Numerical optimization for frictional contact problems

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THESIS
for obtaining the degree of **MASTER OF APPLIED MATHEMATICS**

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Limoges, France
2019

Acknowledgements

This internship was the best opportunity where I could see how people can do the research work in a very good manner and also could observe high professionalism of the whole team where I did my internship. First of all, I would like to thank two great persons which were teaching me during this time - professors Paul Armand and Vincent Acary.

I am immensely grateful to professor Paul Armand for his professionalism in teaching activity and ability to explain difficult things in the easy way. I really thank for this time of working together with Pr. Armand since I also studied how to teach people in efficient way. It is important for me because I would like to spend some part of my life also for teaching.

This internship gives me the diverse knowledge. Professor Armand taught me many important things from optimization theory and practice in general, where professor Vincent Acary showed how to apply this knowledge to solve the real-life problem. He spent a lot of time to teach me about the origin of discovered problem, showed how to simulate the mechanical problem in siconos software where I could see a real application of my research topic. Many practical stuff in programming and in general about numerical simulation and optimization I learned from Pr. Acary. The time when I worked at INRIA in a TRIPOP team under Pr. Acary supervision was the best time of my student life where I felt as a little but true specialist in applied mathematics.

I would like also to thank all professors in University of Limoges which took a part in M2 ACSYON program. They did a very good job and gave a true an quite useful knowledge where we (students) can apply in our future career.

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Introduction

We consider the mechanical system which involves unilateral contact between its parts. In addition to this constraint, Coulomb's friction arises at each contact point in this system. It is shown in [1] that constraints for the mechanical system coincide with optimality conditions for convex quadratic optimization problem with the second order cone constraints. Thus, optimization problem can be formulated and solved numerically. The difficulty of this approach is that there exist conic constraints while the variables must belong to the Lorentz cone at the solution.

This work is devoted to solve the friction contact problem using convex quadratic optimization formulation with the second order cone constraints applying Interior-point methods. In addition, because of the conic constraints, Euclidean Jordan algebra can be applied to derive the Newton system which is a core of IPM. The goal of this work is to answer on two questions:

1. Does Interior-point method able to solve the frictional contact problem?

2. Does Interior-point method able to solve it efficiently?

The structure of this work is following: in Chapter 1 we put general results for formulation the convex quadratic problem and its relation to original mechanical system; in Chapter 2 we derive the framework for solving convex quadratic optimization problems with the second order cone constraints and not a full rank constraint matrix A, quadratic form matrix Q, the core of this chapter is primal-dual regularization [2] and Nesterov and Todd Jacobian rescaling [3]; in Chapter 3 we apply this framework to solve friction problem in convex optimization reformulation; in Chapter 4 we do numerical experiments to test the theory of the framework which is defined in Chapter 2 in practice.

Chapter 1

A Coulomb friction problem formulation via convex quadratic programming

This chapter is devoted to the formulation of the mechanical system which involves unilateral contact between its parts. In addition to this constraint, Coulomb's friction arises at each contact point. After formulation, it is shown how to move from mechanical system to convex quadratic optimization problem and how the constraints at each contact point can be transformed into constraints for optimization problem.

1.1 Mechanical system

We consider a mechanical system in 3-dimensional space [1]. Let us write the equation of motion

$$M(q(t))\frac{dv}{dt}(t) = F(t, q(t), v(t)) + \Lambda(t), \tag{1.1}$$

where $M : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is the mass matrix at time t, $v(t) \in \mathbb{R}^m$ is a vector of generalized velocities, $q(t) \in \mathbb{R}^m$ is a vector of generalized coordinates, $F : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a vector which includes internal and external forces and $\Lambda : \mathbb{R} \to \mathbb{R}^m$ is a vector of generalized reaction forces which includes Coulomb's friction contact model.

Each body in mechanical system (1.1) has not only generalized velocities (direction of movement) but also relative velocities (velocity which is related to the contact point). The generalized velocities v are related to the n relative velocities $u := (u_1, \dots, u_n) \in \mathbb{R}^{nd}$ by the formula

$$u(t) = H(q(t))v(t) + w(t). \tag{1.2}$$

Using the duality theory in mechanics, local reaction forces $r = (r_1, \dots, r_n)$ relates to generalized reaction forces by the formula

$$\Lambda(t) = H^\top(q(t))r(t). \tag{1.3}$$

Then, we can build the time-discretization scheme and obtain the following equations

$$\begin{cases} M(q_k)(v_{k+1} - v_k) = hF(t_k, q_k, v_k) + H^\top(q_k)r_{k+1} \\ u_{k+1} = H(q_k)v_{k+1} + w(q_k) \end{cases} \tag{1.4}$$

Let us denote by $u = u_{k+1} \in \mathbb{R}^{nd}$, $v = v_{k+1} \in \mathbb{R}^m$ and $r = r_{k+1} \in \mathbb{R}^{nd}$. Then, after algebraic reformulations from (1.4) we obtain for given values of v_k and q_k

$$\begin{cases} Mv + f = H^\top r \\ u = Hv + w \end{cases} \tag{1.5}$$

where $H = H(q_k) \in \mathbb{R}^{nd\times m}$, $w = w(t_k) \in \mathbb{R}^{nd}$, $M = M(q_k) \in \mathbb{R}^{m\times m}$ and $f = Mv_k + hF(t_k, q_k, v_k) \in \mathbb{R}^m$ are known at the step $k+1$. In this work we assume that M is symmetric and positive definite.

In addition to these equations, at each step we have the constraints on u and r which comes from Coulomb's law. To define this set we need to decompose vectors u, r into he normal and tangent part $r = r_N + r_T$, $u = u_N + u_T$ where r_T is orthogonal to normal vector e^i at contact as well as u_T. Thus on each step at each i-th contact point $(u_i, r_i) \in \mathcal{C}(e^i, \mu^i)$ where set $\mathcal{C}(e^i, \mu^i) \subset \mathbb{R}^{d\times d}$ is defined by

$$(u, r) \in \mathcal{C}(e^i, \mu^i) \iff \begin{cases} \text{either:} & r = 0 \text{ and } u_N \geq 0 \quad \text{(take off)} \\ \text{or:} & \|r_T\| \geq \mu r_N \text{ and } u = 0 \quad \text{(sticking)} \\ \text{or:} & \begin{cases} 0 < \|t_T\| = \mu r_N \\ \exists \alpha > 0 : r_T = -\alpha u \end{cases} \quad \text{(sliding)} \end{cases} \tag{1.6}$$

Connecting all together we obtain the following incremental problem that is the main target of current work.

$$\begin{cases} Mv + f = H^\top r \\ u = Hv + w \\ (u_i, r_i) \in \mathcal{C}(e^i, \mu^i) \text{ for all } i \in \{1, \ldots, n\} \end{cases} \tag{1.7}$$

Then, let us define the second order cone which plays important role in current work

$$\mathcal{K}_{e,\mu} := \{x \in \mathbb{R}^k : \|x_T\| \leq \mu x_N\}. \tag{1.8}$$

It can be shown that

$$(\mathcal{K}_{e,\mu})^* = \mathcal{K}_{e,\frac{1}{\mu}}, \tag{1.9}$$

where $(\mathcal{K}_{e,\mu})^*$ is the dual cone for $\mathcal{K}_{e,\mu}$. Recall, that the dual cone to S is defined by

$$S^* := \{x \in \mathbb{R}^k : s^\top x \geq 0,\ \forall s \in S\}. \tag{1.10}$$

The cone $\mathcal{K}$ for which $\mathcal{K} = \mathcal{K}^*$ is called auto-dual or self-dual. Note that for $\mu = 1$ cone (1.9) is auto-dual.

1.2 Conic complementarity constraints

Let us provide a change of variable

$$\tilde{u}^i := u^i + \mu^i \|u^i_T\| e^i, \text{ for } i \in \{1, \ldots, n\}, \tag{1.11}$$

which is inspired by so-called bipotential [7]. Then let us define set of indexes for which $\mu^i \neq 0$.

$$I := \{i \in \{1, \ldots, n\} : \mu^i \neq 0\} \quad \text{and} \quad n_I := \text{Card } I. \tag{1.12}$$

For those contact points where $i \in I$, the change $u^i \leftarrow \tilde{u}^i$ can be written using new variable s^i:

$$s^i := \|u^i_T\| = \|\tilde{u}^i_T\| \quad \text{and} \quad \tilde{u}^i = u^i + \mu^i s^i e^i. \tag{1.13}$$

After, we introduce matrix $E \in \mathbb{R}^{nd\times n_I}$ to write this change ob variable in matrix form. Thus E is constructed by concatenating n_I columns $E_i \in \mathbb{R}^{nd}$, where E_i is a concatenation of n vectors of $\mathbb{R}^d$, all zero except for the i-th which is $\mu^i e^i$. Thus (1.11) rewrites as

$$\tilde{u} = u + Es, \tag{1.14}$$

and then the second equation of (1.7) writes as

$$\tilde{u} = Hv + w + Es. \tag{1.15}$$

Now, let us formulate the key lemma of this section

Lemma 1.2.1 (Reformulation of Coulomb's law [1])**.** *The couple (u^i, r^i) satisfies Coulomb's law (1.6) if and only if the couple $(\tilde{u}^i, r^i)$ - with $\tilde{u}^i$ defined by (1.11) - satisfies*

$$(\mathcal{K}_{e^i,\mu^i})^* \ni \tilde{u}^i \perp r^i \in \mathcal{K}_{e^i,\mu^i} \tag{1.16}$$

Thus we introduce the product-cone

$$L := \mathcal{K}_{e^1,\mu^1} \times \cdots \times \mathcal{K}_{e^n,\mu^n} \subset \mathbb{R}^{nd}, \tag{1.17}$$

and then conic complementarity constraints can be written as

$$L^* \ni \tilde{u} \perp r \in L, \tag{1.18}$$

where L^* is defined by

$$L^* = (\mathcal{K}_{e^1,\mu^1})^* \times \cdots \times (\mathcal{K}_{e^n,\mu^n})^* = \mathcal{K}_{e^1,\frac{1}{\mu^1}} \times \cdots \times \mathcal{K}_{e^n,\frac{1}{\mu^n}}. \tag{1.19}$$

To conclude, the incremental problem (1.7) is equivalent to the following system

$$\begin{cases} Mv + f = H^\top r \\ \tilde{u} = Hv + w + Es \\ L^* \ni \tilde{u} \perp r \in L \\ s^i = \|\tilde{u}^i_T\|, \text{ for } i \in I \end{cases} \tag{1.20}$$

where the variables are $(v, r, \tilde{u}, s) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times \mathbb{R}^{nd} \times \mathbb{R}^{n_I}$.

The idea [1] is to extract from (1.20) the following conic complementarity problem

$$\begin{cases} Mv + f = H^\top r \\ \tilde{u} = Hv + w + Es \\ L^* \ni \tilde{u} \perp r \in L \end{cases} \tag{1.21}$$

where $s \in \mathbb{R}^{n_I}$ is a fixed parameter. The motivation to extract constraints (1.21) is that they turn out to be the optimality conditions of an optimization problem with good theoretical results [1].

1.3 Fixed-point formulation

Let us consider the following optimization problem

$$\begin{array}{ll} \min_{(v,u)\in\mathbb{R}^m\times\mathbb{R}^{nd}} & \frac{1}{2}v^\top Mv + f^\top v \\ \text{s.t.} & u = Hv + w + Es, \\ & u \in L^* \end{array} \tag{1.22}$$

and its dual

$$\begin{array}{ll} \max_{(v,r)\in\mathbb{R}^m\times\mathbb{R}^{nd}} & -\frac{1}{2}v^\top Mv - (w + Es)^\top r \\ \text{s.t.} & Mv - H^\top r = -f, \\ & r \in L \end{array} \tag{1.23}$$

Proposition 1.3.1. The optimization problems (1.22) and (1.23) are primal-dual problems pair.

Proof. Follows directly from the Theorem A.0.1 after reformulation the problem (1.22) in to the form

$$\begin{aligned} \min \quad & \tfrac{1}{2} \begin{bmatrix} v^+ \\ v^- \\ u \end{bmatrix}^\top \begin{bmatrix} M & -M & 0 \\ -M & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v^+ \\ v^- \\ u \end{bmatrix} + \begin{bmatrix} f \\ -f \\ 0 \end{bmatrix}^\top \begin{bmatrix} v^+ \\ v^- \\ u \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} H & -H & -I \end{bmatrix} \xi = -(w + Es), \\ & \xi := \begin{bmatrix} v^+ \\ v^- \\ u \end{bmatrix} \in K := \mathbb{R}^{2m}_+ \times L^*, \end{aligned} \tag{1.24}$$

where substitution $v \leftarrow v^+ - v^-$, $v^+ \in \mathbb{R}^m_+$, $v^- \in \mathbb{R}^m_+$ was made. □

Lemma 1.3.2 (Optimality conditions)**.** The first order optimality conditions for (1.22) are

$$\begin{cases} Mv + f - H^\top r = 0 \\ u - Hv - w - Es = 0 \\ L \ni r \perp u \in L^* \end{cases} \tag{1.25}$$

Proof. Follows immediately from Proposition 1.3.1. □

Thus we observe that the solution the optimization problem (1.22) for some fixed s coincides with the solution for (1.21). In the following chapters we suppose that $s = 0$ and concentrate on defining the framework which is able to solve a convex quadratic optimization problem with the second order cone constraints.

Chapter 2

Second order cone programming for convex quadratic problem

In this chapter we consider the convex quadratic programming with the linear equality constraints $Ax = b$ and the second order cone constraints $x \in K$, where A can be not a full rank matrix and cone $\mathcal{K}$ is the Lorentz cone $\mathcal{K} = \{x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 \geq \|\bar{x}\|\}$. The main goal of this chapter is to apply conic programming theory with the framework of Jordan algebra and the results of primal-dual regularization of Friedlander and Orban [2] to develop a well-defined optimization algorithm from the family of Interior-point methods.

