Polynomial-degree-robust a posteriori error estimation for Maxwell's equations

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Outline

A posteriori error estimation 101

A posteriori error estimation

Consider a PDE problem set in a domain Ω, with solution u.

The FEM discretizes the problem using a mesh $\mathcal{T}_h$ of Ω.

This process leads to an approximation u_h of u.

The goal of *a posteriori* estimates is to assess discretization error

$$|||u - u_h|||$$

measured in an "appropriate" norm $|||\cdot|||$.

A posteriori error estimators

Typically, we attach with each element $K \in \mathcal{T}_h$ a computable number η_K.

We call η_K the "error estimator" associated with K.

In this talk we are especially interested with two properties of η_K:

"Reliability" and "Efficiency"

Reliability and efficiency

A posteriori error estimators provide a global error upper bound

Reliability

$$\| u - u_h \|_{\Omega}^2 \leq C_{\mathrm{rel}}^2 \sum_{K \in \mathcal{T}_h} \eta_K^2,$$

and local lower bounds

Efficiency

$$\| u - u_h \|_{\widetilde{K}} \geq C_{\mathrm{eff}} \eta_K.$$

These two properties allow for efficient adaptive mesh refinements.

Polynomial-degree-robustness

The constants C_{rel} and C_{eff} are required to be independent of h.

However, they "usually" depend on the polynomial degree p of the FEM.

Residual- and recovery-based estimator suffer from this for instance.

Here, the goal is to derive p-robust estimates for Maxwell's equations.

This is espacially important for hp-adaptive algorithm.

We propose a novel idea called "quasi-equilibration", to achieve this goal.

Two simplifying assumptions

Throughout the talk, the right-hand sides are piecewise polynomials.

This is only for the sake of simplicity.

General right-hand sides are dealt with using usual "oscillation terms".

Interior and exterior vertices (or edges) require distinct treatments.

When this happens, I will only treat the "interior case".

This is only to save time, and the exterior treatment is always similar.

Poisson's problem

Poisson's problem

The model problem

The Poisson problem

Consider a Lipschitz polyhedral domain $\Omega \subset \mathbb{R}^3$.

Given $f : \Omega \to \mathbb{R}$ we search $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

More formally, if $f \in L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ s.t.

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

FEM discretization

Fix $p \geq 0$, and consider the (Lagrange) discrete space

$$V_h := \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_0^1(\Omega),$$

for piecewise polynomials of degree $p+1$.

There exists a unique $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

We say that u_h is the FEM approximation to u.

Poisson's problem

The concept of equilibrated flux

The key idea behind flux-equilibration

Consider a vector field $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f$. We have

$$(f, v) = (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, v) = -(\boldsymbol{\sigma}, \boldsymbol{\nabla} v) \quad \forall v \in H_0^1(\Omega).$$

In particular, one sees that for $v \in H_0^1(\Omega)$

$$\begin{aligned}(\boldsymbol{\nabla}(u - u_h), \boldsymbol{\nabla} v) &= (f, v) - (\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v) \\ &= (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}, v) - (\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v) \\ &= -(\boldsymbol{\sigma} + \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v).\end{aligned}$$

Selecting $v := u - u_h$, it follows that

$$\|\boldsymbol{\nabla}(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma} + \boldsymbol{\nabla} u_h\|_\Omega.$$

Equilibrated flux and Prager-Synge theorem

Equilibrated flux

$$\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega) \qquad \nabla \cdot \boldsymbol{\sigma} = f$$

Prager-Synge theorem

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma} + \nabla u_h\|_\Omega$$

Equilibrated fluxes provide fully computable (global) upper bounds!

W. Prager and J.L. Synge, 1947

Practical construction of an equilibrated flux

We have assumed that $f \in \mathcal{P}_p(\mathcal{T}_h)$ is piecewise polynomial.

As a result, we can construct *discrete* equilibrated fluxes $\boldsymbol{\sigma}_h$.

The appropriate tool is then the Raviart-Thomas finite element space $\boldsymbol{X}_h$.

Ideal flux

$$\boldsymbol{\sigma}_h := \operatorname*{argmin}_{\substack{\boldsymbol{\tau}_h \in \boldsymbol{X}_h \\ \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_h = f}} \|\boldsymbol{\tau}_h + \boldsymbol{\nabla} u_h\|_\Omega$$

Unfortunately, this leads to global (expensive) saddle-point problem.

