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Abstract In the paper, a new class of quasi-variational inequalities is introduced which can be applied in the study of Lur'e set-valued dynamical systems and game theory. An efficient method to solve the problem and the convergence analysis are provided.

Keywords Quasi-variational inequalities; Variational inequalities; Maximal monotone operators; Lur'e set-valued dynamical systems

1 Introduction

Our aim is to consider the following problem: find $x \in H$ such that

$$0 \in f(x) + (N_K^{-1} + D)^{-1}(x) \tag{1}$$

where D, f are monotone Lipschitz mappings, K is a closed convex set in a Hilbert space H and N_K denotes the normal cone operator to K. When $D = 0$ then (1) becomes the standard variational inequalities problem

$$0 \in f(x) + N_K(x) \tag{2}$$

which is the core of many constrained optimization problems and has been intensively investigated in the literature (see, e.g., [2,18,19,22,21,25,30,31,36]). On the other hand, (1) can be rewritten as follows (see Theorem 1)

$$0 \in f(x) + N_K(x + Df(x)). \tag{3}$$

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The term Df can play the role of a perturbation that affects the state variable under the normal cone operator. If we let $K(x) := K - Df(x)$ then one obtains a class of quasi-variational inequalities

$$0 \in f(x) + N_{K(x)}(x). \tag{4}$$

This famous problem was proposed by Bensoussan, Goursat and Lions in [8] and its applications can be found largely in economics, transportations, mechanics, electrical circuits etc (see, e.g., [9,23,26,32,35]). It is known that solving quasi-variational inequalities is far more difficult than solving variational inequalities and can be addressed under quite restrictive conditions, usually based on the strong monotonicity. One of the nice conditions [5,33] requires that f is μ-strongly monotone, L-Lipschitz continuous, K is l-Lipschitz continuous and $l < \mu/L$. Then the quasi-variational inequality (4) can be approximated by a sequence of variational inequalities [33] or can be reduced into a new variational inequality [5] to obtain the linear convergence. We show that in our class, these restrictive conditions can be removed. The main tool is to reduce (1) into a variational inequality if f is strongly monotone or to compute the resolvent of the operator $(N_K^{-1} + D)^{-1}$ for the general case.

On the other hand, the monotone inclusion (1) plays an important role to analyze the asymptotic behavior of set-valued Lur'e dynamical systems, which have the following form

$$(\mathcal{L}) \begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \text{ a.e. } t \in [0, +\infty); & \text{(5a)} \\ y(t) = Cx(t) + D\lambda(t), & \text{(5b)} \\ \lambda(t) \in -N_K(y(t)), \ t \geq 0; & \text{(5c)} \\ x(0) = x_0, & \text{(5d)} \end{cases}$$

where H_1, H_2 are two Hilbert spaces, $A : H_1 \to H_1, B : H_2 \to H_1, C : H_1 \to H_2$, $D : H_2 \to H_2$ are linear bounded operators, K is a closed convex subset of H_2 and $\lambda, y : \mathbb{R}_+ \to H_2$ are unknown connected mappings. Set-valued Lur'e dynamical systems make a fundamental model in control theory, engineerings and applied mathematics (see, e.g., [1,3,4,11,12,14,15,17,27,28] and references therein). We can rewrite $(\mathcal{L})$ as follows

$$\dot{x} \in -\mathcal{F}(x), \quad x(0) = x_0, \tag{6}$$

where $\mathcal{F}(x) = -Ax + B(N_K^{-1} + D)^{-1}Cx$. One usually finds conditions for A, B, C, D (usually based on the passivity) such that $\mathcal{F}$ is a maximal monotone operator (see, e.g., [3,11,14,17]). Then there exists uniquely a solution $x(\cdot)$ of $(\mathcal{L})$ and when the time is large the trajectory usually converges weakly to an equilibrium point x^* satisfying [6,10]

$$0 \in Ax^* - B(N_K^{-1} + D)^{-1}(Cx^*) \tag{7}$$

which has the form of (1) if B and C are the identity operators. The more general case $PB = C^T$ where P is a linear symmetric and strongly monotone operator can be analyzed similarly. Finally the implicit discretization of $(\mathcal{L})$

$$\frac{x_{n+1} - x_n}{h} \in Ax_{n+1} - B(N_K^{-1} + D)^{-1}(Cx_{n+1}). \tag{8}$$

can be also reduced into (1) with strongly monotone mapping f. This scheme was considered in [12] when K is a convex cone and $\dot{x}$ in $(5a)$ is multiplied by a singular matrix.