The structure of this chapter is following:

- In Sect. 1 we formulate a primal and dual problems for convex quadratic programming with linear equality and conic constraints.
- Sect. 2 is devoted to Jordan algebra framework which is used in conic optimization theory.
- In Sect. 3 we show how to obtain the first order optimality conditions with the framework of Jordan algebra and formulate a Newton system which is used in IPM.
- Sect. 4 is devoted to deriving a well-defined interior-point algorithm where we show:
  - how to apply the regularization technique of Friedlander and Orban [2] and when it is necessary;
  - how to apply Nesterov and Todd scaling technique in the second order cone programming (SOCP);
- In Sect. 5 we show how to derive the perturbed optimality conditions and the Newton system;
- Sect. 6 is devoted to define a way of estimation the barrier parameter in perturbed Newton system.
- In Sect. 7 we show how to compute the step length in IPM with respect to the second ordred cone (SOC) constrains.
- Sect. 8 shows the Interior-point algorithm for solving convex quadratic program with SOC constraints.

At the end, we obtain the set of tools and results which are used in IPM for SOCP with a degenerate constraints matrix and after we describe some crucial details about the application of these tools in practice.

2.1 Primal-dual problem formulation

We consider the primal-dual pair of convex quadratic optimization problem with linear equality and the second order cone constraints in the form

$$\begin{array}{ll} \min_{x\in\mathbb{R}^{nd}} & \frac{1}{2}x^\top Qx + c^\top x \\ \text{s.t.} & Ax = b, \\ & x \in L \end{array} \tag{P}$$

and the dual one

$$\begin{array}{ll} \max_{(x,y,z)\in\mathbb{R}^{nd}\times\mathbb{R}^m\times\mathbb{R}^{nd}} & -\frac{1}{2}x^\top Qx + b^\top y \\ \text{s.t.} & Qx - A^\top y - z = -c, \\ & z \in L^* \end{array} \tag{D}$$

where

- $x = (x_1, \dots, x_n) \in \mathbb{R}^{nd}$, $x_i \in \mathbb{R}^d$ for $i \in \{1, \dots, n\}$ are primal variables.
- $y \in \mathbb{R}^m$, $z = (z_1, \dots, z_n) \in \mathbb{R}^{nd}$ $z_i \in \mathbb{R}^d$ for $i \in \{1, \dots, n\}$ are dual variables.
- $Q \in \mathbb{S}^{nd}_+$ ($\mathbb{S}^{nd}_+$ - the cone of symmetric positive definite matrices), $c \in \mathbb{R}^{nd}$, $A \in \mathbb{R}^{m\times nd}$, $b \in \mathbb{R}^m$ are given data.
- The cone $L = \mathcal{K}_1 \times \dots \times \mathcal{K}_n$ is the cartesian product of the second order cones in $\mathbb{R}^d$

$$\mathcal{K}_i = \{x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{d-1} : x_0 \geq \|\bar{x}\|\} \text{ for } i \in \{1, \dots, n\}.$$

- The dual cone $L^* = \mathcal{K}_1^* \times \dots \times \mathcal{K}_n^*$ is the product of the dual second order cones where $\mathcal{K}_i = \mathcal{K}_i^*$ for $i \in \{1, \dots, n\}$ since Lorentz cone is an autodual cone and thus $L^* = L$.

The proof that (P) and (D) are primal dual problems is presented in Theorem A.0.1.

2.2 Euclidean Jordan algebra

The Euclidean Jordan algebra is a framework which generalizes the algebraic properties of symmetric matrices. This framework has several branches as Jordan algebra for SDP and for SOCP. We will concentrate on the second one since it is exactly our case. Here and further we will use notation *Jordan algebra* for SOCP branch of Euclidean Jordan algebra.

Let us introduce the core of this algebra which is a Jordan product of two vectors $x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $y = (y_0, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

$$x \circ y := \begin{pmatrix} x^\top y \\ x_0\bar{y} + y_0\bar{x} \end{pmatrix}. \tag{2.1}$$

This product is commutative $x \circ y = y \circ x$ but not associative $x \circ (y \circ z) \neq (x \circ y) \circ z$. Following Alizadeh [3] let us define Arrow matrix.

$$\text{Arw}(x) = \begin{pmatrix} x_0 & \bar{x} \\ \bar{x}^\top & x_0 I \end{pmatrix}. \tag{2.2}$$

Then it can be shown that

$$x \circ y = \text{Arw}(x)y = \text{Arw}(x)\text{Arw}(y)e, \tag{2.3}$$

where $e = (1, 0, \dots, 0)$ is the identity element of Jordan algebra.

This product has a property that the square element by the Jordan product belongs to Lorentz cone i.e. if

$$\mathcal{K} = \{x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 \geq \|\bar{x}\|\}, \quad \mathcal{J} = \{x \circ x : x \in \mathbb{R}^n\},$$

then $\mathcal{K} = \mathcal{J}$ [3].

Each element of this algebra has a spectral decomposition like a matrices in linear algebra. It means that elements have eigenvalues and eigenvectors with respect to the Jordan product. It is shown in paper of Alizadeh [3] that each element of this algebra can be represented as a linear combination of the two eigenvalues and two eigenvectors.

$$x = \lambda_1 c_1 + \lambda_2 c_2,$$

where $\lambda_1 = x_0 + \|\bar{x}\|$, $\lambda_2 = x_0 - \|\bar{x}\|$ are eigenvalues and $c_1 = \frac{1}{2}\begin{pmatrix} 1 \\ \frac{\bar{x}}{\|\bar{x}\|} \end{pmatrix}$, $c_2 = \frac{1}{2}\begin{pmatrix} 1 \\ -\frac{\bar{x}}{\|\bar{x}\|} \end{pmatrix}$ are eigenvectors. It can be shown [3] that λ_1 and λ_2 are the greatest and the lowest eigenvalue of $\text{Arw}(x)$. This fact shows that $\text{Arw}(x) \succ 0$ if and only if $x \in \text{int}(\mathcal{K})$ and $\text{Arw}(x) \succcurlyeq 0$ if and only if $x \in \text{bd}(\mathcal{K})$.

Definition 2.2.1. Let $x \in \mathbb{R}^n$, then we have

- $x^{-1} = \lambda_1^{-1} c_1 + \lambda_2^{-1} c_2$;
- $x^{1/2} = \lambda_1^{1/2} c_1 + \lambda_2^{1/2} c_2$ for $x \in \mathcal{K}$;
- $\text{tr}(x) = \lambda_1 + \lambda_2 = 2x_0$;
- $\det(x) = \lambda_1 \lambda_2 = x_0^2 - \|\bar{x}\|^2$;
- $\nabla_x \log \det(x) = 2x^{-1}$.

Definition 2.2.2. Let variable $x = (x_1, \dots, x_n) \in \mathbb{R}^{nd}$, $x_i \in \mathbb{R}^d$ for $i \in \{1, \dots, n\}$ and variable $y = (y_1, \dots, y_n) \in \mathbb{R}^{nd}$, $y_i \in \mathbb{R}^d$ for $i \in \{1, \dots, n\}$ then

- $x \circ y = (x_1 \circ y_1, \dots, x_n \circ y_n)$;
- $x^{-1} = (x_1^{-1}, \dots, x_n^{-1})$;
- $\text{Arw}(x) = \text{Arw}(x_1) \oplus \dots \oplus \text{Arw}(x_n)$.

where $\oplus$ is a direct product, $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

The main result which will be used to derive the first order optimality conditions is the lemma (2.2.1). It becomes very important since in LP complementarity conditions $z^\top x = 0$ satisfies if and only if $x_i z_i = 0$ for $i \in \{1, \dots n\}$. But in SOCP $z^\top x = 0$ where $x = (x_1, \dots, x_n) \in \mathcal{K} \subset \mathbb{R}^{nd}$, $z = (z_1, \dots, z_n) \in \mathcal{K} \subset \mathbb{R}^{nd}$ either $x_i z_i = 0$, $x_i \in \mathbb{R}^d$, $z_i \in \mathbb{R}^d$ or $x_i \in \mathbb{R}^d$ and $z_i \in \mathbb{R}^d$ lies on the opposite sides of the boundary of cone. Thus result is given by the following lemma

Lemma 2.2.1 (**Complementarity conditions** [3])**.** Let $x \in \mathcal{K}$, $z \in \mathcal{K}$. Then $x^\top z = 0$ iff

$$x \circ z = \text{Arw}(x)\text{Arw}(z)e = 0.$$

2.3 The first order optimality conditions

Using the framework of Jordan algebra for SOCP the first order optimality conditions can be derived. Also, using these conditions, the Newton system can be built.

Theorem 2.3.1 (**Optimality conditions**)**.** If (P) and (D) have feasible solutions x^*, z^* such that $x^* \in \text{int}(L)$, $z^* \in \text{int}(L)$, then (x, y, z) is an optimal solution pair if and only if

$$\begin{cases} Qx + c - A^\top y - z & = 0 \\ Ax - b & = 0 \\ L \ni z \perp x \in L \end{cases} \tag{2.4}$$

Proof. Following directly from the Theorem A.0.1. □

The last condition in (2.4) can be rewritten using Lemma 2.2.1.

$$\text{Arw}(x)\text{Arw}(z)e = 0, \quad x \in L, z \in L. \tag{2.5}$$

Then with (2.5), (2.4) can be rewritten as

$$\begin{cases} Qx + c - A^\top y - z & = 0 \\ Ax - b & = 0 \\ \text{Arw}(x)\text{Arw}(z)e & = 0 \\ x \in L,\ z \in L \end{cases} \tag{2.6}$$

2.4 Newton system

Using (2.6) we can derive the Newton system to be solved during IPM iterations. Thus, from (2.6) we can deduce

$$F(w) = \begin{bmatrix} Qx + c - A^\top y - z \\ Ax - b \\ \text{Arw}(x)\text{Arw}(z)e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ x \in L, z \in L, \tag{2.7}$$

where $w = (x, y, z)$.

Then we can obtain Newton system (2.8).

$$\begin{bmatrix} Q & -A^\top & -I \\ A & 0 & 0 \\ \text{Arw}(z) & 0 & \text{Arw}(x) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - z \\ Ax - b \\ \text{Arw}(x)\text{Arw}(z)e \end{bmatrix}. \tag{2.8}$$

This system can be reduced to the 2 by 2 matrix in the following way

$$\begin{bmatrix} Q + \text{Arw}(x)^{-1}\text{Arw}(z) & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y \\ Ax - b \end{bmatrix}. \tag{2.9}$$

2.4.1 Not well-defined systems

System (2.9) can be solved separately for variables Δx and Δy using the condensed system

$$A(Q + \text{Arw}(x)^{-1}\text{Arw}(z))^{-1}A^\top \Delta y = A(Q + \text{Arw}(x)^{-1}\text{Arw}(z))^{-1} r_x + r_y, \tag{2.10}$$

$$(Q + \text{Arw}(x)^{-1}\text{Arw}(z))\Delta x = A^\top \Delta y - r_x, \tag{2.11}$$

where $r_x = Qx + c - A^\top y$ and $r_y = Ax - b$. From the first system we deduce that either rank deficiency of A or singularity of $Q + \text{Arw}(x)^{-1}\text{Arw}(z)$ can give a problem with obtaining the solution of the Newton system. Also, note that $Q + \mathrm{Arw}(x)^{-1}\mathrm{Arw}(z)$ is non symmetric matrix which make impossible an application of symmetric solvers to system (2.9).

Also, let us provide an example in p143 of [5] where it is shown that even if $\mathrm{Arw}(x)$ and $\mathrm{Arw}(z)$ are strictly positive definite, whole matrix $Q + \mathrm{Arw}(x)^{-1}\mathrm{Arw}(z)$ can be singular.

Example 1. *Consider the SOCP problem with autodual cone $\mathcal{K}$ for convex quadratic function*

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} & \frac{1}{2}x^\top Qx + c^\top x \\ \textit{s.t.} & Ax = b, \\ & x \in \mathcal{K}, \end{array} \tag{2.12}$$

where A is a full rank matrix. The optimality conditions are

$$Qx + c - A^\top y - z = 0 \tag{2.13}$$

$$Ax - b = 0 \tag{2.14}$$

$$Arw(x)Arw(z)e = 0 \tag{2.15}$$

$$x \in \mathcal{K}, z \in \mathcal{K} \tag{2.16}$$

The linearization of this system gives:

$$\begin{bmatrix} Q & -A^\top & -I \\ A & 0 & 0 \\ Arw(z) & 0 & Arw(x) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - z \\ Ax - b \\ Arw(x)Arw(z)e \end{bmatrix}. \tag{2.17}$$

It clear from (2.10) that the Jacobian matrix of this system is non-singular if and only if the matrix $(Q + Arw(x)^{-1}Arw(z))^{-1}$ is non-singular. The example below shows that this matrix can be singular while $Arw(z)$ and $Arw(x)$ are positive definite and Q - positive semi-definite.

Let $x = (1.0, 0.8, 0.5)$, $z = (1.0, 0.7, 0.7)$ and matrix $Q = diag([0.3, 1.0]^\top)$. Then

$$\det(Q + Arw(x)^{-1}Arw(z)) = 0. \tag{2.18}$$

2.5 Well-defined Interior-point method

In the previous chapter we showed cases which produce numerical difficulties in solving Newton systems. Here we provide solutions which resolve these difficulties. The first way is to apply the regularization.

Regularization is the way to modify Jacobian of Newton system by modifying the objective function. There are two types of regularization: proximal-point (PP) and augmented-Lagrangian (AL). The first type modifies block (1,1) of (2.8) in the way of adding some diagonal matrix ρI for $\rho > 0$ which gives block $Q + \rho I$. The meaning of the second type is to modify block (2,2) of (2.8) by adding δI. Below we explain the origin of these regularization types and theirs influence on the system.

2.5.1 Proximal-point regularization

Proximal-point is the first kind of regularization which can be applied when numerical difficulties come from the fact that matrix Q is positive semi-definite.

$$\begin{array}{ll} \min_{x\in\mathbb{R}^{nd}} & \frac{1}{2}x^\top Qx + c^\top x + \frac{1}{2}\rho\|x - x_k\|^2 \\ \text{s.t.} & Ax = b, \\ & x \in L \end{array} \tag{PP}$$

where x_k is the current estimate of the vector of primal solution, $\rho > 0$.

Proposition 2.5.1. The dual problem of (PP) is problem

$$\begin{array}{ll} \max_{(x,y,z)\in\mathbb{R}^{nd}\times\mathbb{R}^m\times\mathbb{R}^{nd}} & -\frac{1}{2}x^\top Qx + b^\top y - \frac{1}{2}\rho\|x\|^2 \\ \text{s.t.} & Qx - A^\top y + \rho(x - x_k) - z = -c, \\ & z \in L^* \end{array} \tag{DPP}$$

where x_k is the current estimate of Lagrangian multipliers, $\rho > 0$.

