Fire Sales, Default Cascades and Complex Financial Networks

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Abstract

We present a general tractable framework for understanding the joint impact of fire sales and default cascades on systemic risk in complex financial networks. Our limit theorems quantify how price mediated contagion across institutions with common asset holding could worsen cascades of insolvencies in a heterogeneous financial network, during a financial crisis. For given prices of illiquid assets, we show that, under some regularity assumptions, the default cascade model could be transferred to a death process problem represented by balls-and-bins model. We model the price impact by a given inverse demand function. We state various limit theorems regarding the total sold shares and the equilibrium price of illiquid assets in a stylized fire sales model. In particular, we show that the equilibrium prices of illiquid assets has asymptotically Gaussian fluctuations. Our numerical studies investigate the effect of heterogeneity in network structure and price impact function on the final size of default cascade and fire sales loss.

Keywords: Fire Sales, Default Contagion, Financial Networks, Systemic Risk.

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1 Introduction

Financial institutions are interconnected in various ways. The global financial crisis of 2007–2009 has simultaneously highlighted the importance of interbank network structure and fire sales on the amplification and transmission of initial shocks across the wider financial system.

This paper studies the joint impact of fire sales and insolvency cascades on systemic risk in complex financial networks, during a financial crisis. Fire sales refer to situations where an institution tries or is forced to sell a large amount of assets in a short period of time.

We consider a financial network, in which institutions hold interbank liabilities, cash, and shares of an (or multiple) illiquid asset(s). When a firm defaults, its counterparties may sell their illiquid assets (deleveraging) in response to losses they face by this default, which may trigger to a lower price for these or related assets. This may lead to contagion of losses across institutions with common asset holdings. Indeed, marking to market of institutions' balance sheets reinforces network contagion: lower asset prices may force other institutions to default on their interbank liabilities. This results in an entanglement of price mediated contagion and interbank network mediated contagion.

To deal with the partial information available on the financial network, as pointed out in e.g., [13, 41, 52], we consider a random network approach. We reduce the dimension of the problem by considering a classification of financial institutions according to different types. This may be calibrated to real-world data by using machine learning techniques for classification. In light of its tractability and interpretability, as well as its potential to be enriched with clustering (see e.g., [32, 55]), we use the configuration model as our base probabilistic model. The configuration model has been previously used to model the pure default cascade process in financial networks, see e.g., [4, 5, 11, 12].

We present a general tractable framework for understanding the joint impact of fire sales and default cascades on systemic risk in a heterogeneous financial network, subject to an exogenous macroeconomic shock. As shown in our recent paper [4], under some regularity assumptions on the financial network, the pure default cascade model could be transferred to a death process problem represented by balls-and-bins model. The balls-and-bins model has been previously used in the economic literature; see e.g., [14] for a balls-and-bins model of international trade.

We first show that our limit theorems in [4] hold for any given price of illiquid assets. Since our model is static in nature, following [7, 26, 43], we assume that all the liquidation happens simultaneously and instantaneously. We model the price impact by a given inverse demand function. We state various limit theorems regarding the total sold shares and the equilibrium price of illiquid assets in a stylized fire sales model. In particular, we show that the equilibrium prices of illiquid assets has asymptotically Gaussian fluctuations. Our numerical studies investigate the effect of heterogeneity in network structure and price impact function on the final size of default cascade and fire sales loss.

Our paper is related to several strands of research in the literature.

The literature on financial networks and systemic risk is vast, see e.g., [25, 44] for two recent surveys and references there. An extensive research in this area focuses on equilibrium approach, to derive recovery rates from some fixed point equations [36, 37, 51]. This relies on the assumption that all debts are instantaneously cleared, unlikely to hold during a financial crisis. Following [5, 9, 28], we model recovery rates as given. The model could be extended to a setup with random recovery rates satisfying some cash-flow consistency conditions as in [9].

Our work is also related to the literature on the impact of network structure and heterogeneity on default contagion and systemic risk, see e.g., [1, 2, 6, 10, 15, 21, 40, 42, 49].

Price mediated contagion and the resulting destabilizing feedback effects have been extensively studied in the literature without the inclusion of interbank liability networks, see e.g., [19, 22, 24, 29, 31, 35]. We refer to e.g., [27, 30, 53] for a more detailed review of the literature on fire sales. Our work complements a number of recent papers that integrate the fire sales loss into the cascades of insolvencies in interbank networks, see e.g., [7, 16, 20, 23, 26, 34, 38, 42, 50, 56]. In particular, [34] uses and extends the methods developed in [5, 8] to provide a resilience condition for the financial network in an integrated model of fire sales and default contagion, in the context of inhomogeneous random graphs.

Our primary contribution to the literature is to provide central limit theorems for the size of default cascade and fire sales loss in a stochastic heterogeneous financial network. This extends our previous central limit theorems in [4] for the pure default cascade process in heterogeneous financial networks. Moreover, as we transfer the default cascade process to a death process problem represented by balls-and-bins model, this allows us to provide the limit theorems for a dynamic default cascade process with fire sales. The closed form interpretable limit theorems that we provide in a heterogeneous financial network could also serve as a mandate for regulators to collect data on those specific network characteristics and assess systemic risk via more intensive computational methods.

Outline. The remainder of the paper is as follows. Section 2 introduces a general model for the network of financial counterparties and describe a mechanism for default cascade in such a network, after an exogenous macroeconomic shock. In this section we also provide a stylized model of fire sales in a financial network with single illiquid asset and describe how the default cascade process could be transferred to a death process problem represented by balls-and-bins model. Section 3 gives our main results on limit theorems for the final size of default cascade, the total sold shares and the equilibrium price of the illiquid asset. In particular, we show that the equilibrium price of illiquid asset has asymptotically Gaussian fluctuation. Numerical case studies in Section 4 investigate the effect of heterogeneity in network structure and price impact function on the final size of default cascade and fire sales loss. Proof of main theorems are given in Section 5. Section 6 concludes. Appendix A contains some auxiliary lemmas, used to prove our main results regarding the central limit theorems. Appendix B provides the extension of our model to the financial network with multiple types of illiquid assets. We provide central limit theorems for default cascade with fire sales in this setup.

Notation. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $c \in \mathbb{R}$ is a constant, we write $X_n \xrightarrow{p} c$ to denote that X_n converges in probability to c. That is, for any $\epsilon > 0$, we have $\mathbb{P}(|X_n - c| > \epsilon) \to 0$ as $n \to \infty$. We write $X_n \xrightarrow{d} X$ to denote that X_n converges in distribution to X. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers that tends to infinity as $n \to \infty$. We write $X_n = o_p(a_n)$, if $|X_n|/a_n \xrightarrow{p} 0$. If E_n is a measurable subset of Ω, for any $n \in \mathbb{N}$, we say that the sequence $\{E_n\}_{n\in\mathbb{N}}$ occurs with high probability (w.h.p.) or almost surely (a.s.) if $\mathbb{P}(E_n) = 1 - o(1)$, as $n \to \infty$. Also, we denote by $\mathsf{Bin}(k, p)$ a binomial distribution corresponding to the number of successes of a sequence of k independent Bernoulli trials each having probability of success p. The notation $\mathbb{1}\{E\}$ is used for the indicator of an event E; this is 1 if E holds and 0 otherwise. We denote by $\mathcal{D}[0, \infty)$ the standard space of right-continuous functions with left limits on $[0, \infty)$ equipped with the Skorokhod topology (see e.g., [45, 46]). We will suppress the dependence of parameters on the size of the network n, if it is clear from the context. We denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non-negative integers.

2 Model

In this section, we describe the financial network and the default cascade model of [4], extended to account for the price impact of the liquidation of an illiquid assets and fire sale effects.

2.1 Financial Network

Consider an economy $\mathcal{E}_n$ consisting of n interlinked financial institutions (banks, companies, hedge funds, etc.) denoted by $[n] := \{1, 2, \ldots, n\}$. Interbank liabilities are represented by a matrix of nominal liabilities $(\ell_{ij})_{i,j \in [n]}$, where, for two financial institutions $i, j \in [n]$, $\ell_{ij} \geqslant 0$ denotes the cash-amount that bank i owes bank j. The total nominal liabilities of bank i is $\ell_i = \sum_{j \in [n]} \ell_{ij}$, and the total incoming cash-amount sum up to $a_i = \sum_{j \in [n]} \ell_{ji}$. In addition to this interbank assets and liabilities, every institution holds claims on end-users and vice versa. The total value of claims held by end-users on bank i (deposits) is denoted by d_i, while the total value of claims held by bank i on end-users (external assets) is denoted by e_i. Bank i holds $k_i \geqslant 0$ units of a liquid asset (cash) and $\gamma_i \in [0, \gamma_{\max}]$ units of an illiquid asset. We assume that all γ_i (for all $i \in [n]$) are bounded from the above by $\gamma_{\max} > 0$. Cash has value one. The illiquid asset has a positive fundamental value $p_0 > 0$. The nominal balance sheet of bank i is then given by:

- Assets: $e_i + k_i + \gamma_i p_0 + a_i$;
- Liabilities: $d_i + \ell_i$ + nominal net worth.

In a stress testing framework, we apply a fractional shock $\epsilon_i \in [0, 1]$ to the external assets of bank i. Table 1 summarizes a stylized balance sheet of bank i after the shock ϵ_i. The capital of bank i after the shock denoted by $c_i = c_i(\epsilon_i; p_0)$ satisfies

$$c_i(\epsilon_i; p_0) = k_i + \gamma_i p_0 + (1 - \epsilon_i) e_i + a_i - \ell_i - d_i, \tag{1}$$

which represents the capacity of bank i to absorb losses while remaining solvent.

The nominal cash balance of bank i is then $k_i + (1 - \epsilon_i) e_i + a_i - d_i - \ell_i$.

Price impact of liquidations. If bank i's nominal cash balance is negative, then it has a liquidity shortfall. In this case, bank i sells some of its shares of the illiquid asset. This has a negative price impact on the illiquid asset. We model this by considering a given inverse demand function g satisfying the following mild technical assumptions.[1] We consider the inverse demand function $g(x)$ which gives the equilibrium price for the illiquid asset when nx units of the asset (in the network of size n) are sold. Let $\bar{\gamma} := \frac{\gamma_1 + \cdots + \gamma_n}{n}$ denote the average institutions' shares of illiquid asset. Let $p_{\min} > 0$.

[1] Similarly to [7, 26], we assume there is an outside market for this illiquid asset that can absorb the total illiquid asset holdings of the banks at a distressed price. It is beyond the scope of this paper to endogenize both the demand function for the illiquid asset and the financial network payments.

External e_i	Deposits d_i
$\epsilon_i e_i$ - loss on assets	Interbank $\ell_i = \sum_{j \in [n]} \ell_{ij}$
Interbank $a_i = \sum_{j \in [n]} \ell_{ji}$	Capital c_i
Liquid k_i	
Illiquid $\gamma_i p_0$	$\epsilon_i e_i$ - loss on capital
Assets	**Liabilities**

Table 1: Stylized balance sheet of bank i after shock.

Assumption 1. *We assume that* $g : [0, \bar{\gamma}] \to [p_{\min}, p_0]$ *satisfies:*

(i) $g(0) = p_0$ *(in absence of liquidations the price is given exogenously by* p_0*).*

(ii) $g(x) \in \mathcal{C}^1$ *and it is a non-increasing function of* $x \in [0, \bar{\gamma}]$ *(the price is non-increasing with the average excess supply* x*).*

(iii) $g(\bar{\gamma}) = p_{\min} > 0$ *(the price when the total illiquid asset holdings of the banks are sold is bounded from below by* $p_{\min} > 0$*).*

We end this section by providing a few examples for the price impact function, satisfying the above regularity assumptions. These examples will be further investigated in our numerical experiments, in Section 4.

Example 2.1 (Linear Price Impact function)**.** *We consider the LPI function with* $g(0) = p_0$ *and* $g(\bar{\gamma}) = p_{\min}$*, that is, for* $y \in [0, \bar{\gamma}]$*, we set:*

$$g^{\mathrm{L}}(y) = p_0 - (p_0 - p_{\min})\big(y/\bar{\gamma}\big).$$

Example 2.2 (Quadratic Price Impact function)**.** *We consider the case of QPI function with* $g(0) = p_0$ *and* $g(\bar{\gamma}) = p_{\min}$*, that is, for* $y \in [0, \bar{\gamma}]$ *and some positive control parameter* α*, we set:*

$$g^{\mathrm{Q}}_{\alpha}(y) = p_0 - (p_0 - p_{\min})\big(y/\bar{\gamma}\big)\frac{1 - \alpha\big(y/\bar{\gamma}\big)}{1 - \alpha}.$$

Example 2.3 (Exponential Price Impact function)**.** *We consider the case of EPI function with* $g(0) = p_0$ *and* $g(\bar{\gamma}) = p_{\min}$*, that is, for* $y \in [0, \bar{\gamma}]$ *and some control parameter* α*, we set:*

$$g^{\mathrm{E}}_{\alpha}(y) = p_0 - (p_0 - p_{\min})\frac{1 - e^{-\alpha(y/\bar{\gamma})}}{1 - e^{-\alpha}}.$$

2.2 Default Cascade

For a given shock scenario $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in [0,1]^n$, if the cash balance of bank $i \in [n]$ is negative and the revenue from selling the total γ_i units of the illiquid asset does not cover the negative cash balance, then bank i defaults on its interbank liabilities.

For a given price $p \in [p_{\min}, p_0]$ of the illiquid asset, we say that the bank i is *p-fundamentally insolvent* if its capital, after the shock and under price p of illiquid asset, is negative, i.e.,

$c_i(\epsilon_i; p) < 0$. We let the set of p-fundamental defaults

$$\mathcal{D}_0(\boldsymbol{\epsilon}; p) = \{i \in [n] : c_i(\epsilon_i; p) < 0\}. \tag{2}$$

We next define the pure default cascade (without fire sales loss) triggered by fundamentally insolvent institutions. Note that, for a given shock scenario $\boldsymbol{\epsilon}$, the price of illiquid asset could be impacted by fire sales and becomes $p \leqslant p_0$, leading to a larger set of fundamentally insolvent institutions $\mathcal{D}_0(\boldsymbol{\epsilon}; p)$. This triggers the default contagion process.

Let us fix the shock $\boldsymbol{\epsilon}$ and the price of illiquid asset $p \in [p_{\min}, p_0]$. We denote by $R_{ij} = R_{ij}(\boldsymbol{\epsilon}; p)$ the recovery rate of the liability of i to j, in case of default of i. The matrix of recovery rates is denoted by $\mathcal{R} = (R_{ij})_{i,j \in [n]}$. Since any bank i cannot pay more than its external assets $(1 - \epsilon_i)e_i$ plus what it recovered from its debtors, the recovery rates of i should satisfy the following cash-flow consistency constraints

$$\gamma_i p + k_i + (1 - \epsilon_i)e_i + \sum_{j=1}^{n} R_{ji}\ell_{ji} \geqslant \sum_{j=1}^{n} R_{ij}\ell_{ij} + d_i.$$

Given the shock scenario $\boldsymbol{\epsilon}$, the price of the illiquid asset p and matrix of recovery rates $\mathcal{R}$, following the set of p-fundamentally insolvent institutions $\mathcal{D}_0(\boldsymbol{\epsilon}; p)$, there is a default cascade that reaches the set $\mathcal{D}^\star$ in equilibrium. This represents the set of financial institutions whose capital is insufficient to absorb losses and should satisfy the following fixed point equation:

$$\mathcal{D}^\star = \mathcal{D}^\star(\boldsymbol{\epsilon}, \mathcal{R}; p) = \Big\{i \in [n] : c_i(\epsilon_i; p) < \sum_{j \in \mathcal{D}^\star} (1 - R_{ji})\ell_{ji}\Big\}.$$

As shown in [9], the above fixed point default cascade set has in general multiple solutions. The smallest fixed point set which corresponds to smallest number of defaults can be obtained by starting from $\mathcal{D}_0 = \mathcal{D}_0(\boldsymbol{\epsilon}; p)$ and setting at step k:

$$\mathcal{D}_k = \mathcal{D}_k(\boldsymbol{\epsilon}, \mathcal{R}; p) = \big\{i \in [n] : c_i < \sum_{j \in \mathcal{D}_{k-1}} (1 - R_{ji})\ell_{ji}\big\}. \tag{3}$$

The cascade ends at the first time k such that $\mathcal{D}_k = \mathcal{D}_{k-1}$. Hence, in a financial network of size n and for a given price p of illiquid asset, the cascade will end after at most $n-1$ steps and $\mathcal{D}_{n-1} = \mathcal{D}_{n-1}(\boldsymbol{\epsilon}, \mathcal{R}; p)$ represents the final set of insolvent institutions.

2.3 Node Classification and Configuration Model

As detailed below, under some regularity assumptions, we encode the information regarding assets (liquid and illiquid), liabilities, capital after exogenous shocks and recovery rates (distributions) in a single probability threshold function; see [4, 5] for the proofs in a similar setup.

For a given illiquid asset price p, shock scenario $\boldsymbol{\epsilon}$ and the matrix of recovery rates $\mathcal{R}$, we introduce the (random) threshold $\Theta_i(p) = \Theta_i^{(n)}(p)$ for any institution $i \in [n]$. This measures how many defaults bank i could tolerate before becoming insolvent, when its counterparties default's order is uniformly at random, i.e., when i's debtors default order environment is chosen uniformly at random among all possible permutations.

We next consider a classification of financial institutions into a countable (finite or infinite) possible set of characteristics $\mathcal{X}$. All observable characteristics for the institution i is encoded in $x_i = (d_i^+, d_i^-, t_i, ...) \in \mathcal{X}$, where d_i^+ denotes the in-degree (number of institutions i is exposed to), d_i^- denotes the out-degree (number of institutions exposed to i) and t_i denotes any other institution's type specific (e.g., credit rating, seniority class, systemically importance, etc.).

To state limit theorems, we consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, indexed by the number of institutions. The characteristic of any institution $i \in [n]$, in the economy $\mathcal{E}_n$, will be

$$x_i^{(n)} = (d_i^{+(n)}, d_i^{-(n)}, t_i^{(n)}, ...) \in \mathcal{X}.$$

Note that, without loss of generality, the institutions in the same class $x \in \mathcal{X}$ are assumed to have the same number of creditors (denoted by d_x^-) and the same number of debtors (denoted by d_x^+). For tractability, we make the following assumption on the probability threshold functions.

Assumption 2. *There exists a classification of the financial institutions into a countable set of possible characteristics $\mathcal{X}$ such that, for each $n \in \mathbb{N}$ and for all $p \in [p_{\min}, p_0]$, the institutions in the same class have the same threshold distribution function (denoted by $q_x^{(n)}$ for the institutions in class $x \in \mathcal{X}$). Namely, for the economy $\mathcal{E}_n, i \in [n]$ and all $\theta \in \mathbb{N}$,*

$$\mathbb{P}(\Theta_i^{(n)}(p) = \theta) = q_{x_i^{(n)}}^{(n)}(\theta; p).$$

In particular, in the network of size n, $q_x^{(n)}(0; p)$ represents the proportion of initially insolvent institutions with type $x \in \mathcal{X}$ and under the illiquid asset price p.

Let us denote by $\mu_x^{(n)}$ the fraction of institutions with characteristic $x \in \mathcal{X}$ in the economy $\mathcal{E}_n$. In order to study the limit theorems, it is natural to assume the following.

Assumption 3. *For some probability distribution functions μ and $q(.; p)$ over the set of characteristics $\mathcal{X}$ (independent of n), we have $\mu_x^{(n)} \to \mu_x$ and $q_x^{(n)}(\theta; p) \to q_x(\theta; p)$ as $n \to \infty$, for all $x \in \mathcal{X}, \theta = 0, 1, \ldots, d_x^+$ and $p \in [p_{\min}, p_0]$. The empirical threshold distributions satisfy $q_x^{(n)}(\theta; p) \in \mathcal{C}^1$ and $q_x(\theta; p) \in \mathcal{C}^1$ on $p \in [p_{\min}, p_0]$. Moreover, as $n \to \infty$, $\frac{\partial q_x^{(n)}}{\partial p}(\theta; p)$ converges uniformly to $\frac{\partial q_x}{\partial p}(\theta; p)$ as a function of p for all $x \in \mathcal{X}$ and $\theta = 0, 1, \ldots, d_x^+$.*

We next provide an important example of liabilities (losses) satisfying the above assumptions.

Example 2.4 (Independent random losses)**.** *We consider the case where the capital of each institution (after shock) is a constant depending on the institution's type and the price of illiquid asset, i.e., $c_i = c_{x_i}(p)$. Let $\{L_{x,k}\}_{k=1}^{\infty}$ be a set of i.i.d. continuous random variables with common cumulative distribution function (cdf) F_x and density f_x for all $x \in \mathcal{X}$. We then set $q_x(0; p) = \bar{q}_x$ explicitly. Further, we set*

$$q_x(1; p) = (1 - \bar{q}_x)\mathbb{P}(c_x(p) \leqslant L_{x,1}) = (1 - \bar{q}_x)\big(1 - F_x(c_x(p))\big),$$

and, for $\theta = 2, \ldots, d_x^+$, we set

$$\begin{aligned} q_x(\theta; p) &= (1 - \bar{q}_x)\mathbb{P}(L_{x,1} + \cdots + L_{x,\theta-1} < c_x(p) \leqslant L_{x,1} + \cdots + L_{x,\theta}) \\ &= (1 - \bar{q}_x) \int_0^{c_x(p)} f^{\star k}(\nu)\big(1 - F_x(c_x(p) - \nu)\big) d\nu, \end{aligned}$$

where $f^{\star k}$ is the k-fold convolution of f. Since the capital $c_x(p)$ is smooth (in fact linear on p) for all $x \in \mathcal{X}$, then the threshold distribution is $\mathcal{C}^1$ on p for all $x \in \mathcal{X}$ and θ. We will reconsider *this example in Section 4, for our numerical experiments, by considering a Pareto distribution for losses. Namely, we set, for some scale and shape parameters $x_m, \alpha \in \mathbb{R}^+$,*

$$f(x) = \alpha x_m^\alpha x^{-(\alpha+1)} \mathbb{1}\{x \geqslant x_m\}.$$

In this paper, we allow for the possibility that some institution will never get infected, i.e., the institution remains solvent even if all its counterparties are in default. This case was excluded in the pure default cascade process studied in [4]. We denote by

$$q_x(\infty; p) := 1 - \sum_{\theta=0}^{d_x^+} q_x(\theta; p),$$

for all $x \in \mathcal{X}$ and $p \in [p_{\min}, p_0]$.

Configuration model. Given a set of institutions $[n] := \{1, \ldots, n\}$, the degree sequences $\mathbf{d}_n^+ = (d_1^+, \ldots, d_n^+)$ and $\mathbf{d}_n^- = (d_1^-, \ldots, d_n^-)$ such that $\sum_{i \in [n]} d_i^+ = \sum_{i \in [n]} d_i^-$, we associate to each institution $i \in [n]$ two sets: $\mathcal{H}_i^+$ the set of incoming half-edges and $\mathcal{H}_i^-$ the set of outgoing half-edges, with $|\mathcal{H}_i^+| = d_i^+$ and $|\mathcal{H}_i^-| = d_i^-$. Let $\mathbb{H}^+ = \bigcup_{i=1}^n \mathcal{H}_i^+$ and $\mathbb{H}^- = \bigcup_{i=1}^n \mathcal{H}_i^-$. A *configuration* is a matching of $\mathbb{H}^+$ with $\mathbb{H}^-$. When an out-going half-edge of institution i is matched with an in-coming half-edge of institution j, a directed edge from i to j appears in the graph. The configuration model is the random directed multigraph which is uniformly distributed across all configurations. The random graph constructed by configuration model will be denoted by $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$. It is then easy to show that conditional on the multigraph being a simple graph, we obtain a uniformly distributed random graph with these given degree sequences denoted by $\mathcal{G}_*^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$. In particular, any property that holds with high probability on the configuration model also holds with high probability conditional on this random graph being simple (for the random graph $\mathcal{G}_*^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$) provided that $\liminf_{n\to\infty} \mathbb{P}(\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-) \text{ simple}) > 0$, see e.g. [54].

2.4 Associated Death Process with Balls-and-Bins Model

In [4], in order to study the pure default cascade process, we used an associated balls-and-bins death process and defined the corresponding virtual interaction time denoted by t; this is not the real calendar time, it is a virtual time which helps us to analyze and find the final state of the default cascade in the random financial network $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$.

For a given stress scenario $\boldsymbol{\epsilon} \in [0,1]^n$ and price of illiquid asset $p \in [p_{\min}, p_0]$, we consider the default contagion process in $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$, starting from the set of p-fundamentally insolvent institutions $\mathcal{D}_0(\boldsymbol{\epsilon}; p)$. We use a coupling argument which allows us to simultaneously run the default cascade process and construct the configuration model. We refer to [4] for a more detailed description.

We call all out half-edges and in half-edges that belong to a defaulted (solvent) institution the *infected* (*healthy*) half-edges. We consider all the institutions as bins and all the (in and out) half-edges as (in and out) balls. Consequently, the bins are called defaulted (**D** type) or solvent (**S** type) according to their states as institutions. Similarly the balls are called infected (**I** type) or healthy (**g** type) when they are infected or healthy as half-edges. Hence, all institutions are of two types and all balls are of four different types. For convenience, we denote them as **S** (solvent), **D** (defaulted) bins, and further $\mathbf{H}^+$ (healthy in), $\mathbf{H}^-$ (healthy out), $\mathbf{I}^+$ (infected in) and $\mathbf{I}^-$ (infected out) balls, respectively.

We start from the set of p-fundamental defaults $\mathcal{D}_0(\boldsymbol{\epsilon}; p)$, which gives the set of initially defaulted bins and infected balls. Consequently, at each step, we first remove a uniformly chosen ball of type $\mathbf{I}^-$ and then a uniformly chosen ball from $\mathbf{H}^+ \cup \mathbf{I}^+$. In this process **S** bins may change to **D** bins and, consequently, **g** balls may change to **I** balls. We continue the above process until there is no more $\mathbf{I}^-$ balls.

We now change the description a little by introducing colors for the $\mathbf{I}^-$ balls and life for all in balls from $\mathbf{H}^+ \cup \mathbf{I}^+$. We let all $\mathbf{I}^-$ balls are white and all in balls from $\mathbf{H}^+ \cup \mathbf{I}^+$ are initially alive. We begin by recoloring one random $\mathbf{I}^-$ ball red. Subsequently, in each removal step, we first kill a random in ball from $\mathbf{H}^+ \cup \mathbf{I}^+$ and at the same moment we also recolor a random white ball red. This is repeated until no more white $\mathbf{I}^-$ balls remain.

