Analysis of the implicit Euler time-discretization of a class of descriptor-variable linear cone complementarity systems

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Abstract

This article is largely concerned with the time-discretization of a class of descriptor-variable systems coupled with complementarity constraints, named descriptor-variable linear complementarity systems (DVLCS). Specifically, the Euler implicit discretization of DVLCS is analysed: the one-step non-smooth problem, which is a generalized equation, is shown to be well-posed under some conditions. Then the convergence of the discretized solutions is studied, and the existence of solutions to the continuous-time system is shown as a consequence. Several circuits examples illustrate the applicability and the theoretical developments.

1 Introduction

The analysis of nonsmooth dynamical systems with set-valued right-hand sides satisfying maximal monotone properties has witnessed a very large number of contributions, *e.g.*, [15, 38, 39, 4, 5, 25, 26, 24, 33, 36, 43, 49, 54, 56, 17, 21, 18, 41, 55]. Within this class one finds linear and nonlinear complementarity dynamical systems, projected systems, differential variational inequalities, differential inclusions with maximal monotone right-hand sides, Moreau's sweeping process, some switching dynamical systems, *etc*, see [20] for details and references. In this article we focus on descriptor-variable linear complementarity systems (DVLCS), which may be viewed either as an extension of descriptor-variable systems, or of linear complementarity systems, or of differential-algebraic equations (DAEs). Generally speaking, systems with nonsmooth constraints arise in chemistry [53, 52, 51], switching DAEs analysis [47, 46], circuits with nonsmooth electronic components [1, section 3.5] [24], mechanical systems with bilateral and unilateral constraints [16].
It has to be noted that the study of such singular nonsmooth dynamical systems has not received a lot of attention yet. The first goal of this article is to analyze the well-posedness of the one-step nonsmooth problem (OSNSP) obtained after an implicit Euler discretization. The convergence of the piecewise-linear approximated discrete-time solutions towards a continuous-time limit which is a solution to the continuous-time differential inclusion, is tackled also. Implicit Euler method has been widely used for DAEs, see, *e.g.*, [35, 40], differential inclusions like Moreau's sweeping process [41] and maximal monotone DIs [6, 9, 45], see [20, section 5] for more references. But its

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application to DVLCS remains an open issue. Passivity and maximal monotonicity properties are pivotal throughout the article.
The class of singular systems and their discretization are introduced in section 2. In section 3 the so-called one-step nonsmooth problem of the implicit Euler scheme is analysed. Section 4 is dedicated to the convergence analysis of piecewise-linear discrete-time solutions. Section 5 is devoted to relax a basic assumption made in section 3. Several examples illustrate the theoretical developments, in section 6. Section 7 outlines a different proof for the convergence of discretized solutions. Conclusions are drawn in section 8. The appendix is dedicated to recall various mathematical tools and to present some proofs.

Notation and definitions: for any vector $x \in I\!R^n$ and any matrix $M \in I\!R^{m\times n}$, $||x||$ is the Euclidean norm and $||M||$ is the Frobenius norm, which are compatible norms [11, Proposition 9.3.5], *i.e.*, $||Mx|| \leq ||M||\ ||x||$. Let $M \in I\!R^{n\times m}$, then $\mathrm{Im}(M)$ is its range, $\mathrm{Ker}(M)$ is its null space. The Moore-Penrose pseudo-inverse is denoted $M^{\dagger}$. We use $\langle x, y\rangle = x^\top y$, so $\langle x, x\rangle = ||x||^2$. Positive definite matrix: $M \succ 0$ if $x^\top Mx > 0$ for all $x \neq 0$, positive semidefinite matrix: $M \succcurlyeq 0$ if $x^\top Mx \geq 0$ for all x (such M is not necessarily symmetric). The maximum singular value is denoted as $\sigma_{\max}(M)$, and the minimum and maximum eigenvalues as $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$. Given $M \in I\!R^{n\times m}$, $M_{i\bullet} \in I\!R^m$ denotes the ith row of M, $M_{\bullet i} \in I\!R^n$ denotes the ith column of M. A set-valued mapping $A : \mathrm{dom}(A) \subseteq I\!R^n \rightrightarrows I\!R^n$ is said monotone if for all $x_1,\ x_2 \in \mathrm{dom}(A)$, $y_1 \in A(x_1)$, $y_2 \in A(x_2)$, one has $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. It is maximal monotone if its graph cannot be enlarged without destroying monotonicity. More definitions are given in Appendices B and C.

2 Descriptor-variable linear cone-complementarity systems

Let us consider the following system, that we may name a descriptor-variable linear cone complementarity system (DVLCS):

$$\begin{cases} P\dot{x}(t) = Ax(t) + B\lambda(t) + E(t), & \text{a.e. } t \geq 0 \\ K \ni \lambda(t) \perp w(t) = Cx(t) + D\lambda(t) + F(t) \in K^\star, & \text{for all } t \geq 0 \end{cases} \tag{1}$$

with $x(t) \in I\!R^n$, $\lambda(t) \in I\!R^m$, A, B, C, D constant matrices of appropriate dimensions, $P \in I\!R^{n\times n}$ has rank $p < n$, $K \subseteq I\!R^m$ is a closed convex nonempty cone, $K^\star$ is its dual cone, $E, F : I\!R_+ \to I\!R$ are functions of time. In the sequel it is assumed always that:

Assumption 1 *There exists $E_{\max} < +\infty$ and $F_{\max} < +\infty$ such that $\|E(t)\| \leq E_{\max}$ and $\|F(t)\| \leq F_{\max}$ for all t.*

Our first objective is the analysis of the well-posedness of the Euler discretization of such nonsmooth systems, in particular under which conditions the OSNSP possesses a solution. This class of singular systems are an extension of the Lur'e set-valued systems analysed, *e.g.*, in [38, 39].

3 Time-discretization with an implicit Euler method

Let us consider the time interval $[0, T]$, $T > 0$, $h = \frac{T}{n}$, $n \in I\!N$, $n > 0$, $t_k = kh$, $k \geq 0$, $t_0 = 0$, $t_n = T$. The implicit Euler discretization of (1) reads as:

$$\begin{cases} Px_{k+1} - Px_k = hAx_{k+1} + hB\lambda_{k+1} + hE_k \\ K \ni \lambda_{k+1} \perp w_{k+1} = Cx_{k+1} + D\lambda_{k+1} + F_k \in K^\star. \end{cases} \tag{2}$$

This is called implicit because w_{k+1} is computed with x_{k+1}. Let $P^h \triangleq P - hA$, (2) is rewritten equivalently as:

$$\begin{cases} P^h x_{k+1} - P x_k - hB\lambda_{k+1} - hE_k = 0 \\ K \ni \lambda_{k+1} \perp w_{k+1} = Cx_{k+1} + D\lambda_{k+1} + F_k \in K^\star. \end{cases} \tag{3}$$

This forms a kind of mixed linear complementarity problem (MLCP). If instead an explicit discretization of the linear terms is chosen, *i.e.*, hAx_k, then (3) is changed to, with $P^h \triangleq P + hA$:

$$\begin{cases} P x_{k+1} - P^h x_k - hB\lambda_{k+1} - hE_k = 0 \\ K \ni \lambda_{k+1} \perp w_{k+1} = Cx_{k+1} + D\lambda_{k+1} + F_k \in K^\star. \end{cases} \tag{4}$$

3.1 Case $D = 0$

Let us consider $D = 0$ in (3) (or in (4)). In this case we have

$$\lambda_{k+1} \in -\mathcal{N}_{K^\star}\left(Cx_{k+1} + F_k\right), \tag{5}$$

from which the generalized equation (GE) with unknowns x_{k+1} and z_{k+1}

$$P^h x_{k+1} - P x_k - hE_k \in -hB\, \mathcal{N}_{K^\star}\left(Cx_{k+1} + F_k\right) \tag{6}$$

follows using (3). This GE is not in the standard form [29] because of both P^h, which may be singular, and of B and C, which destroy in general the monotonicity of the set-valued part. It is neither in the canonical form used in [3]. Let us make the following passivity-like assumption:

Assumption 2 *There exists $X = X^\top \succ 0$ such that $XB = C^\top$.*

Mimicking [15] and many articles afterwards (see references in [20, Section 3.4]), let us define $R = R^\top \succ 0$, $R^2 = X$, and the variable change $\xi = Rx$. This allows us to transform (6) into the equivalent GE:

$$RP^h R^{-1}\xi_{k+1} - RPR^{-1}\xi_k - hRE_k \in -hR^{-1}C^\top\, \mathcal{N}_{K^\star}(CR^{-1}\xi_{k+1} + F_k). \tag{7}$$

Let us define $f(\cdot, t) = \psi_{K^\star - F(t)} \circ CR^{-1}(\cdot)$, then this is equivalently rewritten with $f_k(\cdot) \triangleq f(\cdot, t_k) = \psi_{K^\star - F_k} \circ CR^{-1}(\cdot)$ as:

$$RP^h R^{-1}\xi_{k+1} - RPR^{-1}\xi_k - hRE_k \in -\partial f_k(\xi_{k+1}) \Leftrightarrow \xi_{k+1} \in (RP^h R^{-1} + \partial f_k)^{-1}(RPR^{-1}\xi_k + hRE_k), \tag{8}$$

where $h > 0$ pre-multiplying the right-hand side in (7) disappears since it multiplies a cone. It is also assumed that there exists ξ_0 such that $CR^{-1}\xi_0 + F_k \in K^\star$ to apply the chain rule of Convex Analysis [48, Theorem 23.9]. The function $f_k(\cdot)$ is proper convex lower semicontinuous for each $k \geq 0$. Again for each $k \geq 0$, the GE in (8) can be rewritten equivalently as the variational inequality of the second kind VI($RP^h R^{-1}, q_k, f_k$): Find $\xi_{k+1} \in \mathrm{dom}(f_k)$ such that

$$\langle RP^h R^{-1}\xi_{k+1} - RPR^{-1}\xi_k - hRE_k, v - \xi_{k+1}\rangle + f_k(v) - f_k(\xi_{k+1}) \geq 0, \quad \text{for all } v \in \mathrm{dom}(f_k), \tag{9}$$

where $\mathrm{dom}(f_k) = \{w \in I\!R^n \mid CR^{-1}w \in K^\star - F_k\}$ and $q_k \triangleq -RPR^{-1}\xi_k - hRE_k$. The GE in (9) can be analysed using the tools in [3]. Noticing that $f_k(\cdot) = \psi_{\tilde{\Gamma}_k}(\cdot)$ with $\tilde{\Gamma}_k = \{\xi \in I\!R^n \mid CR^{-1}\xi \in K^\star - F_k\}$, one sees that (8) is also an inclusion into a normal cone.

Proposition 1 *Suppose that $\tilde{\Gamma}_k$ is nonempty compact, then the set of solutions ξ_{k+1} to the GE in (8) is nonempty and compact.*

The proof follows from [29, Corollary 2.2.5], since $\tilde{\Gamma}_k$ is convex. Going further necessitates that P^h possesses some positivity property. It is noteworthy that due to its structure, there is no reason that P^h be a symmetric matrix in general. Let us make the following assumption.

Assumption 3 *For all $h > 0$: $RP^hR^{-1} \succcurlyeq 0$.*

Let us now apply [3, Corollary 3] to analyse the GE, without the compactness assumption of Proposition 1.

Proposition 2 *Let Assumptions 2 and 3 hold, $h > 0$ and $k \geq 0$ be given. Assume that $K^\star = \{\xi \in \mathbb{R}^m \mid G\xi \geq 0\}$ for some $G \in \mathbb{R}^{m\times m}$. Let us consider the set:*

$$\begin{aligned}\mathcal{S} \triangleq\ & \underbrace{\{\xi \in \mathbb{R}^n \mid GCR^{-1}\xi \geq 0\}}_{\triangleq \mathcal{S}_1} \cap \{\xi \in \mathbb{R}^n \mid RP^hR^{-1}\xi = \textstyle\sum_{i=1}^m \lambda_i (GCR^{-1})_{i\bullet}^\top\ ,\ \lambda_i \geq 0\} \\ & \cap \operatorname{Ker}(RP^hR^{-1} + R^{-1}(P^h)^\top R).\end{aligned} \tag{10}$$

1. *If $\mathcal{S} = \{0\}$, then the VI(RP^hR^{-1}, q_k, f_k) has at least one solution.*
2. *If $\mathcal{S} \neq \{0\}$, and if there exists $\xi_0 \in \tilde{\Gamma}_k$ such that*

 $$\langle q_k - (RP^hR^{-1})^\top \xi_0, v\rangle > 0 \text{ for all } v \in \mathcal{S},\ v \neq 0, \tag{11}$$

 then the VI(RP^hR^{-1}, q_k, f_k) has at least one solution.
3. *If ξ_{k+1}^1 and ξ_{k+1}^2 are two solutions of the VI(RP^hR^{-1}, q_k, f_k), then $\xi_{k+1}^1 - \xi_{k+1}^2 \in \operatorname{Ker}(RP^hR^{-1} + R^{-1}(P^h)^\top R)$.*
4. *Let RP^hR^{-1} be symmetric. If ξ_{k+1}^1 and ξ_{k+1}^2 are two solutions of the VI(RP^hR^{-1}, q_k, f_k), then $\langle q_k, \xi_{k+1}^1 - \xi_{k+1}^2\rangle = 0$.*
5. *Let RP^hR^{-1} be symmetric. Then any solution of the VI(RP^hR^{-1}, q_k, f_k) is also a solution of the optimization problem:*

 $$\min_{\xi \in \tilde{\Gamma}_k} \frac{1}{2}\xi^\top RP^hR^{-1}\xi + \langle q_k, \xi\rangle. \tag{12}$$

Proof: The proof relies on [3, Corollaries 3 and 4], see Proposition 10 in Appendix B, noting that $f_k(\cdot) = \psi_{\tilde{\Gamma}_k}(\cdot)$ is proper, convex and lower semicontinuous, while $RP^hR^{-1} \succcurlyeq 0$ (this plays the role of the matrix $\mathbf{M}$ in Proposition 10). First one calculates $(\operatorname{dom}(f_k))_\infty$ (the recession cone of $\operatorname{dom}(f_k)$) as $(\operatorname{dom}(f_k))_\infty = (\operatorname{dom}(\psi_{\tilde{\Gamma}_k}))_\infty = (\tilde{\Gamma}_k)_\infty = \{\xi \in \mathbb{R}^n \mid GCR^{-1}\xi \geq 0\} = \mathcal{S}_1$ (applying the rule of calculation of recession cones for polyhedral sets, see [33, Example 17] [48, p.62], and Proposition 9 item e) in Appendix B). The second step is to calculate $(\operatorname{dom}((f_k)_\infty))^\star$, which is the dual cone of $\operatorname{dom}((f_k)_\infty)$, where $(f_k)_\infty(\cdot)$ is the recession function of $f_k(\cdot)$. Indeed Proposition 10 involves the set $\mathcal{K}(RP^hR^{-1}, f_k) = \{\xi \in \mathbb{R}^n \mid RP^hR^{-1}\xi \in (\operatorname{dom}((f_k)_\infty))^\star\}$ in (63). Due to the fact that $f_k(\cdot) = \psi_{\tilde{\Gamma}_k}(\cdot)$ and $\tilde{\Gamma}_k$ is closed convex nonempty, one has $(f_k)_\infty(\cdot) = \psi_{(\tilde{\Gamma}_k)_\infty}(\cdot)$ hence $\operatorname{dom}((f_k)_\infty) = (\tilde{\Gamma}_k)_\infty$. One has $(\tilde{\Gamma}_k)_\infty = \{\xi \in \mathbb{R}^n \mid GCR^{-1}\xi \geq 0\}$, so that

$((\tilde{\Gamma}_k)_\infty)^\star = \{w \in I\!\!R^n \mid w = \sum_{i=1}^m \lambda_i (GCR^{-1})_{i\bullet}^\top, \lambda_i \geq 0\}$ [33, Example 15]. The third set is equal to $\mathrm{Ker}(RP^hR^{-1} + R(P^h)^\top R^{-1})$ directly from Proposition 10 (a). Then items 1 and 2 are a direct consequence of (a) and (b) in Proposition 10: note that $\varphi(\cdot) = \psi_{\tilde{\Gamma}_k}(\cdot)$, hence $\varphi_\infty(\cdot) = \psi_{(\tilde{\Gamma}_k)_\infty}(\cdot)$, and it follows that $\psi_{(\tilde{\Gamma}_k)_\infty}(v) = 0$ for all $v \in \mathcal{S}_1 \subset \mathcal{S}$ (see (64) in Proposition 10). Item 3 is a consequence of (c) in Proposition 10. Item 4 stems from item d) in Proposition 10, since $f_k(\cdot) = \psi_{\tilde{\Gamma}_k}(\cdot)$ (and $\xi_{k+1} \in \tilde{\Gamma}_k$). Item 5 follows from item f) in Proposition 10. ⊠

Remark 1 *As alluded to above, if one uses instead the scheme in (4), then Proposition 2 applies, replacing VI(RP^hR^{-1}, q_k, f_k) by VI(RPR^{-1}, q_k, f_k) and $q_k = -RP^hR^{-1}\xi_k - hRE_k$, $P^h = P + hA$, and modiying Assumption 3 accordingly.*

If a solution exists, then one recovers from item 5 that if $\tilde{\Gamma}_k$ is compact then the solution is uniformly bounded, as in Proposition 1. Now we have that $P^h = P - hA$. Consider (12) and replace P^h by P. It is interesting to notice that $\xi^\top RPR^{-1}\xi = \xi^\top R^{-1}R^2Px = x^\top XPx$. Consequently the fundamental matrix for the studied class of DVLCS is $XP \succcurlyeq 0$. In case of LCS with full rank P one recovers the fundamental matrix X for the storage function.

3.2 Case $D \succcurlyeq 0$

Let us adapt the ideas developed in [54] to our purpose. Indeed due to the presence of a nonzero D in the complementarity condition, a straightforward extension of the material in section 3.1 is not possible. One has to use a particular decomposition so as to recover the method used in section 3.1.

