Backstepping for Uncertain Nonlinear Systems with a Delay in the Control

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Abstract: The recent new backstepping control design strategy based on the introduction of artificial delays and/or dynamic extensions is adapted to a family of systems. That way, globally asymptotically stabilizing control laws for fundamental systems which cannot be handled by other techniques are determined.

Keywords: Backstepping, delay, control, nonlinear

1. INTRODUCTION

Backstepping is probably the most popular technique of construction of stabilizing control laws for nonlinear systems with a triangular structure. For thirty years, it has been developed in plethora of works in many research areas e.g. adaptive control, output feedback stabilization, stabilization of time-varying systems with delay, incremental stabilization, Coron and Praly (1991), Freeman and Kokotovic (1996), Khalil (2002), Zhou et al. (2009), Zhang et al. (2003), Postoyan et al. (2009), (Karafyllis and Jiang, 2011, Chapt. 6), Bekiaris and Krstic (2010). It is worth mentioning that some contributions are devoted to the problem of designing bounded feedbacks via the backstepping technique e.g. Freeman and Praly (1998), which is motivated by size constraints of the control which are inherent to many models arising from applications.

Since the pioneering recent work Mazenc and Malisoff (2016), new advances of the backstepping approach have been obtained via a fundamentally new variant of backstepping Mazenc et al. (2018c), Mazenc et al. (2016), Mazenc et al. (2019b). A crucial aspect of it is that it relies on the introduction of either *'artificial' delays* in the control or *dynamic extensions*. The new control laws designed by this approach offer, in many circumstances, decisive advantages. In particular, the bounded feedbacks with artificial delays are given by formulas much simpler than those provided by the classical backstepping approach, at each step of the backstepping, fictitious feedbacks that are not of class C^1 can be used and output feedback problems can be solved in cases where no accurate measurement of some state variables is available Mazenc et al. (2018b), Mazenc et al. (2018a), Mazenc et al. (2017a).

In the present work, we solve stabilizing problems for a broad family of non-linear time-varying systems with a pointwise constant delay in the input. Our control design uses operators which are reminiscent of those introduced in Mazenc et al. (2019a). A strong motivation of our work arises from the fact that we propose control laws for systems with uncertain terms so that we cover cases which cannot be handled using the main results of Mazenc et al. (2004). It is worth mentioning that the result we propose applies when a delay in the input is present, but is also of interest for systems without delay.

Finally, we point out that the expressions of the feedbacks we build are very different from those of papers as Mazenc et al. (2011), where the classical backstepping approach is adapted to the case of time-invariant systems with one integrator and a delay in the input under a condition on the size of the delay.

1.1 Notation and classical definitions

We omit arguments of functions when they are clear. Given any constant $T > 0$, we let C_{in} denote the set of all continuous functions $\phi : [-T, 0] \to \mathbb{R}^a$, which we call the set of all *initial functions*. We define $\Xi_t \in C_{\text{in}}$ by $\Xi_t(s) = \Xi(t+s)$ for all choices of Ξ, $s \leq 0$, and $t \geq 0$ for which the equality is defined. Given $L > 0$, σ_L denotes the classical symmetric saturation function i.e.

$$\sigma_L(s) = \max\{-L, \min\{L, s\}\} \tag{1}$$

for all $s \in \mathbb{R}$. For a function $\varphi : \mathbb{R} \to \mathbb{R}$ differentiable p times, we denote by $\varphi^{(j)}$ it's j's derivative for all $j \in \{0, ..., p\}$. We adopt the convention $\varphi^{(0)} = \varphi$.
Let us consider a system

$$\dot{x}(t) = \mathcal{A}(t, x_t, w_1(t), ..., w_m(t)) \tag{2}$$

with a finite delay $\tau \geq 0$, $x \in \mathbb{R}^n$ and $w = (w_1, ..., w_m) \in \mathbb{R}^m$.

Definition 1. We say that the system (2) is bounded-input bounded-state (BIBS) with input w if there exist α, γ of class $\mathcal{K}$ such that a solution x of (2) with $\phi_{x_0} \in C_{\text{in}}$ as initial condition at the instant $t_0 \geq 0$ and $w \in \mathcal{L}_\infty$ satisfies

$$|x(t)| \leq \alpha(|\phi_{x_0}|) + \gamma\left(\sup_{m\in[t_0,t]} |w(m)|\right) \tag{3}$$

for all $t \geq t_0 \geq 0$.

Definition 2. We say that the system (2) is converging-input converging-state (CICS) with input w if it is forward complete and when

$$\lim_{t\to+\infty} |w(t)| = 0 \quad \text{then} \quad \lim_{t\to+\infty} |x(t)| = 0. \tag{4}$$

Definition 3. The system (2) is Input-to-State-Stable (ISS) if there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that any solution x of it satisfies:

$$|x(t)| \leq \beta\left(\sup_{r\in[s-\tau,s]} |x(r)|, t-s\right) + \gamma\left(\sup_{r\in[s,t]} |w(r)|\right) \tag{5}$$

for all $t \geq s \geq 0$.

2. OPERATORS: DEFINITIONS AND LEMMAS

In this section, we introduce operators and establish results that will be instrumental when we establish the main result of our contribution.

Let $k > 0$, $h > 0$ and $\tau > 0$ be real numbers. Throughout the work, we can let k be equal to 1, or any other positive constant. However, we keep k as a tuning parameter helping to improve the performances of the control laws we will propose. Let us introduce the constant

$$\mathfrak{j} = \frac{ke^{kh}}{e^{kh}-1} \tag{6}$$

and consider a system

$$\dot{x}(t) = \mathcal{R}(t, x_t), \tag{7}$$

with $x(t) \in \mathbb{R}^q$ that is forward complete and has a finite delay $\tau \geq 0$.