Poisson's problem

Localization techniques

Main ideas behind localization

The idea is then to compute the flux locally. Assume we can decompose

$$f = \sum_{\boldsymbol{a}} f_{\boldsymbol{a}},$$

where each $f_{\boldsymbol{a}}$ as a "small" support $\omega_{\boldsymbol{a}}$.

We could then (try to) define locally $\boldsymbol{\sigma}_h^{\boldsymbol{a}}$ such that

$$\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = f_{\boldsymbol{a}} \text{ in } \omega_{\boldsymbol{a}} \qquad \boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n} = 0 \text{ on } \partial\omega_{\boldsymbol{a}},$$

and the sum

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \Omega)$$

would be an equilibrated flux.

It is trickier than it looks!

But... There is a catch! Stoke's theorem says that

$$\int_{\omega_{\boldsymbol{a}}} f_{\boldsymbol{a}} = \int_{\omega_{\boldsymbol{a}}} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = \int_{\partial\omega_{\boldsymbol{a}}} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n} = 0.$$

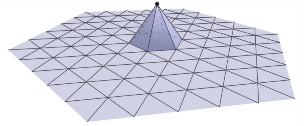
Hence, we cannot use *any* decomposition.

Actually, it is hard, but there is a solution:

P. Destuynder and B. Métivet, 1999

Hat functions and vertex patches

We denote by $\mathcal{V}_h$ the vertices of the mesh.
$\mathcal{T}_h^{\boldsymbol{a}}$ is the set of elements sharing $\boldsymbol{a} \in \mathcal{V}_h$.
$\omega_{\boldsymbol{a}}$ is the associated domain.



For each $\boldsymbol{a} \in \mathcal{V}_h$, there is a "hat function" $\psi_{\boldsymbol{a}} \in \mathcal{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$ s.t.

$$\psi_{\boldsymbol{a}}(\boldsymbol{a}') = \delta_{\boldsymbol{a},\boldsymbol{a}'} \quad \forall \boldsymbol{a}' \in \mathcal{V}_h$$

They are locally supported, $\operatorname{supp} \psi_{\boldsymbol{a}} = \omega_{\boldsymbol{a}}$, and

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi_{\boldsymbol{a}} = 1.$$

Compatible decomposition of the right-hand side

At this point, it is tempting to define $f_{\boldsymbol{a}} := \psi_{\boldsymbol{a}} f$, but it does not work!

The key idea is to observe that

$$0 = (f, \psi_{\boldsymbol{a}}) - (\nabla u_h, \nabla \psi_{\boldsymbol{a}}) = \int_{\omega_{\boldsymbol{a}}} \psi_{\boldsymbol{a}} f - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h$$

because $\psi_{\boldsymbol{a}} \in V_h$, and then define

$$f_{\boldsymbol{a}} := \psi_{\boldsymbol{a}} f - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h.$$

Poisson's problem

Flux-equilibrated error estimator

Flux-equilibrated error estimator

Let $\boldsymbol{X}_h^{\boldsymbol{a}}$ denote the Raviart-Thomas space on the vertex patch $\mathcal{T}_h^{\boldsymbol{a}}$.

As we saw, the definition

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \underset{\substack{\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{X}_h^{\boldsymbol{a}} \\ \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_h^{\boldsymbol{a}} = \psi_{\boldsymbol{a}} f - \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \cdot \boldsymbol{\nabla} u_h \text{ in } \omega_{\boldsymbol{a}} \\ \boldsymbol{\tau}_h^{\boldsymbol{a}} \cdot \boldsymbol{n} = 0 \text{ on } \partial \omega_{\boldsymbol{a}}}}{\operatorname{argmin}} \|\boldsymbol{\tau}_h^{\boldsymbol{a}} + \psi_{\boldsymbol{a}} \boldsymbol{\nabla} u_h\|_{\omega_{\boldsymbol{a}}}$$

makes sense. And correspond to local (cheap) saddle-point problems.

Gathering the patch contributions, we construct the equilibrated flux

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}.$$

A. Ern and M. Vohralík, 2015

Efficiency of the estimator

The reliability of the estimator follows from Prager-Synge theorem.

What about efficiency?

Unfortunately, I do not have time to show proofs.

Still, we can discuss the main results.

Efficiency of the estimator

Efficiency

$$\|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{cont},\boldsymbol{a}} C_{\mathrm{st},\boldsymbol{a}} \|\nabla(u - u_h)\|_{\omega_{\boldsymbol{a}}}$$

In the efficiency estimate:

$C_{\mathrm{cont},\boldsymbol{a}}$ only depends on the geometry of the patch $\omega_{\boldsymbol{a}}$,

$C_{\mathrm{st},\boldsymbol{a}}$ depends on the geometry, but also involve the discrete space V_h.