Assumption 4 *There exists $X = X^\top \succ 0$ such that $\mathrm{Ker}(D) \subseteq \mathrm{Ker}(XB - C^\top)$, and $D = D^\top \succcurlyeq 0$.*

This is satisfied by passive quadruples (A, B, C, D) with symmetric D [19, Proposition 3.62] [25, 23], and it extends Assumption 2. Similarly as in section 3.1, let us define $R = R^\top \succ 0$, $R^2 = X$, with X such that Assumption 4 holds. Following [54, Equations (13)-(15)], let us rewrite (1) as:

$$P\dot{x}(t) \in Ax(t) - B\Phi\,(t, Cx(t)) + E(t), \tag{13}$$

with $\Phi\,(t, \zeta) \stackrel{\Delta}{=} (\partial\sigma_{\Gamma(t)} + D)^{-1}(\zeta)$, $\Gamma(t) = K^\star - F(t)$ is closed and convex for each $t \geq 0$. Let us now state two assumptions.

Assumption 5 *For all $t \geq 0$, one has $\mathrm{Im}(C) \cap \mathrm{rint}(\mathrm{Im}(\partial\sigma_{\Gamma(t)} + D)) \neq \emptyset$.*

Assumption 6 *For all $t \geq 0$ and each $v \in \mathrm{Im}(C) \cap \mathrm{Im}(\partial\sigma_{\Gamma(t)} + D)$, it holds that $\mathrm{Im}(D + D^\top) \cap (\partial\sigma_{\Gamma(t)} + D)^{-1}(v) \neq \emptyset$.*

Remind that since $D \succcurlyeq 0$ then $\mathrm{Im}(D + D^\top) = \mathrm{Im}(D) = \mathrm{Im}(D^\top)$ [30] (D needs not be symmetric for this). Following the developments in [54, section 3.2] where the meaning of the assumptions is explained, one obtains the next result.

Proposition 3 *Let Assumptions 4, 5, 6 hold. Let $\xi = Rx$. Then the DVLCS in (1) is rewritten equivalently as:*

$$RPR^{-1}\,\dot{\xi}(t) \in g(t, \xi(t)) - \Psi(t, \xi(t)), \tag{14}$$

with: $g(t, \xi) = RAR^{-1}\xi + RE(t) - (R^{-1}C^\top - RB)\mathcal{P}_D(\lambda_\alpha(t))$, $\lambda_\alpha(t)$ is any element of the set $\Phi\,(t, Cx(t))$, $\Psi(t, \xi) = R^{-1}C^\top\Phi(t, CR^{-1}\xi)$, and $\mathcal{P}_D$ denotes the projection on $\mathrm{Im}(D + D^\top)$ (which is unique for any element $\lambda_\alpha(t)$).

3.2.1 Calculation of sets

Notice that $\Psi(t,\cdot) = \Phi(t,\cdot) \circ CR^{-1}$, with $\Phi(t,\zeta) \stackrel{\Delta}{=} (\partial\sigma_{\Gamma(t)} + D)^{-1}(\zeta)$. By [48, Theorem 23.8] and since $D = D^\top$ (Assumption 4), one has $\partial(\sigma_{\Gamma(t)}(z) + \frac{1}{2}z^\top Dz) = \partial\sigma_{\Gamma(t)}(z) + Dz$, hence denoting $h(t,z) = \sigma_{\Gamma(t)}(z) + \frac{1}{2}z^\top Dz$, so that $h(t,\cdot)$ is convex proper lower semicontinuous for each $t \geq 0$, one has $\Phi(t,\cdot) = (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot) = \partial h^\star(t,\cdot)$ [37, Corollary 1.4.4], where $h^\star(t,\cdot)$ is the conjugate function of $h(t,\cdot)$. Therefore $\Psi(t,\cdot) = \partial f(t,\cdot)$, with

$$f : \xi \mapsto (h^\star(t,\cdot) \circ CR^{-1})(\xi) \tag{15}$$

a convex, proper and lower semicontinuous function. One has $\mathrm{dom}(f) = \{\xi \in I\!R^n \mid CR^{-1}\xi \in \mathrm{dom}(h^\star)\}$, and $\mathrm{dom}(\partial h^\star) \subseteq \mathrm{dom}(h^\star) \subseteq \overline{\mathrm{dom}(\partial h^\star)}$ [7, Theorem 2, Chapter 10, section 3], and $\mathrm{dom}(\partial h^\star) = \mathrm{Im}(\partial\sigma_{\Gamma(t)} + D)$. Characterizing this range may be done as follows, see for instance [18]. Here both $\partial\sigma_{\Gamma(t)}(\cdot)$ and D are monotone, they both satisfy the property (*) stated in [14, section 1] (because they are both subdifferentials of convex proper lower semicontinuous functions [14, p.167]), and $\partial\sigma_{\Gamma(t)} + D$ is also monotone because $(\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$ is monotone [54, Lemma 1]. Hence from [14, Theorems 3, 4] one finds $\overline{\mathrm{Im}(\partial\sigma_{\Gamma(t)} + D)} = \overline{\mathrm{Im}(\partial\sigma_{\Gamma(t)}) + \mathrm{Im}(D)} = \overline{\Gamma(t) + \mathrm{Im}(D)}$ and $\mathrm{Int}(\mathrm{Im}(\partial\sigma_{\Gamma(t)} + D)) = \mathrm{Int}(\mathrm{Im}(\partial\sigma_{\Gamma(t)}) + \mathrm{Im}(D)) = \mathrm{Int}(\Gamma(t) + \mathrm{Im}(D))$. Thus $\mathrm{dom}(\partial h^\star) = \Gamma(t) + \mathrm{Im}(D)$, and

$$\mathrm{dom}(f) = \{\xi \in I\!R^n \mid CR^{-1}\xi \in \Gamma(t) + \mathrm{Im}(D)\} \tag{16}$$

is a closed convex set, which is nonempty by Assumption 5. Let us state two lemmae that will be useful for the generalisation of Proposition 2.

Lemma 1 *Let $K^\star = \{\xi \in I\!R^m \mid G\xi \geq 0\}$ for some $G \in I\!R^{m\times m}$ as in Proposition 2, and let the assumptions of Proposition 3 hold true. (i) If D has full rank, then* $\mathrm{dom}(f) = I\!R^n$ *and* $(\mathrm{dom}(f))_\infty = I\!R^n$. *(ii) If $D = 0$ then* $\mathrm{dom}(f) = \tilde{\Gamma}(t) = \{\xi \in I\!R^n \mid GCR^{-1}\xi + GF(t) \geq 0\}$ *and* $(\mathrm{dom}(f))_\infty = \{\xi \in I\!R^n \mid GCR^{-1}\xi \geq 0\}$. *(iii) In general the recession cone of* $\mathrm{dom}(f)$ *is* $(\mathrm{dom}(f))_\infty = \{\xi \in I\!R^n \mid CR^{-1}\xi \in \mathrm{Im}(D) + K^\star\}$.

Proof: (i) Follows from (16) with $\mathrm{Im}(D) = I\!R^m$, and Proposition 9 d). (ii) Follows from (16) with $\mathrm{Im}(D) = \{0\}$, and Proposition 9 e). (iii) One has $(\mathrm{dom}(f))_\infty = \bigcap_{\lambda>0} \frac{1}{\lambda}(\mathrm{dom}(f) - \xi_0)$ for some $\xi_0 \in \mathrm{dom}(f)$. Let $w \in \frac{1}{\lambda}(\mathrm{dom}(f) - \xi_0) \Leftrightarrow \lambda w \in \mathrm{dom}(f) - \xi_0 \Leftrightarrow \lambda CR^{-1}w = D\zeta + y - F(t) - CR^{-1}\xi_0$ for some $\zeta \in I\!R^m$ and $y \in K^\star$, equivalently $CR^{-1}w = \frac{1}{\lambda}D\zeta + \frac{1}{\lambda}y - \frac{1}{\lambda}F(t) - \frac{1}{\lambda}CR^{-1}\xi_0$. Letting λ vary in $(0,+\infty)$ and since $K^\star$ is a cone, the conclusion follows. ⊠

To extend Proposition 2 we will also need to characterize $(\mathrm{dom}(f_\infty))^\star$, that is the dual cone of the domain of the recession function $f_\infty(\cdot)$. Let us remind that $f(t,\cdot) = (h^\star(t,\cdot) \circ CR^{-1})(\cdot)$, with $h^\star(t,w) = \sup_{z\in I\!R^m}(w^\top z - \sigma_{\Gamma(t)}(z) - \frac{1}{2}z^\top Dz)$. Let us better characterize the latter conjugate function. Using [37, Theorem 2.3.1, Chapter E], one finds that $h^\star(t,w) = \mathrm{cl}\left(\sigma^\star_{\Gamma(t)}(w) \overset{+}{\vee} \left(\frac{1}{2}w^\top Dw\right)^\star\right)$, both functions being closed since they are convex and lower semicontinuous, and $\overset{+}{\vee}$ stands for the inf-convolution operation [37, p.92]. Thus $h^\star(t,w) = \inf_{y\in I\!R^m}\left(\psi_{\Gamma(t)}(y) + g(w-y)\right)$ [37, Definition 2.3.1], since $\sigma^\star_{\Gamma(t)}(\cdot) = \psi_{\Gamma(t)}(\cdot)$, being $\Gamma(t)$ closed convex for each t [48, Theorem 13.2], where

$$g(w-y) = \begin{cases} \frac{1}{2}(w-y)^\top D^\dagger(w-y) & \text{if } w-y \in \mathrm{Im}(D) \\ +\infty & \text{if } w-y \notin \mathrm{Im}(D), \end{cases} \tag{17}$$

where $D^\dagger$ is the Moore-Penrose generalized inverse (when $D = D^\top \Leftrightarrow D^\dagger = (D^\dagger)^\top$, then $D^\dagger \succcurlyeq 0 \Leftrightarrow D \succcurlyeq 0$, and $\mathrm{Im}(D) = \mathrm{Im}(D^\dagger)$ [11, Proposition 6.1.6 , vi), xxxi)], and if $D = 0$ then $D^\dagger = 0$). It follows that $h^\star(t,w)=\inf_{y\in\Gamma(t)} g(w-y)$. Therefore:

$$
\begin{array}{rl}
f(t,\xi) & = \inf_{y\in\Gamma(t)} \left\{ \begin{array}{ll} \frac{1}{2}(CR^{-1}\xi-y)^\top D^\dagger (CR^{-1}\xi-y) & \text{if } CR^{-1}\xi-y\in \mathrm{Im}(D^\dagger) \\ +\infty & \text{if } CR^{-1}\xi-y\notin \mathrm{Im}(D^\dagger) \end{array}\right. \\
& = \left\{ \begin{array}{ll} \inf_{y\in\Gamma(t)} \frac{1}{2}(CR^{-1}\xi-y)^\top D^\dagger (CR^{-1}\xi-y) & \text{if } CR^{-1}\xi-y\in \mathrm{Im}(D^\dagger) \\ +\infty & \text{if } CR^{-1}\xi-y\notin \mathrm{Im}(D^\dagger). \end{array}\right.
\end{array}
\tag{18}
$$

where we recall that $y\in\Gamma(t)\Leftrightarrow G(y+F(t))\geq 0$, and $f(t,\cdot)=(h^\star(t,\cdot)\circ CR^{-1})(\cdot)$.

The next lemma is useful to calculate the set in (63).

Lemma 2 *Assume that $D=D^\top\succcurlyeq 0$, and let it be unitarily similar to $\begin{pmatrix}\bar{D} & 0\\ 0 & 0\end{pmatrix}$ with unitary matrix U, $\bar{D}\succ 0$. Assume also that $K^\star=\{\xi\in I\!R^m \mid G\xi\geq 0\}$ with $G\in I\!R^{m\times m}$ full rank. Let $\Gamma\stackrel{\Delta}{=}\{\xi\in I\!R^n \mid UGCR^{-1}\xi\in I\!R^m_+\}$. Then $f_\infty(t,\xi)=f_\infty(\xi)=\psi_\Gamma(\xi)$.*

Proof: The proof is in Appendix D. ⊠

Therefore the following holds (see the proof of Proposition 2 and [33, Example 15]).

Corollary 1 *Let Γ be as in Lemma 2. It follows that $(\mathrm{dom}(f_\infty))^\star=\Gamma^\star=\{w\in I\!R^n \mid w=\sum_{i=1}^m \lambda_i (UGCR^{-1})^\top_{i\bullet}, \lambda_i\geq 0\}$.*

3.2.2 Analysis of the OSNSP

Let us now come back to the differential inclusion (14). If $D=D^\top$, one can rewrite it equivalently as the VI: Find $\xi(t)\in\mathrm{dom}(f)$ such that

$$
\langle RPR^{-1}\,\dot{\xi}(t)-g(t,\xi(t)),v-\xi(t)\rangle+f(t,v)-f(t,\xi(t))\geq 0,\ \text{for all } v\in\mathrm{dom}(f), \tag{19}
$$

where it is reminded that $f(t,\cdot)=(h^\star(t,\cdot)\circ CR^{-1})(\cdot)$. Let us now pass to the time-discretization of (19). Find $\xi_{k+1}\in\mathrm{dom}(f_k)$ such that

$$
\langle RPR^{-1}\,\xi_{k+1}-RPR^{-1}\,\xi_k-h\,g(t_k,\xi_k),v-\xi_{k+1}\rangle+f_k(v)-f_k(\xi_{k+1})\geq 0 \text{ for all } v\in\mathrm{dom}(f_k), \tag{20}
$$

where $g(t_k,\xi_k)=RAR^{-1}\xi_k+RE_k-(R^{-1}C^\top-RB)\mathcal{P}_D(\lambda_{\alpha,k})$, $\lambda_{\alpha,k}$ is any element of the set $\Phi(t_k,CR^{-1}\xi_k)$ (see (13) and Proposition 3 for the definition of this set). The VI in (20) has quite the same structure as the VI in (9), excepted for the function $f_k(\cdot)$ which is now more general. Notice that we chose the discretization in (4) instead of (3), this does not modify much the developments excepted that P^h is replaced by P in the term $RPR^{-1}\xi_{k+1}$ in (20). The next results hold, and extend Proposition 2.

Proposition 4 *Let Assumptions 3 (with P^h replaced by P), 4, 5, 6 hold, $h>0$ and $k\geq 0$ be given and $D=D^\top$ with same properties as in Lemma 2. Let $\xi=Rx$, $R^2=X$, $R=R^\top\succ 0$. Let the DVLCS in (1) be discretized as in (20), and $q_k\stackrel{\Delta}{=}-RPR^{-1}\,\xi_k-h\,g(t_k,\xi_k)$. Let us consider the set:*

$$
\mathcal{S}\stackrel{\Delta}{=}(\mathrm{dom}(f_k))_\infty\cap\{\xi\in I\!R^n \mid RPR^{-1}\xi=\sum_{i=1}^m \lambda_i (UGCR^{-1})^\top_{i\bullet},\ \lambda_i\geq 0\}\cap\mathrm{Ker}(RPR^{-1}+R^{-1}P^\top R), \tag{21}
$$

where $(\mathrm{dom}(f_k))_\infty$ is given in Lemma 1. Then:

1. *If $\mathcal{S}=\{0\}$, then the VI(RPR^{-1},q_k,f_k) has at least one solution.*

2. *If $\mathcal{S}\neq\{0\}$, and if there exists $\xi_0\in\mathrm{dom}(f_k)$ such that*

$$\langle q_k-(RPR^{-1})^\top\xi_0\rangle>0 \text{ for all } v\in\mathcal{S},\ v\neq 0, \tag{22}$$

   then the VI(RPR^{-1},q_k,f_k) has at least one solution.

3. *If ξ^1_{k+1} and ξ^2_{k+1} are two solutions of the VI(RPR^{-1},q_k,f_k), then $\xi^1_{k+1}-\xi^2_{k+1}\in\mathrm{Ker}(RPR^{-1}+R^{-1}P^\top R)$.*

4. *Let RPR^{-1} be symmetric. If ξ^1_{k+1} and ξ^2_{k+1} are two solutions of the VI(RPR^{-1},q_k,f_k), then $\langle q_k,\xi^1_{k+1}-\xi^2_{k+1}\rangle=0$.*

5. *Let RPR^{-1} be symmetric. Then any solution of the VI(RPR^{-1},q_k,f_k) is also a solution of the optimization problem:*

$$\min_{\xi\in\mathbb{R}^n}\frac{1}{2}\xi^\top RPR^{-1}\xi+\langle q_k,\xi\rangle+f_k(\xi). \tag{23}$$

Proof: The proof is similar to that of Proposition 2. It follows from Proposition 3 (which itself is a consequence of the material in [54]), Proposition 10 as well as Lemmae 1 and 2 and Corollary 1. In item 2 it is used that $\psi_{(\mathrm{dom}(f_k))_\infty}(v)=0$ for all $v\in\mathcal{S}$. ⊠

One sees that computing the set $\mathcal{S}$ in (21) boils down to calculating the set of vectors $\xi\in\mathbb{R}^n$ such that:

$$\begin{cases} CR^{-1}\xi\in\mathrm{Im}(D)+\mathbb{R}^m_+\\ (P+P^T)R^{-1}\xi=0\\ RPR^{-1}\xi=\sum_{i=1}^m\lambda_i(UGCR^{-1})^\top_{i\bullet}\ ,\ \lambda_i\geq 0.\end{cases} \tag{24}$$

It follows from item 5 that solutions are in the set $\mathrm{dom}(f_k)=\{\xi\in\mathbb{R}^n\mid CR^{-1}\xi\in\Gamma_k+\mathrm{Im}(D)\}$ with $\Gamma_k=\{\zeta\in\mathbb{R}^n\mid G(\zeta+F_k)\geq 0\}$. One sees from Lemma 1 that if D is full rank ($\Rightarrow D\succ 0$ since $D\succcurlyeq 0$), then $\psi_{(\mathrm{dom}(f))_\infty}(\cdot)=0$, so the first line in (24) and the first set in $\mathcal{S}$ in (21) can be ignored. The multiplier verifies $\lambda_{k+1}=(\mathcal{N}_K+D)^{-1}(-Cx_{k+1}-F_k)$, where the operator $(\mathcal{N}_K+D)^{-1}(\cdot)$ is single-valued Lipschitz continuous [17, Proposition 1], and $\mathrm{dom}((\mathcal{N}_K+D)^{-1})=\mathrm{Im}(\mathcal{N}_K+D)=\mathbb{R}^m$ so that $\mathrm{dom}((\mathcal{N}_K+D)^{-1}(C\cdot))=\mathbb{R}^n$. Then the implicit Euler discretization (5) is written equivalently as (compare with (6)):

$$Px_{k+1}-hB(\mathcal{N}_K+D)^{-1}(-Cx_{k+1}-F_k)=P^hx_k+hE_k, \tag{25}$$

where $P^h=P+hA$.