2.1 Definitions

Let $\mathcal{U} : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$ be a function such that there are a constant $L_{\mathcal{U}} \geq 0$ and a function $\mathcal{B}$ of class $\mathcal{K}$ such that

$$|\mathcal{U}(a, b_1) - \mathcal{U}(a, b_2)| \leq L_{\mathcal{U}}|b_1 - b_2| \tag{8}$$

for all $(a, b_1, b_2) \in \mathbb{R}^{1+2q}$ and

$$|\mathcal{U}(a_1, b) - \mathcal{U}(a_2, b)| \leq |a_1 - a_2|\mathcal{B}(|b|) \tag{9}$$

for all $(a_1, a_2, b) \in \mathbb{R}^{2+q}$.

i) Let $\Gamma_{\mathcal{U},i} : C_{\text{in}} \to \mathbb{R}$ with $i \in \mathbb{N}$ denote the operators such that along the trajectories of (7),

$$\Gamma_{\mathcal{U},0}(t, x_t) = \mathcal{U}(t-\tau, x(t-\tau)) \tag{10}$$

and, for $i \geq 1$,

$$\Gamma_{\mathcal{U},i}(t, x_t) = \mathfrak{j}\int_{t-h}^{t} e^{k(s-t)}\Gamma_{\mathcal{U},i-1}(s, x_s)ds, \tag{11}$$

where $\mathfrak{j}$ is the constant defined in (6).

ii) For all $j \in \mathbb{N}$, $j > 0$, $i \in \{0, ..., j\}$, we let $\Omega_{\mathcal{U},j,i} : C_{\text{in}} \to \mathbb{R}$ denote the operators such that along the trajectories of (7),

$$\Omega_{\mathcal{U},j,i}(t, x_t) = \Gamma_{\mathcal{U},j}^{(i)}(t, x_t). \tag{12}$$

iii) Let $\zeta : C_{\text{in}} \to \mathbb{R}^q$ be the operator defined by:

$$\zeta(\phi) = \frac{k}{e^{kh}-1}\left[e^{kh}\phi(0) - \phi(-h)\right] \tag{13}$$

for all $\phi \in C_{\text{in}}$.

2.2 Estimate and upper bounds for the operators

In this section, we give technical results whose proofs are omitted. Lemmas 1 and 2 are direct consequences of the definitions of the operators $\Gamma_{\mathcal{U},j}$ and Lemmas 3 and 4 can be established by induction.

Basically, the following lemma shows that the smaller the constant h is selected, the closer $\Gamma_{\mathcal{U},j}(t, x_t)$ is to $\mathcal{U}(t-\tau, x(t-\tau))$.

Lemma 1 Along the trajectories of the system (7), for all $j \in \mathbb{N}$, the inequalities

$$\begin{aligned} &|\Gamma_{\mathcal{U},j}(t, x_t) - \mathcal{U}(t-\tau, x(t-\tau))| \leq \\ &L_{\mathcal{U}}\int_{t-\tau-jh}^{t-\tau} |\dot{x}(s)|ds + jh\mathcal{B}(|x(t-\tau)|) \end{aligned} \tag{14}$$

hold for all $t \geq \tau + jh$.

Now, we determine upper bounds for the operators $\Gamma_{\mathcal{U},j}$ and $\Omega_{\mathcal{U},j,i}$.

Lemma 2 Let us consider the system (7). For all $j \in \mathbb{N}$, $j > 0$,

$$|\Gamma_{\mathcal{U},j}(t, x_t)| \leq \mathfrak{j}\int_{t-\tau-jh}^{t-\tau} |\mathcal{U}(s, x(s))|ds \tag{15}$$

for all $t \geq \tau + jh$.

Lemma 3 For all $j \in \mathbb{N}$, $j \geq 1$, $i \in \{0, ..., j-1\}$, the inequalities

$$|\Omega_{\mathcal{U},j,i}(t, x_t)| \leq 2^i\mathfrak{j}^{i+1}\int_{t-\tau-jh}^{t-\tau} |\mathcal{U}(s, x(s))|ds \tag{16}$$

are satisfied for all $t \geq \tau + jh$.

Lemma 4 Let the function $\mathcal{U}$ by bounded by a constant $\overline{\mathcal{U}} \geq 0$. Then for all $j \in \mathbb{N}$,

$$|\Gamma_{\mathcal{U},j}|_\infty \leq \overline{\mathcal{U}} \tag{17}$$

and for all $j \in \mathbb{N}$, $j > 0$, and $m \in \{1, ..., j\}$,

$$|\Omega_{\mathcal{U},j,m}|_\infty \leq 2^m\mathfrak{j}^m\overline{\mathcal{U}}. \tag{18}$$

3. STABILIZATION OF NONLINEAR TIME-VARING SYSTEMS WITH DELAY

This section is devoted to the main result of the paper, which is the design of a globally asymptotically stabilizing control law for time-varying nonlinear systems with delay in the input.