It is a lot of work, but one may show that $C_{\mathrm{st},\boldsymbol{a}}$ is independent of p:

D. Braess, V. Pillwein, J. Schöberl, 2009

A. Ern and M. Vohralík, 2019

I will get back to these constants when discussing Maxwell's equations...

Summary on flux-equilibrated estimators

Definition

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \underset{\substack{\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{X}_h^{\boldsymbol{a}} \\ \nabla \cdot \boldsymbol{\tau}_h^{\boldsymbol{a}} = \psi_{\boldsymbol{a}} f - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h \text{ in } \omega_{\boldsymbol{a}} \\ \boldsymbol{\tau}_h^{\boldsymbol{a}} \cdot \boldsymbol{n} = 0 \text{ on } \partial \omega_{\boldsymbol{a}}}}{\operatorname{argmin}} \|\boldsymbol{\tau}_h^{\boldsymbol{a}} + \psi_{\boldsymbol{a}} \nabla u_h\|_{\omega_{\boldsymbol{a}}}$$

Reliability

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_\Omega$$

Efficiency

$$\|\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \nabla u_h\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{cont},\boldsymbol{a}} C_{\mathrm{st},\boldsymbol{a}} \|\nabla(u - u_h)\|_{\omega_{\boldsymbol{a}}}$$

Poisson's problem

Quasi-equilibrated error estimators

Definition of the estimator

For the equilibrated error estimator, we considered

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \underset{\substack{\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{X}_h^{\boldsymbol{a}} \\ \nabla \cdot \boldsymbol{\tau}_h^{\boldsymbol{a}} = \psi_{\boldsymbol{a}} f - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h \text{ in } \omega_{\boldsymbol{a}} \\ \boldsymbol{\tau}_h^{\boldsymbol{a}} \cdot \boldsymbol{n} = 0 \text{ on } \partial\omega_{\boldsymbol{a}}}}{\operatorname{argmin}} \|\boldsymbol{\tau}_h^{\boldsymbol{a}} + \psi_{\boldsymbol{a}} \nabla u_h\|_{\omega_{\boldsymbol{a}}},$$

where the BC and decomposition of f are important to build $\boldsymbol{\sigma}_h$.

For quasi-equilibration, we instead set

$$\widetilde{\boldsymbol{\sigma}}_h^{\boldsymbol{a}} := \underset{\substack{\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{X}_h^{\boldsymbol{a}} \\ \nabla \cdot \boldsymbol{\tau}_h^{\boldsymbol{a}} = f \text{ in } \omega_{\boldsymbol{a}}}}{\operatorname{argmin}} \|\boldsymbol{\tau}_h^{\boldsymbol{a}} + \nabla u_h\|_{\omega_{\boldsymbol{a}}},$$

where the boundary conditions, and the $\psi_{\boldsymbol{a}}$ have disappeared.

Quasi-equilibration does not provide an equilibrated flux

The resulting field $\widetilde{\boldsymbol{\sigma}}_h^{\boldsymbol{a}}$ are "locally" equilibrated,

but cannot be assembled into a globally equilibrated object.

Yet, we may associate with each $\boldsymbol{a} \in \mathcal{V}_h$ the quantity

$$\eta_{\boldsymbol{a}} := \min_{\substack{\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{X}_h^{\boldsymbol{a}} \\ \nabla \cdot \boldsymbol{\tau}_h^{\boldsymbol{a}} = f \text{ in } \omega_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h^{\boldsymbol{a}} + \boldsymbol{\nabla} u_h\|_{\omega_{\boldsymbol{a}}}.$$

It turns out that $\eta_{\boldsymbol{a}}$ makes a good estimator,

even if we cannot use Prager-Theorem anymore.

Summary of quasi-equilibrated estimator

Definition

$$\eta_{\boldsymbol{a}} := \min_{\substack{\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{X}_h^{\boldsymbol{a}} \\ \nabla\cdot\boldsymbol{\tau}_h^{\boldsymbol{a}} = f \text{ in } \omega_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h^{\boldsymbol{a}} + \nabla u_h\|_{\omega_{\boldsymbol{a}}}.$$

Reliability

$$\|\nabla(u - u_h)\|_\Omega \leq 2 \left(\sum_{\boldsymbol{a} \in \mathcal{T}_h} C_{\mathrm{cont},\boldsymbol{a}}^2 \eta_{\boldsymbol{a}}^2 \right)^{1/2}$$

Efficiency

$$\eta_{\boldsymbol{a}} \leq C_{\mathrm{st},\boldsymbol{a}} \|\nabla(u - u_h)\|_{\omega_{\boldsymbol{a}}}$$

The estimates involve the same constants, but placed differently!