The paper is organized as follows. In Section 2 we recall some definitions and useful results in the theory of monotone operators. An efficient way to solve (1) is provided in Section 3. Similar results can be obtained if we extend the normal cone operators to maximal monotone operators in Section 4. The paper ends in Section 5 with some perspectives.

2 Notations and preliminaries

Let be given a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. Let K be a closed convex subset of H. One defines the distance and the projection from a point s to K as follows

$$d(s,K) := \inf_{x\in K}\|s-x\|,\ \ \mathrm{proj}_K(s) := \bar{x}\in K \ \text{ such that } d(s,K)=\|s-\bar{x}\|.$$

The normal cone to K at $x\in K$ is defined by

$$N_C(x) := \{x^*\in H : \langle x^*, y-x\rangle \le 0,\ \ \forall\, y\in C\}. \tag{9}$$

It is easy to see that if $\mathrm{proj}_K(s) := \bar{x}$ then $s-\bar{x}\in N_K(\bar{x})$. A mapping $f : H\to H$ is called L-Lipschitz continuous $(L>0)$ provided

$$\|f(x)-f(y)\| \le L\|x-y\| \ \ \forall\, x,y\in H. \tag{10}$$

If $L\le 1$ then f is called non-expansive. It is called μ-strongly monotone $(\mu>0)$ if

$$\langle f(x)-f(y), x-y\rangle \ge \mu\|x-y\|^2 \ \ \forall\, x,y\in H. \tag{11}$$

It follows that if f is μ-strongly monotone then

$$\|f(x)-f(y)\| \ge \mu\|x-y\|. \tag{12}$$

The domain, the range and the graph of a set-valued mapping $F : H \rightrightarrows H$ are defined respectively by

$$\mathrm{dom}(F) = \{x\in H :\ F(x)\neq\emptyset\},\ \ \mathrm{rge}(F) = \bigcup_{x\in H} F(x)$$

and

$$\mathrm{gph}(F) = \{(x,y) : x\in H, y\in F(x)\}.$$

The inverse of F is defined by

$$F^{-1}(y) = \{x\in H : y\in F(x)\}. \tag{13}$$

The resolvent of F is defined as follows

$$J_F := (I+F)^{-1} \tag{14}$$

where I denotes the identity operator.

The mapping F is called monotone if

$$\langle x^*-y^*, x-y\rangle \ge 0 \ \ \forall\, x,y\in \mathrm{dom}(F)\subset H, x^*\in F(x) \text{ and } y^*\in F(y).$$

In addition, it is called maximal monotone if there is no monotone operator G such that the graph of F is strictly included in the graph of G.

3 Main results

We suppose that the followings are satisfied throughout this section.

Assumption 1 : The operators $D, f : H \to H$ are monotone, Lipschitz continuous with Lipschitz constants d and L respectively.

Assumption 2 : K is a nonempty closed convex subset of H.

The following fact states that the inverse of a strongly monotone, Lipschitz continuous mapping is again strongly monotone (see, e. g., [24]) and Lipschitz continuous.

