On Homogeneous Approximations, Stability and Robustness of Infinite Dimensional Systems

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Stability and Robustness of Locally Homogeneous Abstract Control Systems

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Abstract. The paper generalizes the concept of homogeneous approximations to a class of unbounded operators satisfying certain regularity assumptions. Stability and robustness of locally homogeneous abstract control systems are studied. The viscous Burgers equation and its nonlinear modifications are considered as illustrative examples.

Key word. abstract control systems; input-to-state stability; distributed parameter systems.

1. Introduction.

By the Noether's Theorem [29] any differential symmetry defines a conservation law, so it is expectable that many physical systems are symmetric. *Homogeneity* is a dilation symmetry well-known in mathematics. A function $x \mapsto f(x)$ satisfying $f(e^s x) = e^{\nu s} f(x)$, $\forall s \in \mathbb{R}, \forall x$, where $\nu \in \mathbb{R}$ is a degree, is called homogeneous. This kind of homogeneity is known today as the *standard (or Euler's) homogeneity* and the scaling of the argument $x \to e^s x$ is referred as the *standard dilation*. A generalized homogeneity is a symmetry with respect to a generalized dilation. The famous example of a generalized dilation in $\mathbb{R}^n$ is the so-called weighted dilation $(x_1, ..., x_n) \to (e^{r_1 s} x_1, ..., e^{r_n s} x_n)$ studied since 1950s.

On the one hand, the generalized homogeneity simplifies an analysis of nonlinear systems modeled by Ordinary Differential Equations (ODEs) [48], [12], [19],[40], [41], [4], [2] as well as non-linear controllers/observers design [18], [6], [11], [21], [30], [1], [32], [23]. In particular, homogeneous system may be finite-time [5] or fixed-time [33] stable dependently of the homogeneity degree (see, e.g. [28], [4]). Asymptotic stability of homogeneous system guarantees its Input-to-State Stability (ISS) with respect to homogeneously involved exogenous perturbations [41], [13], [1]. ISS is an important characteristic of both finite dimensional [43], [45] and infinite dimensional [16], [25], [17], [26] control systems.

On the other hand, the homogeneity is a rather fragile property in the sense that adding a small nonlinearity to a homogeneous function may destroy the dilation symmetry. Even more, a sum of two homogeneous functions with different degrees is a non-homogeneous function. However, it is expectable that dynamical systems, which are close (in some sense) to homogeneous systems, should have similar properties. For the finite dimensional case, this issue was rigorously studied in [48], [9], [1], [3] by means of the so-called homogeneous approximations. In this paper, the corresponding results about robustness/stability analysis of *locally homogeneous control systems* are obtained for infinite dimensional case. More precisely, we deal with a semilinear evolution equation in a real Banach space with a nonlinear operator satisfying some regularity assumptions. Such a model includes various Time-Delay Systems and Partial Differential Equations (PDEs) as particular cases. An introduction to infinite dimensional homogeneous control systems can be found in [39], [38], [37]. Algebraic properties of homogeneous differential operators are studied in [10].

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The paper is organized as follows. First, model description and basic assumptions are presented. Next, elements of the theory of generalized homogeneous systems are revised and some new results are proven. After that the concept of homogeneous approximation in a Banach space is introduced and utilized for stability and robustness analysis of nonlinear evolution equations. Finally, concluding remarks are given.

Notation. $\mathbb{R}$ is the field of real numbers; $\mathbb{R}_+ = [0,+\infty)$; $x \cdot y = \sum x_i y_i$ is the dot product; $\mathbb{B}$ is a real Banach space with a norm $\|\cdot\|_{\mathbb{B}}$ (we omit the subindex $\mathbb{B}$ in the notation of the norm if the context is clear); for Banach spaces we use also notations $\mathbb{X}, \mathbb{Y}$ and $\mathbb{V}$; $\mathbf{0}$ is the zero element of a Banach space; I (resp. O) denotes the identity (resp. the zero) operator; $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ denotes the Banach space of linear bounded operators $\mathbb{B}_1 \mapsto \mathbb{B}_2$ with the norm $\|A\| = \sup\limits_{x\neq 0} \frac{\|Ax\|_{\mathbb{B}_2}}{\|x\|_{\mathbb{B}_1}}$; $S_{\mathbb{B}}$ is the unit sphere in $\mathbb{B}$; $B_{\mathbb{B}}(r)$ is the ball in $\mathbb{B}$ of the radius $r>0$ centered at $\mathbf{0}$; for $r>1$ the set $K_{\mathbb{B}}(r) \subset \mathbb{B}$ is defined as follows $K_{\mathbb{B}}(r) := \{x \in \mathbb{B} : 1/r < \|x\|_{\mathbb{B}} < r\}$; $\mathcal{D}(A)$ denotes a domain of an operator $A : \mathcal{D}(A) \subset \mathbb{B} \mapsto \mathbb{B}$; $\nabla = (\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}), z \in \mathbb{R}^n$ is the ∇-operator; $\Delta = \nabla \cdot \nabla = \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ is the Laplace operator; $C([t_1,t_2],\mathbb{B})$ is the space of continuous functions $x : [t_1,t_2] \mapsto \mathbb{B}$ with the uniform norm $\|x\|_C = \max_{t\in[t_1,t_2]} \|x(t)\|$ with $-\infty < t_1 < t_2 < +\infty$; $f_1 \circ f_2$ and $f_1 f_2$ denote a composition of operators f_1 and f_2; $C_c^\infty(\Omega, \mathbb{R}^m)$ is a set of infinitely smooth functions $\mathbb{R}^n \mapsto \mathbb{R}^m$ with compact supports in an open connected set $\Omega \subset \mathbb{R}^n$ with a smooth boundary (or $\Omega = \mathbb{R}^n$); $H^p(\Omega, \mathbb{R}^m)$ is a Sobolev space and H_0^p is a completion of C_c^∞ in the norm of H^p; $L^1((t_1,t_2),\mathbb{B})$ is the space of Bochner integrable functions $(t_1,t_2) \mapsto \mathbb{B}$, where $-\infty \le t_1 < t_2 \le +\infty$; $L^\infty((t_1,t_2),\mathbb{B})$ is a space uniformly essentially bounded Bochner measurable functions with the norm $\|q\|_\infty := \operatorname{ess\,sup}_{s\in(t_1,t_2)} \|q(t)\|$; $\mathcal{K}$ is a set of strictly increasing continuous functions $\sigma : [0,+\infty) \mapsto [0,+\infty)$ such that $\sigma(0) = 0$; $\mathcal{K}^\infty := \{\sigma \in \mathcal{K} : \sigma(s) \to +\infty \text{ as } s \to +\infty\}$; the function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot,t) \in \mathcal{K}$ and $t \mapsto \beta(s,t)$ is strictly decreasing to zero; $\dot{x}(t) = \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}$ is a time derivative of the function $x : \mathbb{R} \mapsto \mathbb{B}$, where the limit is understood in the strong topology of $\mathbb{B}$; $\overline{D}^+ V(t) = \limsup_{h\to 0^+} \frac{v(t+h)-v(t)}{h}$ denotes the right-hand upper Dini derivative of the function $v : \mathbb{R} \mapsto \mathbb{R}$; $\overline{D}^+ V(x;g) = \limsup_{h\to 0^+} \frac{V(x+hg)-V(x)}{h}$ denotes the right-hand upper directional derivative of the functional $V : \mathbb{B} \mapsto \mathbb{R}$ in the direction $g \in \mathbb{B}$.

2. Model Description, Basic Assumptions and Main Results.

Let us consider the following nonlinear system

$$\dot{x} = Ax + f(x,q), \quad t > t_0, \quad x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in \mathbb{X}$ is the system state at the time instant $t \ge t_0$ and $\mathbb{X}$ is a real Banach space; $t_0 \in \mathbb{R}$ is an initial instant of time and $x_0 \in \mathbb{X}$ is an initial state; $q \in L^\infty(\mathbb{R},\mathbb{V})$ is a time-varying exogenous input which can be treated as a control or a perturbation; $\mathbb{V}$ is a real Banach space as well; $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear (*possibly unbounded*) closed densely defined operator which generates a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{X}$; $f : \mathcal{D}_f \times \mathbb{V} \mapsto \mathbb{X}$ is a non-linear (*possibly unbounded*) closed operator such that $f(\mathbf{0},\mathbf{0}) = \mathbf{0}$ and $\mathcal{D}_f \subset \mathbb{X}$ is a linear subspace dense in $\mathbb{X}$.

The system (2.1) with $q = \mathbf{0}$ is referred below as *unperturbed system*. The non-linear evolution equations are well-studied in the literature using the theory of strongly continuous (C_0-) semigroups (see, [31, 8, 47] and references therein). We follow this classical framework.

Definition 2.1. *A continuous function* $x : [t_0, t_1) \mapsto \mathbb{X}$ *is said to be a mild solution of* (2.1) *if* $f(x(\cdot), q(\cdot)) \in L^1((t_0, t_1), \mathbb{X})$ *and*

$$x(t) = \Phi(t - t_0)x_0 + \int_{t_0}^{t} \Phi(t - s)f(x(s), q(s))ds, \quad \forall t \in [t_0, t_1).$$

If this mild solution satisfies (2.1) *for (almost) all* $t \in (t_0, t_1)$ *then* x *is called classical (strong) solution of* (2.1).

The assumption $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ implies the existence of the zero solution of the unperturbed system (2.1) with $x_0 = \mathbf{0}$. The class of nonlinear operators f considered in the paper is specified by the following definition.

Definition 2.2. *Let* $\mathbb{B}, \mathbb{X}, \mathbb{V}$ *be Banach spaces. A non-linear operator* $g : \mathcal{D}_g \times \mathbb{V} \mapsto \mathbb{B}$, $\mathcal{D}_g \subset \mathbb{X}$ *is said to be* M*-regular if there exists a linear closed densely defined operator* $M : \mathcal{D}_g \subset \mathbb{X} \mapsto \mathbb{X}$ *with a bounded inverse* $M^{-1} : \mathbb{X} \mapsto \mathcal{D}_g$ *such that the nonlinear mapping* $x \mapsto g(M^{-1}x, q)$ *is locally Lipschitz continuous in* $x \in \mathbb{X}\backslash\{\mathbf{0}\}$ *uniformly in* q. *More precisely, for any* $r > 1$ *and for any* $\bar{q} > 0$ *there exists* $L_{r,\bar{q}} > 0$ *such that*

$$\|g(M^{-1}x_1, q) - g(M^{-1}x_2, q)\|_{\mathbb{B}} \leq L_{r,\bar{q}} \|x_1 - x_2\|_{\mathbb{X}} \tag{2.2}$$

for all $x_1, x_2 \in K_{\mathbb{X}}(r)$ *and all* $q \in B_{\mathbb{V}}(\bar{q})$.

To guarantee the existence of mild solutions of (2.1) with $x_0 \neq \mathbf{0}$, we assume that the operator f admits some "M-regularization" consistent with A.

Assumption 1. *Let* f *be* M*-regular and continuous in the second argument,* M *commutes with* Φ:

$$\Phi(t)Mx_0 = M\Phi(t)x_0, \quad \forall t \geq 0 \quad \textit{and} \quad \forall x_0 \in \mathcal{D}_f,$$

the linear operator $M\Phi(t) : \mathbb{X} \mapsto \mathbb{X}$ *is bounded for any* $t > 0$ *and there exists a continuous function* $\omega : (0, +\infty) \mapsto \mathbb{R}_+$ *such that* $\|M\Phi(t)\|_{\mathbb{X}} \leq \omega(t)$ *and* $\int_0^t \omega(\sigma)d\sigma < +\infty, \forall t > 0$.

If $\mathcal{D}_f = \mathbb{X}$ then $M = I$ is the identity operator and $\omega(t) = Ce^{\bar{\omega}t}$ with $C \geq 1$ and $\bar{\omega} \in \mathbb{R}$ (see, [31], page 4), i.e. Assumption 1 simply asks a regularity (local Lipschitz continuity) of f. If A is a generator of an analytic semigroup Φ then to fulfill Assumption 1 the operator M could be a fractional power of the operator A. In latter case, $\omega(t) = \frac{C}{t^{\alpha}}$ with $C \geq 1$ and $\alpha \in (0, 1)$ (see, e.g. [31], page 195 for more details).

Notice, the closedness of M implies that *the linear subspace* $\mathbb{Y} = \mathcal{D}_f$ *with the norm* $\|y\|_{\mathbb{Y}} = \|y\| + \|My\|$ *is a Banach space* as well. Assumption 1 guarantees the local-in-time existence and uniqueness of mild solutions of (2.1) on $\mathbb{Y}\backslash\{\mathbf{0}\}$ as well as their continuous dependence on initial conditions (see, [36]for more details).

The robustness (ISS) means a special continuous dependence of global-in-time solutions of (2.1) on the magnitude of the exogenous input q. This additionally constrains a class of admissible nonlinearities. Below we prove ISS of (2.1) under the following assumption.

Assumption 2. *There exist* $\chi \in \mathcal{K}^{\infty}$ *and* $\xi \in C([0, +\infty), \mathbb{R}_+)$ *such that*

$$\|f(x, q) - f(x, 0)\|_{\mathbb{Y}} \leq \xi(\|x\|_{\mathbb{Y}})\chi(\|q\|_{\mathbb{V}}), \quad \forall x \in \mathbb{Y}, \quad \forall q \in \mathbb{V}.$$

Notice that Assumption 2 impies that $\delta f = f(x,q) - f(x,\mathbf{0})$ maps $\mathbb{Y}\times\mathbb{V}$ into $\mathbb{Y}\times\mathbb{V}$. The latter implicitely means that components f, which are defined by unbounded operators, are not perturbed by the input q. For instance, one can be shown that $f(x,q) = \left(\frac{\partial x}{\partial z}+q\right)x$ satisfies Assumption 2 with $\mathbb{Y}=\mathbb{V}=H^1(\mathbb{R},\mathbb{R})$, but $f(x,q)=\left(\frac{\partial x}{\partial z}+x\right)q$ does not.

The aim of this paper is to analyze stability and robustness (ISS) properties of (2.1) assuming that this system is locally or globally homogeneous in a generaralized sense explained below. This paper generalizes the main resuls of [13] and [1] to the evolution equation (2.1) with *unbounded nonlinear operators* satisfying Assumptions 1 and 2. Namely, we prove that

- a homogeneous system (2.1) is ISS if the unperturbed system ($q=\mathbf{0}$) is uniformly asymptotically stable;
- stability properties of the system (2.1) are defined by homogeneous approximations;
- a homogeneous in the bi-limit system (2.1) is ISS if its homogenenous approximations are ISS and the unperturbed system (2.1) is globally uniformly asymptotically stable;
- a uniformly asymptotically stable system (2.1) is finite-/fixed-time (input-to-state) stable provided that its homogeneous approximations at 0 and at ∞ satisfy certain conditions.

3. Homogeneous Systems.

3.1. Dilations in Banach Spaces.

Elements of the theory of generalized dilations in finite-dimensional and infinite-dimensional spaces can be found in [48], [20], [15], [19], [39].

Recall [35, Chapter 6] that a one-parameter group of linear bounded operator $\mathfrak{d}(s)\in\mathcal{L}(\mathbb{B},\mathbb{B})$ is a linear *dilation* in a Banach space $\mathbb{B}$ if it satisfies the *limit property*: $\lim\limits_{s\to-\infty}\|\mathfrak{d}(s)\|=0$ and $\lim\limits_{s\to+\infty}\|\mathfrak{d}(s)\|=+\infty$. Since we study group with respect to a composition of operators: $\mathfrak{d}(0)=I$, $\mathfrak{d}(s+t)=\mathfrak{d}(s)\circ\mathfrak{d}(t), \forall s,t\in\mathbb{R}$ then $\mathfrak{d}(s)$ is invertible and $\mathfrak{d}(-s)=(\mathfrak{d}(s))^{-1}, \forall s\in\mathbb{R}$.

Example 1. *[38] Let us consider a one-parameter group of linear bounded invertible operators on the Lebesgue space $L^p(\mathbb{R}^n,\mathbb{R}^m)$ or the Sobolev space $H^p(\mathbb{R}^n,\mathbb{R}^m)$ given by*

$$(\mathfrak{d}(s)x)(z) = e^{\alpha s}x(e^{\beta s}z), \quad s\in\mathbb{R}, \quad x\in L^p(\mathbb{R}^n,\mathbb{R}^m), \quad z\in\mathbb{R}^n, \tag{3.1}$$

where $\alpha,\beta\in\mathbb{R}$ are given. Making the change of the variable in the Lebesgue integral we derive $\|\mathfrak{d}(s)x\|_{L^p}=e^{(\alpha-\beta n/p)s}\|x\|_{L^p}$. Hence, $\mathfrak{d}$ is a dilation in $L^p(\mathbb{R}^n,\mathbb{R}^m)$ provided that $\alpha-\beta n/p>0$. On the other hand, since $\|\mathfrak{d}(s)x\|^2_{H^p}=\sum_{i=0}^p e^{2\alpha-\beta n+2i\beta}\|\nabla^i x\|^2_{L_2}$, then $\mathfrak{d}$ is a dilation in $H^p(\mathbb{R}^n,\mathbb{R}^m)$ provided that $\alpha>\beta(0.5n/2-i), i=0,1,...,p$.

In this paper, we deal only with strongly continuous dilations. This means, by definition, that the function $s\mapsto\mathfrak{d}(s)u$ is continuous for each $u\in\mathbb{B}$.

Example 2. *The dilation $\mathfrak{d}$ given by (3.1) is strongly continuous in $H^p(\mathbb{R}^n,\mathbb{R}^m)$. Indeed, for $p=0$ and $x_\infty\in C_c^\infty(\mathbb{R}^n,\mathbb{R}^m)$ we derive $\langle\mathfrak{d}(s)x_\infty-x_\infty,x_\infty\rangle_{L^2}=\int_{B_{\mathbb{R}^n}(c)}\left(e^{\alpha s}x_\infty(e^{\beta s}z)-x_\infty(z)\right)\cdot x_\infty(z)dz, \quad |s|\le s_0$ for some finite $c\ge0$ (dependent of x_∞ and $s_0>0$). Since $x_\infty\in C_c^\infty$ is a uniformly continuous function then $\|x_\infty(e^{\beta s}z)-x_\infty(z)\|_{\mathbb{R}^m}\le\sigma(\|e^{\beta s}z-z\|_{\mathbb{R}^n})\le\sigma(c(e^{\beta s}-1))$ for all $z\in B_{\mathbb{R}^n}(c)$ and some $\sigma\in\mathcal{K}^\infty$. Hence $\langle\mathfrak{d}(s)x_\infty-x_\infty,x_\infty\rangle_{L_2(\mathbb{R}^n,\mathbb{R}^m)}\to0$ as $s\to0$ and $\|\mathfrak{d}(s)x_\infty-x_\infty\|^2_{L^2}=\|\mathfrak{d}(s)x_\infty\|^2_{L^2}-\|x_\infty\|^2_{L^2}-2\langle\mathfrak{d}(s)x_\infty-x_\infty,x_\infty\rangle_{L^2}\to0$ as $s\to0$. Taking into account that C_c^∞ is dense in H^p, the continuity of $s\to\mathfrak{d}(s)x$ can be shown for any $x\in H^p(\mathbb{R}^n,\mathbb{R}^m)$, $p=0,1,2,\ldots$.*

Being a strongly continuous group of linear bounded operators, the linear dilation always has an infinitesimal generator [31, Chapter 1] that is a closed densely defined linear operator $G_{\mathfrak{d}}:\mathcal{D}(G_{\mathfrak{d}})\subset\mathbb{B}\to\mathbb{B}$ given by $G_{\mathfrak{d}}u=\lim_{s\to 0}\frac{\mathfrak{d}(s)u-u}{s}$, $u\in\mathcal{D}(G_{\mathfrak{d}})$.

