratio ε/d as in $\alpha = s(\varepsilon/d)$ from (1). Let there be no further inflow to the cell after time-step $t = 0$ and no restrictions on the outflow from the cell.

The CTM solution. If we instead apply the CTM ((4),(3)) to the above scenario and assume $\alpha < 1$, then the results are quite different than in (i)-(ii) above. The CTM yields cell occupancies and cell outflow rates that continue for all time $t > 0$ in a geometrically decreasing tail, as follows.
Let the discretisation of time and space in the CTM equations ((4),(3)) be refined to the continuous limit, while holding the discretisation ratio ε/d constant. Then, in the resulting continuous-time solution:

(a) the cell density $k(t)$ and outflow rate $v(t)$ decrease exponentially over time, i.e. $k(t) = k(0)e^{-\alpha t}$ and $v(t) = v(0)e^{-\alpha t}$ where $k(0)$ is the cell density at time $t = 0$, and
(b) since the discretisation ratio ε/d is held fixed, so that $\alpha = s(\varepsilon/d)$, then this also yields $k(t) = k(0)e^{-s(\varepsilon/d)t}$ and $v(t) = v(0)e^{-s(\varepsilon/d)t}$, where $v(0)$ is the cell outflow rate in time-step $t = 0$.

Proof. (a). As there is no further inflow to the cell, (3) reduces to $x^{t+1} = x^t - v^t$ and, from (4), in free-flow conditions $v^t = \alpha x^t$ hence $x^{t+1} = x^t - \alpha x^t$. If $\alpha = 1$ this reduces to $x^{t+1} = 0$ hence we will consider only $\alpha < 1$. Equation $x^{t+1} = x^t - \alpha x^t$ can be rewritten as $x^{t+1} - x^t = -\alpha x^t$. By definition, $x^t = k^t d$ where k^t is the cell density and d is the cell length. Substituting this into $x^{t+1} - x^t = -\alpha x^t$, d cancels, leaving $k^{t+1} - k^t = -\alpha k^t$. It is now convenient to restate the time-step length as a time interval Δt so that the change in density (i.e. $k^{t+1} - k^t$) can be written as $k(t + \Delta t) - k(t)$ = $\Delta k(t)$. The traffic speed α in cells per time-step can then be re-written as $(\Delta t)\alpha$ cells per time interval Δt. With these changes of notation, the above CTM conservation equation $k^{t+1} - k^t = -\alpha k^t$ can be re-written as $\Delta k(t)$ = $(\Delta t)\alpha k(t)$ hence $\Delta k(t)/\Delta t = \alpha k(t)$. In the continuous time limit as $\Delta t \to 0$ this goes to $dk(t)/dt = -\alpha k(t)$. Solving this gives density $k(t) = k(0)e^{-\alpha t}$ at time t at the current location. (Recall that this location is assumed to be a point at which a road segment is in free-flow state but with no further inflows to it.)

Also, recall from (4) that $v^t = \alpha x^t$, and $x^t = k^t d$ by definition, hence $v^t = \alpha k^t d$ and $k(t) = v(t)/\alpha d$. Substituting this for $k(t)$ in $\Delta k(t)/\Delta t = \alpha k(t)$ above gives $\Delta v(t)/\Delta t = \alpha v(t)$. In the continuous time limit as $\Delta t \to 0$ this goes to $dv(t)/dt = -\alpha v(t)$ and solving this gives the outflow rate $v(t) = v(0)e^{-\alpha t}$ at time t at the current location.

(b). The above continuous time equations for $k(t)$ and $v(t)$ use the free-flow speed α in cells per time-step but this can be converted to speed in natural units (e.g. metres per second) by using $\alpha = s(\varepsilon/d)$ from (1). Making this substitution gives $k(t) = k(0)e^{-s(\varepsilon/d)t}$ and $v(t) = v(0)e^{-s(\varepsilon/d)t}$. ■

Corollary. The cell density $k(t) = k(0)e^{-\alpha t}$ is $k(t) = k(0)$ when $t = 0$ and $k(t) \to +0$ as $t \to +\infty$. Similarly, the cell outflow $v(t) = v(0)e^{-\alpha t}$ is $v(t) = v(0)$ when $t = 0$ and $v(t) \to +0$ as $t \to +\infty$. ■

The paragraph that appears just after Proposition 1A and its corollary also applies here, to Proposition 3 and its corollary. That is, in the CTM solution set out in (a) and (b) above, the cell occupancy at time-step t depends only on the initial occupancy of the cell, and does not depend on occupancies or inflows from any upstream cells, because we have assumed that there are no further inflows to the cell from upstream.

