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# An equilibrated a posteriori error estimator for the curl–curl problem

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# Outline

- 1 Equilibration for the Poisson problem
- 2 Equilibration for the curl–curl problem

# The Poisson problem

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# Poisson problem

Consider a Lipschitz polyhedral domain  $\Omega$  and  $f \in L^2(\Omega)$ .

Our first model problem is to find  $u \in H_0^1(\Omega)$  such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega$$

for all  $v \in H_0^1(\Omega)$ .

$u_h$  is the Lagrange FEM approximation of  $u$  with degree  $p + 1$ .

For the sake of simplicity, I will assume that  $f = f_h \in \mathcal{P}_p(\mathcal{T}_h)$ .

# The idea of flux equilibration

Assume that we have a field  $\sigma_h \in \mathbf{H}(\text{div}, \Omega)$  such that

$$\nabla \cdot \sigma_h = f_h \text{ in } \Omega$$

at our disposal.

Then, we have

$$\begin{aligned}(\nabla(u - u_h), \nabla v)_\Omega &= (f_h, v)_\Omega - (\nabla u_h, \nabla v) \\ &= (\nabla \cdot \sigma_h, v)_\Omega - (\nabla u_h, \nabla v) \\ &= -(\sigma_h + \nabla u_h, \nabla v),\end{aligned}$$

for all  $v \in H_0^1(\Omega)$  and in particular

$$\|\nabla(u - u_h)\|_\Omega \leq \|\sigma_h + \nabla u_h\|_\Omega.$$

# Prager-Synge theorem

## Equilibrated flux

$$\boldsymbol{\sigma}_h \in \mathbf{H}(\operatorname{div}, \Omega); \quad \nabla \cdot \boldsymbol{\sigma}_h = f_h \text{ in } \Omega$$

## Error estimate

$$\|\nabla(u - u_h)\|_{\Omega} \leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_{\Omega}.$$



W. Prager and J.L. Synge, 1947

The particular choice  $\boldsymbol{\sigma} := -\nabla u$  saturates the bound.

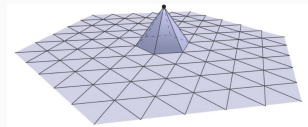
# Local flux constructions

The idea flux would be  $\sigma := -\nabla u$ .

Let's characterize it locally and cook up a discrete computable version.

Let us set

$$\sigma^a := \psi_a \sigma$$



where  $\psi_a$  is the “hat function” associated with the vertex  $a$ , so that

$$\sigma = \sum_{a \in \mathcal{V}_h} \sigma^a.$$



# Characterization of the local contributions

We know that  $\boldsymbol{\sigma}^a := -\psi_a \nabla u$ . So that in particular

$$\boldsymbol{\sigma}^a \in \mathbf{H}_0(\text{div}, \omega_a)$$

where  $\omega_a$  is the set of tetrahedra  $K \in \mathcal{T}_h$  sharing the vertex  $\mathbf{a}$ , and

$$\nabla \cdot \boldsymbol{\sigma}^a = \psi_a f_h - \nabla \psi_a \cdot \nabla u.$$

As a result, we have the characterization

$$\boldsymbol{\sigma}^a = \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v} = \psi_a f_h - \nabla \psi_a \cdot \nabla u}} \|\mathbf{v} + \psi_a \nabla u\|_{\omega_a},$$

since the minimum is zero and achieved when  $\mathbf{v} = -\psi_a \nabla u$ .

# Discrete construction

Recall that

$$\sigma^a = \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\operatorname{div}, \omega_a) \\ \nabla \cdot \mathbf{v} = \psi_a f_h - \nabla \psi_a \cdot \nabla u}} \|\mathbf{v} + \psi_a \nabla u\|_{\omega_a}.$$

As a discrete counterpart, we set

$$\sigma_h^a := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{RT}_{p+1}(\mathcal{T}_h^a) \cap \mathbf{H}_0(\operatorname{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = \psi_a f_h - \nabla \psi_a \cdot \nabla u_h}} \|\mathbf{v}_h + \psi_a \nabla u_h\|_{\omega_a}.$$

This is indeed well-defined since

$$(\psi_a f - \nabla \psi_a \cdot \nabla u_h, 1)_{\omega_a} = (f, \psi_a)_{\Omega} - (\nabla u_h, \nabla \psi_a)_{\omega_a} = 0$$

whenever  $a \notin \partial\Omega$ .

Summation provides an equilibrated flux:

$$\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}},$$

and reliability follows from the Prager–Synge theorem.