Example 3 ([35], Lemma 6.4). *The generator $G_{\mathfrak{d}}$ of the group $\mathfrak{d}$ from Example 1 is*

$$(G_{\mathfrak{d}}x)(z)=\alpha x(z)+\beta(z\cdot\nabla)x(z),\quad z\in\mathbb{R}^n,\quad x\in\mathcal{D}(G_{\mathfrak{d}})\subset H^p(\mathbb{R}^n,\mathbb{R}^m),\tag{3.2}$$

where $z\cdot\nabla=z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+...+z_n\frac{\partial}{\partial z_n}$, the domain $\mathcal{D}(G_{\mathfrak{d}})$ is a completion of C_c^∞ with respect to the norm $\|x\|_{H^p}+\|G_{\mathfrak{d}}x\|_{H^p}$.

3.2. Homogeneous Spheres, Balls and Cones. Let us consider some basic geometric objects induced by dilations.

Definition 3.1. *The set $S_{\mathfrak{d}}(r,z_0)=\{z\in\mathbb{B}:\|\mathfrak{d}\left(-\ln(r)\right)(z-z_0)\|=1\}$ is a $\mathfrak{d}$-homogeneous sphere of the radius $r>0$ centered at $z_0\in\mathbb{B}$. An open $\mathfrak{d}$-homogeneous ball of the radius $r>0$ centered at $z_0\in\mathbb{B}$ is given by $B_{\mathfrak{d}}(r,z_0)=\{z\in\mathbb{B}:\|\mathfrak{d}\left(-\ln(r)\right)(z-z_0)\|<1\}$.*

For spheres and balls centered at the origin we use the notations $S_{\mathfrak{d}}(r):=S_{\mathfrak{d}}(r,\mathbf{0})$ and $B_{\mathfrak{d}}(r):=B_{\mathfrak{d}}(r,\mathbf{0})$, respectively. Obviously, the unit homogeneous sphere $S_{\mathfrak{d}}(1)$ coincides with the unit sphere in $\mathbb{B}$ and the unit homogeneous ball $B_{\mathfrak{d}}(1)$ coincides with the unit ball in $\mathbb{B}$, but $S_{\mathfrak{d}}(r)=\mathfrak{d}(\ln r)S_{\mathfrak{d}}(1)$ and $B_{\mathfrak{d}}(r)=\mathfrak{d}(\ln r)B_{\mathfrak{d}}(1)$, $r>0$. The homogeneous spheres of different radius have no intersections provided that the dilation is monotone (see below).

Definition 3.2. *A set $\mathcal{D}\subseteq\mathbb{B}$ is a $\mathfrak{d}$-homogeneous cone in $\mathbb{B}$ if $\mathfrak{d}(s)u\in\mathcal{D},\forall u\in\mathcal{D},\forall s\in\mathbb{R}$.*

If $\mathfrak{d}$ is the standard dilation $\mathfrak{d}(s)=e^sI, s\in\mathbb{R}$, then the set $\mathcal{D}$ is a usual positive cone in $\mathbb{B}$.

3.3. Monotone dilations. The monotonicity of dilation simplifies an analysis and design of generalized homogeneous control systems [38], [37]. Recall that a linear dilation $\mathfrak{d}$ in $\mathbb{B}$ is *monotone* if $s\to\|\mathfrak{d}(s)x\|$ is a non-decreasing function. A *strict monotonicity* [38] asks for the existence of a number $\omega>0$ such that $\|\mathfrak{d}(s)\|\leq e^{\omega s}$ for all $s\leq 0$. In the latter case, $\mathfrak{d}(s)$ with $s\leq 0$ defines the so-called C_0-semigroup of contractions. By definition, the monotonicity is linked with the norm in $\mathbb{B}$. It is known [34] that any continuous linear dilation in $\mathbb{R}^n$ is strictly monotone under a proper selection of the weighted Euclidean norm. The following theorem extends this result to Banach spaces.

Theorem 3.3. *If $\mathfrak{d}$ is a strongly continuous linear dilation in a real Banach space $\mathbb{B}$ then there exist an equivalent norm $\|\cdot\|_*$ in $\mathbb{B}$ and an a number $\omega>0$ such that $\|\mathfrak{d}(s)\|_*\leq e^{\omega s},\forall s<0$.*

Proof. 1) On the one hand, since $\mathfrak{d}(s)$ is a bounded invertible operator for any $s\in\mathbb{R}$ and the group $\mathfrak{d}$ satisfies the limit property then there exists $s_0<0$ such that $0<\|\mathfrak{d}(s_0)\|<1$. On the other hand, since $\mathfrak{d}(s)$ is a strongly continuous group then there exists $M_0\in[1,+\infty)$: $\sup_{s\in[0,s_0]}\|\mathfrak{d}(s)\|\leq M_0$ (see [31, Theorem 2.2, page 4]). Notice that any $s\leq 0$ admits the representation $s=ns_0+\delta$, where $n\in\{0,1,2,...\}$ and $\delta\in(s_0,0]$. Denoting $\omega=\frac{\ln\|\mathfrak{d}(s_0)\|}{s_0}>0$ and using the group property of $\mathfrak{d}$ we derive $\|\mathfrak{d}(s)\|=\|\mathfrak{d}(\delta)\mathfrak{d}^n(s_0)\|\leq M_0\|\mathfrak{d}(s_0)\|^n=M_0e^{s_0n\omega}=M_0e^{s\omega-\delta\omega}\leq Me^{\omega s}$ where $M=M_0e^{-s_0\omega}\geq 1$.

2) Let the norm $\|\cdot\|_*$ in $\mathbb{B}$ be defined as follows $\|x\|_*=\sup_{s\leq 0}e^{-\omega s}\|\mathfrak{d}(s)x\|$. On the one hand, the function $\|\cdot\|_*$ satisfies all properties of the norm: $\|x\|_*>0$ for $x\neq 0$, $\|\mathbf{0}\|_*=0$; $\|e^sx\|_*=e^s\|x\|_*$; $\|-x\|_*=\|x\|_*$ and sub-additivity $\|x+y\|_*\leq\|x\|_*+\|y\|_*$. On the other

hand, we have $\|x\| \leq \|x\|_* \leq \sup_{s\leq 0} e^{-\omega s}\|\mathfrak{d}(s)\|\|x\| \leq M\|x\|, \forall x \in \mathbb{B}$. Therefore, $\|\cdot\|_*$ is, indeed, an equivalent norm in $\mathbb{B}$ and for any $t \geq 0$ we have

$$\|\mathfrak{d}(t)\|_* = \sup_{x\neq 0} \frac{\|\mathfrak{d}(t)x\|_*}{\|x\|_*} = \sup_{x\neq 0} \frac{\sup\limits_{s\leq 0} e^{-\omega s}\|\mathfrak{d}(t+s)x\|}{\sup\limits_{s\leq 0} e^{-\omega s}\|\mathfrak{d}(s)x\|} = e^{\omega t} \sup_{x\neq 0} \frac{\sup\limits_{t+s\leq t} e^{-\omega(t+s)}\|\mathfrak{d}(t+s)x\|}{\sup\limits_{s\leq 0} e^{-\omega s}\|\mathfrak{d}(s)x\|} \leq e^{\omega t}. \blacksquare$$

The latter theorem proves that any strongly continuous linear dilation in $\mathbb{B}$ is strictly monotone with respect to an equivalent norm in $\mathbb{B}$. This generalizes the main results of the papers [38], [37] and [36] obtained under the assumption of strict monotonicity.

Example 4. *The dilation (3.1) is strictly monotone in $L^p(\mathbb{R}^n, \mathbb{R}^m)$ if $\alpha - \beta n/p > 0$ and strictly monotone in $H^p(\mathbb{R}^n, \mathbb{R}^m)$ if $\alpha > \beta(0.5n/2 - i), i = 0, 1, ..., p$ (see Example 1).*

Strict monotonicity of a dilation in a Hilbert space can be established using the generator.

Proposition 3.4. *[35, Proposition 6.5] A strongly continuous linear dilation $\mathfrak{d}$ in a real Hilbert space $\mathbb{H}$ is strictly monotone if and only if there exists $\gamma > 0$ and a $\mathfrak{d}$-homogeneous cone $\mathcal{D}$ dense in $\mathcal{D}(G_\mathfrak{d})$ such that*

$$\langle G_\mathfrak{d} x, x\rangle \geq \gamma\|z\|^2 \quad \text{for any} \quad x \in \mathcal{D}. \tag{3.3}$$

Example 5. *The generator of the dilation (3.1) is given by (3.2). Using integration by parts, we derive $\langle G_\mathfrak{d} x, x\rangle_{L_2} = \langle \alpha x + \beta(z\cdot\nabla)x, x\rangle_{L_2} = \left(\alpha - \frac{\beta n}{2}\right)\langle x, x\rangle_{L_2}, \forall x \in \mathcal{D}(G_\mathfrak{d})$.*

An analog of Theorem 3.3 can be obtained for Hilbert spaces as follows.

Corollary 3.5. *If $\mathfrak{d}$ is a strongly continuous linear dilation in a real Hilbert Space $\mathbb{H}$ then the inequality (3.3) is fulfilled for the inner product $\langle x, y\rangle_* = \int_{-\infty}^0 e^{-2\gamma s}\langle \mathfrak{d}(s)x, \mathfrak{d}(s)y\rangle ds$, where $\gamma \in (0, \omega)$ is an arbitrary number and $\omega > 0$ is defined in the proof of Theorem 3.3.*

Proof. 1) Let us show that $\langle x, y\rangle_*$ is an inner product in $\mathbb{H}$. On the one hand, since $\mathfrak{d}$ is a strongly continuous group in $\mathbb{H}$ then the function $s \mapsto \langle \mathfrak{d}(s)x, \mathfrak{d}(s)y\rangle$ is a continuous for any fixed $x, y \in \mathbb{H}$. On the other hand, since $\gamma \in (0, \omega)$ and $\|\mathfrak{d}(s)z\|^2 \leq M^2 e^{2\omega s}\|z\|^2$ for all $s \leq 0$ and for all $z \in \mathbb{H}$, where $M \geq 1$ is defined in the proof of Theorem 3.3, then

$$\int_{-\infty}^0 e^{-2\gamma s}|\langle \mathfrak{d}(s)x, \mathfrak{d}(s)y\rangle| ds \leq M^2\|x\|\|y\| \int_{-\infty}^0 e^{2(\omega-\gamma)s} ds = \frac{M^2}{2(\omega-\gamma)}\|x\|\|y\|.$$

Hence, $\langle x, y\rangle_*$ is well-defined for any $x, y \in \mathbb{H}$. The linearity $\langle \mu x, z\rangle_* = \langle x, y\rangle_*, \langle x+y, z\rangle_* = \langle x, z\rangle_* + \langle y, z\rangle_*, \forall x, y, z \in \mathbb{H}, \forall \mu \in \mathbb{H}$ and the Hermitian symmetry $\langle x, y\rangle_* = \langle y, x\rangle_*, \forall x, y \in \mathbb{H}$ of $\langle\cdot,\cdot\rangle_*$ comes from the linearity and the Hermitian symmetry of $\langle\cdot,\cdot\rangle$. Since the function $s \mapsto \|\mathfrak{d}(s)x\|^2$ is continuous and $\|\mathfrak{d}(0)x\|^2 = \|x\|^2$ then $\langle x, x\rangle_* > 0$ for any $x \neq 0$.

2) Let us show that the inequality (3.3) holds for the inner product $\langle\cdot,\cdot\rangle_*$. Since $\frac{d}{ds}\mathfrak{d}(s)x = \mathfrak{d}(s)G_\mathfrak{d} x$ for all $x \in \mathcal{D}(G_\mathfrak{d})$ (see [31, Theorem 2.4, page 4]) then

$$\langle G_\mathfrak{d} x, x\rangle_* = \int_{-\infty}^0 e^{-2\gamma s}\langle \mathfrak{d}(s)G_\mathfrak{d} x, \mathfrak{d}(s)x\rangle ds = \int_{-\infty}^0 e^{-2\gamma s}\left\langle \frac{d}{ds}\mathfrak{d}(s)x, \mathfrak{d}(s)x\right\rangle ds$$
$$= \int_{-\infty}^0 \frac{e^{-2\gamma s}}{2}\frac{d}{ds}\|\mathfrak{d}(s)x\|^2 ds = \frac{e^{-2\gamma s}}{2}\|\mathfrak{d}(s)x\|^2\Big|_{-\infty}^0 + \gamma\int_{-\infty}^0 e^{-2\gamma s}\|\mathfrak{d}(s)x\|^2 ds.$$

Since $\gamma \in (0, \omega)$ and $\|\mathfrak{d}(s)x\|^2 \leq M^2 e^{2\omega s}\|x\|^2, \forall s \leq 0$ then $e^{-2\gamma s}\|\mathfrak{d}(s)x\|^2 \to 0$ as $s \to -\infty$ and $\langle G_\mathfrak{d} x, x\rangle_* = \frac{\|x\|^2}{2} + \gamma\|x\|_*^2 \geq \gamma\|x\|_*^2$ for all $x \in \mathcal{D}(G_\mathfrak{d})$. $\blacksquare$

3.4. The canonical homogeneous norm. In the view of Theorem 3.3, without loss of generality, we can always assume that any strongly continuous linear dilation in $\mathbb{B}$ is strictly monotone. Such a dilation introduces an alternative norm topology in $\mathbb{B}$ by means of the so-called *canonical homogeneous norm* $\|\cdot\|_{\mathfrak{d}}:\mathbb{B}\to\mathbb{R}_+$ defined as follows: $\|\mathbf{0}\|_{\mathfrak{d}}=0$ and

$$\|u\|_{\mathfrak{d}}=e^{s_u},\quad \text{where } s_u\in\mathbb{R}:\|\mathfrak{d}(-s_u)u\|=1,\quad u\neq\mathbf{0} \tag{3.4}$$

(see, [38] for more details). By construction, $\|\mathfrak{d}(s)u\|_{\mathfrak{d}}=e^s\|u\|_{\mathfrak{d}}$, $\|u\|_{\mathfrak{d}}=\|-u\|_{\mathfrak{d}}$ for $\forall u\in\mathbb{B},\forall s\in\mathbb{R}$ and $\|u\|_{\mathfrak{d}}=1\Leftrightarrow\|u\|=1$. Notice that $\|\cdot\|_{\mathfrak{d}}=\|\cdot\|$ provided that $\mathfrak{d}$ is the standard dilation $\mathfrak{d}(s)=e^sI$, $s\in\mathbb{R}$. The following result was originally proven in [38].

Theorem 3.6 ([35], Lemmas 7.1, 7.2). *If $\mathfrak{d}$ is a strongly continuous strictly monotone linear dilation then $\|\cdot\|_{\mathfrak{d}}$ is single-valued, positive definite, locally Lipschitz continuous on $\mathbb{B}\backslash\{\mathbf{0}\}$ and there exist $\eta\geq\omega>0$, $C_{\mathfrak{d}}\geq 1$ such that $\|\mathfrak{d}(s)\|\leq e^{\omega s},\forall s\leq 0$, $\|\mathfrak{d}(s)\|\leq C_{\mathfrak{d}}e^{\eta s},\forall s\geq 0$, $\frac{1}{C_{\mathfrak{d}}}\|u\|_{\mathfrak{d}}^{\eta}\leq\|u\|\leq\|u\|_{\mathfrak{d}}^{\omega}$ if $u\in B_{\mathfrak{d}}(1)$ and $\|u\|_{\mathfrak{d}}^{\omega}\leq\|u\|\leq C_{\mathfrak{d}}\|u\|_{\mathfrak{d}}^{\eta}$ if $u\in\mathbb{B}\backslash B_{\mathfrak{d}}(1)$.*

The latter means that there exist $\underline{\sigma},\overline{\sigma}\in\mathcal{K}^\infty:\underline{\sigma}(\|u\|)\leq\|u\|_{\mathfrak{d}}\leq\overline{\sigma}(\|u\|)$, $\forall u\in\mathbb{B}$. In [35, Theorem 7.1] it is also shown that $\|\cdot\|_{\mathfrak{d}}$ is a norm (in the classical sense) for a Banach space $\tilde{\mathbb{B}}$ homeomorphic to $\mathbb{B}$, where $\tilde{\mathbb{B}}$ consists of elements of $\mathbb{B}$, but the rules for addition and scalar multiplication are modified. This justifies the name "norm" for the functional $\|\cdot\|_{\mathfrak{d}}$.

Example 6 ([36]). *Since $\|\mathfrak{d}(s)x\|_{H^p}^2=\sum_{i=0}^p e^{s(\alpha-\beta n/2+i\beta)}\|\nabla^i x\|_{L^2}^2, x\in H^p(\mathbb{R}^n,\mathbb{R}^m)$ for the dilation (3.1) then for $x\neq\mathbf{0}$ the canonical homogeneous norm in H^p is defined as $\|x\|_{\mathfrak{d},H^p}=1/V$, where $V>0$ is a unique real positive root of the following equation $1=\sum_{i=0}^p V^{\alpha-\beta n/2+i\beta}a_i$, where $a_i=\|\nabla^i x\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^m)}$, $x\neq\mathbf{0}$, $\alpha>\beta n/2-i\beta, i=0,1...,p$. Hence, $x\mapsto\|x\|_{\mathfrak{d},H^p}$ is a continuously differentiable function for $x\neq 0$.*

It is known [24] that the weighted Euclidean norm is a Lyapunov function for any stable linear system in $\mathbb{R}^n$. The canonical homogeneous norm is a Lyapunov function for many homogeneous systems [34]. The differentiability of the homogeneous norm is important for the corresponding analysis. In the case of an abstract Hilbert space $\mathbb{H}$, the canonical homogeneous norm is Frechét differentiable, at least, on the domain of the generator $G_{\mathfrak{d}}$.