4 Convergence analysis

Well-posedness results have been stated in [24] for DVLCS with passive quintuple (P,A,B,C,D) (the set-valued feedback term is more general in [24] than in (1)), and also in [53, 52, 51] (the Lipschitz continuous nonsmooth constraints considered in [53, 52, 51] embed the complementarity constraints in (1) when $D\succ 0$). However the discrete-time solutions have not been analysed in these articles. The first step is to analyse the limits as $h\to 0$ of the piecewise-linear functions constructed with the above iterates. We consider the time interval $[0,T]$, $T>0$, $h=\frac{T}{n}$, $n \in I\!N$, $n>0$, $t_k = kh$, $k \geq 0$, $t_0 = 0$, $t_n = T$. In this section we always assume that the conditions for existence of a solution to the OSNSP, *i.e.*, the VIs in (9) and in (20), are satisfied for all $h>0$ (see Propositions 2 and 4).

4.1 The case $D = 0$

When $D = 0$, using the material in section C, and still assuming that $K^\star = \{\xi \in I\!R^n \mid G\xi \geq 0\}$, the DVLCS in (1) can be rewritten equivalently as the differential inclusion:

$$P\dot{x}(t) - Ax(t) \in -B\mathcal{N}_{\Gamma(t)}(Cx(t)) + E(t), \tag{26}$$

with $\Gamma(t) = \{w \in I\!R^m \mid G(w + F(t)) \geq 0\}$. Since $\Gamma(t)$ is convex polyhedral for each t, then using as above the variable change matrix R from Assumption 2, this can be equivalently rewritten using the Convex Analysis chain rule as

$$RPR^{-1}\dot{\xi}(t) - RAR^{-1}\xi(t) \in -\mathcal{N}_{\tilde{\Gamma}(t)}(\xi(t)) + RE(t), \tag{27}$$

where $\tilde{\Gamma}(t) = \{\xi \in I\!R^n \mid GCR^{-1}\xi + GF(t) \geq 0\}$. The differential inclusion in (27) is a singular first order sweeping process (FOSwP [20]). It is noteworthy that singular zero order sweeping processes (ZOSwP [20]) have been studied [56, 4, 5], but they form another class of singular systems than (27). The analysis in the foregoing sections can therefore be thought of as that of a singular FOSwP. A time-discretisation of (27) is in (7).

Assumption 7 *The time-function $F(\cdot)$ in (1) satisfies $F(t) = F$ for some constant F and for all $t \geq 0$.*

Then we can denote $f(\cdot,t) = \psi_{K^\star - F} \circ CR^{-1}(\cdot) = \psi_\Gamma \circ CR^{-1}(\cdot) = f(\cdot)$ since $f(\cdot)$ no longer depends explicitly on time, and the differential inclusion in (27) is no longer a FOSwP.

Proposition 5 *Let Assumptions 2, 3, 7 and the following conditions be satisfied:*

1. $D = 0$.
2. $P = P^\top$.
3. *The polyhedral set* $\tilde{\Gamma} = \{\xi \in I\!R^n \mid G(CR^{-1}\xi + F) \geq 0\}$ *is compact.*
4. *There exists* $U \in I\!R^{n\times n}$, $UU^\top = I_n$, $U = (U_1 \; U_2)$, $U_1 \in I\!R^{n\times p}$, $U_2 \in I\!R^{n\times(n-p)}$, *such that* $A = R^{-1}U \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} U^\top R$, $U^\top RPR^{-1}U = \begin{pmatrix} \bar{P} & 0 \\ 0 & 0 \end{pmatrix}$, *where* $\bar{P} = \mathrm{diag}(\lambda_i(P))$, $\lambda_i(P) > 0$, $1 \leq i \leq p$ *are all the positive eigenvalues of* P, $\bar{P} \in I\!R^{p\times p}$, $M_{11} \in I\!R^{p\times p}$, $M_{22} \in I\!R^{(n-p)\times(n-p)}$.
5. $M_{22} + \frac{1}{2}||(M_{12}^\top + M_{21})||^2 I_{n-p} + \frac{1}{2}I_{n-p} \preccurlyeq 0$.
6. $M_{11} + I_p \preccurlyeq 0$.
7. *$E(\cdot)$ is Lipschitz continuous,* i.e., $||E(t_1) - E(t_2)|| \leq k_E|t_1 - t_2|$ *for all* t_1, $t_2 \geq 0$ *and some bounded* $k_E > 0$.

Consider the following piecewise-linear approximations $[0,T] \to I\!R^n$*:*

$$\begin{cases} \xi^h(t) \triangleq \xi_{k+1} + \frac{t_{k+1}-t}{h}(\xi_k - \xi_{k+1}) \\ \dot{\xi}^h(t) \triangleq \frac{\xi_{k+1}-\xi_k}{h} \;\; \textit{almost everywhere,} \end{cases} \quad \textit{for all } t \in [t_k, t_{k+1}). \tag{28}$$

Then $\xi^h(\cdot)$ *is uniformly bounded on* $[0,T]$*, and* $\eta^1_h(\cdot) \triangleq U_1^\top \xi^h(\cdot) \to \eta^1(\cdot)$ *as* $h \searrow 0$*, uniformly in* $\mathcal{C}^0([0,T]; I\!R^p)$*, for some continuous function* $\eta^1 : [0,T] \to I\!R^p$*.*

Proof: See section E. ⊠

Item 6 guarantees that $\bar{P} - 2hM_{11} - 2hI_p \succ 0$ for all $h > 0$, and $\bar{P} \preccurlyeq \bar{P} - 2hM_{11} - 2hI_p$, which are both used in the proof, see (80). The role of Assumption 7, which is a mild constraint on exogenous signals, becomes clear in the proof. The dynamics after the transformation in item 4 of the proposition, takes the form:

$$\begin{cases} \begin{pmatrix} \bar{P} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\eta}^1(t) \\ \dot{\eta}^2(t) \end{pmatrix} = M \begin{pmatrix} \eta^1(t) \\ \eta^2(t) \end{pmatrix} + U^\top RB\lambda(t) + U^\top RE(t) \\ K \ni \lambda(t) \perp CR^{-1}U \begin{pmatrix} \eta^1(t) \\ \eta^2(t) \end{pmatrix} \;\; (+D\lambda(t) + F(t)) \in K^\star. \end{cases} \tag{29}$$

The DVLCS in (29) is under the semi-implicit form of a DAE coupled to linear complementarity constraints, and may be named a DALCS. We may also see (29) as a kind of Weierstrass form [22] for (1), with the nipoltency degree equal to 1 (but M is not in the Weierstrass canonical form that implies $M_{22} = I_{n-p}$). The convergence analysis in this section is an extension of the case with full-rank P analysed for instance in [9]. Clearly if M_{22} is nonsingular, then (29) can be rewritten as the LCCS:

$$\begin{cases} \bar{P}\dot{\eta}^1(t) = (M_{11} - M_{12}M_{22}^{-1}M_{21})\eta^1(t) + (U_1^\top RB - M_{22}^{-1}U_2^\top RB)\lambda(t) + (U_1^\top R - M_{22}^{-1}U_2^\top R)E(t) \\ K \ni \lambda(t) \perp w(t) = (CR^{-1}U_1 - CR^{-1}U_2M_{22}^{-1}M_{21})\eta^1(t) + (D - M_{22}^{-1}U_2^\top RB)\lambda(t) + F(t) \in K^\star. \end{cases} \tag{30}$$

It is unclear how the passivity assumptions 2 and 4 on (A, B, C, D), and the properties of the LCCS quadruple $(\bar{P}^{-1}(M_{11} - M_{12}M_{22}^{-1}M_{21}), \bar{P}^{-1}(U_1^\top - M_{22}^{-1}U_2^\top)RB, CR^{-1}U_1 - CR^{-1}U_2M_{22}^{-1}M_{21}, D - M_{22}^{-1}U_2^\top RB)$ in (30), are related one to each other in the general case. The convergence analysis in this section is fitted to DVLCS, not to any equivalent reduced order LCS. Moreover it is usually not desired to reduce DALCS to LCS [57]. Let us state the following, which allows us to relax the compactness assumption in item 3 of Proposition 5.

Proposition 6 *Let Assumption 7 hold,* $D = 0$*,* $P = P^\top$*,* M_{22} *be non singular, and:*

1. *all initial data be bounded,*

2. $E(t)$ *be uniformly bounded for all* $t \geq 0$*,*

3. $\bar{P}$ *be defined as in Proposition 5 item 4 (*$\Rightarrow \bar{P} \succ 0$*),*

4. $M_{\rm sch} + I_P \preccurlyeq 0$*, where* $M_{\rm sch} \triangleq M_{11} - M_{12}M_{22}^{-1}M_{21}$*,*

5. $U^\top R\,\mathrm{Im}(P) \supseteq U^\top R\,\mathrm{Im}(B)$.

Then η_k^1 and η_k^2 are uniformly bounded (independently of n) on $k \in \{1, n\}$.

Proof: See section F. ⊠

Item 5 implies that the quadruple of the LCCS in (30) is given by $(\bar{P}^{-1}M_{\rm sch}, \bar{P}^{-1}U_1^\top RB, CR^{-1}U_1 - CR^{-1}U_2M_{22}^{-1}M_{21}, D)$. Thus again it is in general not possible to conclude about any nice property for this quadruple, using the above assumptions 2 and 4 as well as items 4 and 5. When $M_{22} = 0$ the DVLCS is as in (59).

Remark 2 *The usefulness of the Schur complement negative definiteness in item 4 of Proposition 6, appears clearly in the proof of the proposition, see (88). However, since $\bar{P} \succ 0$, it is also possible to guarantee in (88) that $\tilde{P} \succ 0$ for $h > 0$ small enough, using Theorem 1 and Corollary 2 in Appendix A. Then the result holds for n large enough. Thus the condition in item 4 is sufficient but by far not necessary, and could be replaced by the less restrictive condition $\tilde{P} \succ 0$ for small enough h.*

It is inferred that each time the conditions of Proposition 6 are satisfied, $||\xi_k - \xi_0||$ and thus $||x_k - x_0||$ are uniformly bounded, as well as the selections ζ_k of $\partial f(\xi_k)$ (without doing the compactness assumption of Proposition 5 item 3; thus Proposition 6 allows us to solve the very first step of the proof of Proposition 5). Then the piecewise-linear approximations $x^h(\cdot)$ are uniformly bounded in $h > 0$ on $[0, T]$. We can redo as in the proof of Proposition 5 to conclude the uniform boundedness of $\dot{\eta}_h^1(\cdot)$, and the existence of a limit function $\eta^1(\cdot)$ follows. Let us denote $\eta_h^{1\star}(\cdot)$ the step function defined as $\eta_h^{1\star}(t) = \eta_{k+1}^1$ if $t \in [t_k, t_{k+1})$, $k \geq 0$. Then the following holds:

Lemma 3 *Assume that the conditions of Proposition 5 (or of Proposition 6 to avoid the compactness in item 3 of Proposition 5) are satisfied. Then $\eta_h^{1\star}(\cdot)$ converges strongly in $\mathcal{L}^2([0,T]; I\!\!R^p)$ towards the limit $\eta^1(\cdot)$, and $\dot{\eta}_h^1(\cdot)$ converges weakly to $\dot{\eta}^1(\cdot)$ in $\mathcal{L}_2([0,T]; I\!\!R^p)$.*

Proof: We know that $\eta_h^1(\cdot) \to \eta^1(\cdot)$ uniformly in $\mathcal{C}^0([0,T]; I\!\!R^p)$ as $h \to 0$. Let $\dot{\eta}_k^1 \triangleq \frac{\eta_{k+1}^1 - \eta_k^1}{h}$, and notice that $||\dot{\eta}_k^1|| \leq C$ for some $C > 0$. Then:

$$
\begin{aligned}
\|\eta_h^1 - \eta_h^{1\star}\|_{\mathcal{L}^2}^2 &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - t)^2 \|\dot{\eta}_k^1\|^2 dt \\
&\leq C^2 \sum_{k=0}^{n-1} \frac{1}{3}(t_{k+1} - t_k)^3 = \frac{C^2 n h^3}{3} = \frac{C^2 T h^2}{3}.
\end{aligned}
\tag{31}
$$

Therefore $\|\eta_h^1 - \eta_h^{1\star}\|_{\mathcal{L}^2} \to 0$ as $h \to 0$. Since $\|\eta_h^1 - \eta^1 + \eta^1 - \eta_h^{1\star}\|_{\mathcal{L}^2} \leq \|\eta_h^1 - \eta^1\| + \| + \eta^1 - \eta_h^{1\star}\|_{\mathcal{L}^2} \to \|\eta^1 - \eta_h^{1\star}\|_{\mathcal{L}^2}$ as $h \to 0$ the proof is done.
The sequence $\{\dot{\eta}_h^1(\cdot)\}_{h>0}$ being uniformly bounded on $[0, T]$ (from the proof of Proposition 5 which holds under the stated assumptions), it follows applying the Banach-Alaoglu-Bourbaki Theorem [13, Theorem III.15] that it possesses a limit $s(\cdot)$ in the weak $\star$ topology $\sigma(\mathcal{L}_\infty([0,T]; I\!\!R^p), \mathcal{L}_1([0,T]; I\!\!R^p))$ [13, Proposition III.12 (i)], *i.e.*,

$$
\int_0^T \dot{\eta}_h^1(t)\varphi(t)dt \underset{h \to 0}{\longrightarrow} \int_0^T s(t)\varphi(t)dt, \text{ for all } \varphi(\cdot) \in \mathcal{L}_1([0,T]; I\!\!R^p). \tag{32}
$$

Being $[0, T]$ bounded one has $\mathcal{L}_1([0,T]; I\!\!R^p) \subset \mathcal{L}_2([0,T]; I\!\!R^p)$ so the weak convergence holds also in $\mathcal{L}_2([0,T]; I\!\!R^p)$. Moreover $\eta_h^1(t) = \eta_h^1(0) + \int_0^t s(\tau)d\tau$ for all $t \in [0, T]$. It follows that $s(\cdot) = \dot{\eta}^1(\cdot)$ almost everywhere. Consequently $\dot{\eta}_h^1(\cdot)$ converges weakly to $\dot{\eta}^1(\cdot)$ in $\mathcal{L}_2([0,T]; I\!\!R^p)$. ⊠

Now we have a rather complete set of results for the convergence of approximations of $\eta^1(\cdot)$. Let us investigate convergence of approximations of $\eta^2(\cdot)$. We remind that $R^2 = X$, $R = R^\top$, with X in Assumption 2, and U is in item 4 of Proposition 5.

Proposition 7 *Let the conditions of Proposition 5 items 1, 2, 4, 5, 6, 7 and of Proposition 6 hold. Then $\eta_h^2(\cdot)$ converges uniformly in $\mathcal{C}^0([0,T];\mathbb{R}^{n-p})$ to $\eta^2(\cdot)$, $\eta_h^{2\star}(\cdot)$ converges strongly in $\mathcal{L}_2([0,T];\mathbb{R}^{n-p})$ to $\eta^2(\cdot)$, $\dot{\eta}_h^2(\cdot)$ converges weakly in $\mathcal{L}_2([0,T];\mathbb{R}^{n-p})$ to $\dot{\eta}^2(\cdot)$.*

Proof: It follows that the results of both Propositions 5 and 6 hold. Consider (74) in section E. Applying the U transformation (item 4 in Proposition 5) we obtain the algebraic equality part:

$$\begin{cases} -M_{12}\eta_{k+1}^1 - M_{22}\eta_{k+1}^2 - U_2^\top RE_k = -U_2^\top \zeta_{k+1} = 0 \\ -M_{12}\eta_k^1 - M_{22}\eta_k^2 - U_2^\top RE_{k-1} = -U_2^\top \zeta_k = 0, \end{cases} \tag{33}$$

where $\zeta_k \in \partial f(\xi_k)$ and $\zeta_{k+1} \in \partial f(\xi_{k+1})$, and item 5 of Proposition 6 was used. Thus it follows that

$$-M_{12}\frac{\eta_{k+1}^1 - \eta_k^1}{h} - M_{22}\frac{\eta_{k+1}^2 - \eta_k^2}{h} - U_2^\top R\frac{\Delta E_k}{h} = 0. \tag{34}$$

Hence $\frac{\eta_{k+1}^2 - \eta_k^2}{h} = \dot{\eta}_h^2(t) = -M_{22}^{-1}(M_{12}\dot{\eta}_h^1(t) + U_2^\top R\dot{E}_h(t))$ for all $t \in [t_k, t_{k+1})$, $k \geq 1$, since M_{22} is non singular due to item 5 of Proposition 5. One has that $||\frac{\Delta E_k}{h}(t)|| \leq k_E h$, hence $\dot{E}_h(t) \stackrel{\Delta}{=} \frac{E_k - E_{k-1}}{h}$ for $t \in [t_k, t_{k+1})$, is uniformly bounded on $[0,T]$. Thus the sequence $\{E_h(\cdot)\}$ is uniformly bounded, continuous with uniformly bounded derivatives almost everywhere, and is thus equicontinuous. So by Ascoli-Arzela Theorem $E_h(\cdot)$ converges uniformly in $\mathcal{C}^0([0,T];\mathbb{R}^n)$ to a limit $E(\cdot)$. Using the proposition's assumptions, we know that $\dot{\eta}_h^1(\cdot)$ is uniformly bounded (using Proposition 6, and then applying Proposition 5 without item 3, see the proof of Proposition 5 and (81)). We infer from (34) and item 5 of Proposition 5 ($\Rightarrow M_{22}$ is full-rank) that $\dot{\eta}_h^2(\cdot)$ is uniformly bounded also. From (33) and the boundedness of $E(\cdot)$ it follows that η_k^2 is uniformly bounded for $k \in \{0, n-1\}$, thus the sequence $\{\eta_h^2(\cdot)\}$ is uniformly bounded on $[0,T]$, continuous with uniformy bounded derivative almost everywhere. Applying the Ascoli-Arzela Theorem we deduce that it converges uniformly in $\mathcal{C}^0([0,T];\mathbb{R}^n)$ towards a limit $\eta^2(\cdot)$. Now we can make the same calculations as in (31) to infer that $||\eta_h^2(t) - \eta_h^{2\star}(t)||_{\mathcal{L}^2} \to 0$ as $h \to 0$. The sequence $\{\dot{\eta}_h^2(\cdot)\}$ being uniformly bounded on $[0,T]$, the same reasoning as in the proof of Lemma 3 can be redone to conclude that $\dot{\eta}_h^2(\cdot)$ converges weakly in $\mathcal{L}_2([0,T];\mathbb{R}^{n-p})$ to $\dot{\eta}^2(\cdot)$. ⊠

Let us now consider (74) in appendix E: $RPR^{-1}\dot{\xi}_k - RAR^{-1}\xi_{k+1} - RE_k \in -\partial f(\xi_{k+1})$. Since we assume that all the assumptions of Proposition 5, Proposition 6, Lemma 3, and Proposition 7 are satisfied, it follows that $RPR^{-1}\dot{\xi}_h(\cdot) - RAR^{-1}\xi_h^\star(\cdot) - RE_h(\cdot) \to RPR^{-1}\dot{\xi}(\cdot) - RAR^{-1}\xi(\cdot) - RE(\cdot)$ weakly in $\mathcal{L}_2([0,T];\mathbb{R}^{n-p})$ as $h \to 0$ (recall that $\xi_h^\star(t) = \xi_{k+1}$ on $t \in [t_k, t_{k+1})$, $k \geq 0$). Since $\xi_h^\star(\cdot)$ converges strongly towards $\xi(\cdot)$ in $\mathcal{L}_2([0,T];\mathbb{R}^{n-p})$, we infer using [8, Proposition 2] and the maximal monotonicity of $\partial f(\cdot)$ that $RPR^{-1}\dot{\xi}(\cdot) - RAR^{-1}\xi(\cdot) - RE(\cdot) \in -\partial f(\xi(\cdot))$. This proves the existence of a $\mathcal{C}^0([0,T];\mathbb{R}^n)$ solution to the DVLCS (1).