3.1 Studied system

We consider the system:

$$\begin{cases} \dot{X}(t) = F(t, X_t, y_1(t) + \mathfrak{r}_1(t)) \\ \dot{Y}(t) = A(t)Y(t) + Bu(t-\tau) + \mathfrak{r}_2(t), \end{cases} \tag{19}$$

with $X = (x_1, ..., x_q) \in \mathbb{R}^q$, $Y = (y_1, ..., y_n) \in \mathbb{R}^n$, $B = (0 \dots 0\, 1)^\top \in \mathbb{R}^n$, $u \in \mathbb{R}$, $\tau \geq 0$, where F is a nonlinear functional, locally Lipschitz with respect to its two last arguments and piecewise-continuous with respect to the first, $\mathfrak{r}_1 : [0, +\infty) \to \mathbb{R}$ and $\mathfrak{r}_2 : [0, +\infty) \to \mathbb{R}^n$ are disturbances, and where $A : [0, +\infty) \to \mathbb{R}^{n\times n}$ is of the form:

$$A(t) =$$

$$\begin{bmatrix} a_{1,1}(t) & a_{1,2}(t) & 0 & ... & 0 \\ a_{2,1}(t) & a_{2,2}(t) & a_{2,3}(t) & ... & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ \vdots & & & & \vdots \\ a_{n-1,1}(t) & a_{n-1,2}(t) & \dots & \ddots & a_{n-1,n}(t) \\ a_{n,1}(t) & a_{n,2}(t) & \dots & \dots & a_{n,n}(t) \end{bmatrix}. \tag{20}$$

The structure of A ensures that the classical backstepping approach can be applied to the system (19) under some smoothness and stabilizability conditions for the X and Y subsystems. Our objective is to design control laws under less restrictive conditions.

Let us introduce 3 assumptions.

Assumption A1. *There are a functional $\mathcal{V}$, a functional φ and a constant $\overline{\mathfrak{d}} \geq 0$ such that the origin of the system*

$$\begin{cases} \dot{\zeta}(t) = A(t)\zeta(t) + B\mathcal{V}(t-\tau, \chi_{t-\tau}, \zeta_{t-\tau}) + \mathfrak{d}(t) \\ \dot{\chi}(t) = \varphi(t, \chi_t, \zeta_t), \end{cases} \tag{21}$$

with $\zeta \in \mathbb{R}^n$, $\chi \in \mathbb{R}^r$, is BIBS and CICS with input $\mathfrak{d}$ when

$$|\mathfrak{d}|_\infty \leq \overline{\mathfrak{d}}. \tag{22}$$

The functional $\mathcal{V}$ is locally Lipschitz and there is a Lipschitz continuous function $\overline{\mathcal{V}}$ such that

$$|\mathcal{V}(t, \phi_1, \phi_2)| \leq \overline{\mathcal{V}}(\phi_1, \phi_2) \tag{23}$$

for all $t \in [0, +\infty)$ and $\phi_1 \in C_{\text{in}}$ and $\phi_2 \in \mathbb{R}$.

Assumption A2. *There are a function $\mathcal{W}$, a functional ϖ and constants $k > 0$ and $h_\star > 0$ such that when $h \in (0, h_\star]$, then the system*

$$\begin{cases} \dot{\xi}(t) = F(t, \xi_t, \Gamma_{\mathcal{W},n}(t, \xi_t, \aleph_t) + \mathfrak{s}(t)) \\ \dot{\aleph}(t) = \varpi(t, \xi_t, \aleph_t), \end{cases} \tag{24}$$

with $\xi \in \mathbb{R}^q$, $\aleph \in \mathbb{R}^u$ and $\Gamma_{\mathcal{W},n}$ defined in (10)-(11) is BIBS and CICS with input $\mathfrak{s}$. The function $\mathcal{W}$ is locally Lipschitz and there is a Lipschitz continuous function $\overline{\mathcal{W}}$ such that

$$|\mathcal{W}(t, \xi, \aleph)| \leq \overline{\mathcal{W}}(\xi, \aleph) \tag{25}$$

for all $t \in [0, +\infty)$ and $\xi \in \mathbb{R}^q$, $\aleph \in \mathbb{R}^u$.

Remark. In the systems (21) and (24), there are dynamic extensions. We introduce them for the sake of generality. Evidently, Assumptions A1 and A2 can be satisfied in cases where there is no dynamic extension.

Assumption A3. *The functions $a_{i,j}$, $i \in \{1, ..., n\}$, $j \in \{1, ..., \min\{i+1, n\}\}$ are of class C^{n-i} and there are constants $\overline{a} > 0$ and $\underline{a} > 0$ such that*

$$|A(t)| \leq \overline{a} \quad , \quad \forall t \geq 0 \tag{26}$$

and

$$|a_{i,j}^{(p)}(t)| \leq \overline{a} \quad , \quad \forall t \geq 0 \tag{27}$$

for all $i \in \{1, ..., n\}$, $j \in \{1, ..., \min\{i+1, n\}\}$, $p \in \{1, ...n-i\}$ and for all $i \in \{1, ..., n-1\}$

$$\underline{a} \leq a_{i,i+1}(t) \quad , \quad \forall t \geq 0. \tag{28}$$

There is a Lipschitz continuous function $\overline{F}$ such that

$$|F(t, \phi_1, \phi_2)| \leq \overline{F}(\phi_1, \phi_2) \tag{29}$$

for all $t \in [0, +\infty)$ and $(\phi_1, \phi_2) \in C_{\text{in}} \times \mathbb{R}$ and

$$\overline{F}(0, 0) = 0. \tag{30}$$

Remark. The assumptions above do not imply that F and $\mathcal{W}$ are of class C^1. Thus, under our assumptions, the classical backstepping approach does not apply to the system (19).