Lemma 1 *If f is μ-strongly monotone and L-Lipschitz continuous then f^{-1} is $\frac{\mu}{L^2}$-strongly monotone and $\frac{1}{\mu}$-Lipschitz continuous.*

Proof It remains to show the Lipschitz continuity of f^{-1}. Indeed, for all $x, y \in H$, one has

$$\|x - y\| = \|f \circ f^{-1}(x) - f \circ f^{-1}y\| \geq \mu \|f^{-1}(x) - f^{-1}(y)\| \tag{15}$$

and the conclusion follows. □

Let S be the solution set of (1). Assumptions $(H_1), (H_2)$ imply the maximal monotonicity of the operator $\Gamma := f + (N_K^{-1} + D)^{-1}$. Thus $S = \Gamma^{-1}(0)$ is closed and convex [6,10]. The following result provides a characterization of a solution of our problem, which is the connection between (1) and the quasi-variational inequality.

Theorem 1 *One has $x^* \in S \Leftrightarrow 0 \in f(x^*) + N_K(x^* + Df(x^*))$. In addition if f is strongly monotone then $x^* \in S \Leftrightarrow y^* = (f^{-1} + D)f(x^*)$ is the unique solution of the strongly monotone variational inequality*

$$0 \in (f^{-1} + D)^{-1}(y^*) + N_K(y^*). \tag{16}$$

Proof We have $x^* \in S \Leftrightarrow 0 \in f(x^*) + \lambda$ where $\lambda \in (N_K^{-1} + D)^{-1}(x^*) \Leftrightarrow \lambda \in N_K(x^* - D\lambda)$. From $\lambda = -f(x^*)$, we obtain the first conclusion.

If f is strongly monotone we have $0 \in f(x^*) + N_K(x^* + Df(x^*)) = f(x^*) + N_K(f^{-1}f(x^*) + Df(x^*)) = (f^{-1} + D)^{-1}(y^*) + N_K(y^*)$. □

Remark 1 i) Computing f^{-1} is easy if f is strongly monotone and Lipschitz continuous (see, e.g., [5]).
ii) Theorem 1 allows us to reduce (1) into a strongly monotone variational inequality under the strong monotonicity of f, which can be solved by any standard algorithms. The inclusion (16) can be considered as the dual form of (1). In general if f is not strongly monotone, we can compute the resolvent of the composition operator $\mathcal{B} := (N_K^{-1} + D)^{-1}$.

Let us begin with a convergence result for the inverse strongly monotone variational inequality.

Lemma 2 *Let $g : H \to H$ be a α-strongly monotone, l-Lipschitz continuous mapping. Then the sequence $(x_n)_{n\geq 0}$ generated by*

$$x_0 \in H, x_{n+1} = \mathrm{proj}_K(x_n - \gamma g^{-1}(x_n)), n = 0, 1, 2 \dots \tag{17}$$

with $\gamma = \alpha$ converges to the unique solution $\tilde{x}$ of the inverse strongly monotone variational inequality

$$0 \in g^{-1}(x) + N_K(x) \tag{18}$$

with linear rate $r_1 := \frac{\sqrt{l^2-\alpha^2}}{l}$.

Proof We have for all $x, y \in H$

$$\|x - y\| = \|g \circ g^{-1}(x) - g \circ g^{-1}(y)\| \leq l\|g^{-1}(x) - g^{-1}(y)\|. \tag{19}$$

In addition

$$\begin{aligned}\langle g^{-1}(x) - g^{-1}(y), x - y\rangle &= \langle g^{-1}(x) - g^{-1}(y), g \circ g^{-1}(x) - g \circ g^{-1}(y)\rangle \\ &\geq \alpha\|g^{-1}(x) - g^{-1}(y)\|^2. \end{aligned} \tag{20}$$

Since the projection operator is non-expansive, one obtains

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &= \|\mathrm{proj}_K(x_n - \gamma g^{-1}(x_n)) - \mathrm{proj}_K(\tilde{x} - \gamma g^{-1}(\tilde{x}))\|^2 \\ &\leq \|(x_n - \tilde{x}) - \gamma(g^{-1}(x_n) - g^{-1}(\tilde{x}))\|^2 \\ &\leq \|x_n - \tilde{x}\|^2 - (2\alpha\gamma - \gamma^2)\|g^{-1}(x_n) - g^{-1}(\tilde{x}))\|^2 \ \ \text{(using (20))} \\ &= \|x_n - \tilde{x}\|^2 - \alpha^2\|g^{-1}(x_n) - g^{-1}(\tilde{x}))\|^2 \\ &\leq (1 - \frac{\alpha^2}{l^2})\|x - \tilde{x}\|^2 \ \ \text{(using (19))}.\end{aligned}$$