Lemma 3.7 ([35], Lemma 7.4). *Let $\mathfrak{d}$ be a strongly continuous strictly monotone linear dilation in a Hilbert space $\mathbb{H}$ then the homogeneous norm $\|\cdot\|_{\mathfrak{d}}$ is differentiable on $\mathcal{D}(G_{\mathfrak{d}})\backslash\{\mathbf{0}\}$ and the Fréchet derivative of $\|\cdot\|_{\mathfrak{d},\mathbb{H}}$ at $u\in\mathcal{D}(G_{\mathfrak{d}})\backslash\{\mathbf{0}\}$ is given by*

$$(D\|u\|_{\mathfrak{d},\mathbb{H}})(\cdot)=\frac{\langle\mathfrak{d}(-\ln\|u\|_{\mathfrak{d},\mathbb{H}})\,\cdot\,,\ \mathfrak{d}(-\ln\|u\|_{\mathfrak{d},\mathbb{H}})u\rangle}{\langle G_{\mathfrak{d}}\mathfrak{d}(-\ln\|u\|_{\mathfrak{d},\mathbb{H}})u,\mathfrak{d}(-\ln\|u\|_{\mathfrak{d},\mathbb{H}})u\rangle}\|u\|_{\mathfrak{d},\mathbb{H}}. \tag{3.5}$$

Example 7 ([36]). *Let us consider again the strongly continuous strictly monotone dilation (3.1) in $H^1(\mathbb{R}^n,\mathbb{R}^m)$ with $\alpha>\beta n/2-i\beta, i=0,1$. On the one hand, from Example 6 we conclude the Fréchet differentiability of $\|\cdot\|_{\mathfrak{d},H^p}$ on $H^p(\mathbb{R}^n,\mathbb{R}^m)\backslash\{\mathbf{0}\}$. On the other hand, for $v\in\mathcal{D}(G_{\mathfrak{d}})\subset H^1(\mathbb{R}^n,\mathbb{R})$ using integration by parts we derive $\langle G_{\mathfrak{d}}v,v\rangle_{H^1}=(\alpha-\beta n/2)\langle v,v\rangle_{L_2}+(\alpha+\beta(1-n/2))\langle\nabla v,\nabla v\rangle_{L_2}$, so the Fréchet derivative of $\|\cdot\|_{\mathfrak{d},H^1}$ at the point $x\in H^1(\mathbb{R}^n,\mathbb{R})$ is a linear functional $D\|x\|_{\mathfrak{d},H^1}:H^1(\mathbb{R}^n,\mathbb{R})\mapsto\mathbb{R}$ defined as follows*

$$\left(D\|x\|_{\mathfrak{d},H^1}\right)(u)=\frac{\left\langle\mathfrak{d}(-\ln\|x\|_{\mathfrak{d},H^1})u,\,v\right\rangle_{H^1}}{(\alpha-\beta n/2)\|v\|_{L_2}^2+(\alpha+\beta(1-n/2))\|\nabla v\|_{L_2}^2}\|x\|_{\mathfrak{d},H^1},\ u\in H^1(\mathbb{R}^n,\mathbb{R}), \tag{3.6}$$

where $v=\mathfrak{d}\left(-\ln\|x\|_{\mathfrak{d},H^1}\right)x$.

3.5. Homogeneous operators. Homogeneous functionals and operators on $\mathbb{B}$ (see [39]) are defined similarly to homogeneous functions and vector fields in $\mathbb{R}^n$ (see e.g. [19], [11]).

Definition 3.8. *Let $\mathfrak{d}$ be a dilation. An operator $f:\mathcal{D}(f)\subset\mathbb{B}\to\mathbb{B}$ (a functional $h:\mathcal{D}(h)\subset\mathbb{B}\to\mathbb{R}$) is said to be $\mathfrak{d}$-homogeneous of a degree $\nu\in\mathbb{R}$ if the domain $\mathcal{D}(f)$ (resp. $\mathcal{D}(h)$) is a $\mathfrak{d}$-homogeneous cone and*

$$\begin{array}{c} e^{\nu s}\mathfrak{d}(s)f(u)=f(\mathfrak{d}(s)u),\quad \forall s\in\mathbb{R},\quad \forall u\in\mathcal{D}(f),\\ (\textit{resp. } h(\mathfrak{d}(s)u)=e^{\nu s}h(u),\quad \forall s\in\mathbb{R},\quad \forall u\in\mathcal{D}(h)).\end{array}\tag{3.7}$$

We say that an evolution equation (inclusion) is $\mathfrak{d}$-homogeneous of degree $\nu\in\mathbb{R}$ if its right-hand side is a $\mathfrak{d}$-homogeneous operator of degree ν.

Formally, the latter definition does not require the dilation $\mathfrak{d}$ to be a linear or strongly continuous. However, we deal only with strongly continuous linear dilations in $\mathbb{B}$. The identity (3.7) can be understood in the week sense (see, [37] for the more details).

Example 8 ([37]). *The Laplace operator $\Delta: H^2(\mathbb{R}^n,\mathbb{R}^m)\subset L^2(\mathbb{R}^n,\mathbb{R}^m)\mapsto L^2(\mathbb{R}^n,\mathbb{R}^m)$ is $\mathfrak{d}$-homogeneous of the degree 2β with respect to the dilation $\mathfrak{d}$ given by (3.1). Indeed,*

$$(\Delta(\mathfrak{d}(s)x)(z)=e^{\alpha s}\sum_{i=1}^{n}\frac{\partial^2 x(e^{\beta s}z)}{\partial z_i^2}=e^{(\alpha+2\beta)s}\sum_{i=1}^{n}\frac{\partial^2 x(y)}{\partial y_i^2}\bigg|_{y=e^{\beta s}z}=(e^{2\beta s}\mathfrak{d}(s)\Delta x)(z),\quad s\in\mathbb{R}, z\in\mathbb{R}^n$$

provided that $x\in\mathcal{D}(A)$. The operator $f(x)=(x\cdot\nabla)x$ is $\mathfrak{d}$-homogeneous as well, since

$$f(\mathfrak{d}(s)x)=((\mathfrak{d}(s)x)\cdot\nabla)(\mathfrak{d}(s)x)=e^{(\alpha+\beta)s}\mathfrak{d}(s)((x\cdot\nabla)x).$$

Many other examples of $\mathfrak{d}$-homogeneous PDEs can be found in [39], [35].

3.6. Symmetry of solutions of homogeneous systems. A C_0-semigroup Φ generated by a closed densely defined linear homogeneous operator in $\mathbb{B}$ is homogeneous as well.

Lemma 3.9 ([35], Lemma 8.1). *Let a linear closed densely defined operator $A:\mathcal{D}(A)\subset\mathbb{X}\mapsto\mathbb{X}$ generate a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{X}$. If the operator A is $\mathfrak{d}$-homogeneous of a degree $\mu\in\mathbb{R}$ then $\Phi(t)\mathfrak{d}(s)=\mathfrak{d}(s)\Phi(e^{\mu s}t)$, $\forall t\geq 0$, $\forall s\in\mathbb{R}$.*

Lemma 3.9 proves the symmetry of solutions of (2.1) for $f\equiv\mathbf{0}$:

$$x_{\mathfrak{d}(s)x_0}(t)=\mathfrak{d}(s)x_{x_0}(e^{\mu s}t),\quad s\in\mathbb{R}, t\geq 0,\tag{3.8}$$

where x_z denotes a solution of (2.1) with the initial data $x(0)=z$. This symmetry of solutions takes a place for any nonlinear $\mathfrak{d}$-homogeneous system (2.1) as well.

Theorem 3.10 ([35], Theorem 8.1). *Let a linear closed densely defined operator $A:\mathcal{D}(A)\subset\mathbb{X}\mapsto\mathbb{X}$ generate a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{X}$ and $f:\mathcal{D}(f)\subset\mathbb{X}\mapsto\mathbb{X}$. Let A and f be $\mathfrak{d}$-homogeneous operators of degree $\mu\in\mathbb{R}$. If $x:[0,T)\mapsto\mathbb{X}$ is a mild solution of*

$$\dot{x}=Ax+f(x),\quad t>0\tag{3.9}$$

and $x(t)\overset{a.e.}{\in}\mathcal{D}(f)$ then for any $s\in\mathbb{R}$ the function $x^s:[0,e^{-\mu s}T)\mapsto\mathbb{X}$ given by $x^s(t):=\mathfrak{d}(s)x(e^{\mu s}t)$, $t\in[0,e^{-\mu s}T)$ is a mild solution of the evolution equation (3.9) and $x^s(t)\overset{a.e.}{\in}\mathcal{D}(f)$.

According to the latter theorem, The dilation symmetry expands globally any local property of solutions, e.g., locally stability of the system (3.9) guarantees its globally stablity, the forward completeness for small initial data implies the forward completeness for large initial data, etc. Theorem 3.10 is generalized below to the system (2.1) with perturbations.

Example 9. *In the view Theorem 3.10 and Example 8, all mild solutions of the system* $\dot{x} = \Delta x - (x \cdot \nabla)x$, $t \geq 0$, $x \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ *are symmetric:* $x_{\mathfrak{d}(s)x_0}(t) = \mathfrak{d}(s)x_{x_0}(e^{\mu s}t)$, $s \in \mathbb{R}$, $t \geq 0$, *where* $\mathfrak{d}$ *is given by* (3.1) *with* $\alpha = \beta$.

3.7. Homogeneous Lyapunov function theorem. Since solutions of (3.9) with the M-regular operator f are well-defined on $\mathbb{Y}\backslash\{\mathbf{0}\}$ then it is reasonable to analyze the stability and robustness of the system (3.9) in $\mathbb{Y}$-topology. In this paper we deal mainly with some notions of uniform stability in $\mathbb{Y}$ typical for homogeneous systems having homogeneous approximations [1, 3, 35]. For shortness we omit the words "globally uniformly" in the stability definitions given below.

Definition 3.11. *The system* (3.9) *is said to be*

- *practically Lyapunov stable in* $\mathbb{Y}$ *if there exist* $c \geq 0$ *and* $\varepsilon \in \mathcal{K}^\infty$ *such that*

$$\|x_{x_0}(t)\|_{\mathbb{Y}} \leq c + \varepsilon(\|x_0\|_{\mathbb{Y}}), \quad \forall t \geq 0 \tag{3.10}$$

for any mild solution $x_{x_0} : \mathbb{R}_+ \to \mathbb{Y}$ *of* (3.9) *with* $x(0) = x_0$*;*
- *Lyapunov stable in* $\mathbb{Y}$ *if it is practically Lyapunov stable in* $\mathbb{Y}$ *with* $c = 0$*;*
- *practically asymptotically stable in* $\mathbb{Y}$ *if it is practically Lyapunov stable in* $\mathbb{Y}$ *with* $c > 0$ *and* $\forall R > c$, $\exists T_c(R) \in \mathbb{R}_+ : \|x_0\|_{\mathbb{Y}} \leq R \Rightarrow \|x_{x_0}(t)\|_{\mathbb{Y}} \leq c$ *for all* $t > T_c(R)$.
- *asymptotically stable in* $\mathbb{Y}$ *if it is practically asymptotically stable for any* $c > 0$
- *finite-time stable in* $\mathbb{Y}$ *if it is practically asymptotically stable in* $\mathbb{Y}$ *with* $c = 0$
- *practically fixed-time stable in* $\mathbb{Y}$ *if it is practically Lyapunov stable with some* $c > 0$ *and there exists* $T_{\max}(c) > 0$ *such that* $\|x_{x_0}(t)\| \leq c$, $\forall t \geq T_{\max}(c)$, $\forall x_0 \in \mathbb{Y}$*;*
- *nearly fixed-time stable in* $\mathbb{Y}$ *if it is practically fixed-time stable for any* $c > 0$*;*
- *fixed-time stable in* $\mathbb{Y}$ *if it is practically fixed-time stable with* $c = 0$

If one of properties mentioned above is fulfilled only for all x_0 from a neighborhood of the origin then the corresponding stability is called local. Below we show that homogeneous systems as well as systems having a stable homogeneous approximation admit stability properties mentioned above. The following theorem is the straightforward corollary of Theorem 3.3 and [36, Theorem 3].

Theorem 3.12 ([36]). *Let* $\mathfrak{d}$ *be a strongly continuous linear dilation in* $\mathbb{Y}$*, the evolution equation* (3.9) *be* $\mathfrak{d}$*-homogeneous of degree* $\nu \in \mathbb{R}$*, and* $f : \mathbb{Y} \subset \mathbb{X} \to \mathbb{X}$ *satisfy Assumption 1. Let* $m > -\nu$ *be an arbitrary number. The origin of* (3.9) *is uniformly asymptotically stable in* $\mathbb{Y}$ *if and only if there exists a continuous positive definite functional* $V : \mathbb{Y} \to \mathbb{R}$ *such that*

1) V *is* $\mathfrak{d}$*-homogeneous of a degree* m *and locally Lipschitz continuous on* $\mathbb{Y}\backslash\{\mathbf{0}\}$*;*
2) $\exists \underline{k}, \overline{k} \in \mathcal{K}^\infty$ *satisfying*

$$\underline{k}(\|x\|_{\mathbb{Y}}) \leq V(x) \leq \overline{k}(\|x\|_{\mathbb{Y}}), \quad \forall x \in \mathbb{Y}, \tag{3.11}$$

3) *for any mild solution* x_{x_0} *of* (3.9) *with* $x(0) = x_0 \in \mathbb{Y}$ *the inequality*

$$\overline{D}^+ V(x_{x_0}(t)) \leq -cV^{\frac{m+\nu}{m}}(x_{x_0}(t)), \quad t > 0 \tag{3.12}$$

holds as long as $x_{x_0}(t)\neq \mathbf{0}$, *where* $c>0$ *is a constant.*

The latter theorem, in particular, implies that any uniformly asymptotically stable $\mathfrak{d}$-homogeneous system (3.9) of negative (resp. positive) degree is globally uniformly finite-time (resp. nearly fixed-time) stable. For reflexive Banach spaces (in, particular, for Hilbert spaces), the time derivative of the Lyapunov function along trajectories of the system can be calculated using the right-hand side of the evolution equation (3.9).

Corollary 3.13 ([36]). *Theorem 3.12 remains true even if the condition 3) is replaced with 3') $\overline{D}^{+}V(x;Ax+f(x))\leq -cV^{\frac{m+\nu}{m}}(x), \quad \forall x\in M^{-1}\mathcal{D}(A)\backslash\{\mathbf{0}\}$, provided that $\mathbb{X}$ is a reflexive Banach space and the set $M^{-1}\mathcal{D}(A)$ is dense in $\mathbb{Y}$.*

The canonical homogeneous norm is a Lyapunov function for some homogeneous PDEs (see, e.g. [38], [36], [35, Chapter 9]). In this case, the time derivative of $\|x\|_{\mathfrak{d},\mathbb{Y}}$ computed along the trajectories of the system satisfies $\frac{d}{dt}\|x\|_{\mathfrak{d},\mathbb{Y}}\leq -c\|x\|_{\mathfrak{d},\mathbb{Y}}^{1+\mu}$. The existence of the positive constant $c>0$ satisfying the latter inequality can always be guaranteed for the case of the *uniform* asymptotic stability. If homogeneous system (3.9) is asymptotically stable (but non-uniformly with respect to initial conditions) we should expect inequality like $\frac{d}{dt}\|x\|_{\mathfrak{d},\mathbb{Y}}< -\tilde{c}(x)\|x\|_{\mathfrak{d},\mathbb{Y}}^{m+\mu}$ for $\forall x\in M^{-1}\mathcal{D}(A)\backslash\{\mathbf{0}\}$, where the positive definite functional $\tilde{c}:\mathbb{Y}\to\mathbb{R}_+$ is $\mathfrak{d}$-homogeneous of degree 0, but $\inf_{x\in S_{\mathbb{Y}}}\tilde{c}(x)=0$. The latter is possible even for continuous positive definite functional $\tilde{c}$, since the unit sphere in $\mathbb{B}$ is non-compact in the general case. Inspired by [36], we present the following example.

Example 10. *Let us consider the viscous Burgers equation*

$$\dot{x}=\frac{\partial^2 x}{\partial z^2}-x\frac{\partial x}{\partial z},\quad t>0,\quad x(0)=x_0, \tag{3.13}$$

where $x(t)\in L^2(\mathbb{R},\mathbb{R})$. This equation can be the represented in the form (3.9) with $A=\frac{\partial^2}{\partial z^2}$, $\mathbb{X}=L^2(\mathbb{R},\mathbb{R})$, $\mathcal{D}(A)=H^2(\mathbb{R},\mathbb{R})$, $f(x)=-x\frac{\partial x}{\partial z}$ and $\mathcal{D}(f)=\mathbb{Y}=H^1(\mathbb{R},\mathbb{R})$. Notice that f is M-regular with $M=\frac{\partial}{\partial z}+I$ and $(M^{-1}y)(z)=\int_{-\infty}^{z}e^{-(z-s)}y(s)ds$, $y\in\mathbb{X}, z\in\mathbb{R}$, respectively. Indeed, the inequality (2.2), obviously, follows from $\|\frac{\partial}{\partial z}M^{-1}\|_{L^2}<\infty$ and

$$\|M^{-1}y\|_{L^\infty}\leq\sup_{z\in\mathbb{R}}\int_{-\infty}^{z}e^{-(z-s)}|y(s)|ds\leq\sup_{z\in\mathbb{R}}\sqrt{\int_{-\infty}^{z}y^2(s)ds}\sqrt{\int_{-\infty}^{z}e^{-2(z-s)}ds}=\frac{\|y\|_{L^2}}{\sqrt{2}},\ \forall y\in\mathbb{X}. \tag{3.14}$$

The C_0-semigroup generated by A is given by $(\Phi(t)y)(z)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(z-s)^2}{4t}}y(s)ds$. Since Φ commutes with M and $\|M\Phi(t)\|\leq C\left(\frac{1}{\sqrt{t}}+1\right),\forall t>0$ for some $C>0$, then the system (3.13) satisfies Assumption 1.

The system (3.13) is $\mathfrak{d}$-homogeneous of the degree $\mu=2$ with respect to the dilation $(\mathfrak{d}(s)x)(z)=e^s x(e^s z)$, $z\in\mathbb{R}, x\in\mathbb{X}$. The dilation $\mathfrak{d}$ is strictly monotone and strongly continuous in $\mathbb{X}$ and $\mathbb{Y}$ (see Example 1).

Let us show that the canonical homogeneous norm $\|\cdot\|_{\mathfrak{d},H^1}$ is a $\mathfrak{d}$-homogeneous Lyapunov function of the system. Indeed, for $v\in M^{-1}\mathcal{D}(A)=H^3(\mathbb{R},\mathbb{R})$ we have

$$\langle v,Av+f(v)\rangle_{H^1}=\left\langle v,\frac{\partial^2 v}{\partial z^2}-v\frac{\partial v}{\partial z}\right\rangle_{L^2}+\left\langle\frac{\partial v}{\partial z},\frac{\partial}{\partial z}\left(\frac{\partial^2 v}{\partial z^2}-v\frac{\partial v}{\partial z}\right)\right\rangle_{L^2}=$$

$$-\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2-\left\|\tfrac{\partial^2 v}{\partial z^2}\right\|_{L^2}^2+\left\langle\tfrac{\partial^2 v}{\partial z^2},v\tfrac{\partial v}{\partial z}\right\rangle=-\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2-\tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2}\right\|_{L^2}^2-\tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2}-v\tfrac{\partial v}{\partial z}\right\|_{L^2}^2+\tfrac{1}{2}\left\|v\tfrac{\partial v}{\partial z}\right\|_{L^2}^2=$$

$$-\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2-\tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2}\right\|_{L^2}^2-\tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2}-v\tfrac{\partial v}{\partial z}\right\|_{L^2}^2-\tfrac{1}{6}\left\langle v^3,\tfrac{\partial^2 v}{\partial z^2}\right\rangle=$$

$$-\left\|\tfrac{\partial v}{\partial z}\right\|_{L^2}^2-\tfrac{5}{12}\left\|\tfrac{\partial^2 v}{\partial z^2}\right\|_{L^2}^2-\tfrac{1}{2}\left\|\tfrac{\partial^2 v}{\partial z^2}-v\tfrac{\partial v}{\partial z}\right\|_{L^2}^2-\tfrac{1}{12}\left\|v^3+\tfrac{\partial^2 v}{\partial z^2}\right\|_{L_2}^2+\tfrac{1}{12}\|v\|_{L_6}^6.$$

Using the Gagliardo-Nirenberg-Sobolev inequality (see [7]) we derive $\|v\|_{L^6}^6\leq\sqrt{\frac{\pi}{2}}\|v\|_{L^2}^4\left\|\frac{\partial v}{\partial z}\right\|_{L^2}^2$ *and conclude* $\langle v,Av+f(v)\rangle_{H^1}\leq-\left\|\frac{\partial v}{\partial z}\right\|_{L^2}^2\left(1-\frac{\sqrt{\pi/2}}{12}\|v\|_{L^2}^4\right)$. *The latter means that,* $\|\cdot\|_{H_1}$ *is a local Lyapunov function of the system. Using* $\mathfrak{d}$*-homogeneity of operators* A, f *and the formula (3.6), for any* $x\in M^{-1}\mathcal{D}(A)$ *we obtain*

$$\left(D\|x\|_{\mathfrak{d},H^1}\right)(Ax+f(x))=\frac{\left\langle\mathfrak{d}\left(-\ln\|x\|_{\mathfrak{d},H^1}\right)(Ax+f(x)),\,v\right\rangle_{H^1}}{0.5\|v\|_{L_2}^2+1.5\left\|\frac{\partial v}{\partial z}\right\|_{L_2}^2}\|x\|_{\mathfrak{d},H^1}=$$

$$\frac{\langle Av+f(v),\,v\rangle_{H^1}}{0.5\|v\|_{L_2}^2+1.5\left\|\frac{\partial v}{\partial z}\right\|_{L_2}^2}\|x\|_{\mathfrak{d},H^1}^3\leq-\tilde{c}(x)\|x\|_{\mathfrak{d},H^1}^{1+\mu},\qquad \tilde{c}(x)=\frac{\left\|\frac{\partial v}{\partial z}\right\|_{L^2}^2\left(1-\frac{\sqrt{\pi/2}}{12}\|v\|_{L^2}^4\right)}{0.5\|v\|_{L_2}^2+1.5\left\|\frac{\partial v}{\partial z}\right\|_{L_2}^2},$$

where $v=\mathfrak{d}(-\ln\|x\|_{\mathfrak{d},H^1})x$ *and* $\tilde{c}\in C(\mathbb{Y}\backslash\{\mathbf{0}\},\mathbb{R}_+)$. *Obviously,* $\tilde{c}$ *is* $\mathfrak{d}$*-homogeneous functional of degree* 0. *Taking into account* $\|v\|_{H^1}=1$, *we conclude* $\|v\|_{L_2}^2+\left\|\frac{\partial v}{\partial z}\right\|_{L_2}^2=1$ *and* $\|v\|_{L^2}^4\leq 1$. *Since* $\left\|\frac{\partial v}{\partial z}\right\|_{L_2}\neq 0$ *for any* $x\in\mathbb{Y}\backslash\{\mathbf{0}\}$ *then* $\tilde{c}(x)$ *is positive definite. Therefore,* $\|\cdot\|_{\mathfrak{d},\mathbb{Y}}$ *is a global* $\mathfrak{d}$*-homogeneous Lyapunov function for the viscous Burgers equation. However, a* ***uniform*** *asymptotic stability of (3.13) cannot be guaranteed since* $\inf_{x\in S_{\mathbb{Y}}}\tilde{w}(x)=0$.