Taking into account the proposition's assumptions (in particular $U^\top R\,\mathrm{Im}(P) \supseteq U^\top R\,\mathrm{Im}(B)$), the initial data for (29) have to satisfy:

$$\begin{aligned} &M_{21}\eta^1(0) + M_{22}\eta^2(0) + U_2^\top RE(0) = 0 \\ &\lambda(0) \in -\mathcal{N}_{K^\star}\left(CR^{-1}U^\top \begin{pmatrix}\eta^1(0) \\ \eta^2(0)\end{pmatrix} + F\right). \end{aligned} \tag{35}$$

The above limits satisfy this generalized equation initially, by construction (see for instance (33)). Let us finally notice that the multiplier $\lambda(\cdot)$ is not continuous in general. Even in the simpler case of LCS, it is known that it may possess infinitely many discontinuity times in finite time-intervals (Zeno phenomenon), excepted if some conditions are satisfied [49].

4.2 The case $D \succcurlyeq 0$

Let us now deal with the framework in section 3.2, with the same basic assumptions 4, 5, 6, 7. Then, one works with (20) instead of (8). The only difference between both VIs is in the term $g(t_k, \xi_k)$, with $-(R^{-1}C^\top - RB)\mathcal{P}_D(\lambda_{\alpha,k})$. It is known from [54, Lemma 3] that $\mathcal{P}_D(\lambda_{\alpha,k}(t_k, \xi_k))$ is Lipschitz continuous in ξ_k for each t_k, *i.e.*, there exists $k_\lambda(t) \geq 0$ such that $||\mathcal{P}_D(\lambda_{\alpha,k}(t_k, \xi_k)) - \mathcal{P}_D(\lambda_{\alpha,k-1}(t_{k-1}, \xi_{k-1}))|| \leq k_\lambda(t)||\xi_k - \xi_{k-1}|| = hk_\lambda(t_k)||\dot{\xi}_k||$, for some $k_\lambda : I\!R \to I\!R_+$. The regularity of $k_\lambda(\cdot)$ depends on the regularity of $F(\cdot)$.

Let us consider (76) in the proof of Proposition 5. In addition to the term $\langle R\,\Delta E_k, \dot{\xi}_k\rangle$, we have to consider $\langle -(R^{-1}C^\top - RB)(\lambda^{\rm im}(t_k, \xi_k) - \lambda^{\rm im}(t_{k-1}, \xi_{k-1})), \dot{\xi}_k\rangle$, where $\lambda^{\rm im}(t,\xi) \stackrel{\Delta}{=} \mathcal{P}_D(\lambda_\alpha(t,\xi))$ for any element $\lambda_\alpha(t,\xi)$ in the set $\Phi(t, CR^{-1}\xi)$. This projection is single-valued and unique [54, Lemma 2]. Let $\tilde{C} \stackrel{\Delta}{=} -(R^{-1}C^\top - RB)$. We have $\langle \tilde{C}(\lambda^{\rm im}(t_k,\xi_k) - \lambda^{\rm im}(t_{k-1},\xi_{k-1})), \dot{\xi}_k\rangle = \langle \tilde{C}(\lambda^{\rm im}(t_k,\xi_k) - \lambda^{\rm im}(t_k,\xi_{k-1}) + \lambda^{\rm im}(t_k,\xi_{k-1}) - \lambda^{\rm im}(t_{k-1},\xi_{k-1})), \dot{\xi}_k\rangle$, and:

$$\begin{array}{rl}\langle \tilde{C}(\lambda^{\rm im}(t_k,\xi_k) - \lambda^{\rm im}(t_k,\xi_{k-1})), \dot{\xi}_k\rangle & = \langle \frac{1}{\sqrt{h}}\tilde{C}(\lambda^{\rm im}(t_k,\xi_k) - \lambda^{\rm im}(t_k,\xi_{k-1})), \sqrt{h}\dot{\xi}_k\rangle \\ & \leq \frac{1}{2h}||\tilde{C}||^2\;||\lambda^{\rm im}(t_k,\xi_k) - \lambda^{\rm im}(t_k,\xi_{k-1})||^2 + \frac{h}{2}||\dot{\xi}_k||^2 \\ & \leq \frac{1}{2h}||\tilde{C}||^2\;k_\lambda^2(t_k)||\xi_k - \xi_{k-1}||^2 + \frac{h}{2}||\dot{\xi}_k||^2 \\ & = \frac{h}{2}||\tilde{C}||^2\;k_\lambda^2(t_k)||\dot{\xi}_k||^2 + \frac{h}{2}||\dot{\xi}_k||^2 = \frac{h}{2}(||\tilde{C}||^2\;k_\lambda^2(t_k)+1)\langle \dot{\xi}_k, \dot{\xi}_k\rangle.\end{array} \tag{36}$$

Let us now analyze the term $\langle \tilde{C}(\lambda^{\rm im}(t_k,\xi_{k-1}) - \lambda^{\rm im}(t_{k-1},\xi_{k-1})), \dot{\xi}_k\rangle$. To that aim let us make the following:

Assumption 8 *The function $\lambda^{\rm im}(\cdot,\xi)$ is differentiable on $[0,T]$, and $||\frac{\partial\lambda^{\rm im}(t,\xi)}{\partial t}|| \leq d < +\infty$ for some d and all $\xi \in I\!R^n$, $t \in [0,T]$.*

Let us recall that $\lambda^{\rm im}(t,\xi)$ is the least-norm element of the set $\Phi(t, CR^{-1}\xi)$ [54, Lemma 3]. Assumption 8 is trivially verified if Assumption 7 holds, since $\Phi(t,x) = (\partial\sigma_{\Gamma(t)} + D)^{-1}(x)$, $\Gamma(t) = K^\star - F(t)$. Now we have:

$$\begin{array}{rl}\langle \tilde{C}(\lambda^{\rm im}(t_k,\xi_{k-1}) - \lambda^{\rm im}(t_{k-1},\xi_{k-1})), \dot{\xi}_k\rangle & = \langle \tilde{C}\int_{t_{k-1}}^{t_k}\frac{\partial\lambda^{\rm im}}{\partial t}(t,\xi_{k-1})dt, \dot{\xi}_k\rangle \\ & \leq ||\tilde{C}||\;hd\;||\dot{\xi}_k|| \leq \frac{1}{2}hd^2||\tilde{C}||^2 + \frac{1}{2}h\langle \dot{\xi}_k, \dot{\xi}_k\rangle.\end{array} \tag{37}$$

Thus grouping (36) and (37) we obtain $\langle \tilde{C}(\lambda^{\rm im}(t_k,\xi_k) - \lambda^{\rm im}(t_{k-1},\xi_{k-1})), \dot{\xi}_k\rangle \leq \frac{h}{2}(||\tilde{C}||^2\;k_\lambda^2(t_k) + 2)\langle \dot{\xi}_k, \dot{\xi}_k\rangle + \frac{1}{2}hd^2||\tilde{C}||^2$. Recalling that $\eta = U^\top\xi$, one sees that the developments in (79) still hold with minor modifications of the constants α_1, α_2, and of item 5, due to the additional terms. Then the conclusions of Proposition 5 hold true (remind that since U is unitary then $\langle U^\top\dot{\xi}_k, U^\top\dot{\xi}_k\rangle = \langle \dot{\eta}_k, \dot{\eta}_k\rangle = \langle \dot{\xi}_k, \dot{\xi}_k\rangle$).

Let us now examine Proposition 6. The additional term in the left-hand side of the inclusion in (84) is $hU^\top R\tilde{C}\lambda^{\rm im}(t_k,\xi_k)$. Let us now have a look at (86). Let us modify w_0 as $w_0 \stackrel{\Delta}{=} U^\top\zeta_0 + \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix} U^\top\xi_0 + U^\top R\tilde{C}\lambda^{\rm im}(0,\xi_0)$. The last term in the left-hand side of (86) is rewritten as:

$$\langle hU^\top RE_k + hU^\top R\tilde{C}(\lambda^{\rm im}(t_k,\xi_k) - \lambda^{\rm im}(t_k,\xi_0) + \lambda^{\rm im}(t_k,\xi_0) - \lambda^{\rm im}(0,\xi_0)), U^\top(\xi_{k+1} - \xi_0)\rangle. \tag{38}$$

Proceeding as above and using the Lipschitz continuity of $\lambda^{\rm im}(t,\xi)$ in the second argument, yields:

$$\begin{array}{ll}\langle \sqrt{h}U_1^\top R\tilde{C}(\lambda^{\rm im}(t_k,\xi_k)-\lambda^{\rm im}(t_k,\xi_0)),\sqrt{h}U_1^\top(\xi_{k+1}-\xi_0)\rangle & \leq \frac{h}{2}k_\lambda^2(t_k)\;||U_1^\top R\tilde{C}||^2\;||\xi_{k+1}-\xi_0||^2 \\ & +\frac{h}{2}||\eta^1_{k+1}-\eta^1_0||^2\end{array} \tag{39}$$

for the first term, and

$$\langle hU^\top R\tilde{C}(\lambda^{\rm im}(t_k,\xi_0)-\lambda^{\rm im}(0,\xi_0)),U^\top(\xi_{k+1}-\xi_0)\rangle \;\; = \langle hU^\top R\tilde{C}\int_0^{t_k}\frac{\partial\lambda^{\rm im}}{\partial t}(t,\xi_0)dt,U^\top(\xi_{k+1}-\xi_0)\rangle \tag{40}$$

for the second term. In view of Assumption 8, one has $||\int_0^{t_k}\frac{\partial\lambda^{\rm im}}{\partial t}(t,\xi_0)dt||\leq t_k d\leq Td$. Hence the second additional term can be treated similarly to $U_1^\top RE_k$. From (39) one sees that the first additional term adds quadratic terms $-\frac{h}{2}(k_\lambda^2(t_k)\;||U_1^\top R\tilde{C}||^2+1)\langle\eta^1_{k+1}-\eta^1_0,\eta^1_{k+1}-\eta^1_0\rangle$ in the right-hand side of (87). Consequently the matrix $\tilde{P}$ in (88) has to be modified as $\tilde{P}=\bar{P}-hM_{\rm sch}-hI_p-\frac{h}{2}(k_\lambda^2(t_k)\;||U_1^\top R\tilde{C}||^2+1)I_p$. Thus item 4 in Proposition 6 has to be changed accordingly to guarantee $\tilde{P}\succ 0$, by augmenting the coercivity of the Schur complement $M_{\rm sch}$ (or of $\bar{P}$, or by imposing small enough time-step $h>0$, see Remark 2).

Remark 3 *The above convergence and existence results rely on Assumption 7 which removes the dependence on time in the set-valued part of the dynamics,* i.e., *the set $\tilde{\Gamma}(t)=\tilde{\Gamma}$ in (27). In the time-varying case the proofs for convergence and existence of solutions have to be modified because (27) is a singular FOSwP. It is possible that the approaches used for non singular FOSwP (*via *Moreau-Yosida regularisation or time-stepping discretization [20, 41]) may be adapted to the singular case.*

4.3 Numerical computations

When $D=0$ the VI(RP^hR^{-1},q_k,f_k) to be solved at each timestep is in (8) with $f_k(\cdot)=\psi_{K^\star-F_k}\circ CR^{-1}(\cdot)=\psi_{\tilde{\Gamma}_k}(\cdot)$, $\tilde{\Gamma}_k=\{\xi\in I\!R^n\mid CR^{-1}\xi\in K^\star-F_k\}$. Thus the inclusion in (7) is equivalent to:

$$RP^hR^{-1}\xi_{k+1}-RPR^{-1}\xi_k-hRE_k\in-\mathcal{N}_{\tilde{\Gamma}_k}(\xi_{k+1}) \tag{41}$$

When a solution is guaranteed to exist according to Proposition 2, efficient numerical algorithms exist to calculate a solution to such VIs using Lemke's algorithm, interior point methods, reformulation as quadratic programs, *etc* [2, 29], a strong property being that $RP^hR^{-1}\succcurlyeq 0$ (most of the aforementioned algorithms allow for nonsymmetry). Now let us consider the case $D\succcurlyeq 0$ in section 3.2 and the VI in (20). First the calculation of the term $g(t_k,\xi_k)$ requires the calculation of an element $\lambda_{\alpha,k}$ of the nonempty set $\Phi(t_k,CR^{-1}\xi_k)=(\partial\sigma_{\Gamma(t_k)}+D)^{-1}(CR^{-1}\xi_k)$, with $\Gamma(t_k)=K^\star-F(t_k)$, then computng its projection onto $\mathrm{Im}(D)$ (recall that by assumption $D=D^\top$), with the projector defined as $DD^\dagger$ [11, Proposition 6.1.6]. This boils down to solving a linear cone complementarity problem (LCCP) with unknown $-\lambda_{\alpha,k}$, as: $K\ni(-\lambda_{\alpha,k})\perp D(-\lambda_{\alpha,k})+CR^{-1}\xi_k+F_k\in K^\star$. Second the VI$(RPR^{-1},q_k,f_k)$ in (20) relies on the calculation of the subdifferential of $f(t,\xi)$ in (18), which may not be obvious. One way to implement the scheme is to assume that the conditions of Proposition 4 for existence are fullfilled, and that the DVLCS can be transformed as in (29) (with $D\neq 0$), using item 4 in Proposition 5. The implicit time-discretization of (29) yields a LCCP with unknown λ_{k+1}, which corresponds to the discretized LCCS in (30). Since $\eta_k=U^\top\xi_k$ with U in item 4 of Proposition 5, the existence of ξ_{k+1} guarantees that of η_{k+1}. This in turn guarantees that the LCCP obtained from (30) has a solution λ_{k+1} (possibly non-unique). Advancing the numerical algorithm to step $k+1$ then boils down to solving this LCCP. It is noteworthy that if $D = D^\top \succ 0$ the mapping $\Phi(t,\zeta)$ is single-valued Lipschitz continuous, using [17, Proposition 1]. In this case using (65) (66) it follows that the DVLCS in (1) is a differential-algebraic equation with nonlinear Lipschitz continuous right-hand side.

5 Relaxation of Assumption 2

Imposing $X \succ 0$ in Assumption 2 may be restrictive. This is however necessary if one wants to apply the coordinate change with R. Let us instead make the following assumption about the DVLCS in (1).

Assumption 9 *There exists a matrix X such that $XB = C^\top$, $XP \succcurlyeq 0$, and $D = 0$.*

The constraints imposed in this assumption are close to those satisfied by positive real descriptor variable systems [19, 31]. Let us now consider a matrix V such that $VV^\top = I_n$. Then from Assumption 9 one has $V^\top XB = V^\top C^\top$. Thus:

$$\begin{array}{ll} P_V V^\top \dot{x}(t) & = V^\top XAVV^\top x(t) + V^\top XB\lambda(t) + V^\top XE(t) \\ & = V^\top XAVV^\top x(t) + V^\top C^\top \lambda(t) + V^\top XE(t) \\ & \in A_V V^\top x(t) - V^\top C^\top \partial\psi_{K^\star}(CVV^\top x(t) + F(t)) + V^\top XE(t), \end{array} \tag{42}$$

where $A_V \triangleq V^\top XV\ V^\top AV$, $P_V \triangleq V^\top XPV \succcurlyeq 0$. Let us define $\varphi \triangleq V^\top x$, then we obtain a descriptor-variable FOSwP:

$$\begin{array}{ll} P_V \dot{\varphi}(t) & \in A_V \varphi(t) - V^\top C^\top \partial\psi_{K^\star}(CV\varphi(t) + F(t)) + V^\top XE(t) \\ & \in A_V \varphi(t) - \partial\psi_{\bar{\Gamma}(t)}(\varphi(t)) + V^\top XE(t), \end{array} \tag{43}$$

where $\bar{\Gamma}(t) \triangleq \{w \in I\!R^n \mid CVw + F(t) \in K^\star\}$ and we assume that there exists w_0 such that $CVw_0 + F(t) \in K^\star$ for any $t \geq 0$ to apply the Convex Analysis chain rule. Let us now discretize the DI in (43) with a fully implicit method:

$$(P_V - hA_V)\varphi_{k+1} - P_V\varphi_k - hV^\top XE_k \in -\partial\psi_{\bar{\Gamma}_k}(\varphi_{k+1}), \tag{44}$$

The inclusion in (44) has the same structure as the one in (8), with different matrices however. Similarly as above a semi-implicit method yields:

$$P_V\varphi_{k+1} - (P_V + hA_V)\varphi_k - hV^\top XE_k \in -\partial\psi_{\bar{\Gamma}_k}(\varphi_{k+1}), \tag{45}$$

This does not modify fundamentally the next analysis, excepted that Proposition 10 requires $P_V - hA_V \succcurlyeq 0$ for (44) and $P_V \succcurlyeq 0$ for (45) (the latter being implied by Assumption 9). It is noteworthy that neither P nor X need to be symmetric and positive semidefinite. The next result is similar to Proposition 2 and is given without proof.