In the above proposition the traffic free-flow speed is assumed to be α, in cells per time-step. In that case, the 'correct' number of time-steps taken to traverse a cell should be $t = 1/\alpha$, hence all traffic in the cell should have exited by time $t = 1/\alpha$. However, it is clear from the above proposition and corollary that this correct solution is very different from the CTM solution, if $\alpha < 1$, even if the discretisation in the CTM is refined to its continuous limit.

5. Modifying the CTM: using CTM for congested cells and simple (non-CTM) travel times for cells in a free-flow state with free-flow speed $\alpha < 1$

5.1 Modifying the CTM for a *single* cell.

As seen above, the approximation error that arises when using the CTM for free-flow traffic with free-flow speed $\alpha < 1$, can not be eliminated by refining the discretisation, while holding the discretisation ratio ε/d fixed. However, it is possible to modify the CTM to reduce or eliminate this problem, as set out below.

In the previous section we considered a very simple free-flow scenario with a flow-occupancy function that has an initial linear free-flow segment, in which case the CTM flow equation (2) reduces to (4), i.e. $v_i^t = \alpha x_i^t$. We showed that if the free-flow speed is $\alpha < 1$ then the CTM (4)-(3) yields a poor approximation to the cell outflow rate and to the remaining occupancy of the cell. In the present section we propose modifying the CTM for this scenario, so as to eliminate the above approximation error.

Basically, for the scenarios considered in this paper (free-flow traffic with free-flow speeds $\alpha < 1$), we propose simply replacing the CTM equations with the corrected outflows and occupancies for the cell, as set out in (i)-(ii) in Propositions 1A and 2A, instead of using the values provided by the CTM. If the cell fftt, $1/\alpha$, happens to be an integer, then computing the correct outflows and occupancies for the cell, as set out in (i)-(ii) in Propositions 1A and 2A, is fairly simple. If the cell fftt $1/\alpha$ is not an integer, then the traffic that enters the cell in say time-step *t* will exit from the cell spread over time-steps *t* +rounddown($1/\alpha$) and *t* + roundup($1/\alpha$), where rounddown(cfftt) and roundup(cfftt) denote cfftt rounded down or up to the nearest integer.

To simplify the presentation here, we will assume that all traffic in the cell *i* in time-step *t* has been travelling at free-flow speed since it entered the cell at time *t*–cfftt. If cfftt is not integer, we will assume that the traffic in cell *i* has been in a free-flow state since the beginning of the time-step in which *t*–cfftt lies, i.e., for time-steps *t*–roundup(cfftt) to *t* inclusive. We refer to this below as Assumption **A1**.

To develop a method for computing the correct outflows and occupancies for the cell we start with the following simple illustrative numerical example and then generalise from that. If the free-flow speed is say $\alpha = 0.8$ this means that it takes exactly 1/0.8 = 1.25 time-steps to traverse the cell. Hence traffic that exits from cell *i* in time-step t_i must have entered the cell between time $t_i - 1.25$ and time $(t_i - 1) - 1.25 = t_i - 2.25$, where t_i is the number of time-steps up to the *end* of time-step t_i and $t_i - 1$ is the number of time-steps up to the end of time-step $t_i - 1$, which is also the *beginning* of time-step t_i.