□

Remark 2 When g is only strongly monotone, the strong convergence for the inverse strongly monotone variational inequality (18) is known (see, e. g., [24,29,39]). If g is additionally Lipschitz continuous, by using Lemma 1, g^{-1} is strongly monotone, Lipschitz continuous and the linear convergence follows. However, if one uses the arguments in [36], the convergence rate is only $\tilde{r}_1 := \frac{1/\alpha}{\sqrt{\alpha^2/l^4+1/\alpha^2}} = \frac{l^2}{\sqrt{l^4+\alpha^4}}$. It is easy to see that $r_1 < \tilde{r}_1$, which means our analysis is sharper in this case.

The following result provides a way to compute the resolvent of $\mathcal{B} := (N_K^{-1}+D)^{-1}$, which is the key for solving our main problem in general.

Theorem 2 *Given $x \in H$ and $\gamma > 0$, let $E = I + D/\gamma$. Then $y = J_{\gamma\mathcal{B}}(x)$ if and only if $w = Ey$ is the unique solution of the inverse strongly monotone variational inequality*

$$0 \in E^{-1}w - E^{-1}x + N_{K_1}(w), \tag{21}$$

where $K_1 = K + \frac{Dx}{\gamma}$. If D is linear symmetric, let U be the square root of E. Then

$$J_{\gamma\mathcal{B}}(x) = U^{-1}\Big(\mathrm{proj}_{U^{-1}(K_1)}(Ux)\Big). \tag{22}$$

Proof We have $y = J_{\gamma\mathcal{B}}(x) = (I+\gamma\mathcal{B})^{-1}(x) \Leftrightarrow x \in y + \gamma(N_K^{-1}+D)^{-1}(y) \Leftrightarrow \frac{x-y}{\gamma} \in (N_K^{-1}+D)^{-1}(y) \Leftrightarrow \frac{x-y}{\gamma} \in N_K(y - D\frac{x-y}{\gamma}) \Leftrightarrow x-y \in N_{K_1}(Ey) \Leftrightarrow$ (21) where $w = Ey$.

If D is linear symmetric, we have

$$\begin{aligned} & x-y \in N_{K_1}(Ey) = N_{K_1}(UUy) \\ \Leftrightarrow\ & Ux - z \in UN_{K_1}(Uz) = N_{K_2}(z) \ \text{ where } z = Uy \text{ and } \ K_2 = U^{-1}(K_1) \\ \Leftrightarrow\ & z = \mathrm{proj}_{K_2}(Ux) \Leftrightarrow y = U^{-1}\Big(\mathrm{proj}_{K_2}(Ux)\Big) \Leftrightarrow (22). \end{aligned}$$

□

Remark 3 One can solve the inverse strongly monotone variational inequality (21) easily by using Lemma 2. Thus (1) can be solved completely.

For clarity, we consider 2 cases: f is strongly monotone and f is only monotone.

3.1 f is strongly monotone

Since $\Gamma := f + (N_K^{-1}+D)^{-1}$ is strongly monotone, combining with Minty's theorem [6,10], it follows that the solution set S contains exactly one element x^*. We can use Theorem 1 or 2 and the forward-backward algorithm [20,31,36] to solve (1), depending on the computation complexity of $J_{\gamma\mathcal{B}}(x)$.

Proposition 1 *Let $\tilde{f} := (f^{-1}+D)^{-1}$. Then the sequence $(y_n)_{n\geq 0}$ generated by*

$$y_0 \in H, y_{n+1} = \mathrm{proj}_K(y_n - \gamma\tilde{f}(y_n)), n = 0,1,2\ldots$$

converges to $y^* = (I+Df)(x^*)$ with linear rate for some $\gamma > 0$.