Proof. The problem (DPP) can be written as following

$$\begin{array}{ll} \min_{x\in\mathbb{R}^{nd}} & \frac{1}{2}x^\top(Q + \rho I)x + (c - \rho x_k)^\top x + \frac{1}{2}\rho\|x_k\|^2 \\ \text{s.t.} & Ax = b, \\ & x \in L \end{array} \tag{2.19}$$

Since $\frac{1}{2}\rho\|x_k\|^2$ is a constant, we can apply directly the Theorem A.0.1 and obtain the dual

$$\begin{array}{ll} \max_{(x,y,z)\in\mathbb{R}^{nd}\times\mathbb{R}^m\times\mathbb{R}^{nd}} & -\frac{1}{2}x^\top(Q + \rho I)x + b^\top y \\ \text{s.t.} & (Q + \rho I)x - A^\top y + \rho(x - x_k) - z = -(c - \rho x_k), \\ & z \in L^* \end{array} \tag{2.20}$$

from where after brackets expanding we obtain (DPP). □

2.5.2 Augmented-Lagrangian regularization

Augmented-Lagrangian is the second kind of regularization which solves a problem with ill-conditioning of constraint matrix A by inserting into the Newton system diagonal matrix δI where $\delta > 0$.

$$\begin{array}{ll} \min_{(x,p)\in\mathbb{R}^{nd\times m}} & \frac{1}{2}x^\top Qx + c^\top x + \frac{1}{2}\delta\|p + y_k\|^2 \\ \text{s.t.} & Ax + \delta p = b, \\ & x \in L \end{array} \tag{ALP}$$

where y_k is the current estimate of the vector of dual variables for equality constraint, $\delta > 0$.

Lemma 2.5.2. The optimization problem (ALP) is the augmented-Lagrangian for (P).

Proof. At first, let us write saddle-point problem for augmented-Lagrangian of (P).

$$\begin{array}{ll} \max_{y\in\mathbb{R}^m}\min_{x\in\mathbb{R}^{nd}} & \frac{1}{2}x^\top Qx + c^\top x + y^\top(b - Ax) + \frac{1}{2\delta}\|b - Ax\|^2 \\ \text{s.t.} & x \in L \end{array} \tag{2.21}$$

We introduce new variable $\bar{p} = b - Ax$ and then (2.21) rewrites as

$$\begin{array}{ll} \max_{y\in\mathbb{R}^m}\min_{(x,\bar{p})\in\mathbb{R}^{nd}\times\mathbb{R}^m} & \frac{1}{2}x^\top Qx + c^\top x + y^\top\bar{p} + \frac{1}{2\delta}\|\bar{p}\|^2 \\ \text{s.t.} & Ax + \bar{p} = b, \\ & x \in L \end{array} \tag{2.22}$$

Then we replace dual multipliers y by its estimation y_k and obtain minimization problem (2.23)

$$\begin{array}{ll} \min_{(x,\bar{p})\in\mathbb{R}^{nd}\times\mathbb{R}^m} & \frac{1}{2}x^\top Qx + c^\top x + \delta y_k^\top\bar{p} + \frac{1}{2\delta}\|\bar{p}\|^2 \\ \text{s.t.} & Ax + \bar{p} = b, \\ & x \in L \end{array} \tag{2.23}$$

After, in accordance with [2], we introduce variable re-scaling $p = \frac{1}{\delta}\bar{p}$ and obtain problem (2.24).

$$\begin{array}{lll} \min_{(x,\bar{p})\in\mathbb{R}^{nd}\times\mathbb{R}^m} & \frac{1}{2}x^\top Qx + c^\top x + \delta y_k^\top p + \frac{1}{2}\delta\|p\|^2 & \\ \text{s.t.} & Ax + \delta p = b & (2.24) \\ & x \in L & \end{array}$$

To obtain problem (ALP) we need to do a following algebraic reformulation.

$$\begin{aligned} \delta y_k^\top p + \frac{1}{2}\delta\|p\|^2 &= y_k^T p + \frac{1}{2}\delta p^\top p \\ &= \frac{1}{2}\delta p^\top p + y_k^\top p + \frac{1}{2}\delta y^\top y - \frac{1}{2}\delta y^\top y \\ &= \frac{1}{2}\delta\|p + y_k\|^2 - \frac{1}{2}\delta\|y_k\|^2. \end{aligned}$$

It is clear that $\frac{1}{2}\delta\|y_k\|^2$ is a constant and can be removed from the objective function. Using this reformulation we obtain minimization problem (ALP). □

Proposition 2.5.3. The dual of (ALP) is proximal-point regularization of (D)

$$\begin{array}{lll} \max_{(x,y,z)\in\mathbb{R}^{nd}\times\mathbb{R}^m\times\mathbb{R}^{nd}} & -\frac{1}{2}x^\top Qx + b^\top y - \frac{1}{2}\delta\|y - y_k\|^2 & \\ \text{s.t.} & Qx - A^\top y - z = -c, & \text{(PPD)} \\ & z \in L^* & \end{array}$$

where y_k - current estimate of dual multipliers, $\delta > 0$.

Proof. The problem (ALP) can be written as following

$$\begin{array}{lll} \min_{(x,p)\in\mathbb{R}^{nd\times m}} & \frac{1}{2}\begin{bmatrix} x \\ p \end{bmatrix}^\top \begin{bmatrix} Q & 0 \\ 0 & \delta I \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} c \\ y_k \end{bmatrix}^\top \begin{bmatrix} x \\ p \end{bmatrix} + \frac{1}{2}\delta\|y_k\|^2 & \\ \text{s.t.} & \begin{bmatrix} A & \delta I \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} = b, & (2.25) \\ & x \in L & \end{array}$$

To apply the Theorem A.0.1 we need to reformulate it in the standard one. For this aim we introduce new variables $p \leftarrow p^+ - p^-$ where $p^+ \in L$ and $p^- \in L$. Then (2.25) writes as

$$\begin{array}{ll} \min_{(x,p^+,p^-)\in\mathbb{R}^{nd+m+m}} & \frac{1}{2}\begin{bmatrix} x \\ p^+ \\ p^- \end{bmatrix}^\top \begin{bmatrix} Q & 0 & 0 \\ 0 & \delta I & -\delta I \\ 0 & -\delta I & \delta I \end{bmatrix}\begin{bmatrix} x \\ p^+ \\ p^- \end{bmatrix} + \begin{bmatrix} c \\ y_k \\ -y_k \end{bmatrix}^\top \begin{bmatrix} x \\ p^+ \\ p^- \end{bmatrix} + \frac{1}{2}\delta\|y_k\|^2 \\ \text{s.t.} & \begin{bmatrix} A & \delta I & -\delta I \end{bmatrix}\begin{bmatrix} x \\ p^+ \\ p^- \end{bmatrix} = b, \\ & \begin{bmatrix} x \\ p^+ \\ p^- \end{bmatrix} \in L \times L \times L \end{array} \quad (2.26)$$

Then we directly apply the Theorem A.0.1 and obtain the dual

$$\begin{array}{ll} \max_{(x,p^+,p^-,y,z)\in\mathbb{R}^{4*nd+m}} & -\frac{1}{2}\begin{bmatrix} x \\ p^+ \\ p^- \end{bmatrix}^\top \begin{bmatrix} Q & 0 & 0 \\ 0 & \delta I & -\delta I \\ 0 & -\delta I & \delta I \end{bmatrix}\begin{bmatrix} x \\ p^+ \\ p^- \end{bmatrix} + b^\top y \\ \text{s.t.} & \begin{bmatrix} Q & 0 & 0 \\ 0 & \delta I & -\delta I \\ 0 & -\delta I & \delta I \end{bmatrix}\begin{bmatrix} x \\ p^+ \\ p^- \end{bmatrix} - \begin{bmatrix} A^\top \\ \delta I \\ -\delta I \end{bmatrix} y - \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} c \\ y_k \\ -y_k \end{bmatrix}, \\ & z \in L \end{array} \quad (2.27)$$

From the constraints of (2.27) we can deduce that $Qx - A^\top y - z = -c$ and $\delta(y - y_k) = \delta p$. Thus if we replace $p = p^+ - p^-$ and put the last equation into objective function we obtain exactly (PPD). $\square$

2.5.3 Primal-dual regularization

We consider the following PP-AL regularized problem which is a convex quadratic program in variables (x, p).

$$\begin{array}{ll} \min_{(x,p)\in\mathbb{R}^{nd}\times\mathbb{R}^m} & \frac{1}{2}x^\top Qx + c^\top x + \frac{1}{2}\rho\|x - x_k\|^2 + \frac{1}{2}\delta\|p + y_k\|^2 \\ \text{s.t.} & Ax + \delta p = b, \\ & x \in L \end{array} \tag{PPALP}$$

where $\rho > 0$, $\delta > 0$, x_k and y_k are current estimate of the primal-dual solution. Its dual can be formulated as

$$\begin{array}{ll} \max_{(x,y,z)\in\mathbb{R}^{nd}\times\mathbb{R}^m\times\mathbb{R}^{nd}} & -\frac{1}{2}x^\top Qx + b^\top y - \frac{1}{2}\delta\|y - y_k\|^2 - \frac{1}{2}\rho\|x\|^2 \\ \text{s.t.} & Qx - A^\top y - z + \rho\big(x - x_k\big) = -c, \\ & z \in L^* \end{array} \tag{DPPAL}$$

Let us explicitly show that (DPPAL) is the dual for (PPALP).

Proposition 2.5.4. (DPPAL) is the dual problem for (PPALP).

Proof. Follows immediately from the Proposition 2.5.1 and Proposition 2.5.3. $\square$

2.5.4 KKT optimality conditions

The aim of this section is to derive the KKT conditions for primal PP-AL regularized problem and show that it is the same for the dual one. The first, let us write the KKT conditions for the primal problem (PPALP). After introducing new variable $q = x - x_k$ and using this condition in the form of $\rho x - \rho(q + x_k) = 0$ we obtain the following optimality conditions.

$$\begin{cases} Qx + c - A^\top y - z + \rho q & = 0 \\ Ax + \delta p - b & = 0 \\ \delta(p + y_k) - \delta y & = 0 \\ \rho x - \rho(q + x_k) & = 0 \\ \text{Arw}(x)\text{Arw}(z)e & = 0 \\ x \in L,\ z \in L \end{cases} \tag{2.28}$$

Let us derive the KKT conditions for problem (DPPAL). Let us write the Lagrangian has for variables $(x, y, z, \lambda) \in \mathbb{R}^{nd} \times \mathbb{R}^m \times \mathbb{R}^{nd} \times \mathbb{R}^{nd}$.

$$\mathcal{L}(x, y, z, \lambda) = \frac{1}{2}x^\top Qx - b^\top y + \frac{1}{2}\delta\|y - y_k\|^2 + \frac{1}{2}\rho\|x\|^2 + \lambda^\top(-c - Qx + A^\top y + z - \rho(x - x_k)).$$

Following the same algorithm as we used in previous sections, the optimality conditions are

$$\nabla_{x,y}\mathcal{L}(x, y, z, \lambda) = 0,$$
$$L \ni \nabla_z\mathcal{L}(x, y, z, \lambda) \perp z \in L^*$$

where

$$\nabla_x\mathcal{L}(x, y, z, \lambda) = Qx + \rho x - Q\lambda - \rho\lambda, \tag{2.29}$$
$$\nabla_y\mathcal{L}(x, y, z, \lambda) = -b + \delta(y - y_k) + A\lambda, \tag{2.30}$$
$$\nabla_z\mathcal{L}(x, y, z, \lambda) = \lambda \tag{2.31}$$

From (2.29) we get that $x = \lambda$, thus setting $p = y - y_k$ we get (2.28).

2.5.5 Newton system

Let us first, use the results which are obtained above to formulate a nonlinear equations system which should be solved on each iteration of the interior-point method.

$$F_k(w) = \begin{bmatrix} Qx + c - A^\top y - z + \rho q \\ \delta(p + y_k) - \delta y \\ \rho x - \rho(q + x_k) \\ Ax + \delta p - b \\ \text{Arw}(x)\text{Arw}(z)e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ x \in L, \ z \in L, \tag{2.32}$$

where $w = (x, p, q, y, z)$.

We obtain that Newton system will have the following form.

$$\begin{bmatrix} Q & 0 & \rho I & -A^\top & -I \\ 0 & \delta I & 0 & -\delta I & 0 \\ \rho I & 0 & -\rho I & 0 & 0 \\ A & \delta I & 0 & 0 & 0 \\ \text{Arw}(z) & 0 & 0 & 0 & \text{Arw}(x) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta p \\ \Delta q \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - z + \rho q \\ \delta(p + y_k) - \delta y \\ \rho x - \rho(q + x_k) \\ Ax + \delta p - b \\ \text{Arw}(x)\text{Arw}(z)e \end{bmatrix}. \tag{2.33}$$

Notice, that we obtain a sparse system of linear equations to solve on each iteration. It is possible to reduce this system and thus to remove some variables. We see that Δp and Δq can be expressed through another variables.

$$\Delta p = -p + \Delta y + (y - y_k),$$
$$\Delta q = -q + \Delta x + (x - x_k)$$

To eliminate these variables from right-hand side we need:

- Add the 3-d row to the 1-st, then obtain $Qx + c - A^\top y - z + \rho(x - x_k)$ in right-hand side;
- Multiply the 2-nd row by -1 and add to the 4-th row, then we obtain $Ax + \delta(y - y_k) - b$ in right-hand side.

Finally, we can reduce the system (2.33) to (2.34).

$$\begin{bmatrix} Q + \rho I & -A^\top & -I \\ A & \delta I & 0 \\ \text{Arw}(z) & 0 & \text{Arw}(x) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - z + \rho(x - x_k) \\ Ax + \delta(y - y_k) - b \\ \text{Arw}(x)\text{Arw}(z)e \end{bmatrix}. \tag{2.34}$$

Then, let us multiply the first row by -1 and put an external minus to the right-hand side.

$$\begin{bmatrix} -(Q + \rho I) & A^\top & I \\ A & \delta I & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} Qx + c - A^\top y - z + \rho(x - x_k) \\ b - Ax - \delta(y - y_k) \\ -\text{Arw}(x)\text{Arw}(z)e \end{bmatrix}. \tag{2.35}$$

Following M.P.Friedlander and D.Orban [2] we put $x_k = x$ and $y_k = y$. Then we can eliminate variable z from the system by multiplying the 3-d row by $-\text{Arw}(x)^{-1}$ and adding to the 1-st row.

$$\begin{bmatrix} -(Q + \text{Arw}(x)^{-1}\text{Arw}(z) + \rho I) & A^\top \\ A & \delta I \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} Qx + c - A^\top y \\ b - Ax \end{bmatrix}. \tag{2.36}$$

And Δz can be calculated by formula $\Delta z = Qx + c - A^\top y - z + (Q + \rho I)\Delta x - A^\top \Delta y$.