We next run the above death process in continuous time. We assume that each ball from $\mathbf{H}^+ \cup \mathbf{I}^+$ has an exponentially distributed random lifetime with mean one, independent of all other balls. Namely, if there are ℓ alive in balls remaining, then we wait an exponential time with mean $1/\ell$ until the next pair of interactions. We stop when we should recolor a white ball red but there is no such ball.

Let us denote by $W_n(t;p)$ the number of white $\mathbf{I}^-$ balls at time t. Hence, the above death process ends at the stopping time $\tau_n^\star(p)$ which is the first time when we need to recolor a white ball but there are no white balls left. However, we pretend that we recolor a (nonexistent) white ball at time $\tau_n^\star(p)$ and write $W_n(\tau_n^\star; p) = -1$.

We denote by $I_n^+(t;p)$, $H_n^+(t;p)$ and $L_n(t;p)$ the number of alive (in) balls in $\mathbf{I}^+$, $\mathbf{H}^+$ and $\mathbf{H}^+ \cup \mathbf{I}^+$ at time t, respectively. For $x \in \mathcal{X}, \theta \in \mathbb{N}, \ell = 0, \ldots, \theta - 1$, we let $S_{x,\theta,\ell}^{(n)}(t;p)$ denote the number of solvent institutions (bins) with type x, threshold θ and ℓ defaulted neighbors at time t. Further, let $S_n(t;p)$ and $D_n(t;p)$ be the numbers of **S** bins and **D** bins at time t. Hence, $S_n(\tau_n^\star; p)$ denotes the final number of solvent institutions. Further, $D_n(\tau_n^\star; p) = n - S_n(\tau_n^\star; p) = |\mathcal{D}_{n-1}|$ will be the final number of defaulted institutions.

2.5 A Simple Model of Fire Sales

We assume that due to each infected incoming link (coming from a defaulted neighbor), the host institution is forced to liquidate parts of his asset holdings in order to comply with regulatory (leverage) constraints. Since the default cascade process is now transferred to a death process problem represented by balls-and-bins, where we only keep track of the type and threshold of each institution, we assume the following: The sold shares units, each time when an institution is exposed to a defaulted neighbor, are independent random variables with a distribution depending on the host's institution type, threshold and also the (equilibrium) price of illiquid asset. Since the default contagion and fire sales are instantaneous in our model, following [7, 26, 43], we consider a conservative approach and assume that all institutions might sell only at the final equilibrium price.

For a fixed $p \in [p_{\min}, p_0]$, let $D_{x,\theta}^{(n)}(t;p)$ denote the total number of defaulted institutions with type x and threshold θ. Then $D_x^{(n)}(t;p) = \sum_\theta D_{x,\theta}^{(n)}(t;p)$ is the total number of defaulted institutions with type $x \in \mathcal{X}$ at time t. Recall also that $S_{x,\theta,\ell}^{(n)}(t;p)$ denotes the number of solvent institutions of type $x \in \mathcal{X}$, threshold $\theta \in \mathbb{N}$ and with ℓ defaulted neighbors at time t.

We define for $x \in \mathcal{X}, p \in [p_0, p_{\min}]$ and $\theta = 1, \ldots, d_x^+$,

$$I_{x,\theta}^{(n)}(t;p) := \theta D_{x,\theta}^{(n)}(t;p) + \sum_{\ell=1}^{\theta-1} \ell S_{x,\theta,\ell}^{(n)}(t;p),$$

the total number of liquidations (infected links) for institutions with type $x \in \mathcal{X}$ and threshold θ up to time t. The first term states that a defaulted institution with type x and threshold $\theta \geqslant 1$ should liquidate θ times. Moreover, a solvent institution with ℓ defaulted neighbors (ℓ infected incoming links) should liquidate ℓ times before time t.

We also need to consider the institutions who never default for such a shock scenario, even if all their counterparties default. Let us denote by $S_{x,\infty,\ell}^{(n)}(t;p)$ as the number of institutions with type x, threshold larger than d_x^+ (which never defaults) and with ℓ defaulted neighbors at time t. We define

$$I_{x,\infty}^{(n)}(t;p) := \sum_{\ell=1}^{d_x^+} \ell S_{x,\infty,\ell}^{(n)}(t;p),$$

as the total number of infected links leading to such institutions with type $x \in \mathcal{X}$ up to time t.

We assume that the amount of liquidation for each initially defaulted institution with type $x \in \mathcal{X}$ is a fixed value $\bar{\gamma}_x$. The number of initially insolvent institutions with type $x \in \mathcal{X}$ is denoted by $\mathcal{D}_{x,0}(p) = nq_x^{(n)}(0;p)$ for a price p of illiquid asset.

Let $\{L_{x,\theta}^{(i)}(p)\}_{i=1}^{\infty}$ be i.i.d. positive bounded random variables with common distribution $F_{x,\theta}(.;p)$, which have expectation $\bar{\ell}_{x,\theta}(p)$ and variance $\varsigma_{x,\theta}^2(p)$ under price $p \in [p_{\min}, p_0]$ of the illiquid asset, for all $x \in \mathcal{X}$ and $\theta \in \{1, \ldots, d_x^+\} \cup \{\infty\}$.

Assumption 4. *The mean $\bar{\ell}_{x,\theta}(p)$ and variance $\varsigma_{x,\theta}^2(p)$ of sold shares for each liquidation are both continuous on p, for all $x \in \mathcal{X}$ and $\theta \in \{0, 1, \ldots, d_x^+\} \cup \{\infty\}$.*

Note that $L_{x,\theta}^{(i)}(p)$ denotes the units of illiquid asset sold at i-th incoming default leading to institutions with type x and threshold θ. Further, $L_{x,\infty}^{(i)}(p)$ denotes the units of illiquid asset sold at i-th incoming default leading to institutions with type x who never default (with threshold larger than d_x^+).

The total shares of illiquid asset sold by time t could be written as (for a given price p)

$$\Gamma_n(t;p) := \sum_{x \in \mathcal{X}} \Big(\bar{\gamma}_x D_{x,0}^{(n)}(p) + \sum_{\theta=1}^{d_x^+} Y_{x,\theta}^{(n)}(t;p), + Y_{x,\infty}^{(n)}(t;p) \Big), \tag{4}$$

where

$$Y_{x,\theta}^{(n)}(t;p) := \sum_{i=1}^{I_{x,\theta}^{(n)}(t;p)} L_{x,\theta}^{(i)}(p) \quad \text{and} \quad Y_{x,\infty}^{(n)}(t;p) := \sum_{i=1}^{I_{x,\infty}^{(n)}(t;p)} L_{x,\infty}^{(i)}(p). \tag{5}$$

The final shares of illiquid asset which has been sold under price p will then be $\Gamma_n(\tau_n^\star(p);p)$.

Since the default contagion and fire sales are instantaneous in our model, we consider a conservative approach and assume that the illiquid asset could be sold only at the final equilibrium price. This results to a price given by our inverse demand function g. Let us denote the price given by inverse demand function as

$$\kappa_n(p) := g(\Gamma_n(\tau_n^\star(p);p)/n).$$

Since, $\tau_n^\star(p)$ is not in general continuous in our model (this will be shown in the next section), the fixed point equation $p = \kappa_n(p)$ may not have a solution in general. This motivates us to define the *equilibrium price* of the illiquid asset as

$$p_n^\star = \sup\{p \in [p_{\min}, p_0] : p \leqslant \kappa_n(p)\}. \tag{6}$$

3 Limit Theorems

In this section we state our main results regarding the limit theorems for the total sold shares and equilibrium price of the illiquid asset in the random financial network $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$.

Let us first define some functions that will be used later. For $z \in [0,1]$, we let

$$\begin{aligned} b(d,z,\ell) :=& \mathbb{P}(\mathsf{Bin}(d,z) = \ell) = \binom{d}{\ell} z^\ell (1-z)^{d-\ell}, \\ \beta(d,z,\ell) :=& \mathbb{P}(\mathsf{Bin}(d,z) \geqslant \ell) = \sum_{r=\ell}^{d} \binom{d}{r} z^r (1-z)^{d-r}, \end{aligned}$$

and $\mathsf{Bin}(d,z)$ denotes the binomial distribution with parameters d and z.

3.1 Asymptotic Magnitude of Default Cascade with Fire Sales

We consider the random financial network $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$ and assume that the average degrees converges to a finite limit.

Assumption 5a. *We assume that, as $n \to \infty$, the average degrees converges and is finite:*

$$\lambda^{(n)} := \sum_{x \in \mathcal{X}} d_x^+ \mu_x^{(n)} = \sum_{x \in \mathcal{X}} d_x^- \mu_x^{(n)} \longrightarrow \lambda := \sum_{x \in \mathcal{X}} d_x^+ \mu_x \in (0, \infty).$$

For $z \in [0,1]$ and $p \in [p_0, p_{\min}]$, we define the functions:

$$\begin{aligned} f_S(z;p) :=& \sum_{x \in \mathcal{X}} \mu_x \Big[\sum_{\theta=1}^{d_x^+} q_x(\theta;p) \beta\big(d_x^+, z, d_x^+ - \theta + 1\big) + q_x(\infty;p) \Big], \quad f_D(z;p) = 1 - f_S(z;p), \\ f_W(z;p) :=& \lambda z - \sum_{x \in \mathcal{X}} \mu_x d_x^- \Big[\sum_{\theta=1}^{d_x^+} q_x(\theta;p) \beta\big(d_x^+, z, d_x^+ - \theta + 1\big) + q_x(\infty;p) \Big]. \end{aligned}$$

The following theorem (proved in our companion paper [4, Theorem 3.1]) states the law of large numbers for the number of solvent/defaulted banks and the total number of existing white balls (controlling the default contagion stopping time) at any time t in the economy $\mathcal{E}_n$ satisfying above regularity assumptions. Note that the theorem in [4] is stated for the fixed threshold distribution. When we fix $p \in [p_0, p_{\min}]$, the threshold distribution is fixed and theorem could be applied. However, since we allow for the possibility that some institution will never get infected, i.e., the institution with ∞ threshold, the forms of limiting functions f_W and f_S are a bit different from those in [4]. We discuss this extension in Section 5.5.

Theorem 3.1 ([4])**.** *Let $\tau_n \leqslant \tau_n^\star(p)$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$.*

For all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+$ and $\ell = 0, \ldots, \theta - 1$, we have (as $n \to \infty$)

$$\sup_{t \leqslant \tau_n} \Big| \frac{S^{(n)}_{x,\theta,\ell}(t;p)}{n} - \mu_x q_x(\theta;p) b\left(d_x^+, 1 - e^{-t}, \ell\right) \Big| \xrightarrow{p} 0.$$

Moreover, as $n \to \infty$,

$$\sup_{t \leqslant \tau_n} \Big| \frac{S_n(t;p)}{n} - f_S(e^{-t};p) \Big| \xrightarrow{p} 0, \quad \sup_{t \leqslant \tau_n} \Big| \frac{D_n(t;p)}{n} - f_D(e^{-t};p) \Big| \xrightarrow{p} 0,$$

and the number of white balls satisfies

$$\sup_{t \leqslant \tau_n} \Big| \frac{W_n(t;p)}{n} - f_W(e^{-t};p) \Big| \xrightarrow{p} 0.$$

We next consider a limit theorem for the total sold shares under price $p \in [p_0, p_{\min}]$ up to time t. We define the following functions which are, respectively, the limiting functions of $I^{(n)}_{x,\theta}(e^{-t};p)/n$, $I^{(n)}_{x,\infty}(e^{-t};p)/n$ and $\Gamma_n(e^{-t};p)/n$:

$$f_{x,\theta}(z;p) := \mu_x q_x(\theta;p)\big(\theta - \sum_{\ell = d_x^+ - \theta + 1}^{d_x^+} \beta(d_x^+, z, \ell)\big), \qquad f_{x,\infty}(z;p) := (1-z)\mu_x q_x(\infty;p) d_x^+,$$

and,

$$f_\Gamma(z;p) := \sum_{x \in \mathcal{X}} \mu_x \Big(\bar{\gamma}_x q_x(0;p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) f_{x,\theta}(z;p) + \bar{\ell}_{x,\infty}(p) f_{x,\infty}(z;p) \Big).$$

By using Theorem 3.1, we prove the following law of large numbers regarding the total sold shares under price $p \in [p_0, p_{\min}]$.

Theorem 3.2. *Let $\tau_n \leqslant \tau_n^\star$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then, as $n \to \infty$ and for all $p \in [p_{\min}, p_0]$,*

$$\sup_{t \leqslant \tau_n} \Big| \frac{\Gamma_n(t;p)}{n} - f_\Gamma(e^{-t};p) \Big| \xrightarrow{p} 0.$$

The proof of theorem is provided in Section 5.1.

We consider now the stopping time $\tau_n^\star$ which is the first time such that $W_n(\tau_n^\star) = -1$ (becomes negative). Let us define

$$z^\star(p) := \sup\{z \in [0,1] : f_W(z;p) = 0\}. \tag{7}$$

Note that for any $p \in [p_0, p_{\min}]$, we have $f_W(1;p) \geqslant 0$ and $f_W(0;p) \leqslant 0$. Hence, since $f_W(z;p)$ is a continuous function, $z^\star(p)$ is well defined.

We have the following lemma from [4, Lemma 3.2], which could be applied for any fixed p.

Lemma 3.3 ([4]). *For any fixed $p \in [p_0, p_{\min}]$, we have (as $n \to \infty$):*

(i) If $z^\star(p) = 0$ then $\tau_n^\star(p) \xrightarrow{p} \infty$.

(ii) If $z^\star(p) \in (0,1]$ and $z^\star(p)$ is a stable solution, i.e., $f'_W(z^\star;p) > 0$, then $\tau_n^\star(p) \xrightarrow{p} -\ln z^\star(p)$.

By applying Theorem 3.2 and Lemma 3.3, we prove the following limit theorem for the final sold shares of illiquid assets $\Gamma_n(\tau_n^\star; p)$.

Theorem 3.4. *For any fixed $p \in [p_{\min}, p_0]$, the final number of sold shares satisfy:*

(i) If $z^\star(p) = 0$ then asymptotically almost all institutions default after shock and (as $n \to \infty$)

$$\frac{\Gamma_n(\tau_n^\star; p)}{n} \xrightarrow{p} \sum_{x \in \mathcal{X}} \mu_x \Big(\bar{\gamma}_x q_x(0; p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) \theta q_x(\theta; p) \Big).$$

(ii) If $z^\star(p) \in (0, 1]$ and $z^\star(p)$ is a stable solution, i.e., $f_W'(z^\star(p); p) > 0$, then, as $n \to \infty$,

$$\frac{\Gamma_n(\tau_n^\star; p)}{n} \xrightarrow{p} f_\Gamma(z^\star(p); p).$$

The proof of Theorem 3.4 is provided in Section 5.2.

Since g is a continuous function from Assumption 1, by using the continuous mapping theorem, we obtain the convergence of $\kappa_n(p)$. As a direct corollary of Theorem 3.2, we have the following result on the price $\kappa_n(p) := g(\Gamma_n(\tau_n^\star(p); p)/n)$.

Corollary 3.5. *For any fixed $p \in [p_{\min}, p_0]$ and as $n \to \infty$, the price $\kappa_n(p)$ given by the inverse demand function satisfies:*

(i) If $z^\star(p) = 0$ then asymptotically almost all institutions default after shock and

$$\kappa_n(p) \xrightarrow{p} g\Big(\sum_{x \in \mathcal{X}} \mu_x \big(\bar{\gamma}_x q_x(0; p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) \theta q_x(\theta; p) \big) \Big).$$

(ii) If $z^\star(p) \in (0, 1]$ and $z^\star(p)$ is a stable solution, i.e., $f_W'(z^\star(p); p) > 0$, then

$$\kappa_n(p) \xrightarrow{p} g\big(f_\Gamma(z^\star(p); p) \big).$$

We next state a limit theorem for the equilibrium price after shock, defined by Equation-Definition (6). The Corollary 3.5 motivates us to define

$$\bar{p} := \sup\big\{ p \in [p_{\min}, p_0] : p \leqslant g\big(f_\Gamma(z^\star(p); p) \big) \big\}. \tag{8}$$

We say that $\bar{p}$ is a *stable* fixed point solution if either $\bar{p} = p_{\min}$ or, $\bar{p} \in (p_{\min}, p_0]$ and there exists some $\epsilon > 0$ such that $p < g\big(f_\Gamma(z^\star(p); p) \big)$ for all $p \in (\bar{p} - \epsilon, \bar{p})$.

Theorem 3.6. *As $n \to \infty$, the equilibrium price satisfies:*

(i) If $z^\star(\bar{p}) = 0$ and $\bar{p}$ is a stable solution, then the equilibrium price $p_n^\star$ converges to $\bar{p}$ in probability. In this case, $\bar{p}$ is the largest solution of the fixed point equation

$$p = g\Big(\sum_{x \in \mathcal{X}} \mu_x \big(\bar{\gamma}_x q_x(0; p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) \theta q_x(\theta; p) \big) \Big).$$

(ii) If $z^\star(\bar{p}) \in (0, 1]$ is a stable solution of $f_W(z; \bar{p})$, i.e., $\frac{\partial f_W}{\partial z}(z^\star; \bar{p}) > 0$, and $\bar{p}$ is a stable

solution of Equation (8), then as $n \to \infty$,

$$p_n^\star \xrightarrow{p} \bar{p}.$$

The proof of the above theorem is provided in Section 5.3.

We end this section by the following remark. The above theorems could be used to provide a resilience condition for default cascade in random financial networks. Namely, in the notations above, starting from a small fraction ϵ of institutions representing the fundamental defaults, the financial network is said to be resilient if $\lim_{\epsilon\to 0} z^\star(\bar{p}) = 0$. We refer to [3, 5, 34] for the resilience conditions.

3.2 Asymptotic Normality of Default Cascade with Fire Sales

In order to study the central limit theorems, as shown in our companion paper [4], we need to restrict our attention to the sparse networks regime. Namely, we consider the random financial network $\mathcal{G}^{(n)}(\mathbf{d}_n^+, \mathbf{d}_n^-)$ and assume that degrees sequences satisfy the following moment condition.

Assumption 5b. *We assume that for every constant $A > 1$, we have*

$$\sum_{i=1}^n A^{d_i^+} = n \sum_{x\in\mathcal{X}} \mu_x^{(n)} A^{d_x^+} = O(n) \quad \text{and} \quad \sum_{i=1}^n A^{d_i^-} = n \sum_{x\in\mathcal{X}} \mu_x^{(n)} A^{d_x^-} = O(n).$$

For $z \in [0,1]$ and $p \in [p_0, p_{\min}]$, we define the functions:

$$f_S^{(n)}(z;p) := \sum_{x\in\mathcal{X}} \mu_x^{(n)} \Big[\sum_{\theta=1}^{d_x^+} q_x^{(n)}(\theta;p)\beta\big(d_x^+, z, d_x^+ - \theta + 1\big) + q_x^{(n)}(\infty;p) \Big], \quad f_D^{(n)}(z;p) = 1 - f_S^{(n)}(z;p),$$

$$f_W^{(n)}(z;p) := \lambda^{(n)} z - \sum_{x\in\mathcal{X}} \mu_x^{(n)} d_x^- \Big[\sum_{\theta=1}^{d_x^+} q_x^{(n)}(\theta;p)\beta\big(d_x^+, z, d_x^+ - \theta + 1\big) + q_x^{(n)}(\infty;p) \Big].$$

For convenience, we set the time transformed functions

$$\widehat{f}_\clubsuit^{(n)}(t;p) = f_\clubsuit^{(n)}(e^{-t};p),$$

for all the functions in this paper including $\clubsuit \in \{S, D, W\}$.

We have the following theorem (from [4, Theorem 3.5 and Theorem 3.6]) regarding the asymptotic normality of the total number of solvent institutions, defaulted institutions and the total number of white balls (controlling the default contagion stopping time) at any time t before the end of default cascade. Note that the results of [4] could be applied to the default cascade process for any fixed p (when the threshold distribution is fixed). However, since we allow for the possibility that some institution will never get infected, i.e., the institution with ∞ threshold, the theorem holds with different forms of covariance functions. We discuss this extension in Section 5.5.

Theorem 3.7 ([4])**.** *Let $\tau_n \leqslant \tau_n^\star(p)$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$.*

(i) For all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+, \ell = 0, \ldots, \theta - 1$ and jointly in $\mathcal{D}[0, \infty)$,

$$n^{-1/2}\left(S^{(n)}_{x,\theta,\ell}(t \wedge \tau_n; p) - n\mu_x^{(n)} q_x^{(n)}(\theta; p) b\big(d_x^+, 1 - e^{-(t\wedge\tau_n)}, \ell\big)\right) \xrightarrow{d} \mathcal{Z}_{x,\theta,\ell}(t \wedge t_0; p),$$

where $\mathcal{Z}_{x,\theta,\ell}(t;p)$ is a Gaussian process with mean 0 and variance $\sigma_{x,\theta,\ell}(t;p)$ which is provided in [4].

(ii) For $\clubsuit \in \{S, D, W\}$, as $n \to \infty$ and jointly in $\mathcal{D}[0, \infty)$,

$$n^{-1/2}\left(\clubsuit_n(t \wedge \tau_n; p) - n\widehat{f}^{(n)}_{\clubsuit}(t \wedge \tau_n; p)\right) \xrightarrow{d} \mathcal{Z}_{\clubsuit}(t \wedge t_0; p), \tag{9}$$

where $\{Z_\clubsuit\}$ are continuous Gaussian processes on $[0, t_0]$ with mean 0 and covariances provided in [4]. In particular, the form of the variance of $\mathcal{Z}_W$ denoted by $\sigma_W(e^{-t}; p) := \mathrm{Var}(\mathcal{Z}_W(t;p))$, is given by (26).

We next consider a central limit theorem for the total sold shares. We first define the following functions which could be interpreted as the limiting functions of $I^{(n)}_{x,\theta}(e^{-t};p)/n$, $I^{(n)}_{x,\infty}(e^{-t};p)/n$ and $\Gamma_n(e^{-t};p)/n$ respectively,

$$f^{(n)}_{x,\theta}(z;p) := \mu_x^{(n)} q_x^{(n)}(\theta;p)\big(\theta - \sum_{\ell=d_x^+-\theta+1}^{d_x^+} \beta(d_x^+, z, \ell)\big), \quad f^{(n)}_{x,\infty}(z;p) := (1-z)\mu_x^{(n)} q_x^{(n)}(\infty;p) d_x^+,$$

and,

$$f^{(n)}_{\Gamma}(z;p) := \sum_{x\in\mathcal{X}} \mu_x^{(n)}\left(\bar{\gamma}_x q_x^{(n)}(0;p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) f^{(n)}_{x,\theta}(z;p) + \bar{\ell}_{x,\infty}(p) f^{(n)}_{x,\infty}(z;p)\right).$$

The time-transformed versions for the above functions are then

$$\widehat{f}^{(n)}_{x,\theta}(t;p) := f^{(n)}_{x,\theta}(e^{-t};p), \qquad \widehat{f}^{(n)}_{\Gamma}(t;p) := f^{(n)}_{\Gamma}(e^{-t};p),$$

and the same for other functions.

We have the following central limit theorem for the total sold shares.

Theorem 3.8. *Let $\tau_n \leqslant \tau_n^\star$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then for any fixed $p \in [p_{\min}, p_0]$ and $t > 0$, as $n \to \infty$,*

$$n^{-1/2}(\Gamma_n(t \wedge \tau_n; p) - n\widehat{f}^{(n)}_{\Gamma}(t \wedge \tau_n; p)) \xrightarrow{d} \mathcal{Z}_\Gamma(t \wedge t_0; p), \tag{10}$$

where $\mathcal{Z}_\Gamma(t;p)$ is a Gaussian random variable with mean 0 and variance

$$\Psi(t;p) := \mathrm{Var}(\mathcal{Z}_\Gamma(t;p)),$$

where the form of $\Psi(t;p)$ is given by (17).

The proof of the above theorem is provided in Section 5.4. Although we do not have the asymptotic normality for the whole process, the asymptotic normality of $\Gamma_n(t \wedge \tau_n^\star; p)$ for any fixed t is sufficient to deduce some important properties for the final total sold shares.

We use the notation

$$f^1_\clubsuit(z;p) := \frac{\partial f_\clubsuit}{\partial z}(z;p), \qquad f^2_\clubsuit(z;p) := \frac{\partial f_\clubsuit}{\partial p}(z;p),$$

and

$$f_{\clubsuit}^{1,(n)}(z;p) := \frac{\partial f_{\clubsuit}^{(n)}}{\partial z}(z;p), \qquad f_{\clubsuit}^{2,(n)}(z;p) := \frac{\partial f_{\clubsuit}^{(n)}}{\partial p}(z;p),$$

for all (bivariate) functions in this paper, including $\clubsuit \in \{\Gamma, W\}$.

Remark 3.9. *We emphasize that under our assumptions (in particular Assumption 3), the bivariate functions $f_W(z;p), f_W^{(n)}(z;p), f_\Gamma(z;p)$ and $f_\Gamma^{(n)}(z;p)$ have all first order partial derivatives with respect to both z and p, and these are all continuous. Moreover, for any couple $(z,p) \in [0,1] \times [p_{\min}, p_0]$, we have as $n \to \infty$,*

$$f_{\clubsuit}^{1,(n)}(z;p) \to f_{\clubsuit}^{1}(z;p) \quad \text{and} \quad f_{\clubsuit}^{2,(n)}(z;p) \to f_{\clubsuit}^{2}(z;p).$$

For any fixed z, the convergence is uniform on p for all $f_{\clubsuit}^{(n)}$ and their derivatives with respect to p. Further, as shown in [4], under Assumption 5b, the convergences become also uniform with respect to z together with all the derivatives with respect to z, for any fixed price p. Thus the convergences are uniform with respect to both variables z and p under Assumption 5b.

Similarly to Definition-Equation (4.1), for the network of size n, we define

$$z_n^\star(p) := \sup\{z \in [0,1] : f_W^{(n)}(z;p) = 0\}. \tag{11}$$

We then let $t^\star(p) := -\ln z^\star(p)$ and $t_n^\star(p) := -\ln z_n^\star(p)$.

Based on the Theorem 3.7 and Theorem 3.8, we have the following theorem regarding the asymptotic normality of the final total sold shares.