Poisson's problem

Equilibration vs Quasi-equilibration

Construction

Equilibration

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \operatorname*{argmin}_{\substack{\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{X}_h^{\boldsymbol{a}} \\ \nabla \cdot \boldsymbol{\tau}_h^{\boldsymbol{a}} = \psi_{\boldsymbol{a}} f - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h \text{ in } \omega_{\boldsymbol{a}} \\ \boldsymbol{\tau}_h^{\boldsymbol{a}} \cdot \boldsymbol{n} = 0 \text{ on } \partial \omega_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h^{\boldsymbol{a}} + \psi_{\boldsymbol{a}} \nabla u_h\|_{\omega_{\boldsymbol{a}}}$$

Quasi-equilibration

$$\eta_{\boldsymbol{a}} := \min_{\substack{\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{X}_h^{\boldsymbol{a}} \\ \nabla \cdot \boldsymbol{\tau}_h^{\boldsymbol{a}} = f \text{ in } \omega_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h^{\boldsymbol{a}} + \nabla u_h\|_{\omega_{\boldsymbol{a}}}.$$

The constructions are similar. The quasi-equilibration is slightly simpler.

Efficiency

Equilibration

$$\eta_{\boldsymbol{a}} \leq C_{\mathrm{cont},\boldsymbol{a}} C_{\mathrm{st},\boldsymbol{a}} \|\nabla(u - u_h)\|_{\omega_{\boldsymbol{a}}}$$

Quasi-equilibration

$$\|\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \nabla u_h\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st},\boldsymbol{a}} \|\nabla(u - u_h)\|_{\omega_{\boldsymbol{a}}}$$

Both estimators are p-robust.

Reliability

Equilibration

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_\Omega$$

Quasi-equilibration

$$\|\nabla(u - u_h)\|_\Omega \leq 2 \left(\sum_{\boldsymbol{a} \in \mathcal{T}_h} C^2_{\mathrm{cont},\boldsymbol{a}} \eta^2_{\boldsymbol{a}} \right)^{1/2}$$

Equilibrated-fluxes provide nicer upper bounds.

Maxwell’s equations

Maxwell's equations

Model problem

Given $\boldsymbol{J} : \Omega \to \mathbb{R}^3$, with $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$, we search $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ such that

$$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{A} &= 0 & \text{in } \Omega, \\ \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} &= \boldsymbol{J} & \text{in } \Omega, \\ \boldsymbol{A} \times \boldsymbol{n} &= \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

More formally, given $\boldsymbol{J} \in \boldsymbol{H}(\mathrm{div}^0, \Omega)$, there exists a unique $\boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ such that

$$(\boldsymbol{\nabla} \times \boldsymbol{A}, \boldsymbol{\nabla} \times \boldsymbol{v}) = (\boldsymbol{J}, \boldsymbol{v}) \qquad (\boldsymbol{A}, \boldsymbol{\nabla} q) = 0$$

for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ and $q \in H_0^1(\Omega)$.

F. Brezzi, 1974

Why is this problem interesting?

In magnetostatics, the magnetic field $\boldsymbol{H}$ satisfies

$$\nabla \cdot \boldsymbol{H} = 0 \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J},$$

and then $\boldsymbol{A}$ is the magnetic potential: $\nabla \times \boldsymbol{A} = \boldsymbol{H}$.

Also, our model problem is the “differential part” of:

$$-k^2 \boldsymbol{E} - \nabla \times \nabla \times \boldsymbol{E} = ik\boldsymbol{J},$$

known as time-harmonic Maxwell's equations, $\boldsymbol{E}$ being the electric field.

T. Chaumont-Frelet, A. Ern, M. Vohralík, 2019

S. Congreve, J. Gedicke, and I. Perugia, 2019

Finite element discretization

We now consider the $\boldsymbol{W}_h$, the Nédélec finite-element space of degree p.

There exists a unique $\boldsymbol{A}_h \in \boldsymbol{W}_h$ such that

$$(\nabla \times \boldsymbol{A}_h, \nabla \times \boldsymbol{v}_h) = (\boldsymbol{J}, \boldsymbol{v}_h) \qquad (\boldsymbol{A}_h, \nabla q_h) = 0$$

for all $\boldsymbol{v}_h \in \boldsymbol{W}_h$ and $q_h \in V_h$.