3.8. Homogeneous systems with perturbations. Certain symmetry of solutions can also be discovered for the perturbed homogeneous system (2.1). Formally, the following result generalizes Theorem 4 from [37] to the case of unbounded operator f. However, its proof literally repeats the proof of the mentioned theorem.

Theorem 3.14 ([37]). *Let* $\mathfrak{d}$ *be a group of linear bounded invertible operators in* $\mathbb{X}$ *and in* $\mathbb{Y}$, *the operator* $A:\mathcal{D}(A)\subset\mathbb{X}\mapsto\mathbb{X}$ *be* $\mathfrak{d}$*-homogeneous of degree* $\mu\in\mathbb{R}$ *closed densely defined generator of a strongly continuous semigroup* Φ *of linear bounded operators on* $\mathbb{X}$, *the operator* $f:\mathcal{D}_f\times\mathbb{V}\mapsto\mathbb{X}$ *satisfies Assumption 1 and*

$$f(\mathfrak{d}(s)x,\tilde{\mathfrak{d}}(s)q)=e^{\mu s}\mathfrak{d}(s)f(x,q),\quad\forall x\in\mathcal{D}_f,\forall q\in\mathbb{V},\forall s\in\mathbb{R},$$

where $\tilde{\mathfrak{d}}$ *is a group of linear bounded operators in* $\mathbb{V}$ *and* $\mathcal{D}_f\subset\mathbb{B}$ *is a* $\mathfrak{d}$*-homogeneous cone.*

If $x_{x_0,q}:[t_0,t_0+T]\mapsto\mathbb{Y}$ *is a mild solution of the system (2.1) then for any* $s\in\mathbb{R}$ *one has*

$$x_{\mathfrak{d}(s)x_0,\tilde{q}}(t)=\mathfrak{d}(s)x_{x_0,q}(t_0+e^{\mu s}(t-t_0)),\quad\forall t\in\left[t_0,t_0+\tfrac{T}{e^{\mu s}}\right],\tag{3.15}$$

where $x_{\mathfrak{d}(s)x_0,\tilde{q}}:[t_0,t_0+e^{-\mu s}T]\mapsto\mathbb{Y}$ *is a solution of the system (2.1) for* $x_0\in\mathbb{Y}$ *replaced with* $\mathfrak{d}(s)x_0\in\mathbb{Y}$ *and* $q\in L^\infty((t_0,t_0+T),\mathbb{V})$ *replaced with* $\tilde{q}\in L^\infty((t_0,t_0+e^{-\mu s}T),\mathbb{V})$ *given by*

$$\tilde{q}(t)=\tilde{\mathfrak{d}}(s)q(t_0+e^{\mu s}(t-t_0)),\quad\forall t\in\left[t_0,t_0+\tfrac{T}{e^{\mu s}}\right].$$

The following definition of the input-to-state stability is inspired by [43].

Definition 3.15 ([43], [17], [27], [26]). *The system* (2.1) *is said to be*

- *practically ISS in* $\mathbb{Y}$ *if there exist* $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ *and* $c \geq 0$ *such that*

$$\|x_{x_0,q}(t)\|_{\mathbb{Y}} \leq \beta(\|x_0\|_{\mathbb{Y}}, t-t_0) + \gamma\left(\|q\|_{L^\infty((t_0,t),\mathbb{V})}\right) + c, \quad \forall t \geq t_0, \tag{3.16}$$

  for any mild solution $x_{x_0,q}$ *of* (2.1) *with* $x_0 \in \mathbb{Y}$;
- *ISS in* $\mathbb{Y}$ *if* (3.16) *holds for* $c = 0$;
- *locally ISS in* $\mathbb{Y}$ *if* $\exists \bar{r} > 0$ *and* $\exists \bar{q} > 0$ *such that* (3.16) *holds for* $c = 0$, $\forall x_0 : \|x_0\|_{\mathbb{Y}} \leq \bar{r}$ *and* $\forall q : \|q\|_{L^\infty((t_0,t),\mathbb{V})} \leq \bar{q}$.

The dilation symmetry simplifies the ISS analysis of the system (2.1) .

Theorem 3.16. *Let Assumption 2 and all conditions of Theorem 3.14 be fulfilled for a strongly continuous linear dilations* $\mathfrak{d}$ *and* $\tilde{\mathfrak{d}}$ *in the Banach spaces* $\mathbb{Y}$ *and* $\mathbb{V}$, *respectively. If the system* (2.1) *with* $q = \mathbf{0}$ *is uniformly asymptotically stable in* $\mathbb{Y}$ *then this system is ISS in* $\mathbb{Y}$.

Proof. First of all, notice that in the view of Theorem 3.3 the dilations $\mathfrak{d}$ and $\tilde{\mathfrak{d}}$ can assumed be strictly monotone without the loss of generality. The locally existence of solutions in $\mathbb{Y}$ and their continuous dependence of initial data follow from [36, Lemma 2 and Corollary 2], which remain valid for the system (2.1).

I. Let $\bar{q} > 0$ be an arbitrary positive number. Let $r > 1$ and $\delta > r$ be defined for $f(\cdot, \mathbf{0})$ as in [36, Corollary 4]. Since for $q = \mathbf{0}$ the origin of (2.1) is assumed to be uniformly asymptotically stable in $\mathbb{Y}$ then exists $T_r > 0$ such that $x_0 \in K_{\mathbb{Y}}(r) \Rightarrow \|x_{x_0,\mathbf{0}}(t)\|_{\mathbb{Y}} \leq 1/r, \forall t > t_0 + T_r$, and $x_0 \in K_{\mathbb{Y}}(r) \Rightarrow \|x_{x_0,\mathbf{0}}(t)\|_{\mathbb{Y}} \leq \delta, \forall t \geq t_0$. Moreover, [36, Lemma 2] yields $x_{x_0,\mathbf{0}}(t) = x^\delta_{x_0,\mathbf{0}}(t)$ as long as $x_{x_0,\mathbf{0}}(t) \in K_{\mathbb{Y}}(\delta)$. Since $x^\delta_{x_0,\mathbf{0}}(t) = \Phi(t)x_0 + \int_{t_0}^t \Phi(t-s) f_\delta(x^\delta_{x_0,\mathbf{0}}(\tau), \mathbf{0}) d\tau, \forall t > t_0$ then, for any $x_0 \in \mathbb{Y}$ we have $x^\delta_{x_0,q}(t) = x^\delta_{x_0,\mathbf{0}}(t) + \int_{t_0}^t \Phi(t-s)\big(f_\delta(x^\delta_{x_0,q}(\tau), q(\tau)) - f_\delta(x^\delta_{x_0,\mathbf{0}}(\tau), \mathbf{0})\big) d\tau$. Let $k = \sup_{s\in[0,T_r]} \|\Phi(s)\|_{\mathbb{X}}$ and the Lipschitz constant $L_{\delta,T_r,\bar{q}} > 0$ be defined as in [36, Corollary 2] for all $q \in L^\infty((t_0, t_0+T_r), \mathbb{V}) : \|q\|_{L^\infty} \leq \bar{q}$. Since $f_\delta(x,q) = \mathbf{0}$ for $x \notin K_{\mathbb{Y}}(\delta)$ then for any $q \in L^\infty((t_0,t_0+T_r),\mathbb{V}) : \|q\|_{L^\infty} \leq \bar{q}$ we have $\|x^\delta_{x_0,q}(t) - x^\delta_{x_0,\mathbf{0}}(t)\|_{\mathbb{Y}} \leq L_{\delta,\bar{q}} \int_{t_0}^t \|\Phi(t-\tau)\|_{\mathbb{X}} \left\|Mx^\delta_{x_0,q}(\tau) - Mx^\delta_{x_0,\mathbf{0}}(\tau)\right\|_{\mathbb{X}} d\tau + L_{\delta,\bar{q}} \int_{t_0}^t \|M\Phi(t-\tau)\|_{\mathbb{X}} \left\|Mx^\delta_{x_0,q}(\tau) - Mx^\delta_{x_0,\mathbf{0}}(\tau)\right\|_{\mathbb{X}} d\tau + k_\delta \int_{t_0}^t \|\Phi(t-\tau)\|_{\mathbb{Y}} \cdot \chi\left(\|q(\tau)\|_{\mathbb{V}}\right) d\tau$ for all $t \in [t_0, t_0+T_r]$, where M-regularity of f and Assumption 2 are utilized on the last step, $k_\delta := \max_{\rho\in[1/\delta,\delta]} \xi(\rho)$. Since $\left\|Mx^\delta_{x_0,q}(\tau) - Mx^\delta_{x_0,\mathbf{0}}(\tau)\right\|_{\mathbb{X}} \leq \left\|x^\delta_{x_0,q}(\tau) - x^\delta_{x_0,\mathbf{0}}(\tau)\right\|_{\mathbb{Y}}$ and $\|\Phi(\tau)\|_{\mathbb{Y}} \leq \|\Phi(\tau)\|_{\mathbb{X}}$, then using Grönwall–Bellman inequality for all $t \in [t_0, t_0+T_r]$ and $\forall q \in L^\infty((t_0,t_0+T_r),\mathbb{V}) : \|q\|_\infty \leq \bar{q}$ we derive $\|x^\delta_{x_0,q}(t) - x^\delta_{x_0,\mathbf{0}}(t)\|_{\mathbb{Y}} \leq \tilde{C}(t,t_0) \int_{t_0}^t \chi\left(\|q(\tau)\|_{\mathbb{V}}\right) d\tau$, where $\|M\Phi(t-\tau)\|_{\mathbb{X}} \leq \omega(t-\tau)$ (see Assumption 1) is utilzed to define the function $\tilde{C}(t,t_0) = k_\delta k L_{\delta,T_r,\bar{q}} e^{L_{\delta,T_r,\bar{q}} \int_{t_0}^t k+\omega(t-\tau)\,d\tau}$.

II. By Theorem 3.12, there exists a $\mathfrak{d}$-homogeneous Lyapunov function such that for any $x_0 \in \mathbb{Y}\backslash\{\mathbf{0}\}$ we have $V(x_{x_0,\mathbf{0}}(h)) - V(x_{x_0,\mathbf{0}}(0)) \leq -\int_0^h \|x_{x_0,\mathbf{0}}(\tau)\|^{\mu+1}_{\mathfrak{d},\mathbb{Y}} d\tau$ as long as $x_{x_0,\mathbf{0}}(h) \neq \mathbf{0}$. Let $x_0 \in K_{\mathbb{Y}}(r)$. If $h > 0 : x_{x_0,\mathbf{0}}(\theta), x_{x_0,q}(\theta) \in K_{\mathbb{Y}}(\delta), \forall \theta \in [0,h]$ then $x_{x_0,\mathbf{0}}(\theta) = x^\delta_{x_0,\mathbf{0}}(\theta), x_{x_0,q}(\theta) = x^\delta_{x_0,q}(\theta), \forall \theta \in [0,h]$ and using local Lipschitz continuity of V we derive $\frac{V(x_{x_0,q}(t_0+h)) - V(x_{x_0,q}(t_0))}{h} = \frac{V(x_{x_0,\mathbf{0}}(t_0+h)) - V(x_{x_0,\mathbf{0}}(t_0))}{h} + \frac{V(x_{x_0,q}(t_0+h)) - V(x_{x_0,\mathbf{0}}(t_0+h))}{h}$
$\leq -\frac{1}{h}\int_0^h \|x_{x_0,\mathbf{0}}(\tau)\|^{\mu+1}_{\mathfrak{d},\mathbb{Y}} d\tau + \frac{L_V \tilde{C}(t_0+h,t_0)}{h} \int_{t_0}^{t_0+h} \chi\left(\|q(\tau)\|_{\mathbb{V}}\right) d\tau$, where L_V is a Lipschitz constant. Therefore, one has $\overline{D}^+ V(x_{x_0,q}(t_0)) \leq -\|x_0\|^{1+\mu}_{\mathfrak{d},\mathbb{Y}} + C(\bar{q})\chi(\bar{q})$ for any $q : \|q\|_{L^\infty((t_0,t_0+\epsilon),\mathbb{V})} \leq \bar{q}$,

where $C(\bar q)=L_V\limsup_{h\to 0^+}\tilde C(h,0)$ and $\epsilon>0$ is arbitrary small number. Assumption 1 guarantees that the function $\bar q\mapsto C(\bar q)$ is uniformly bounded on any compact from $\mathbb{R}_+$. Using the semigroup property of solutions we conclude

$$\overline{D}^+V(x_{x_0,q}(t))\le -\|x_{x_0,q}(t)\|^{\mu+1}_{\mathfrak{d},\mathbb{Y}}+C(\bar q)\chi(\bar q)\quad \text{if}\quad \|q\|_{L^\infty((t,t+\epsilon),\mathbb{V})}\le\bar q,\quad x_{x_0,q}(t)\in K_{\mathbb{Y}}(\delta).$$

III. Let $s\in\mathbb{R}$ and let $\mathfrak{d}(s)x_{x_0,q}(t)\in K_{\mathbb{Y}}(r)$ (or, equivalently, $x_{x_0,q}(t)\in\mathfrak{d}(-s)K_{\mathbb{Y}}(r)$) for all $t\in[t_0,t_0+T]$ and some $T>0$. From Theorem 3.14 we derive $x_{\mathfrak{d}(s)x_0,\tilde q}(\tilde t)=\mathfrak{d}(s)x_{x_0,q}(t_0+e^{\mu s}(\tilde t-t_0))$, $\forall\tilde t\in\left[t_0,t_0+\frac{T}{e^{\mu s}}\right]$, where $\tilde t:=t_0+e^{-\mu s}(t-t_0)$, $x_{\mathfrak{d}(s)x_0,\tilde q}:[t_0,t_0+e^{-\mu s}T]\to\mathbb{B}$ is a solution of the system (2.1) for x_0 replaced with $\mathfrak{d}(s)x_0$ and $q\in L^\infty((t_0,t_0+T),\mathbb{V})$ replaced with $\tilde q\in L^\infty((t_0,t_0+e^{-\mu s}T),\mathbb{V})$ given by $\tilde q(\tilde t)=\tilde{\mathfrak{d}}(s)q(t_0+e^{\mu s}(\tilde t-t_0))$, $\forall\tilde t\in\left[t_0,t_0+\frac{T}{e^{\mu s}}\right]$. Since $x_{\mathfrak{d}(s)x_0,\tilde q}(\tilde t)=\mathfrak{d}(s)x_{x_0,q}(t_0+e^{\mu s}(\tilde t-t_0))\in K_{\mathbb{Y}}(r)$ then $\overline{D}^+V(x_{\mathfrak{d}(s)x_0,\tilde q}(\tilde t))\le-\|x_{\mathfrak{d}(s)x_0,\tilde q}(\tilde t)\|^{1+\mu}_{\mathfrak{d},\mathbb{Y}}+C(\bar q)\chi(\bar q)$ for all $\tilde t\in[t_0,t_0+e^{-\mu s}T]$ and $\forall\tilde q:\|\tilde q\|_{L^\infty((t_0,t_0+\epsilon),\mathbb{V})}\le\bar q$. Taking into account the $\mathfrak{d}$-homogeneity of V and $\|\cdot\|_{\mathfrak{d}}$ we derive $V(x_{\mathfrak{d}(s)x_0,\tilde q}(\tilde t))=V(\mathfrak{d}(s)x_{x_0,q}(t_0+e^{\mu s}(\tilde t-t_0)))=e^sV(x_{x_0,q}(t_0+e^{\mu s}(\tilde t-t_0)))$. and $\|x_{\mathfrak{d}(s)x_0,\tilde q}(\tilde t)\|^{\mu+1}_{\mathfrak{d},\mathbb{Y}}=e^{(\mu+1)s}\|x_{x_0,q}(t_0+e^{\mu s}(\tilde t-t_0))\|_{\mathfrak{d},\mathbb{Y}}$. The latter means that for any $s\in\mathbb{R}$ and $\forall x_0\in\mathfrak{d}(-s)K_{\mathbb{Y}}(r)$ we have

$$\overline{D}^+V(x_{x_0,q}(t))\le-\|x_{x_0,q}(t)\|^{1+\mu}_{\mathfrak{d},\mathbb{Y}}+e^{-(\mu+1)s}C(\bar q)\xi(\bar q)$$

as long as $x_{x_0,q}(t)\in\mathfrak{d}(-s)K_{\mathbb{Y}}(r)$ and $q\in L^\infty(\mathbb{R},\mathbb{V}):\|\tilde{\mathfrak{d}}(s)q\|_{L^\infty((t,t+\varepsilon),\mathbb{V})}\le\bar q$. Since $C:\mathbb{R}_+\mapsto\mathbb{R}_+$ is locally uniformly bounded then $\exists\bar q>0:C(\bar q)\chi(\bar q)\le 0.5$. Selecting $s=-\ln\|x_{x_0,q}(t)\|_{\mathfrak{d},\mathbb{Y}}$ we derive $\|\mathfrak{d}(s)x_{x_0,q}(t)\|_{\mathbb{Y}}=1$ and $x_{x_0,q}(t)\in\mathfrak{d}(-s)K_{\mathbb{Y}}(r)$ for $x_{x_0,q}(t)\ne\mathbf{0}$. The dilation $\tilde{\mathfrak{d}}$ is strictly monotone in $\mathbb{V}$, so by Theorem 3.6 there exist $\underline{\sigma},\overline{\sigma}\in\mathcal{K}^\infty:\underline{\sigma}\left(\|q\|_{\tilde{\mathfrak{d}},\mathbb{V}}\right)\le\|q\|_{\mathbb{V}}\le\overline{\sigma}\left(\|q\|_{\tilde{\mathfrak{d}},\mathbb{V}}\right),\forall q\in\mathbb{V}$ and the inequality $\|x_{x_0,q}(t)\|_{\mathfrak{d},\mathbb{Y}}\ge\frac{\underline{\sigma}^{-1}\left(\|\tilde q\|_{L^\infty((t_0,t+\varepsilon),\mathbb{V})}\right)}{\overline{\sigma}^{-1}(\bar q)}$ implies that $\left\|\tilde{\mathfrak{d}}(-\ln\|x_{x_0,q}(t)\|_{\mathfrak{d},\mathbb{Y}})q\right\|_{L^\infty((t_0,t+\varepsilon),\mathbb{V})}\le\bar q$. Therefore, we derive

$$\overline{D}^+V(x_{x_0,q}(t))\le-0.5\|x_{x_0,q}(t)\|^{1+\mu}_{\mathfrak{d},\mathbb{Y}}\quad\text{if}\quad\|x_{x_0,q}(t)\|_{\mathfrak{d},\mathbb{Y}}\ge\zeta\left(\|\tilde q\|_{L^\infty((t,t+\varepsilon),\mathbb{V})}\right).\tag{3.17}$$

as long as $x_{x_0,q}(t)\ne\mathbf{0}$, where $\epsilon>0$ is an arbitrary small number, $x_{x_0,q}$ is any solution of (2.1) with an arbitrary $q\in L^\infty(\mathbb{R},\mathbb{V})$ and $\zeta=\frac{1}{\overline{\sigma}^{-1}(\bar q)}\underline{\sigma}^{-1}\in\mathcal{K}^\infty$. Repeating the conventional arguments from [43], [45], [44], [17],[26] we conclude that there exist $\beta\in\mathcal{KL}$, $\gamma\in\mathcal{K}$ such that $\|x_{x_0,q}(t)\|_{\mathbb{Y}}\le\beta(\|x_0\|_{\mathbb{Y}},t-t_0)+\gamma\left(\|q\|_{L^\infty((t_0,t+\epsilon),\mathbb{V})}\right),\forall t\ge t_0$, for any mild solution $x_{x_0,q}$ of (2.1) with $x_0\in\mathbb{Y}$. Using continuity of $x_{x_0,q}$ in time we derive (3.16) by tending $\epsilon\to 0$. ■