Theorem 3.10. *For any fixed $p \in [p_{\min}, p_0]$, as $n \to \infty$, the final total sold shares satisfies:*

(i) If $z^\star(p) = 0$ then (similarly to Theorem 3.4) asymptotically almost all institutions default after shock and (as $n \to \infty$)

$$\frac{\Gamma_n(\tau_n^\star;p)}{n} \xrightarrow{p} \sum_{x \in \mathcal{X}} \mu_x \Big(\bar{\gamma}_x q_x(0;p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p)\theta q_x(\theta;p)\Big).$$

(ii) If $z^\star(p) \in (0,1]$ and $z^\star(p)$ is a stable solution, i.e., $\alpha(p) := f_W^1(z^\star(p);p) > 0$, then

$$n^{-1/2}(\Gamma_n(\tau_n^\star;p) - n f_\Gamma^{(n)}(t_n^\star(p);p)) \xrightarrow{d} \mathcal{Z}_\Gamma(t^\star(p);p) - \alpha(p)^{-1} f_\Gamma^1(z^\star(p);p)\mathcal{Z}_W(t^\star(p)).$$

The proof of the above theorem is provided in Section 5.6.

As a corollary of Theorem 3.10, we show the following theorem on the price given by inverse demand function $\kappa_n(p) := g(\Gamma_n(\tau_n^\star(p);p)/n)$.

Theorem 3.11. *For any $p \in [p_{\min}, p_0]$ fixed and as $n \to \infty$, the price $\kappa_n(p)$ given by the inverse demand function satisfies:*

(i) If $z^\star(p) = 0$ then asymptotically almost all institutions default after shock and

$$\kappa_n(p) \xrightarrow{p} g\Big(\sum_{x \in \mathcal{X}} \mu_x \big(\bar{\gamma}_x q_x(0;p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p)\theta q_x(\theta;p)\big)\Big).$$

(ii) If $z^\star(p) \in (0,1]$ and $z^\star(p)$ is a stable solution, i.e., $\alpha(p) := f_W^1(z^\star(p);p) > 0$, then

$$n^{1/2}\big(\kappa_n(p) - g\big(f_\Gamma^{(n)}(t_n^\star(p);p)\big)\big) \xrightarrow{d} g'\big(f_\Gamma(z^\star(p);p)\big)\Big[\mathcal{Z}_\Gamma(t^\star(p);p) - \alpha(p)^{-1} f_\Gamma^1(z^\star(p);p)\mathcal{Z}_W(t^\star(p))\Big],$$

where g' denotes the first derivative of g.

The proof of the above theorem is provided in Section 5.7.

We next state a central limit theorem for the equilibrium price after shock, defined by Equation (6). Similarly to Definition-Equation (8), for the network of size n, we define

$$\bar{p}_n := \sup\big\{p \in [p_{\min}, p_0] : p \leqslant g\big(f_\Gamma^{(n)}(z_n^\star(p);p)\big)\big\}. \tag{12}$$

Recall that $\bar{p}$ is a *stable* fixed point solution if either $\bar{p} = p_{\min}$ or, $\bar{p} \in (p_{\min}, p_0]$ and there exists some $\epsilon > 0$ such that $p < g\big(f_\Gamma(z^\star(p);p)\big)$ for all $p \in (\bar{p} - \epsilon, \bar{p})$.

Theorem 3.12. *As $n \to \infty$, the equilibrium price satisfy:*

(i) If $z^\star(\bar{p}) = 0$ and $\bar{p}$ is a stable solution, then the equilibrium price converges to $p_n^\star \xrightarrow{p} \bar{p}$, where $\bar{p}$ is the largest solution of the fixed point equation

$$p = g\Big(\sum_{x\in\mathcal{X}} \mu_x\big(\bar{\gamma}_x q_x(0;p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p)\theta q_x(\theta;p)\big)\Big).$$

(ii) If $z^\star(\bar{p}) \in (0,1]$ is a stable solution of $f_W(z;\bar{p}) = 0$, i.e., $\alpha(\bar{p}) := f_W^1(z^\star;\bar{p}) > 0$, and $\bar{p}$ is a stable solution of (8), then

$$n^{1/2}(p_n^\star - \bar{p}_n) \xrightarrow{d} -\rho^{-1}(\bar{p})\mathcal{Z}_V(\bar{p}),$$

where

$$\rho(p) := 1 - g\big(f_\Gamma(z^\star(p);p)\big)\Big[f_\Gamma^1(z^\star(p);p) - \alpha(p)^{-1} f_W^2(z^\star(p);p) + f_\Gamma^2(z^\star(p);p)\Big],$$

and,

$$\mathcal{Z}_V(p) := -g'\big(f_\Gamma(z^\star;p)\big)\Big[\mathcal{Z}_\Gamma(t^\star(p);p) - \alpha(p)^{-1} f_\Gamma^1(z^\star;p)\mathcal{Z}_W(t^\star(p);p)\Big]$$

is a Gaussian random variable with mean 0.

The proof of the above theorem is provided in Section 5.8.

4 Numerical Experiments

Empirical studies on network topology of banking systems show that we may have very different structures; from centralized networks as in [48] to core-periphery structures [33, 39, 47] and scale-free structures as in [18, 28]. In this section we study the effect of heterogeneity in network structure and price impact function on the final size of default cascade and fire sales loss.

In our numerical experiments, we assume that the in-degree and out-degree are equal for each institution, i.e., $d_x^+ = d_x^- = d_x$ for all $x \in \mathcal{X}$. We further set (normalize) the price of illiquid asset to be between $p_{\min} = 1$ and $p_0 = 2$.

We assume that all institutions with the same type (characteristics) have the same capital structure. For $x \in \mathcal{X}$, we use the capital vector $\mathbf{h}_x$ to describe the capital structure of institutions with type x, given by

$$\mathbf{h}_x := [\gamma_x \quad k_x + a_x \quad \ell_x + d_x \quad e_x].$$

In our stress testing framework, we assume that the initial fraction of defaults is fixed over all types and $q_x(0;p) = \epsilon$ for all $x \in \mathcal{X}$.

We further set $\epsilon_i = \epsilon$ for illustration purposes, so that each initially solvent institution looses a fraction $\epsilon \in [0,1]$ of its external assets. Each time an institution defaults, its incoming counterparties face a loss which are assumed to be i.i.d. random variables with Pareto distribution for some (type-dependent) scale and shape parameters $x_m, \alpha \in \mathbb{R}^+$ to be specified. Then the threshold distributions could be calculated as provided in Example 2.4.

We let the initially defaulted institutions liquidate all their shares of illiquid assets so that (since institutions with same type are assumed to have the same amount of illiquid asset) $\bar{\gamma}_x = \gamma_x$. We let the mean liquidations be of the following linear form

$$\bar{\ell}_{x,\theta}(p) = \frac{\gamma_x}{p\theta} \quad \text{for} \quad \theta = 1, \ldots, d_x,$$

and we further set $\bar{\ell}_{x,\infty}(p) = \frac{\gamma_x}{2pd_x}$ for all $p \in [1,2]$.

In order to study the effect of price impact function on the final size of default cascade and fire sales loss, we consider three different price impact functions, as provided in Examples 2.1, 2.2 and 2.3, with the following concrete forms:

- Linear price impact (LPI): $g^L(y) = 2 - (y/\bar{\gamma})$,
- Quadratic price impact (QPI): $g^Q_\infty(y) = 2 - (y/\bar{\gamma})^2$,
- Exponential price impact (EPI): $g^E_1(y) = 2 - \frac{1-e^{-(y/\bar{\gamma})}}{1-e^{-1}}$,

for all $y \in [0, \bar{\gamma}]$. Note that the LPI function decreases by the same rate for all y. The QPI function drops slowly at the beginning (for small y) and then more and more quickly, as y increases. On the contrary, the EPI function drops rapidly at the beginning and then more and more slowly, as y increases.

To measure how much losses the fire sales brings to the financial system subject to the exogenous shock ϵ, we use the so-called Fire Sales Losses indicator denoted by FSL. Let $p^\star_n(\epsilon)$ be the equilibrium price of illiquid asset after shock ϵ in the above financial system. We define the FSL indicator by setting

$$\text{FSL}(\epsilon) = \frac{p_0 - p^\star_n(\epsilon)}{p_0}.$$

4.1 Regular Financial Networks

In a regular network, all institutions are assumed to be of the same type, and hence same degree d. We investigate how the connectivity and the fire sales affect the default cascade size and how much loss it could bring to the financial system during a crisis. We will compare two situations where the network has high connectivity and low connectivity. One main result of the financial network literature is that, for the regular homogeneous financial networks, when the shocks are small, a higher connectivity leads to a lowest risk of contagion, see e.g., [1] for a comparison of ring and complete network structures. Our results are of this flavor too. However, our results show that, the fire sales loss in the two financial networks with high and low connectivity are very close to each other.

For a d-regular financial network, the limiting function of white ball process simplifies to

$$f_W(z;p) = d\big(z - \sum_{\theta=1}^{d} q(\theta;p)\beta(d,z,d-\theta+1) - q(\infty;p)\big),$$

and hence in this case,

$$z^\star(p) := \sup\big\{z \in [0,1] : z = \sum_{\theta=1}^{d} q(\theta;p)\beta(d,z,d-\theta+1) + q(\infty;p)\big\}.$$

The limiting function of total liquidation also simplifies to

$$f_\Gamma(z;p) = \gamma q(0;p) + \frac{\gamma}{p}\sum_{\theta=1}^{d} \frac{q(\theta;p)}{\theta}\big(\theta - \sum_{\ell=d-\theta+1}^{d} \beta(d,z,\ell)\big) + \frac{\gamma}{2p}(1-z)q(\infty;p).$$

Recall that, from Theorem 3.6, $\bar{p} = \bar{p}(\epsilon)$ given by (8) is the limit for the price of illiquid asset in equilibrium after shock ϵ. Then the (limiting) fire sales loss could be written as

$$\mathrm{FSL}(\epsilon) = \frac{p_0 - \bar{p}(\epsilon)}{p_0}.$$

The final fraction of defaulted institutions under fire sales is given by

$$f_D(z^\star(\bar{p});\bar{p}) = 1 - \sum_{\theta=1}^{d} q(\theta;\bar{p})\beta(d,z,d-\theta+1) - q(\infty;\bar{p}).$$

Moreover, the final fraction of defaults without fire sales (with the initial price $p_0 = 2$) is

$$f_D(z^\star(2);2) = 1 - \sum_{\theta=1}^{d} q(\theta;2)\beta(d,z,d-\theta+1) - q(\infty;2).$$

For the financial network with low connectivity, we set $d_L = 2$ and let the capital vector be $\mathbf{h} = [50 \quad 100 \quad 250 \quad 300]$. For the financial network with high connectivity, we set $d_H = 12$ and, for a comparison, we let the capital vector be the same as the one with low connectivity. However, we let the interbank liabilities depend on the degree. Namely, the expectation of liabilities is assumed to be proportional to $1/d$. This is because we want to keep the total (expected) interbank liabilities to be the same, since we took the same capital vector for the two financial networks with low and high connectivity. We further set $x_m = 160$ and $\alpha = 2$ for $d = 2$, and correspondingly, $x_m = 26.7$ and $\alpha = 2$ for $d = 12$.

In Figure 1 we plot the final fraction of defaulted institutions for the above two regular financial networks with low and high connectivity, and the above three price impact functions, for the linear (LPI) g^L, fully quadratic (QPI) g^Q_∞ and exponential (EPI) g^E_1. In particular, the figure illustrates the following two points. First, as expected, we observe that among the three different price impact functions, EPI causes the largest fraction of defaults for both low and high connectivity, while the QPI causes the smallest default cascade size. This happens because, for the same amount of sold shares, the EPI function always gives the lowest price while the QPI function gives the highest. Second, we can clearly observe that there exists a critical value for shock (depending on the connectivity and price impact function) such that above this critical value, all institutions default. Interestingly, the default cascade size increases smoothly in low connectivity network. In contrary, for the high connectivity financial network, we can see a sharper phase transition at the critical point. Moreover, when the shock is smaller than the critical value, the fraction of defaults increase slowly and is smaller than the value in the low connectivity financial network. But once the shock surpasses the critical value, the fraction of defaults jumps to a higher level than that in the low connectivity one. This phenomenon corresponds well to the existing literature on homogeneous financial network, see e.g., [1]. Namely, for a small shock, the high connectivity network is more resilient, but once the shock is large enough, the default propagates to a larger fraction through its higher connectivity.

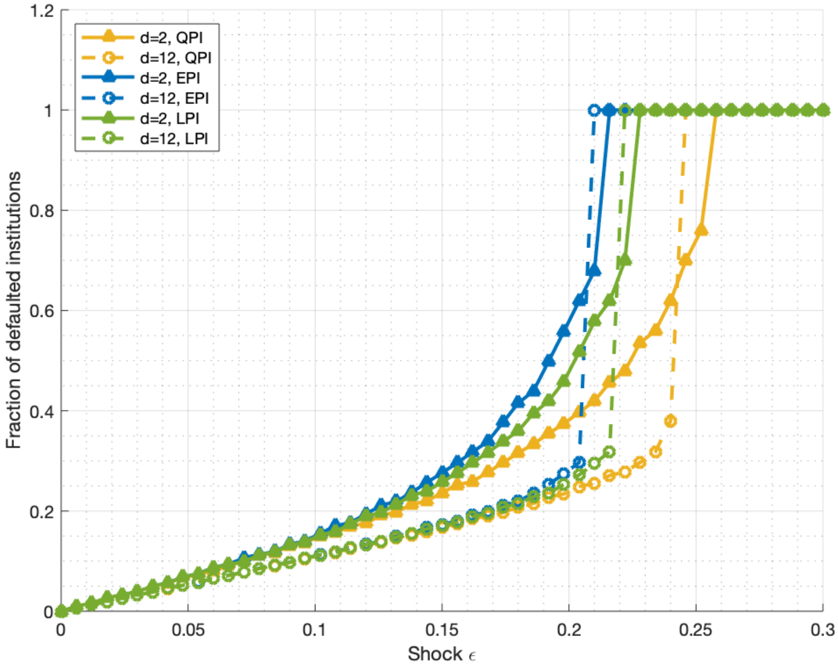

Figure 1: Final fraction of defaulted institutions for two regular financial networks with $d_L = 2$ and $d_H = 12$, under three different price impact functions LPI, QPI and EPI.

In Figure 2 we plot the fire sales loss for the two regular financial networks with low and high connectivity, and the above three price impact functions. Since the fraction of defaults reflects somehow the amount of liquidations, the curves are quite similar to those in Figure 1. In particular, we observe that the EPI function always makes the largest fire sales loss and the QPI makes the smallest one. More interestingly, we observe that the fire sales loss in the two financial networks with high and low connectivity are very close to each other. Indeed, when the shock is small (< 0.15), the high connectivity network could even trigger more fire sales loss than the low connectivity one, which is in contrary to the situation in Figure 1. This happens since the institutions could liquidate while they are solvent. Indeed, for a higher connectivity network, some institutions with high thresholds remain solvent with liquidating around $80 - 90\%$ of their total illiquid assets. In contrary, for a lower connectivity network, the amount of liquidations is much less among the solvent institutions. As the shock goes larger but still smaller than the critical value, the fire sales loss for the low connectivity network could surpass the loss in the high connectivity finacial network, under the three price impact functions.

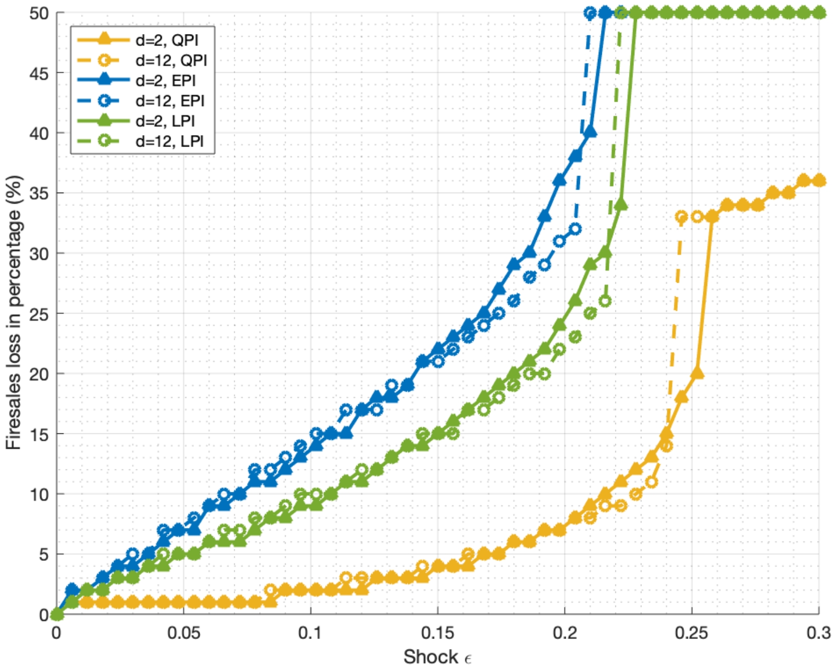

Figure 2: Fire sales loss for two regular financial networks with $d_L = 2$ and $d_H = 12$, under three different price impact functions LPI, QPI and EPI.

4.2 Core-Periphery Financial Networks

Financial networks often involve significant asymmetries, such as the presence of a core-periphery structure. This affects the default cascade size. Large core institutions can be resistant to small shocks, but can trigger the default catastrophically in the financial system when hit with a large shock. Here we do not impose special inter structure for the core and periphery banks but let them connect with each other uniformly at random. We assume two different types $\mathcal{X} = \{C, P\}$, which stand for the core institutions and periphery institutions respectively.

We set the fraction of core and periphery institutions to be respectively $\mu_C = 0.3$ and $\mu_P = 0.7$. For the core institutions, we set the degree $d_C = 12$ with illiquid asset holdings $\gamma_C = 160$, and for the periphery type institutions $d_P = 2$ with $\gamma_P = 60$. Correspondingly, the capital structure vector for core type institutions is set to

$$\mathbf{h}_C = [160 \quad 320 \quad 800 \quad 960],$$

while the capital structure vector for the periphery type institutions is

$$\mathbf{h}_P = [60 \quad 120 \quad 300 \quad 360].$$

The average degree is thus given by $\lambda = 0.3 \times 12 + 0.7 \times 2 = 5$.

In our numerical experiments, we compare the above core-periphery financial network with a 5-regular financial network which has the same average degree. We further set the capital structure to be the same as average core-periphery network. Namely, we consider the 5-regular network where the capital structure of all institutions is fixed to

$$\bar{\mathbf{h}} = 0.3\mathbf{h}_P + 0.7\mathbf{h}_C = [90 \quad 180 \quad 450 \quad 540].$$

We let all interbank liabilities be i.i.d. with Pareto distribution, as in Example 2.4, with scale and shape parameters $x_m = 65$ and $\alpha = 2$.

Let q_C and q_P denote the probability threshold distributions for the core type and periphery type institutions, respectively. In our core-periphery financial network example, the limiting function f_W simplifies to the following form

$$\begin{aligned} f_W(z;p) =& 5z - 3.6\Big(\sum_{\theta=1}^{12} q_C(\theta;p)\beta(12,z,12-\theta+1) + q_C(\infty;p)\Big) \\ & - 1.4\Big(\sum_{\theta=1}^{2} q_P(\theta;p)\beta(2,z,2-\theta+1) + q_P(\infty;p)\Big), \end{aligned}$$

and the limiting function for the total liquidations simplifies to

$$\begin{aligned} f_\Gamma(z;p) =& 48q_C(0;p) + \sum_{\theta=1}^{12} \frac{48}{p\theta} q_C(\theta;p)\Big(\theta - \sum_{\ell=12-\theta+1}^{12} \beta(12,z,\ell)\Big) + \frac{24}{p}(1-z)q_C(\infty;p) \\ & + 42q_P(0;p) + \sum_{\theta=1}^{2} \frac{42}{p\theta} q_P(\theta;p)\Big(\theta - \sum_{\ell=2-\theta+1}^{2} \beta(2,z,\ell)\Big) + \frac{21}{p}(1-z)q_C(\infty;p). \end{aligned}$$

Let $\bar{p}_{cp} = \bar{p}_{cp}(\epsilon)$ given by (8) be the limit for the price of illiquid asset in equilibrium after shock ϵ, by Theorem 3.6, in the above core-periphery financial network. Then the limiting fire sales loss could be written as

$$\text{FSL}(\epsilon) = \frac{p_0 - \bar{p}_{cp}(\epsilon)}{p_0},$$

and the final fraction of defaulted institutions under fire sales is given by

$$\begin{aligned} f_D(z^\star(\bar{p}_{cp});\bar{p}_{cp}) =& 1 - 0.3\Big(\sum_{\theta=1}^{12} q_C(\theta;\bar{p}_{cp})\beta(12,z,12-\theta+1) + q_C(\infty;\bar{p}_{cp})\Big) \\ & - 0.7\Big(\sum_{\theta=1}^{2} q_P(\theta;\bar{p}_{cp})\beta(2,z,2-\theta+1) + q_P(\infty;\bar{p}_{cp})\Big). \end{aligned}$$

In Figure 3 we plot the fire sales loss for the above core-periphery financial network and compare it with the average regular financial network, under the above three different price impact functions LPI, QPI and EPI. In particular, we observe that under each price impact function, regular and core-periphery financial networks perform very closely when the shock is small (smaller than 0.15). When the shock is larger, the core-periphery financial network has less fire sales loss compared to the regular financial network. This happens since a large part of periphery institutions liquidate less than average level. On the other hand, the regular financial network has a larger critical shock value (beyond which all institutions default) compared to the core-periphery network. Smaller critical value for the core-periphery financial network is caused by the core institutions since their degree is very high and they are more likely to trigger a larger default cascade.

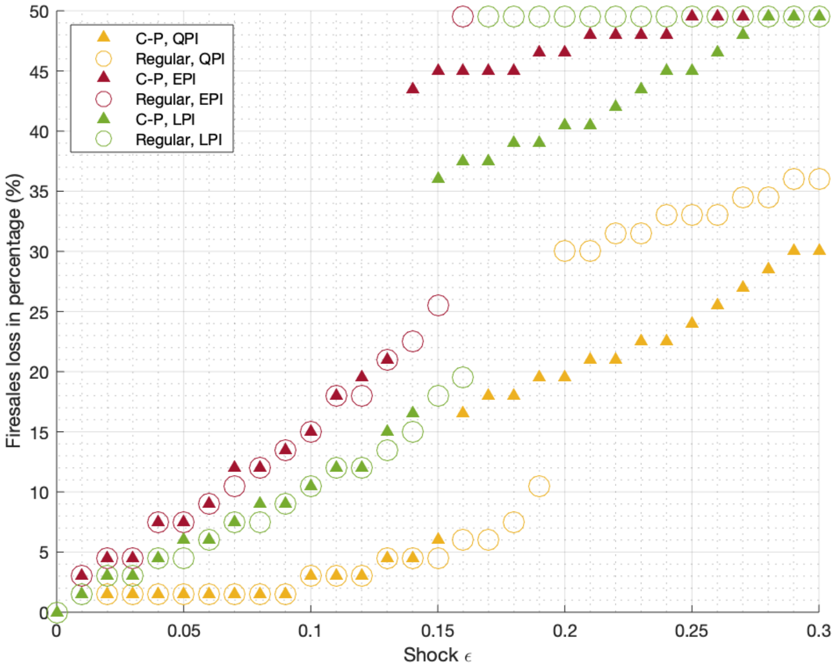

Figure 3: Fire sales loss for core-periphery (C-P) and (average) regular financial networks, under three different price impact functions LPI, QPI and EPI.

In Figure 4 we plot the final fraction of defaulted institutions for the above core-periphery financial network and compare it with the average regular financial network, for the case without fire sales and with fire sales, by considering the above linear (LPI) and exponential (EPI) price impact fucntions. We can clearly observe that the fire sales make both financial networks more vulnerable. Without the fire sales, the core-periphery financial network has a critical shock value around 0.16 (beyond which all institutions default), while for the regular financial network, the critical shock value is around 0.21. By considering the fire sales, both financial networks have smaller critical value for the shock. As expected, we observe that the EPI function gives a smaller critical value compared to the LPI function for both financial networks. This is also interesting to note that the fire sales makes the gap between the two critical shock values (for core-periphery and regular financial networks) smaller. This gap is about 0.05 without fire sales, but with fire sales (under both LPI and EPI functions), the gap is merely around 0.01. This phenomenon could be interpreted by the fact that under fire sales, institutions have smaller thresholds θ to default, since $q_x(\theta; p)$ (stochastically) decreases with price p.

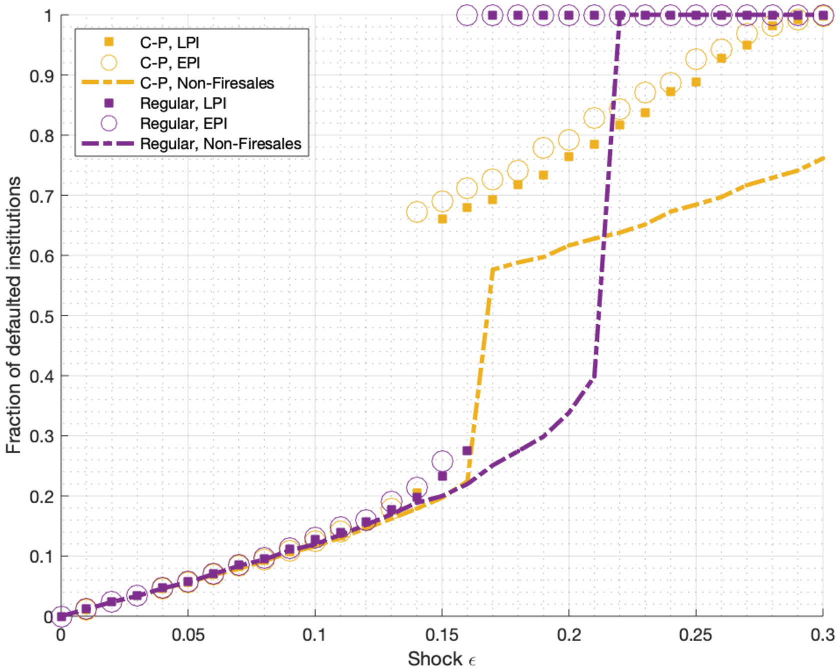

Figure 4: Final fraction of defaults for core-periphery (C-P) and (average) regular financial networks, under three different price impact functions LPI, QPI and EPI.