3.2 Feedback stabilization

Associating the dynamic extension in (24) to the system (19), we obtain:

$$\begin{cases} \dot{X}(t) = F(t, X_t, y_1(t) + \mathfrak{r}_1(t)) \\ \dot{Y}(t) = A(t)Y(t) + Bu(t-\tau) + \mathfrak{r}_2(t) \\ \dot{\pounds}(t) = \varpi(t, X_t, \pounds_t). \end{cases} \tag{31}$$

We define now some functionals. First we let

$$y_{\dagger,1}(t, X_t, \pounds_t) = \Gamma_{\mathcal{W},n}(t, X_t, \pounds_t) \tag{32}$$

and by induction, we define the functionals $y_{\dagger,i}$ as the functionals such that for $i \in \{1, ..., n-1\}$, along the trajectories of the system (31),

$$y_{\dagger,i+1}(t, X_t, \pounds_t) = \frac{1}{a_{i,i+1}(t)} \left[\dot{y}_{\dagger,i}(t, X_t, \pounds_t) - \sum_{l=1}^{i} a_{i,l}(t) y_{\dagger,l}(t, X_t, \pounds_t) \right]. \tag{33}$$

Assumption A3 ensures that they are well-defined. Now, one can prove by induction that there are continuous and bounded functions $b_{i,s}(t)$ such that, for $i \in \{1, ..., n\}$,

$$y_{\dagger,i}(t, X_t, \pounds_t) = \sum_{s=n-i+1}^{n} b_{i,s}(t)\Gamma_{\mathcal{W},s}(t, X_t, \pounds_t). \tag{34}$$

We deduce that there are continuous and bounded functions $c_s(t)$ such that

$$\begin{aligned} &\sum_{l=1}^{n} a_{n,l}(t) y_{\dagger,l}(t, X_t, \pounds_t) - \dot{y}_{\dagger,n}(t, X_t, \pounds_t) \\ &= \sum_{s=0}^{n} c_s(t)\Gamma_{\mathcal{W},s}(t, X_t, \pounds_t). \end{aligned} \tag{35}$$

Now, let us introduce:

$$\tilde{y}_i(t) = y_i(t) - y_{\dagger,i}(t, X_t, \pounds_t) \quad , \quad i = 1, ..., n \tag{36}$$

and

$$\tilde{Y}(t) = (\tilde{y}_1(t), \; .... \;, \tilde{y}_n(t)). \tag{37}$$

We are ready to state and prove the following result:

Theorem 5 Let the system (19) satisfy Assumptions A1 to A3. Then the system (19) in closed-loop with the dynamic feedback:

$$\begin{aligned} u(t-\tau) &= -\sum_{s=0}^{n} c_s(t)\Gamma_{\mathcal{W},s}(t, X_t, \pounds_t) \\ &\quad + \mathcal{V}\left(t-\tau, \chi_{t-\tau}, \tilde{Y}_{t-\tau}\right) \\ \dot{\pounds}(t) &= \varpi(t, X_t, \pounds_t) \\ \dot{\chi}(t) &= \varphi\left(t, \chi_t, \tilde{Y}_t\right), \end{aligned} \tag{38}$$

with $\tilde{Y}$ defined in (37) is BIBS and CICS with input $(\mathfrak{r}_1(t), \mathfrak{r}_2(t))$ when

$$|\mathfrak{r}_2|_\infty \leq \overline{\mathfrak{d}}, \tag{39}$$

where $\overline{\mathfrak{d}}$ is the constant in (22).

Discussion of Theorem 5.

1) Since $\Gamma_{\mathcal{W},s}(t, X_t, \pounds_t)$ and $\mathcal{V}\left(t-\tau, \chi_{t-\tau}, \tilde{Y}_{t-\tau}\right)$ depend on values of the various variables involved at instants smaller than $t-\tau$, the feedback in (38) is well-defined. For any system (19), an explicit expression for the functions c_s in (35) can be determined. Thus the control in (38) can be used in practice.

2) Assumptions A1 to A3 can be checked even when F is not accurately known and the control law (38) does not incorporate F in its expression. It follows that one of the advantages of Theorem 5 is that it can be applied in cases where there are uncertainties in F.

3) When the functions $\mathcal{V}$ and $\mathcal{W}$ are bounded, then the feedback u defined in (38) is bounded because the functions c_s and $\Gamma_{\mathcal{W},s}$ are bounded.

4) The main differences between the system (19) and the one studied in Mazenc et al. (2019b) are the following: (i) A depends on t, (ii) the delay τ is present in u, (iii) delays can be present in the X-subsystem. The approach in Mazenc et al. (2019b) uses dynamic extensions with high gains and no artificial delay. It is worth mentioning that we conjecture that the approach of Mazenc et al. (2019b) can be adapted to the system (19). Doing this would lead to a variant of Theorem 5 which would be by no means direct and we conjecture it would involve complicated mathematical developments. Notice also that the operators we use are simpler than those introduced in Mazenc et al. (2018c) and we consider a family of systems different from the one studied in Mazenc et al. (2018c).

5) Under the additional mild condition that the system (19) is forward complete, one can deduce from Theorem 5 the expression of a globally asymptotically stabilizing control law with pointwise delays instead of distributed delays. Indeed, let us introduce the dynamic extension:

$$\dot{w}_1(t) = -kw_1(t) + \mathcal{W}(t-\tau, X(t-\tau)) \tag{40}$$

and, for all $j \in \mathbb{N}$, $j \geq 2$

$$\dot{w}_j(t) = -kw_j(t) + \zeta(w_{j-1,t}) \tag{41}$$

with ζ defined in (13). Then one can prove that for all positive integer m, for all $t \geq mh$, the equality

$$\zeta(w_{m,t}) = \Gamma_{\mathcal{W},m}(t, X_t, \pounds_t) \tag{42}$$

is satisfied. Thus

$$\begin{aligned} &v(t-\tau) = -\sum_{s=0}^{n} c_s(t)\zeta(w_{s,t}) + \mathcal{V}\left(t-\tau, \pounds_{t-\tau}, \tilde{Y}_{t-\tau}\right) \\ &\dot{\pounds}(t) = \varpi(t, X_t, \pounds_t) \\ &\dot{\chi}(t) = \varphi\left(t, \chi_t, \tilde{Y}_t\right) \end{aligned} \tag{43}$$

is such that

$$v(t-\tau) = u(t-\tau) \tag{44}$$

for all $t \geq nh$ where u is the feedback defined in (38). From a practical point of view, implementing v may be easier than implementing u.