## Efficiency

$$\|\boldsymbol{\sigma}_h + \nabla u_h\|_{\mathcal{K}} \lesssim \|\nabla(u - u_h)\|_{\tilde{\mathcal{K}}}$$

The hidden constant does not depend on  $p$ .

# The curl–curl problem

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# The curl–curl problem

For the sake of simplicity, assume that  $\Omega$  is simply connected, and consider a divergence-free right-hand side  $\mathbf{J}_h \in \mathbf{RT}_p(\mathcal{T}_h)$ .

Our model problem is then to find  $\mathbf{A} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$  such that

$$(\nabla \times \mathbf{A}, \nabla \times \mathbf{v})_\Omega = (\mathbf{J}_h, \mathbf{v})_\Omega, \quad (\mathbf{A}, \nabla q)_\Omega = 0,$$

for all  $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$  and  $q \in H_0^1(\Omega)$ .

$\mathbf{A}_h \in \mathbf{N}_p(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl}, \Omega)$  is the Nédélec approximation of  $\mathbf{A}$ .

# Prager-Synge theorem

## Equilibrated flux

$$\mathbf{B}_h \in \mathbf{H}(\text{curl}, \Omega); \quad \nabla \times \mathbf{B}_h = \mathbf{J}_h$$

## Error estimate

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{\Omega} \leq \|\mathbf{B}_h - \nabla \times \mathbf{A}_h\|_{\Omega}$$

The “ideal” flux  $\mathbf{B} := \nabla \times \mathbf{A}$  saturates the bound.

## The issue with localization (1/2)

We follow the same steps than for the Poisson problem.

The ideal flux is  $\mathbf{B} := \nabla \times \mathbf{A}$ . If we set  $\mathbf{B}^a := \psi_a \nabla \times \mathbf{A}$ , then

$$\nabla \times \mathbf{B}^a = \psi_a \mathbf{J}_h + \nabla \psi_a \times \nabla \times \mathbf{A},$$

so that

$$\mathbf{B}^a = \arg \min_{\substack{\mathbf{v} \in H_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v} = \psi_a \mathbf{J} + \nabla \psi_a \times \nabla \times \mathbf{A}}} \|\mathbf{v} - \psi_a \nabla \times \mathbf{A}\|_{\omega_a}.$$

## The issue with localization (2/2)

Recall that

$$B^a = \arg \min_{\substack{\mathbf{v} \in H_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v} = \psi_a \mathbf{J}_h + \nabla \psi_a \times \nabla \times \mathbf{A}}} \|\mathbf{v} - \psi_a \nabla \times \mathbf{A}\|_{\omega_a}.$$

Unfortunately, we can not set

$$B_h^a := \arg \min_{\substack{\mathbf{v}_h \in N_{p+1}(\mathcal{T}_h^a) \cap H_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v}_h = \psi_a \mathbf{J}_h + \nabla \psi_a \times \nabla \times \mathbf{A}_h}} \|\mathbf{v}_h - \psi_a \nabla \times \mathbf{A}_h\|_{\omega_a}$$

as the minimization set is empty: the field

$$\psi_a \mathbf{J}_h + \nabla \psi_a \times \nabla \times \mathbf{A}_h$$

is not divergence-free!



# Possible solutions

An initial idea for lowest-order elements:



D. Braess and J. Schöberl, 2008

Recently developed extensions:



J. Gedicke, S. Geeveres and I. Perugia, 2019



J. Gedicke, S. Geeveres, I. Perugia and J. Schöberl, 2020



T. Chaumont-Frelet and M. Vohralík, 2021

Here, I will detail the last construction.

# The idea

Let  $\boldsymbol{\theta}^a := \nabla\psi_a \times \nabla \times \mathbf{A}$ . There are two important properties:

$$\sum_{a \in \mathcal{V}_h} \boldsymbol{\theta}^a = \mathbf{0}, \quad \nabla \cdot \boldsymbol{\theta}^a = -\nabla\psi_a \cdot \mathbf{J}_h.$$

At the discrete level, we have

$$\sum_{a \in \mathcal{V}_h} (\nabla\psi_a \times \nabla \times \mathbf{A}_h) = \mathbf{0}, \quad \nabla \cdot (\nabla\psi_a \times \nabla \times \mathbf{A}_h) \neq -\nabla\psi_a \cdot \mathbf{J}_h.$$

It is tempting to set

$$\boldsymbol{\theta}_h^a := \arg \min_{\substack{\mathbf{v} \in \mathbf{RT}_p(\mathcal{T}_h^a) \cap \mathbf{H}_0(\operatorname{div}, \omega_a) \\ \nabla \cdot \mathbf{v} = -\nabla\psi_a \cdot \mathbf{J}_h}} \|\mathbf{v} - \nabla\psi_a \times \nabla \times \mathbf{A}_h\|_{\omega_a},$$

but it does not sum up to zero.