Proposition 8 *Consider (45). Let Assumption 9 hold true, $h > 0$ be given, and $q_k = -(P_V + hA_V)\varphi_k - hV^\top XE_k$. Let us consider the set:*

$$\mathcal{S} \triangleq \{\xi \in I\!R^n \mid CV\xi \geq 0\} \cap \{\xi \in I\!R^n \mid P_V\xi = \sum_{i=1}^{m} \lambda_i (CV)_{i\bullet}^\top,\ \lambda_i \geq 0\} \cap \mathrm{Ker}(P_V + P_V^\top). \tag{46}$$

1. *If* $\mathcal{S} = \{0\}$*, then the VI(*P_V, q_k, ψ_{Φ_k}*) has at least one solution.*

2. *If* $\mathcal{S} \neq \{0\}$*, and if there exists* $\xi_0 \in \bar{\Gamma}_k$ *such that*

$$\langle q_k - P_V^\top \xi_0, v\rangle > 0 \textit{ for all } v \in \mathcal{S},\ v \neq 0, \tag{47}$$

   then the VI(P_V, q_k, ψ_{Φ_k}*) has at least one solution.*

3. *If* ξ_{k+1}^1 *and* ξ_{k+1}^2 *are two solutions of the VI(*P_V, q_k, ψ_{Φ_k}*), then* $\xi_{k+1}^1 - \xi_{k+1}^2 \in \mathrm{Ker}(P_V + P_V^\top)$*.*

4. *Let* P_V *be symmetric. If* ξ_{k+1}^1 *and* ξ_{k+1}^2 *are two solutions of the VI(*P_V, q_k, ψ_{Φ_k}*), then* $\langle q_k, \xi_{k+1}^1 - \xi_{k+1}^2\rangle = 0$*.*

5. *Let* P_V *be symmetric. Then any solution of the VI(*P_V, q_k, ψ_{Φ_k}*) is also a solution of the optimization problem:* $\min_{\xi \in \bar{\Gamma}_k} \frac{1}{2}\xi^\top P_V \xi + \langle q_k, \xi\rangle$*.*

The above results hold with $V = I_n$, however V may be used to transform P_V (into a block diagonal matrix if XP is symmetric). It is noteworthy that A_V can be at most positive semi-definite, hence the results in section 4 do not apply because they require strong coercivity properties (see items 4, 5, 6 in Proposition 5). Therefore Assumption 9 allows us to extend the state transformation $\xi = Rx$ introduced in [15] (see [20, section 3.4] for a bibliography) and to show the well-posedness of the OSNSP (44) (45), but the well-posedness of the DVLCS (43) remains an open issue.

6 Examples

This section is devoted to present various examples which illustrate the applicability and the limitations of the above developments. Let us remind that electrical circuits with ideal diodes can serve equally well as models for hydraulic circuits with check valves [42, 34] [32, Chapter 1], thus extending the range of applications.

6.1 First example

These are academic examples which illustrate the developments of section 3.1. Let $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ (hence $\mathrm{rank}(P) = 2$), $D = 0$, $A = \mathrm{diag}(a_1, a_2, a_3)$, $B = C^\top \Rightarrow X = I_3$ in Assumption 2. If $\det(P^h + (P^h)^\top)) = (1 - ha_1)(ha_2a_3 - a_2 - a_3) - 2(1 - ha_3) \neq 0$, one gets $\mathrm{Ker}(P^h + (P^h)^\top)) = \{0\}$. Then the OSNSP for the fully implict scheme (3) always has a solution. Let us now examine the semi-implicit scheme in (4). One has $\det(P + P^\top) = -2$ so $\mathrm{Ker}(P + P^\top) = \{0\}$, hence the OSNSP always has a solution.

Consider now $P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$ and the unitary matrix $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$, then $U^\top PU = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$. Let $P = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & 2 & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$, then $U^\top PU = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *i.e.*, $\bar{P} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \succ 0$. The convergence results in section 4.1 hold provided A and B are suitably chosen.

Let now $P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A = \mathrm{diag}(a_1, a_2, a_3) \succ 0$, $B = C^\top \Rightarrow X = I_3$. Then $\mathrm{Ker}(P^h + (P^h)^\top) = \{\xi \in I\!R^3 \mid 2\xi_1 + \xi_2 = 0, \xi_1 + 2\xi_2 = 0, \xi_3 \in I\!R\} = \{\xi \in I\!R^3 \mid \xi_1 = 0, \xi_2 = 0, \xi_3 \in I\!R\}$. Let $C = \begin{pmatrix} c_{11} & c_{12} & 1 \\ c_{21} & c_{22} & 1 \end{pmatrix}$. We have $P_h\xi = \begin{pmatrix} (2 - ha_1)\xi_1 + \xi_2 \\ \xi_1 + (2 - ha_2)\xi_2 \\ -ha_3\xi_3 \end{pmatrix} = \lambda_1 C_1^\top + \lambda_2 C_2^\top$. In the set $\mathcal{S}$ of Proposition 2 we obtain $-ha_3\xi_3 = \lambda_1 + \lambda_2 \geq 0$, hence $\xi_3 \leq 0$. Also $\mathcal{S}_1 = \{\xi \in I\!R^3 \mid c_{11}\xi_1 + c_{12}\xi_2 + \xi_3 \geq 0, c_{21}\xi_1 + c_{22}\xi_2 + \xi_3 \geq 0\}$, hence inside $\mathcal{S}$ we get $\mathcal{S}_1 = \{\xi \in I\!R^3 \mid \xi_3 \geq 0\}$. It follows that $\xi_3 = 0$ in $\mathcal{S}$, hence $\xi_1 = \xi_2 = \xi_3 = 0$ and $\mathcal{S} = \{0\}$. This example shows that the kernel of $P^h + (P^h)^\top$ may be non trivial, but still $\mathcal{S} = \{0\}$ due to the inequalities.

6.2 Second example

Let us consider the circuit with an ideal diode in Figure 1. Its dynamics may be written as:

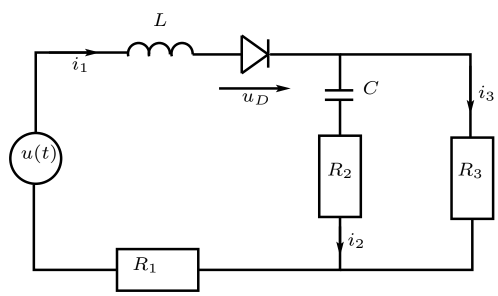

Figure 1: An $RLCD$ passive circuit.

$$\begin{cases} \dot{\xi}_1(t) = -\frac{R_1}{L}\xi_1(t) - \frac{R_3}{L}\xi_3(t) + \frac{1}{L}u(t) + \frac{1}{L}u_D(t) \\ \dot{\xi}_2(t) = \frac{R_3}{R_2}\xi_3(t) - \frac{1}{R_2\mathbf{C}}\xi_2(t) \\ 0 = \xi_1(t) + \frac{1}{R_2\mathbf{C}}\xi_2(t) - (1 + \frac{R_3}{R_2})\xi_3(t) \;\; (= \xi_1(t) - \xi_3(t) - \dot{\xi}_2(t)) \\ 0 \leq u_D(t) \perp \xi_1(t) \geq 0, \end{cases} \tag{48}$$

where $\xi_1 = i_1$, $\xi_2(t) = \int_0^t i_2(s)ds$, $\xi_3 = i_3$. Obviously this dynamical system is not minimal, since we could eliminate ξ_3 using the equality constraint. However on one hand our goal is to illustrate the above developments, one the other hand designers are not always interested by a reduction of the coordinates. According to the notation in (1) we obtain: $x^\top = (\xi_1, \xi_2, \xi_3)$, $\lambda = u_D$, $m = 1$, and:

$$P = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} -\frac{R_1}{L} & 0 & -\frac{R_3}{L} \\ 0 & -\frac{1}{R_2\mathbf{C}} & \frac{R_3}{R_2} \\ 1 & \frac{1}{R_2\mathbf{C}} & -(1 + \frac{R_3}{R_2}) \end{pmatrix} \quad B = \begin{pmatrix} \frac{1}{L} \\ 0 \\ 0 \end{pmatrix} \quad E(t) = \begin{pmatrix} \frac{1}{L}u(t) \\ 0 \\ 0 \end{pmatrix} \tag{49}$$

$$D = 0 \qquad F(t) = 0 \; (\forall t \geq 0) \qquad C = (1 \;\; 0 \;\; 0).$$

Notice that we could also work with $P = \begin{pmatrix} L & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It follows that Assumption 2 is satisfied with $X = \mathrm{diag}(L, \star, \star)$ where $\star$ is any positive real, hence $R = \mathrm{diag}(\sqrt{L}, \sqrt{\star}, \sqrt{\star})$. Also $RPR^{-1} \not\succcurlyeq 0$

(Assumption 3). To simplify the calculations let us take $L = 1$H (though such a value may not possess a strong physical meaning) so that $X = R = I_3$. One has:

$$P^h = P - hA = \begin{pmatrix} 1 + h\frac{R_1}{L} & 0 & h\frac{R_3}{L} \\ 0 & 1 + \frac{h}{R_2\mathbf{C}} & -h\frac{R_3}{R_2} \\ -h & -\frac{h}{R_2\mathbf{C}} & h(1+\frac{R_3}{R_2}) \end{pmatrix}. \tag{50}$$

To apply Proposition 2 with scheme (3), we need to compute the set $\mathcal{S}$ in (10). One has $\det(P^h + (P^h)^\top) > 0$ for $h > 0$ small enough. Then we need to check that $P^h \succcurlyeq 0$ in (50) (Assumption 3). It can be verified that for $R_1 = R_2 = R_3 = 1\Omega$, $L = 1$H, $\mathbf{C} = 1$F [1], then $P^h \succ 0$ for all $0 < h < 1$. Hence $\mathcal{S} = \{0\}$. This shows that there exists circuits as in Figure 1 such that Proposition 2 item 1, applies. Let us now investigate the convergence properties of the scheme (3). To that aim let us check the conditions of Proposition 6. Here Assumption 7 holds, and $\text{Im}(B) \subset \text{Im}(P)$. The matrix U is used to set P in its block diagonal form, and can thus be chosen $U = I_3$ here. The matrix A can be written as $A = \begin{pmatrix} -\frac{R_1}{L} & 0 & 0 \\ 0 & -\frac{1}{R_2\mathbf{C}} & 0 \\ 0 & 0 & -(1+\frac{R_3}{R_2}) \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{R_3}{L} \\ 0 & 0 & \frac{R_3}{R_2} \\ 1 & \frac{1}{R_2\mathbf{C}} & 0 \end{pmatrix}$. Corollary 2 can be used to determine conditions that guarantee that $A \prec 0$. This is the case for the above choice of parameters. Here we find that $M_{11} = \begin{pmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2\mathbf{C}} \end{pmatrix}$, $M_{22} = -(1 + \frac{R_3}{R_2})$. One has also:

$$M_{\text{sch}} = \begin{pmatrix} \frac{-R_1}{L} - \frac{R_3}{L}\alpha & \frac{-R_3}{LR_2\mathbf{C}}\alpha \\ \frac{R_3}{R_2}\alpha & \frac{R_3}{R_2^2\mathbf{C}}\alpha - \frac{1}{R_2\mathbf{C}} \end{pmatrix}, \tag{51}$$

with $\alpha = (1 + \frac{R_3}{R_2})^{-1}$. For such a choice, and if all the foregoing conditions hold, convergence as proved in section 4 follows.

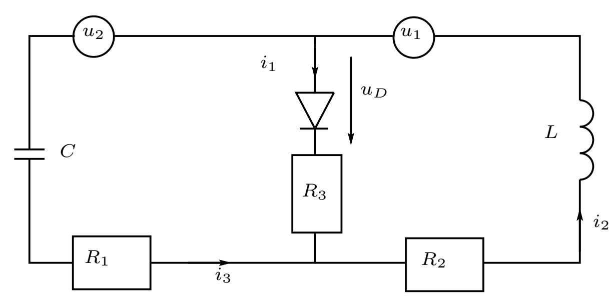

Figure 2: An $RLCD$ passive circuit.

[1] Again, such values may not be very realistic, but our goal is merely to show that there exists circuits to which our tools apply.

6.3 Third example

Let us consider the circuit in Figure 2, with three positive resistances. Its dynamics is given by:

$$\begin{cases} \dot{\xi}_1(t) = -\frac{R_2}{L}\xi_1(t) - \frac{R_3}{L}\xi_3(t) + \frac{1}{L}u_D + \frac{1}{L}u_1(t) \\ \dot{\xi}_2(t) = \xi_1(t) - \xi_3(t) \\ 0 = \xi_2(t) + R_1\mathbf{C}\xi_1(t) - (R_1+R_3)\mathbf{C}\xi_3(t) + \mathbf{C}u_D(t) + \mathbf{C}u_2(t) \\ 0 \le u_D \perp \xi_3(t) \ge 0. \end{cases} \tag{52}$$

One has $C = (0\ 0\ 1)$, $B = (\frac{1}{L}\ 0\ 1)^\top$, hence there is no $X \succ 0$ such that $XB = C^\top$ and Assumption 2 fails. However $X = \mathrm{diag}(0, \alpha, \beta)$, $\alpha > 0$, $\beta > 0$, satisfies Assumption 9. Let us choose $V = I_3$. One has $\mathcal{S} = \{\xi \in I\!R^3 \ \ \xi_1 \in I\!R, \xi_2 = 0, \xi_3 \ge 0\} \neq \{0\}$. Calculations show that item 2 in Proposition 8 may hold if at step k one has $\varphi_k^1 = -\frac{R_3}{R_2}\varphi_k^3$, $h > 0$ small enough and $E_k^3 < 0$. One has to remind that Proposition 10 provides sufficient conditions only, hence it is possible that existence holds even if these conditions are not satisfied. In addition different V could be used.

6.4 Fourth example

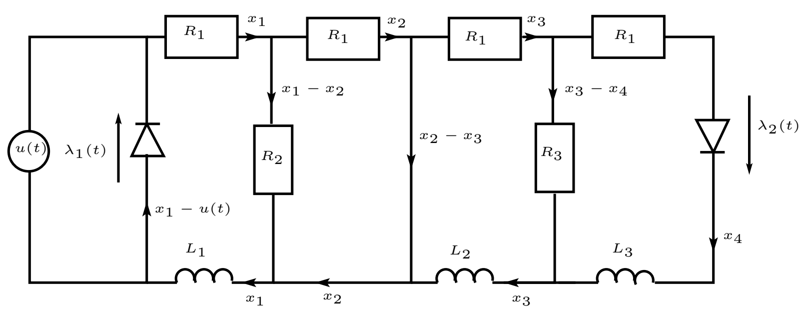

Figure 3: An $RLCD$ passive circuit.

Let us consider the circuit in Figure 3 made of resistances, inductors, ideal diodes and a source of voltage, the dynamics of which are developed in [44]. The variables x_1, x_2, x_3, x_4 are currents through the resistances, λ_1 and λ_2 are the voltages accross the diodes, $u(t)$ is an exogenous source of current. The dynamics can be expressed as:

$$\begin{cases} L_1\dot{x}_1(t) + R_1x_1(t) + R_2(x_1(t) - x_4(t)) - \lambda_1(t) = 0 \\ L_2\dot{x}_2(t) + R_1x_2(t) + R_3(x_2(t) - x_3(t)) = 0 \\ L_3\dot{x}_3(t) + R_1x_3(t) - R_3(x_2(t) - x_3(t)) - \lambda_2(t) = 0 \\ 0 = R_1x_4(t) - R_2(x_1(t) - x_4(t)) \\ 0 \le \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} x_1(t) - u(t) \\ x_3(t) \end{pmatrix} \ge 0. \end{cases} \tag{53}$$

One has $P = \begin{pmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} -(R_1+R_2) & 0 & 0 & R_2 \\ 0 & -(R_1+R_3) & R_3 & 0 \\ 0 & R_3 & -(R_1+R_3) & 0 \\ R_2 & 0 & 0 & -(R_1+R_2) \end{pmatrix}$,
$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = B^\top$, $D = 0$, $F(t) = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}$, $E(t) = 0$. One infers that for R_1 large enough

(compared with R_2 and R_3), then $A = A^\top \prec 0$. Assumption 2 holds with $X = I_4$, Assumption 3 a) and b) hold. In fact provided that $A \prec 0$ then $P^h \succ 0$ for all $h > 0$. It follows that the set $\mathcal{S} = \{0\}$ and using items 1 and 3 in Proposition 2 the OSNSP has a unique solution. Now one sees that the system is in the canonical form of Proposition 5 (see (29)), with $\eta_1 = (x_1, x_2, x_3)^\top$, $\eta_2 = x_4$, $M_{11} = \begin{pmatrix} -(R_1+R_2) & 0 & 0 \\ 0 & -(R_1+R_3) & R_3 \\ 0 & R_3 & -(R_1+R_3) \end{pmatrix}$, $M_{21} = M_{12}^\top = (R_2, 0, 0)$, $M_{22} = -(R_1+R_2)$. One has $\mathrm{Im}(B) \subset \mathrm{Im}(P)$, and $M_{\mathrm{sch}} = M_{11} + \frac{1}{R_1+R_2}\begin{pmatrix} R_2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus provided that R_1 is large enough (which implies that $-(R_1+R_2)+\frac{R_2^2}{R_1+R_2} < -1$), one has $M_{\mathrm{sch}}+I_3 \prec 0$ (item 4 in Proposition 6). It is also verified that item 5 in Proposition 5 holds for large enough R_1. One infers that Lemma 3 and Proposition 7 apply.