Proof Using Lemma 1, it is inferred that $f^{-1}+D$ is $\mu_1 := \frac{\mu}{L^2}$-strongly monotone and $L_1 := (\frac{1}{\mu}+d)$-Lipschitz continuous, and thus $\tilde{f}$ is $\mu_2 := \frac{\mu_1}{L_1^2}$-strongly monotone and $L_2 := \frac{1}{\mu_1}$-Lipschitz continuous. Thus y_n converges to y^* with linear rate $r_2 := \frac{L_2}{\sqrt{L_2^2+\mu_2^2}}$ when $\gamma = \frac{\mu_2}{L_2^2}$. □

Proposition 2 *The sequence $(x_n)_{n\geq 0}$ generated by*

$$x_0 \in H, x_{n+1} = J_{\gamma\mathcal{B}}(x_n - \gamma f(x_n)), n = 0,1,2\ldots$$

converges to the unique solution x^* of (1) with linear rate $r_3 := \frac{L}{\sqrt{L^2+\mu^2}}$ when $\gamma = \frac{\mu}{L^2}$.

Proof The operator $\mathcal{B} = (N_K^{-1}+D)^{-1}$ is maximal monotone. Theorem 2 allows us to compute $J_{\gamma\mathcal{B}}$ and the convergence follows. □

Remark 4 We can also apply the algorithms proposed in [5,33] if the Lipschitz constant of Df is less than μ/L.

3.2 f is only monotone

One can use the forward-backward algorithm combining Tseng's technique to obtain the strong convergence for solving (1), see e.g., [37,40,41].

Remark 5 From Theorem 1, if D and f are linear and $0 \in K$ then $x^* = 0$ is a solution of (1). In general if f is only monotone, the non-emptiness of S is not assured. For example in one dimension, let us take $f(x) \equiv 1, D = I$ and $K = (-\infty, 0]$.

4 Extensions

We can similarly consider the following monotone inclusion

$$0 \in f(x) + (F^{-1} + D)^{-1}(x) \tag{23}$$

where $D, f : H \to H$ are monotone Lipschitz continuous and $F : H \rightrightarrows H$ is a maximal monotone operator. First it enriches the well-known problem of finding a zero of sum of two maximal monotone operators [20,31,36]

$$0 \in Ax + \mathcal{B}x, \tag{24}$$

where $\mathcal{B} := (F^{-1} + D)^{-1}$ is not the usual composition of F and D. Note that it is different from the parallel sum of F and D, defined by $(F^{-1} + D^{-1})^{-1}$ (see [7, Section 24.4]). Secondly, (23) can be reduced into a class of state-dependent inclusion [5]

$$0 \in f(x) + F(x + Df(x)). \tag{25}$$

Thanks to this particular form, restricted conditions used in [5] are not required. Finally in the set-valued Lur'e dynamical systems, not only normal cone operators but also general maximal monotone operators have been intensively considered (see, e.g., [16] for a survey). Let us recall that the Sign function defined by

$$\mathrm{Sign}(x) = \begin{cases} -1 & \text{if } \ x < 0, \\ [-1, 1] & \text{if } \ x = 0, \\ 1 & \text{if } \ x > 0, \end{cases}$$

and Sign-like functions are maximal monotone operators which play an important role in mechanical and electrical engineering (see, e.g., [1,15]).

Let us provide some examples to show that (1) and (23) can appear not only in Lur'e dynamical systems but also in game theory. We consider a game of m players. Let $x^i \in \mathbb{R}^n$ be the strategy vector of the player $i \in S := \{1, 2, \dots, m\}$ and x^{-i} be the vector $(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^m)$. The cost function $g_i(x^i, x^{-i})$ of the player i depends on both his strategy x^i and the strategies of the other players x^{-i}.