4.3 Scale-Free Financial Networks

Many empirically observed interbank networks have much more heterogeneity than the core-periphery financial network studied in the previous section. In order to study the effect of heterogeneity in network structure on the final size of default cascade and fire sales loss, we compare the following three financial networks: Regular network (without heterogeneity); Erdös Renyi random network (with low heterogeneity; the vast majority of institutions have a degree close to the average degree) and the Scale-free financial networks (with high heterogeneity). To compare, we let all these financial networks to have the same average degree, which we set it to be $\lambda = 5$.

For the Erdös-Rényi financial network, denoted by $\mathrm{ER}(n; p_n)$, where each pair of nodes (a potential directed link) is independently connected with a fixed probability p_n with $np_n \to \lambda$ as $n \to \infty$, the degree distribution converges to a Poisson distribution with parameter λ, i.e., the (in- and out-) degree of a randomly chosen institution dented by D satisfies

$$\mathbb{P}(D = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

On the other hand, for the scale-free financial network, is given by a power law distribution

$$\mathbb{P}(D = k) \sim ck^{-\eta},$$

where $c > 0$ is a normalizing constant and $\eta > 1$ is a control parameter.

We set the parameters $\lambda = 5$ and $\eta = 1.2$. Moreover, to reduce the simulation complexity, we assume that the degrees are upper bounded by $d_{\max} = 23$. By choosing these parameters and setting the normalizing constant c, both the scale-free and Erdös-Rényi financial networks have an average degree very close to 5. Correspondingly, we will also compare these networks with a regular financial network with degree 5.

We will also consider the heterogeneity on the interbank liabilities for all these networks. We consider the i.i.d. Pareto distributed liabilities, as in Example 2.4, with scale and shape parameters $x_m = 55$ and $\alpha = 2$. We allow for institutions with different degrees to have different capital structures. Namely, we let the capital to be proportional to the degree of each institution. Basically, we impose the capital vector $\mathbf{h}_1$ for the institutions with degree 1 as

$$\mathbf{h}_1 = [50 \quad 100 \quad 250 \quad 300],$$

and then for degrees $d = 2, \ldots, 23$, we set

$$\mathbf{h}_d = [10d + 40 \quad 20d + 80 \quad 50d + 200 \quad 60d + 240].$$

In particular, for the regular financial network, since all institutions have degree 5, the capital structure for each institution is given by $\mathbf{h}_5 = [90 \quad 180 \quad 450 \quad 540]$, as in the previous section.

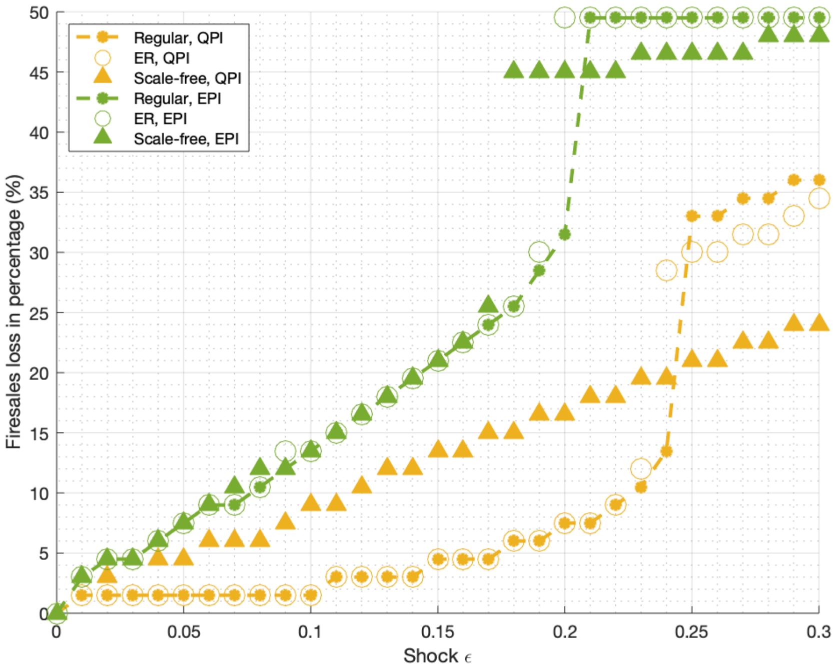

Figure 5: Fire sales loss for regular, Erdös-Rényi (ER) and scale-free financial networks, under the fully quadratic (QPI) and exponential price impact (EPI) functions.

In Figure 5 we plot the fire sales loss for the above regular, Erdös-Rényi (ER) and scale-free financial networks, under the fully quadratic (QPI) and exponential price impact (EPI) functions. We observe that for the EPI function, when the shock is small (less than 0.17 which is the critical shock value for the scale-free network), the heterogeneity does not have a significant influence to the fire sales loss. On the other hand, for the QPI function, when the shock is small, the fire sales loss in scale-free network is larger than the fire sales loss in the two other financial networks. Note that when the shock is small (less than 0.1), the fire sales loss is only about 0.2% for both ER and regular financial networks. This happens since, by choosing a ϵ-fraction of initially defaulted institutions at random among all institutions, a small fraction of initial defaults for high degree institutions might lead to a considerable fraction of defaults among low degree institutions, bringing more fire sales loss. This will be particularly significant under the slow dropping price impact function.

Moreover, as we can observe in Figure 5, a financial network with a higher heterogeneity has smaller critical value for shock (beyond which a large fraction of institutions default). When the shock is larger than the critical value for regular financial network (around 0.2 under EPI and 0.24 under QPI), the regular network turns out to have the largest fire sales loss while the scale-free network has the smallest loss. This is quite reasonable as in the scale-free network, we have a larger proportion of institutions with a low degree (such as 1 and 2) which have more chances to survive for a large value of shock. This makes the scale-free network more resilient for a large shock, compared to the other two networks.

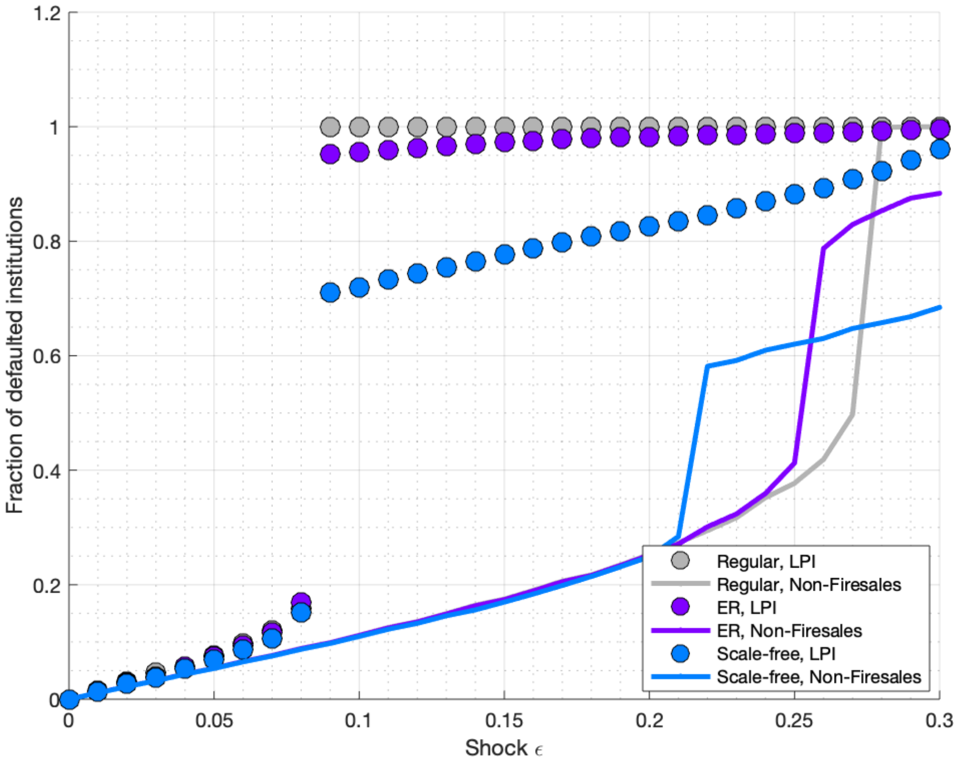

Figure 6: Final fraction of defaults under linear price impact (LPI) and original fraction of defaults without fire sales, for regular, Erdös-Rényi (ER) and scale-free financial networks.

In Figure 6 we plot the final fraction of defaulted institutions for the above regular, Erdös-Rényi (ER) and scale-free financial networks, for the case without fire sales and with the linear price impact (EPI) function fire sales. One could observe quite similar results as those in Figure 4. When the shock is small, the fire sales do not have so much impact on the resilience of financial network. On the other hand, fire sales make the critical values for shocks much smaller than those without fire sales, for all the three considered financial networks. The fire sales also drags the critical values closer. Originally the critical values of regular, ER and scale-free networks are around 0.27, 0.25 and 0.21 respectively. They become around the same value 0.11 under the linear price impact function. Among the three financial networks, the scale-free network has the smallest critical value for shock with and without fire sales, then follows the ER network. The regular financial network has the largest critical value for shock. Moreover, the resistance to a large shock also grows as heterogeneity increases, especially under the fire sales impact. The scale-free network has the smallest fraction of defaults for a shock larger than 0.1.

Therefore, as one can observe from Figure 4 and Figure 6, the financial networks with a higher heterogeneity seems to have a smaller critical value for the shock beyond which a large fraction of institutions default, both with and without fire sales. On the other hand, for the small shocks, the most heterogeneous network could be the least resilient.

5 Proofs

This section contains the proofs of all the theorems in previous sections.

5.1 Proof of Theorem 3.2

By Theorem 3.1 for all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+, \ell = 0, \ldots, \theta - 1$ and $p \in [p_0, p_{\min}]$, as $n \to \infty$,

$$\sup_{t \leqslant \tau_n} \Big| \frac{S^{(n)}_{x,\theta,\ell}(t;p)}{n} - \mu_x q_x(\theta;p) b\left(d_x^+, 1 - e^{-t}, \ell\right) \Big| \xrightarrow{p} 0,$$

and,

$$\sup_{t \leqslant \tau_n} \Big| \frac{S_n(t;p)}{n} - f_S(e^{-t};p) \Big| \xrightarrow{p} 0, \quad \sup_{t \leqslant \tau_n} \Big| \frac{D_n(t;p)}{n} - f_D(e^{-t};p) \Big| \xrightarrow{p} 0.$$

Consider now the death process as described in Section 2.4. We denote by $U^{(n)}_{x,\theta,s}(t;p)$ the number of bins (institutions) with type $x \in \mathcal{X}$, threshold θ and s alive (in-) balls at time t. Further we denote by $N^{(n)}_{x,\theta}(p)$ the number of bins with type x and threshold θ, under price p. Note that the number of bins with type x is (not random) $n\mu_x^{(n)}$. By the construction of balls-and-bins model, every in-ball have an exponentially distributed with parameter one, i.e., exp(1), lifetime independently from others. Based on the Glivenko-Cantelli theorem, in [4] we show the following convergence results of $U^{(n)}_{x,\theta,\ell}(t;p)$, for all possible triple (x, θ, ℓ) and the summation of them; see [4, Lemma 6.4]. We first state this lemma.

Lemma 5.1 ([4]). *Let $\tau_n \leqslant \tau_n^\star(p)$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Under Assumption 5a, for all $x \in \mathcal{X}, \theta = 1, \ldots, d_x^+$ and $\ell = 0, \ldots, \theta - 1$, we have (as $n \to \infty$)*

$$\sup_{t \leqslant \tau_n} \Big| \frac{U^{(n)}_{x,\theta,\ell}(t;p)}{n} - \mu_x q_x(\theta;p) b\left(d_x^+, e^{-t}, \ell\right) \Big| \xrightarrow{p} 0.$$

Further,

$$\sup_{t\leqslant\tau_n}\sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+1}^{d_x^+}\left|U_{x,\theta,s}^{(n)}(t;p)/n-\mu_x q_x(\theta;p)b(d_x^+,e^{-t},s)\right|\xrightarrow{p}0.$$

Consider now $\{L_{x,\theta}^{(i)}(p)\}_{i=1}^{\infty}$ which are i.i.d. positive bounded random variables with expectation $\bar{\ell}_{x,\theta}(p)$ and variance $\varsigma_{x,\theta}^2(p)$ under price $p\in[p_{\min},p_0]$ for the illiquid asset, for all $x\in\mathcal{X}$ and $\theta\in\{1,\dots,d_x^+\}\cup\{\infty\}$. Since all these random losses are assumed to be bounded, we denote by C the common upper bound. From Section 2.5, the total shares of illiquid asset sold by time t could be written as

$$\Gamma_n(t;p):=\sum_{x\in\mathcal{X}}\Big(\bar{\gamma}_x D_{x,0}^{(n)}(p)+\sum_{\theta=1}^{d_x^+}Y_{x,\theta}^{(n)}(t;p),+Y_{x,\infty}^{(n)}(t;p)\Big),$$

where

$$Y_{x,\theta}^{(n)}(t;p):=\sum_{i=1}^{I_{x,\theta}^{(n)}(t;p)}L_{x,\theta}^{(i)}(p),\quad Y_{x,\infty}^{(n)}(t;p):=\sum_{i=1}^{I_{x,\infty}^{(n)}(t;p)}L_{x,\infty}^{(i)}(p),$$

and,

$$I_{x,\theta}^{(n)}(t;p):=\theta D_{x,\theta}^{(n)}(t;p)+\sum_{\ell=1}^{\theta-1}\ell S_{x,\theta,\ell}^{(n)}(t;p),\quad I_{x,\infty}^{(n)}(t;p):=\sum_{\ell=1}^{d_x^+}\ell S_{x,\infty,\ell}^{(n)}(t;p).$$

By Assumption 3 and applying dominated convergence theorem, the first term in $\Gamma_n(t;p)$ converges to

$$\sum_{x\in\mathcal{X}}\bar{\gamma}_x D_{x,0}^{(n)}(p)\xrightarrow{p}\sum_{x\in\mathcal{X}}\bar{\gamma}_x\mu_x q_x(0;p).$$

Note that by definition $S_{x,\theta,\ell}^{(n)}(t;p)=U_{x,\theta,d_x^+-\ell}^{(n)}(t;p)$, which implies that

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\bar{\ell}_{x,\theta}\sum_{\ell=1}^{\theta-1}\ell S_{x,\theta,\ell}^{(n)}(t;p)=\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\bar{\ell}_{x,\theta}\sum_{s=d_x^+-\theta+1}^{d_x^+}(d_x^+-s)U_{x,\theta,s}^{(n)}(t;p),$$

and,

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\theta\bar{\ell}_{x,\theta}D_{x,\theta}^{(n)}(t;p)=\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\theta\bar{\ell}_{x,\theta}\big(N_{x,\theta}^{(n)}(p)-\sum_{s=d_x^+-\theta+1}^{d_x^+}U_{x,\theta,s}^{(n)}(t;p)\big).$$

So for $\theta=1,\dots,d_x^+$, we have

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\bar{\ell}_{x,\theta}I_{x,\theta}^{(n)}(t;p)=\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\theta\bar{\ell}_{x,\theta}N_{x,\theta}^{(n)}(p)-\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\bar{\ell}_{x,\theta}\sum_{s=d_x^+-\theta+1}^{d_x^+}(s-d_x^++\theta)U_{x,\theta,s}^{(n)}(t;p).$$

Notice now that

$$\sum_{s=d_x^+-\theta+1}^{d_x^+}\beta(d_x^+,e^{-t},s)=\sum_{s=d_x^+-\theta+1}^{d_x^+}(s-d_x^++\theta)b(d_x^+,e^{-t},s),$$

and, from the definition $f_{x,\theta}(z;p) := \mu_x q_x(\theta;p)\big(\theta - \sum_{\ell=d_x^+-\theta+1}^{d_x^+}\beta(d_x^+,z,\ell)\big)$, it follows that

$$
\begin{aligned}
&\Big|\frac{1}{n}\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+} Y_{x,\theta}^{(n)}(t;p) - \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\bar{\ell}_{x,\theta} f_{x,\theta}(e^{-t};p)\Big| \\
&\quad\leqslant \Big|\sum_{x\in\mathcal{X}} d_x^+ \sum_{\theta=1}^{d_x^+}\bar{\ell}_{x,\theta}\sum_{s=d_x^+-\theta+1}^{d_x^+}\big(U_{x,\theta,s}^{(n)}(t;p)/n - \mu_x q_x(\theta;p) b(d_x^+,e^{-t},s)\big)\Big| \\
&\qquad + \frac{1}{n}\Big|\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}(Y_{x,\theta}^{(n)}(t;p) - \bar{\ell}_{x,\theta} I_{x,\theta}^{(n)}(t;p))\Big| + \Big|\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\theta\bar{\ell}_{x,\theta}N_{x,\theta}^{(n)}(p)/n - \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\theta\bar{\ell}_{x,\theta}\mu_x q_x(\theta;p)\Big| \\
&\quad\leqslant C\sum_{x\in\mathcal{X}}(d_x^+ + d_x^-)\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+1}^{d_x^+}\big|U_{x,\theta,s}^{(n)}(t;p)/n - \mu_x q_x(\theta;p) b(d_x^+,e^{-t},s)\big| \\
&\qquad + \frac{1}{n}\Big|\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}(Y_{x,\theta}^{(n)}(t;p) - \bar{\ell}_{x,\theta} I_{x,\theta}^{(n)}(t;p))\Big| + C\Big|\sum_{x\in\mathcal{X}} d_x^+\sum_{\theta=1}^{d_x^+}\big(N_{x,\theta}^{(n)}(p)/n - \mu_x q_x(\theta;p)\big)\Big|.
\end{aligned}
$$

The first term converges to 0, as $n\to\infty$, by Lemma 5.1. For the second term, notice that for all $n, x\in\mathcal{X}$ and $t\leqslant\tau_n$, the term $\sum_{\theta=1}^{d_x^+}\big(Y_{x,\theta}^{(n)}(t;p) - \bar{\ell}_{x,\theta}I_{x,\theta}^{(n)}(t;p)\big)$ is a martingale. Combining this with the independency between any two different types in $\mathcal{X}$, by using Cauchy-Schwarz inequality and Doob's L^2-inequality, we have that for some constant C_0, as $n\to\infty$,

$$
\mathbb{E}\Big[\sup_{t\leqslant\tau_n}\frac{1}{n}\Big|\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}(Y_{x,\theta}^{(n)}(t;p) - \bar{\ell}_{x,\theta}I_{x,\theta}^{(n)}(t;p))\Big|\Big]^2 \leqslant \frac{4C_0^2}{n^2}\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+} I_{x,\theta}^{(n)}(\tau_n) \leqslant \frac{4C_0^2}{n}\sum_{x\in\mathcal{X}}\mu_x^{(n)} d_x^+ \to 0.
$$

The second inequality follows from $\sum_{\theta=1}^{d_x^+} I_{x,\theta}^{(n)}(t;p)/n \leqslant \mu_x^{(n)} d_x^+$ for all $t\leqslant\tau_n$. The final convergence holds from Assumption 5a. We next analyze the convergence result for the third term. First notice that, by the law of large numbers and Assumption 3, $N_{x,\theta}^{(n)}(p)/n \xrightarrow{p} \mu_x q_x(\theta;p)$. Let $\mathcal{X}_K^+$ be the set of all characteristic $x\in\mathcal{X}$ such that $d_x^+ \geqslant K$. Since by Assumption 5a, $\lambda\in(0,\infty)$, for arbitrary small $\varepsilon>0$, there exists K_ε such that $\sum_{x\in\mathcal{X}_{K_\varepsilon}}\mu_x d_x^+ < \varepsilon$. Then by dominated convergence theorem, we obtain for n large enough,

$$
\sum_{x\in\mathcal{X}_{K_\varepsilon}} d_x^+\sum_{\theta=1}^{d_x^+} N_{x,\theta}^{(n)}(p)/n \xrightarrow{p} \sum_{x\in\mathcal{X}_{K_\varepsilon}} d_x^+\sum_{\theta=1}^{d_x^+}\mu_x q_x(\theta) \leqslant \sum_{x\in\mathcal{X}_{K_\varepsilon}} d_x^+\mu_x < \varepsilon..
$$

It therefore follows that

$$
C\Big|\sum_{x\in\mathcal{X}} d_x^+\sum_{\theta=1}^{d_x^+}\big(N_{x,\theta}^{(n)}(p)/n - \mu_x q_x(\theta;p)\big)\Big| \leqslant C\sum_{x\in\mathcal{X}\setminus\mathcal{X}_{K_\varepsilon}} d_x^+\sum_{\theta=1}^{d_x^+}\big|N_{x,\theta}^{(n)}(p)/n - \mu_x q_x(\theta;p)\big| + C\varepsilon = o_p(1) + C\epsilon.
$$

We thus conclude that

$$
\sup_{t\leqslant\tau_n}\Big|\frac{1}{n}\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+} Y_{x,\theta}^{(n)}(t;p) - \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\bar{\ell}_{x,\theta} f_{x,\theta}(e^{-t};p)\Big| \xrightarrow{p} 0.
$$

It thus only remains to prove the convergence for the third term (infinite sum) in $\Gamma_n(t;p)$.

First notice that

$$I^{(n)}_{x,\infty}(t;p) = \sum_{s=0}^{d_x^+} (d_x^+ - s) U^{(n)}_{x,\infty,s}(t;p).$$

By using Lemma 5.1, for any type $x \in \mathcal{X}$, we have that

$$\sup_{t \leqslant \tau_n} \left| I^{(n)}_{x,\infty}(t;p)/n - \sum_{s=1}^{d_x^+} (d_x^+ - s)\mu_x q_x(\infty;p) b(d_x^+, e^{-t}, s) \right| \xrightarrow{p} 0.$$

Moreover,

$$\sum_{s=1}^{d_x^+} (d_x^+ - s)\mu_x q_x(\infty) b(d_x^+, e^{-t}, s) = \mu_x q_x(\infty)(d_x^+ - z d_x^+).$$

Then, by following a similar argument as above, one can show that

$$\sup_{t \leqslant \tau_n} \left| \frac{1}{n} \sum_{x \in \mathcal{X}} Y^{(n)}_{x,\infty}(t;p) - \sum_{x \in \mathcal{X}} \bar{\ell}_{x,\infty} f_{x,\infty}(e^{-t};p) \right| \xrightarrow{p} 0.$$

Putting all these convergence results together, we conclude that

$$\sup_{t \leqslant \tau_n} \left| \frac{\Gamma_n(t;p)}{n} - f_\Gamma(e^{-t};p) \right| \xrightarrow{p} 0,$$

as desired.

5.2 Proof of Theorem 3.4

Fix $p \in [p_{\min}, p_0]$. The theorem follows from Theorem 3.2 and Lemma 3.3. Indeed, for $z^\star(p) = 0$, by Lemma 3.3, $\tau_n^\star(p) \xrightarrow{p} \infty$. Note also that $z^\star(p) = 0$ indicates that almost all institutions default during the cascade. In this case, for all $x \in \mathcal{X}$, we have $q_x(\infty;p) = 0$. Otherwise $z^\star(p)$ can not be 0, since if $q_x(\infty;p) > 0$ for some $x \in \mathcal{X}$, then we have $f_W(0;p) < 0$. Then $e^{-\tau_n^\star} \xrightarrow{p} 0$, and we have

$$f_\Gamma(0;p) = \sum_{x \in \mathcal{X}} \mu_x \Big(\bar{\gamma}_x q_x(0;p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) \theta q_x(\theta;p) \Big).$$

Thus it follows by the continuity of f_Γ that

$$f_\Gamma(e^{-\tau_n^\star(p)};p) = \sum_{x \in \mathcal{X}} \mu_x \Big(\bar{\gamma}_x q_x(0;p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) \theta q_x(\theta;p) \Big) + o_p(1).$$

We therefore have by Theorem 3.2 that

$$\frac{\Gamma_n(\tau_n^\star(p))}{n} \xrightarrow{p} \sum_{x \in \mathcal{X}} \mu_x \Big(\bar{\gamma}_x q_x(0;p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) \theta q_x(\theta;p) \Big).$$

To prove the point (ii), again by Lemma 3.3, it follows that $\tau_n^\star(p) \xrightarrow{p} -\ln z^\star(p)$, then $e^{-\tau_n^\star(p)} \xrightarrow{p} z^\star(p)$. By a similar argument and applying Theorem 3.2, we conclude that

$$\Gamma_n(\tau_n^\star(p)) \xrightarrow{p} f_\Gamma(z^\star(p);p).$$

5.3 Proof of Theorem 3.6

By Theorem 3.4 and Corollary 3.5, for all $p \in [p_{\min}, p_0]$,

$$\kappa_n(p) \xrightarrow{p} g\big(f_\Gamma(z^\star(p); p)\big).$$

Let us define

$$\Phi_n(p) := p - g(\Gamma_n(\tau_n^\star; p)/n),$$

so that a fixed $p_1 > \bar{p}$, we have

$$\Phi_n(p_1) = p_1 - g\big(f_\Gamma(z^\star(p_1); p_1)\big) - o_p(1).$$

From Definition (8) for $\bar{p}$, it thus follows that, for n large enough, $\mathbb{P}(p_n^\star > p_1) \to 0$.

On the other hand, since $\bar{p}$ is a stable solution, if $\bar{p} = p_{\min}$, then by taking p_1 arbitrarily close to $p_{\min}$, it follows that $p_n^\star \xrightarrow{p} \bar{p}$. Moreover, if $\bar{p} \in (0,1]$, there exists some $\epsilon > 0$ such that $p < g\big(f_\Gamma(z^\star(p); p)\big)$ for all $p \in (\bar{p} - \epsilon, \bar{p})$. Similarly, for any $\bar{p} - \epsilon < p_2 < \bar{p}$, we have $\Phi_n(p_2) < 0$ with high probability, i.e., as $n \to \infty$, $\mathbb{P}(p_n^\star < p_2) \to 0$.

Then by taking p_1 and p_2 arbitrarily close to $\bar{p}$, we conclude that $p_n^\star \xrightarrow{p} \bar{p}$.