2.5.6 Nesterov and Todd scaling

It is shown in Example 1 that even if $\mathrm{Arw}(x)$ and $\mathrm{Arw}(z)$ are strictly positive definite, matrix $Q + \mathrm{Arw}(x)^{-1}\mathrm{Arw}(z)$ can be singular. To solve this issue people do rescaling of Jacobian in Newton system. The main idea of rescaling is an appropriate change of variable, where new variables have a good properties in some sense.

Let us introduce the quadratic representation operator for $x \in \mathbb{R}^d$

$$Q_p := 2p^\top p - \det(p)R, \tag{2.37}$$

where $\det(p)$ is a determinant of Jordan algebra and R is a reflection matrix which is defined as

$$R = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 \end{bmatrix} \subset \mathbb{R}^{d\times d}. \tag{2.38}$$

This representation has a fundamental importance in matrix scaling.

Let $p = (p_1, \dots, p_n)$, $p_i \in \mathbb{R}^d$ for $i \in \{1, \dots, n\}$, then

$$Q_p := Q_{p_1} \oplus \dots \oplus Q_{p_n}. \tag{2.39}$$

We use a change of variable [3] in the following form. Let $p \in K$, then we define

$$\bar{x} = Q_p x, \tag{2.40}$$

$$\underline{x} = Q_{p^{-1}} x \tag{2.41}$$

Using this notation and the property that $Q_{p^{-1}} Q_p = I$ we do a change of variable in (2.4) and obtain the following systems

$$\begin{cases} \underline{\bar{Q}}\bar{x} + \underline{c} - \underline{A}^\top y - \underline{z} & = 0 \\ \underline{A}\bar{x} - b & = 0 \\ \bar{x}^\top \underline{z} & = 0 \\ \bar{x} \in L, \underline{z} \in L \end{cases} \tag{2.42}$$

Proposition 2.5.5. Optimality conditions (2.42) is equivalent to (2.4)

Proof. Let us multiply x by $Q_{p^{-1}} Q_p$ and z by $Q_p Q_{p^{-1}}$.

$$\begin{cases} QQ_{p^{-1}}Q_p x + c - A^\top y - Q_p Q_{p^{-1}} z & = 0 \\ AQ_{p^{-1}}Q_p x - b & = 0 \\ Q_{p^{-1}}Q_p x Q_p Q_{p^{-1}} z & = 0 \\ x \in L, z \in L \end{cases}$$

Then, using definitions (2.40), (2.41) obtain

$$\begin{cases} QQ_{p^{-1}}\bar{x} + c - A^\top y - Q_p \underline{z} & = 0 \\ AQ_{p^{-1}}\bar{x} - b & = 0 \\ Q_{p^{-1}}\bar{x} Q_p \underline{z} & = 0 \\ x \in L, z \in L \end{cases}$$

Multiplying the first equation by $Q_{p^{-1}}$ obtain

$$\begin{cases} Q_{p^{-1}}QQ_{p^{-1}}\bar{x} + Q_{p^{-1}}c - Q_{p^{-1}}A^\top y - \underline{z} & = 0 \\ AQ_{p^{-1}}\bar{x} - b & = 0 \\ Q_{p^{-1}}\bar{x} Q_p \underline{z} & = 0 \\ x \in L, z \in L \end{cases}$$

Defining $\underline{A} = AQ_{p^{-1}}$, $\bar{\underline{Q}} = Q_{p^{-1}}QQ_{p^{-1}}$, $\underline{c} = Q_{p^{-1}}$ obtain

$$\begin{cases} \bar{\underline{Q}}\bar{x} + \underline{c} - \underline{A}^\top y - \underline{z} & = 0 \\ \underline{A}\bar{x} - b & = 0 \\ Q_{p^{-1}}\bar{x}Q_p\underline{z} & = 0 \\ x \in L, z \in L \end{cases}$$

The last equation can be transformed in following way

$$\begin{aligned} x^\top z &= (Q_{p^{-1}}Q_p x)^\top z \\ &= (Q_{p^{-1}}\bar{x})^\top z \\ &= (Q_{p^{-1}}\bar{x})^\top (Q_p Q_{p^{-1}} z) \\ &= (Q_{p^{-1}}\bar{x})^\top (Q_p \underline{z}) \\ &= \bar{x}^\top Q_{p^{-1}}^\top Q_p \underline{z} \\ &= \bar{x}^\top \underline{z}. \end{aligned}$$

Conic constraints $x \in L$, $z \in L$ remain since $Q_p(\mathcal{K}) = \mathcal{K}$ and $Q_{p^{-1}}(\mathcal{K}) = \mathcal{K}$ and conversely [3]. Thus we showed that (2.42) is equivalent to (2.4). □

Note that to return to original variables we just need to expand the notation with $\bar{x}$ and $\underline{x}$ and then will obtain system (2.4).

Now, using this Proposition 2.5.5 we can deduce that rescaled optimality conditions (2.6) using Lemma 2.2.1 will have a form

$$\begin{cases} \bar{\underline{Q}}\bar{x} + \underline{c} - \underline{A}^\top y - \underline{z} & = 0 \\ \underline{A}\bar{x} - b & = 0 \\ \mathrm{Arw}(\bar{x})\mathrm{Arw}(\underline{z})e & = 0 \\ \bar{x} \in L,\ \underline{z} \in L \end{cases} \tag{2.43}$$

This step is very important since from now if we want to return to original variables we will not obtain the same system as (2.4). Thus the following lemma has a key value.

Lemma 2.5.6. $(\bar{\Delta x}, \Delta y, \underline{\Delta z})$ *solves the linearized system of equations of (2.43)*

$$\begin{aligned} \bar{\underline{Q}}\bar{\Delta x} - \underline{A}^\top \Delta y - \underline{\Delta z} &= -\bar{\underline{Q}}\bar{x} - \underline{c} + \underline{A}^\top y + \underline{z}, \\ \underline{A}\bar{\Delta x} &= b - \underline{A}\bar{x}, \\ Arw(\underline{z})\bar{\Delta x} + Arw(\bar{x})\underline{\Delta z} &= -Arw(\bar{x})Arw(\underline{z})e \end{aligned}$$

if and only if $(\Delta x, \Delta y, \Delta z)$ *solves*

$$\begin{aligned} Q\Delta x - A^\top \Delta y - \Delta z &= -Qx - c + A^\top y + z, \\ A\Delta x &= b - Ax, \\ \big(Arw(Q_{p^{-1}}z)Q_p\big)\Delta x + \big(Arw(Q_p x)Q_{p^{-1}}\big)\Delta z &= -Arw(Q_p x)Arw(Q_{p^{-1}}z)e \end{aligned}$$

Thus, applying this Lemma to system (2.9) we obtain new rescaled system

$$\begin{bmatrix} Q & -A^\top & -I \\ A & 0 & 0 \\ \mathrm{Arw}(Q_{p^{-1}}z)Q_p & 0 & \mathrm{Arw}(Q_p x)Q_{p^{-1}} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - z \\ Ax - b \\ \mathrm{Arw}(Q_p x)\mathrm{Arw}(Q_{p^{-1}}z)e \end{bmatrix}, \tag{2.44}$$

and after eliminating variable Δz we obtain reduced system with 2 by 2 blocks

$$\begin{bmatrix} Q + Q_p \text{Arw}(\bar{x})^{-1}\text{Arw}(\underline{z})Q_p & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y \\ Ax - b \end{bmatrix}. \tag{2.45}$$

The one way to define vector p was presented by Nesterov and Todd, where

$$p = \left[Q_{x^{1/2}} \left(Q_{x^{1/2}} z \right)^{-1/2} \right]^{-1/2} = \left[Q_{z^{-1/2}} \left(Q_{z^{1/2}} x \right)^{1/2} \right]^{-1/2}. \tag{NT}$$

Lemma 2.5.7. [3] *Let p is defined in (NT), then*

$$\bar{x} = Q_p x = Q_{p^{-1}} z = \underline{z}.$$

Using the Observation 2.5.7, the system (2.45) can be transformed in the following one

$$\begin{bmatrix} Q + Q_{p^2} & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y \\ Ax - b \end{bmatrix}, \tag{2.46}$$

where Q_{p^2} is symmetric positive definite matrix. This direction is interesting for two reasons:

- the Jacobian of the system (2.46) is symmetric.
- If Q is positive (semi-)definite, then $Q + Q_{p^2}$ is non singular.

Thus this kind of direction solves the issue illustrated in Example 1 and also symmetrize Jacobian which makes possible an application of symmetric solvers.

2.6 Perturbed optimality conditions

Let us consider perturbed optimization problem (P)

$$\begin{array}{ll} \min_{x \in \mathbb{R}^{nd}} & \frac{1}{2} x^\top Q x + c^\top x - \mu \sum_{i=1}^n \log \det x_i \\ \text{s.t.} & Ax = b, \\ & x \in \text{int}(L) \end{array} \tag{PrP}$$

where $\mu > 0$ is a barrier parameter and det is defined by the Jordan algebra. Recall that $\det x > 0$ if $x \in \text{int}(L)$ thus the objective function of (PrP) is not defined for x which outside the cone and on the boundary.

Lemma 2.6.1 (Perturbed optimality conditions). *The KKT optimality conditions for (PrP) are*

$$\begin{aligned} Qx + c - A^\top y - 2\mu x^{-1} &= 0 \\ Ax - b &= 0 \\ x &\in \mathit{int}(L) \end{aligned} \tag{2.47}$$

where $x^{-1} = (x_1^{-1}, \dots, x_n^{-1})$ *and* x_i^{-1} *for* $i \in \{1, \dots, n\}$ *is defined in the Jordan algebra.*

Proof. Let us define the Lagrangian for variables $(x, y) \in \mathbb{R}^{nd} \times \mathbb{R}^m$

$$\mathcal{L}(x, y) = \frac{1}{2} x^\top Q x + c^\top x - \mu \sum_{i=1}^n \log \det x_i + y^\top (b - Ax). \tag{2.48}$$

Since function is not defined outside and on the boundary of the cone, then KKT optimality conditions are

$$\begin{aligned} \nabla_{x,y} \mathcal{L}(x, y) &= 0, \\ x &\in \text{int}(L), \end{aligned}$$

where $\nabla_x \mathcal{L}(x, y) = Qx + c - 2\mu x^{-1} - A^\top y$ and $\nabla_y \mathcal{L}(x, y) = b - Ax$. Thus we obtain (2.47). □

Set $z = 2\mu x^{-1}$ in (2.47) and multiplying from the right by x using the Jordan product, we obtain

$$z \circ x = 2\mu e. \tag{2.49}$$

Note, that after this action we obtain the conic constraint on variable z since $x^{-1} \in \text{int}(L)$ and thus $2\mu x^{-1} \in \text{int}(L)$. Thus the perturbed optimality conditions can be written in the following form

$$\begin{cases} Qx + c - A^\top y - z & = 0 \\ Ax - b & = 0 \\ \text{Arw}(x)\text{Arw}(z)e - 2\mu e & = 0 \\ x \in \text{int}(L), z \in \text{int}(L) \end{cases} \tag{2.50}$$

Then, let us consider perturbed optimization problem (D) and show that its optimality conditions coincide with the primal one.

$$\begin{array}{lll} \min_{(x,y,z)\in\mathbb{R}^{nd\times m\times nd}} & \frac{1}{2}x^\top Qx - b^\top y - \mu \sum_{i=1}^n \log\det z_i & \\ \text{s.t.} & Qx + c - A^\top y - z = 0, & \quad\text{(PrD)} \\ & z \in \text{int}(L^*) & \end{array}$$

Lemma 2.6.2. *If matrix Q is a positive definite then the KKT optimality conditions for (PrD) are*

$$\begin{aligned} Ax - b &= 0 \\ x - 2\mu z^{-1} &= 0 \\ Qx + c - A^\top y - z &= 0 \\ z &\in \textit{int}(L) \end{aligned} \tag{2.51}$$

Proof. Let us define the Lagrangian for variables $(x, y, z, \lambda) \in \mathbb{R}^{nd} \times \mathbb{R}^m \times \mathbb{R}^{nd} \times \mathbb{R}^{nd}$

$$\mathcal{L}(x, y, z, \lambda) = \frac{1}{2}x^\top Qx - b^\top y - \mu \sum_{i=1}^n \log\det z_i + \lambda^\top(-c - Qx + A^\top y + z). \tag{2.52}$$

Since function is not defined outside and on the boundary of the cone, then KKT optimality conditions are

$$\begin{aligned} \nabla_{x,y,z,\lambda}\mathcal{L}(x, y, z, \lambda) = 0, \\ z \in \text{int}(L), \end{aligned}$$

where $\nabla_x\mathcal{L}(x, y, z, \lambda) = Qx - Q\lambda$, $\nabla_y\mathcal{L}(x, y, z, \lambda) = -b - A\lambda$, $\nabla_z\mathcal{L}(x, y, z, \lambda) = -2\mu z^{-1} + \lambda$. By assumption Q is positive definite, then from $Q(x - \lambda) = 0$ we deduce that $x = \lambda$. Thus we obtain (2.51). □

By multiplying from the right on z with the Jordan product the condition (2.51) can be transformed into $x \circ z - 2\mu e = 0$ and it is equivalent to (2.49) since Jordan product is commutative.

Thus perturbed optimality conditions can be written in the following form

$$\begin{cases} Qx + c - A^\top y - z & = 0 \\ Ax - b & = 0 \\ \text{Arw}(x)\text{Arw}(z)e - 2\mu e & = 0 \\ x \in \text{int}(L),\ z \in \text{int}(L) \end{cases} \tag{2.53}$$

which coincide with (2.50).