7 Preliminary results towards another approach

A central property for the existence of solutions is the positive definiteness conditions of Propositions 5 and 6. In particular $-M_{22} \succ 0$ may be a stringent assumption. Another stringent assumption is the decoupling one in item 5 of Proposition 6, that is used to avoid the compactness assumption in item 3 of Proposition 5. Let us outline in this section a different way of analysing the boundedness of the discrete-time solutions, using minimal norm elements. First of all let us consider the discretization of (29) with $K = \mathbb{R}^m_+$ (recall that this dynamics is obtained with the assumptions made in Proposition 5). We obtain the LCP: $0 \le \lambda_{k+1} \perp w_{k+1} = CR^{-1}U\eta_{k+1} + D\lambda_{k+1} + F_k \ge 0$. It follows from [28, Theorem 3.1.7] and [36, Lemma 1] that there exists a constant $\gamma > 0$ depending only on D such that for any $CR^{-1}U\eta_{k+1} + F_k$ such that the set of solutions is not empty, the least-norm solution $\lambda_{k+1}^{\min}$ satisfies $||\lambda_{k+1}^{\min}|| \le \gamma\, ||CR^{-1}U\eta_{k+1}+F_k|| \le \gamma\, ||CR^{-1}U||\; ||\eta_{k+1}||+\gamma\, ||F_k||$. Assume that the conditions for the well-posedness of the OSNSP as stated in Propositions 2 and 4 hold. This guarantees that the above LCP is solvable (because the iterates η_{k+1} guarantee it even in case $D = 0$) and the minimum element in the set of solutions exists. Assume that $\lambda_{k+1}^{\min}$ is used in the discretized scheme (which we may name the *minimal norm implicit discretization*), yielding:

$$\left\{\begin{array}{l} (\bar{P} - hM_{11})\eta_{k+1}^1 = \bar{P}\eta_k^1 + hM_{12}\eta_{k+1}^2 + hU_1^\top RB\lambda_{k+1}^{\min} + hU_1^\top RE_k \\ M_{21}\eta_{k+1}^1 + M_{22}\eta_{k+1}^2 + U_2^\top RB\lambda_{k+1}^{\min} + U_2^\top RE_k = 0 \\ 0 \le \lambda_{k+1}^{\min} \perp w_{k+1} = CR^{-1}U\eta_{k+1} + D\lambda_{k+1}^{\min} + F_k \ge 0. \end{array}\right. \tag{54}$$

It is noteworthy that this scheme is equivalent to the one in (2) as long as $h > 0$. This is rewritten as:

$$\underbrace{\begin{pmatrix} \bar{P} - hM_{11} & -hM_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{\triangleq \tilde{M}_h}\eta_{k+1} = \begin{pmatrix} \bar{P} & 0 \\ 0 & 0 \end{pmatrix}\eta_k + \underbrace{\begin{pmatrix} hU_1^\top \\ U_2^\top \end{pmatrix}}_{\triangleq \tilde{U}_h^\top} RB\lambda_{k+1}^{\min} + \begin{pmatrix} hU_1^\top \\ U_2^\top \end{pmatrix} RE_k, \tag{55}$$

Since the OSNSP is solvable, there exists $\bar{\eta}_k \in \mathbb{R}^n$ such that [11, Proposition 6.1.7]:

$$\eta_{k+1} = \tilde{M}_h^\dagger\left(\begin{pmatrix} \bar{P} & 0 \\ 0 & 0 \end{pmatrix}\eta_k + \tilde{U}_h^\top RB\lambda_{k+1}^{\min} + \tilde{U}_h^\top RE_k\right) + (I_n - \tilde{M}_h^\dagger\tilde{M}_h)\bar{\eta}_k, \tag{56}$$

which implies (using compatible norms):

$$\begin{array}{rl} \|\eta_{k+1}\| & \leq \|\tilde{M}_h^{\dagger}\begin{pmatrix} \bar{P} & 0 \\ 0 & 0 \end{pmatrix}\| \|\eta_k\| + \|\tilde{M}_h^{\dagger}\tilde{U}_h^{\top} RE_k\| + \|(I_n - \tilde{M}_h^{\dagger}\tilde{M}_h)\bar{\eta}_k\| \\ & +\|\tilde{M}_h^{\dagger}\tilde{U}_h^{\top} RB\|(\gamma\ ||CR^{-1}U||\ ||\eta_{k+1}|| + \gamma\ ||F_k||), \end{array} \quad (57)$$

thus:

$$\begin{array}{rl} (1-\gamma\|\tilde{M}_h^{\dagger}\tilde{U}_h^{\top} RB\|\|CR^{-1}U\|)\|\eta_{k+1}\| & \leq \|\tilde{M}_h^{\dagger}\begin{pmatrix} \bar{P} & 0 \\ 0 & 0 \end{pmatrix}\| \|\eta_k\| + \|\tilde{M}_h^{\dagger}\tilde{U}_h^{\top} RE_k\| \\ & +\|(I_n - \tilde{M}_h^{\dagger}\tilde{M}_h)\bar{\eta}_k\| + \gamma\|\tilde{M}_h^{\dagger}\tilde{U}_h^{\top} RB\|\|F_k||. \end{array} \quad (58)$$

Assume that $\gamma\|\tilde{M}_h^{\dagger}\tilde{U}_h^{\top} RB\|\|CR^{-1}U\| < 1$ for all $h > 0$. Then (58) can be rewritten as $||\eta_{k+1}|| \leq \alpha||\eta_k|| + \beta_k$ for some $\alpha > 0$, and $\beta_k \geq 0$ depends only on F_k and E_k, the external signals, as well as on $\bar{\eta}_k$. If $\tilde{M}_h$ is invertible, which does not require M_{22} nor M_{11} to be definite, then $\bar{\eta}_k = 0$. It is inferred that $||\eta_{k+1}|| \leq \alpha^{k+1}||\eta_0|| + \sum_{i=0}^{k}\alpha^i\beta_{k-i}$. Some further assumptions have to be made to conclude about the uniform boundedness of the iterates as $h \to 0$ (or as $k \to +\infty$), for instance to secure that $\alpha = \frac{\|\tilde{M}_h^{\dagger}\begin{pmatrix} \bar{P} & 0 \\ 0 & 0 \end{pmatrix}\|}{1-\gamma\|\tilde{M}_h^{\dagger}\tilde{U}_h^{\top} RB\|\|CR^{-1}U\|} \leq 1 \Leftrightarrow \|\tilde{M}_h^{\dagger}\begin{pmatrix} \bar{P} & 0 \\ 0 & 0 \end{pmatrix}\| + \gamma\|\tilde{M}_h^{\dagger}\tilde{U}_h^{\top} RB\|\|CR^{-1}U\| \leq 1$. Roughly speaking, $\tilde{M}_h$ should be "big" while γ should be small.
For the sake of briefness we do not push forward the developments. However it is noteworthy that this approach allows us to relax some assumptions made in the foregoing sections (coercivity conditions with $-M_{22} \succ 0$ and $-M_{11} \succ 0$, decoupling with $U_2^{\top} RB = 0$, time-invariance with $F(t)$ constant), at the price of doing different assumptions, however. Permitting a varying $F(t)$ allows to encompass some classes of singular FOSwP [20].

Remark 4 *All the above convergence results are obtained under the condition that the matrix M_{22} is invertible, which is sufficient only and linked to the tools used for the proofs of Propositions 5 and 6. This is interpreted as an index 1 on the algebraic part of the DVLCS in (29). Let us consider (29) with $M_{22} = 0$, the equality becomes $M_{21}\eta^1(t) + U_2^{\top} RB\lambda(t) + U_2^{\top} RE(t) = 0$, with $\lambda(t) \in (D + \partial\psi_K)^{-1}(-CR^{-1}U\eta(t) - F(t))$. Therefore (29) is rewritten equivalently as:*

$$\begin{array}{ll} (a) & \bar{P}\dot{\eta}^1(t) \in M_{11}\eta^1(t) + M_{12}\eta^2(t) + U_1^{\top} RE(t) + U_1^{\top} RB(D + \partial\psi_K)^{-1}(-CR^{-1}U\eta(t) - F(t)) \\ (b) & 0 \in M_{21}\eta^1(t) + U_2^{\top} RE(t) + U_2^{\top} RB(D + \partial\psi_K)^{-1}(-CR^{-1}U\eta(t) - F(t)) \end{array} \quad (59)$$

Proposition 6 does not apply to (59). There are various ways to analyse (59), depending on the rank of M_{21}, D, and several other properties. The inclusion (59) (b) is a generalized equation for $\eta^1(t)$, parameterised with $\eta^2(t)$. In particular $D = 0$ yields a differential inclusion that could embed singular FOSwP. This is not tackled in this paper and is left as a future research work. It is noteworthy that passive descriptor-variable LCS [24, 31] possess a Weierstrass structure with $M_{22} = 0$ but with specific properties that may make them amenable for analysis.

8 Conclusion

This article deals with well-posedness issues in a class of singular nonsmooth set-valued systems, named descriptor-variable linear complementarity systems (DVLCS). The implicit Euler time-discretization is analysed. First, conditions which guarantee the existence and the uniqueness of solutions to the one-step nonsmooth problem, are stated. Then the convergence of the approximate piecewise-linear discrete solutions is studied. This article leaves many issues open, the above results could be generalized in various directions. The set-valued operator considered in this work (represented through the complementarity conditions, equivalently a normal cone to a convex set) could be extended to the subdifferential of proper lower semicontinuous convex functions as initiated in [17, 54], or to maximal monotone operators [26], or to normal cones to prox-regular sets, see [20] for more references. The singular matrix P could be supposed to be state-dependent $P(x, z)$ (with applications in circuits with nonlinear resistors, inductances and capacitors [20] [1, section 3.5]). Most importantly the problem of bounded-variation solutions with discontinuities, which is well understood in DAEs, in sweeping processes and in Linear Complementarity Systems, when considered separately, has not been tackled and deserves future analysis. In that same vein the well-known notions of differentiation index in DAEs, and of relative degree in LCS, have to be further analysed for DVLCS.

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A Preservation of positive definiteness with perturbation

We give here an excerpt of [27, Theorem 2.11], and a corollary of it. Let us recall that for a given $M \in \mathbb{R}^{n\times n}$, $||M||_{2,2}$ is the induced matricial norm such that $||M||_{2,2} = \sigma_{\max}(M)$ (the largest singular value).

Theorem 1 *[27] Let $M \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Then every matrix*

$$A \in \{A \in \mathbb{R}^{n\times n} \mid \left|\left|\left(\frac{M+M^\top}{2}\right)^{-1}\right|\right|_{2,2} ||M - A||_{2,2} < 1\}$$

is positive definite.

Corollary 2 *Let $D = P+N$, where D, P and N are $n\times n$ real matrices, and $P \succ 0$, not necessarily symmetric. If*

$$||N||_2 < \frac{1}{||\left(\frac{P+P^\top}{2}\right)^{-1}||_{2,2}} \tag{60}$$

then $D \succ 0$.

B Well-posedness of Variational Inequalities

The next results use the notions of recession functions and cones, which we briefly introduce now [48, 33, 58]. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function, we denote by $\mathrm{dom}(f) \stackrel{\Delta}{=} \{x \in \mathbb{R}^n |\ f(x) < +\infty\}$ the domain of the function $f(\cdot)$. The Fenchel transform $f^\star(\cdot)$ of $f(\cdot)$ is the proper, convex and lower semicontinuous function defined by

$$(\text{for all } z \in \mathbb{R}^n)|\ f^\star(z) = \sup_{x\in\ \mathrm{dom}(f)} \{\langle x, z\rangle - f(x)\}. \tag{61}$$

The subdifferential $\partial f(x)$ of $f(\cdot)$ at $x \in I\!R^n$ is defined by

$$\partial f(x) = \{\omega \in I\!R^n |\ f(v) - f(x) \geq \langle \omega, v - x \rangle, \forall v \in I\!R^n\}.$$

We denote by $\mathrm{Dom}(\partial f) \triangleq \{x \in I\!R^n | \partial f(x) \neq \emptyset\}$ the domain of the subdifferential operator $\partial f : I\!R^n \to I\!R^n$. Recall that (see *e.g.* Theorem 2, Chapter 10, Section 3 in [7]): $\mathrm{Dom}(\partial f) \subset \mathrm{dom}(f)$.

Let x_0 be any element in the domain $\mathrm{dom}(f)$ of $f(\cdot)$, the recession function $f_\infty(\cdot)$ of $f(\cdot)$ is defined by

$$(\text{for all } x \in I\!R^n): \ f_\infty(x) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} f(x_0 + \lambda x).$$

The function $f_\infty : I\!R^n \to I\!R \cup \{+\infty\}$ is a proper convex and lower semicontinuous function. Let $K \subset I\!R^n$ be a nonempty closed convex set. Let x_0 be any element in K. The recession cone of K is defined by [48] [58, Definition 1.11]:

$$K_\infty = \bigcap_{\lambda > 0} \frac{1}{\lambda}(K - x_0) = \{u \in I\!R^n | x + \lambda u \in K, \forall\, \lambda \geq 0, \forall x \in K\}.$$

The set K_∞ is a nonempty closed convex cone that is described in terms of the directions which recede from K. Let us here recall some important properties of the recession function and recession cone (see *e.g.*, [12, Proposition 1.4.8]):

Proposition 9 *The following statements hold:*

a) Let $f_1 : I\!R^n \to I\!R \cup \{+\infty\}$ and $f_2 : I\!R^n \to I\!R \cup \{+\infty\}$ be two proper, convex and lower semicontinuous functions. Suppose that $f_1 + f_2$ is proper. Then for all $x \in I\!R^n$: $(f_1 + f_2)_\infty(x) = (f_1)_\infty(x) + (f_2)_\infty(x)$.

b) Let $f : I\!R^n \to I\!R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and let K be a nonempty closed convex set, such that $f + \Psi_K$ is proper (equivalently dom(f) $\cap K$ is nonempty). Then for all $x \in I\!R^n$: $(f + \Psi_K)_\infty(x) = f_\infty(x) + (\Psi_K)_\infty(x)$.

c) Let $K \subset I\!R^n$ be a nonempty, closed and convex set. Then for all $x \in I\!R^n$: $(\Psi_K)_\infty(x) = \Psi_{K_\infty}(x)$. Moreover for all $x \in K$ and $e \in K_\infty$: $x + e \in K$.

d) If $K \subseteq I\!R^n$ is a nonempty closed and convex cone, then $K_\infty = K$.

e) Let $K = \mathcal{P}(A, b) \triangleq \{x \in I\!R^n | Ax \geq b\}$ for $A \in I\!R^{m \times n}$ and $b \in I\!R^m$. If $K \neq \emptyset$ then $K_\infty = \mathcal{P}(A, 0) = \{x \in I\!R^n | Ax \geq 0\}$.

f) $K \subset I\!R^n$ is a nonempty closed convex bounded set if and only if $K_\infty = \{0_n\}$.

g) Let $f : I\!R^n \to I\!R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Then $\mathrm{epi}(f_\infty) = (\mathrm{epi}(f))_\infty$.

Sets as in item e) are called $\mathcal{H}$-polyhedra, and there is an equivalence between sets $\mathcal{P}(A, 0)$ and finitely generated convex cones [58, Theorem 1.3]. Let us now concatenate [3, Theorem 3, Corollaries 3 and 4]. They concern variational inequalities (VIs) of the form: Find $u \in I\!R^n$ such that

$$\langle \mathbf{M}u + \mathbf{q}, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for all } v \in I\!R^n, \tag{62}$$

where $\mathbf{M} \in I\!R^{n \times n}$ is a real matrix, $\mathbf{q} \in I\!R^n$ a vector and $\varphi : I\!R^n \to I\!R \cup \{+\infty\}$ a proper convex and lower semicontinuous function. The VI in (62) is equivalent to the inclusion $\mathbf{M}u + \mathbf{q} \in -\partial\varphi(u) \Leftrightarrow$

$u \in (\mathbf{M} + \partial\varphi)^{-1}(-\mathbf{q})$. The problem in (62) is denoted as $VI(\mathbf{M}, \mathbf{q}, \varphi)$ in the next proposition. We also set:

$$\mathcal{K}(\mathbf{M}, \varphi) = \{x \in I\!R^n | \mathbf{M}x \in (\mathrm{dom}(\varphi_\infty))^\star\}. \tag{63}$$

Note that $(\mathrm{dom}(\varphi_\infty))^\star$ is the dual cone of the domain of the recession function φ_∞ while $(\mathrm{dom}(\varphi))_\infty$ is the recession cone of $\mathrm{dom}(\varphi)$.

Proposition 10 *[3, Theorem 3, Corollaries 3 and 4] Let $\varphi : I\!R^n \to I\!R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function with closed domain, $\mathbf{M} \in I\!R^{n\times n}$, and suppose that $\mathbf{M} \succcurlyeq 0$ (not necessarily symmetric).*

a) If $(\mathrm{dom}(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) = \{0\}$ then for each $\mathbf{q} \in I\!R^n$, problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.

b) Suppose that $(\mathrm{dom}(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi) \neq \{0\}$. If there exists $x_0 \in \mathrm{dom}(\varphi)$ such that

$$\langle \mathbf{q} - \mathbf{M}^\top x_0, v\rangle + \varphi_\infty(v) > 0,\ \forall v \in\ (dom(\varphi))_\infty \cap \ker\{\mathbf{M} + \mathbf{M}^\top\} \cap \mathcal{K}(\mathbf{M}, \varphi),\ v \neq 0, \tag{64}$$

then problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$ has at least one solution.

b') If $\mathbf{M} = \mathbf{M}^\top$ then one can take $x_0 = 0$ in b).

c) If u_1 and u_2 denote two solutions of problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$ then $u_1 - u_2 \in \ \ker\{\mathbf{M} + \mathbf{M}^\top\}$.

d) If $\mathbf{M} = \mathbf{M}^\top$ and u_1 and u_2 denote two solutions of problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$, then $\langle \mathbf{q}, u_1 - u_2\rangle = \varphi(u_2) - \varphi(u_1)$.

e) If $\mathbf{M} = \mathbf{M}^\top$ and $\varphi(x + z) = \varphi(x)$ for all $x \in \mathrm{dom}(\varphi)$ and $z \in\ \ker\{\mathbf{M}\}$ and $\langle \mathbf{q}, e\rangle \neq 0$ for all $e \in\ \ker\{\mathbf{M}\}, e \neq 0$, then problem $VI(\mathbf{M}, \mathbf{q}, \varphi)$ has at most one solution.

f) If $\mathbf{M} = \mathbf{M}^\top$, then u is a solution of $VI(\mathbf{M}, \mathbf{q}, \varphi)$ if and only if it is a solution of the optimization problem $\min_{x\in I\!R^n} \frac{1}{2}x^\top \mathbf{M}x + \langle \mathbf{q}, x\rangle + \varphi(x)$.