Notice in particular that

$$(\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h), \nabla \times \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{W}_h.$$

F. Brezzi, 1974

Maxwell's equations

Flux equilibration?

Prager-Synge theorem

Equilibrated flux

$$\boldsymbol{B} \in \boldsymbol{H}(\mathbf{curl}, \Omega) \quad \nabla \times \boldsymbol{B} = \boldsymbol{J}$$

Prager-Synge theorem

$$\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \|\boldsymbol{B} - \nabla \times \boldsymbol{A}_h\|_\Omega$$

Global construction

$$\boldsymbol{B}_h := \underset{\substack{\boldsymbol{M}_h \in \boldsymbol{W}_h \\ \nabla \times \boldsymbol{M}_h = \boldsymbol{J}}}{\operatorname{argmin}} \|\boldsymbol{M}_h - \nabla \times \boldsymbol{A}_h\|_\Omega$$

The issue with localization

We can (in principle) follow the same steps than for Poisson's problem:

Decompose the right-hand side

$$\boldsymbol{J} = \sum_{\boldsymbol{a}} \boldsymbol{J}_{\boldsymbol{a}},$$

and find locally $\boldsymbol{B}_h^{\boldsymbol{a}}$ such that

$$\boldsymbol{\nabla} \times \boldsymbol{B}_h^{\boldsymbol{a}} = \boldsymbol{J}_{\boldsymbol{a}} \text{ in } \omega_{\boldsymbol{a}} \qquad \boldsymbol{B}_h^{\boldsymbol{a}} \times \boldsymbol{n} = \boldsymbol{o} \text{ on } \partial\omega_{\boldsymbol{a}}.$$

The problem is that the compatibility conditions are "hard".

The "natural choice" $\boldsymbol{J}_{\boldsymbol{a}} := \psi_{\boldsymbol{a}}\boldsymbol{J} + \boldsymbol{\nabla}\psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h$ fails: $\boldsymbol{\nabla} \cdot \boldsymbol{J}_{\boldsymbol{a}} \neq 0$.

Possible solutions

An initial idea for lowest-order elements:

- D. Braess and J. Schöberl, 2008

An extension developed in the same time than the present work:

- J. Gedicke, S. Geevers and I. Perugia, 2019
- J. Gedicke, S. Geevers, I. Perugia and J. Schöberl, 2020

The construction is complex and requires 4 mesh traversals.

An alternative, quasi-equilibration:

- T. Chaumont-Frelet, A. Ern and M. Vohralík, 2020

Maxwell's equations

Quasi-equilibration!

Quasi-equilibration

Quasi-equilibration does not require any compatible decomposition.

Actually, we can readily apply it to Maxwell's equations.

The resulting estimators are attached to the edges $\ell \in \mathcal{E}_h$ of the mesh

$$\eta_\ell := \min_{\substack{\boldsymbol{B}_h^\ell \in \boldsymbol{W}_h^\ell \\ \boldsymbol{\nabla} \times \boldsymbol{B}_h^\ell = \boldsymbol{J}}} \|\boldsymbol{B}_h^\ell - \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega_\ell},$$

where ω_ℓ is the set of tetrahedra $K \in \mathcal{T}_h$ sharing the edge ℓ.

Key properties

Reliability

$$\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \sqrt{6} C_{\mathrm{lift},\Omega} \left(\sum_{\ell \in \mathcal{E}_h} C_{\mathrm{cont},\ell}^2 \eta_\ell^2 \right)^{1/2}$$

Efficiency

$$\eta_\ell \leq C_{\mathrm{st},\ell} \|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\omega_\ell}$$

The estimator is p-robust, all the constants are independent of p.

We do not have time for the proofs, but I can tell you where the constants come from, and what are the key ideas.

Constants in the reliability estimate

Reliability

$$\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \sqrt{6} C_{\mathrm{lift},\Omega} \left(\sum_{\ell \in \mathcal{E}_h} C_{\mathrm{cont},\ell}{}^2 \eta_\ell^2 \right)^{1/2}$$

Constants in the reliability estimate

Reliability

$$\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \sqrt{6} C_{\mathrm{lift},\Omega} \left(\sum_{\ell \in \mathcal{E}_h} C_{\mathrm{cont},\ell}{}^2 \eta_\ell^2 \right)^{1/2}$$

There are 6 edges in each tetrahedron!

This factor comes from "covering" arguments.