Example 11 (ISS analysis of a nonlinear viscous Burgers equation using homogeneity). *Let us consider the viscous Burgers equation with nonlinearities and perturbations*

$$\dot x=\frac{\partial^2 x}{\partial z^2}-\left(\frac{\partial x}{\partial z}+q_1\right)x-\rho(\|x\|^4_{L_2}+q_2)x+q_3,\quad t>0,\quad z\in\mathbb{R},\quad x(0)=x_0,\tag{3.18}$$

where $x(t)\in L^2(\mathbb{R},\mathbb{R})$, $q_1(t)\in H^1(\mathbb{R},\mathbb{R})$, $q_2(t)\in\mathbb{R}$ and $q_3(t)\in L^2(\mathbb{R},\mathbb{R})$. The considered equation admits the representation (2.1) with $A=\frac{\partial^2}{\partial z^2},\mathbb{X}=L^2(\mathbb{R},\mathbb{R}),\mathbb{V}=H^1(\mathbb{R},\mathbb{R})\times\mathbb{R}\times L^2(\mathbb{R},\mathbb{R})$, and $f(x,q)=-\left(\frac{\partial x}{\partial z}+q_1\right)x-\rho(\|x\|_{L_2}+q_2)x+q_3,x\in\mathbb{Y}=H^1(\mathbb{R},\mathbb{R}),q=(q_1,q_2,q_3)\in\mathbb{V}$. Repeating considerations of Example 10, one can be shown that the considered system *satisfies Assumption 1 with* $M = \frac{\partial}{\partial z} + I$. *Since due to the inequality* (3.14) *we have* $\|q_1\|_{L^\infty} \leq 2^{-\frac{1}{2}}\|Mq_1\|_{L^2}$ *for any* $q_1 \in H^1$, *then* $\|f(x,q) - f(x,\mathbf{0})\|_{\mathbb{Y}} = \|q_1 x - \rho q_2 x\|_{\mathbb{Y}} \leq 2\|q_1\|_{L^\infty}\|x\|_{\mathbb{Y}} + \|Mq_1\|_{\mathbb{X}}\|x\|_{L^\infty} + \rho|q_2|\|x\|_{\mathbb{Y}} \leq \max\left\{\rho, 2^{\frac{1}{2}}\right\}\|q\|_{\mathbb{V}}\|x\|_{\mathbb{Y}}$, *i.e., Assumption 2 holds as well.*

For $q_1(t) \equiv \mathbf{0}, q_2(t), \equiv 0, q_3(t) \equiv \mathbf{0}$ *and* $\rho = 0$ *the latter system becomes the viscous Burgers equation studied in Example 10, where it is shown that the canonical homogeneous norm* $\|\cdot\|_{\mathfrak{d},\mathbb{Y}}$ *is the Lyapunov function for the Burgers equation provided that the dilation* $\mathfrak{d}$ *is given by* $(\mathfrak{d}(s)x)(z) = e^s x(e^s z)$, $z \in \mathbb{R}, x \in \mathbb{X}, s \in \mathbb{R}$.

Repeating considerations of Example 10, for $q_1(t) \equiv \mathbf{0}, q_2(t) \equiv 0, q_3(t) \equiv \mathbf{0}$ *and* $\rho > 0$, *we derive that the unperturbed system is* $\mathfrak{d}$*-homogeneous of degree* 2 *with* $(\mathfrak{d}(s)x)(z) = e^s x(e^s z)$ *and* $\|\cdot\|_{\mathfrak{d},\mathbb{Y}}$ *is the strict Lyapunov function for the unperturbed system:*

$$\left(D\|x\|_{\mathfrak{d},H^1}\right)(Ax + f(x,\mathbf{0})) \leq \frac{-\rho\|v\|_{L^2}^6 - \left\|\frac{\partial v}{\partial z}\right\|_{L^2}^2\left(1 - \frac{\sqrt{\pi/2}}{12}\|v\|_{L^2}^4 + \rho\right)}{0.5\|v\|_{L_2}^2 + 1.5\left\|\frac{\partial v}{\partial z}\right\|_{L_2}^2}\|x\|_{\mathfrak{d},H^1}^3 \leq -\tfrac{1}{3}\rho\|x\|_{\mathfrak{d},H^1}^3 \text{ for all}$$

$x \in M^{-1}\mathcal{D}(A)\backslash\{\mathbf{0}\}$, *where* $v = \mathfrak{d}(-\ln\|x\|_{\mathfrak{d},H^1})x$ *and* $\|v\|_{L_2}^2 + \left\|\frac{\partial v}{\partial z}\right\|_{L_2}^2 = 1$.

Let the dilation in V *be defined as follows* $\tilde{\mathfrak{d}}(s)q = (\mathfrak{d}(s)q_1, e^{2s}q_2, e^{2s}\mathfrak{d}(s)q_3)$. *Then* $\forall s \in \mathbb{R}, \forall x \in \mathbb{Y}, \forall q \in \mathbb{V}$. *one has* $f(\mathfrak{d}(s)x, \tilde{\mathfrak{d}}(s)q) = e^{2s}\mathfrak{d}(s)f(x,q)$ *and, by Theorem 3.16, the non-linear viscous Burgers equation* (3.18) *is ISS in* $\mathbb{Y}$ *if* $\rho > 0$.

ISS with exponential, finite-time and fixed-time convergence rates is studied in the papers [14], [42], [22], [26] where additional restrictions to the function $\beta \in \mathcal{KL}$ are suggested.

Definition 3.17. *An ISS system* (2.1) *is said to be*
- *exponentially ISS if* $\exists C \geq 1, \lambda > 0 : \beta(r,t) \leq Ce^{-\lambda t}, \forall r,t \in \mathbb{R}_+$;
- *finite-time ISS if* $\forall r > 0, \exists T_r > 0 : \beta(r, T_r) = 0$;
- *nearly fixed-time ISS if* $\forall \beta_0 > 0, \exists T_{\beta_0} > 0 : \beta(r, T_{\beta_0}) \leq \beta_0, \forall r \in \mathbb{R}_+$;
- *fixed-time ISS if it is both finite-time and nearly fixed-time ISS.*

Corollary 3.18. *If the system* (2.1) *satisfies all condition of Theorem 3.16 then it is*
- *finite-time ISS provided that* $\mu < 0$;
- *exponentially ISS provided that* $\mu = 0$;
- *nearly fixed-time ISS provided that* $\mu > 0$;

Proof. The inequality (3.17) yields $\underline{k}\|x_{x_0,q}(t)\|_{\mathfrak{d},\mathbb{Y}} \leq \left((\overline{k}\|x_0\|_{\mathfrak{d},\mathbb{Y}})^{-\mu} + \frac{0.5\mu}{\overline{k}^{1+\mu}}(t - t_0)\right)^{-\frac{1}{\mu}}$ for $\mu \neq 0$, and $\underline{k}\|x_{x_0,q}(t)\|_{\mathfrak{d},\mathbb{Y}} \leq \overline{k}\|x_0\|_{\mathfrak{d},\mathbb{Y}}e^{-\frac{0.5}{\overline{k}}t}, t \geq 0$ for $\mu = 0$, where $\underline{k}, \overline{k} > 0$ are such that $\underline{k}\|x\|_{\mathfrak{d},\mathbb{Y}} \leq V(x) \leq \overline{k}\|x\|_{\mathfrak{d},\mathbb{Y}}, \forall x \in \mathbb{Y}$ (see Theorem 3.12). ■

4. Homogeneous Approximations.

4.1. Definition and basic properties. To apply homogeneity-based methods of ISS analysis (such as Theorem 3.16) to a non-homogeneous system, the so-called homogeneous approximation can be utilized. The concept of homogeneous approximation can be explained using the following scalar function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x + x^2 - x^3$ with $x \in \mathbb{R}$. According to [48], [1], it has the the following standard homogeneous approximations $f_0(x) = x$ and $f_\infty = x^3$ at zero and at infinity limits, respectively. Indeed, the cubic and quadratic term can be neglected for local stability analysis (close to zero), while the cubic term dominates as $x \to \infty$. Similar considerations can be repeated for M-regular operators in Banach spaces.

Definition 4.1. *Let $\tilde{\mathbb{X}}$ be a Banach space and $\mathfrak{d}_L$ be a strongly continuous linear dilation in a Banach space $\tilde{\mathbb{Y}} \subset \tilde{\mathbb{X}}$, where $L=0$ or $L=\infty$. An operator $f: \tilde{\mathbb{Y}} \subset \mathbb{B} \mapsto \tilde{\mathbb{X}}$ (a functional $h: \tilde{\mathbb{Y}} \subset \tilde{\mathbb{X}} \mapsto \mathbb{R}$) is said to be $\mathfrak{d}_L$-homogeneous in the L-limit if there exists a $\mathfrak{d}_L$-homogeneous operator $f_L: \tilde{\mathbb{Y}} \subset \tilde{\mathbb{X}} \mapsto \tilde{\mathbb{X}}$ ($\mathfrak{d}_L$-homogeneous functional $h_L: \tilde{\mathbb{Y}} \subset \tilde{\mathbb{X}} \mapsto \mathbb{R}$) with degree $\nu_L \in \mathbb{R}$:*

$$\lim_{r\to L} \ \sup_{x\in S_{\tilde{\mathbb{Y}}}} \|r^{-\nu}\mathfrak{d}_L(-\ln r) f(\mathfrak{d}_L(\ln r)x) - f_L(x)\|_{\tilde{\mathbb{Y}}} = 0 \tag{4.1}$$

(resp., $\lim_{r\to L}\sup_{x\in S_{\tilde{\mathbb{Y}}}} |r^{-\nu}h(\mathfrak{d}_L(\ln r)x) - h_L(x)| = 0$), where $S_{\tilde{\mathbb{Y}}}$ is the unit sphere in $\tilde{\mathbb{Y}}$. The operator f_L (resp. the functional h_L) is called ***a $\mathfrak{d}_L$-homogeneous approximation*** *of the operator f (resp. the functional h). If the latter holds for both $L=0$ and $L=\infty$ then the operator f (resp. the functional h) is said to be* ***homogeneous in the bi-limit****.*

The condition (4.1) can be rewritten as $\lim\limits_{r\to L} \sup_{x\in S_{\mathfrak{d}_L,\tilde{\mathbb{Y}}}(r)} \|r^{-\nu}\mathfrak{d}_L(-\ln r)(f(x) - f_L(x))\|_{\mathbb{Y}} = 0$, where $S_{\mathfrak{d}_L,\tilde{\mathbb{Y}}}(r) = \mathfrak{d}(\ln r)S_{\tilde{\mathbb{Y}}}$ is the $\mathfrak{d}_L$-homogeneous sphere in $\tilde{\mathbb{Y}}$ of the radius $r>0$. Without loss of generality, a linear dilation $\mathfrak{d}_L$ can be assumed strictly monotone (see, Theorem 3.3). In this case, the inclusion $x \in S_{\mathfrak{d}_L,\tilde{\mathbb{Y}}}(r)$ simply means that $\|x\|_{\mathfrak{d},\tilde{\mathbb{Y}}} = r$.

In the finite dimensional case, if $f = f_0 + f_\infty$ is a sum of $\mathfrak{d}$-homogeneous operators with degrees $\nu_0 \in \mathbb{R}$ and $\nu_\infty > \nu_0$ then f_0 is $\mathfrak{d}$-homogeneous approximation of f in the 0-limit and f_∞ is $\mathfrak{d}$-homogeneous approximation of f in the ∞-limit. This may not hold for unbounded nonlinear operators in Banach spaces.

Example 12. *Let $\tilde{\mathbb{X}} = L^2(\mathbb{R},\mathbb{R})$, the operator $f: \tilde{\mathbb{Y}} \subset \tilde{\mathbb{X}} \mapsto \tilde{\mathbb{X}}$ be defined as follows $f(x) = \frac{\partial x}{\partial z}x + \|x\|_{L^2}x + x$ with $x \in \tilde{Y} = H^1(\mathbb{R},\mathbb{R})$. The first two terms of f are standard homogeneous of degree 1: $\frac{\partial \mathfrak{d}_1(s)x}{\partial z}\mathfrak{d}_1(s)x = e^s\mathfrak{d}_1(s)\left(\frac{\partial x}{\partial z}x\right)$ and $\|\mathfrak{d}_1(s)x\|_{L^2}\mathfrak{d}_1(s)x = e^s\mathfrak{d}_1(s)(\|x\|_{L^2}x)$, where $\mathfrak{d}_1(s) = e^sI, s\in\mathbb{R}$. The last term in $f(x)$ is linear, so it is standard homogeneous of degree 0, so $f_\infty(x) = \frac{\partial x}{\partial z}x + \|x\|_{L^2}x$ is the $\mathfrak{d}_1$-homogeneous approximation of the operator f at ∞. Indeed, $\sup_{x\in S_{\tilde{\mathbb{Y}}}}\|r^{-1}\mathfrak{d}_1(-\ln r)f(\mathfrak{d}_1(\ln r)x) - f_\infty(x)\|_{\tilde{\mathbb{Y}}} = \sup_{x\in S_{\tilde{\mathbb{Y}}}} r^{-1}\|x\|_{\tilde{\mathbb{Y}}} = r^{-1} \to 0$ as $r\to\infty$.*

In the finite-dimensional case, the term with smallest homogeneity degree usually defines the homogeneous approximation in the 0-limit [1]. However, $f_0(x) = x$ does not satisfy Definition 4.1. This is reasonable, since usually an unbounded nonlinear operator cannot be consistently approximated by a bounded linear one. To obtain an approximation of f at 0, let us consider another dilation in $\mathbb{B}$ defined as follows (see Example 1): $(\mathfrak{d}_2(s)x)(z) = e^s x(e^{-s}z)$. Since $\left(\frac{\partial \mathfrak{d}_2(s)x}{\partial z}\mathfrak{d}_2(s)x\right)(z) = \left(\mathfrak{d}_2(s)\left[\frac{\partial x}{\partial z}x\right]\right)(z)$ and $\|\mathfrak{d}_2(s)x\|_{L^2}\mathfrak{d}_2(s)x = e^{\frac{3}{2}s}\mathfrak{d}_2(s)(\|x\|_{L^2}x)$ then $f_0(x) = \frac{\partial x}{\partial z}x + x$ is the $\mathfrak{d}_2$-homogeneous approximation of f at 0 with degree 0. Indeed, $\sup\limits_{x\in S_{\tilde{\mathbb{Y}}}}\|\mathfrak{d}_2(-\ln r)f(\mathfrak{d}_2(\ln r)x) - f_0(x)\|_{\tilde{\mathbb{Y}}} = \sup\limits_{x\in S_{\tilde{\mathbb{Y}}}}\|\mathfrak{d}_2(\ln r)x\|_{L^2}\|x\|_{\tilde{\mathbb{Y}}} \le \|\mathfrak{d}_2(\ln r)\|_{L_2} = r^{\frac{3}{2}} \to 0$ as $r\to 0$.

A precision of a $\mathfrak{d}$-homogeneous approximation is characterized by the following theorem.

Theorem 4.2. *If an operator $f: \tilde{\mathbb{Y}} \subset \tilde{\mathbb{X}} \mapsto \tilde{\mathbb{X}}$ (resp., a functional $h: \tilde{\mathbb{Y}} \subset \tilde{\mathbb{X}} \mapsto \mathbb{R}$) is $\mathfrak{d}_L$-homogeneous of degree $\nu_L \in \mathbb{R}$ in the L-limit, where $L=0$ of $L=\infty$ and $\mathfrak{d}_L$ is a strongly continuous strictly monotone linear dilation in $\tilde{\mathbb{Y}}$, then there exist $\tilde{r}_L > 0$ and $\sigma_L \in \mathcal{K}$ such that*
1) $\|f(x) - f_0(x)\|_{\tilde{\mathbb{Y}}} \le \sigma_0(\rho)\rho^{\nu_L+\omega_L}$ (resp., $|h(x) - h_0(x)| \le \sigma_0(\rho)\rho^{\nu_L}$), $\rho = \|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}$, $\|x\|_{\mathbb{Y}} \le \tilde{r}_0$,
2) $\|f(x) - f_\infty(x)\|_{\tilde{\mathbb{Y}}} \le \sigma_\infty\left(\frac{1}{\rho}\right)\rho^{\nu_L+\eta_L}$ (or $|h(x) - h_\infty(x)| \le \sigma_\infty\left(\frac{1}{\rho}\right)\rho^{\nu_L}$), $\rho = \|x\|_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}$, $\|x\|_{\mathbb{Y}} \ge \tilde{r}_\infty$,
where $\mathbb{B}_{\tilde{\mathbb{Y}}}(1)$ is the unit ball in $\tilde{\mathbb{Y}}$, $\|\cdot\|_{\mathfrak{d}_L,\tilde{\mathbb{Y}}}$ is the canonical $\mathfrak{d}_L$-homogeneous norm in $\mathbb{Y}$ and the positive parameters $\eta_L \ge \omega_L > 0$, $C_{\mathfrak{d}_L} \ge 1$ are defined in Theorem 3.6.