2.6.1 Perturbed Newton system

Using that we can derive Newton system

$$\begin{bmatrix} Q & -A^\top & -I \\ A & 0 & 0 \\ \mathrm{Arw}(z) & 0 & \mathrm{Arw}(x) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - z \\ Ax - b \\ \mathrm{Arw}(x)\mathrm{Arw}(z)e - 2\mu e \end{bmatrix},$$

and after eliminating variable z we obtain 2 by 2 block system

$$\begin{bmatrix} Q + \mathrm{Arw}(x)^{-1}\mathrm{Arw}(z) & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - 2\mu \mathrm{Arw}(x)^{-1} e \\ Ax - b \end{bmatrix}. \tag{2.54}$$

2.6.2 Nesterov and Todd rescaling for perturbed system

We apply Nesterov and Todd rescaling to Newton system (2.6.1) and obtain the following scaled system

$$\begin{bmatrix} Q & -A^\top & -I \\ A & 0 & 0 \\ \mathrm{Arw}(Q_{p^{-1}}z)Q_p & 0 & \mathrm{Arw}(Q_p x)Q_{p^{-1}} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - z \\ Ax - b \\ \mathrm{Arw}(Q_p x)\mathrm{Arw}(Q_{p^{-1}}z)e - 2\mu e \end{bmatrix},$$

and after eliminating variable z we obtain 2 by 2 block system

$$\begin{bmatrix} Q + Q_{p^2} & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} Qx + c - A^\top y - 2\mu Q_p \mathrm{Arw}(Q_p x)^{-1} e \\ Ax - b \end{bmatrix}. \tag{2.55}$$

2.7 Duality gap

To solve perturbed Newton system we need to define a formula for computing barrier parameter μ. Usually, people use the value of the duality gap divided by number of variables to estimate this parameter.

$$\begin{aligned} n\mu &= \frac{1}{2}x^\top Qx + c^\top x + \frac{1}{2}x^\top Qx - b^\top y \\ &= x^\top Qx + c^\top x - b^\top y. \end{aligned}$$

The value of $x^\top Qx + c^\top x - b^\top y$ can be obtained from the following equation

$$Qx + c - A^\top y - z = 0,$$

multiplying by x from the left we obtain

$$x^\top Qx + c^\top x - (Ax)^\top y - z^\top x = 0,$$

and thus using the equality $Ax = b$ and moving $z^\top x$ to the right hand side part, obtain

$$x^\top Qx + c^\top x - b^\top y = z^\top x.$$

We conclude that

$$\mu = \frac{z^\top x}{n}. \tag{2.56}$$

2.8 Step length computation

The important part of solving an optimization problem using interior-point method is a step length computation. With linear inequality constraints we usually apply the *fraction to the boundary* rule. But since we have SOCP, we need to use another rule to select step length which maintain variables inside the cone. In our algorithm we select the step length as it was done in SDPT3 software [4].

Let us consider the second-order code $\mathcal{K} = \{x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : \ x_0 \geq \|\bar{x}\|\}$. It means that $x \in \mathcal{K}$ if and only if $x_0 - \|\bar{x}\| \geq 0$. Then, on each iteration we need to select α to maintain

$$x_0 + \alpha\Delta x_0 - \|\bar{x} + \alpha\Delta\bar{x}\| \geq 0.$$

Then raise to the power 2 we obtain

$$(x_0 + \alpha\Delta x_0)^2 - \|\bar{x} + \alpha\Delta\bar{x}\|^2 \geq 0,$$

which is the same as if we use determinant from the Jordan algebra

$$\det(x + \alpha\Delta x) = (x_0 + \alpha\Delta x_0)^2 - \|\bar{x} + \alpha\Delta\bar{x}\|^2 \geq 0.$$

Expanding brackets and regrouping elements with respect to α we obtain

$$\left(\Delta x_0^2 - \|\Delta\bar{x}\|^2\right)\alpha^2 + 2\left(x_0\Delta x_0 - \bar{x}^\top\Delta\bar{x}\right)\alpha + \left(x_0^2 - \|\bar{x}\|^2\right) \geq 0,$$

or

$$a\alpha^2 + 2b\alpha + c \geq 0,$$

where $a = \det(\Delta x)$, $b = x_0\Delta x_0 - \bar{x}^\top\Delta\bar{x}$ and $c = \det(x)$. Thus to select the largest positive α we need to get either second root of this one-dimensional quadratic equation or the single root of linear equation (if $a = 0$). If there are no roots (parabola does not intersect the Ox axis), then step length can be 1. To formalize that we write the following rule to select the α.

$$\alpha = \begin{cases} \frac{-b-\sqrt{d}}{a} & \text{if } a < 0 \text{ or } b < 0, a \leq b^2/c, \\ \frac{-c}{2b} & \text{if } a = 0, b < 0, \\ 1 & \text{otherwise} \end{cases} \tag{2.57}$$

where $d = b^2 - ac$.

2.9 Practical algorithm

We defined some tools which can be used for solving SOCP using IPM. Here we define how to use it in practice. Our algorithm bases on the predictor-corrector scheme of Mehrotra and uses some modifications for calculating centering parameter σ which have taken from SDPT3 implementation [4]. In this algorithm we use both: regularization and Nesterov and Todd scaling. Regularization parameter selects dynamically on each iteration by a rule.

Let us define a system which should be solved on each iteration

$$\begin{bmatrix} Q + Q_{p^2} + \rho I & -A^\top \\ A & \delta I \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\begin{bmatrix} Qx + c - A^\top y - 2\mu_k\sigma_k Q_p \mathrm{Arw}(Q_p x)^{-1}e \\ Ax - b \end{bmatrix}. \tag{2.58}$$

The modified right-hand side will be

$$\tilde{F}_k(x, y) = -\begin{bmatrix} Qx + c - A^\top y - 2\mu_k\sigma_k Q_p \mathrm{Arw}(Q_p x)^{-1}e + (Q_p\Delta x) \circ (Q_{p^{-1}}\Delta z) \\ Ax - b \end{bmatrix}. \tag{2.59}$$

Algorithm 1 Primal-dual regularized NT interior-point algorithm

Set regularization parameters $\rho = 1$, $\delta = 1$
Set starting point for $(x, y, z) \in \times L \times \mathbb{R}^m \times L$
Set barrier parameter $\mu_0 = 1$
while True do
    pinfeas $\leftarrow \|b - Ax\|_F/(1 + \|b\|_2)$
    dinfeas $\leftarrow \|Qx + c - A^\top y - z\|_F/(1 + \|c\|_2)$
    $p \leftarrow$ NT(x, z) ▷ Compute p for NT direction (NT)
    $Q_p \leftarrow$ quad_repr(p) ▷ Compute quadratic representation (2.37)
    ###### PREDICTOR STEP #######
    $d \leftarrow -\nabla F_k(x, y)^{-1} F_k(x, y)$ ▷ Solve system (2.58) with $\sigma_k = 0$
    $\Delta x, \Delta y \leftarrow d$ ▷ Parse solve output
    $\Delta z = c + Qx - A^\top y - z + (Q + \rho I)\Delta x - A^\top \Delta y$
    $\alpha_p \leftarrow$ get_step_length(x,Δx) ▷ Compute primal step length
    $\alpha_d \leftarrow$ get_step_length(z,Δz) ▷ Compute dual step length
    $\gamma \leftarrow 0.9 + 0.09 \min\{\alpha_p, \alpha_d\}$
    ###### CORRECTOR STEP #######
    $\mu_a \leftarrow (x + \alpha_p \Delta x)^\top (z + \alpha_d \Delta z)/n$
    if $\mu < 10^{-5}$ **then**
        $e \leftarrow \max\{1, 3\min\{\alpha_p, \alpha_d\}\}$
    else
        $e \leftarrow 1$
    $\sigma \leftarrow \min\{1, (\mu_a/\mu)^e\}$
    $d \leftarrow -\nabla F_k(x, y)^{-1} \tilde{F}_k(x, y)$ ▷ Solve system (2.58) with modified right-hand side (2.59)
    $\Delta x, \Delta y \leftarrow d$ ▷ Parse solve output
    $\Delta z = c + Qx - A^\top y - z + (Q + \rho I)\Delta x - A^\top \Delta y$
    $\alpha_p \leftarrow \gamma$ get_step_length(x,Δx) ▷ Compute primal step length
    $\alpha_d \leftarrow \gamma$ get_step_length(z,Δz) ▷ Compute dual step length
    ###### UPDATE VARIABLES #######
    $x \leftarrow x + \alpha_p \Delta x$
    $y \leftarrow y + \alpha_d \Delta y$
    $z \leftarrow z + \alpha_d \Delta z$
    ###### UPDATE REGULARIZATION PARAMETERS #######
    $\rho, \delta \leftarrow \rho/5, \delta/5$
    ###### UPDATE BARRIER PARAMETER #######
    $\mu \leftarrow x^\top z/n$
    ###### CHECK EXIT CONDITIONS #######
    if $\max\{\mu, \mathit{pinfeas}, \mathit{dinfeas}\} < 10^{-8}$ **then**
        break
return (x, y, z)

Chapter 3

Interior-point method for contact friction problem

In this chapter we try apply the framework that is derived in the previous chapter. At first we formulate the primal dual problems derived in Chapter 1, the optimality conditions and the perturbed variant of the problem. After we discuss the necessity of application the regularization and Nesterov and Todd rescaling.

We consider the primal-dual pair of convex quadratic optimization problem with linear equality and second-order cone constraints in the form

$$\begin{array}{ll} \min_{(v,\tilde{u})\in\mathbb{R}^m\times\mathbb{R}^{nd}} & \frac{1}{2}v^\top Mv + f^\top v \\ \text{s.t.} & \tilde{u} = \tilde{H}v + \tilde{w} \\ & \tilde{u} \in \tilde{L}^* \end{array} \tag{FPP*}$$

$$\begin{array}{ll} \max_{\tilde{r}\in\mathbb{R}^{nd}} & -\frac{1}{2}v^\top Mv - w^\top \tilde{r} \\ \text{s.t} & Mv - H^\top \tilde{r} = -f \\ & \tilde{r} \in \tilde{L} \end{array} \tag{FPD*}$$

where $M \in \mathbb{S}^n_+$, $f \in \mathbb{R}^m$, $\tilde{H} \in \mathbb{R}^{nd\times m}$, $\tilde{w} \in \mathbb{R}^{nd}$ are given data. $d = 3$, m - size of generalized velocities, n - number of contact points and variables $v \in \mathbb{R}^m$, $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n) \in \mathbb{R}^{nd}$, $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_n) \in \mathbb{R}^{nd}$ where $\tilde{u}_i = (\tilde{u}_N, \tilde{u}_T) \in \mathbb{R}\times\mathbb{R}^{d-1}$, $\tilde{r}_i = (\tilde{r}_N, \tilde{r}_T) \in \mathbb{R}\times\mathbb{R}^{d-1}$ for $i \in \{1, \dots, n\}$. The cone $\tilde{L} = \mathcal{K}_{\mu^1} \times \cdots \times \mathcal{K}_{\mu^n}$, where

$$\mathcal{K}_{\mu^i} := \{\tilde{u} = (\tilde{u}_N, \tilde{u}_T) \in \mathbb{R}\times\mathbb{R}^{d-1} : \ \mu^i \tilde{u}_N \geq \|\tilde{u}_T\|\}. \tag{3.1}$$

It is proved in Chapter 1 that $\mathcal{K}^*_{\mu^i} = \mathcal{K}_{\frac{1}{\mu^i}}$ and thus $\tilde{L}^* = \mathcal{K}_{\frac{1}{\mu^1}} \times \cdots \times \mathcal{K}_{\frac{1}{\mu^n}}$. Since the Jordan algebra framework which is defined in Chapter 2 can be applied to auto-dual cones we need to do a change of variable and obtain a conic constraints where the cone is an auto-dual. Let us introduce a new variable

$$u = P_\mu \tilde{u}, \tag{3.2}$$

where $u = (u_1, \dots, u_n)$, $u_i = P_{\mu^i}\tilde{u}_i$, $P_{\mu^i} = \begin{bmatrix} \frac{1}{\mu^i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and then matrix P_μ will be a composition of matrices $P_{\mu^i} \in \mathbb{R}^{d\times d}$ on the main diagonal

$$P_\mu = \begin{bmatrix} P_{\mu^1} & 0 & \dots & 0 \\ 0 & P_{\mu^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{\mu^n} \end{bmatrix}$$

and $P_\mu \in \mathbb{R}^{nd \times nd}$. Then (FPP*) and (FPD*) rewrites as

$$\begin{aligned} \min_{(v,u)\in\mathbb{R}^m\times\mathbb{R}^{nd}} \quad & \tfrac{1}{2}v^\top M v + f^\top v \\ \text{s.t.} \quad & u = Hv + w, \\ & u \in L^* \end{aligned} \tag{FPP}$$

$$\begin{aligned} \max_{(r,v)\in\mathbb{R}^{nd}\times\mathbb{R}^m} \quad & -\tfrac{1}{2}v^\top M v - w^\top r \\ \text{s.t} \quad & Mv - H^\top r = -f, \\ & r \in L \end{aligned} \tag{FPD}$$

where $r = P_\mu^{-\top}\tilde{r}$, $H = P_\mu \tilde{H}$, $w = P_\mu \tilde{w}$, $L = \mathcal{K} \times \cdots \times \mathcal{K}$ and cone $\mathcal{K}$ is a Lorentz cone

$$\mathcal{K} = \{u = (u_N, u_T) \in \mathbb{R} \times \mathbb{R}^{d-1} : u_N \geq \|u_T\|\}, \tag{3.3}$$

and since Lorentz cone is a self-dual cone, then $\mathcal{K} = \mathcal{K}^*$ and thus

$$L^* = \mathcal{K}^* \times \cdots \times \mathcal{K}^* = \mathcal{K} \times \cdots \times \mathcal{K} = L. \tag{3.4}$$

Then, the Jordan algebra framework can be applied.

3.1 The first order optimality conditions

Let us derive the first order optimality conditions for primal and dual problems and formulate it in terms of the Jordan algebra.