# Trick #1: Over-constrained minimization

We instead set

$$\hat{\boldsymbol{\theta}}_h^a := \arg \min_{\substack{\mathbf{v} \in \mathbf{RT}_p(\mathcal{T}_h^a) \cap \mathbf{H}_0(\operatorname{div}, \omega_a) \\ (\mathbf{v} - \nabla \psi_a \times \nabla \times \mathbf{A}_h, \mathbf{r})_{\omega_a} = 0 \quad \forall \mathbf{r} \in \mathcal{P}_0(\mathcal{T}_h^a) \\ \nabla \cdot \mathbf{v} = -\nabla \psi_a \cdot \mathbf{J}_h}} \|\mathbf{v} - \nabla \psi_a \times \nabla \times \mathbf{A}_h\|_{\omega_a},$$

where the mean-value constrain works because  $\mathbf{A}_h$  solves the discrete problem. We then set

$$\hat{\boldsymbol{\theta}}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \hat{\boldsymbol{\theta}}_h^a.$$

Importantly, we have

$$(\hat{\boldsymbol{\theta}}_h, \mathbf{r})_{\Omega} = 0 \quad \forall \mathbf{r} \in \mathcal{P}_0(\mathcal{T}_h).$$

## Trick #2: Divergence-free decomposition of $\widehat{\boldsymbol{\theta}}$ (1/2)

The  $\widehat{\boldsymbol{\theta}}_h^a$  have the correct divergence, but does not sum up to zero.  
We need a divergence-free decomposition of  $\widehat{\boldsymbol{\theta}}_h$  into contributions  $\widetilde{\boldsymbol{\theta}}_h^a$ .

We set for each  $\mathbf{a} \in \mathcal{V}_h$  and  $K \in \mathcal{T}_h^a$

$$\widetilde{\boldsymbol{\theta}}_h^a|_K := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{RT}_{p+1}(K) \\ \nabla \cdot \mathbf{v}_h = 0 \\ \mathbf{v}_h \cdot \mathbf{n}_K = \psi_a \widehat{\boldsymbol{\theta}}_h \cdot \mathbf{n}_K \text{ on } \partial K}} \|\mathbf{v}_h - \psi_a \widehat{\boldsymbol{\theta}}_h\|_K$$

This is indeed well-posed since

$$\begin{aligned} (\psi_a \widehat{\boldsymbol{\theta}}_h \cdot \mathbf{n}_K, 1)_{\partial K} &= (\psi_a, \widehat{\boldsymbol{\theta}}_h \cdot \mathbf{n}_K)_{\partial K} \\ &= (\nabla \psi_a, \widehat{\boldsymbol{\theta}}_h)_K + (\nabla \cdot \widehat{\boldsymbol{\theta}}_h, \psi_a)_K \\ &= 0. \end{aligned}$$

## Trick #2: Divergence-free decomposition of $\widehat{\theta}$ (2/2)

We can then actually show that

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \widetilde{\theta}_h^{\mathbf{a}} = \widehat{\theta}_h,$$

so that setting

$$\theta_h^{\mathbf{a}} := \widehat{\theta}_h^{\mathbf{a}} - \widetilde{\theta}_h^{\mathbf{a}} \in \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}),$$

we have

$$\nabla \cdot \theta_h^{\mathbf{a}} = -\nabla \psi_{\mathbf{a}} \cdot \mathbf{J}_h, \quad \sum_{\mathbf{a} \in \mathcal{V}_h} \theta_h^{\mathbf{a}} = \mathbf{0}.$$

# Local flux construction and efficiency

Since the  $\theta_h^a$  have all the required properties, we can now set

$$B_h^a := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{N}_{p+1}(\mathcal{T}_h^a) \cap \mathbf{H}_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v}_h = \psi_a \mathbf{J}_h + \theta_h^a}} \|\mathbf{v}_h - \psi_a \nabla \times \mathbf{A}_h\|_{\omega_a},$$

and we construct an equilibrated flux as

$$B_h := \sum_{a \in \mathcal{V}_h} B_h^a.$$

## Efficiency

$$\|B_h - \nabla \times A_h\|_{\mathcal{K}} \lesssim \|\nabla \times (A - A_h)\|_{\tilde{\mathcal{K}}}$$

The hidden constant does not depend on  $p$ .



T. Chaumont-Frelet, A. Ern and M. Vohralík, 2021