Time $t_i - 1.25$ is 0.25 of a time-step before the beginning of time-step t_i hence is 0.75 of a time-step through time-step $t_i - 1$. Similarly, time $t_i - 2.25$ is 1.25 time-steps before the beginning of time-step t_i hence is 0.25 time-steps before the end of time-step $t_i - 2$. Hence, in free-flow conditions, the outflow from the cell *i* in time-step t_i is the inflow to cell *i* in the above time interval (from 0.25 time-steps before the end of time-step $t_i - 2$ to 0.75 of a time-step through time-step $t_i - 1$). That is,

$$\text{if } \alpha = 0.8 \text{ then } v_i^t = 0.75\, u_i^{t-1} + 0.25\, u_i^{t-2}. \tag{6}$$

We can generalise from the above example as follows. For the second term on the right-hand side of (6), the coefficient 0.25 = frac($1/\alpha$) where frac($1/\alpha$) is the fractional part of $1/\alpha$ and, for the first term, the coefficient is 0.75 = ($1 -$ frac($1/\alpha$)). The time indices on u_i^{t-1} and u_i^{t-2} are $t_i - int(1/\alpha)$

and $t_i - int(1/\alpha) - 1$ respectively, where $int(1/\alpha)$ is the integer part of $1/\alpha$ hence (6) can be generalised to

$$v_i^t = (1 - \text{frac}(1/\alpha))u_i^{t-int(1/\alpha)} + \text{frac}(1/\alpha)u_i^{t-int(1/\alpha)-1}. \tag{7}$$

Thus, when Assumption **A1** holds, we can replace (4) with (7). This can also be stated as: set u_{i+1}^t equal to the traffic estimated, by linear interpolation, to have entered cell *i* in the time interval from $(t_i - 1) - 1/\alpha$ to $t_i - 1/\alpha$. Note that if $\alpha = 1$ then (7) reduces to $v_i^t = u_i^{t-1}$, and if the free-flow inflows are constant over time then this also implies $v_i^t = u_i^t$.

5.2 Modifying the CTM for *a network* with some cells in a free-flow state.

5.2.1 The modified CTM for a one-pass network loading.

Daganzo (1995a, 1995b) sets out the CTM ((2)-(3)) as a one-pass process, rolling forward over time, one time-step at a time, using the cell occupancies in each time step to compute the flows from cell-to-cell in that time step and hence the cell occupancies in the next time step, and so on for all time steps. In Propositions 1A and 2A above we revised that process, for cells in which the cell occupancies correspond to the upward sloping part of the flow-occupancy curve. For such cells, instead of computing cell flows using the CTM equations (2)-(3), we assume a simple constant free-flow rate.

It might be wondered if there may be a problem in determining in advance whether a cell will be in a free-flow state. Fortunately, that is not a problem in the CTM or in the proposed modified CTM. The network loading proceeds is a one-pass process, rolling forward one time-step at a time, regardless of whether some or all cells are in free-flow state or are in a congested state described by the CTM. At each time step, the cell occupancies are already known, having been updated from the previous time step by using eq. (3). Since the cell occupancy is known, we immediately know whether the corresponding cell outflow is on the free-flow (upward-sloping) part of the flow-occupancy curve. Hence, when considering each cell in each time step, we also immediately know which of the two methods, stated in the preceding paragraph, to use to compute the cell outflow, i.e., to use the CTM or constant free-flow.

5.2.2 Using the CTM to obtain a user equilibrium.

In the CTM, in each time step the inflows to each *spatial* path linking each origin-destination (OD) pair are taken as given. The CTM then loads these given spatial path flows onto the network *over time*, which is achieved in a one-pass process. Various authors (e.g., Lo (1999), Carey and Ge (2012)) have investigated and demonstrated how to obtain a user equilibrium by repeated use the above one-pass CTM. A general method for achieving this (achieving a user equilibrium) can be outlined as follows. Load initial OD path flows onto the network as outlined above. Then, for each OD pair, compute and compare the travel times on the various paths between the OD pair. These travel times are likely to be different on each path between an OD pair. In that case, for each OD pair, take some of the inflow that was allocated to paths that yielded higher travel times and reallocate it to paths that yielded lower travel times. Then repeat the CTM network loading using these new allocations of path inflows and again compute and compare the new path travel times, and so on.

Keep repeating this process until, for each OD pair, the travel times have converged sufficiently to satisfy a criterion for user equilibrium. Lo (1999) and Carey and Ge (2012) investigated several different schemes for such reallocating of traffic flows between spatial paths. Carey and Ge (2012) found that, while all of the reallocation schemes that they considered, eventually converged to a user equilibrium, the speed of convergence was strongly affected by the which reallocation scheme was used. For example, perhaps not surprisingly, schemes that performed better were those in which the amount reallocated between spatial paths was decreased when the difference in the path travel times reduced.