Item d) is [3, Equation (53)], item f) is [3, Equation (50)]. Notice that the function $\varphi(\cdot)$ will never be strictly convex in the case studied in this article (it is an indicator function) so that the strict convexity argument of [3, Theorem 5] which applies when $\mathbf{M}$ is a P_0-matrix never holds. The study of VIs as in (62) can be traced back to [50].

C Some Convex Analysis and Complementarity Theory tools

If $K \subset I\!R^n$ is a set, then $K^\star = \{z \in I\!R^n | \langle z, x\rangle \geq 0$ for all $x \in K\}$ is its dual cone. Its closure is denoted $\bar{K}$. The indicator function of a set $K \subseteq I\!R^n$ is $\psi_K(x) = 0$ if $x \in K$, $\psi_K(x) = +\infty$ if $x \notin K$. If K is closed nonempty convex, we have $\partial\psi_K(x) = \mathcal{N}_K(x)$, the so-called normal cone to K at x, defined as $\mathcal{N}_K(x) = \{v \in I\!R^n \mid v^\top(s - x) \leq 0$ for all $s \in K\}$. When K is finitely represented, *i.e.*, $K = \{x \in I\!R^n \mid k_i(x) \geq 0, 1 \leq i \leq m\}$, and if the functions $k_i(\cdot)$ satisfy some constraint qualification (like, independency, or extensions like the MFCQ), then $\mathcal{N}_K(x)$ is generated by the outwards normals at the active constraints $k_i(x) = 0$, *i.e.*, $\mathcal{N}_K(x) = \{v \in I\!R^n \mid v = -\lambda_i \nabla k_i(x), k_i(x) = 0, \lambda_i \geq 0\}$. Let K be a closed convex cone, then:

$$K^\star \ni x \perp y \in K \ \Leftrightarrow\ x \in -\mathcal{N}_K(y) \iff y \in -\mathcal{N}_{K^\star}(x). \tag{65}$$

Let $M = M^\top \succ 0$, x and y two vectors, then

$$M(x-y) \in -\mathcal{N}_K(x) \Leftrightarrow x = \mathrm{proj}_M[K;y] \Leftrightarrow x = \mathrm{argmin}_{z\in K} \frac{1}{2}(z-y)^\top M(z-y). \tag{66}$$

The first equivalence in (66) is [10, Proposition 6.46].

D Proof of Lemma 2

It is known that $f_\infty(t,\xi) = \liminf_{\lambda\to+\infty, v\to\xi} \frac{f(t,\lambda v)}{\lambda}$ for each fixed $t \geq 0$ [33, Proposition 7]. Therefore using (18):

$$\begin{aligned}
&\frac{f(t,\lambda v)}{\lambda} \\
&= \inf_{G(y+F(t))\geq 0} \begin{cases} \frac{1}{2\lambda}(\lambda CR^{-1}v - y)^\top D^\dagger(\lambda CR^{-1}v-y) & \text{if } \lambda CR^{-1}v - y \in \mathrm{Im}(D^\dagger) \\ +\infty & \text{otherwise} \end{cases} \\
&= \begin{cases} \inf_{G(y+F(t))\geq 0} \frac{\lambda}{2}(CR^{-1}v - \frac{y}{\lambda})^\top D^\dagger(CR^{-1}v - \frac{y}{\lambda}) & \text{if } CR^{-1}v - \frac{y}{\lambda} \in \mathrm{Im}(D^\dagger) \\ +\infty & \text{otherwise.} \end{cases} \\
&= \begin{cases} \frac{\lambda}{2}\inf_{G(y+F(t))\geq 0}(GCR^{-1}v - G\frac{y}{\lambda})^\top G^{-\top}D^\dagger G^{-1}(GCR^{-1}v - G\frac{y}{\lambda}) & \text{if } GCR^{-1}v - G\frac{y}{\lambda} \in G\mathrm{Im}(D^\dagger) \\ +\infty & \text{otherwise.} \end{cases} \\
&= \begin{cases} \frac{\lambda}{2}\inf_{z+GF(t))\geq 0}(GCR^{-1}v - \frac{z}{\lambda})^\top G^{-\top}D^\dagger G^{-1}(GCR^{-1}v - \frac{z}{\lambda}) & \text{if } GCR^{-1}v - \frac{z}{\lambda} \in G\mathrm{Im}(D^\dagger) \\ +\infty & \text{otherwise.} \end{cases}
\end{aligned} \tag{67}$$

Clearly the limit as $\lambda \to \infty$ of the quantity in the first line, is either zero or infinity, since the function to be minimized is a quadratic function and $G^{-\top}D^\dagger G^{-1} \succcurlyeq 0$. If $D = 0$ then the infimum is zero, $\frac{z}{\lambda} \geq \frac{-GF(t)}{\lambda}$, so as $\lambda \to +\infty$ one has $\frac{z}{\lambda} \geq 0$ since $GF(t)$ is bounded, and we obtain that $GCR^{-1}v = \frac{z}{\lambda}$ so $GCR^{-1}v \geq 0$. Hence $f_\infty(t,\xi) = \psi_\Gamma(\xi)$ (which is in agreement with Lemma 1), noting that $U = I_m$ is suitable in this case.

Notice that $GCR^{-1}v - \frac{z}{\lambda} \in G\mathrm{Im}(D^\dagger) \Leftrightarrow \exists \eta \in I\!R^m$ such that $GCR^{-1}v - \frac{z}{\lambda} = GD^\dagger\eta$. Thus the minimization problem $\inf_{z+GF(t)\geq 0} \frac{1}{2}(GCR^{-1}v - \frac{z}{\lambda})^\top G^{-\top}D^\dagger G^{-1}(GCR^{-1}v - \frac{z}{\lambda})$ if $GCR^{-1}v - \frac{z}{\lambda} \in G\mathrm{Im}(D^\dagger)$, is equivalent to

$$\inf_{GD^\dagger\eta = GCR^{-1}v - \frac{z}{\lambda}, z\geq -GF(t)} \frac{1}{2}(\eta^\top D^\dagger)D^\dagger(D^\dagger\eta) \;\Leftrightarrow\; \inf_{GD^\dagger\eta - GCR^{-1}v - G\frac{F(t)}{\lambda}\leq 0} \frac{1}{2}(\eta^\top D^\dagger)D^\dagger(D^\dagger\eta) \tag{68}$$

where $v \in I\!R^m$ is given. Let us now state the necessary and sufficient KKT conditions [48, Theorem 28.3]: η solves the minimization problem if and only if there exists a multiplier $\gamma \in I\!R^m$ such that $0 \leq \gamma \perp GD^\dagger\eta - GCR^{-1}v - \frac{GF(t)}{\lambda} \leq 0$ and $(D^\dagger)^3\eta + D^\dagger G^\top\gamma = 0$. Assume first that $D \succ 0$ (hence we can take $U = I_m$). In this case $\frac{f(t,\lambda v)}{\lambda} = \frac{\lambda}{2}\inf_{z+GF(t)\geq 0}(GCR^{-1}v - \frac{z}{\lambda})^\top G^{-\top}D^\dagger G^{-1}(GCR^{-1}v - \frac{z}{\lambda}) = \frac{\lambda}{2}\inf_{GD^{-1}\eta - GCR^{-1}v - \frac{GF(t)}{\lambda}\leq 0}(\eta^\top D^{-1})D^{-1}(D^{-1}\eta)$. The (necessary and sufficient) KKT conditions yield $\eta = -D^2G^\top\gamma$, and $0 \leq \gamma \perp GDG^\top\gamma + GCR^{-1}v + \frac{GF(t)}{\lambda} \geq 0 \Leftrightarrow \gamma = \mathrm{proj}_{GDG^\top}[I\!R^m_+; -(GDG^\top)^{-1}(GCR^{-1}v + \frac{GF(t)}{\lambda})]$. Also $(\eta^\top D^{-1})D^{-1}(D^{-1}\eta) = \gamma^\top GDG^\top\gamma$. Therefore:

$$\frac{f(t,\lambda v)}{\lambda} = \frac{\lambda}{2}\mathrm{proj}^\top_{GDG^\top}[I\!R^m_+; -(GDG^\top)^{-1}(GCR^{-1}v - \frac{GF(t)}{\lambda})]\, GDG^\top \times \\ \times\, \mathrm{proj}_{GDG^\top}[I\!R^m_+; -(GDG^\top)^{-1}(GCR^{-1}v - \frac{GF(t)}{\lambda})]. \tag{69}$$

Using the complementarity conditions it follows that $GCR^{-1}v + \frac{GF(t)}{\lambda} > 0 \Rightarrow \gamma = 0$ and this is the unique solution since $GDG^\top \succ 0$. Thus the projection vanishes and we infer that $\frac{f(t,\lambda v)}{\lambda} = 0$ for any $\lambda \in I\!R$. Thus taking $\lambda \to +\infty$ and $v \to \xi$ implies that $f_\infty(t,\xi) = 0$ if $GCR^{-1}\xi \geq 0 \Leftrightarrow \xi \in \Gamma$. Now if $\xi \notin \Gamma$ then the limit is infinite. Thus we have proved that $f_\infty(t,\xi) = \psi_\Gamma(\xi)$.

Let now $D \succcurlyeq 0$, so that there exists a unitary matrix U, $UU^\top = U^\top U = I_m$, such that $UDU^\top = \begin{pmatrix} \bar{D} & 0 \\ 0 & 0 \end{pmatrix}$, $\bar{D} = \mathrm{diag}(\lambda_i) \in I\!R^{\bar{m}\times\bar{m}}$, where $\lambda_i > 0$ are the positive eigenvalues of D. Then $D^\dagger = U^\top \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & 0 \end{pmatrix} U$. Let us assume for the moment that $U = I_m$. It follows that $D^\dagger = \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. Let us define $\eta_I = (\eta_1, \ldots, \eta_{\bar{m}})^\top$, $\eta_{II} = (\eta_{\bar{m}+1}, \ldots, \eta_m)^\top$, $\tilde{m} = m - \bar{m}$, $(GCR^{-1})_{I\bullet}$ are the first $\bar{m}$ rows of GCR^{-1}, $(GCR^{-1})_{II\bullet}$ are its last $\tilde{m}$ rows, so that $GCR^{-1} = \begin{pmatrix} (GCR^{-1})_{I\bullet} \\ (GCR^{-1})_{II\bullet} \end{pmatrix}$. The minimization problem (68) can be equivalently rewritten as:

$$\begin{cases} \inf \frac{1}{2}\eta_I^\top \bar{D}^{-3}\eta_I \\ \text{subject to: } \begin{pmatrix} G_{\bar{m}\bar{m}} \\ G_{\tilde{m}\bar{m}} \end{pmatrix} \bar{D}^{-1}\eta_I - GCR^{-1}v - G\frac{F(t)}{\lambda} \leq 0. \end{cases} \tag{70}$$

Also, $GCR^{-1}v - \frac{z}{\lambda} \in G\mathrm{Im}(D^\dagger) = G\mathrm{Im}\begin{pmatrix} \bar{D} & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow (GCR^{-1})_{I\bullet}v - \frac{z_I}{\lambda} \in G_{\bar{m}\bar{m}}\mathrm{Im}(\bar{D}) = G_{\bar{m}\bar{m}}I\!R^{\bar{m}}$ and $(GCR^{-1})_{II\bullet}v - \frac{z_{II}}{\lambda} \in G_{\tilde{m}\bar{m}}I\!R^{\bar{m}} \subset I\!R^{\tilde{m}}$, with $z \geq -GF(t)$. Therefore:

$$\frac{f(t,\lambda v)}{\lambda} = \\ \begin{cases} \begin{cases} \inf \frac{1}{2\lambda}\eta_I^\top \bar{D}^{-3}\eta_I \\ \text{subject to: } \begin{pmatrix} G_{\bar{m}\bar{m}} \\ G_{\tilde{m}\bar{m}} \end{pmatrix} \bar{D}^{-1}\eta_I - GCR^{-1}v - G\frac{F(t)}{\lambda} \leq 0 \end{cases} & \text{if } \begin{array}{l} (GCR^{-1})_{I\bullet}v - \frac{z_I}{\lambda} \in G_{\bar{m}\bar{m}}I\!R^{\bar{m}} \\ \text{and } (GCR^{-1})_{II\bullet}v - \frac{z_{II}}{\lambda} \in G_{\tilde{m}\bar{m}}I\!R^{\bar{m}} \end{array} \\ +\infty & \text{otherwise.} \end{cases} \tag{71}$$

If $GCR^{-1}v + G\frac{F(t)}{\lambda} \geq 0$ then the mimimum of the quadratic function is attained at $\eta_I = 0$ which is an admissible value for the inequality constraints, and in this case the limit as $\lambda \to +\infty$ is null. Oterwise the minimum is attained at some $\eta_I \neq 0$ and the limit is infinity. Thus again $f_\infty(t,\xi) = f_\infty(\xi) = \psi_\Gamma(\xi)$. To finish the proof, let us assume that $U \neq I_m$, the minimization problem in (67) s rewritten equivalently as:

$$\frac{f(t,\lambda v)}{\lambda} = \\ \begin{cases} \frac{\lambda}{2}\inf_{y+F(t)\geq 0}(UGCR^{-1}v - U\frac{y}{\lambda})^\top \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & 0 \end{pmatrix} (UGCR^{-1}v - U\frac{y}{\lambda}) & \text{if } \begin{cases} GCR^{-1}v - \frac{y}{\lambda} \in \mathrm{Im}(D^\dagger) \\ = \mathrm{Im}(U^\top \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & 0 \end{pmatrix} U) \end{cases} \\ +\infty & \text{otherwise.} \end{cases} \tag{72}$$

One has $\mathrm{Im}(D^\dagger) = U^\top \mathrm{Im}(\begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & 0 \end{pmatrix} U) \subseteq U^\top \mathrm{Im}(\begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & 0 \end{pmatrix})$ [11, p.102]. Thus the condition becomes $UGCR^{-1}v - U\frac{Gy}{\lambda} \subseteq \mathrm{Im}(\begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & 0 \end{pmatrix})$. Following the same steps as above we infer that when $UGCR^{-1}\xi \in \mathbb{R}^m_+$ then the optimal value of the minimization problem vanishes, otherwise it equals $+\infty$. Thus $f_\infty(\xi) = \psi_\Gamma(\xi)$.

E Proof of Proposition 5

Item 3 implies that the iterates ξ_{k+1}, which belong to $\tilde{\Gamma}$ (because $\xi_k \in \mathrm{dom}(\tilde{\Gamma})$ for all $k \geq 0$), are uniformly bounded, and so is the sequence $\{\xi^h(\cdot)\}$. Let us rewrite (8) now as

$$RP^hR^{-1}\xi_{k+1} - RPR^{-1}\xi_k - hRE_k = \zeta_{k+1}, \quad \zeta_{k+1} \in -\partial f(\xi_{k+1}). \tag{73}$$

It is inferred that the selection ζ_{k+1} is uniformly bounded (thus $B\lambda_{k+1} = R^{-1}\zeta_{k+1}$ is bounded as well, said otherwise, any unbounded part of the multiplier $\lambda_{k+1} \in \mathrm{Ker}(B)$). Let $\dot{\xi}_k \stackrel{\Delta}{=} \frac{\xi_{k+1}-\xi_k}{h}$, we have:

$$\begin{cases} RPR^{-1}\dot{\xi}_k - RAR^{-1}\xi_{k+1} - RE_k \in -\partial f(\xi_{k+1}) \\ RPR^{-1}\dot{\xi}_{k-1} - RAR^{-1}\xi_k - RE_{k-1} \in -\partial f(\xi_k), \end{cases} \tag{74}$$

hence substracting both inclusions and multiplying both sides by $\dot{\xi}_k$ we obtain:

$$\langle RPR^{-1}(\dot{\xi}_k - \dot{\xi}_{k-1}), \dot{\xi}_k\rangle - \langle hRAR^{-1}\dot{\xi}_k, \dot{\xi}_k\rangle - \langle R(E_k - E_{k-1}), \dot{\xi}_k\rangle \in -\langle \zeta_{k+1} - \zeta_k, \dot{\xi}_k\rangle \leq 0, \tag{75}$$

where the inequality is obtained from the monotonicity of $\partial f(\cdot)$[2]. From (75) we get:

$$\langle R(P - hA)R^{-1}\dot{\xi}_k, \dot{\xi}_k\rangle - \langle RPR^{-1}\dot{\xi}_{k-1}, \dot{\xi}_k\rangle - \langle R\,\Delta E_k, \dot{\xi}_k\rangle \leq 0 \tag{76}$$

where $\Delta E_k = E_k - E_{k-1}$. Using item 4 in the proposition's assumptions, it follows that $\langle R(P - hA)R^{-1}\dot{\xi}_k, \dot{\xi}_k\rangle = \langle U^\top R(P-hA)R^{-1}UU^\top\dot{\xi}_k, U^\top\dot{\xi}_k\rangle = \langle \begin{pmatrix} \bar{P} - hM_{11} & -hM_{12} \\ -hM_{21} & -hM_{22} \end{pmatrix} U^\top\dot{\xi}_k, U^\top\dot{\xi}_k\rangle$. Define $\eta_k = U^\top\xi_k$, so that $\dot{\eta}_k = U^\top\dot{\xi}_k = \begin{pmatrix} U_1^\top\dot{\xi}_k \\ U_2^\top\dot{\xi}_k \end{pmatrix} = \begin{pmatrix} \dot{\eta}_k^1 \\ \dot{\eta}_k^2 \end{pmatrix}$, $\dot{\eta}_k^1 \in \mathbb{R}^p$, $\dot{\eta}_k^2 \in \mathbb{R}^{n-p}$. Thus we obtain

$$\langle R(P - hA)R^{-1}\dot{\xi}_k, \dot{\xi}_k\rangle = \langle (\bar{P} - hM_{11})\dot{\eta}_k^1, \dot{\eta}_k^1\rangle - \langle hM_{22}\dot{\eta}_k^2, \dot{\eta}_k^2\rangle - \langle \dot{\eta}_k^1, h(M_{12}^\top + M_{21})\dot{\eta}_k^2\rangle. \tag{77}$$