Proof. We prove estimates for the operator f. The functional h can be studied similarly.

The case $L=0$. If f_0 is a $\mathfrak{d}_0$-homogeneous approximation of f in the 0-limit then the function $\tilde{\sigma}:[0,+\infty)\to\mathbb{R}_+\cup\{+\infty\}$ given by $\tilde{\sigma}_0(0)=0$ and

$$\tilde{\sigma}_0(\rho)=\sup_{x\in B_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}(\rho)\backslash\{\mathbf{0}\}}\|\rho^{-\nu_0}\mathfrak{d}_0(-\ln\rho)(f(x)-f_0(x))\|_{\tilde{\mathbb{Y}}},\quad \rho>0$$

is continuous at 0 and there exists $\tilde{r}_0>0$ such that σ is finite-valued on $[0,\tilde{r}_0]$, where $B_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}(\rho)$ is the $\mathfrak{d}_0$-homogeneous ball in $\tilde{\mathbb{Y}}$ of the radius $\rho>0$. By construction σ is non-decreasing on $[0,\tilde{r}_0]$. Then there exist $\sigma_0\in\mathcal{K}$ such that $\tilde{\sigma}_0(\rho)\le\sigma_0(\rho)$ for all $\rho\in[0,\tilde{r}_0]$. On the other hand, $\sup\limits_{x\in S_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}(r)}\left\|\frac{\mathfrak{d}_0(-\ln(r))(f(x)-f_L(x))}{r^{\nu_0}}\right\|_{\tilde{\mathbb{Y}}}\ge\frac{\inf\limits_{y\neq0}\frac{\|\mathfrak{d}_0(-\ln r)y\|_{\tilde{\mathbb{Y}}}}{\|y\|_{\tilde{\mathbb{Y}}}}\sup\limits_{x\in S_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}(r)}\|f(x)-f_L(x)\|_{\tilde{\mathbb{Y}}}}{r^{\nu_0}}\ge\frac{\sup\limits_{x\in S_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}(r)}\|f(x)-f_L(x)\|_{\tilde{\mathbb{Y}}}}{r^{\nu_0}\|\mathfrak{d}_0(\ln r)\|}$, where the inequality $\|\mathfrak{d}_0(\ln r)\|_{\tilde{\mathbb{Y}}}\|\mathfrak{d}_0(-\ln r)y\|_{\tilde{\mathbb{Y}}}\ge\|\mathfrak{d}_0(\ln r)\mathfrak{d}_0(-\ln r)y\|_{\tilde{\mathbb{Y}}}=\|y\|_{\tilde{\mathbb{Y}}}$ is utilized on the last step. Using the monotonicity of $\mathfrak{d}_0$ (i.e., $\exists\omega_0>0:\|\mathfrak{d}_0(s)\|_{\tilde{\mathbb{Y}}}\le e^{\omega_0 s},\forall s\le0$) we derive $r^{-(\nu_0+\omega_0)}\sup_{x\in S_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}(r)}\|f(x)-f_L(x)\|_{\tilde{\mathbb{Y}}}\le\sup_{x\in S_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}(r)}\|r^{-\nu_0}\mathfrak{d}_0(-\ln(r))(f(x)-f_L(x))\|_{\tilde{\mathbb{Y}}}\le\sigma_0(r)$, and $\|f(x)-f_0(x)\|_{\tilde{\mathbb{Y}}}\le\sup_{y\in S_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}(\|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}})}\|f(y)-f_0(y)\|_{\tilde{\mathbb{Y}}}\le\sigma_0(\|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}})\|x\|^{\nu_0+\omega_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}$. In **the case** $L=\infty$, the function $\tilde{\sigma}_\infty$ can be defined as follows: $\tilde{\sigma}_\infty(0)=0$ and $\tilde{\sigma}_\infty\left(r^{-1}\right)=\sup_{x\in\mathbb{Y}\backslash B_{\mathfrak{d}_\infty,\mathbb{Y}}(r)}\|r^{-\nu_\infty}\mathfrak{d}_\infty(-\ln r)(f(x)-f_L(x))\|_{\tilde{\mathbb{Y}}},\quad r\ge r_\infty$, where $\tilde{r}_\infty>0$ is such that the function $\tilde{\sigma}_\infty$ finite-valued on $[\tilde{r}_\infty,+\infty)$. The existence of such a number $\tilde{r}_\infty$ follows form the $\mathfrak{d}_\infty$-homogeneity in the ∞-limit. A strictly increasing continuous function $\sigma_\infty:\tilde{\sigma}_\infty(\rho)\le\sigma_\infty(\rho),\rho\ge0$ can be constructed similarly to σ_0. We have $\sup_{x\in S_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}(r)}\left\|\frac{\mathfrak{d}_\infty(\ln(r))(f(x)-f_L(x))}{r^{\nu_\infty}}\right\|_{\tilde{\mathbb{Y}}}\ge\frac{\inf\limits_{y\neq0}\frac{\|\mathfrak{d}_\infty(-\ln r)y\|_{\tilde{\mathbb{Y}}}}{\|y\|_{\tilde{\mathbb{Y}}}}\sup\limits_{x\in S_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}(r)}\|f(x)-f_L(x)\|_{\tilde{\mathbb{Y}}}}{r^{\nu_\infty}}\ge\frac{\sup\limits_{x\in S_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}(r)}\|f(x)-f_L(x)\|_{\tilde{\mathbb{Y}}}}{r^{\nu_\infty}\|\mathfrak{d}_\infty(\ln r)\|_{\tilde{\mathbb{Y}}}}\ge\frac{r^{-(\nu_\infty+\eta_\infty)}\sup\limits_{x\in S_{\mathfrak{d}_\infty}(r)}\|f(x)-f_L(x)\|_{\tilde{\mathbb{Y}}}}{C_{\mathfrak{d}_\infty}}$, $\forall r\in[\tilde{r}_\infty,+\infty)$, where the inequality $\|\mathfrak{d}_L(s)\|_{\tilde{\mathbb{Y}}}\le C_{\mathfrak{d}_\infty}e^{\eta_\infty s}$, $s>0$ (see, Theorem 3.6) is utilized on the last step. Hence, for all $x\in\mathbb{Y}\backslash\{B_{\tilde{\mathbb{Y}}}(r_0)\}$ we derive $C^{-1}_{\mathfrak{d}_\infty}r^{-(\nu_\infty+\eta_\infty)}\sup\limits_{x\in S_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}(r)}\|f(x)-f_L(x)\|_{\tilde{\mathbb{Y}}}\le\sup_{x\in S_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}(r)}\|r^{-\nu_\infty}\mathfrak{d}_\infty(-\ln(r))(f(x)-f_L(x))\|_{\tilde{\mathbb{Y}}}\le\sigma_\infty(r^{-1})$. Consequently, $\|f(x)-f_0(x)\|_{\tilde{\mathbb{Y}}}\le\sup_{y\in S_{\mathfrak{d}_\infty}(\|x\|_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}})}\|f(y)-f_0(y)\|_{\tilde{\mathbb{Y}}}\le C_{\mathfrak{d}_\infty}\sigma_\infty(\|x\|^{-1}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}})\|x\|^{\nu_\infty+\eta_\infty}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}$. ∎

Corollary 4.3. *Let a functional $h:\tilde{\mathbb{Y}}\subset\tilde{\mathbb{X}}\to\mathbb{R}_+$ be homogeneous in the bi-limit. If*

$$\inf_{x\in S_{\tilde{\mathbb{Y}}}}h_0(x)=\underline{h}_0>0,\quad \inf_{x\in S_{\tilde{\mathbb{Y}}}}h_\infty(x)=\underline{h}_\infty>0,\quad 0<\inf_{x\in K_{\tilde{\mathbb{Y}}}(r)}h(x)\le\sup_{x\in K_{\tilde{\mathbb{Y}}}(r)}h(x)<+\infty,\ \forall r>0,\tag{4.2}$$

then there exist positive real numbers $\underline{\rho}_0$, $\overline{\rho}_0$, $\underline{\rho}_\infty$, $\overline{\rho}_\infty$: $\underline{\rho}_0h_0(x)\le h(x)\le\overline{\rho}_0h_0(x)$, $\forall x\in B_{\tilde{\mathbb{Y}}}(1)\backslash\{\mathbf{0}\}$, and $\underline{\rho}_\infty h_\infty(x)\le h(x)\le\overline{\rho}_\infty h_\infty(x)$, $\forall x\in\tilde{\mathbb{Y}}\backslash B_{\tilde{\mathbb{Y}}}(1)$.

Moreover, if $\nu_0\ge0$ and $\nu_\infty\ge0$ then there exist $\underline{c}>0$ and $\overline{c}>0$ such that

$$\underline{c}\,\mathfrak{H}\left(h_0(x),h_\infty(x)\right)\le h(x)\le\overline{c}\,\mathfrak{H}\left(h_0(x),h_\infty(x)\right),\quad \forall x\in\tilde{\mathbb{Y}}\backslash\{\mathbf{0}\},\tag{4.3}$$

where $\mathfrak{H}(a,b)=a^{\frac{1+b}{1+a}},\quad a,b\in\mathbb{R}_+$.

Proof. In the view of Theorem 3.3, without loss of generality, the strongly continuous linear dilations $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ can be assumed to be strictly monotone. The condition (4.2) implies that

Theorem 4.2 is fulfilled for any finite $\tilde{r}_0 > 0$ and $\tilde{r}_\infty > 0$, in particular, for $\tilde{r}_0 = \tilde{r}_\infty = 1$. Notice also that $\|x\|_{\mathfrak{d},\tilde{\mathbb{Y}}} \leq 1$ (resp. ≥ 1) is equivalent $\|x\|_{\tilde{\mathbb{Y}}} \leq 1$ (resp. ≥ 1) for any strictly monotone strongly continuous dilation $\mathfrak{d}$ in $\tilde{\mathbb{Y}}$. Using the homogeneity we derive

$$h_0(x) = \|x\|^{\nu_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} h(-\ln\|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} x), \quad h_\infty(x) = \|x\|^{\nu_\infty}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} h(-\ln\|x\|_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} x), \quad \forall x \in \tilde{\mathbb{Y}}\backslash\{\mathbf{0}\},$$

$$\underline{h}_0\|x\|^{\nu_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} \leq h_0(x) \leq \|x\|^{\nu_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}\bar{h}_0, \quad \underline{h}_\infty\|x\|^{\nu_\infty}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} \leq h_\infty(x) \leq \|x\|^{\nu_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}\bar{h}_\infty, \quad \forall x \in \tilde{\mathbb{Y}}\backslash\{\mathbf{0}\},$$

where $\underline{h}_0 = \inf_{x\in S_{\tilde{\mathbb{Y}}}} h_0(x)$, $\bar{h}_0 = \sup_{x\in S_{\tilde{\mathbb{Y}}}} h_0(x)$, $\underline{h}_\infty = \inf_{x\in S_{\tilde{\mathbb{Y}}}} h_\infty(x)$, $\bar{h}_\infty = \sup_{x\in S_{\tilde{\mathbb{Y}}}} h_\infty(x)$ and $\bar{h}_0 < +\infty$ and $\bar{h}_\infty < +\infty$ due to the condition (4.2) and Theorem 4.2.

I. On the one hand, Theorem 4.2 gives

$$h(x) \leq \delta_0(1)\|x\|^{\nu_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} + h_0(x),\ \forall x \in B_{\tilde{\mathbb{Y}}}(1)\backslash\{\mathbf{0}\}, \quad h(x) \leq \delta_\infty(1)\|x\|^{\nu_\infty}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} + h_\infty(x),\ \forall x \in \tilde{\mathbb{Y}}\backslash B_{\tilde{\mathbb{Y}}}(1),$$

so using the homogeneity-based estimates for h_0 and h_∞ we derive $\overline{\rho}_0 = 1 + \frac{\delta_0(1)}{\underline{h}_0}, \overline{\rho}_\infty = 1 + \frac{\delta_\infty(1)}{\underline{h}_\infty}$. On the other hand, Theorem 4.2 implies that

$$\left(-\delta_0(\|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}})\underline{h}_0^{-1} + 1\right) h_0(x) \leq -\delta_0(\|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}})\|x\|^{\nu_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} + h_0(x) \leq h(x), \quad \forall x \in B_{\tilde{\mathbb{Y}}}(1)\backslash\{\mathbf{0}\}.$$

$$\left(-\delta_\infty(\|x\|^{-1}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}})\underline{h}_\infty^{-1} + 1\right) h_\infty(x) \leq -\delta_\infty(\|x\|^{-1}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}})\|x\|^{\nu_\infty}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} + h_\infty(x) \leq h(x), \quad \forall x \in \tilde{\mathbb{Y}}\backslash B_{\tilde{\mathbb{Y}}}(1).$$

Since $\delta_0 \in \mathcal{K}$ then there exists $r_0 \in (0,1)$ such that $1 - \delta_0(r_0)\underline{h}_0^{-1} > 0$, $1 - \delta_0(r)\underline{h}_0^{-1} > 1 - \delta_0(r_0)\underline{h}_0^{-1}$, $\forall r \in [0, r_0]$. Notice that $\|x\|^{\nu_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} \leq \max\{1, r_0^{\nu_0}\}$, $\forall x : r_0 \leq \|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} \leq 1$. By assumption, we have $h_{r_0} = \inf_{r_0 \leq \|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} \leq 1} h(x) > 0$ and $\frac{h_{r_0}h_0(x)}{\bar{h}_0\max\{1,r_0^{\nu_0}\}} \leq \frac{h_{r_0}h_0(x)}{\|x\|^{\nu_0}_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}\bar{h}_0} \leq h_{r_0} \leq h(x)$, $\forall x : r_0 \leq \|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}} \leq 1$, $\underline{\rho}_0 h_0(x) \leq h(x)$, $\forall x \in B_{\tilde{\mathbb{Y}}}(1)\backslash\{\mathbf{0}\}$, $\underline{\rho}_0 = \min\left\{1 - \frac{\delta_0(r_0)}{\underline{h}_0}, \frac{h_{r_0}}{\bar{h}_0\max\{1,r_0^{\nu_0}\}}\right\}$. Since $\delta_\infty \in \mathcal{K}$ then there exists $r_\infty > 1$ such that $1 - \delta_\infty\left(r_\infty^{-1}\right)\underline{h}_\infty^{-1} > 0$ and $1 - \delta_\infty\left(r^{-1}\right)\underline{h}_\infty^{-1} \geq 1 - \delta_\infty\left(r_\infty^{-1}\right)\underline{h}_\infty^{-1}$, $\forall r \geq r_\infty$. Notice that $\|x\|^{\nu_\infty}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} \leq \max\{1, r_\infty^{\nu_\infty}\}$, $\forall x : 1 \leq \|x\|_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} \leq r_\infty$. By assumption, we have $h_{r_\infty} = \inf_{1 \leq \|x\|_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} \leq r_\infty} h(x) h_{r_\infty} > 0$ and $\frac{h_{r_\infty}h_\infty(x)}{\bar{h}_\infty\max\{1,r_\infty^{\nu_\infty}\}} \leq \frac{h_{r_\infty}h_\infty(x)}{\|x\|^{\nu_\infty}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}\bar{h}_\infty} \leq h_{r_\infty} \leq h(x)$, $\forall x : 1 \leq \|x\|_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}} \leq r_\infty$, $\underline{\rho}_\infty h_\infty(x) \leq h(x)$, $\forall x \in B_{\tilde{\mathbb{Y}}}(1)\backslash\{\mathbf{0}\}$, $\underline{\rho}_\infty = \min\left\{1 - \frac{\delta_\infty(r_\infty^{-1})}{\underline{h}_\infty}, \frac{h_{r_\infty}}{\bar{h}_\infty\max\{1,r_\infty^{\nu_\infty}\}}\right\}$.

II. For $\nu_0 \geq 0$ we have $0 \leq h_0(x) \leq \bar{h}_0, \forall x \in B_{\tilde{\mathbb{Y}}}(1)\backslash\{\mathbf{0}\}$, so $\mathfrak{H}\left(h_0(x), h_\infty(x)\right) = \frac{1+h_\infty(x)}{1+h_0(x)}h_0(x) \geq \frac{h_0(x)}{1+\bar{h}_0}$, $\forall x \in B_{\tilde{\mathbb{Y}}}(1)\backslash\{\mathbf{0}\}$. Since $\mathfrak{H}\left(h_0(x), h_\infty(x)\right) \geq \frac{h_\infty(x)h_0(x)}{1+h_0(x)} \geq h_\infty(x)\frac{\|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}\underline{h}_0}{1+\|x\|_{\mathfrak{d}_0,\tilde{\mathbb{Y}}}\bar{h}_0} \geq h_\infty(x)\frac{\underline{h}_0}{1+\bar{h}_0}$, $\forall x \in \tilde{\mathbb{Y}}\backslash B_{\tilde{\mathbb{Y}}}(1)$ then (4.3) holds for $\bar{c} = \bar{\max}\left\{\overline{\rho}_0(1+\bar{h}_0), \frac{\overline{\rho}_\infty(1+\bar{h}_0}{\underline{h}_0}\right\}$. For $\nu_\infty \geq 0$ we have $0 \leq h_\infty(x) \leq \bar{h}_\infty$, so $\mathfrak{H}\left(h_0(x), h_\infty(x)\right) = \frac{1+h_\infty(x)}{1+h_0(x)}h_0(x) \leq (1+\bar{h}_0)h_0(x)$, $\forall x \in B_{\tilde{\mathbb{Y}}}(1)\backslash\{\mathbf{0}\}$. Since $\mathfrak{H}\left(h_0(x), h_\infty(x)\right) \leq 1 + h_\infty(x) \leq \frac{h_\infty(x)}{\underline{h}_\infty\|x\|^{\nu_\infty}_{\mathfrak{d}_\infty,\tilde{\mathbb{Y}}}} + h_\infty(x)$, $\forall x \in \tilde{\mathbb{Y}}\backslash B_{\tilde{\mathbb{Y}}}(1)$ then $\mathfrak{H}\left(h_0(x), h_\infty(x)\right) \leq \left(\underline{h}_\infty^{-1} + 1\right) h_\infty(x)$, $\forall x \in \tilde{\mathbb{Y}}\backslash B_{\tilde{\mathbb{Y}}}(1)$. The inequality (4.3) holds for $\underline{c} = \min\left\{\frac{\underline{\rho}_0}{1+\bar{h}_0}, \frac{\underline{\rho}_\infty}{\underline{h}_\infty^{-1}+1}\right\}$. ■

The estimates obtained above simplify stability and robustness analysis of the system (2.1) using homogeneous approximations.

4.2. Stability analysis by means of homogeneous approximations. *The system* (3.9) *is said to be $\mathfrak{d}_L$-homogeneous in the L-limit if f is homogeneous in the L-limit with degree ν_L and the linear operator A is $\mathfrak{d}_L$-homogeneous of degree ν_L.* The following theorem shows that stability of a $\mathfrak{d}$-homogeneous approximation guarantees local stability (close to 0 or close to ∞) of the original system.