Notice that for point (i), when $z^\star(\bar{p}) = 0$, this indicates that almost all institutions default during the cascade and for all $x \in \mathcal{X}$, we necessarily have $q_x(\infty; \bar{p}) = 0$. Otherwise $z^\star(\bar{p}) \neq 0$ since if $q_x(\infty; \bar{p}) > 0$ for some $x \in \mathcal{X}$, then $f_W(0; \bar{p}) < 0$. Hence, it follows that for $z^\star(\bar{p}) = 0$,

$$g(f_\Gamma(z^\star(\bar{p}); \bar{p})) = g\Big(\sum_{x \in \mathcal{X}} \mu_x \big(\bar{\gamma}_x q_x(0; \bar{p}) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(\bar{p}) \theta q_x(\theta; \bar{p})\big)\Big).$$

Moreover, as this will be shown by Lemma 5.4 in Section 5.8, the function ϕ is locally continuous at $\bar{p}$. It thus follows that $\bar{p}$ is the largest solution of the fixed point equation

$$p = g\Big(\sum_{x \in \mathcal{X}} \mu_x \big(\bar{\gamma}_x q_x(0; p) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) \theta q_x(\theta; p)\big)\Big).$$

This completes the proof of Theorem 3.6.

5.4 Proof of Theorem 3.8

We first state some lemmas from [4] which will be served for the proof of Theorem 3.8.

Recall that $U_{x,\theta,s}^{(n)}(t)$ denotes the number of bins (institutions) with type $x \in \mathcal{X}$, threshold θ and s alive (in-) balls at time t. Further, we let $V_{x,\theta,s}^{(n)}(t)$ denote the number of bins with type $x \in \mathcal{X}$, threshold θ and at least s alive balls at time t, so that $V_{x,\theta,s}^{(n)}(t) = \sum_{\ell \geqslant s} U_{x,\theta,\ell}^{(n)}(t)$.

We next define

$$V_{x,\theta,s}^{*(n)}(t; p) := n^{-1/2}\big(V_{x,\theta,s}^{(n)}(t; p) - n\mu_x^{(n)} q_x^{(n)}(\theta; p)\beta(d_x^+, e^{-t}, s)\big),$$

and

$$N_{x,\theta}^{*(n)}(p) := n^{-1/2}\big(N_{x,\theta}^{(n)}(p)(p) - n\mu_x^{(n)} q_x^{(n)}(\theta; p)\big).$$

We need the following lemma from [4, Lemma 6.5] which shows the joint convergence of

$N^{*(n)}_{x_1,\theta_1}$ and $V^{*(n)}_{x_2,\theta_2,s}$ for all possible (x_1,θ_1) and (x_2,θ_2,s). Notice that in this paper we allow for the threshold to be $\theta=\infty$ (see Section 5.5) and the results depend on p. But the lemma stays valid fo any fixed $p\in[p_{\min},p_0]$.

Lemma 5.2. *([4]) Let $\tau_n \leqslant \tau^\star_n$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0>0$. Under Assumption 5b and for any fixed $p\in[p_{\min},p_0]$, we have that for all couple $x\in\mathcal{X}$ and $\theta\in\{1,\dots,d^+_x\}\cup\{\infty\}$, jointly as $n\to\infty$,*

$$N^{*(n)}_{x,\theta}(p) \xrightarrow{d} \mathcal{Y}^*_{x,\theta}(p),$$

where all $\mathcal{Y}^*_{x,\theta}(p)$ are Gaussian random variables with mean 0 and covariance

$$\mathrm{Cov}(\mathcal{Y}^*_{x_1,\theta_1}(p),\mathcal{Y}^*_{x_2,\theta_2}(p)) = \psi_{x_1,\theta_1,\theta_2}(p)\mathbb{1}\{x_1=x_2\},$$

where

$$\psi_{x,\theta,\theta}(p) := \mu_x q_x(\theta;p)(1-q_x(\theta;p)), \quad \psi_{x,\theta_1,\theta_2}(p) := -\mu_x q_x(\theta_1;p)q_x(\theta_2;p) \quad \textit{for all} \quad \theta_1\neq\theta_2.$$

Further, for all triple (x,θ,s), jointly in $\mathcal{D}[0,\infty)$ and as $n\to\infty$,

$$V^{*(n)}_{x,\theta,s}(t\wedge\tau_n;p) \xrightarrow{d} \mathcal{Z}^*_{x,\theta,s}(t\wedge t_0;p),$$

where all $\mathcal{Z}^*_{x,\theta,s}(t;p)$ are continuous Gaussian processes with mean 0 and covariances

$$\begin{aligned}
\mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,s_1}(t;p),\mathcal{Z}^*_{x_2,\theta_2,s_2}(t;p)\big) &=0, \quad \textit{for all} \quad x_1\neq x_2,\\
\mathrm{Cov}\big(\mathcal{Z}^*_{x,\theta_1,s_1}(t;p),\mathcal{Z}^*_{x,\theta_2,s_2}(t;p)\big) &=\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(e^{-t};p), \quad \textit{for all} \quad \theta_1\neq\theta_2,\\
\mathrm{Cov}\big(\mathcal{Z}^*_{x,\theta,s_1}(t;p),\mathcal{Z}^*_{x,\theta,s_2}(t;p)\big) &=\widehat{\sigma}_{x,\theta,\theta,s_1,s_2}(e^{-t};p)+\widetilde{\sigma}_{x,\theta,s_1,s_2}(e^{-t};p),
\end{aligned}$$

where

$$\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(e^{-t};p) := \beta(d^+_x,e^{-t},s_1)\beta(d^+_x,e^{-t},s_2)\psi_{x,\theta_1,\theta_2}(p),$$

with $\widetilde{\sigma}_{x,\theta,s_1,s_2}=\widetilde{\sigma}_{x,\theta,s_2,s_1}$ and

$$\widetilde{\sigma}_{x,\theta,s,s+k}(y;p) := \frac{1}{2}y^{2s+k}\sum_{j=s+k}^{d^+_x}\binom{j-1}{s-1}\binom{j-1}{s+k-1}\int_y^1 (v-y)^{2j-2s-k}v^{-2j}d\varphi_{x,\theta,j}(v;p),$$

with $\varphi_{x,\theta,j}(y;p) := \mu_x q_x(\theta;p)\beta(d^+_x,y,j)$.

Moreover, the covariance between $\mathcal{Z}^*_{x_1,\theta_1,s}(t;p)$ and $\mathcal{Y}^*_{x_2,\theta_2}(p)$ is given by

$$\mathrm{Cov}\big(\mathcal{Z}^*_{x_1,\theta_1,s}(t;p),\mathcal{Y}^*_{x_2,\theta_2}(p)\big) = \beta(d^+_{x_1},e^{-t},s)\psi_{x_1,\theta_1,\theta_2}(p)\mathbb{1}\{x_1=x_2\}.$$

By using the above lemma, we first show the following result regarding the asymptotic normality for $I_{x,\theta}(t;p)$, the total number of liquidations (infected links) for institutions with type $x\in\mathcal{X}$ and threshold θ up to time t and under price p .

Lemma 5.3. *Let $\tau_n \leqslant \tau^\star_n$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0>0$. Under Assumption 5b and for any fixed $p\in[p_{\min},p_0]$, for all $x\in\mathcal{X}$, $\theta\in\{1,\dots,d^+_x\}\cup\{\infty\}$, we have* *the following joint convergence in* $\mathcal{D}[0,\infty)$ *as* $n \to \infty$,

$$n^{-1/2}(I_{x,\theta}^{(n)}(t \wedge \tau_n; p) - n\hat{f}_{x,\theta}^{(n)}(t \wedge \tau_n; p)) \xrightarrow{d} \mathcal{Z}_{I_{x,\theta}}(t \wedge t_0; p), \tag{13}$$

where all $\mathcal{Z}_{I_{x,\theta}}(t;p)$ *are Gaussian processes with mean 0 and covariances*

$$\mathrm{Cov}(\mathcal{Z}_{I_{x_1,\theta_1}}(t;p), \mathcal{Z}_{I_{x_2,\theta_2}}(t;p)) = \sigma^I_{x_1,\theta_1,\theta_2}(e^{-t};p)\mathbb{1}\{x_1 = x_2\},$$

where the form of $\sigma^I_{x,\theta_1,\theta_2}(y;p)$ *is given by* (20)-(23).

For the sake of readability, we postpone the proof of lemma to the end of this section.

We next consider the total liquidations, given by

$$Y_{x,\theta}^{(n)}(t;p) := \sum_{i=1}^{I_{x,\theta}^{(n)}(t;p)} L_{x,\theta}^{(i)}(p) \quad \text{and} \quad Y_{x,\infty}^{(n)}(t;p) := \sum_{i=1}^{I_{x,\infty}^{(n)}(t;p)} L_{x,\infty}^{(i)}(p),$$

where $\{L_{x,\theta}^{(i)}(p)\}_{i=1}^{\infty}$ are i.i.d. positive bounded random variables with expectation $\bar{\ell}_{x,\theta}(p)$ and variance $\varsigma^2_{x,\theta}(p)$ for $p \in [p_{\min}, p_0], x \in \mathcal{X}$ and $\theta \in \{1, \dots, d_x^+\} \cup \{\infty\}$.

Note that conditioned on $I_{x_1,\theta_1}^{(n)}$ and $I_{x_2,\theta_2}^{(n)}$, the processes $Y_{x_1,\theta_1}^{(n)}(t;p)$ and $Y_{x_2,\theta_2}^{(n)}(t;p)$ are independent for $(x_1,\theta_1) \neq (x_2,\theta_2)$. In particular, from Lemma 5.3, for $x_1 \neq x_2$ we have that

$$\mathrm{Cov}\big(Y_{x_1,\theta_1}^{(n)}(t;p), Y_{x_2,\theta_2}^{(n)}(t;p)\big) = 0.$$

Consider now the decomposition

$$Y_{x,\theta}^{(n)}(t;p) - \hat{f}_{x,\theta}^{(n)}(t;p) = \big(Y_{x,\theta}^{(n)}(t;p) - \bar{\ell}_{x,\theta}(p)I_{x,\theta}^{(n)}(t;p)\big) + \big(\bar{\ell}_{x,\theta}(p)I_{x,\theta}^{(n)}(t;p) - \hat{f}_{x,\theta}^{(n)}(t;p)\big),$$

which implies that

$$\mathrm{Cov}\big(Y_{x,\theta_1}^{(n)}(t;p), Y_{x,\theta_2}^{(n)}(t;p)\big) = \bar{\ell}_{x,\theta_1}(p)\bar{\ell}_{x,\theta_2}(p)\mathrm{Cov}\big(I_{x,\theta_1}^{(n)}(t;p), I_{x,\theta_2}^{(n)}(t;p)\big),$$

and the same holds for their limit processes.

We now proceed to the proof of Theorem 3.8. The proof is based on some auxiliary results regarding a central limit theorem for processes which could be written as $Y_n(t) := \sum_{i=1}^{\lfloor X_n(t) \rfloor} G_i$, where $X_n(t)$ is a non-decreasing stochastic process satisfying $X_n(t) = O(n)$ for all $t > 0$ and $\{G_i\}_{i \geqslant 1}$ are i.i.d. positive bounded random variables. This is provided in Appendix A.

Notice that the processes $I_{x,\theta}^{(n)}(t \wedge \tau_n^\star; p)$ for all $x \in \mathcal{X}, p \in [p_{\min}, p_0]$ and $\theta \in \{1, \dots, d_x^+\} \cup \{\infty\}$ satisfy the conditions for $X_n(t)$ in Proposition A.2. Indeed, $\hat{f}_{x,\theta}^{(n)}(t;p) \to \hat{f}_{x,\theta}(t;p)$ uniformly on $[0,\infty)$. Combining with the continuity of $\hat{f}_{x,\theta}(t;p)$, it follows that $\hat{f}_{x,\theta}^{(n)}(t \wedge \tau_n; p) \xrightarrow{p} \hat{f}_{x,\theta}(t \wedge t_0)$, as $n \to \infty$. Then using the Skorokhod coupling theorem [46, Theorem 3.30], we can assume that $\tau_n \to t_0$ a.s. in a new common probability space. It follows then a.s. $\hat{f}_{x,\theta}^{(n)}(t \wedge \tau_n; p) \to \hat{f}_{x,\theta}(t \wedge t_0)$. Thus, by Lemma 5.3, we have that for each ω outside a probability null set, for all $x \in \mathcal{X}$ and $\theta \in \{1, \dots, d_x^+\} \cup \{\infty\}$, the process $I_{x,\theta}^{(n)}(t \wedge \tau_n; p)$ satisfies the conditions for $X_n(t)$ in Proposition A.2, with $f_n(t) = \hat{f}_{x,\theta}^{(n)}(t \wedge \tau_n(\omega))$ for different ω, but common $f(t) = \hat{f}_{x,\theta}(t \wedge t_0)$ and $\mathcal{V} = \mathcal{Z}_{x,\theta}(t \wedge t_0)$. This leads to the same limit distribution up to a probability null set.

For convenience, we let

$$\Delta_{x,\theta}^{(n)}(t;p) := n^{-1/2}\big(Y_{x,\theta}^{(n)}(t;p) - n\bar{\ell}_{x,\theta}\widehat{f}_{x,\theta}^{(n)}(t;p)\big).$$

By Proposition A.2, we have that for all $x \in \mathcal{X}$, $\theta \in \{1,\ldots,d_x^+\} \cup \{\infty\}$ and a fixed $t > 0$, as $n \to \infty$, the following convergence holds

$$\Delta_{x,\theta}^{(n)}(t \wedge \tau_n; p) \xrightarrow{d} \mathcal{Z}_{x,\theta}(t \wedge t_0; p),$$

where $\mathcal{Z}_{x,\theta}(t;p)$ is Gaussian with mean 0 and variance

$$\Psi_{x,\theta}(t;p) := \widehat{f}_{x,\theta}(t;p)\varsigma_{x,\theta}^2(p) + \bar{\ell}_{x,\theta}^2(p)\sigma_{x,\theta,\theta}^I(e^{-t};p). \tag{14}$$

Moreover, from the above arguments, the covariances between two different types $x_1 \neq x_2$ will be 0 and for $\theta_1 \neq \theta_2$, we have

$$\mathrm{Cov}(\mathcal{Z}_{x,\theta_1}(t;p), \mathcal{Z}_{x,\theta_2}(t;p)) = \bar{\ell}_{x,\theta_1}(p)\bar{\ell}_{x,\theta_2}(p)\sigma_{x,\theta_1,\theta_2}^I(e^{-t};p).$$

We next consider the convergence of the following infinite sum

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\Delta_{x,\theta}^{(n)}(t \wedge \tau_n; p).$$

Recall that $\mathcal{X}_s^+$ denotes the collection of all characteristics $x \in \mathcal{X}$ with the in-degree $d_x^+ \geqslant s$. Recall also that all random variables $L_{x,\theta}^{(i)}(p)$ are assumed to be bounded. Then there exists some constant C such that $L_{x,\theta}^{(i)}(p) < C$, for all $x \in \mathcal{X}, p \in [p_{\min}, p_0], \theta \in \{1,\ldots,d_x^+\} \cup \{\infty\}$ and $i \in \mathbb{N}$. Thus we have for any fixed $T > 0$,

$$\mathbb{E}\Big[\sup_{t\leqslant T}\Big|\sum_{x\in\mathcal{X}_s^+}\sum_{\theta=1}^{d_x^+}\Delta_{x,\theta}^{(n)}(t;p)\Big|\Big] \leqslant C\mathbb{E}\Big[\sup_{t\leqslant T}\Big|\sum_{x\in\mathcal{X}_s^+}\sum_{\theta=1}^{d_x^+}n^{-1/2}(I_{x,\theta}^{(n)}(t;p) - n\widehat{f}_{x,\theta}^{(n)}(t;p))\Big|\Big] \tag{15}$$

$$+ \mathbb{E}\Big[\sup_{t\leqslant T}\Big|\sum_{x\in\mathcal{X}_s^+}\sum_{\theta=1}^{d_x^+}n^{-1/2}(Y_{x,\theta}^{(n)}(t;p) - \bar{\ell}_{x,\theta}I_{x,\theta}^{(n)}(t;p))\Big|\Big]. \tag{16}$$

We first show that the first term on RHS converges to 0 as $s \to \infty$ for n large enough. Indeed, [4, Lemma 6.7] implies that when n is large enough, for any $T > 0$, as $\ell \to \infty$,

$$\mathbb{E}\Big[\sup_{t\leqslant T}\Big|\sum_{x\in\mathcal{X}_\ell^+}\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+1}^{d_x^+}V_{x,\theta,s}^{*(n)}(t \wedge \tau_n; p)\Big|\Big] \to 0.$$

Moreover, as shown in the proof of Lemma 5.3,

$$\sum_{x\in\mathcal{X}_s^+}\sum_{\theta=1}^{d_x^+}n^{-1/2}(I_{x,\theta}^{(n)}(t;p) - n\widehat{f}_{x,\theta}^{(n)}(t;p)) = \sum_{x\in\mathcal{X}_\ell^+}\sum_{\theta=1}^{d_x^+}N_{x,\theta}^{*(n)} - \sum_{x\in\mathcal{X}_\ell^+}\sum_{\theta=1}^{d_x^+}\sum_{s=d_x^+-\theta+1}^{d_x^+}V_{x,\theta,s}^{*(n)}(t \wedge \tau_n; p),$$

and by the Cauchy-Schwarz inequality,

$$\mathbb{E}\Big|\sum_{x\in\mathcal{X}_\ell^+}\sum_{\theta=1}^{d_x^+} N_{x,\theta}^{*(n)}\Big| \leqslant \sum_{x\in\mathcal{X}_\ell^+} q_x^{(n)}(\infty)(1-q_x^{(n)}(\infty)),$$

which goes again to 0 as $\ell\to\infty$ uniformly in n. We thus conclude that the first term on RHS of (15) converges to 0.

For the second term on RHS of (15), first note that each term of the sum inside the expectation is a martingale. Then, by the Doob inequality, we can control its L^2-norm by $4C^2\sum_{x\in\mathcal{X}_s^+} d_x^+\mu_x^{(n)}$. Hence, using again Assumption 5b, the L^2-bound converges to 0 as $s\to\infty$ for n large enough. We therefore have that the second term converges to 0 as $s\to\infty$ for n large enough, as desired. We can then take the limit under the infinite sum, by using e.g., [17, Theorem 4.2]. It therefore follows that

$$\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\Delta_{x,\theta}^{(n)}(t\wedge\tau_n;p) \xrightarrow{d} \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\mathcal{Z}_{x,\theta}(t\wedge t_0).$$

For the second and third term of $n^{-1/2}(\Gamma_n(t\wedge\tau_n;p)-n\widehat{f}_\Gamma^{(n)}(t\wedge\tau_n;p))$, by using similar arguments as above, we obtain

$$\sum_{x\in\mathcal{X}} n^{-1/2}\big(Y_{x,\infty}^{(n)}(t;p)-n\bar{\ell}_{x,\infty}\widehat{f}_{x,\infty}^{(n)}(t\wedge\tau_n;p)\big) \xrightarrow{d} \sum_{x\in\mathcal{X}}\mathcal{Z}_{x,\infty}(t\wedge t_0),$$

and,

$$\sum_{x\in\mathcal{X}}\bar{\gamma}_x N_{x,0}^{*(n)} \xrightarrow{d} \sum_{x\in\mathcal{X}}\bar{\gamma}_x\mathcal{Y}_{x,0}^*.$$

Hence we have

$$\mathcal{Z}_\Gamma(t\wedge t_0) := \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\mathcal{Z}_{x,\theta}(t\wedge t_0) + \sum_{x\in\mathcal{X}}\mathcal{Z}_{x,\infty}(t\wedge t_0) + \sum_{x\in\mathcal{X}}\bar{\gamma}_x\mathcal{Y}_{x,0}^*,$$

which is a centered Gaussian random variable with mean 0. By Lemma 5.2, Lemma 5.3 and above arguments, the variance is given by

$$\begin{aligned}\Psi(t;p) = &\sum_{x\in\mathcal{X}}\Big(\sum_{\theta=1}^{d_x^+}\Psi_{x,\theta}(t;p)+\Psi_{x,\infty}(t;p)+\bar{\gamma}_x^2\psi_{x,0,0}(p)\Big) + \sum_{x\in\mathcal{X}}\sum_{\theta_1,\theta_2=1}^{d_x^+}\sigma_{x,\theta_1,\theta_2}^I(e^{-t};p)\\ &+\sum_{x\in\mathcal{X}}\Big(2\sum_{\theta=1}^{d_x^+}\sigma_{x,\theta,\infty}^I(e^{-t};p)+2\sum_{\theta=1}^{d_x^+}\psi_{x,\theta,0}(p)\sum_{s=d_x^+-\theta+1}^{d_x^+}\beta(d_x^+,e^{-t},s)\Big)\\ &+\sum_{x\in\mathcal{X}}\psi_{x,\infty,0}(p)\sum_{s=1}^{d_x^+}\beta(d_x^+,e^{-t},s),\end{aligned} \tag{17}$$

where $\psi_{x,\theta_1,\theta_2}$ is defined in Lemma 5.2, $\sigma_{x,\theta_1,\theta_2}^I$ is given by (20)-(23) and $\Psi_{x,\theta}$ is defined by (14).

At the present we are only left to prove Lemma 5.3.

Proof of Lemma 5.3. Recall that $V_{x,\theta,s}^{(n)}$ denotes the number of bins with type x, threshold θ and with at least s in-balls at time t. We thus have

$$\begin{aligned} I_{x,\theta}^{(n)}(t;p) =& \theta N_{x,\theta}^{(n)}(p) - \sum_{s=d_x^+-\theta+1}^{d_x^+} (s - d_x^+ + \theta) U_{x,\theta,s}^{(n)}(t;p) \\ =& \theta N_{x,\theta}^{(n)}(p) - \sum_{s=d_x^+-\theta+1}^{d_x^+} V_{x,\theta,s}^{(n)}(t;p). \end{aligned}$$

It then follows that

$$n^{-1/2}(I_{x,\theta}^{(n)}(t \wedge \tau_n;p) - n\widehat{f}_{x,\theta}^{(n)}(t \wedge \tau_n)) = \theta N_{x,\theta}^{*(n)} - \sum_{s=d_x^+-\theta+1}^{d_x^+} V_{x,\theta,s}^{*(n)}(t \wedge \tau_n;p), \tag{18}$$

and,

$$n^{-1/2}(I_{x,\infty}^{(n)}(t \wedge \tau_n;p) - n\widehat{f}_{x,\infty}^{(n)}(t \wedge \tau_n)) = d_x^+ N_{x,\infty}^{*(n)} - \sum_{s=1}^{d_x^+} V_{x,\theta,s}^{*(n)}(t \wedge \tau_n;p). \tag{19}$$

By Lemma 5.2, we have the joint convergence of $N_{x_1,\theta_1}^{*(n)}$ and $V_{x_2,\theta_2,s}^{*(n)}$ for all possible (x_1,θ_1) and (x_2,θ_2,s). We therefore have for $\theta \in \{1,\ldots,d_x^+\}$,

$$\mathcal{Z}_{I_{x,\theta}}(t;p) := \theta \mathcal{Y}_{x,\theta}^* - \sum_{s=d_x^+-\theta+1}^{d_x^+} \mathcal{Z}_{x,\theta,s}^*(t;p),$$

and for the threshold $\theta = \infty$,

$$\mathcal{Z}_{I_{x,\infty}}(t;p) := d_x^+ \mathcal{Y}_{x,\theta}^* - \sum_{s=1}^{d_x^+} \mathcal{Z}_{x,\theta,s}^*(t;p).$$

By using the covariance formulas in Lemma 5.2 and some basic calculations, we have the following formulas for the covariance $\sigma_{x,\theta_1,\theta_2}^I(e^{-t};p)$.

- For $\theta_1 = \theta_2 = \theta \in \{1,\ldots,d_x^+\}$:

$$\begin{aligned} \sigma_{x,\theta,\theta}^I(y;p) =& \theta^2 \psi_{x,\theta,\theta}(p) + \sum_{s_1,s_2=d_x^+-\theta+1}^{d_x^+} (\widehat{\sigma}_{x,\theta,\theta,s_1,s_2}(y;p) + \widetilde{\sigma}_{x,\theta,s_1,s_2}(y;p)) \\ & - 2\theta\psi_{x,\theta,\theta}(p) \sum_{s=d_x^+-\theta+1}^{d_x^+} \beta(d_x^+,y,s), \end{aligned} \tag{20}$$

- For $\theta_1,\theta_2 \in \{1,\ldots,d_x^+\}$ and $\theta_1 \neq \theta_2$:

$$\begin{aligned} \sigma_{x,\theta_1,\theta_2}^I(y;p) =& \theta_1\theta_2\psi_{x,\theta_1,\theta_2}(p) + \sum_{s_1=d_x^+-\theta_1+1}^{d_x^+} \sum_{s_2=d_x^+-\theta_2+1}^{d_x^+} \widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(y;p) \\ & - \theta_1\psi_{x,\theta_1,\theta_2}(p) \sum_{s=d_x^+-\theta_2+1}^{d_x^+} \beta(d_x^+,y,s) - \theta_2\psi_{x,\theta_1,\theta_2}(p) \sum_{s=d_x^+-\theta_1+1}^{d_x^+} \beta(d_x^+,y,s), \end{aligned} \tag{21}$$

- For $\theta_1 = \theta_2 = \infty$:

$$\begin{aligned}\sigma^I_{x,\infty,\infty}(y;p) =& (d_x^+)^2\psi_{x,\infty,\infty}(p) + \sum_{s_1,s_2=1}^{d_x^+}\left(\widehat{\sigma}_{x,\theta,\theta,s_1,s_2}(y;p) + \widetilde{\sigma}_{x,\theta,s_1,s_2}(y;p)\right) \\ &- 2d_x^+\psi_{x,\infty,\infty}(p)\sum_{s=1}^{d_x^+}\beta(d_x^+,y,s),\end{aligned} \tag{22}$$

- For $\theta_1 = \infty$ and $\theta_2 = \theta \in \{1,\dots,d_x^+\}$:

$$\begin{aligned}\sigma^I_{x,\infty,\theta}(y;p) =& d_x^+\theta\psi_{x,\infty,\theta}(p) + \sum_{s_1=1}^{d_x^+}\sum_{s_2=d_x^+-\theta+1}^{d_x^+}\widehat{\sigma}_{x,\infty,\theta,s_1,s_2}(y;p) \\ &- d_x^+\psi_{x,\infty,\theta}(p)\sum_{s=d_x^+-\theta+1}^{d_x^+}\beta(d_x^+,y,s) - \theta\psi_{x,\infty,\theta}(p)\sum_{s=1}^{d_x^+}\beta(d_x^+,y,s),\end{aligned} \tag{23}$$

where the forms of $\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}$, $\widetilde{\sigma}_{x,\theta,s_1,s_2}$ and $\psi_{x,\theta_1,\theta_2}$ for all $\theta_1,\theta_2 \in \{1,\dots,d_x^+\}\cup\infty$ are provided in Lemma 5.2. This completes the proof of Lemma 5.3. □

5.5 Proof of (Generalization) Theorem 3.1 and Theorem 3.7

When we fix $p \in [p_0, p_{\min}]$, the threshold distribution is fixed and the results of [4] could be applied. We discuss in this section how to extend the theorems in [4] to allow for the possibility that some institution will never get infected, i.e., the institution with ∞ threshold.