One also has $\langle RPR^{-1}\dot{\xi}_{k-1}, \dot{\xi}_k\rangle = \langle U^\top RPR^{-1}UU^\top\dot{\xi}_{k-1}, U^\top\dot{\xi}_k\rangle = \langle U^\top RPR^{-1}U\dot{\eta}_{k-1}, \dot{\eta}_k\rangle = \langle \bar{P}\dot{\eta}_{k-1}^1, \dot{\eta}_k^1\rangle$ and $\langle R\,\Delta E_k, \dot{\xi}_k\rangle = \langle U^\top R\,\Delta E_k, U^\top\dot{\xi}_k\rangle = \langle U^\top R\,\Delta E_k, \dot{\eta}_k\rangle$. Therefore (76) is rewritten equivalently as:

$$\langle (\bar{P} - hM_{11})\dot{\eta}_k^1, \dot{\eta}_k^1\rangle - \langle hM_{22}\dot{\eta}_k^2, \dot{\eta}_k^2\rangle - \langle \dot{\eta}_k^1, h(M_{12}^\top + M_{21})\dot{\eta}_k^2\rangle - \langle \bar{P}\dot{\eta}_{k-1}^1, \dot{\eta}_k^1\rangle - \langle U^\top R\,\Delta E_k, \dot{\eta}_k\rangle \leq 0, \tag{78}$$

[2] This would not hold in the time-varying case if Assumption 7 does not hold.

equivalently:

$$\begin{array}{rl}\langle(\bar{P}-hM_{11})\dot{\eta}_k^1,\dot{\eta}_k^1\rangle & \leq \langle hM_{22}\dot{\eta}_k^2,\dot{\eta}_k^2\rangle+\langle\dot{\eta}_k^1,h(M_{12}^\top+M_{21})\dot{\eta}_k^2\rangle+\langle\bar{P}\dot{\eta}_{k-1}^1,\dot{\eta}_k^1\rangle-\langle U^\top R\,\Delta E_k,\dot{\eta}_k\rangle \\ & \leq \langle hM_{22}\dot{\eta}_k^2,\dot{\eta}_k^2\rangle+\frac{1}{2}h||\dot{\eta}_k^1||^2+\frac{1}{2}h||(M_{12}^\top+M_{21})||^2||\dot{\eta}_k^2||^2 \\ & \quad +\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_{k-1}^1||^2+\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_k^1||^2+|\langle(U^\top R\,\Delta E_k)^{1,\top}\dot{\eta}_k^1\rangle|+|\langle(U^\top R\,\Delta E_k)^{2,\top}\dot{\eta}_k^2\rangle| \\ & \leq \langle h\left(M_{22}+\frac{1}{2}||(M_{12}^\top+M_{21})||^2\right)\dot{\eta}_k^2,\dot{\eta}_k^2\rangle+\frac{1}{2}h||\dot{\eta}_k^1||^2+\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_{k-1}^1||^2+\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_k^1||^2 \\ & \quad +||(U^\top R\,\Delta E_k)^1||\;||\dot{\eta}_k^1||+||(U^\top R\,\Delta E_k)^2||\;||\dot{\eta}_k^2|| \\ & \leq \langle h\left(M_{22}+\frac{1}{2}||(M_{12}^\top+M_{21})||^2\right)\dot{\eta}_k^2,\dot{\eta}_k^2\rangle+\frac{1}{2}h||\dot{\eta}_k^1||^2+\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_{k-1}^1||^2+\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_k^1||^2 \\ & \quad +\alpha_1 h||\dot{\eta}_k^1||+\alpha_2 h||\;||\dot{\eta}_k^2|| \\ & \leq \langle h\left(M_{22}+\frac{1}{2}||(M_{12}^\top+M_{21})||^2\right)\dot{\eta}_k^2,\dot{\eta}_k^2\rangle+\frac{1}{2}h||\dot{\eta}_k^1||^2+\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_{k-1}^1||^2+\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_k^1||^2 \\ & \quad +\frac{1}{2}\alpha_1^2 h+\frac{1}{2}h||\dot{\eta}_k^1||^2+\frac{1}{2}\alpha_2^2 h+\frac{1}{2}h||\dot{\eta}_k^2||^2,\end{array} \tag{79}$$

where the inequalities $|x^\top y|\leq||x||\;||y||\leq\frac{1}{2}||x||^2+\frac{1}{2}||y||^2$ for any real vectors x and y are used, and item 7 guarantees the existence of bounded $\alpha_1>0$ and $\alpha_2>0$ (remind that $t_{k+1}-t_k=h$). Hence:

$$\begin{array}{rl}\langle(\frac{1}{2}\bar{P}-hM_{11}-hI_p)\dot{\eta}_k^1,\dot{\eta}_k^1\rangle & \leq\langle h\left(M_{22}+\frac{1}{2}||(M_{12}^\top+M_{21})||^2 I_{n-p}+\frac{1}{2}I_{n-p}\right)\dot{\eta}_k^2,\dot{\eta}_k^2\rangle+\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_{k-1}^1||^2 \\ & \leq\frac{1}{2}||\bar{P}^{\frac{1}{2}}\dot{\eta}_{k-1}^1||^2+(\alpha_1^2+\alpha_2^2)h=\langle\frac{1}{2}\bar{P}\dot{\eta}_{k-1}^1,\dot{\eta}_{k-1}^1\rangle+(\alpha_1^2+\alpha_2^2)h \\ & \leq\langle(\frac{1}{2}\bar{P}-hM_{11}-hI_p)\dot{\eta}_{k-1}^1,\eta_{k-1}^1\rangle+(\alpha_1^2+\alpha_2^2)h,\end{array} \tag{80}$$

where item 5 has been used to obtain the second inequality and item 6 has been used for the third inequality to secure that $\frac{1}{2}\bar{P}\preccurlyeq\frac{1}{2}\bar{P}-hM_{11}-hI_p$. Denote $V(\dot{\eta}_k^1)\stackrel{\Delta}{=}\langle(\frac{1}{2}\bar{P}-hM_{11}-hI_n)\dot{\eta}_k^1,\dot{\eta}_k^1\rangle$, we obtain from (80)

$$V(\dot{\eta}_k^1)\leq V(\dot{\eta}_0^1)+(\alpha_1^2+\alpha_2^2)\frac{(k+1)T}{n}. \tag{81}$$

Therefore $V(\dot{\eta}_k^1)$ and consequently $\dot{\eta}_k^1$ is uniformly bounded for all $k\in\{0,n\}$, independently of n. Thus $\dot{\eta}_h^1(t)=U_1^\top\dot{\xi}^h(t)$ is bounded in $t\in[0,T]$ for any bounded initial data. The sequence $\{\eta_h^1\}$ is thus uniformly bounded (because $\{\xi^h(\cdot)\}$ is), continuous with uniformly bounded derivatives almost everywhere, and thus equicontinuous. From Arzelà-Ascoli Theorem, $\{\eta_h^1\}$ stays in a compact subset of $\mathcal{C}^0([0,T];\mathbb{R}^p)$, and there exists subsequences (which we still denote $\eta_h^1(\cdot)$) that converge uniformly in $\mathcal{C}^0([0,T];\mathbb{R}^p)$ to a limit $\eta^1(\cdot)$.

F Proof of Proposition 6

Let us start with

$$RPR^{-1}(\xi_{k+1}-\xi_0)-hRAR^{-1}\xi_{k+1}-RPR^{-1}(\xi_k-\xi_0)-hRE_k\in-h\,\partial f(\xi_{k+1}), \tag{82}$$

equivalently

$$\begin{array}{r}U^\top RPR^{-1}UU^\top(\xi_{k+1}-\xi_0)-hU^\top RAR^{-1}UU^\top\xi_{k+1}-U^\top RPR^{-1}UU^\top(\xi_k-\xi_0) \\ -hU^\top RE_k\in-hU^\top\,\partial f(\xi_{k+1}),\end{array} \tag{83}$$

that is

$$\begin{pmatrix}\bar{P} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta^1_{k+1}-\eta^1_0\\ \eta^2_{k+1}-\eta^2_0\end{pmatrix} - h\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\eta^1_{k+1}\\ \eta^2_{k+1}\end{pmatrix} - \begin{pmatrix}\bar{P} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta^1_{k}-\eta^1_0\\ \eta^2_{k}-\eta^2_0\end{pmatrix}$$
$$-hU^\top RE_k \in -hU^\top\,\partial f(\xi_{k+1}). \tag{84}$$

The condition $U^\top R\ \mathrm{Im}(P) \supseteq U^\top R\ \mathrm{Im}(B)$ implies that $U_2^\top RB = 0 \ \Rightarrow U_2^\top\zeta_k = 0$ for any $\zeta_k \in \partial f(\xi_{k+1})$. Indeed $U^\top R\ \mathrm{Im}(P) = \mathrm{Im}(U^\top RPR^{-1}U) = \mathrm{Im}(\begin{pmatrix}\bar{P} & 0\\ 0 & 0\end{pmatrix})$ and $U^\top R\ \mathrm{Im}(B) = \mathrm{Im}(U^\top RB) = \mathrm{Im}(\begin{pmatrix}U_1^\top RB\\ U_2^\top RB\end{pmatrix})$. Thus using the second line in the left-hand side of (84) and item 5, we obtain $\eta^2_{k+1} = M_{22}^{-1}(-M_{21}\eta^1_{k+1} + U_2^\top RE_k)$. Inserting this in the first line of the second term in the left-hand side of (84) gives $-hM_{11}\eta^1_{k+1} - hM_{12}\eta^2_{k+1} = -hM_{\mathrm{sch}}\eta^1_{k+1} - hM_{12}M_{22}^{-1}U_2^\top RE_k$, where M_{sch} is in item 4 of the proposition. Now using (84) we obtain:

$$\begin{aligned}
&\langle\begin{pmatrix}\bar{P} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta^1_{k+1}-\eta^1_0\\ \eta^2_{k+1}-\eta^2_0\end{pmatrix}, U^\top(\xi_{k+1}-\xi_0)\rangle - \langle h\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\eta^1_{k+1}-\eta^1_0\\ \eta^2_{k+1}-\eta^2_0\end{pmatrix}, U^\top(\xi_{k+1}-\xi_0)\rangle\\
&\quad -\langle\begin{pmatrix}\bar{P} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta^1_{k}-\eta^1_0\\ \eta^2_{k}-\eta^2_0\end{pmatrix}, U^\top(\xi_{k+1}-\xi_0)\rangle - \langle hU^\top RE_k, U^\top(\xi_{k+1}-\xi_0)\rangle\\
&\quad -\langle h\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\eta^1_0\\ \eta^2_0\end{pmatrix}, U^\top(\xi_{k+1}-\xi_0)\rangle\\
&= \langle h(U^\top\zeta_{k+1} - U^\top\zeta_0 + U^\top\zeta_0), U^\top(\xi_{k+1}-\xi_0)\rangle \le \langle U^\top\zeta_0, U^\top(\xi_{k+1}-\xi_0)\rangle.
\end{aligned} \tag{85}$$

Let us now define $w_0 \triangleq U^\top\zeta_0 + \begin{pmatrix}M_{11} & M_{21}\\ M_{21} & M_{22}\end{pmatrix}U^\top\xi_0$, with $\zeta_0 \in \partial f(\xi_0)$. Using the monotonicity of $\partial f(\cdot)$, the definition of w_0 and $\zeta_{k+1} \in \partial f(\xi_{k+1})$:

$$\begin{aligned}
&\langle\begin{pmatrix}\bar{P} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta^1_{k+1}-\eta^1_0\\ \eta^2_{k+1}-\eta^2_0\end{pmatrix}, U^\top(\xi_{k+1}-\xi_0)\rangle - \langle h\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\eta^1_{k+1}-\eta^1_0\\ \eta^2_{k+1}-\eta^2_0\end{pmatrix}, U^\top(\xi_{k+1}-\xi_0)\rangle\\
&\quad -\langle\begin{pmatrix}\bar{P} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta^1_{k}-\eta^1_0\\ \eta^2_{k}-\eta^2_0\end{pmatrix}, U^\top(\xi_{k+1}-\xi_0)\rangle - \langle h(U^\top RE_k + w_0), U^\top(\xi_{k+1}-\xi_0)\rangle \le 0
\end{aligned} \tag{86}$$

Using the calculations made just above (85) for $M_{11}\eta^1_{k+1} + M_{12}\eta^2_{k+1}$ with the Schur complement M_{sch}, the fact that $U^\top(\xi_{k+1}-\xi_0) = \begin{pmatrix}\eta^1_{k+1}-\eta^1_0\\ \eta^2_{k+1}-\eta^2_0\end{pmatrix}$, it follows from (86) that:

$$\begin{aligned}
\langle(\bar{P} - hM_{\mathrm{sch}})(\eta^1_{k+1}-\eta^1_0), \eta^1_{k+1}-\eta^1_0\rangle \ &\le \langle\bar{P}(\eta^1_k - \eta^1_0), \eta^1_{k+1}-\eta^1_0\rangle\\
&\quad + h\langle U_1^\top RE_k + w^1_0 - M_{11}\eta^1_0 - M_{12}\eta^2_0, \eta^1_{k+1}-\eta^1_0\rangle\\
&\quad + h\langle U_2^\top\zeta_0, M_{12}M_{22}^{-1}U_2^\top RE_k\rangle - h\langle M_{12}M_{22}^{-1}U_2^\top\zeta_0, \eta^1_{k+1}-\eta^1_0\rangle\\
&\quad - h\langle M_{12}M_{22}^{-1}U_2^\top\zeta_0, \eta^1_0\rangle\\
&\le \langle\bar{P}(\eta^1_k - \eta^1_0), \eta^1_{k+1}-\eta^1_0\rangle\\
&\quad + \tfrac{h}{2}||U_1^\top RE_k + w^1_0 - M_{11}\eta^1_0 - M_{12}\eta^2_0||^2 + \tfrac{h}{2}||\eta^1_0||^2\\
&\quad + \tfrac{h}{2}||M_{12}M_{22}^{-1}U_2^\top\zeta_0||^2 + \tfrac{h}{2}||U_2^\top\zeta_0||^2 + \tfrac{h}{2}||M_{12}M_{22}^{-1}U_2^\top RE_k||^2\\
&\quad + h||\eta^1_{k+1}-\eta^1_0||^2.
\end{aligned} \tag{87}$$

Using items 3 and 4 we have $\tilde{P} \triangleq \bar{P} - hM_{\text{sch}} - hI_p \succ 0$, then:

$$\begin{array}{ll} \langle \tilde{P}(\eta^1_{k+1} - \eta^1_0), \eta^1_{k+1} - \eta^1_0 \rangle & \leq \langle \bar{P}(\eta^1_k - \eta^1_0), \eta^1_{k+1} - \eta^1_0 \rangle + \frac{h}{2}\Upsilon(E_k, \zeta_0, \eta_0) \\ & \leq \langle \tilde{P}(\eta^1_k - \eta^1_0), \eta^1_{k+1} - \eta^1_0 \rangle + \frac{h}{2}\Upsilon(E_k, \zeta_0, \eta_0) \end{array} \quad (88)$$

where $\Upsilon(E_k, \zeta_0, \eta_0)$ is easy to define. From the assumptions of the proposition, $\frac{1}{2}\sum_{j=1}^{n-1} h\Upsilon(E_k, \zeta_0, \eta_0) \leq C\frac{h}{2}(n-1) = C\frac{T}{2}\frac{n-1}{n} \leq \frac{CT}{2}$ for some constant $C > 0$. Thus $\langle \tilde{P}(\eta^1_{k+1} - \eta^1_0), \eta^1_{k+1} - \eta^1_0 \rangle \leq \frac{CT}{2} + \langle \tilde{P}(\eta^1_1 - \eta^1_0), \eta^1_1 - \eta^1_0 \rangle$ for all $k \in \{1, n-1\}$, while $\langle \tilde{P}(\eta^1_1 - \eta^1_0), \eta^1_1 - \eta^1_0 \rangle \leq \frac{CT}{2}$. We conclude that $||\eta^1_{k+1} - \eta^1_0||$ is bounded uniformly in k, thus η^1_k and η^2_k are uniformly bounded as well. This ends the proof.

Remark 5 *Item 5 in Proposition 6 guarantees a decoupling between the algebraic equation and the multiplier* λ*. If this assumption is relaxed, one has to deal with the term* $\langle U_2^\top \zeta_0, M_{12}M_{22}^{-1}U_2^\top \zeta_k \rangle$ *in the right-hande side of (87). Another way to decouple is thus to assume that* $M_{12} = 0$*. Item 5 could be replaced by:* $U^\top R \operatorname{Im}(P) \subseteq U^\top R \operatorname{Im}(P)$ *or* $M_{12} = 0$*. If none of these conditions is satisfied, then one has to resort to another kind of proof, as outlined in section 7.*

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Answers to the Reviewer's comments

Thank you very much for your positive review. Concerning the numerical computations aspects, I have added section 4.3 to point out some possible paths for the implementation of the implicit Euler scheme. Indeed the case $D \neq 0$ is more tricky in all aspects. Normally and under some assumptions it is possible to solve the one-step nonsmooth problem to advance the scheme, using linear cone complementarity problems. Globally it is clear that the numerical implementation of the scheme, in the general case, may be a delicate matter that deserves future investigations.