The computational schemes referred to above were all designed, developed and tested in scenarios in which the free-flow speeds α were assumed to be $\alpha = 1$ for all cells, while here we are also considering scenarios with $\alpha < 1$ for some cells. Whether the introduction of cells with free-flow speeds $\alpha < 1$ may affect the performance and speed of convergence of the above algorithms, for reallocating flows on spatial paths for OD pairs, is a topic still to be explored.

6. Concluding remarks

The contribution of this paper has been summarised in the abstract and in the introduction. In the original papers, and later papers, on the CTM, it is assumed that the free-flow speed in each cell in each time step is $\alpha = 1$ cell per time step (CPTS). Section 2 sets out several scenarios in which the free-flow speed will be $\alpha < 1$ CPTS. In general, the reason for this is that the free-flow speed α may vary across the network or over time hence it can not have the same value ($\alpha = 1$) everywhere. As the maximum value of α is 1, it follows that for some cells in some time steps it must be that $\alpha < 1$. For these scenarios (i.e., when some cells or links are in a free-flow state with free-flow speeds $\alpha < 1$ CPTS), Section 4 in the paper, Propositions 1A and 2A,

(i) shows that the CTM is likely to be an unnecessarily inaccurate method for approximating traffic flows, occupancies and travel times in cells and links,

(ii) shows that more accurate estimates of these variables are obtained by replacing the CTM equations with simple free-flow travel times, and

(iii) by comparing (i) and (ii), derives measures of the inaccuracies involved in using the CTM to model traffic in a free-flow state with free-flow speeds $\alpha < 1$ CPTS.

The free-flow state considered above refers to the situation in which the flows and occupancies are on the initial upward sloping straight line part of the flow-occupancy curve, as in Fig. 1. In Propositions 1B and 2B we note that when the cell occupancies are beyond that part of the flow-occupancy curve, and are on the flat top of the flow-occupancy curve or on the downward sloping part, as in Fig. 1, then the flows, occupancies and travel times continue to be described by the usual CTM equations from Section 3. The free-flow speeds $\alpha < 1$ CPTS affect the results only for the upward sloping free-flow part of the flow-occupancy curve.
Section 4.3 considers whether the above inaccuracy in the CTM solutions, when $\alpha < 1$, are caused by the discretisation of time and space in the CTM, and shows that even if the discretisation is refined to its continuous limit, the CTM still does not yield the correct the solution, if $\alpha < 1$.

Section 5 generalises the above results by considering a network in which some cells are in a free-flow state with free-flow speeds $\alpha < 1$ CPTS and the remaining cells are congested, hence not in a free-flow state.

The results in this paper also apply to revised or extended versions of the CTM introduced or proposed by Daganzo and others. For example, Daganzo (1999) introduced a lagged CTM (LCTM) by introducing lags in the "receiving capacity" term in the CTM flow equation. Also, Daganzo (1994) Section 5.3, using a trapezoidal flow-occupancy equation in the CTM, proposed modifying the "receiving capacity" term to avoid spreading of shock waves. He proposed replacing a w/v coefficient with 1 if cell occupancy) ≤ (cell capacity) and leaving it as w/v if cell occupancy) > (cell capacity). The results in this paper continue to hold for both of these extensions since they affect only the formulation of the "receiving capacity" term in the CTM flow function, which does not enter into the free-flow component of the CTM or into any of the arguments in this paper.

Acknowledgement: We wish to thank the reviewers and associate editor for their helpful comments and queries which have motivated significant extensions and improvements in the paper. I would also like to thank Professor David Watling with whom I have discussed this project and paper over the years.

Appendix 1. Some terms or acronyms used in this paper.

The following terms are defined when they were first used in this paper, but for reference we also state their definitions here.

ffs denotes the free-flow speed α of traffic in a cell, that is, the number of cells traversed per time step (cpts). By construction, $\alpha \leq 1$. In the standard presentation of the CTM, in Daganzo (1994, 1995a), etc., the cell lengths are chosen so that the ffs α is exactly $\alpha = 1$. In this paper we also consider $0 < \alpha < 1$.

cfftt denotes the cell free-flow travel time, that is, the time (measured in time steps) needed to traverse a cell when it is in a free-flow state. cfftt = 1/ffs = $1/\alpha$.

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