Proof. Let us consider the system (19) and $\tilde{Y}$ defined in (37). Since, according to (33),

$$\dot{y}_{\dagger,i}(t, X_t, \pounds_t) = \sum_{l=1}^{i+1} a_{i,l}(t) y_{\dagger,l}(t, X_t, \pounds_t) \tag{45}$$

for all $i = 1$ to $n-1$, we have

$$\dot{\tilde{y}}_i(t) = \sum_{l=1}^{i+1} a_{i,l}(t)\tilde{y}_l(t) \tag{46}$$

and

$$\begin{aligned} \dot{\tilde{y}}_n(t) &= \sum_{l=1}^{n} a_{n,l}(t) y_l(t) + u(t-\tau) - \dot{y}_{\dagger,n}(t, X_t, \pounds_t) \\ &= \sum_{l=1}^{n} a_{n,l}(t)\tilde{y}_l(t) + u(t-\tau) \\ &\quad + \sum_{l=1}^{n} a_{n,l}(t) y_{\dagger,l}(t, X_t, \pounds_t) - \dot{y}_{\dagger,n}(t, X_t, \pounds_t). \end{aligned} \tag{47}$$

It follows from (46) and (35) that

$$\dot{\tilde{Y}}(t) = A(t)\tilde{Y}(t) + B\left[u(t-\tau) + \sum_{s=0}^{n} c_s(t)\Gamma_{\mathcal{W},s}(t, X_t, \pounds_t)\right]. \tag{48}$$

From (36), it follows that $y_1(t) = \Gamma_{\mathcal{W},n}(t, X_t, \pounds_t) + \tilde{y}_1(t)$. Thus we have:

$$\left\{\begin{aligned} \dot{X}(t) &= F\left(t, X_t, \Gamma_{\mathcal{W},n}(t, X_t, \pounds_t) + \tilde{y}_1(t) + \mathfrak{r}_1(t)\right) \\ \dot{\tilde{Y}}(t) &= A(t)\tilde{Y}(t) \\ &\quad + B\left[u(t-\tau) + \sum_{s=0}^{n} c_s(t)\Gamma_{\mathcal{W},s}(t, X_t, \pounds_t)\right] \\ &\quad + \mathfrak{r}_2(t). \end{aligned}\right. \tag{49}$$

Applying the feedback $u(t-\tau)$ defined in (38), we obtain

$$\left\{\begin{aligned} \dot{X}(t) &= F\left(t, X_t, \Gamma_{\mathcal{W},n}(t, X_t, \pounds_t) + \tilde{y}_1(t) + \mathfrak{r}_1(t)\right) \\ \dot{\pounds}(t) &= \varpi(t, X_t, \pounds_t) \\ \dot{\tilde{Y}}(t) &= A(t)\tilde{Y}(t) + B\mathcal{V}\left(t-\tau, \chi_{t-\tau}, \tilde{Y}_{t-\tau}\right) + \mathfrak{r}_2(t) \\ \dot{\chi}(t) &= \varphi(t, \chi_t, \tilde{Y}_t). \end{aligned}\right. \tag{50}$$

Assumption A1 and (39) ensure that the $(\tilde{Y}, \chi)$-subsystem of (50) is BIBS and CICS with input $\mathfrak{r}_2(t)$. Next, Assumption A2 allows us to conclude.

3.3 Checking the assumptions

In general, checking that Assumption A1 is satisfied is a standard problem. Indeed, this assumption basically means that a linear time-varying system with a delay in the input is stabilizable by a dynamic feedback. Assumption A3 is a boundedness condition on the functions of the system with respect to t. It can be easily checked. Assumption A2 is more problematic because it consists in a stabilizability condition in which the unusual operator $\Gamma_{\mathcal{W},n}$ is involved. However, one can check that it is satisfied using a strategy in 2 steps:

(i) First, one determines functions $\mathcal{W}$ and ϖ such that

$$\left\{\begin{aligned} \dot{\xi}(t) &= F(t, \xi_t, \mathcal{W}(t-\tau, \xi(t-\tau), \aleph(t-\tau)) \\ &\quad + \mathfrak{s}(t)) \\ \dot{\aleph}(t) &= \varpi(t, \xi_t, \aleph_t) \end{aligned}\right. \tag{51}$$

is BIBS and CICS with input $\mathfrak{s}$.

(ii) Next, one establishes that the system (24) is BIBS and CICS when the tuning parameter h is sufficiently small.

To prove this, various approaches can be tried, according to the type of systems that are studied. In particular, one can adopt a Lyapunov Krasovskii functional technique, as done for instance in Mazenc et al. (2018c) or the trajectory based approach presented for instance in Mazenc et al. (2017b). This technique can be applied in the particular case where the following assumption is satisfied:

Assumption A4. *There are constants* $\overline{f}_i \geq 0$, $i = 1, 2$ *such that*

$$|F(t, \phi, z)| \leq \overline{f}_1 \sup_{s\in[-\tau,0]} |\phi(s)| + \overline{f}_2|z| \qquad (52)$$

for all $t \geq 0, \phi \in C_{\text{in}}$ *and* $z \in \mathbb{R}$. *There are a function* $\mathcal{W} : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$, *two constants* $K_{\mathcal{W}} \geq 0$ *and* $B_{\mathcal{W}} \geq 0$ *such that*

$$|\mathcal{W}(a, b_1) - \mathcal{W}(a, b_2)| \leq K_{\mathcal{W}}|b_1 - b_2| \qquad (53)$$

for all $a \in \mathbb{R}$, $b_1 \in \mathbb{R}^q$, $b_2 \in \mathbb{R}^q$ *and*

$$|\mathcal{W}(a_1, b) - \mathcal{W}(a_2, b)| \leq B_{\mathcal{W}}|a_1 - a_2||b| \qquad (54)$$

for all $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$, $b \in \mathbb{R}^q$ *and constants* $T > 0$ *and* $t_\sharp \geq T + \tau$ *such that the solutions of the system:*

$$\dot{\mathcal{Y}}(t) = F(t, \mathcal{Y}_t, \mathcal{W}(t-\tau, \mathcal{Y}(t-\tau)) + \mathfrak{s}(t)) \qquad (55)$$

satisfy

$$|\mathcal{Y}(t)| \leq \iota_1 \sup_{s\in[t-T,t]} |\mathcal{Y}(s)| + \iota_2 \sup_{s\in[t-T,t]} |\mathfrak{s}(s)| \qquad (56)$$

with

$$0 \leq \iota_1 < 1 \qquad (57)$$

and $\iota_2 > 0$ *for all* $t \geq t_\sharp$.