Theorem 3.1.1 (Optimality conditions). If (FPP) and (FPD) have feasible solutions u^*, r^* such that $u^* \in \text{int}(L)$, $r^* \in \text{int}(L)$, then (v, u, r) is an optimal solution pair if and only if

$$\begin{cases} Mv + f - H^\top r & = 0 \\ u - Hv - w & = 0 \\ r^\top u & = 0 \\ u \in L, \ r \in L \end{cases} \tag{3.5}$$

Proof. Following directly from the Theorem A.0.1. □

The complementarity constraint $r^\top u = 0$ can be replaced by the Jordan product (Lemma 2.2.1)

$$r \circ u = \text{Arw}(r)\text{Arw}(u)e = 0$$

Thus (3.5) rewrites as

$$\begin{cases} Mv + f - H^\top r & = 0 \\ u - Hv - w & = 0 \\ \text{Arw}(r)\text{Arw}(u)e & = 0 \\ u \in L, \ r \in L \end{cases} \tag{3.6}$$

3.2 Perturbed optimality conditions

3.2.1 Primal problem

Let us derive the KKT optimality conditions for perturbed fixed-point problem. At first, we note that there is a variable u which belongs to the product of second-order cone. Let us recall, that SDP problems also have conic constraints $X \succcurlyeq 0$ which means that X belongs to the cone of positive semi-definite matrices. A classical approach to introduce a perturbation and thus transform the conic constrain to the strict one $X \succ 0$ is to add the *logarithmic barrier term* $-\mu \sum_i \ln \det(X)$ and thus make function not defined outside the cone. We need to do the same for the case with SOCP.

The canonical barrier function for Lorentz cone is $-\log(u_N^2 - \|u_T\|^2)$ [6]. Note that using Euclidean Jordan algebra, we have $\det(u) = u_N^2 - \|u_T\|^2$, then perturbed variant of (FPP) rewrites as

$$\begin{array}{lll} \min_{(v,u)\in\mathbb{R}^m\times\mathbb{R}^{nd}} & \frac{1}{2}v^\top Mv + f^\top v - \gamma \sum_{i=1}^n \log\det(u_i) \\ \text{s.t.} & u = Hv + w \\ & u_i \in \text{int}(\mathcal{K}), \quad i \in \{1,\dots,n\} \end{array} \tag{3.7}$$

Let us derive the optimality conditions for perturbed problem (3.7). Let us define the Lagrangian for variables $(v, u, r) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times \mathbb{R}^{nd}$.

$$\mathcal{L}(v,u,r) = \frac{1}{2}v^\top Mv + f^\top v - \gamma \sum_{i=1}^n \log\det(u_i) + r^\top(u - Hv - w).$$

The optimality conditions are

$$\nabla_{v,u}\mathcal{L}(v,u,r) = 0, \tag{3.8}$$

where $\nabla_v \mathcal{L}(v,u,r) = Mv + f - H^\top r$ and using the fact that $\nabla \log\det(u) = 2u^{-1}$ we have $\nabla_u \mathcal{L}(v,u,r) = -2\gamma u_i^{-1} + r$. Than, KKT conditions will have a form

$$\begin{cases} Mv + f - H^\top r & = 0 \\ r - 2\gamma u^{-1} & = 0 \\ u - Hv - w & = 0 \\ u_i \in \text{int}(\mathcal{K}), \quad i \in \{1,\dots,n\} \end{cases} \tag{3.9}$$

where $r \in \mathbb{R}^{nd}$, $r = (r_1, \dots, r_n)$ is a dual multiplier for the equality constraint. Then, to eliminate the inverting of the element u we multiply the second equality from the right by u using Jordan product and transform the second equality to the following one.

$$r_i \circ u_i - 2\gamma e = 0, \quad r_i \in \text{int}(\mathcal{K}),\ i \in \{1,\dots,n\} \tag{3.10}$$

Note, that new conic constrain arose for variable r, it is so since the second equation of system (3.9) leads to the fact that $r_i = 2\gamma u_i^{-1}$ for $i \in \{1,\dots,n\}$. From Jordan algebra we know that if $u_i \in \text{int}(\mathcal{K})$ then $u_i^{-1} \in \text{int}(\mathcal{K})$ as well. Thus we obtain the following perturbed KKT conditions

$$\begin{cases} Mv + f - H^\top r & = 0 \\ u - Hv - w & = 0 \\ r \circ u - 2\gamma e & = 0 \\ u \in \text{int}(L),\ z \in \text{int}(L) \end{cases} \tag{3.11}$$

where $L = \mathcal{K} \times \cdots \times \mathcal{K}$, $r = (r_1, \dots, r_n) \in \mathbb{R}^{nd}$, $u = (u_1, \dots, u_n) \in \mathbb{R}^{nd}$.

3.2.2 Dual problem

Let us also derive the perturbed KKT conditions for the dual problem (FPD) which is transformed to the following minimization problem

$$\begin{array}{lll} \min_{(r,v)\in\mathbb{R}^{nd}\times\mathbb{R}^m} & \frac{1}{2}v^\top Mv + w^\top r - \gamma \sum_{i=1}^n \log\det r_i \\ \text{s.t} & Mv - H^\top r = -f, \\ & r_i \in \text{int}(\mathcal{K}),\ i \in \{1,\dots,n\} \end{array} \tag{FPD}$$

Let us define the Lagrangian for the variables $(v, r, y) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times \mathbb{R}^m$.

$$\mathcal{L}(v, r, y) = \frac{1}{2} v^\top M v + w^\top r - \gamma \sum_{i=1}^{n} \log \det r_i + y^\top(-f - Mv + H^\top r).$$

The optimality conditions are

$$\nabla_{v,r,y} \mathcal{L}(v, r, y) = 0, \tag{3.12}$$

where $\nabla_v \mathcal{L}(v, r, y) = Mv - My$ and using the fact that $\nabla \log \det r = 2r^{-1}$ we have

$$\nabla_r \mathcal{L}(v, r, y) = -2\gamma r^{-1} + w + Hy.$$

Thus we obtain that $y = v$ and after introducing new variable $u = Hv + w$ obtain the following KKT conditions

$$\begin{cases} Mv + f - H^\top r & = 0 \\ u - 2\gamma r^{-1} & = 0 \\ u - Hv - w & = 0 \\ u \in \text{int}(L),\ r \in \text{int}(L) \end{cases} \tag{3.13}$$

where $r \in \mathbb{R}^{nd}$, $r = (r_1, \dots, r_n)$ is a dual multiplier for the equality constraint. Then, to eliminate the inverting of the element r we multiply the second equality from the right by r using the Jordan product and transform the second equality to the following one

$$u \circ r - 2\gamma e = 0. \tag{3.14}$$

Note, that new conic constrain arose for variable u, it is so since the second equation of system (3.13) leads to the fact that $u_i = 2\gamma r_i^{-1}$ for $i \in \{1, \dots, n\}$. From Jordan algebra we know that if $r_i \in \text{int}(\mathcal{K})$ then $r_i^{-1} \in \text{int}(\mathcal{K})$ as well. Thus we obtain the following pertrubated KKT conditions

$$\begin{cases} Mv + f - H^\top r & = 0 \\ u - Hv - w & = 0 \\ r \circ u - 2\gamma e & = 0 \\ u \in \text{int}(L),\ r \in \text{int}(L) \end{cases} \tag{3.15}$$

where $L = \mathcal{K} \times \dots \times \mathcal{K}$, $r = (r_1, \dots, r_n) \in \mathbb{R}^{nd}$, $u = (u_1, \dots, u_n) \in \mathbb{R}^{nd}$. Thus we showed that optimality conditions for primal and dual problem coincide.

3.3 Newton's system for fixed-point formulation

Recall that Jordan product is equivalent to product of Arw(x) matrices. Then we have, that

$$r \circ u = \text{Arw}(r)\text{Arw}(u)e, \tag{3.16}$$

where $\text{Arw}(r) := \text{Arw}(r_1) \oplus \dots \oplus \text{Arw}(r_n)$ and $\text{Arw}(u) := \text{Arw}(u_1) \oplus \dots \oplus \text{Arw}(u_n)$.

Using all which we defied above, the Newton system for primal perturbed problem (3.7) computes as

$$\begin{bmatrix} M & 0 & -H^\top \\ 0 & \text{Arw}(r) & \text{Arw}(u) \\ -H & I & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ \text{Arw}(r)\text{Arw}(u)e - 2\gamma_k \sigma_k e \\ u - Hv - w \end{bmatrix} \tag{3.17}$$

where γ_k is the current value of barrier parameter and σ_k is the centering parameter.

3.4 Primal-dual regularization

Before applying regularization we need to understand that will it be useful or not. Recall that regularization is being applied either matrix Q is positive semi-definite or constraint matrix A is degenerate. Let us check this conditions for fixed-point problem (FPP). To do that we need to formulate this problem in the form of (P). We can observe that the objective function remains and $Q = M$. The constraints are not in the same form, thus we do next

$$u = Hv + w \Rightarrow Hv - u = -w \Rightarrow \begin{bmatrix} H & -I \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = -w.$$

Then we obtain that $A = \begin{bmatrix} H & -I \end{bmatrix}$.

In current work we consider the fixed-point problems where M is strictly positive definite, thus proximal-point regularization is not necessary to get an invertible Jacobian matrix. Then we see that constraints are $A = \begin{bmatrix} H & -I \end{bmatrix}$ where H is always rank deficient but it is clear that A is a full rank matrix even if H is not. Thus we conclude that augmented-Lagrangian regularization is idem as well.

3.5 Nesterov and Todd scaling

In this section we shall show that Nesterov and Todd scaling is necessary to define well-defined IP algorithm for solving fixed-point problem. To do that let us consider the system (3.17). We write it in condensed kind with respect to variable Δr.

Let us multiply the first row by HM^{-1} and sum with the third row to eliminate variable v. Thus we obtain

$$\begin{bmatrix} I & -HM^{-1}H^\top \\ \text{Arw}(r) & \text{Arw}(u) \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} u - (w - HM^{-1}f) - HM^{-1}H^\top r \\ \text{Arw}(r)\text{Arw}(u)e - 2\gamma_k\sigma_k e \end{bmatrix}.$$

After, we multiply the the second row by $-\text{Arw}(r)^{-1}$ and sum with the first one to eliminate variable u. Defining $W = HM^{-1}H^\top$ and $q = w - HM^{-1}f$ we obtain

$$\begin{aligned} -\big(W + \text{Arw}(r)^{-1}\text{Arw}(u)\big)\Delta r &= -(u - q - Wr - \text{Arw}(r)^{-1}(\text{Arw}(r)\text{Arw}(u)e - 2\gamma_k\sigma_k e)) \\ &= -(u - q - Wr - \text{Arw}(u)e + 2\gamma_k\sigma_k\text{Arw}(r)^{-1}e) \\ &= Wr + q - 2\gamma_k\sigma_k\text{Arw}(r)^{-1}e, \end{aligned}$$

and finally

$$\big(W + \text{Arw}(r)^{-1}\text{Arw}(u)\big)\Delta r = -\big(Wr + q - 2\gamma_k\sigma_k\text{Arw}(r)^{-1}e\big). \tag{3.18}$$

Note that W is always degenerate since H is always rank deficient matrix thus using Example 1 we conclude that system (3.18) is not well-defined and IP algorithm as well.

In opposite to that let us show that Nesterow and Todd rescaled Newton system is well defined. Let us use the results in Chapter 2 and apply Lemma 2.5.6 to system (3.17). Thus we obtain new system

$$\begin{bmatrix} M & 0 & -H^\top \\ 0 & \text{Arw}(Q_{p^{-1}}r)Q_p & \text{Arw}(Q_p u)Q_{p^{-1}} \\ -H & I & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ \text{Arw}(Q_{p^{-1}}r)\text{Arw}(Q_p u)e - 2\gamma_k\sigma_k e \\ u - Hv - w \end{bmatrix}.$$

Then consequently multiplying the second row by $\text{Arw}(Q_p u)^{-1}$ and Q_p obtain the following system

$$\begin{bmatrix} M & 0 & -H^\top \\ 0 & Q_p \operatorname{Arw}(\underline{r})\operatorname{Arw}(\bar{u})^{-1}Q_p & I \\ -H & I & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ r - 2\gamma_k \sigma_k Q_p \bar{u}^{-1} \\ u - Hv - w \end{bmatrix}.$$

Then, using the fact that $\underline{z}$ and $\bar{u}$ points to the same variable, we obtain $\operatorname{Arw}(\underline{z})\operatorname{Arw}(\bar{u})^{-1} = I$ and thus system rewrites as

$$\begin{bmatrix} M & 0 & -H^\top \\ 0 & Q_{p^2} & I \\ -H & I & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ r - 2\gamma_k \sigma_k Q_p \bar{u}^{-1} \\ u - Hv - w \end{bmatrix}. \tag{3.19}$$

Then, let us multiply the first row by HM^{-1} and sum with the third row to eliminate variable v.

$$\begin{bmatrix} I & -HM^{-1}H^\top \\ Q_{p^2} & I \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} -HM^{-1}H^\top r + (w - HM^{-1}f) + u \\ r - 2\gamma_k \sigma_k Q_p \bar{u}^{-1} \end{bmatrix}.$$

After, we multiply the second row by $-Q_{p^2}^{-1} = -Q_{p^{-2}}$ and sum with the first row obtain

$$-(HM^{-1}H^\top + Q_{p^{-2}})\Delta r = -HM^{-1}H^\top r + (w - HM^{-1}f) + u - Q_{p^{-2}}r + 2\gamma_k\sigma_k Q_{p^{-1}}\bar{u}^{-1}.$$

Then we observe that $HM^{-1}H^\top$ is symmetric and positive semi-definite, $Q_{p^{-2}}$ is symmetric and positive definite thus the sum of these matrices will be positive definite matrix.

Finally, we conclude that Nesterov and Todd rescaling is mandatory for defining a well-defined IPM for fixed-point problem.

Chapter 4

Numerical experiments

In this chapter we present the results obtained from numerical experiments. The goal of these experiments was to understand the numerical behavior of Nesterov and Todd scaled and not scaled Interior-point method. Also we describe some results about properties of reduced and not reduced Newton systems.