Constants in the reliability estimate

Reliability

$$\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \sqrt{6} C_{\mathrm{lift},\Omega} \left(\sum_{\ell \in \mathcal{E}_h} C_{\mathrm{cont},\ell}{}^2 \eta_\ell^2 \right)^{1/2}$$

For all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ such that

$$\nabla \times \boldsymbol{w} = \nabla \times \boldsymbol{v} \qquad \|\nabla \boldsymbol{w}\|_\Omega \leq C_{\mathrm{lift},\Omega} \|\nabla \times \boldsymbol{v}\|_\Omega.$$

This is a usually trick, also employed in residual-based estimator:

R. Beck, R. Hiptmair, H.W. Hoppe and B. Wohlmuth, 2000

S. Nicaise and E. Creusé, 2003

$C_{\mathrm{lift},\Omega} = 1$ for convex domains, but it is “hard” to compute in general.

Constants in the reliability estimate

Reliability

$$\|\boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \sqrt{6} C_{\mathrm{lift},\Omega} \left(\sum_{\ell \in \mathcal{E}_h} C_{\mathrm{cont},\ell}{}^2 \eta_\ell^2 \right)^{1/2}$$

We have

$$C_{\mathrm{cont},\ell} := \|\boldsymbol{\psi}_\ell\|_{L^\infty(\omega_\ell)} + C_{\mathrm{P},\ell} h_\ell \|\boldsymbol{\nabla} \times \boldsymbol{\psi}_\ell\|_{L^\infty(\omega_\ell)} \simeq 1$$

where

$C_{\mathrm{P},\ell}$ is the (computable) "Poincaré constant" of the patch ω_ℓ

h_ℓ is the diameter of the patch ω_ℓ

$\boldsymbol{\psi}_\ell$ is the "edge function" associated with ℓ.

Under the hood, we use a vectorial edge-based partition of unity.

Constant in the efficiency estimate

Efficiency

$$\eta_\ell := \min_{\substack{\boldsymbol{B}_h^\ell \in \boldsymbol{W}_h^\ell \\ \nabla \times \boldsymbol{B}_h^\ell = \boldsymbol{J}}} \|\boldsymbol{B}_h^\ell - \nabla \times \boldsymbol{A}_h\|_{\omega_\ell} \leq C_{\mathrm{st},\ell} \|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\omega_\ell}$$

Stable discrete minimization

$$\min_{\substack{\boldsymbol{B}_h^\ell \in \boldsymbol{W}_h^\ell \\ \nabla \times \boldsymbol{B}_h^\ell = \boldsymbol{j}_h^\ell}} \|\boldsymbol{B}_h^\ell - \boldsymbol{\theta}_h\|_{\omega_\ell} \leq C_{\mathrm{st},\ell} \min_{\substack{\boldsymbol{B}^\ell \in \boldsymbol{H}(\mathbf{curl},\omega_\ell) \\ \nabla \times \boldsymbol{B}^\ell = \boldsymbol{j}_h^\ell}} \|\boldsymbol{B}^\ell - \boldsymbol{\theta}_h\|_{\omega_\ell}$$

The tough part is to show that $C_{\mathrm{st},\ell}$ does not depend on p!

A. Ern and M. Vohralík, 2019

T. Chaumont-Frelet, A. Ern and M. Vohralík, 2020

Summary

We design an edge-based p-robust estimator

$$\eta_\ell := \min_{\substack{\boldsymbol{B}_h^\ell \in \boldsymbol{W}_h^\ell \\ \boldsymbol{\nabla} \times \boldsymbol{B}_h^\ell = \boldsymbol{J}}} \|\boldsymbol{B}_h^\ell + \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega_\ell}.$$

It is cheap to compute, and only uses standard FEM matrices for its implementation.

We obtain efficiency estimate on "tight" edge patches.

But the constant in the upper bound is not always practically computable.