We have the following result, whose proof is omitted:

Proposition 1 Let the system (19) satisfy Assumption A3. Let F be a functional and $\mathcal{W}$ be a function such that Assumption A4 is satisfied. Then Assumption A2 is satisfied.

4. ILLUSTRATIONS OF THEOREM 5

4.1 TORA system

We illustrate our theory using the system

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = -z_1(t) + \epsilon \sin(\varsigma_1(t)) \\ \dot{\varsigma}_1(t) = \varsigma_2(t) \\ \dot{\varsigma}_2(t) = u(t), \end{cases} \qquad (58)$$

with $(z_1, z_2, \varsigma_1, \varsigma_2) \in \mathbb{R}^4$, $u \in \mathbb{R}$ and $\epsilon > 0$. This is the TORA model (whose stabilization is studied for instance in Escobar et al. (1999)) after a preliminary change of feedback. This change of feedback contains an unbounded term. Thus we do not claim that the control law we propose in this section stabilizes a trajectory of the TORA model by bounded feedback. Our objective is merely to illustrate how our technique applies and can produce bounded feedbacks. In (Malisoff and Mazenc, 2002, p. 90), it is explained that, when $\epsilon = \frac{3}{4}$, then tracking the trajectory:

$$\begin{aligned} &(z_{1,r}(t), z_{2,r}(t), \varsigma_{1,r}(t), \varsigma_{2,r}(t)) \\ &\qquad = \left(\sin\left(\frac{t}{2}\right), \frac{1}{2}\cos\left(\frac{t}{2}\right), \frac{t}{2}, \frac{1}{2}\right) \end{aligned} \qquad (59)$$

results in the problem of globally asymptotically stabilizing the origin of the system:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + \frac{3}{4}\cos\left(\frac{t}{2}\right)\sin(y_1(t)) \\ \qquad + \frac{3}{4}\left(\sin\left(\frac{t}{2}\right)(\cos(y_1(t)) - 1)\right) \\ \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = u(t). \end{cases} \qquad (60)$$

The triangular structure of this system ensures that the bounded backstepping results developed in Freeman and Praly (1998) and Mazenc et al. (2011) apply. But this strategy leads to rather complicated feedback laws. Let us observe that the forwarding approach Mazenc and Praly (2000) applies too and leads to rather complicated feedbacks laws too.

4.2 Control design

Let us check that Theorem 5 applies to the system (60).

Through a simple proof using the Lyapunov function $\mho(\zeta) = \int_0^{\zeta_1+\zeta_2} \sigma_1(\ell)d\ell + \frac{1}{2}\zeta_2^2$, one can prove that Assumption A1 is satisfied with

$$\mathcal{V}(\zeta) = -\sigma_1(\zeta_1 + \zeta_2) - \sigma_1(\zeta_2). \qquad (61)$$

One can check easily that Assumption A3 is satisfied. Now, let us check that Assumption A2 is satisfied with

$$\mathcal{W}(t, X) = -\varepsilon \cos\left(\frac{t}{2}\right)\sigma_1(x_2), \qquad (62)$$

with $\varepsilon \in \left(0, \frac{1}{2}\right)$. The equation which corresponds to the system (24) is

$$\begin{cases} \dot{\xi}_1(t) = \xi_2(t) \\ \dot{\xi}_2(t) = -\xi_1(t) + \frac{3}{4}\left[\cos\left(\frac{t}{2}\right)\sin(\Gamma_{\mathcal{W},2}(t, \xi_t) + \mathfrak{s}(t))\right] \\ \qquad \frac{3}{4}\left[+\sin\left(\frac{t}{2}\right)(\cos(\Gamma_{\mathcal{W},2}(t, \xi_t) + \mathfrak{s}(t)) - 1)\right]. \end{cases} \qquad (63)$$

It can be rewritten as:

$$\begin{cases} \dot{\xi}_1(t) = \xi_2(t) \\ \dot{\xi}_2(t) = -\xi_1(t) + \mathcal{C}_1(t, \xi_t) + \mathcal{C}_2(t, \mathfrak{s}(t), \xi(t)) \\ \qquad + \frac{3}{4}\left[\cos\left(\frac{t}{2}\right)\sin\left(-\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\right] \\ \qquad + \frac{3}{4}\left[\sin\left(\frac{t}{2}\right)\left(\cos\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right) - 1\right)\right] \end{cases} \qquad (64)$$

with

$$\begin{aligned} &\mathcal{C}_1(t, \xi_t) = \\ &\frac{3}{4}\left[\sin(\Gamma_{\mathcal{W},2}(t, \xi_t))\cos\left(\frac{t}{2}\right)\right. \\ &\left. - \sin\left(-\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\cos\left(\frac{t}{2}\right)\right] \\ &-\frac{3}{4}\left[(\cos(\Gamma_{\mathcal{W},2}(t, \xi_t)) - 1)\sin\left(\frac{t}{2}\right)\right. \\ &\left. - \left(\cos\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right) - 1\right)\sin\left(\frac{t}{2}\right)\right] \end{aligned} \qquad (65)$$

and

$$\mathcal{C}_2(t,\mathfrak{s}(t),\xi_t) = \frac{3}{4}\left[\left(\sin\left(\Gamma_{\mathcal{W},2}(t,\xi_t)+\mathfrak{s}(t)\right)-\sin\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)\right)\right]\cos\left(\frac{t}{2}\right) + \frac{3}{4}\left[\cos\left(\Gamma_{\mathcal{W},2}(t,\xi_t)+\mathfrak{s}(t)\right)-\cos\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)\right]\sin\left(\frac{t}{2}\right) \tag{66}$$