Theorem 4.4. *Let the system* (3.9) *satisfy Assumption 1 with some operator M. Let $\mathfrak{d}_L$ be strongly continuous linear dilation in $\mathbb{Y}$. Let the system* (3.9) *be $\mathfrak{d}_L$-homogeneous in the L-limit of degree ν_L, where $L=0$ or $L=\infty$, such that $\tilde{\mathfrak{d}}(s)=e^{\nu s}\mathfrak{d}(s), s\in\mathbb{R}$ is a dilation in $\mathbb{Y}$ and the system*

$$\dot{x} = Ax + f_L(x), \quad t>0 \tag{4.4}$$

satisfy Assumption 1 with the same operator M. If the system (4.4) *is uniformly asymptotically stable in $\mathbb{Y}$ then the system* (3.9) *is uniformly locally asymptotically (finite-time) stable in $\mathbb{Y}$ if $L=0$ (and $\nu_0<0$) or uniformly globally practically asymptotically (fixed-time) stable in $\mathbb{Y}$ if $L=\infty$ (and $\nu_\infty>0$).*

Proof. Let us consider the system

$$\dot{x} = Ax + f_L(x) + q, \quad t>t_0, \quad x(t_0)=x_0, \tag{4.5}$$

where $q\in L^\infty(\mathbb{R},\mathbb{V})$. Assumption 2 is fulfilled for (4.5) with $\xi(r)=1$ and $\chi(r)=r$ provided that $\mathbb{V}=\mathbb{Y}$.

Notice that if $x_{x_0}^f:[0,T]\to\mathbb{Y}$ is a solution (3.9) then $x_{x_0}^f(t)=x_{x_0,q}(t), \forall t\in[t_0,t_0+T]$, where $x_{x_0,q}$ is the solution of (4.5) with

$$q(t) = f\left(x_{x_0}^f(t)\right) - f_L\left(x_{x_0}^f(t)\right) \tag{4.6}$$

Due to Theorem 4.2 we have $q(t)\in\mathbb{Y}$ for all $t\geq t_0$.

Let us consider an arbitrary $q\in L^\infty(\mathbb{R},\mathbb{Y})$ and repeat the proof of Theorem 3.16 to derive the following inequality $\overline{D}^+V_L(x_{x_0,q}(t)) \leq -\|x_{x_0,q}(t)\|_{\mathfrak{d}_L,\mathbb{Y}}^{1+\nu_L} + \frac{C_L(\bar{q})\bar{q}}{e^{(\nu_L+1)s}}$ as long as $x_{x_0,q}(t)\in \mathfrak{d}_L(-s)K_{\mathbb{Y}}(r)$ and $q\in L^\infty(\mathbb{R},\mathbb{Y}):\|\tilde{\mathfrak{d}}_L(s)q\|_{L^\infty((t,t+\epsilon),\mathbb{Y})}\leq\bar{q}$, where $s\in\mathbb{R}$ is an arbitrary real number, $\epsilon>0$ is an arbitrary small number, $V_L:\mathbb{Y}\to\mathbb{R}_+$ is a $\mathfrak{d}_L$-homogeneous Lyapunov functional of degree 1 for (4.4) and $\bar{q}\mapsto C_L(\bar{q})$ is defined in the proof of Theorem 3.16.

Let us consider **the case** $L=0$. By Theorem 4.2, there exist $r_0>0$ and $\sigma_0\in\mathcal{K}$ such that $\sup_{x\in S_{\mathfrak{d}_0,\mathbb{Y}}(\rho)}\|\rho^{-\nu}\mathfrak{d}_0(-\ln\rho)(f(x)-f_L(x))\|_{\mathbb{Y}}\leq\sigma_0(\rho), \forall\rho\in[0,r_0]$. Let $\bar{q}\in(0,\sigma_0(r_0)):$ $C(\bar{q})\bar{q}<0.5$. Since for $s=-\ln\|x_{x_0,q}(t)\|_{\mathfrak{d}_0,\mathbb{Y}}$, $\|x_{x_0,q}(t)\|_{\mathfrak{d}_0,\mathbb{Y}}\leq\min\{1,r_0\}$ and $q(t)$ given by (4.6) we have $\|\tilde{\mathfrak{d}}_0(s)q(t)\|\leq\sigma_0(\|x_{x_0,q}(t)\|_{\mathfrak{d}_0,\mathbb{Y}})$ then $\overline{D}^+V_L(x_{x_0,q}(t))\leq -0.5\|x_{x_0,q}(t)\|_{\mathfrak{d}_0,\mathbb{Y}}^{1+\nu_0}$ as long as $0<\|x_{x_0,q}(t)\|_{\mathfrak{d},\mathbb{Y}}\leq\delta_0^{-1}$, where δ_0 is a constant. The system (3.9) is locally uniformly asymptotically stable in $\mathbb{Y}$. Moreover, for $\nu_0<0$ it is locally finite-time stable in $\mathbb{Y}$. **The case** $L=\infty$ can be analyzed similarly to derive the following estimate $\overline{D}^+V_\infty(x_{x_0,q}(t))\leq -0.5\|x_{x_0,q}(t)\|_{\mathfrak{d}_\infty,\mathbb{Y}}^{1+\nu_\infty}$ as long as $\|x_{x_0,q}(t)\|_{\mathfrak{d}_\infty,\mathbb{Y}}\geq\delta_\infty$, where δ_∞ is a constant. ■

Example 13. *The system*

$$\dot{x} = \frac{\partial^2 x}{\partial z^2} - \left(\frac{\partial x}{\partial z}\right)x - \rho\|x\|_{L_2}^4 x + x^2, \quad t>0, \quad z\in\mathbb{R}, \quad x(0)=x_0, \tag{4.7}$$

is $\mathfrak{d}_\infty$-homogeneous in the ∞-limit with the degree 2, where $\mathbb{X}, \mathbb{Y}$ and the dilation $\mathfrak{d}_\infty$ are defined as in Example 10. Indeed, $f_\infty(x) = -\left(\frac{\partial x}{\partial z}\right)x - \rho\|x\|_{L_2}^4 x$ is $\mathfrak{d}$-homogeneous of degree 2 and $\sup_{x\in S_\mathbb{Y}}\left\|\frac{\mathfrak{d}(-\ln r)f(\mathfrak{d}(\ln r)x)-f_\infty(x))}{r^2}\right\|_\mathbb{Y} = \sup_{x\in S_\mathbb{Y}}\left\|\frac{\mathfrak{d}(-\ln r)(\mathfrak{d}(\ln r)x)^2}{r^2}\right\|_\mathbb{Y} = \frac{\sup_{x\in S_\mathbb{Y}}\|x^2\|_\mathbb{Y}}{r} \to 0$ as $r \to +\infty$. By Theorem 4.4 the system (4.7) is global uniformly practically fixed-time stable.

Corollary 3.18 can be extended to the case of local homogeneity as follows.

Corollary 4.5. *If, under conditions of Theorem 4.4, the system (3.9) is globally uniformly asymptotically stable then the system (3.9) is globally uniformly finite-time stable for $L = 0$ and $\nu_0 < 0$, or globally uniformly nearly fixed-time stable for $L = \infty$ and $\nu_\infty > 0$.*

The proof of this corollary is straightforward.

Example 14. *The system*

$$\dot{x} = \frac{\partial^2 x}{\partial z^2} - \left(\frac{\partial x}{\partial z}\right)x - \rho(\|x\|_{L^2}^4 + \|x\|_{L^2})x, \quad t > 0, \quad z \in \mathbb{R}, \quad x(0) = x_0, \tag{4.8}$$

is $\mathfrak{d}_\infty$-homogeneous in the ∞-limit (see Example 4.7). Repeating the constructions of Example 10, one can be shown that the system (4.8) globally uniformly asymptotically stable. The latter means that the system (4.8) globally uniformly nearly fixed-time stable.

An analog of the homogeneous Lyapunov function theorem can be proven for evolution equations homogeneous the bi-limit. *The system (3.9) is said to be homogeneous in the bi-limit if f is homogeneous in the the bi-limit and the linear operator A is $\mathfrak{d}_L$-homogeneous of degree ν_L for both $L = 0$ and $L = \infty$.*

Theorem 4.6. *Let the system (3.9) be homogeneous in the bi-limit with linear dilations $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ in $\mathbb{Y}$ such that $\tilde{\mathfrak{d}}_0(s) = e^{\nu_0 s}\mathfrak{d}_0(s)$ and $\tilde{\mathfrak{d}}_\infty(s) = e^{\nu_\infty s}\mathfrak{d}_\infty(s)$ are dilations in $\mathbb{Y}$ as well. If the system (3.9) and its homogeneous approximations in the 0- and ∞-limits are is globally uniformly asymptotically stable then $\forall m_0 > \max\{0, -\nu_0\}$ and $\forall m_\infty > \max\{0, -\nu_\infty\}$ there exists a positive definite* ***homogeneous in the bi-limit*** *functional $V : \mathbb{Y} \mapsto \mathbb{R}$ such that*

1) *V is locally Lipschitz continuous on $\mathbb{Y}\backslash\{\mathbf{0}\}$;*
2) *$\exists \underline{k}, \overline{k} \in \mathcal{K}^\infty$: $\underline{k}(\|x\|_\mathbb{Y}) \le V(x) \le \overline{k}(\|x\|_\mathbb{Y}), \forall x \in \mathbb{Y}$;*
3) *for any mild solution x_{x_0} of (3.9) with $x(0) = x_0 \in \mathbb{Y}$ the inequality*

$$\overline{D}^+ V(x_{x_0}(t)) \le -c\,\mathfrak{H}\left(V^{\frac{m_0+\nu_0}{m_0}}(x_{x_0}(t)), V^{\frac{m_\infty+\nu_\infty}{m_\infty}}(x_{x_0}(t))\right), \quad t > 0 \tag{4.9}$$

holds as long as $x_{x_0}(t) \neq \mathbf{0}$, where $c > 0$, the function $\mathfrak{H} : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is given in Corollary 4.3, m_L is the homogeneity degree of V in the L-limit for $L = 0$ or $L = \infty$.

Proof. In the proof ot Theorem 4.4 it was shown that there exists a locally Lipschitz continuous $\mathfrak{d}_0$-homogeneous Lypapunov function $V_0 : \mathbb{Y} \mapsto \mathbb{R}_+$ of degree 1: $\exists \underline{k}_0, \overline{k}_0, c_0 > 0$

$$\underline{k}_0\|x\|_{\mathfrak{d}_0,\mathbb{Y}} \le V_0(x) \le \overline{k}_0\|x\|_{\mathfrak{d}_0,\mathbb{Y}}, \qquad \overline{D}^+ V_0(x_{x_0}(t)) \le -c_0\|x_{x_0}(t)\|_{\mathfrak{d}_0,\mathbb{Y}}^{1+\nu_0}, \tag{4.10}$$

as long as $0 < \|x_{x_0}(t)\|_{\mathfrak{d}_0,\mathbb{Y}} \le \frac{1}{\delta_0}$, and there exists a locally Lipschitz continuous $\mathfrak{d}_\infty$-homogeneous Lypapunov function $V_\infty : \mathbb{Y} \mapsto \mathbb{R}_+$ of degree 1: $\exists \underline{k}_\infty, \overline{k}_\infty, c_\infty > 0$

$$\underline{k}_\infty\|x\|_{\mathfrak{d}_\infty,\mathbb{Y}} \le V_\infty(x) \le \overline{k}_\infty\|x\|_{\mathfrak{d}_\infty,\mathbb{Y}}, \qquad \overline{D}^+ V_\infty(x_{x_0}(t)) \le -c_\infty\|x_{x_0}(t)\|_{\mathfrak{d}_\infty,\mathbb{Y}}^{1+\nu_\infty}, \tag{4.11}$$

as long as $\|x_{x_0}(t)\|_{\mathfrak{d}_\infty,\mathbb{Y}} \geq \delta_\infty$.

Notice that, in the view of Theorem 3.6, there exists $\tilde{\delta}_0 > 1$ such that $\|x\|_{\mathbb{Y}} \leq \frac{1}{\tilde{\delta}_0} \Rightarrow \|x\|_{\mathfrak{d}_0,\mathbb{Y}} \leq \frac{1}{\delta_0}$ and there exists $\tilde{\delta}_\infty > 1$ such that $\|x\|_{\mathbb{Y}} \geq \tilde{\delta}_\infty \Rightarrow \|x\|_{\mathfrak{d}_\infty,\mathbb{Y}} \geq \delta_\infty$.

Let the functions $\hat{T} : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $\varepsilon \in \mathcal{K}^\infty$ be defined by Definition 3.11 and let $r > \max\left\{\tilde{\delta}_0, \tilde{\delta}_\infty, r_0, r_\infty\right\}$, where $r_0 = \frac{1}{\varepsilon^{-1}\left(\frac{1}{2\tilde{\delta}_0}\right)}$, $r_\infty = 2\varepsilon(\tilde{\delta}_\infty)$ and $\varepsilon^{-1} \in \mathcal{K}^\infty$ is the inverse function to ε. Consider the functional $V_r : \mathbb{Y} \mapsto \mathbb{R}_+$ given by $V_r(x_0) = \sup_{t\geq 0} \|x_{x_0}(t)\|_{\mathbb{Y}} + \int_0^{+\infty} a\left(\|x_{x_0}(\tau)\|_{\mathbb{Y}}\right) \|x_{x_0}(\tau)\|_{\mathbb{Y}} d\tau$ where $a \in C^\infty$ such that $a(\rho) \in \left(0, \hat{T}\left(r, \frac{1}{r}\right)\right)$ for $\rho \in \left(\frac{1}{r}, r\right)$ and $a(\rho) = 0$ otherwise. Since the system (3.9) is globally uniformly asymptotically stable then $\|x\|_{\mathbb{Y}} \leq V_r(x) \leq 2\varepsilon(\|x\|_{\mathbb{Y}})$. Notice that, by construction, $2\varepsilon\left(\frac{1}{r}\right) \leq V_r(x) \leq \frac{1}{\tilde{\delta}_0} \quad \Rightarrow \quad \frac{1}{r} \leq \|x\|_{\mathbb{Y}} \leq \frac{1}{\tilde{\delta}_0}$ and $2\varepsilon(\tilde{\delta}_\infty) \leq V_r(x) \leq r \quad \Rightarrow \quad \tilde{\delta}_\infty \leq \|x\|_{\mathbb{Y}} \leq r$.

The functional V_r satisfies Lipschitz condition on $K(r)$ in the view of [36, Corollary 4]. Repeating considerations of the proof of Theorem 3 from [36] we derive $\overline{D}^+ V_r(x_{x_0}(t)) \leq -W_r(x_{x_0}(t))$, as long as $x_{x_0}(t) \in K(r)$, where $W : \mathbb{Y} \to \mathbb{R}_+$ satisfies $\exists c_r \in \mathcal{K}^\infty : \quad W_r(x) \geq c_r(\|x\|)$ for all $x \in K(r)$. Inspired by [1], let us consider the functional $V : \mathbb{Y} \mapsto \mathbb{R}_+$ defined as $V(x) = w_\infty \phi_\infty(V_r(x)) V_\infty^{m_\infty}(x) + (1 - \phi_\infty(V_r(x)))\, \phi_0(V_r(x)) V_r(x) + w_0(1 - \phi_0(V_r(x))) V_0^{m_0}(x)$, where $\phi_0, \phi_\infty \in C^\infty$ are such that $\phi_0(\rho) = 0$ if $\rho \leq 2\varepsilon\left(\frac{1}{r}\right)$, $\phi_0'(\rho) > 0$ if $\rho \in \left(2\varepsilon\left(\frac{1}{r}\right), \frac{1}{\tilde{\delta}_0}\right)$, and $\phi_0(\rho) = 1$ if $\rho \geq \frac{1}{\tilde{\delta}_0}$ $\phi_\infty(\rho) = 0$ if $\rho \leq 2\varepsilon\left(\tilde{\delta}_\infty\right)$, $\phi_\infty'(\rho) > 0$ if $\rho \in \left(2\varepsilon\left(\tilde{\delta}_\infty\right), r\right)$, $\phi_\infty(\rho) = 1$ if $\rho \geq r$ and $w_0 > \sup_{1/r \leq \|x\|_{\mathbb{Y}} \leq 1/\tilde{\delta}_0} \frac{V_r(x)}{V_0^{m_0}(x)}$ and $w_\infty > \sup_{\tilde{\delta}_\infty \leq \|x\|_{\mathbb{Y}} \leq r} \frac{V_r(x)}{V_\infty^{m_\infty}(x)}$. Since V_0, V_r and V_∞ are bounded from above and from below by class K^∞ functions of the norm $\|x\|_{\mathbb{Y}}$, then w_0 and w_∞ are finite positive numbers. In this case, we derive $\overline{D}^+ V(x_{x_0}(t)) \leq -W(x_{x_0}(t))$ as long as $x_{x_0}(t) \neq 0$, where

- $W(x) = m_\infty w_\infty c_\infty \|x\|^{1+\nu_\infty}_{\mathfrak{d}_\infty,\mathbb{Y}} V_\infty^{m_\infty - 1}(x)$ if $V_r(x) \geq r$;
- $W(x) = m_\infty w_\infty c_\infty \phi(V_r(x)) \|x\|^{1+\nu_\infty}_{\mathfrak{d}_\infty,\mathbb{Y}} V_\infty^{m_\infty - 1}(x) + \Psi_\infty(x) W_r(x)$ if $2\varepsilon(\tilde{\delta}_\infty) < V_r(x) < r$, where $\Psi_\infty(x) := \phi_\infty'(V_r(x))\,(w_\infty V_\infty^{m_\infty}(x) - V_r(x)) + 1 - \phi_\infty(V_r(x)) > 0$;
- $W(x) = W_r(x)$ if $\frac{1}{\tilde{\delta}_0} \leq V_r(x) \leq 2\varepsilon(\tilde{\delta}_\infty)$;
- $W(x) = \Psi_0(x) W_r(x) + m_0 w_0 c_0 \phi_0(V_r(x)) \|x\|^{1+\nu_0}_{\mathfrak{d}_0,\mathbb{Y}} V_0^{m_0 - 1}(x)$ if $2\varepsilon\left(\frac{1}{r}\right) \leq V_r(x) \leq \frac{1}{\tilde{\delta}_0}$, where $\Psi_0(x) := \phi_0'(V_r(x))\,(V_r(x) - w_0 V_0(x)) + \phi_0(V_r(x)) > 0$;
- $W(x) = m_0 c_0 \|x_{x_0}(t)\|^{1+\nu_0}_{\mathfrak{d}_0,\mathbb{Y}} V_0^{m_0 - 1}(x)$ if $0 < V_r(x) \leq 2\varepsilon\left(\frac{1}{r}\right)$.