Numerical examples

Numerical examples

Smooth solution in a cube

Smooth solution in a cube

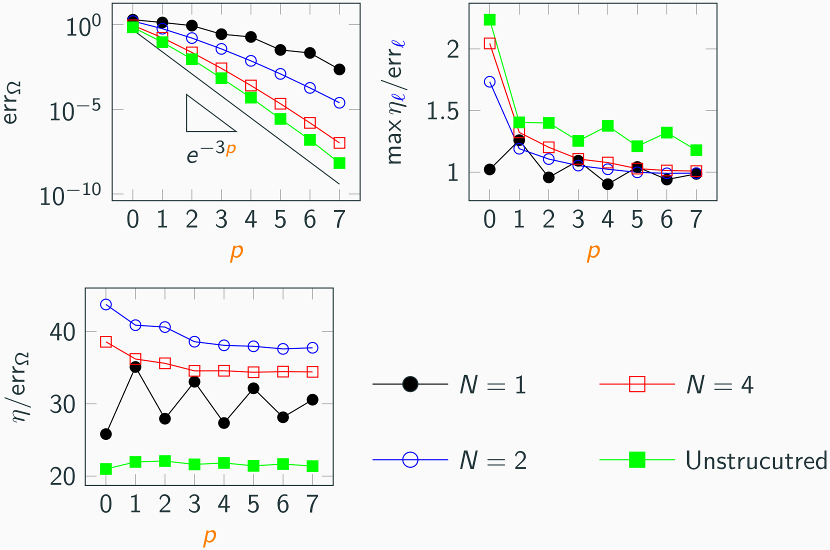
Consider $\Omega := (0,1)^3$, and

$$\boldsymbol{J}(\boldsymbol{x}) := 3\pi^3 \begin{pmatrix} \sin(\pi \boldsymbol{x}_1)\cos(\pi \boldsymbol{x}_2)\cos(\pi \boldsymbol{x}_3) \\ -\cos(\pi \boldsymbol{x}_1)\sin(\pi \boldsymbol{x}_2)\cos(\pi \boldsymbol{x}_3) \\ 0 \end{pmatrix}.$$

Then, the solution reads

$$\boldsymbol{A}(\boldsymbol{x}) := 3\pi^3 \begin{pmatrix} \sin(\pi \boldsymbol{x}_1)\cos(\pi \boldsymbol{x}_2)\cos(\pi \boldsymbol{x}_3) \\ -\cos(\pi \boldsymbol{x}_1)\sin(\pi \boldsymbol{x}_2)\cos(\pi \boldsymbol{x}_3) \\ 0 \end{pmatrix}.$$

p-robustness in the unit-cube experiment

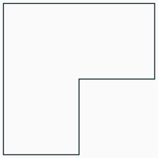


Numerical examples

Singular solution in an “L brick”

L brick experiment

We now consider an "L brick" $\Omega = L \times (0,1)$.

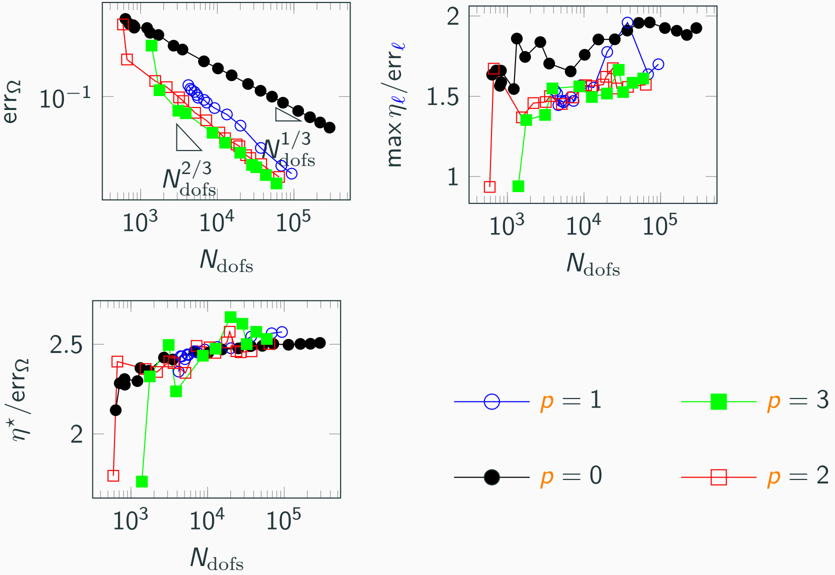


The right-hand side $\boldsymbol{J} \in \boldsymbol{H}(\mathrm{div}^0, \Omega)$ is chosen such that

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ 0 \\ \chi(r) r^\alpha \sin(\alpha\theta) \end{pmatrix},$$

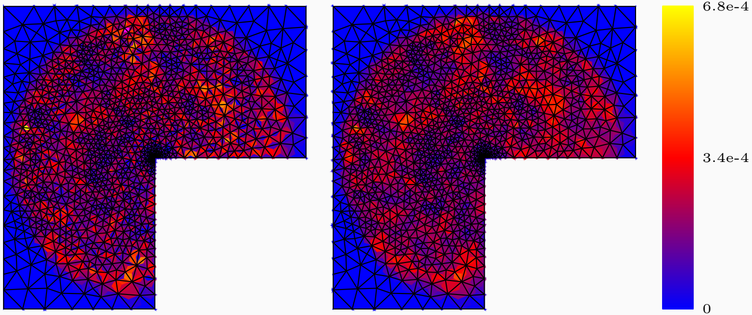
where (r, θ) the polar coordinates in L, χ is a cutoff and $\alpha = 2/3$.