Let us consider the positive definite quadratic function:

$$Q(\xi_1,\xi_2) = \frac{2}{3}(\xi_1^2+\xi_2^2). \tag{67}$$

Its derivative along the trajectories of the system (64) satisfies

$$\begin{aligned}\dot{Q}(t) \leq\ & -\cos\left(\frac{t}{2}\right)\xi_2(t)\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\\ &+|x_2(t)|\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\\ &\times\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\\ &+\xi_2(t)\left[\mathcal{C}_1(t,\xi_t)+\mathcal{C}_2(t,\mathfrak{s}(t),\xi(t))\right],\end{aligned} \tag{68}$$

where the last inequality is a consequence of the fact that $1-\cos(a)\leq a\sin(a)$ for all $a\in\left[0,\frac{\pi}{2}\right]$. Since $\varepsilon\in\left(0,\frac{1}{2}\right)$, we have:

$$\begin{aligned}\dot{Q}(t) \leq\ & -\cos\left(\frac{t}{2}\right)\xi_2(t)\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\\ &+\frac{1}{2}|\xi_2(t)|\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\\ &\times\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\\ &+\xi_2(t)\left[\mathcal{C}_1(t,\xi_t)+\mathcal{C}_2(t,\mathfrak{s}(t),\xi(t))\right]\\ \leq\ & -\frac{1}{2}\cos\left(\frac{t}{2}\right)\xi_2(t)\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\\ &+\xi_2(t)\left[\mathcal{C}_1(t,\xi_t)+\mathcal{C}_2(t,\mathfrak{s}(t),\xi(t))\right].\end{aligned} \tag{69}$$

Since the system (64) is periodic, we deduce from the LaSalle Invariance Principle that this system would be globally uniformly asymptotically stable if $\mathcal{C}_1$ and $\mathcal{C}_2$ were not present. Now, let us investigate what is the impact of these functions. The inequalities

$$\begin{aligned}&|\mathcal{C}_1(t,\xi_t)| \leq\\ &\frac{3}{4}\left|\sin\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)-\sin\left(-\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\right|\\ &+\frac{3}{4}\left|\cos\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)-\cos\left(-\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\right|\\ &\leq\frac{3}{2}\left|\Gamma_{\mathcal{W},2}(t,\xi_t)+\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right|\end{aligned} \tag{70}$$

and

$$|\mathcal{C}_2(t,\mathfrak{s}(t),\xi_t)| \leq \frac{3}{2}|\mathfrak{s}(t)| \tag{71}$$

are satisfied. Then one can prove that there is a constant $h_\star>0$ such that when $h\in(0,h_\star]$ this system is ISS with restriction with input $\mathfrak{s}(t)$. Thus Assumption A2 is satisfied. Then

$$y_{\dagger,1}(t,X_t) = \Gamma_{\mathcal{W},2}(t,X_t) \tag{72}$$

$$\begin{aligned}y_{\dagger,2}(t,X_t) &= \dot{y}_{\dagger,1}(t,X_t)\\ &= -k\Gamma_{\mathcal{W},2}(t,X_t)+\frac{ke^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t)\\ &\quad-\frac{k}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h})\end{aligned} \tag{73}$$

and

$$\begin{aligned}\dot{y}_{\dagger,2}(t,X_t) =\ & k^2\Gamma_{\mathcal{W},2}(t,X_t)-k\frac{ke^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t)\\ &+k\frac{k}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h})\\ &+\frac{ke^{kh}}{e^{kh}-1}\dot{\Gamma}_{\mathcal{W},1}(t,X_t)\\ &-\frac{k}{e^{kh}-1}\dot{\Gamma}_{\mathcal{W},1}(t-h,X_{t-h})\\ =\ & k^2\Gamma_{\mathcal{W},2}(t,X_t)-k\frac{ke^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t)\\ &+k\frac{k}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h})\\ &+\frac{ke^{kh}}{e^{kh}-1}\Big[-k\Gamma_{\mathcal{W},1}(t,X_t)\\ &+\frac{ke^{kh}}{e^{kh}-1}\mathcal{W}(t,x_2(t))\\ &-\frac{k}{e^{kh}-1}\mathcal{W}(t-h,x_2(t-h))\Big]\\ &-\frac{k}{e^{kh}-1}\Big[-k\Gamma_{\mathcal{W},1}(t-h,X_{t-h})\\ &+\frac{ke^{kh}}{e^{kh}-1}\mathcal{W}(t-h,x_2(t-h))\\ &-\frac{k}{e^{kh}-1}\mathcal{W}(t-2h,x_2(t-2h))\Big].\end{aligned} \tag{74}$$

By grouping the terms,

$$\begin{aligned}\dot{y}_{\dagger,2}(t,X_t) =\ & k^2\Gamma_{\mathcal{W},2}(t,X_t)-\frac{2k^2e^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t)\\ &+\frac{2k^2}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h})\\ &+\left(\frac{ke^{kh}}{e^{kh}-1}\right)^2\mathcal{W}(t,x_2(t))\\ &-2e^{kh}\left(\frac{k}{e^{kh}-1}\right)^2\mathcal{W}(t-h,x_2(t-h))\\ &+\left(\frac{k}{e^{kh}-1}\right)^2\mathcal{W}(t-2h,x_2(t-2h)).\end{aligned} \tag{75}$$