By construction, V and W satisfy Corollary 4.3 then there exist positive constants $\bar{\rho}_{0,V}$, $\overline{\rho}_{0,V}$, $\bar{\rho}_{\infty,V}$, $\overline{\rho}_{\infty,V}$, $\bar{\rho}_{0,W}$, $\overline{\rho}_{0,W}$, $\bar{\rho}_{\infty,W}$ and $\overline{\rho}_{0,W}$ such that $\underline{\rho}_{0,V} \tilde{V}_0(x) \leq V(x) \leq \overline{\rho}_{0,V} \tilde{V}_0(x)$, $\underline{\rho}_{0,W} W_0(x) \leq W(x) \leq \overline{\rho}_{0,W} W_0(x), \forall x \in B_{\mathbb{Y}}(1)\backslash\{\mathbf{0}\}$, $\underline{\rho}_{\infty,V} \tilde{V}_\infty(x) \leq V(x) \leq \overline{\rho}_{\infty,V} \tilde{V}_\infty(x)$, $\underline{\rho}_{\infty,W} W_\infty(x) \leq W(x) \leq \overline{\rho}_{\infty,W} W_\infty(x), \forall x \in \mathbb{Y}\backslash B_{\mathbb{Y}}(1)$, where $\tilde{V}_0 = w_0 V_0^{m_0}, \tilde{V}_\infty = w_\infty V_\infty^{m_\infty}$, and $W_0(x) = w_0 m_0 c_0 \|x_{x_0}(t)\|^{1+\nu_0}_{\mathfrak{d}_0,\mathbb{Y}} V_0^{m_0-1}(x)$, $W_\infty(x) = w_\infty m_\infty c_\infty \|x\|^{1+\nu_\infty}_{\mathfrak{d}_\infty,\mathbb{Y}} V_\infty^{m_\infty - 1}(x)$ are homogeneous approximations of V and W at the 0-limit and the ∞-limit, respectively. Using (4.10) and (4.11) we derive that there exist $c_1, c_2 > 0$: $W(x) \geq c_1 V^{\frac{m_0+\nu_0}{m_0}}(x), \forall x \in B_{\mathbb{Y}}(1)\backslash\{\mathbf{0}\}$ and $W(x) \geq c_2 V^{\frac{m_\infty+\nu_\infty}{m_\infty}}(x), \quad \forall x \in \mathbb{Y}\backslash B_{\mathbb{Y}}(1)$, and the inequality (4.9) holds for a constant $c > 0$ (see, the proof of Corollary 4.3 for more details about computation of the constant c). ■

Remark 4.7. Similarly to Corollary 3.13 for reflexive Banach space $\mathbb{X}$, the time derivative of the Lyapunov function V along trajectories of the system can be computed using the right-hand side of the system (3.9) only.

The following claim is the straightforward consequence of the Theorem 4.6.

Corollary 4.8. *Under conditions of Theorem 4.6, the system (3.9) is globally uniformly fixed-time stable provided that $\nu_0 < 0 < \nu_\infty$.*

4.3. ISS analysis based on homogeneous approximations. The following result shows that a local and a practical ISS of (2.1) follows from ISS of a homogeneous approximation.

Theorem 4.9. *Let $f : \mathcal{D}_f \times \mathbb{X} \mapsto \mathbb{X}$ satisfy Assumptions 1, 2 with some operator M and the operator $\tilde{f} : \tilde{\mathbb{Y}} \mapsto \tilde{\mathbb{X}}$ be defined as follows $\tilde{f}(x,q) = \begin{pmatrix} f(x,q) & \mathbf{0} \end{pmatrix} \in \tilde{\mathbb{X}}, \quad x \in \mathbb{X}, \quad q \in \mathbb{V}$, where $\tilde{\mathbb{X}} = \mathbb{X} \times \mathbb{V}$ is a Banach space with the norm $\|(x,q)\|_{\tilde{\mathbb{X}}} = \|x\|_{\mathbb{X}} + \|q\|_{\mathbb{V}}$ and $\tilde{\mathbb{Y}} = \mathbb{Y} \times \mathbb{V}$ is a Banach space with the norm $\|(x,q)\|_{\tilde{\mathbb{X}}} = \|x\|_{\mathbb{Y}} + \|q\|_{\mathbb{V}}$. Let $\mathfrak{d}_L$ and $\tilde{\mathfrak{d}}_L$ be strongly continuous strictly monotone dilations in $\mathbb{Y}$ and $\mathbb{V}$, respectively, and the operator A be $\mathfrak{d}_L$-homogeneous of degree $\nu_L \in \mathbb{R}$ such that $\hat{\mathfrak{d}}(s) = e^{\nu_L s}\mathfrak{d}(s)$ is a dilation in $\mathbb{Y}$ as well. Let a linear dilation $\bar{\mathfrak{d}}_L$ in $\tilde{\mathbb{X}}$ be defined as follows*

$$\bar{\mathfrak{d}}_L(s)(x,q) = (\mathfrak{d}_L(s)x, \tilde{\mathfrak{d}}_L(s)q), \quad s \in \mathbb{R}, \quad x \in \mathbb{X}, \quad q \in \mathbb{V}.$$

Let the operator $\tilde{f}$ be $\bar{\mathfrak{d}}_L$-homogeneous in the L-limit with degree ν_L and its $\mathfrak{d}_L$-homogeneous approximation in the L-limit be defined as follows $\tilde{f}_L(x,q) = \begin{pmatrix} f_L(x,q) & \mathbf{0} \end{pmatrix}$, where $L = 0$ or $L = \infty$. If f_L satisfies Assumptions 1,2 with the same operator M and the system

$$\dot{x} = Ax + f_L(x, \mathbf{0}) \tag{4.12}$$

is uniformly asymptotically stable then the system (2.1) is practically ISS for $L = \infty$ and locally ISS for $L = 0$.

Proof. By Theorem 3.3, the dilations $\mathfrak{d}$ and $\tilde{\mathfrak{d}}$ can be assumed strictly monotone. This implies that $\bar{\mathfrak{d}}$ is strictly monotone as well. Let us consider the system

$$\dot{x} = Ax + f_L(x,q) + \hat{q}, \quad t > t_0, \quad x(t_0) = x_0, \tag{4.13}$$

where $\hat{q} \in L^\infty(\mathbb{R}, \mathbb{Y})$ and the pair $(\hat{q}, q) \in \tilde{V}$ defines the exogenous input. Since $\hat{\mathfrak{d}}_L$ is a dilation in $\mathbb{Y}$ then by Theorem 3.16 the considered system is ISS. Let $x_{x_0,\hat{q}}$ denote a mild solution of (4.13). Repeating the proof of Theorem 3.16 we derive $\bar{D}^+ V_L(x_{x_0,\hat{q}}(t)) \leq -\|x_{x_0,\hat{q}}(t)\|^{1+\nu_L}_{\mathfrak{d}_L,\mathbb{Y}} + \frac{C_L(\bar{q})(\chi(\bar{q})+\bar{q})}{e^{(\nu_L+1)s}}$ as long as $x_{x_0,\hat{q}}(t) \in \mathfrak{d}_L(-s)K_{\mathbb{Y}}(r)$, $\hat{q} \in L^\infty(\mathbb{R},\mathbb{Y}) : \|\hat{\mathfrak{d}}_L(s)\hat{q}\|_{L^\infty((t,t+\epsilon),\tilde{\mathbb{Y}})} \leq \bar{q}$, and $q \in L^\infty(\mathbb{R},\mathbb{V}) : \|\tilde{\mathfrak{d}}_L(s)q\|_{L^\infty((t,t+\epsilon),\mathbb{V})} \leq \bar{q}$, where $s \in \mathbb{R}$ is an arbitrary real number, $\epsilon > 0$ is an arbitrary small number, $V_L : \mathbb{Y} \to \mathbb{R}_+$ is a $\mathfrak{d}_L$-homogeneous Lyapunov functional of degree 1 for the unperturbed system (4.13), the function $\bar{q} \mapsto C_L(\bar{q})$ is defined in the proof of Theorem 3.16. Let $\bar{q} > 0$ be *fixed* such that $C_L(\bar{q})(\chi(\bar{q}) + \bar{q}) \leq 0.5$. If $x_{x_0,q} : [0,T] \to \mathbb{Y}$ is a solution (2.1) then $x_{x_0,q}(t) = x_{x_0,\hat{q}}(t), \forall t \in [t_0, t_0+T]$, where $x_{x_0,\hat{q}}$ is the solution of (4.13) with

$$\hat{q}(t) = f\left(x_{x_0}(t), q(t)\right) - f_L\left(x_{x_0}(t), q(t)\right). \tag{4.14}$$

The case $L = 0$. On the other hand, by Theorem 3.6 there exists $\bar{\sigma}_0^* \in \mathcal{K}^\infty$ such that $\|(x,q)\|_{\tilde{\mathfrak{d}}_0,\tilde{\mathbb{Y}}} \leq \bar{\sigma}_0^*(\|x\|_{\mathbb{Y}} + \|q\|_{\mathbb{V}})$ for all $x \in \mathbb{Y}, q \in \mathbb{V}$, so by Theorem 4.2 $\exists \sigma_0 \in \mathcal{K}$ and $\exists r_0 > 0$:

$$\|\hat{\mathfrak{d}}_0(s)(f(x,q) - f_0(x,q))\|_{\mathbb{Y}} \leq \left\|\hat{\mathfrak{d}}_0\left(\ln \bar{\sigma}_0^*(\|\mathfrak{d}_0(s)x\|_{\mathbb{Y}} + \|\tilde{\mathfrak{d}}_0(s)q\|_{\mathbb{V}})\right)\right\|_{\mathbb{Y}} \sigma_0(\bar{\sigma}_0^*(\|x\|_{\mathbb{Y}} + \|q\|_{\mathbb{V}}))$$

for any $x, q : \|(x,q)\|_{\tilde{\mathfrak{d}}_0,\tilde{\mathbb{Y}}} \leq r_0$ and any $\tilde{s} \in \mathbb{R}$. Since $x_{x_0,q}$ is continuous function of time then for any $t : x_{x_0,q} \neq \mathbf{0}$ there exists $\epsilon = \epsilon(t) > 0$ such that $\|\mathfrak{d}_0(-\ln \|x_{x_0,q}(t)\|_{\mathfrak{d}_0,\mathbb{Y}})x_{x_0,q}(\tau)\|_{\mathbb{Y}} \leq 1 + \bar{q}$ for all $\tau \in [t, t+\epsilon]$. Hence, for $\hat{q}$ given by (4.14) and for $s = -\ln \|x_{x_0,q}(t)\|_{\mathfrak{d}_0,\mathbb{Y}}$ we derive that the inequality $\|\hat{\mathfrak{d}}_0(s)\hat{q}\|_{L^\infty((t,t+\epsilon(t)),\mathbb{V})} \leq \bar{q}$ holds if $\|\tilde{\mathfrak{d}}_0(s)q\|_{L^\infty((t,t+\epsilon(t)),\mathbb{V}))} \leq \bar{q}$ and $\left\|\hat{\mathfrak{d}}_0\left(\ln \bar{\sigma}_0^*(1+2\bar{q})\right)\right\|_{\mathbb{Y}} \sigma_0(\bar{\sigma}_0^*(\|x_{x_0,q}\|_{L^\infty((t,t+\epsilon(t)),\mathbb{Y})} + \|q\|_{L^\infty((t,t+\epsilon(t)),\mathbb{V})})) \leq \bar{q}$. Notice that the latter inequality holds for $\|x_{x_0,q}(t)\| \leq \tilde{r}_0$ and $\|q\|_{L^\infty((t,t+\epsilon(t)),\mathbb{V})} \leq \tilde{r}_0$, where $\tilde{r}_0$ is a sufficiently small constant, which exists since $\sigma \in \mathcal{K}$ and $\bar{\sigma}^* \in \mathcal{K}^\infty$. Therefore, we conclude that $\bar{D}^+ V_L(x_{x_0,\hat{q}}(t)) \leq -0.5\|x_{x_0,\hat{q}}(t)\|_{\mathfrak{d}_L,\mathbb{Y}}^{1+\nu_L}$ as long as $\|\tilde{\mathfrak{d}}_0(-\ln \|x_{x_0,q}(t)\|_{\mathfrak{d}_0,\mathbb{Y}})q\|_{L^\infty((t,t+\epsilon(t)),\mathbb{V}))} \leq \bar{q}$ and $0 < \|x_{x_0,q}(t)\| \leq \tilde{r}_0$ and $\|q\|_{L^\infty((t,t+\epsilon(t)),\mathbb{V})} \leq \tilde{r}_0$.

For **the case** $L = \infty$, we derive that $\exists r_\infty > 0$ and $\exists \sigma_\infty \in \mathcal{K}$ (see, Theorem 4.2) such that

$$\|\hat{\mathfrak{d}}_\infty(s)(f(x,q) - f_0(x,q))\|_{\mathbb{Y}} \leq \left\|\hat{\mathfrak{d}}_\infty\left(\ln \bar{\sigma}_\infty^*(\|\mathfrak{d}_\infty(s)x\|_{\mathbb{Y}} + \|\tilde{\mathfrak{d}}_\infty(s)q\|_{\mathbb{V}})\right)\right\|_{\mathbb{Y}} \sigma_\infty\left(\frac{1}{\underline{\sigma}_\infty^*(\|x\|_{\mathbb{Y}} + \|q\|_{\mathbb{V}})}\right)$$

for any $(x,q) : \|(x,q)\|_{\tilde{\mathfrak{d}}_\infty,\tilde{\mathbb{Y}}} \geq r_\infty$, where $\underline{\sigma}_\infty^*(\|x\|_{\mathbb{Y}} + \|q\|_{\mathbb{V}}) \leq \|(x,q)\|_{\tilde{\mathfrak{d}}_\infty,\tilde{\mathbb{Y}}} \leq \bar{\sigma}_\infty^*(\|x\|_{\mathbb{Y}} + \|q\|_{\mathbb{V}})$, $\forall x \in \mathbb{Y}, \forall q \in \mathbb{V}$ (see, Theorem 3.6) with $\bar{\sigma}_\infty^*, \underline{\sigma}_\infty^* \in \mathcal{K}^\infty$. Let $\tilde{r}_\infty$ be such that $\sigma_\infty(1/\underline{\sigma}_\infty^*(\tilde{r}_\infty)) \leq \bar{q}/\left\|\hat{\mathfrak{d}}_\infty\left(\ln \bar{\sigma}_\infty^*(1+2\bar{q})\right)\right\|_{\mathbb{Y}}$ and let $\epsilon = \epsilon(t)$ be such that $\|\mathfrak{d}_\infty(-\ln \|x_{x_0,q}(t)\|_{\mathfrak{d}_\infty,\mathbb{Y}})x_{x_0,q}(\tau)\|_{\mathbb{Y}} \leq 1 + \bar{q}$ for all $\tau \in [t, t+\epsilon]$. Then, for $\hat{q}$ given by (4.14) and for $s = -\ln \|x_{x_0,q}(t)\|_{\mathfrak{d}_\infty,\mathbb{Y}} \geq 1$ we derive that the inequality $\|\hat{\mathfrak{d}}_\infty(s)\hat{q}\|_{L^\infty((t,t+\epsilon(t)),\mathbb{V})} \leq \bar{q}$ holds if $\|\tilde{\mathfrak{d}}_\infty(s)q\|_{L^\infty((t,t+\epsilon(t)),\mathbb{V}))} \leq \bar{q}$ and $\|x_{x_0,q}(t)\|_{\mathbb{Y}} \geq \tilde{r}_0$. Repeating the last part of the proof of Theorem 3.16 we derive local ISS for $L = 0$ and practical ISS for $L = \infty$. ■

Example 15. *The system* $\dot{x} = \frac{\partial^2 x}{\partial z^2} - \left(\frac{\partial x}{\partial z}\right)x - \rho\|x\|_{L_2}^4 x + x^2 + q, \quad t > 0, \quad z \in \mathbb{R}, \quad x(0) = x_0,$ *is* $\mathfrak{d}_\infty$*-homogeneous in the* ∞*-limit with the degree* 2*, where* $\mathbb{X}, \mathbb{Y}$ *and the dilation* $\mathfrak{d}_\infty$ *are defined as in Example 10,* $q \in \mathbb{V} = H^1(\mathbb{R},\mathbb{R})$*,* $\tilde{\mathfrak{d}}(s) = e^{2s}\mathfrak{d}(s)$*, and* $f_\infty(x,q) = -\left(\frac{\partial x}{\partial z}\right)x - \rho\|x\|_{L_2}^4 x + q$*. By Theorem 4.9 the system (15) is practically ISS.*

ISS of a system homogeneous in the bi-limit can be studied the following

Corollary 4.10. *Let conditions of Theorem 4.9 hold for both* $L = 0$ *and* $L = \infty$*. The system (2.1) is ISS provided that the unperturbed system (2.1) is globally uniformly asymptotically stable. If additionally* $\nu_0 < 0 < \nu_\infty$ *then the system (2.1) is fixed-time ISS .*

Proof. Theorem 4.6 gives $\overline{D}^+ V(x_{x_0,q}(t)) \leq -c\,\mathfrak{H}\left(V^{\frac{m_0+\nu_0}{m_0}}(x_{x_0,q}(t)), V^{\frac{m_\infty+\nu_\infty}{m_\infty}}(x_{x_0,q}(t))\right) + C(\bar{q})\chi(q)$, as long as $\hat{r}_0 \leq \|x_{x_0,q}(t)\| \leq \hat{r}_\infty$ and $q \in L^\infty : \|q\|_{L^\infty((t_0,t+\epsilon),\mathbb{V})} \leq \bar{q}$, where $\epsilon > 0$ is arbitrary small, the function $\bar{q} \mapsto C(\bar{q})$ is defined in the proof of Theorem 3.16 and the parameters $0 < \hat{r}_0 < \hat{r}_\infty < +\infty$ are defined in the proof of Theorem 4.9 such that $V(x) = V_0(x)$ for $\|x\|_{\mathbb{Y}} \leq \hat{r}_0$ and $V(x) = V_\infty(x)$ for all $\|x\|_{\mathbb{Y}} \geq \hat{r}_\infty$. Such a selection of $\hat{r}_0$ and $\hat{r}_\infty$ is always possible (see, the proof Theorem 4.6). The obtained inequality (as well as the properties of V and $\mathfrak{H}$) implies that there exists $\hat{\chi}_{\hat{r}_0,\hat{r}_\infty} \in \mathcal{K}^\infty$ such that $\overline{D}^+ V(x_{x_0,q}(t)) \leq -0.5c\,\mathfrak{H}\left(V^{\frac{m_0+\nu_0}{m_0}}(x_{x_0,q}(t)), V^{\frac{m_\infty+\nu_\infty}{m_\infty}}(x_{x_0,q}(t))\right)$ as long as $\max\{\hat{r}_0, \hat{\chi}_{\hat{r}_0,\hat{r}_\infty}(\|q\|_{L^\infty(t_0,t+\epsilon),\mathbb{V})}\} \leq$

$\|x_{x_0,q}(t)\| \leq \hat{r}_\infty$, where $\epsilon > 0$ is arbitrary small. The obtained inequality together with local and practical ISS proven by Theorem 4.9 yield ISS of the system (2.1). ■

5. Conclusions. In this paper, some important results, about stability and robustness (ISS) of homogeneous and locally homogeneous systems known before for finite dimensional dynamical systems [1], are extended to a class of abstract evolution equations in Banach spaces with unbounded nonlinear operators. Homogeneity of the system is understood in a generalized sense as a symmetry with respect to a *linear dilation* being a strongly continuous group of bounded linear operators satisfying certain limit property. All linear and some nonlinear models of mathematical physics are homogeneous this sense, e.g., heat, wave, KdV, Saint-Venant, Burgers and Navier-Stokes equations. It is shown that ISS of a homogeneous and locally homogeneous system follows from uniform asymptotic stability of the unperturbed system (see, Theorem 3.16, Theorem 4.9 and Corollary 4.10). Moreover, the homogeneity degree of the system (or its homogeneous approximation) specifies the convergence rate (see, Corollary 3.18, Theorem 4.4, Corollary 4.8). Theorem 3.16 for the case of locally Lipschitz nonlinearity f was proven in [37]. This paper clarify, simply and extend the corresponding constructions to unbounded operators f. The obtained results may simply the stability and robustness of nonlinear evolution equation in some cases. A few examples are given for the viscous Burger equation and its nonlinear modifications, but similar considerations can be repeated, for example, for heat, wave and KdV equations. All proves are given for a dynamical system in an abstract space under Assumption 1, 2, which may be too restrictive for some concrete PDE models. For example, the Saint-Venant equation being generalized homogeneous does not satisfy Assumption 1. An extension of the class of evolution equations allowing homogeneity-based stability and robustness analysis seems to be a promising direction for the future research.

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