We only discuss the proof for $W_n(t;p)$ and the generalizations for $S_n(t;p)$ and $D_n(t;p)$ are quite similar. We denote by $L_n(t;p)$ and $H_n^-(t;p)$ the number of alive (not removed) out-balls at time t and the number of healthy out-balls at time t respectively. From the definition of white balls process $W_n(t;p)$, it is clear that $W_n(t;p) = L_n(t;p) - H_n^-(t;p)$. Further,

$$H_n^-(t;p) = \sum_{x\in\mathcal{X}} d_x^-\big(\sum_{\theta=1} V^{(n)}_{x,\theta,d_x^+-\theta+1}(t;p) + N^{(n)}_{x,\infty}(p)\big).$$

We further denote by $\widetilde{W}_n(t;p)$ and $\widetilde{f}_W$ the white balls process and corresponding limiting function as in [4]. It is shown in [4] that

$$\widetilde{W}_n(t;p) = L_n(t;p) - \sum_{x\in\mathcal{X}} d_x^- \sum_{\theta=1} V^{(n)}_{x,d_x^+-\theta+1}(t;p),$$

and,

$$\widetilde{f}_W(z;p) = \lambda z - \sum_{x\in\mathcal{X}}\mu_x d_x^- \sum_{\theta=1}^{d_x^+} q_x(\theta;p)\beta(d_x^+, z, d_x^+ - \theta + 1).$$

We thus have

$$W_n(t;p) = \widetilde{W}_n(t;p) - \sum_{x\in\mathcal{X}} d_x^- N^{(n)}_{x,\infty} \quad \text{and} \quad f_W(z;p) = \widetilde{f}_W(z;p) - \sum_{x\in\mathcal{X}} d_x^-\mu_x q_x(\infty).$$

Further, as shown in Section 5.1,

$$\sum_{x\in\mathcal{X}} d_x^- N_{x,\infty}^{(n)}/n \xrightarrow{p} \sum_{x\in\mathcal{X}} d_x^- \mu_x q_x(\infty).$$

Together with [4, Theorem 3.1], which gives

$$\sup_{t\leqslant\tau_n}\Big|\frac{\widetilde{W}_n(t;p)}{n} - \widetilde{f}_W(e^{-t};p)\Big| \xrightarrow{p} 0,$$

we obtain

$$\sup_{t\leqslant\tau_n}\Big|\frac{W_n(t;p)}{n} - f_W(e^{-t};p)\Big| \xrightarrow{p} 0,$$

which shows how to generalize the limit result of $W_n(t;p)$ in Theorem 3.1.

We next show how to generalize the asymptotic normality of $W_n(t;p)$, as in Theorem 3.7. We have

$$\begin{aligned} n^{-1/2}\left(W_n(t\wedge\tau_n;p) - n\widehat{f}_W^{(n)}(t\wedge\tau_n;p)\right) =& n^{-1/2}\left(\widetilde{W}_n(t\wedge\tau_n;p) - n\widehat{\widetilde{f}}_W^{(n)}(t\wedge\tau_n;p)\right) \\ & - \sum_{x\in\mathcal{X}} d_x^- n^{-1/2}\big(N_{x,\infty}^{(n)} - n\mu_x^{(n)} q_x^{(n)}(\infty)\big). \end{aligned}$$

By Lemma 5.2 and following similar arguments as in the proof of Theorem 3.8, one can show that the second term on RHS of the above formula is asymptotically Gaussian. The first term is also asymptotically Gaussian as shown in [4, Theorem 3.6]. Moreover, they are jointly asymptotically Gaussian. It thus only remains to calculate the form of the variance function for $\sigma_W(e^{-t};p)$ of the limit white ball process. To do this, we write the limit process as

$$\mathcal{Z}_W(t;p) = \mathcal{Z}_L(t;p) - \sum_{x\in\mathcal{X}} d_x^- \big(\sum_{\theta=1}^{d_x^+} \mathcal{Z}^*_{x,\theta,d_x^+-\theta+1}(t;p) + \mathcal{Y}^*_{x,\infty}\big),$$

where $\mathcal{Z}_L(t;p)$ is the limit process for $n^{-1/2}(L_n(t;p) - n\lambda^{(n)}e^{-t})$. Moreover, as shown in [4], $\mathcal{Z}_L(t;p)$ is asymptotically Gaussian jointly with $\mathcal{Z}^*_{x,\theta,s}$ for all possible (x,θ,s) and jointly with $\mathcal{Y}^*_{x,\theta}$ for all (x,θ). Further, the covariances w.r.t. $\mathcal{Z}_L$ are also given in [4] by

$$\sigma_L(y) := \mathrm{Var}(\mathcal{Z}_L(-\ln y)) = \lambda(y-y^2)/2, \tag{24}$$

and

$$\begin{aligned} \sigma^L_{x,\theta,s}(y;p) :&= \mathrm{Cov}\big(\mathcal{Z}_L(-\ln y), \mathcal{Z}^*_{x,\theta,s}(-\ln y)\big) \\ &= y^{s+1}\sum_{j=s}^{d_x^+}\binom{j-1}{s-1}\int_y^1 (v-y)^{j-s} v^{-(j+1)} d\varphi_{x,\theta,j}(v;p). \end{aligned} \tag{25}$$

Notice that $\mathcal{Z}_L$ is independent of $\mathcal{Y}^*_{x,\theta}$ for all (x,θ). Then combining with the covariances given in Lemma 5.2, we conclude that

$$\begin{aligned}\sigma_W(y;p) = &\sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\Big[(d_x^-)^2\widetilde{\sigma}_{x,\theta,\pi_x(\theta;p),\pi_x(\theta;p)}(y;p) - 2d_x^-\sigma^L_{x,\theta,\pi_x(\theta;p)}(y;p)\Big] + \sigma_L(y)\\ &+ \sum_{x\in\mathcal{X}}(d_x^-)^2\sum_{\theta_1=1}^{d_x^+}\sum_{\theta_2=1}^{d_x^+}\widehat{\sigma}_{x,\theta_1,\theta_2,\pi_x(\theta_1),\pi_x(\theta_2)}(y;p) + \sum_{x\in\mathcal{X}}(d_x^-)^2\psi_{x,\infty,\infty}(p)\\ &+ \sum_{x\in\mathcal{X}}\sum_{\theta=1}^{d_x^+}\beta(d_x^+,y,\pi_x(\theta;p))\psi_{x,\theta,\infty}(p),\end{aligned} \tag{26}$$

where $\pi_x(\theta;p) := d_x^+ - \theta + 1$, $\sigma_L(y)$ and $\sigma^L_{x,\theta,s}(y;p)$ are given by (24) and (25) respectively. Moreover, $\psi_{x,\theta,\infty}$, $\widetilde{\sigma}_{x,\theta,s_1,s_2}(y;p)$ and $\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}(y;p)$ are defined as in Lemma 5.2.

5.6 Proof of Theorem 3.10

The first point for the case $z^\star(p) = 0$ follows from Theorem 3.4. Consider now the case when $z^\star(p) \in (0,1]$ and $z^\star(p)$ is a stable solution, i.e., $\alpha(p) := f^1_W(z^\star(p);p) > 0$.

First note that the variance $\Psi(t;p)$ of $\mathcal{Z}_\Gamma(t;p)$ is continuous on t. Indeed, from the explicit forms of $\widetilde{\sigma}_{x,\theta,s_1,s_2}$ and $\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}$ in Lemma 5.2, we have the following inequalities,

$$|\widehat{\sigma}_{x,\theta_1,\theta_2,s_1,s_2}| \leqslant \mu_x$$

and,

$$|\widetilde{\sigma}_{x,\theta,s_1,s_2}| \leqslant \sum_{j=0}^{d_x^+}\int_y^1 \frac{y^2}{v^2}d\varphi_{x,\theta,j}(v;p) \leqslant 2d_x^+\mu_x q_x(\theta;p),$$

for all $(x,\theta_1,\theta_2,s_1,s_2)$. Thus, we obtain that for all $x\in\mathcal{X}$, $\theta_1,\theta_2\in\{1,\dots,d_x^+\}\cup\{\infty\}$,

$$\sigma^I_{x,\theta_1,\theta_2} \leqslant 4(d_x^+)^3\mu_x.$$

By the definition of $\Psi(t;p)$ as in (14), we have that for some constant C, the infinite tail sum of the first term in Ψ in (17) satisfies that

$$\sum_{x\in\mathcal{X}_\ell^+}\Big(\sum_{\theta=1}^{d_x^+}\Psi_{x,\theta}(t;p) + \Psi_{x,\infty}(t;p) + \bar{\gamma}_x^2\psi_{x,0,0}(p)\Big) \leqslant C\sum_{x\in\mathcal{X}_\ell^+}(d_x^+)^4\mu_x,$$

which goes to 0 as $\ell\to\infty$ by Assumption 5b. One can also show by a similar argument that the other sum terms in Ψ have the same tail convergence property. Since each single term is continuous in t, again we can pass the continuity in the infinite sum. Moreover, since $\mathcal{Z}_\Gamma(t;p)$ is a centered Gaussian random variable, its distribution is determined by $\Psi(t;p)$. Thus for a sequence $\{t_n\}_n$ which converges to t, we have that as $n\to\infty$,

$$\mathcal{Z}_\Gamma(t_n;p) \xrightarrow{d} \mathcal{Z}_\Gamma(t;p). \tag{27}$$

Then we can use the Skorokhod representation theorem, which shows that one can change the probability space where all the random variables are well defined and all the convergence results of Theorem 3.8, Lemma 3.7, Lemma 3.3 ($\tau_n^\star \to t^\star$) and (27) hold almost surely. Taking $t = \tau_n^\star$ and $t_0 = t^\star$, we obtain by Lemma 3.7 and by continuity of $\mathcal{Z}_W$ that

$$\begin{aligned} W_n(\tau_n^\star; p) &= n\widehat{f}_W^{(n)}(\tau_n^\star; p) + n^{1/2}\mathcal{Z}_W(\tau_n^\star \wedge t^\star; p) + o(n^{1/2}) \\ &= n\widehat{f}_W^{(n)}(\tau_n^\star; p) + n^{1/2}\mathcal{Z}_W(t^\star; p) + o(n^{1/2}). \end{aligned}$$

Since $W_n(\tau_n^\star; p) = -1$, then

$$\widehat{f}_W^{(n)}(\tau_n^\star; p) = -n^{-1/2}\mathcal{Z}_W\big(t^\star; p) + o(n^{-1/2}\big).$$

Since, as $n \to \infty$, $\tau_n^\star \to t^\star$ and $t_n^\star \to t^\star$ hold a.s., there exists some ξ_n in the interval between $t_n^\star$ and $\tau_n^\star$ such that $\xi_n \to t^\star$. Further, as $n \to \infty$,

$$(\widehat{f}_W^{(n)})'(\xi_n; p) \to \widehat{f}_W'(t^\star; p) = -z^\star(p)\alpha(p).$$

It follows then by Mean-Value theorem that

$$\widehat{f}_W^{(n)}(\tau_n^\star; p) = \widehat{f}_W^{(n)}(\tau_n^\star; p) - \widehat{f}_W^{(n)}(t_n^\star; p) = (\widehat{f}_W^{(n)})'(\xi_n)(\tau_n^\star - t_n^\star) = (-z^\star(p)\alpha(p) + o(1))(\tau_n^\star - t_n^\star).$$

Hence we have

$$\tau_n^\star - t_n^\star = \Big(-\frac{1}{z^\star(p)\alpha(p)} + o(1)\Big)\widehat{f}_W^{(n)}(\tau_n^\star; p) = n^{-1/2}\frac{1}{z^\star(p)\alpha(p)}(Z_W(t^\star; p) + o(1)). \tag{28}$$

On the other hand, it follows by Theorem 3.8 that

$$n^{-1/2}\Gamma_n(\tau_n^\star; p) = n^{1/2}\widehat{f}_\Gamma^{(n)}(\tau_n^\star; p) + \mathcal{Z}_\Gamma(\tau_n^\star \wedge t^\star; p) + o(1).$$

Since, as $n \to \infty$, $\tau_n^\star \wedge t^\star \to t^\star$ a.s., we obtain that a.s. $\mathcal{Z}_\Gamma(t_n) \to \mathcal{Z}_\Gamma(t; p)$. It then follows that, for some $\xi_n' \to t^\star$ as $n \to \infty$, that

$$\begin{aligned} n^{-1/2}\Gamma_n(\tau_n^\star; p) &= n^{1/2}\widehat{f}_\Gamma^{(n)}(\tau_n^\star; p) + \mathcal{Z}_\Gamma(t^\star; p) + o(1) \\ &= n^{1/2}\widehat{f}_\Gamma^{(n)}(\tau_n^\star; p) + n^{1/2}(\widehat{f}_\Gamma^{(n)})'(\xi_n'; p)(\tau_n^\star - t_n^\star) + \mathcal{Z}_\Gamma(t^\star; p) + o(1). \end{aligned}$$

Then by plugging (28) into the above formula and some simplification, it follows that

$$n^{-1/2}\Gamma_n(\tau_n^\star; p) = n^{1/2}f_\Gamma^{(n)}(z_n^\star; p) - \frac{f_\Gamma'(z^\star; p)}{\alpha}\mathcal{Z}_W(t^\star; p) + \mathcal{Z}_\Gamma(t^\star; p) + o(1).$$

This completes the proof of Theorem 3.10.

5.7 Proof of Theorem 3.11

The case $z^\star(p) = 0$ is a direct consequent of point (i) of Theorem 3.10, since f is continuous. Consider now the case when $z^\star(p) \in (0, 1]$ and $z^\star(p)$ is a stable solution, i.e., $\alpha(p) := f_W^1(z^\star(p); p) > 0$. We have by Theorem 3.10 that $\Gamma_n(\tau_n^\star; p)$ is asymptotic normal

$$n^{1/2}(\Gamma_n(\tau_n^\star; p)/n - f_\Gamma^{(n)}(z_n^\star; p)) \xrightarrow{d} \mathcal{Z}_\Gamma(t^\star; p) - \alpha(p)^{-1}f_\Gamma'(z^\star; p)\mathcal{Z}_W(t^\star; p). \tag{29}$$

Since for any fixed $p \in [p_{\min}, p_0]$, $f_W^{(n)}(z;p)$ converges to $f_W(z;p)$ uniformly on $[0,1]$, we have that $z_n^\star(p) \to z^\star(p)$ as $n \to \infty$ in probability. Moreover, by continuity of f_Γ and $f_\Gamma^{(n)}$ for all n and the uniformly convergence of $f_\Gamma^{(n)}(\cdot;p)$ to $f_\Gamma(\cdot;p)$ for any fixed p, we can conclude that

$$f_\Gamma^{(n)}(z_n^\star;p) \to f_\Gamma(z^\star;p)$$

in probability for any $p \in [p_{\min}, p_0]$.

Since the inverse demand function g is in $\mathcal{C}^1$ by Assumption 1, we have

$$g'\big(f_\Gamma^{(n)}(z_n^\star;p)\big) \xrightarrow{p} g'\big(f_\Gamma(z^\star;p)\big).$$

By the mean-value theorem, there exists some ξ_n between $\Gamma_n(\tau_n^\star;p)/n$ and $f_\Gamma^{(n)}(z_n^\star;p)$ such that

$$g(\Gamma_n(\tau_n^\star;p)/n) - g\big(f_\Gamma^{(n)}(z_n^\star;p)\big) = g'(\xi_n)(\Gamma_n(p)/n - f_\Gamma^{(n)}(z_n^\star;p)). \tag{30}$$

Note that $\Gamma_n(\tau_n^\star;p)/n \xrightarrow{p} f_\Gamma(z^\star;p)$, thus we also have

$$g'(\xi_n) \xrightarrow{p} g'\big(f_\Gamma(z^\star;p)\big).$$

Multiplying both side of (30) by $n^{1/2}$ gives

$$n^{1/2}\big(h(\Gamma_n(\tau_n^\star;p)/n) - g(f_\Gamma^{(n)}(z_n^\star;p))\big) = n^{1/2}g'(\xi_n)(\Gamma_n(\tau_n^\star;p)/n - f_\Gamma^{(n)}(z_n^\star;p)).$$

Then by the asymptotic normality in (42) and Slutsky's theorem we have finally

$$n^{1/2}(\kappa_n(p) - g(f_\Gamma^{(n)}(z_n^\star;p))) \xrightarrow{d} g'(f_\Gamma(z^\star;p))\big(\mathcal{Z}_\Gamma(t^\star;p) - \alpha^{-1}f_\Gamma^1(z^\star;p)\mathcal{Z}_W(t^\star;p)\big).$$

This completes the proof of Theorem 3.11.

5.8 Proof of Theorem 3.12

We first state a lemma which will be used in the proof of Theorem 3.12. Let us define

$$\phi(p) := p - g \circ f_\Gamma(z^\star(p);p) \quad \text{and} \quad \phi_n(p) := p - g \circ f_\Gamma^{(n)}(z_n^\star(p);p).$$

Lemma 5.4. *Under Assumption 3 and Assumption 4, the following holds:*

(a) For any fixed $p \in (p_{\min}, p_0)$, if $z^\star(p) = 0$ or $z^\star(p) \in (0,1)$ and $\alpha(p) > 0$, then there exists some small $\delta > 0$ and N large enough, such that $z^\star(\cdot)$ and all $z_n^\star(\cdot)$ for $n > N$ are continuous on the interval $(p-\delta, p+\delta)$;

(b) For $p \in \{p_{\min}, p_0\}$, with the same conditions as in (a), the continuities hold but on a semi-interval $[p_{\min}, p_{\min}+\delta)$ for $p = p_{\min}$ and $(p_0 - \delta, p_0]$ for $p = p_0$.

(c) If $\bar{p}$ is a stable fixed point solution, then under the same conditions as in (a), we have that, for N large enough, $\bar{p}$ and all $\{\bar{p}_n, n > N\}$ are continuity points of ϕ and ϕ_n, respectively. Moreover, as $n \to \infty$, $\bar{p}_n \to \bar{p}$.

For the sake of readability, we postpone the proof of lemma to the end of this section. We now proceed with the proof of Theorem 3.12.

The first point, when $z^\star(\bar p) = 0$ and $\bar p$ is a stable solution, has been covered in Theorem 3.6. We now consider the second point, when $z^\star(\bar p) \in (0,1]$ is a stable solution of $f_W(z;\bar p) = 0$, i.e., $\alpha(\bar p) := f_W^1(z^\star;\bar p) > 0$, and $\bar p$ is a stable solution of Equation (8).

By Lemma 5.4, we know that, for n large enough, $\bar p_n$ exists and converges to $\bar p$ as $n \to \infty$. Moreover, by Theorem 3.11, we have that as $n \to \infty$,

$$\Phi_n(p) \xrightarrow{d} \mathcal{Z}_V(p).$$

$\mathcal{Z}_V(p)$ is a centered Gaussian random variable, and its distribution is determined uniquely by its variance. By the analysis in the proof of Theorem 3.10, the variance function of $\mathcal{Z}_\Gamma(t;p)$ is continuous in p. By similar arguments, the variance function of $\mathcal{Z}_W$ is also continuous in p. Then by Cauchy-Schwarz inequality, we can control the covariance between $\mathcal{Z}_\Gamma$ and $\mathcal{Z}_W$ by their variances. Thus the variance function of $\mathcal{Z}_V(p)$ is continuous in p. We therefore have that

$$\mathcal{Z}_V(p_n) \xrightarrow{d} \mathcal{Z}_V(p), \tag{31}$$

for any sequence $\{p_n\}_n$ which converges to p as $n \to \infty$.

We next take advantage of the Skorokhod representation theorem which shows that one can change the probability space where all the random variables are well defined and, all the convergence results of Theorem 3.11, the convergence in probability $p_n^\star \to \bar p$ and (31) hold a.s.. Then we can write

$$\begin{aligned} \Phi_n(p_n^\star) &= \phi_n(p_n^\star) + n^{-1/2}\mathcal{Z}_V(p_n^\star) + o(n^{-1/2}) \\ &= \phi_n(p_n^\star) + n^{-1/2}\mathcal{Z}_V(\bar p) + o(n^{-1/2}), \end{aligned} \tag{32}$$

where the second equality follows from $\mathcal{Z}_V(p_n^\star) \to \mathcal{Z}_V(\bar p)$ a.s.. From $\Phi_n(p_n^\star) = 0$, we have

$$\phi_n(p_n^\star) = -n^{-1/2}\mathcal{Z}_V(\bar p) + o(n^{-1/2}). \tag{33}$$

Moreover, as $n \to \infty$, we have a.s. $p_n^\star \to \bar p$ and $\bar p_n \to \bar p$. Combining the continuity of $f_\Gamma^{(n)}$ and the local continuity of $z^\star(\cdot)$, we have that both $f_\Gamma^{(n)}(z_n^\star(p_n^\star);p_n^\star)$ and $f_\Gamma^{(n)}(z_n^\star(\bar p_n);\bar p_n)$ converge a.s. to $f_\Gamma(z^\star(\bar p),\bar p)$.

Thus, by the Mean-Value theorem, there exists some sequence $\{\xi_n\}$ with $\xi_n \to f_\Gamma(z^\star(\bar p);\bar p)$ a.s. in the interval between $f_\Gamma^{(n)}(z_n^\star(p_n^\star);p_n^\star)$ and $f_\Gamma^{(n)}(z_n^\star(\bar p_n);\bar p_n)$ such that

$$g\big(f_\Gamma^{(n)}(z_n^\star(p_n^\star);p_n^\star)\big) - g\big(f_\Gamma^{(n)}(z_n^\star(\bar p_n);\bar p_n)\big) = g'(\xi_n)(f_\Gamma^{(n)}(z_n^\star(p_n^\star);p_n^\star) - f_\Gamma^{(n)}(z_n^\star(\bar p_n);\bar p_n)). \tag{34}$$

We next analyze $f_\Gamma^{(n)}(z_n^\star(p_n^\star);p_n^\star) - f_\Gamma^{(n)}(z_n^\star(\bar p_n);\bar p_n)$. By the Mean-Value theorem and Lemma 5.4, there exists a sequence $\{\xi_n^z\}$ and $\{\xi_n^p\}$ with $\xi_n^z \to z^\star(\bar p)$ a.s and $\xi_n^p \to \bar p$ a.s. such that

$$f_\Gamma^{(n)}(z_n^\star(p_n^\star);p_n^\star) - f_\Gamma^{(n)}(z_n^\star(\bar p_n);\bar p_n) = f_\Gamma^{1,(n)}(\xi_n^z;p_n^\star)(z_n^\star(p_n^\star) - z_n^\star(\bar p_n)) + f_\Gamma^{2,(n)}(z_n^\star(\bar p_n);\xi_n^p)(p_n^\star - \bar p_n). \tag{35}$$

It only remains to analyze $z_n^\star(p_n^\star) - z_n^\star(\bar p_n)$. Notice that, by definition, $f_W^{(n)}(z_n^\star(p);p) = 0$ for any $p \in [p_{\min}, p_0]$. By using again the Mean-Value theorem, we have the following two relations

$$-f_W^{(n)}(z_n^\star(\bar p_n);p_n^\star) = f_W^{(n)}(z_n^\star(p_n^\star);p_n^\star) - f_W^{(n)}(z_n^\star(\bar p_n);p_n^\star) = f_W^{1,(n)}(\alpha_n^z;p_n^\star)(z_n^\star(p_n^\star) - z_n^\star(\bar p_n)),$$

and,

$$f_W^{(n)}(z_n^\star(\bar{p}_n);p_n^\star) = f_W^{(n)}(z_n^\star(\bar{p}_n);p_n^\star) - f_W^{(n)}(z_n^\star(\bar{p}_n;\bar{p}_n) = f_W^{2,(n)}(z_n^\star(\bar{p}_n);\alpha_n^p)(p_n^\star - \bar{p}_n),$$

where $\alpha_n^z \to z^\star(\bar{p})$ a.s. and $\alpha_n^p \to \bar{p}$ a.s. as $n \to \infty$. Then by above two equations we have

$$z_n^\star(p_n^\star) - z_n^\star(\bar{p}_n) = -(f_W^{1,(n)}(\alpha_n^z;p_n^\star))^{-1} f_W^{2,(n)}(z_n^\star(\bar{p}_n);\alpha_n^p)(p_n^\star - \bar{p}_n). \tag{36}$$

Now combine (34), (35) and (36) and by Remark 3.9, we conclude that

$$\begin{aligned}
\phi_n(p_n^\star) &= \phi_n(p_n^\star) - \phi_n(\bar{p}_n) \\
&= p_n^\star - \bar{p}_n - (g(f_\Gamma(z^\star(\bar{p});\bar{p}) + o(1))[(f_\Gamma^1(z^\star(\bar{p});\bar{p}) + o(1)) \\
&\qquad (-(f_W^1(z^\star(\bar{p});\bar{p})^{-1} f_W^2(z^\star(\bar{p});\bar{p}) + o(1)) + (f_\Gamma^2(z^\star(\bar{p});\bar{p}) + o(1))](p_n^\star - \bar{p}_n) \\
&= (\rho + o(1))(p_n^\star - \bar{p}_n).
\end{aligned}$$

Hence combining with (33), we finally obtain

$$p_n^\star - \bar{p}_n = \Big(\frac{1}{\rho} + o(1)\Big)\phi_n(p_n^\star) = -n^{-1/2}\frac{1}{\rho}\mathcal{Z}_V(\bar{p}) + o(n^{-1/2}).$$

This completes the proof of Theorem 3.12.

At the present we are only left to prove Lemma 5.4.

Proof of Lemma 5.4. From the definition of the threshold distribution, $q_x^{(n)}(\theta;p)$ are (stochastically) non-decreasing on p for every (x,θ) and every n. Thus, for an increasing sequence p_n converging to some $p \in [p_{\min}, p_0]$, we can show that for any fixed $z \in [0,1]$, the sequence $\{f_W(z;p_n)\}_n$ is monotone and converges to $f_W(z;p)$. In addition, $f_W(z;p)$ and $f_W^{(n)}(z;p)$ are continuous on z for all n. It therefore follows by Dini's theorem that $\{f_W(\cdot;p_n)\}_n$ converges uniformly to $f_W(z;p)$ on $[0,1]$. Hence the largest root $z^\star(p_n)$ must also converge to $z^\star(p)$. The same argument for a decreasing sequence p_n gives the same uniform convergence. Thus $\{f_W(\cdot;p_n)\}_n$ converges uniformly to $f_W(\cdot;p)$ for any sequence converging to p.