This leads us to the bounded control law:

$$\begin{aligned}u(t) =\ & -\sigma_1(\tilde{y}_1(t)+\tilde{y}_2(t))-\sigma_1(\tilde{y}_2(t))\\ &+k^2\Gamma_{\mathcal{W},2}(t,X_t)-\frac{2k^2e^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t)\\ &+\frac{2k^2}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h})\\ &+\left(\frac{ke^{kh}}{e^{kh}-1}\right)^2\mathcal{W}(t,x_2(t))\\ &-2e^{kh}\left(\frac{k}{e^{kh}-1}\right)^2\mathcal{W}(t-h,x_2(t-h))\\ &+\left(\frac{k}{e^{kh}-1}\right)^2\mathcal{W}(t-2h,x_2(t-2h))\end{aligned} \tag{76}$$

with

$$\begin{aligned}\tilde{y}_1(t) &= y_1(t)-\Gamma_{\mathcal{W},2}(t,X_t)\\ \tilde{y}_2(t) &= y_2(t)+k\Gamma_{\mathcal{W},2}(t,X_t)-\frac{ke^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t)\\ &\quad+\frac{k}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h}).\end{aligned} \tag{77}$$

4.3 Simulations

The stabilizing control law (76) with $h = 1$, $\varepsilon = 0.49$ and $k = 0.1$ gives the following behaviors to the control law and states X, Y.

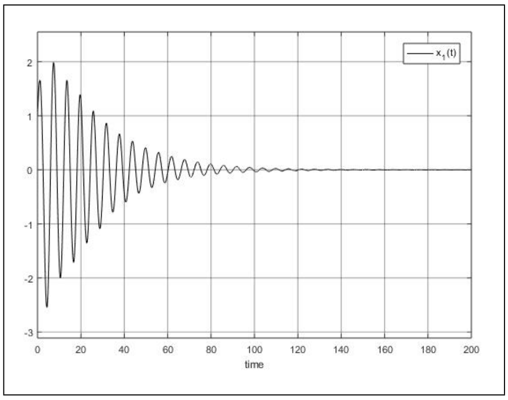

Fig. 1. $x_1(t)$

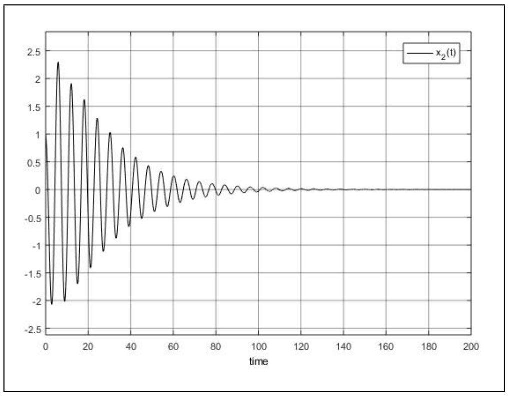

Fig. 2. $x_2(t)$

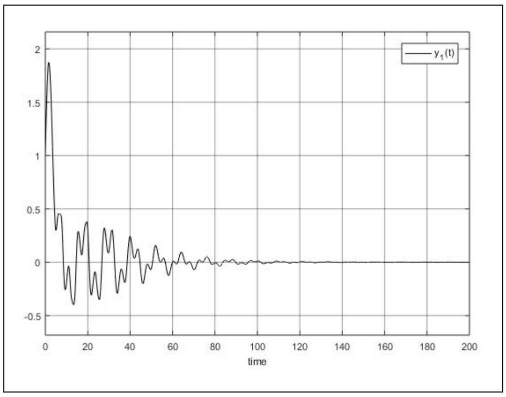

Fig. 3. $y_1(t)$

5. CONCLUSIONS

We proposed a new backstepping design of control laws for a family of nonlinear continuous time-varying systems with delay. This design relies on a family of operators which can be replaced by terms generated by dynamic extensions, with only pointwise delays. The main result we have proposed can be extended in many directions, which

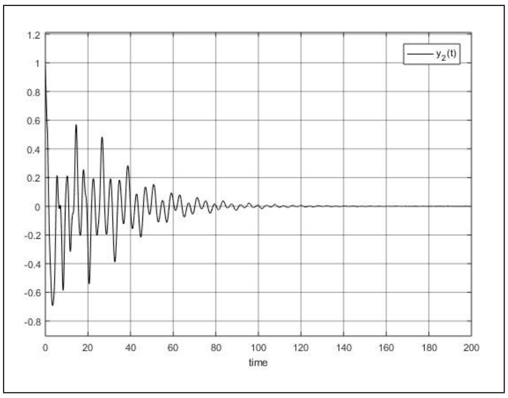

Fig. 4. $y_2(t)$

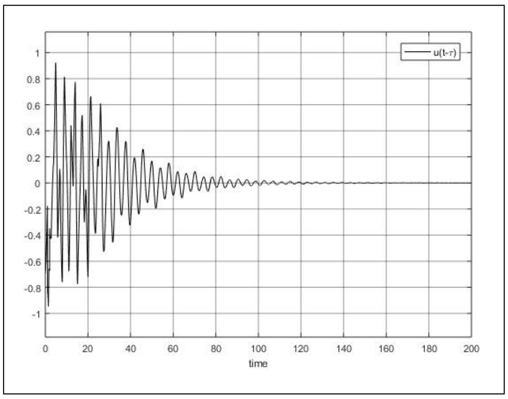

Fig. 5. $u(t)$

include in particular designs for systems with time-varying delays, systems with distributed delays and systems with a nonlinear Y-subsystem. The case where there are delays in the measurements can be considered too.

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