All experiments were provided for 4 types of Newton systems:

- **Original not scaled**

$$\begin{bmatrix} M & 0 & -H^\top \\ 0 & \text{Arw}(r) & \text{Arw}(u) \\ -H & I & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ \text{Arw}(r)\text{Arw}(u)e - 2\gamma_k \sigma_k e \\ u - Hv - w \end{bmatrix}, \tag{4.1}$$

- **Reduced not scaled**

$$\begin{bmatrix} M & 0 & -H^\top \\ 0 & \text{Arw}(u)^{-1}\text{Arw}(r) & I \\ -H & I & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ r - 2\gamma_k \sigma_k e \\ u - Hv - w \end{bmatrix}, \tag{4.2}$$

- **Original scaled**

$$\begin{bmatrix} M & 0 & -H^\top \\ 0 & \text{Arw}(\underline{r})Q_p & \text{Arw}(\bar{u})Q_{p^{-1}} \\ -H & I & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ \text{Arw}(\underline{r})\text{Arw}(\bar{u})e - 2\gamma_k \sigma_k e \\ u - Hv - w \end{bmatrix}, \tag{4.3}$$

- **Reduced scaled**

$$\begin{bmatrix} M & 0 & -H^\top \\ 0 & Q_{p^2} & I \\ -H & I & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta u \\ \Delta r \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ r - 2\gamma_k \sigma_k Q_p \bar{u}^{-1} \\ u - Hv - w \end{bmatrix}. \tag{4.4}$$

”Scaled” means that we apply Nesterov and Todd rescaling technique; ”Reduced” means that we try to symmetrize the Jacobian by multiplying on the inverse of some matrix.

All these systems were tested on 4 types of contact friction problem: Capsules, Spheres, BoxStacks and KaplasTower which were taken from FCLib open library. These problems describe two kind of contact friction problems: with sliding and without sliding. This information says that in the first case the solution is on the boundary of cone, but not in the second. In the following sections we present the performance profiles [9] for each of these problems with different stopping tolerance: 10^{-8}, 10^{-10}, 10^{-12}.

Also, for each of these problems, the data characterization is following:

- Matrix $M \in \mathbb{R}^{m\times m}$ is a sparse symmetric block-diagonal positive definite matrix, where m is a size of vector of generalized velocities v.
- Matrix $H \in \mathbb{R}^{nd\times m}$ is a rectangular sparse matrix where n is a number of contacts, $d = 3$.

4.1 Capsules

We consider ”Capsules” problem and present the profiles for different tolerances for 4 types of solvers.

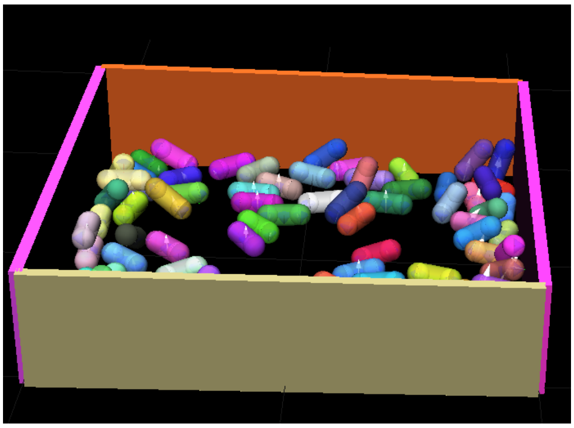
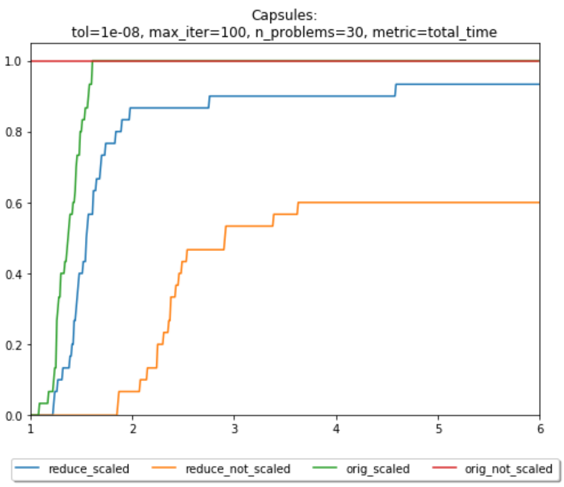
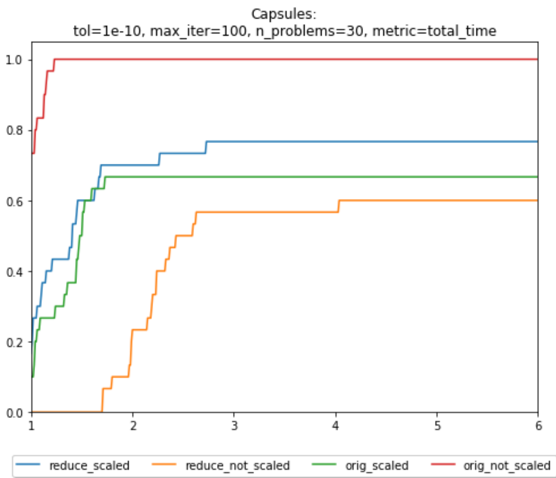


Figure 4.1: Capsules problem (sliding).



(a) Capsules profile for **tol** $= 10^{-8}$



(b) Capsules profile for **tol** $= 10^{-10}$

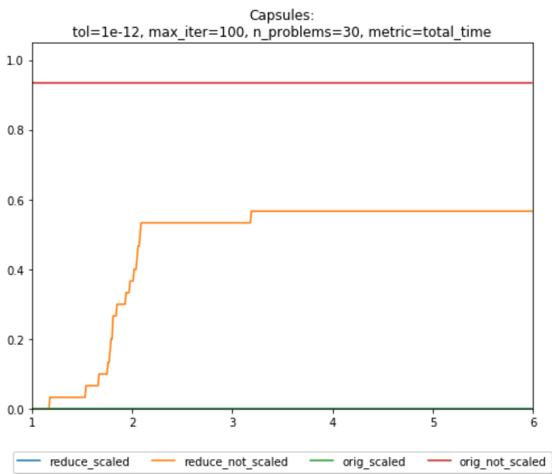


Figure 4.3: (c) Capsules profile for **tol** $= 10^{-12}$

4.2 Spheres

We consider "Spheres" problem and present the profiles for different tolerances for 4 types of solvers.

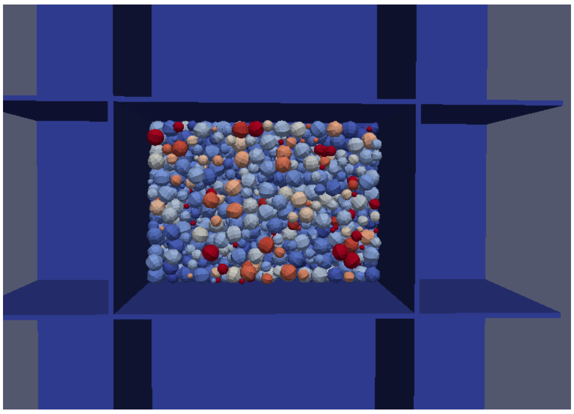
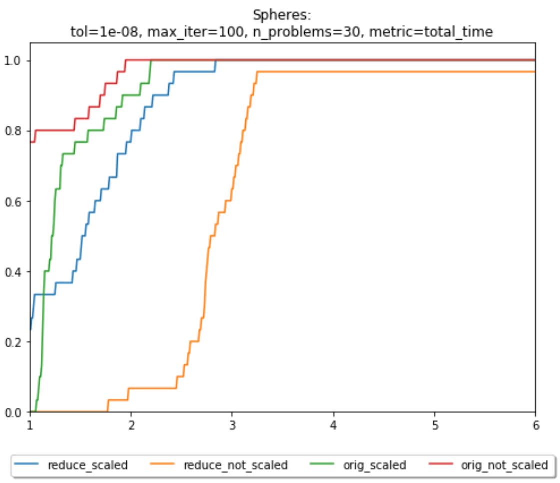
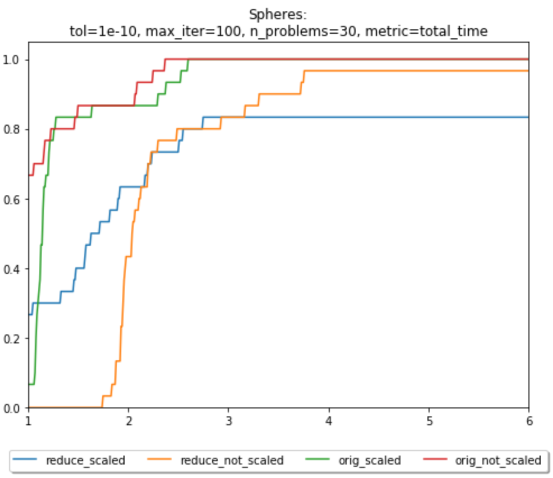


Figure 4.4: Spheres problem (sliding).



(a) Spheres profile for **tol** $= 10^{-8}$



(b) Spheres profile for **tol** $= 10^{-10}$

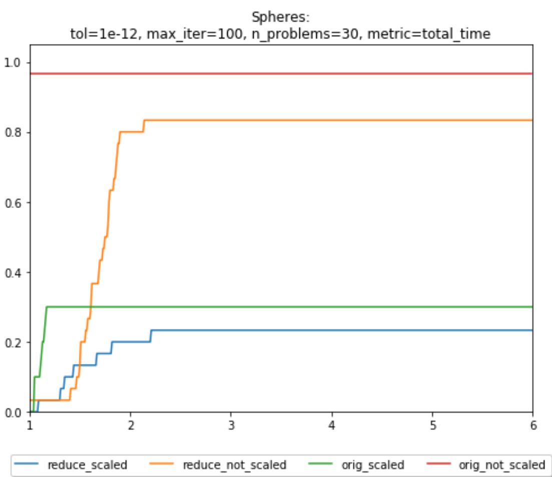


Figure 4.6: (c) Spheres profile for **tol** $= 10^{-12}$

4.3 BoxStacks

We consider ”BoxStacks” problem and present the profiles for different tolerances for 4 types of solvers.

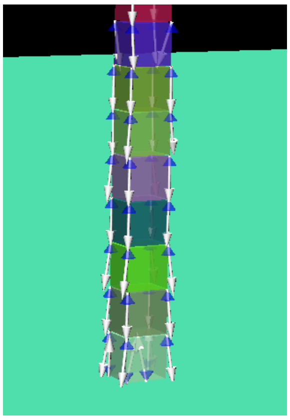
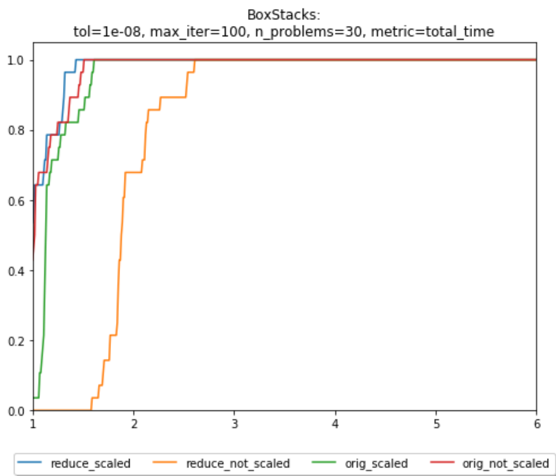
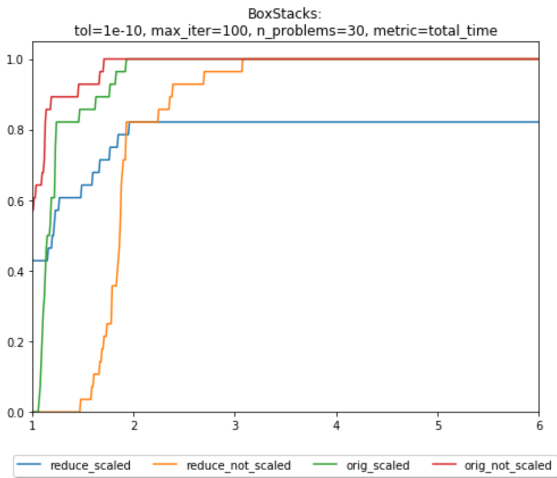


Figure 4.7: BoxStacks problem (no sliding).



(a) BoxStacks profile for **tol** $= 10^{-8}$



(b) BoxStacks profile for **tol** $= 10^{-10}$

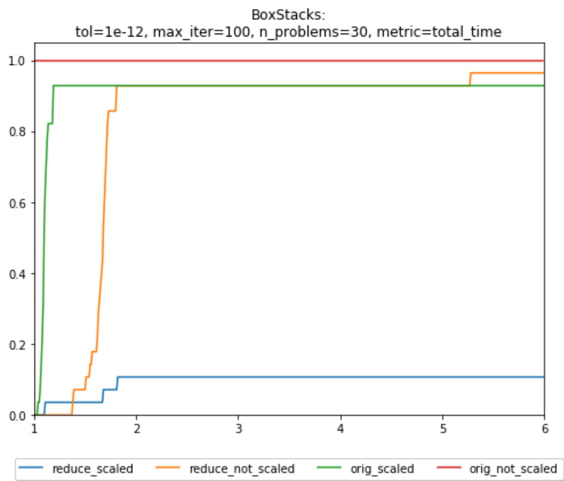


Figure 4.9: (c) BoxStacks profile for **tol** $= 10^{-12}$

4.4 KaplasTower

We consider "KaplasTower" problem and present the profiles for different tolerances for 4 types of solvers.

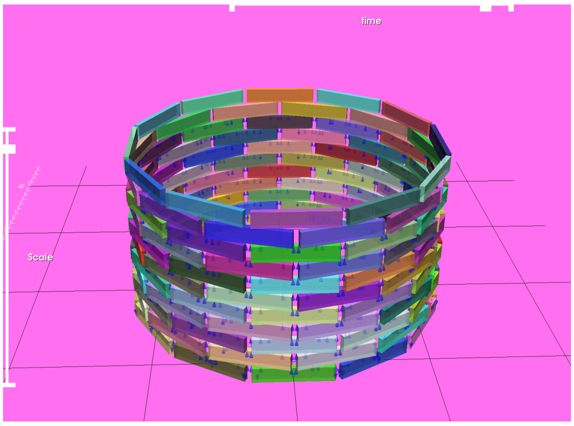
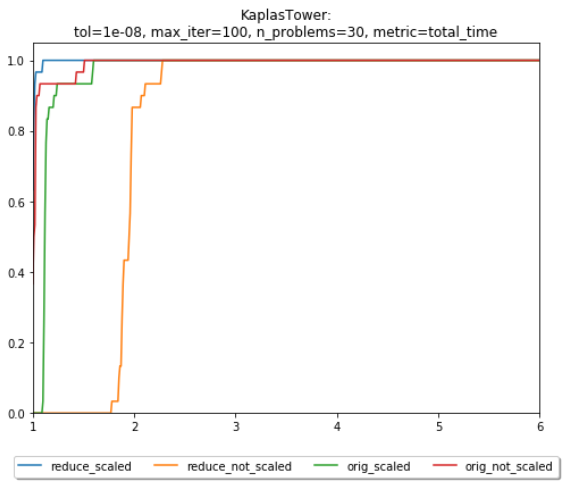
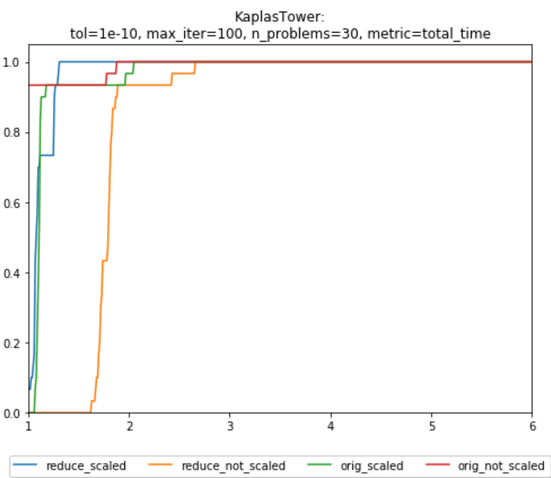


Figure 4.10: KaplasTower problem (no sliding).