If $z^\star(p) = 0$ or $z^\star(p) \in (0,1)$ and $\alpha(p) > 0$, then for some $\epsilon < \delta$ small enough, we have $f_W(z^\star(p) + \epsilon;p) > 0$ and $f_W(z^\star(p) - \epsilon;p) < 0$. Then for n large enough, it follows that $f_W(z^\star(p)+\epsilon;p_n) > 0$ and $f_W(z^\star(p)-\epsilon;p_n) < 0$. We therefore have $z^\star(p_n) \in (z^\star(p)-\epsilon, z^\star(p)+\epsilon)$, and ϵ can be arbitrarily small, thus $z^\star(p_n) \to z^\star(p)$ as $p_n \to p$. If $z^\star(p) = 0$, we have for some $\epsilon > 0$ that $f_W(z;p) > 0$ for all $z \geqslant \epsilon$. By the uniform convergence of p_n to p, for n large enough, we also have $f_W(z;p_n) > 0$ for $z \geqslant \epsilon$. Thus $z^\star(p_n) \in [0,\epsilon)$. Taking ϵ arbitrarily small, we conclude that $z^\star(p_n) \to z^\star(p)$ as $p_n \to p$. This continuity holds on a small interval $(p-\delta, p+\delta)$ for some δ small enough. A similar argument gives the same conclusion for the point (b).

It is also clear that for any fixed p, $f_W^{(n)}(z;p)$ converges to $f_W(z;p)$ point wisely on z. Since for any $z \in [0,1]$,

$$f_W^{(n)}(z;p) \leqslant \lambda^{(n)} + \sum_{x\in\mathcal{X}} \mu_x^{(n)} d_x^- \Big[\sum_{\theta=1}^{d_x^+} q_x^{(n)}(\theta;p) + q_x^{(n)}(\infty;p)\Big],$$

by Assumption 4 and applying dominated convergence theorem, we have further that $f_W^{(n)}(z;p)$

converges to $f_W(z;p)$ uniformly on z in $[0,1]$. The same argument applied to $f_\Gamma^{(n)}$ can give us the uniform convergence of $f_\Gamma^{(n)}$ to f_Γ on z. By the uniform convergence of $f_W^{(n)}$ to f_W, it is obvious that we can choose $\epsilon < \delta$ such that the local continuity of $z^\star(\cdot)$ and of all $z_n^\star(\cdot)$ hold on $(\bar{p}-\epsilon, \bar{p}+\epsilon)$ for n large enough. This completes the proof of point (a) and (b).

We next proceed with the proof of point (c) of the lemma. We first prove the local continuity of ϕ on an interval where we assume that $z^\star(\cdot)$ is continuous on p. Recall that $\mathcal{X}_s^+$ is the collection of all characteristics $x \in \mathcal{X}$ with the in-degree $d_x^+ \geqslant s$. Since all $\bar{\ell}_{x,\theta}(p)$ and $q_x(\theta;p)$ are continuous on p, we have that for any fixed $s \in \mathbb{Z}^+$, the partial sum

$$\sum_{x\in\mathcal{X}\setminus\mathcal{X}_s^+} \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) f_{x,\theta}(z_n^\star;p)$$

is continuous on p. On the other hand, we have $f_{x,\theta}(z^\star;p) \leqslant d_x^+ \mu_x q_x(\theta;p)$. Let C be a common upper bound for all $\bar{\ell}_{x,\theta}$. We thus have that

$$\sum_{x\in\mathcal{X}_s^+} \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta}(p) f_{x,\theta}(z_n^\star;p) \leqslant C \sum_{x\in\mathcal{X}_s^+} d_x^+ \mu_x,$$

which goes to zero as $s \to \infty$ by Assumption 4. Hence $f_\Gamma(z^\star;p)$ is continuous on p and combining with the continuity of the inverse demand function g, it follows that $\phi(p)$ is continuous on p. The same argument for ϕ_n also leads to the continuity of ϕ_n on p, given the continuity of $z_n^\star(\cdot)$.

We next prove the case where $\bar{p} \in (p_{\min}, p_0)$ and there exists some small $\epsilon > 0$ such that $\phi(\bar{p}+\epsilon) > 0$ and $\phi(\bar{p}-\epsilon) < 0$. Notice that ϕ_n converges uniformly to ϕ since $f_\Gamma^{(n)}(z_n^\star(p);p)$ converges uniformly to $f_\Gamma(z^\star(p);p)$ on $[p_{\min}, p_0]$. So we have that for some n large enough, $\phi_n(\bar{p}+\epsilon) > 0$ and $\phi_n(\bar{p}-\epsilon) < 0$. We can choose $\epsilon < \delta$ such that the local continuity of $z^\star(\cdot)$ and all $z_n^\star(\cdot)$ hold on $(\bar{p}-\epsilon, \bar{p}+\epsilon)$. By taking ϵ arbitrarily small, we can conclude that $\bar{p}_n \to \bar{p}$ as $n \to \infty$. A similar argument gives the conclusion for $\bar{p} = p_{\min}$. This completes the proof of Lemma 5.4. □

6 Concluding Remarks

We have proposed a stochastic framework for quantifying the impact of a macroeconomic shock on the resilience of a banking network to fire sales and insolvency cascades. Our limit theorems provide quantitative evidence for the importance of fire sales and indirect contagion in the financial system. We have quantified how price mediated contagion across institutions with common asset holding could worsen cascades of insolvencies in a heterogeneous financial network, during a financial crisis. Under suitable assumptions on the degree and threshold distributions, we have shown that the default cascade model could be transferred to a death process problem represented by balls-and-bins model. This allows us to provide the limit theorems for a dynamic default cascade process with fire sales. We have stated various limit theorems regarding the total sold shares and the equilibrium price of illiquid assets in a stylized fire sales model. In particular, the equilibrium prices of illiquid assets has asymptotically Gaussian fluctuations.

In our numerical experiments, we have investigated the effect of heterogeneity in network structure and price impact function on the final size of default cascade and fire sales loss. For a regular financial network, we found that for a small shock, the high connectivity network is more resilient, but once the shock is large enough, the default propagates to a larger fraction through its higher connectivity. On the other hand, the fire sales loss in the two financial networks with high and low connectivity are very close to each other. We also found that the financial networks with a higher heterogeneity may have a smaller critical value for the shock beyond which a large fraction of institutions default, both with and without fire sales. On the other hand, for the smaller shocks, the most heterogeneous network could be the least resilient.

Our theoretical analysis sheds light on several aspects related to financial stability and systemic risk. A financial network is acceptable if it does not allow for large cascades for a set of stress scenarios in which certain characteristics, such as capital or liquidity reserves are stressed. Higher capital requirements could be imposed on the financial institutions, depending on their types, to ensure that the danger of phase transitions as detailed above is avoided. Moreover, the closed form interpretable limit theorems that we provide in a heterogeneous financial network could serve as a mandate for regulators to collect data on those specific network characteristics and assess systemic risk via more intensive computational methods.

Several directions emerge from the current study. In particular, we have assumed in this paper an exogenous inverse demand function. A much more challenging extension of the model is to endogenize the demand function, or to endogenize the financial network payments. We leave these and some related issues for a future work.

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A Some Auxiliary Lemmas

We first provide (under some regularity conditions) a central limit theorem for functions which could be written as $Y_n(t) := \sum_{i=1}^{\lfloor X_n(t) \rfloor} G_i$, where $X_n(t)$ is a non-decreasing stochastic process satisfying $X_n(t) = O(n)$ for all $t > 0$ and $\{G_i\}_{i\geqslant 1}$ are i.i.d. positive bounded random variables with mean g and variance σ^2.

Lemma A.1. *Using the notations above and for fixed $t > 0$, if $X_n(t) := f_n(t)n + \mathcal{V}n^{1/2}$ with $(f_n(t))_{n=1}^{\infty}$ a positive sequence converging to $f(t)$, and $\mathcal{V}$ a bounded real-valued random variable, then as $n \to \infty$, conditioned on $\{\mathcal{V} = x\}$ for some x on supp($\mathcal{V}$), we have*

$$\Big(\frac{Y_n(t) - gX_n(t)}{\sqrt{nf(t)}\sigma} \mid \mathcal{V} = x\Big) \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. Conditioned on the event $\{\mathcal{V} = x\}$, $X_n(t) = f_n(t)n + xn^{1/2}$ which is non-random. Hence, by standard central limit theorem (CLT), we have

$$\Big(\frac{Y_n(t) - g\lfloor X_n(t)\rfloor}{\sqrt{\lfloor nf_n(t) + xn^{1/2}\rfloor}\sigma} |\mathcal{V} = x\Big) \xrightarrow{d} \mathcal{N}(0,1).$$

Further, we have the decomposition

$$\begin{aligned}\frac{Y_n(t) - gX_n(t)}{\sqrt{nf(t)}\sigma} &= \frac{\sqrt{\lfloor nf_n(t) + xn^{1/2}\rfloor}}{\sqrt{nf(t)}} \cdot \frac{Y_n(t) - g\lfloor X_n(t)\rfloor}{\sqrt{\lfloor nf_n(t) + xn^{1/2}\rfloor}\sigma} + \frac{g\lfloor X_n(t)\rfloor - gX_n(t)}{\sqrt{nf(t)}\sigma} \\ &= \sqrt{1 + O(n^{-1/2})}\frac{Y_n(t) - g\lfloor X_n(t)\rfloor}{\sqrt{\lfloor nf(t) + xn^{1/2}\rfloor}\sigma} + O(n^{-1/2}).\end{aligned}$$

It follows thus by Slutsky's theorem that as $n \to \infty$,

$$\Big(\frac{Y_n(t) - gX_n(t)}{\sqrt{nf(t)}\sigma}|\mathcal{V} = x\Big) \xrightarrow{d} \mathcal{N}(0,1).$$

□

Using the above lemma, we prove the following proposition.

Proposition A.2. *Using the notations above and for fixed $t > 0$, let $X_n(t) := f_n(t)n + \mathcal{V}_n n^{1/2}$ with $\{f_n(t)\}_{n=1}^{\infty}$ a positive sequence converging to $f(t)$ and $\mathcal{V}_n$ a sequence of random variables which converges to a Gaussian random variable $\mathcal{V} \sim \mathcal{N}(0, \upsilon^2)$ in distribution. Then we have, as $n \to \infty$,*

$$\frac{Y_n(t) - ngf_n(t)}{\sqrt{n(f(t)\sigma^2 + \upsilon^2 g^2)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. We first consider the following integration

$$A(z;p) := \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{f(t)}\sigma r} \exp\Big\{-\frac{u^2}{2\upsilon^2} - \frac{(z-gu)^2}{2f(t)\sigma^2}\Big\}\, du. \tag{37}$$

Let us denote by $a := \upsilon^2 g^2 + f(t)\sigma^2$. Then by a change of variable $y = \frac{\sqrt{a}}{r\sigma\sqrt{f(t)}}u - \frac{rgz}{\sigma\sqrt{af(t)}}$, we obtain

$$\begin{aligned}
A(z;p) &= \frac{1}{2\pi\sqrt{f(t)}\sigma r}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2f(t)\sigma^2\upsilon^2}(f(t)\sigma^2u^2+z^2\upsilon^2-2gzu\upsilon^2+\upsilon^2g^2u^2)\right\}du\\
&= \frac{1}{2\pi\sqrt{f(t)}\sigma r}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2f(t)\sigma^2\upsilon^2}((\sqrt{a}u-\frac{\upsilon^2gz}{\sqrt{a}})^2+\frac{f(t)\sigma^2\upsilon^2z^2}{a})\right\}du\\
&= \frac{1}{2\pi\sqrt{f(t)}\sigma r}e^{-\frac{z^2}{2a}}\int_{-\infty}^{\infty}\frac{r\sigma\sqrt{f(t)}}{\sqrt{a}}e^{-\frac{y^2}{2}}dy\\
&= \frac{1}{\sqrt{2\pi a}}e^{-\frac{z^2}{2a}}.
\end{aligned}$$

On the other hand, define a function

$$h_z(x) := \frac{1}{\sqrt{2\pi f(t)}\sigma}\exp\left\{-\frac{(z-gx)^2}{2f(t)\sigma^2}\right\},$$

which is continuous and bounded. Thus by $\mathcal{V}_n \xrightarrow{d} \mathcal{V}$, we have (as $n \to \infty$)

$$A_n(z;p) := \mathbb{E}[h_z(\mathcal{V}_n)] \longrightarrow \mathbb{E}[h_z(\mathcal{V})] = A(z;p).$$

We denote

$$Z_n(t) := \frac{Y_n(t) - ngf_n(t)}{\sqrt{n}},$$

and let $Z(t)$ be a random variable with distribution

$$Z(t) \sim \mathcal{N}(0, \sigma^2 f(t)).$$

Let further μ_n be the probability measure of $\mathcal{V}_n$ and μ be that of $\mathcal{V}$. For convenience, we also denote by

$$\Phi_x(B) := \mathbb{P}(Z(t) - gx \in B),$$

and

$$G_{\mathcal{V}_n}(B|x) := \mathbb{P}(Z_n(t) \in B|\mathcal{V}_n = x).$$

Then for any Borel set $B \subset \mathbb{R}$, we have

$$\begin{aligned}
\big|\mathbb{P}(Z(t)\in B) - \mathbb{P}(Z_n(t)\in B)\big| =& \Big|\int_{\mathbb{R}}\Phi_x(B)d\mu(x) - \int_{\mathbb{R}}G_{\mathcal{V}_n}(B|x)d\mu_n(x)\Big|\\
\leqslant& 2\epsilon + \Big|\int_{[-K,K]}G_{\mathcal{V}_n}(B|x)d\mu_n(x) - \int_{[-K,K]}\Phi_x(B)d\mu_n(x)\Big|\\
&+ \Big|\int_{\mathbb{R}}\Phi_x(B)d\mu_n(x) - \int_{\mathbb{R}}\Phi_x(B)d\mu(x)\Big|,
\end{aligned}$$

where we take K large enough such that $\int_{\mathbb{R}\setminus[-K,K]} 1 d\mu_n(x) \leqslant \epsilon$, uniformly on n.

Next we check the right hand side of the above inequality term by term. For the second term, we have

$$\Big|\int_{[-K,K]}G_{\mathcal{V}_n}(B|x)d\mu_n(x) - \int_{[-K,K]}\Phi_x(B)d\mu_n(x)\Big| \to 0.$$

Since any Borel set is a continuity set of Gaussian distribution, for every $x \in \text{supp}(\mathcal{V}_n) \cap [-K, K]$, $G_{\mathcal{V}_n}(B|x) \to \Phi_x(B)$ by Lemma A.1. Then the result follows by the dominant convergence theorem.

For the third term, we have

$$\Big|\int_{\mathbb{R}} \Phi_x(B) d\mu_n(x) - \int_{\mathbb{R}} \Phi_x(B) d\mu(x)\Big| \leqslant \int_B |\mathbb{E}[h_z(\mathcal{V}_n)] - \mathbb{E}[h_z(\mathcal{V})]| dz$$
$$\leqslant \int_{\mathbb{R}} |A_n(z;p) - A(z;p)| dz \to 0,$$

where the first inequality follows by Fubini's theorem and the third line is by Scheffé's lemma since $\int_{\mathbb{R}} A_n(z;p)dz = \int_{\mathbb{R}} A(z;p)dz = 1$ and $A_n(z;p) \to A(z;p)$ for every $z \in \mathbb{R}$.

Since we can choose ϵ is arbitrarly, we finally get for any borel set $B \in \mathbb{R}$,

$$\mathbb{P}(Z_n(t) \in B) \to \int_B A(z;p) dz.$$

On the other hand, since $A(z;p)$ is the density of $\mathcal{N}(0,a)$ and all Borel sets are continuity set of $\mathcal{N}(0,a)$, it follows that $Z_n(t) \xrightarrow{d} \mathcal{N}(0,a)$, which is equivalent to the statement of proposition. The proof is complete. □

B Extension to Multiple Illiquid Assets

In this section we extend our model to the financial network setup with multiple types of illiquid assets. We next state central limit theorems for default cascade with fire sales in this setup.

B.1 Model

We consider K different illiquid assets $[K] := \{1, 2, \ldots, K\}$. Every institution holds a portfolio of illiquid assets $\boldsymbol{\gamma}_i = (\gamma_{i,1}, \ldots, \gamma_{i,K})^T$. We denote the average assets holdings by the vector $\bar{\boldsymbol{\gamma}} = (\bar{\gamma}_1, \ldots, \bar{\gamma}_K)^T$.

For the initial price vector $\mathbf{p}_0 = (p_{0,1}, \ldots, p_{0,K})^T$ of the illiquid assets and given $\mathbf{p}_{\min} := (p_{\min,1}, \ldots, p_{\min,K})^T \leqslant \boldsymbol{p}_0$, we assume that there exists an exogenously given positive continuous inverse demand function for the multiple illiquid assets

$$\mathbf{g} := (g_1, \ldots, g_K)^T : [\mathbf{0}, \bar{\boldsymbol{\gamma}}] \to [\mathbf{p}_{min}, \mathbf{p}_0],$$

with $g_k : [0, \bar{\gamma}_k] \to [p_{\min,k}, p_{0,k}]$, which satisfies Assumption 1, i.e.,

(i) $\mathbf{g}(\mathbf{0}) = \boldsymbol{p}_0$ (in absence of liquidations the price is given exogenously by $\boldsymbol{p}_0$).

(ii) For all $k \in [K], g_k(x) \in \mathcal{C}^1$ and it is a non-increasing function of $x \in [0, \bar{\gamma}_k]$ (the price is non-increasing with the average excess supply $\boldsymbol{x}$).

(iii) $\boldsymbol{g}(\bar{\boldsymbol{\gamma}}) = \boldsymbol{p}_{\min} > 0$ (the price when the total illiquid asset holdings of the banks are sold is bounded from below by $\boldsymbol{p}_{\min} > 0$).

Similarly, for a given shock scenario $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in [0,1]^n$ and a given price $\boldsymbol{p} \in \mathbb{R}_+^K$ of the illiquid asset, we say that the bank i is ***p**-fundamentally insolvent* if its capital, after the shock and under price $\boldsymbol{p}$ of illiquid assets, is negative. We let the set of $\boldsymbol{p}$-fundamental defaults

$$\mathcal{D}_0(\boldsymbol{\epsilon}; \boldsymbol{p}) = \{i \in [n] : c_i(\epsilon_i; \boldsymbol{p}) < 0\}.$$

We next systemically replace p by $\boldsymbol{p}$ for all definitions and assumptions of Section 2 and Section 3. In particular, for a given price $\boldsymbol{p}$, the default threshold distribution is now $q_x(\theta; \boldsymbol{p})$ for all $x \in \mathcal{X}$ and $\theta \in \{1, \ldots, d_x^+\} \cup \{\infty\}$.

For each asset $k \in [K]$, we let $\{L^{(i)}_{x,\theta,k}(\mathbf{p})\}_{i=1}^{\infty}$ be i.i.d. positive bounded random variables with common distribution $F_{x,\theta,k}(.; \mathbf{p})$, which has expectation $\bar{\ell}_{x,\theta,k}(\mathbf{p})$ and variance $\varsigma^2_{x,\theta,k}(\mathbf{p})$ under price $p \in [p_{\min}, p_0]$ for illiquid asset, for all $x \in \mathcal{X}$ and $\theta \in \{1, \ldots, d_x^+\} \cup \{\infty\}$.

Similarly to Assumption 4, we assume that the mean $\bar{\ell}_{x,\theta,k}(\mathbf{p})$ and variance $\varsigma^2_{x,\theta,k}(\mathbf{p})$ of sold shares for each liquidation are both continuous on $\mathbf{p}$ (on each p_k), for all $x \in \mathcal{X}$ and θ.

We recall that $L^{(i)}_{x,\theta,k}(\mathbf{p})$ denotes the units of k-th illiquid asset sold at i-th incoming default to institutions with type x and threshold θ. Further, $L^{(i)}_{x,\infty,k}(\mathbf{p})$ denotes the units of k-th illiquid asset sold at i-th incoming default to institutions with type x who never defaults.

For $k \in [K]$, the total sold shares of k-th illiquid asset at time t is given by (for price $\mathbf{p}$)

$$\Gamma_k^{(n)}(t; \mathbf{p}) := \sum_{x \in \mathcal{X}} \Big(\bar{\gamma}_{x,k} D^{(n)}_{x,0}(\mathbf{p}) + \sum_{\theta=1}^{d_x^+} Y^{(n)}_{x,\theta,k}(t; \mathbf{p}), + Y^{(n)}_{x,\infty,k}(t; \mathbf{p}) \Big), \tag{38}$$

where

$$Y^{(n)}_{x,\theta,k}(t; \mathbf{p}) := \sum_{i=1}^{I^{(n)}_{x,\theta}(t;p)} L^{(i)}_{x,\theta,k}(\mathbf{p}) \quad \text{and} \quad Y^{(n)}_{x,\infty,k}(t; \mathbf{p}) := \sum_{i=1}^{I^{(n)}_{x,\infty}(t;p)} L^{(i)}_{x,\infty,k}(\mathbf{p}). \tag{39}$$

The final shares of illiquid assets which have been sold under price $\mathbf{p}$ will be

$$\boldsymbol{\Gamma}^{(n)}(\tau_n^\star(\mathbf{p}); \mathbf{p}) = \big(\Gamma_1^{(n)}(\tau_n^\star(\mathbf{p}); \mathbf{p}), \ldots, \Gamma_K^{(n)}(\tau_n^\star(\mathbf{p}); \mathbf{p}) \big)^T,$$

where $\Gamma_k^{(n)}(\tau_n^\star(\mathbf{p}); \mathbf{p})$ denotes the final sold shares of illiquid asset k under price $\mathbf{p}$.

We next set the prices given by inverse demand function as

$$\boldsymbol{\kappa}_n(\mathbf{p}) := \boldsymbol{g}(\boldsymbol{\Gamma}^{(n)}(\tau_n^\star(\mathbf{p}); \mathbf{p})/n).$$

Similarly, we define the *equilibrium prices* of the illiquid assets as

$$\mathbf{p}_n^\star = \sup\{\mathbf{p} \in [\mathbf{p}_{\min}; \mathbf{p}_0] : \mathbf{p} \leqslant \boldsymbol{\kappa}_n(\mathbf{p})\}, \tag{40}$$

where we take the supremum according to the K-dimensional Euclidean distance from $\mathbf{0}$.

B.2 Central Limit Theorems

In this section, we discuss how our central limit theorem results of Section 3.2 could be extended to the case of multiple illiquid assets in financial system. It would be then easy to extend the other limit theorems (law of large numbers) for this setup.

For each asset $k \in [K]$ and $z \in [0,1]$, we define

$$f_{\Gamma,k}(z;\mathbf{p}) := \sum_{x\in\mathcal{X}} \mu_x \Big(\bar{\gamma}_{x,k} q_x(0;\mathbf{p}) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta,k}(\mathbf{p}) f_{x,\theta,k}(z;\mathbf{p}) + \bar{\ell}_{x,\infty,k}(\mathbf{p}) f_{x,\infty,k}(z;\mathbf{p}) \Big),$$

$$f^{(n)}_{\Gamma,k}(z;\mathbf{p}) := \sum_{x\in\mathcal{X}} \mu^{(n)}_x \Big(\bar{\gamma}_{x,k} q^{(n)}_x(0;\mathbf{p}) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta,k}(\mathbf{p}) f^{(n)}_{x,\theta,k}(z;\mathbf{p}) + \bar{\ell}_{x,\infty,k}(\mathbf{p}) f^{(n)}_{x,\infty,k}(z;\mathbf{p}) \Big),$$

where

$$f_{x,\theta}(z;\mathbf{p}) := \mu_x q_x(\theta;\mathbf{p}) \big(\theta - \sum_{\ell=d_x^+-\theta+1}^{d_x^+} \beta(d_x^+, z, \ell)\big), \quad f_{x,\infty}(z;\mathbf{p}) := (1-z)\mu_x q_x(\infty;\mathbf{p}) d_x^+,$$

$$f^{(n)}_{x,\theta}(z;\mathbf{p}) := \mu^{(n)}_x q^{(n)}_x(\theta;\mathbf{p}) \big(\theta - \sum_{\ell=d_x^+-\theta+1}^{d_x^+} \beta(d_x^+, z, \ell)\big), \quad f^{(n)}_{x,\infty}(z;\mathbf{p}) := (1-z)\mu^{(n)}_x q^{(n)}_x(\infty;\mathbf{p}) d_x^+.$$

We also set the vectors

$$\boldsymbol{f}_\Gamma(z;\mathbf{p}) = \big(f_{\Gamma,1}(z;\mathbf{p}), \dots, f_{\Gamma,K}(z;\mathbf{p})\big)^T \quad \text{and} \quad \boldsymbol{f}^{(n)}_\Gamma(z;\mathbf{p}) = \big(f^{(n)}_{\Gamma,1}(z;\mathbf{p}), \dots, f^{(n)}_{\Gamma,K}(z;\mathbf{p})\big)^T.$$

Note that for any $k \in [K]$, the total sold shares for asset k, i.e., $\Gamma^{(n)}_k$, has the same shape as that of Γ_n in the uni-asset case. Hence, it is not hard to generalize our limit theorem on Γ_n in Section 3.2 to the multi-type case under the same assumptions. In particular, the following two theorems hold (under Assumption 5b for degree sequences), by systematically replacing p by $\mathbf{p}$ and considering each asset separately.

Theorem B.1. *Let $\tau_n \leqslant \tau^\star_n(\mathbf{p})$ be a stopping time such that $\tau_n \xrightarrow{p} t_0$ for some $t_0 > 0$. Then for any fixed $k \in [K]; \mathbf{p} \in [\mathbf{p}_{\min}; \mathbf{p}_0]$ and $t > 0$, as $n \to \infty$,*

$$n^{-1/2}(\Gamma^{(n)}_k(t \wedge \tau_n;\mathbf{p}) - n\widehat{f}^{(n)}_{\Gamma,k}(t \wedge \tau_n;\mathbf{p})) \xrightarrow{d} \mathcal{Z}_{\Gamma,k}(t \wedge t_0;\mathbf{p}), \tag{41}$$

where $\mathcal{Z}_{\Gamma,k}(t;\mathbf{p})$ is a Gaussian random variable with mean 0 and variance

$$\Psi_k(t;\mathbf{p}) := \mathrm{Var}(\mathcal{Z}_{\Gamma,k}(t;\mathbf{p})),$$

where the form of $\Psi_k(t;\mathbf{p})$ is given by (44).