The solution feature an "edge singularity": $\boldsymbol{\nabla} \times \boldsymbol{A} \in \boldsymbol{H}^{2/3}(\Omega)$.



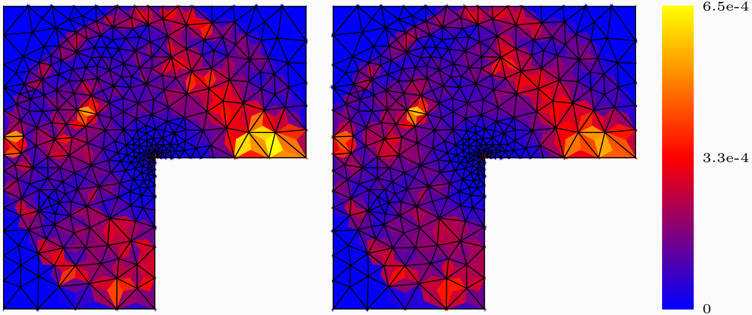
err_Ω
10^{-1}
$N_{\mathrm{dofs}}^{2/3}$
$N_{\mathrm{dofs}}^{1/3}$
10^3 10^4 10^5
N_{dofs}
$\max \eta_\ell / \mathrm{err}_\ell$
2
1.5
1
10^3 10^4 10^5
N_{dofs}
$\eta^\star / \mathrm{err}_\Omega$
2.5
2
10^3 10^4 10^5
N_{dofs}
$p = 1$
$p = 3$
$p = 0$
$p = 2$

Error distribution in the L brick experiment $p = 0$



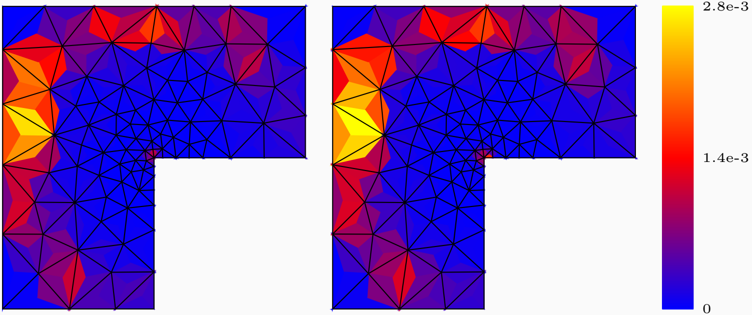
η_ℓ $\quad$ $\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\omega_\ell}$

Error distribution in the L brick experiment $p = 1$



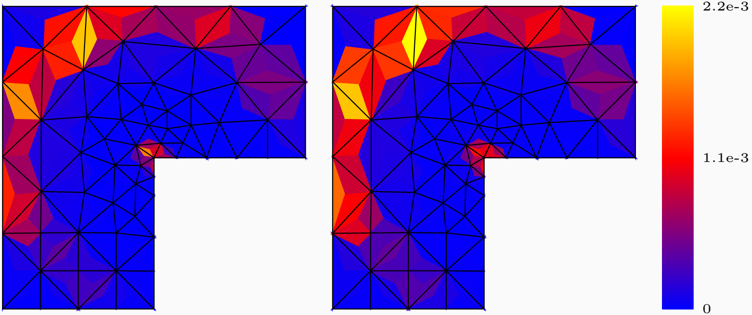
η_ℓ $\qquad\qquad$ $\|\nabla \times (A - A_h)\|_{\omega_\ell}$

Error distribution in the L brick experiment $p = 2$



η_ℓ $\qquad$ $\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\omega_\ell}$

Error distribution in the L brick experiment $p = 3$



η_ℓ $\qquad$ $\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\omega_\ell}$

Conclusion

Conclusion

Quasi-equilibration is an alternative to flux-equilibration.

Both methods provide polynomial-degree-robust estimates.

Flux-equilibration leads to constant-free upper bounds.

Quasi-equilibration relies on simpler local problems.

Importantly, quasi-equilibration bypass "compatible decompositions"

of the right-hand side which is especially handy for Maxwell's equations.

I believe that the present estimator is the "simplest" p-robust estimator currently available for Maxwell's equations in the literature.

T. Chaumont-Frelet, A. Ern and M. Vohralík, 2020