Definition 1 A Nash equilibrium of the game is a vector $x = (x^1, x^2, \dots, x^m)$ such that

$$x^i \quad \text{minimizes} \quad g_i(\cdot, x^{-i}). \tag{26}$$

Example 1 Assume that for each $i \in S$, the function $g_i : \mathbb{R}^{n\times m} \to \mathbb{R}$ is such that $g_i(\cdot, x^{-i})$ is convex and differentiable. We consider the case when S can be divided into two groups: $S = A \cup B$ where $A = \{1, 2, \ldots, k\}$, $B = \{k+1, k+2, \ldots, m\}$ for some $k > 0$ and for each $i \in A$, the function g_i is linear with respect to x_i. Suppose that for each $x^{-i} \in \mathbb{R}^{n\times(m-1)}$, the strategy x^i is restricted in some convex set $C^i(x^{-i}) := K_i - c_i \nabla_{x^1} g_i(x^i, x^{-i})$ where $c_i > 0$ if $i \in A$ and $c_i = 0$ if $i \in B$. Let D be the diagonal matrix $D = \mathrm{diag}(c_1, c_2, \ldots, c_m)$, $K = \prod_{i=1}^m K_i$ and

$$f(x) = (\nabla_{x^1} g_1(x^1, x^{-1}), \ldots, \nabla_{x^m} g_m(x^m, x^{-m})).$$

Then (26) becomes

$$0 \in f(x) + N_K(x + Df(x)) \tag{27}$$

which is in the form of (3).

Example 2 Next assume that g_i can be decomposed as

$$g_i(x^i, x^{-i}) = f_i(x^i, x^{-i}) + h_i(x^i + c_i(x^{-i})) \tag{28}$$

where

- $f_i : \mathbb{R}^{n\times m} \to \mathbb{R}$ is such that $f_i(\cdot, x^{-i})$ is convex and differentiable;
- $c_i : \mathbb{R}^{n\times(m-1)} \to \mathbb{R}^n$ is a Lipschitz continuous mapping which plays a role as perturbation term of the strategies x^{-i} of other players affecting the strategy x^i;
- $h_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

Then (26) is equivalent to

$$0 \in \nabla_{x^i} f_i(x^i, x^{-i}) + \partial h_i(x^i + c_i(x^{-i})), \;\; i = 1, 2 \ldots, m. \tag{29}$$

Let

$$f(x) = (\nabla_{x^1} f_1(x^1, x^{-1}), \ldots, \nabla_{x^m} f_m(x^m, x^{-m})),$$
$$F(x) = (\partial h_1(x^1), \ldots, \partial h_m(x^m)),$$

and

$$c(x) = (c_1(x^{-1}), \ldots, c_m(x^{-m})).$$

Then (29) can be rewritten as

$$0 \in f(x) + F(x + c(x)). \tag{30}$$

We consider the case when S can be divided into two groups: $S = A \cup B$ where $A = \{1, 2, \ldots, k\}$, $B = \{k+1, k+2, \ldots, m\}$ for some $k > 0$ and for each $i \in A$, the function f_i is linear with respect to x_i. In addition, suppose that $c(x^{-i}) = c_i \nabla_{x^1} f_1(x^1, x^{-1})$ for some $c_i > 0$ if $i \in A$ and $c(x^{-i}) = 0$ if $i \in B$. It means that if $i \in A$ the perturbation term affecting the strategy x_i is proportional to the rate of change of the partial cost function f_i with respect to x_i and if $i \in B$ the corresponding perturbation term is zero.

Let D be the diagonal matrix $D = \mathrm{diag}(c_1, c_2, \ldots, c_m)$ with $c_i = 0$ if $i \in B$. Then (30) becomes

$$0 \in f(x) + F(x + Df(x)), \tag{31}$$

which is in the form of (25).

5 Conclusions

In this note, an efficient method to solve a new class of quasi-variational inequalities, which appears in engineering and game theory, is proposed. Quasi-variational inequalities and state-dependent monotone inclusions are sophisticated models which can capture appropriately the behavior of complex phenomena in the real world. However, solving them is still an open interesting question in general and requires more techniques as well as innovations.

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