We also have the following theorem for the asymptotic normality of the final total sold shares.

Theorem B.2. *Let $t^\star(\mathbf{p}) := -\ln z^\star(\mathbf{p})$. For any fixed $\mathbf{p} \in [\mathbf{p}_{\min}; \mathbf{p}_0]$, as $n \to \infty$, the final total sold shares for asset $k \in [K]$ satisfies:*

(i) If $z^\star(\mathbf{p}) = 0$ then asymptotically almost all institutions default after shock and (as $n \to \infty$)

$$\frac{\Gamma^{(n)}_k(\tau^\star_n; p)}{n} \xrightarrow{p} \sum_{x\in\mathcal{X}} \mu_x \Big(\bar{\gamma}_{x,k} q_x(0;\mathbf{p}) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta,k}(\mathbf{p}) \theta q_x(\theta;\mathbf{p}) \Big).$$

(ii) If $z^\star(\mathbf{p}) \in (0,1]$ and $z^\star(\mathbf{p})$ is a stable solution, i.e., $\alpha(\mathbf{p}) := f^1_W(z^\star(\mathbf{p});\mathbf{p}) > 0$, then

$$n^{-1/2}(\Gamma^{(n)}_k(\tau^\star_n;\mathbf{p}) - n f^{(n)}_{\Gamma,k}(z^\star_n(p);p)) \xrightarrow{d} \mathcal{Z}_{\Gamma,k}(t^\star(\mathbf{p});\mathbf{p}) - \alpha(\mathbf{p})^{-1} f^1_{\Gamma,k}(z^\star(\mathbf{p});\mathbf{p}) \mathcal{Z}_W(t^\star(\mathbf{p})),$$

where $f^1_{\Gamma,k}$ denotes the partial derivative of $f_{\Gamma,k}$ with respect to the first variate z.

We next show a central limit theorem on the price $\boldsymbol{\kappa}_n(\mathbf{p}) := \boldsymbol{g}(\boldsymbol{\Gamma}^{(n)}(\tau^\star_n(\mathbf{p});\mathbf{p})/n)$. For convenience, we denote the vectors

$$\boldsymbol{\mathcal{Z}}_\Gamma(t;\mathbf{p}) := \big(\mathcal{Z}_{\Gamma,1}(t;\mathbf{p}),\ldots,\mathcal{Z}_{\Gamma,K}(t;\mathbf{p})\big)^T \quad \text{and} \quad \boldsymbol{f}^1_\Gamma(z;\mathbf{p}) = \big(f^1_{\Gamma,1}(z;\mathbf{p}),\ldots,f^1_{\Gamma,K}(z;\mathbf{p})\big)^T.$$

Theorem B.3. *Let $t^\star(\mathbf{p}) := -\ln z^\star(\mathbf{p})$. For any $\mathbf{p} \in [\mathbf{p}_{\min};\mathbf{p}_0]$ fixed and as $n \to \infty$, the price $\boldsymbol{\kappa}_n(p)$ given by the inverse demand function satisfies:*

(i) If $z^\star(\mathbf{p}) = 0$ then asymptotically almost all institutions default after shock and

$$\boldsymbol{\kappa}_n(p) \xrightarrow{p} \boldsymbol{g}\Big(\bar{\boldsymbol{\Gamma}}(\mathbf{p})\Big),$$

where $\bar{\boldsymbol{\Gamma}}(\mathbf{p}) := \big(\bar{\Gamma}_1(\mathbf{p}),\ldots,\bar{\Gamma}_K(\mathbf{p})\big)^T$ is given by setting, for all $k \in [K]$,

$$\bar{\Gamma}_k(\mathbf{p}) := \sum_{x\in\mathcal{X}} \mu_x\big(\bar{\gamma}_{x,k}q_x(0;\mathbf{p}) + \sum_{\theta=1}^{d_x^+} \bar{\ell}_{x,\theta,k}(\mathbf{p})\theta q_x(\theta;\mathbf{p})\big).$$

(ii) If $z^\star(\mathbf{p}) \in (0,1]$ and $z^\star(\mathbf{p})$ is a stable solution, i.e., $\alpha(\mathbf{p}) := f^1_W(z^\star(\mathbf{p});\mathbf{p}) > 0$, then

$$n^{-1/2}(\boldsymbol{\kappa}_n(\mathbf{p}) - \mathbf{g}(\mathbf{f}^{(n)}_\Gamma(z^\star(\mathbf{p});\mathbf{p})) \xrightarrow{d} \mathbf{J}_\mathbf{g}\big(\mathbf{f}_\Gamma(z^\star(\mathbf{p});\mathbf{p})\big)\Big[\boldsymbol{\mathcal{Z}}_\Gamma(t^\star(\mathbf{p});\mathbf{p}) - \alpha(\mathbf{p})^{-1}\boldsymbol{f}^1_\Gamma(z^\star(\mathbf{p});\mathbf{p})\mathcal{Z}_W(t^\star(\mathbf{p});\mathbf{p})\Big],$$

where $\mathbf{J}_\mathbf{g}$ is the Jacobian matrix of $\mathbf{g}$.

The proof of the above theorem is provided in Section B.3.

We could now state a central limit theorem for the equilibrium price after shock, defined by Equation (40). Let us define

$$\bar{\mathbf{p}} := \sup\big\{\mathbf{p} \in [\mathbf{p}_{min},\mathbf{p}_0] : \mathbf{p} \leqslant \mathbf{g}(\mathbf{f}_\Gamma(z^\star(\mathbf{p});\mathbf{p})\big\},$$

and correspondingly for the network of size n, we set

$$\bar{\mathbf{p}}_n := \sup\big\{\mathbf{p} \in [\mathbf{p}_{min},\mathbf{p}_0] : \mathbf{p} \leqslant \mathbf{g}(\mathbf{f}^{(n)}_\Gamma(z^\star(\mathbf{p});\mathbf{p})\big\},$$

We say that $\bar{\mathbf{p}}$ is a *stable* fixed point solution if either $\mathbf{p} = \mathbf{p}_{\min}$ or, $\mathbf{p} \in (\mathbf{p}_{\min};\mathbf{p}_0]$ and there exists some $\boldsymbol{\epsilon} > 0$ such that $\mathbf{p} < \mathbf{g}(\mathbf{f}_\Gamma(z^\star(\mathbf{p});\mathbf{p})$ for all $\mathbf{p} \in (\bar{\mathbf{p}} - \boldsymbol{\epsilon};\bar{\mathbf{p}})$.

We define some notations here. Let ∇f be the row vector of the gradient of f. Further, for any function $f(z;\mathbf{p})$ and $k = 1,\ldots,K+1$, we define the notation $f^k(z;\mathbf{p})$ be the partial derivative with respect to the k-th variate (z or p_{k-1}).

Theorem B.4. *As $n \to \infty$, the equilibrium price satisfy:*

(i) If $z^\star(\bar{\mathbf{p}}) = 0$ and $\bar{\mathbf{p}}$ is a stable solution, then the equilibrium price converges to $\mathbf{p}^\star_n \xrightarrow{p} \bar{\mathbf{p}}$ and $\bar{\mathbf{p}}$ is the largest solution of the fixed point equation

$$\mathbf{p} = \mathbf{g}(\bar{\boldsymbol{\Gamma}}(\mathbf{p})),$$

where $\bar{\boldsymbol{\Gamma}}(\bar{p})$ is the same vector as defined in Theorem B.3.

(ii) If $z^\star(\bar{\mathbf{p}}) \in (0,1]$ is a stable solution of $f_W(z;\bar{\mathbf{p}}) = 0$, i.e., $\alpha(\bar{\mathbf{p}}) := f_W^1(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}) > 0$, and $\bar{\mathbf{p}}$ is a stable fixed point solution in $(\mathbf{p}_{\min};\mathbf{p}_0)$, then

$$n^{-1/2}(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n) \xrightarrow{d} -(\mathbf{I}_{K\times K} - \mathbf{A}\cdot\mathbf{B})^{-1}\boldsymbol{\mathcal{Z}}_V(\bar{\mathbf{p}}),$$

if the matrix $\mathbf{I}_{K\times K} - \mathbf{A}\cdot\mathbf{B}$ is non-singular, where $\mathbf{I}_{K\times K}$ is the $K\times K$ identity matrix, $\mathbf{A} = \mathbf{J}_{\mathbf{g}}\big(\mathbf{f}_\Gamma(z^\star(\mathbf{p});\mathbf{p})\big)$ is the Jacobian matrix, $\mathbf{B}$ is also a $K\times K$ matrix with entry

$$\mathbf{B}_{ij} := -f_{\Gamma,i}^1(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}})\alpha(\bar{\mathbf{p}})^{-1}f_W^{j+1}(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}) + f_{\Gamma,i}^{j+1}(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}})$$

for all $i,j \in [K]$, and $\boldsymbol{\mathcal{Z}}_V(\bar{\mathbf{p}}) := \big(\mathcal{Z}_{V,1}(\bar{\mathbf{p}}),\ldots,\mathcal{Z}_{V,K}(\bar{\mathbf{p}})\big)^T$ with (for $k \in [K]$)

$$\mathcal{Z}_{V,k}(\bar{\mathbf{p}}) := -\nabla g_k(\mathbf{f}_\Gamma(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}))\Big(\mathcal{Z}_{\Gamma,k}(t^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}) - \alpha(\bar{\mathbf{p}})^{-1}f_{\Gamma,k}^1(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}})\mathcal{Z}_W(t^\star(\bar{\mathbf{p}});\bar{\mathbf{p}})\Big)$$

is a centered Gaussian random variable with mean 0.

The proof of the above theorem is provided in Section B.4.

B.3 Proof of Theorem B.3

The case $z^\star(\mathbf{p}) = 0$ is a direct consequent of point (i) of Theorem B.2, since for all $k \in [K]$, g_k is continuous. Consider now the case when $z^\star(\mathbf{p}) \in (0,1]$ and $z^\star(\mathbf{p})$ is a stable solution, i.e., $\alpha(\mathbf{p}) := f_W^1(z^\star(\mathbf{p});\mathbf{p}) > 0$. Since the liquidations are independent for different types of assets, we have as a consequent of point (ii) of Theorem 3.10 that for all $k \in [K]$, $\Gamma_k^{(n)}(\tau_n^\star;\mathbf{p})$ is asymptotically normal and

$$n^{1/2}(\Gamma_k^{(n)}(\tau_n^\star;\mathbf{p})/n - f_{\Gamma,k}^{(n)}(z_n^\star;\mathbf{p})) \xrightarrow{d} \mathcal{Z}_k(t^\star;\mathbf{p}) - \alpha^{-1}f'_{\Gamma,k}(z^\star;\mathbf{p})\mathcal{Z}_W(t^\star;\mathbf{p}). \tag{42}$$

By a similar argument, we have that $z_n^\star(\mathbf{p}) \to z^\star(\mathbf{p})$ in probability and, for all $k \in [K]$,

$$f_{\Gamma,k}^{(n)}(z_n^\star;\mathbf{p}) \xrightarrow{p} f_{\Gamma,k}(z^\star;\mathbf{p}),$$

as $n \to \infty$, for any fixed $\mathbf{p} \in [\mathbf{p}_{min},\mathbf{p}_0]$. Further, since the inverse demand function $\mathbf{g}$ is $\mathcal{C}^1$, we have for all $k \in [K]$, as $n \to \infty$,

$$g'_k \circ f_{\Gamma,k}^{(n)}(z_n^\star;\mathbf{p}) \xrightarrow{p} g'_k \circ f_{\Gamma,k}(z^\star;\mathbf{p}).$$

Hence, using again the Mean-Value theorem, we have for all $k \in [K]$, there exists some $\boldsymbol{\Xi}_{k,n} := (\xi_{k,n}^{(1)},\ldots,\xi_{k,n}^{(K)})$ converging to $\mathbf{f}_\Gamma(z^\star;\mathbf{p}) = (f_{\Gamma,1}(z^\star;\mathbf{p}),\ldots,f_{\Gamma,K}(z^\star;\mathbf{p}))$ such that

$$g_k(\boldsymbol{\Gamma}^{(n)}(\tau_n^\star;\mathbf{p})/n) - g_k(\boldsymbol{f}_\Gamma^{(n)}(z_n^\star;\mathbf{p})) = \nabla g_k(\boldsymbol{\Xi}_{k,n})\cdot(\boldsymbol{\Gamma}^{(n)}(\tau_n^\star;\mathbf{p})/n - \mathbf{f}_\Gamma^{(n)}(z_n^\star;\mathbf{p})). \tag{43}$$

Multiply both side of (43) by $n^{1/2}$, we obtain

$$n^{1/2}g_k(\boldsymbol{\Gamma}^{(n)}(\tau_n^\star;\mathbf{p})/n) - g_k(_\Gamma^{(n)}(z_n^\star;\mathbf{p})) = n^{1/2}\nabla g_k(\boldsymbol{\Xi}_{k,n})\cdot(\boldsymbol{\Gamma}^{(n)}(\tau_n^\star;\mathbf{p})/n - \mathbf{f}_\Gamma^{(n)}(z_n^\star;\mathbf{p})).$$

Then by the asymptotic normality of point (ii) in theorem B.2, we can generalize to our multi-dimensional case. The random vector $n^{1/2}(\boldsymbol{\Gamma}^{(n)}(\mathbf{p})/n - \mathbf{f}_\Gamma^{(n)}(z_n^\star,\mathbf{p}))$ converges in distribution to a centered Gaussian vector $\boldsymbol{\mathcal{Z}}_{end}(\mathbf{p}) := (\mathcal{Z}^{(1)}_{end}(\mathbf{p}), \dots, \mathcal{Z}^{(K)}_{end}(\mathbf{p}))^T$, where

$$\mathcal{Z}^{(k)}_{end}(\mathbf{p}) := \mathcal{Z}_{\Gamma,k}(t^\star; \mathbf{p}) - \alpha^{-1} f^1_{\Gamma,k}(z^\star; \mathbf{p}) \mathcal{Z}_W(t^\star; \mathbf{p}).$$

Then combining Slutsky's theorem, we have finally for all $k \in [K]$, as $n \to \infty$,

$$n^{1/2}(\kappa^{(k)}_n(\mathbf{p}) - g_k(\boldsymbol{f}^{(n)}_\Gamma(z^\star_n, \mathbf{p}))) \xrightarrow{d} \nabla g_k(\mathbf{f}_\Gamma(z^\star; \mathbf{p})) \cdot \boldsymbol{\mathcal{Z}}_{end}(\mathbf{p}).$$

This completes the proof of Theorem B.3.

B.4 Proof of Theorem B.4

We first provide the variance function of $\mathcal{Z}_{\Gamma,k}(t; \mathbf{p})$. Using the same arguments as in Section 5.4, one can show that for all $k \in [K]$, $\Psi_k(t; \mathbf{p})$ has the same structure as that in the uni-asset case. By replacing the corresponding mean $\bar{\ell}_{x,\theta,k}$, variance $\varsigma^2_{x,\theta,k}$ and $\bar{\gamma}_{x,k}$ for each type k, we can get the variance function for $\mathcal{Z}_{\Gamma,k}(t; \mathbf{p})$, i.e.,

$$\begin{aligned}\Psi_k(t; \mathbf{p}) = &\sum_{x \in \mathcal{X}} \Big(\sum_{\theta=1}^{d^+_x} \Psi_{x,\theta,k}(t; \mathbf{p}) + \Psi_{x,\infty,k}(t; \mathbf{p}) + \bar{\gamma}^2_{x,k} \psi_{x,0,0}(\mathbf{p}) \Big) + \sum_{x \in \mathcal{X}} \sum_{\theta_1,\theta_2=1}^{d^+_x} \sigma^I_{x,\theta_1,\theta_2}(e^{-t}; \mathbf{p}) \\ &+ \sum_{x \in \mathcal{X}} \Big(2 \sum_{\theta=1}^{d^+_x} \sigma^I_{x,\theta,\infty}(e^{-t}; \mathbf{p}) + 2 \sum_{\theta=1}^{d^+_x} \psi_{x,\theta,0}(\mathbf{p}) \sum_{s=d^+_x-\theta+1}^{d^+_x} \beta(d^+_x, e^{-t}, s) \Big) \\ &+ \sum_{x \in \mathcal{X}} \psi_{x,\infty,0}(\mathbf{p}) \sum_{s=1}^{d^+_x} \beta(d^+_x, e^{-t}, s),\end{aligned} \tag{44}$$

where for all $\theta \in \{1, \dots, d^+_x\} \cup \{\infty\}$

$$\Psi_{x,\theta,k}(t; \mathbf{p}) := \widehat{f}_{x,\theta}(t; \mathbf{p}) \varsigma^2_{x,\theta,k}(\mathbf{p}) + \bar{\ell}^2_{x,\theta,k}(\mathbf{p}) \sigma^I_{x,\theta,\theta}(e^{-t}; \mathbf{p}).$$

We now proceed with the proof of Theorem B.4. The case $z^\star(\mathbf{p}) = 0$ is a direct generalization of the corresponding situations in Theorem 3.12 and can be proved by a similar argument, using Theorem B.3.

Consider now the case when $z^\star(\mathbf{p}) \in (0, 1]$ and $z^\star(\mathbf{p})$ is a stable solution, i.e., $\alpha(\mathbf{p}) := f^1_W(z^\star(\mathbf{p}); \mathbf{p}) > 0$. First of all, Lemma 5.4 could be generalized to the multi-dimensional case and show that $\bar{\mathbf{p}}_n \to \bar{\mathbf{p}}$. Further, we also have that

$$\boldsymbol{\mathcal{Z}}_V(\mathbf{p}_n) \xrightarrow{d} \boldsymbol{\mathcal{Z}}_V(\mathbf{p}), \tag{45}$$

for any sequence $\{\mathbf{p}_n\}_n$ converging to $\mathbf{p}$ as $n \to \infty$.

Let us denote

$$\Phi^{(k)}_n(\mathbf{p}) := p_k - g_k(\mathbf{\Gamma}^{(n)}(\tau^\star_n; \mathbf{p})/n).$$

We use again the Skorokhod representation theorem. All the convergence results of Theorem B.3,

$\mathbf{p}_n^\star \to \bar{\mathbf{p}}$ and (45) hold a.s., by changing the probability space. Then we can write

$$\begin{aligned}\Phi_n^{(k)}(\mathbf{p}_n^\star) &= \phi_n^{(k)}(\mathbf{p}_n^\star) + n^{-1/2}\mathcal{Z}_{V,k}(\mathbf{p}_n^\star) + o(n^{-1/2}) \\ &= \phi_n^{(k)}(\mathbf{p}_n^\star) + n^{-1/2}\mathcal{Z}_{V,k}(\bar{\mathbf{p}}) + o(n^{-1/2}),\end{aligned} \tag{46}$$

where the second equality is because that we have a.s. $\mathcal{Z}_{V,k}(\mathbf{p}_n^\star) \to \mathcal{Z}_{V,k}(\bar{\mathbf{p}})$. Notice also that for all $k \in [K]$, $\Phi_n^{(k)}(\mathbf{p}_n^\star) = 0$, then we have

$$\phi_n^{(k)}(\mathbf{p}_n^\star) = -n^{-1/2}\mathcal{Z}_{V,k}(\bar{\mathbf{p}}) + o(n^{-1/2}). \tag{47}$$

Next, we proceed to approximate the difference between $\mathbf{p}_n^\star$ and $\bar{\mathbf{p}}_n$ by the Mean-Value theorem. The arguments are similar to the uni-asset case and, in order to avoid the repeatability, we omit some detail and highlight the difference from the one asset situation. We denote $o(\mathbf{1})$ the K-column vector of $o(1)$. Firstly, for all $k \in [K]$, we have

$$\begin{aligned}&g_k(\mathbf{f}_\Gamma^{(n)}(z_n^\star(\mathbf{p}_n^\star);\mathbf{p}_n^\star)) - g_k(\mathbf{f}_\Gamma^{(n)}(z_n^\star(\bar{\mathbf{p}}_n);\bar{\mathbf{p}}_n)) \\ &\quad = (\nabla g_k(\mathbf{f}_\Gamma(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}})) + o(\mathbf{1}))(\mathbf{f}_\Gamma^{(n)}(z_n^\star(\mathbf{p}_n^\star);\mathbf{p}_n^\star) - \mathbf{f}_\Gamma^{(n)}(z_n^\star(\bar{\mathbf{p}}_n);\bar{\mathbf{p}}_n)).\end{aligned} \tag{48}$$

Next we analyze $\mathbf{f}_\Gamma^{(n)}(z_n^\star(\mathbf{p}_n^\star);\mathbf{p}_n^\star) - \mathbf{f}_\Gamma^{(n)}(z_n^\star(\bar{\mathbf{p}}_n);\bar{\mathbf{p}}_n)$. Again by the Mean-Value theorem, we have for all $k \in [K]$,

$$\begin{aligned}f_k^{(n)}(z_n^\star(\mathbf{p}_n^\star);\mathbf{p}_n^\star) - f_k^{(n)}(z_n^\star(\bar{\mathbf{p}}_n);\bar{\mathbf{p}}_n) =&(f_k^1(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}) + o(1))(z_n^\star(\mathbf{p}_n^\star) - z_n^\star(\bar{\mathbf{p}}_n)) \\ &+ (\nabla^{(2)} f_k(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}) + o(\mathbf{1}))\cdot(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n).\end{aligned} \tag{49}$$

We next approximate $z_n^\star(\mathbf{p}_n^\star) - z_n^\star(\bar{\mathbf{p}}_n)$. Notice that $f_W^{(n)}(z_n^\star(\mathbf{p});\mathbf{p}) = 0$ for any $\mathbf{p} \in [\mathbf{p}_{min}, \mathbf{p}_0]$. Using again the Mean-Value theorem, we have the following two equations

$$f_W^{(n)}(z_n^\star(\mathbf{p}_n^\star);\mathbf{p}_n^\star) - f_W^{(n)}(z_n^\star(\bar{\mathbf{p}}_n);\mathbf{p}_n^\star) = f_W^{1,(n)}(\xi_n^z;\mathbf{p}_n^\star)(z_n^\star(\mathbf{p}_n^\star) - z_n^\star(\bar{\mathbf{p}}_n)),$$

and,

$$f_W^{(n)}(z_n^\star(\bar{\mathbf{p}}_n);\mathbf{p}_n^\star) - f_W^{(n)}(z_n^\star(\bar{\mathbf{p}}_n);\bar{\mathbf{p}}_n) = \nabla^{(2)} f_W^{(n)}(z_n^\star(\bar{\mathbf{p}}_n),\boldsymbol{\zeta}_n)\cdot(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n),$$

where $\xi_n^z \to z^\star(\bar{\mathbf{p}})$ a.s., $\boldsymbol{\zeta}_n \to \bar{\mathbf{p}}$ a.s. and the notation $\nabla^{(2)}$ is defined by setting

$$\nabla^{(2)} F(z;\mathbf{p}) = (F^2(z;\mathbf{p}),\ldots,F^{K+1}(z;\mathbf{p})),$$

Then by the above two equations we have

$$z_n^\star(\mathbf{p}_n^\star) - z_n^\star(\bar{\mathbf{p}}_n) = -((f_W^1(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}))^{-1} + o(\mathbf{1}))(\nabla^{(2)} f_W(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}) + o(\mathbf{1}))\cdot(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n). \tag{50}$$

Now by (49) and (50), we have that for all $k \in [K]$,

$$\begin{aligned}&f_{\Gamma,k}^{(n)}(z_n^\star(\mathbf{p}_n^\star);\mathbf{p}_n^\star) - f_{\Gamma,k}^{(n)}(z_n^\star(\bar{\mathbf{p}}_n);\bar{\mathbf{p}}_n) \\ &\quad = -f_k^1(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}})(f_W^1(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}}))^{-1}\nabla^{(2)} f_W(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}})\cdot(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n) \\ &\qquad + \nabla^{(2)} f_k(z^\star(\bar{\mathbf{p}});\bar{\mathbf{p}})\cdot(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n) + o(\mathbf{1})\cdot(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n) \\ &\quad = (\mathbf{B}_k + o(\mathbf{1})^T)(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n),\end{aligned} \tag{51}$$

where $\mathbf{B}_k$ is the k-th row vector of $\mathbf{B}$. Hence by (48) and (51), for all $k \in [K]$, we can conclude

$$\begin{aligned}\phi_n^{(k)}(\mathbf{p}_n^\star) &= \phi_n^{(k)}(\mathbf{p}_n^\star) - \phi_n^{(k)}(\bar{\mathbf{p}}_n) \\ &= p_n^{\star(k)} - \bar{p}_n^{(k)} - (g_k(\mathbf{f}_\Gamma^{(n)}(z_n^\star(\mathbf{p}_n^\star); \mathbf{p}_n^\star)) - g_k(\mathbf{f}_\Gamma^{(n)}(z_n^\star(\bar{\mathbf{p}}_n); \bar{\mathbf{p}}_n))) \\ &= p_n^{\star(k)} - \bar{p}_n^{(k)} - (\mathbf{A}_k \cdot \mathbf{B} + o(\mathbf{1})^T)(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n),\end{aligned}$$

where $\boldsymbol{A}_k$ is the k-th row vector of $\boldsymbol{A}$.

Combining Equation (47), it then follows that

$$\boldsymbol{\phi}_n(\mathbf{p}_n^\star) = (\mathbf{I}_{K\times K} - \mathbf{A} \cdot \mathbf{B} - o(\mathbf{1}_{K\times K}))(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n) = -n^{-1/2}(\boldsymbol{\mathcal{Z}}_V(\bar{\mathbf{p}}) + o(\mathbf{1})).$$

We therefore obtain that

$$n^{1/2}(\mathbf{p}_n^\star - \bar{\mathbf{p}}_n) = -(\mathbf{I}_{K\times K} - \mathbf{A} \cdot \mathbf{B})^{-1} \boldsymbol{\mathcal{Z}}_V(\bar{\mathbf{p}}) + o(\mathbf{1}),$$

provided that the matrix $\mathbf{I}_{K\times K} - \mathbf{A} \cdot \mathbf{B}$ is non-singular. This completes the proof of Theorem B.4.