(a) KaplasTower profile for **tol** $= 10^{-8}$



(b) KaplasTower profile for **tol** $= 10^{-10}$

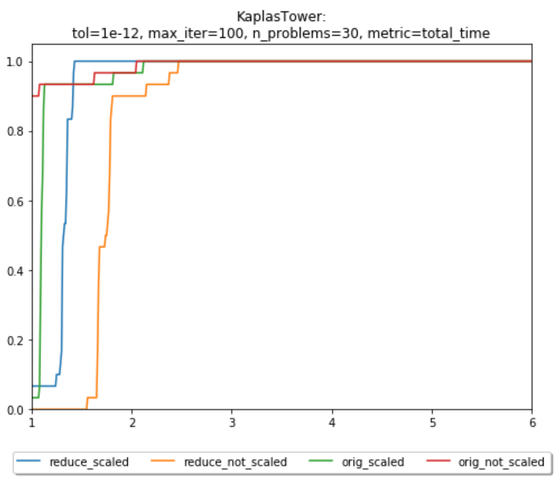


Figure 4.12: (c) KaplasTower profile for **tol** $= 10^{-12}$

4.5 Performance profiles analysis

After analyzing all these profiles we can conclude the following statements:

1. When we try to reduce system by multiplying on the inverse of some matrix it reduces the number of problems which can be solved with high accuracy. It can be observed on each of profiles when we look at the red (original) and yellow (reduced) profiles.

2. For the tolerance equal to 10^{-8} the difference between (4.1) and (4.3) for speed and total computational time, are comparatively small. But with the increase of tolerance, (4.3) loses the ability to solve a big part of problems. It is true for ”Capsules” and ”Spheres” problems. We observe this behavior because more constraints are on the on the boundary of the cone for these kind of problems.

3. IP which is based on system (4.4) can solve from 0% to 20% problems with accuracy 10^{-12} for problems with sliding (Capsules, Spheres) and even for problem with no sliding (BoxStacks). If we compare the solvers with systems (4.4) and (4.3) we observe that in each case (4.3) is better than (4.4). It also confirms the fact that reducing system is a bad idea.

4. It can be observed that solver which is based on Newton system (4.1) is the best solver which can solve more that 90% of all problems with accuracy 10^{-12}. However, theoretically, it is not well defined that was illustrated in Example 1.

5. Systems (4.3) and (4.4) are theoretically well-defined but in practice these solvers can not obtain a solution for high stopping tolerance.

To conclude about these solvers we can say that if for some purposes the tolerance 10^{-8} is enough, people should use solver which is based on Newton system (4.3). This solver is well defined and can efficiently solve 100% problems. But if this tolerance is not enough, people should used solver with system (4.1). This solver is not well-defined but in practice it can solve more than 90% of all problems with accuracy 10^{-12}.

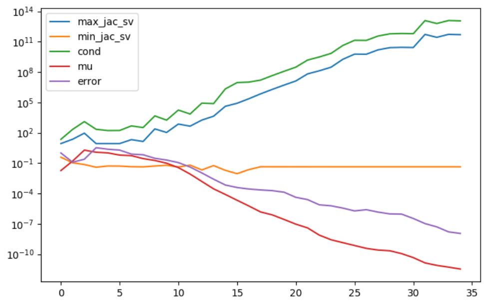
4.6 Newton system conditioning

Then, let us present the plots with a behavior of some representative parameters which characterize the Newton systems and converging process during iterations. These parameters are minimal and maximal singular value of the Jacobian, conditioning number, duality gap and the norm of the residuals. From the performance profiles we can deduce that ”Capsules” is the hardest problem for ”Scaled” system since no problem can be solved with the small tolerance. So, it is interesting to observe the above parameters for this problem with ”Scaled” (4.4) and ”Not scaled” (4.1) system. We present the following plots for ”Capsules” problem with $n = 83$, $n = 101$ and $n = 128$ number of contacts. The plots show the value of chosen parameters on each iteration. The stopping tolerance is 10^{-8}.

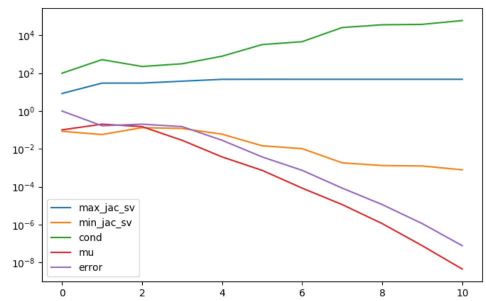
Thus we can observe that

1. The conditioning number tends to infinity much faster for solver with the ”Scaled” system than for ”Not scaled”.

2. The number of iterations which is required for convergence for ”Not scaled” solver is twice less than for ”Scaled”.

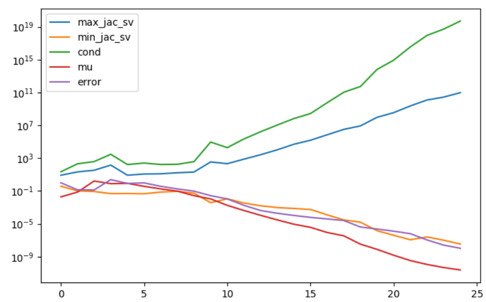


(a) System (4.4)

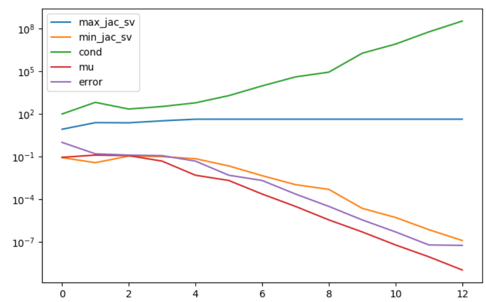


(b) System (4.1)

Figure 4.13: Parameters' behaviors for "Capsules" problem with $n = 83$ contacts.

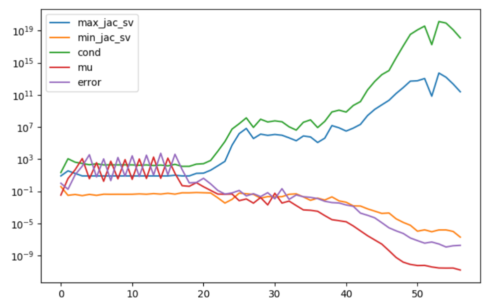


(a) System (4.4)

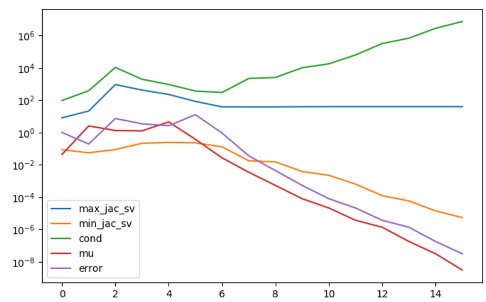


(b) System (4.1)

Figure 4.14: Parameters' behaviors for "Capsules" problem with $n = 101$ contacts.



(a) System (4.4)



(b) System (4.1)

Figure 4.15: Parameters' behaviors for "Capsules" problem with $n = 128$ contacts.

Conclusion

In this work we focus on the application of Interior-point method for solving convex quadratic optimization problem (fixed-point problem) with the second order cone constraints. The origin is a mechanical system which describes the contact friction under the Coulomb's law. The goal of this work was to apply Interior-point method under the framework of Euclidean Jordan algebra [3] and primal-dual regularization [2] for solving fixed-point problem. Inside the Jordan algebra, we tried to understand if with Nesterov and Todd rescaling [3] we can build a well-defined algorithm which is also a numerically stable for contact friction problems.

The results are following:

1. We applied Nesterov and Todd rescaling for this kind of problem and observed some difficulties. From theoretical point of view, after applying NT, we obtain well-defined algorithm with good theoretical properties. However, in practice (see Section 4.6 Newton system conditioning), we observe that with this kind of rescaling, the Jacobian becomes badly conditioned very fast during the iterations. It is so since during computation of rescaling matrix we do a Jordan inverse and then put this element to the square. Also, problems comes from the fact that if solution is on the boundary, then algorithm can not converge with a high accuracy.

2. The main reason of NT rescaling is that algorithm is not well-defined without it (see Example 1). However in practice, from performance profiles in Chapter 4, we observe that if we don't do rescaling and moreover we don't reduce Newton system to obtain condensed form, algorithm converges much faster (the count of iterations is twice less) and with a high accuracy ($\epsilon = 10^{-12}$) even if the solution is on the boundary (”Capsules” and ”Spheres” problems).

3. We adapted a framework of [2] to conic constraints problems and tried to apply it to fixed-point optimization problem. We observed (see Section 3.4 Primal-dual regularization) that regularization gives no effect on this formulation of the problem.

Appendix A

Duality result for convex quadratic problem with conic constraints

Theorem A.0.1. Let $Q \in \mathbb{R}^{ntimesn}$ be a symmetric and positive semi0definite, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Let us denote by $\mathcal{K}$ a regular convex cone of $\mathbb{R}^n$. Let us denote by K^* its dual, i.e., $\mathcal{K}^* = \{z \in \mathbb{R}^n : x^\top z \geq 0, \forall x \in \mathcal{K}\}$. Consider the convex QP

$$\begin{aligned} \min \quad & q(x) := \tfrac{1}{2} x^\top Q x + c^\top x \\ \text{s.t.} \quad & Ax = b, \\ & x \in \mathcal{K} \end{aligned} \tag{A.1}$$

The dual of (A.1) is the problem

$$\begin{aligned} \max \quad & p(x,y) := -\tfrac{1}{2} x^\top Q x + b^\top y \\ \text{s.t.} \quad & Qx - A^\top y + c \in \mathcal{K}^* \end{aligned} \tag{A.2}$$

More precisely, the following properties are satisfied:

(i) (Weak duality) Suppose that $(x, y) \in \mathbb{R}^{n+m}$ is primal-dual feasible i.e. $Ax = b$, $x \in \mathcal{K}$, $Qx - A^\top y + c \in \mathcal{K}^*$, then $p(x, y) \leq q(x)$. In particular if (A.1) (resp. (A.2)) is unbounded, then (A.2) (resp. (A.1)) is infeasible.

(ii) If $\bar{x}$ is an optimal solution of (A.1) and if (A.1) is strictly feasible (i.e. there exists $\bar{x} \in \text{int}(\mathcal{K})$ such that $Ax = b$), then there exists $\bar{y} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{y})$ is an optimal solution (A.2).

(iii) Conversely, if $(\bar{x}, \bar{y})$ is an optimal solution for (A.2) and (A.2) is strictly feasible (i.e there exists $(x, y) \in \mathbb{R}^{n+m}$ such that $Qx - A^\top y + c \in \text{int}(\mathcal{K}^*)$), then there exists an optimal solution $x^* \in \mathbb{R}^n$ to (A.1) such that $Qx^* = Q\bar{x}$.

Proof. Let us prove (i). Let $(x, y) \in \mathbb{R}^{n+m}$ be a primal-dual feasible solution. Because $x \in \mathcal{K}$, $(Qx - A^\top y + c) \in \mathcal{K}^*$ and $Ax = b$, we have

$$\begin{aligned} 0 &\leq x^\top (Qx - A^\top y + c) \\ &= x^\top Q x - b^\top y + c^\top x. \end{aligned}$$

We deduce that

$$\begin{aligned} p(x, y) &= -\frac{1}{2} x^\top Q x + b^\top y \\ &\leq -\frac{1}{2} x^\top Q x + x^\top Q x + c^\top x \\ &= q(x) \end{aligned} \tag{A.3}$$

To prove (ii), let $\bar{x} \in \mathbb{R}^n$ be an optimal solution of (A.1). Let us consider the linear problem

$$\begin{array}{ll} \min & f(x) := -\frac{1}{2}\bar{x}^\top Q\bar{x} + \bar{x}^\top Qx + c^\top x \\ \text{s.t.} & Ax = b, \\ & x \in \mathcal{K} \end{array} \tag{A.4}$$

Because the problems (A.1) and (A.4) are convex, $\bar{x}$ is optimal for (A.1) and $\nabla q(\bar{x}) = \nabla f(\bar{x}) = Q\bar{x} + c$, $\bar{x}$ is also an optimal solution for (A.4).

By SOCP duality, it can be easily shown (see, e.g. [8] Theorem 11.23) that the dual of the problem (A.4) is

$$\begin{array}{ll} \max & g(y, z) := -\frac{1}{2}\bar{x}^\top Q\bar{x} + b^\top y \\ \text{s.t.} & Q\bar{x} - A^\top y + c \in \mathcal{K}^* \end{array} \tag{A.5}$$

Since it is assumed that there exists an optimal solution $\bar{x}$ of (A.4) and that this problem is strictly feasible, the problem (A.5) has an optimal solution $\bar{y} \in \mathbb{R}^m$ and the strong duality holds, that is $f(\bar{x}) = g(\bar{y})$, see [8] Theorem 11.23. This implies that

$$b^\top \bar{y} = \bar{x}^\top Q\bar{x} + c^\top \bar{x}. \tag{A.6}$$

Let us show that $(\bar{x}, \bar{y})$ is optimal for (A.2). Let (x, y) be a feasible solution for (A.2). By using the convexity of the quadratic form defined by the matrix Q, we have

$$\frac{1}{2}x^\top Qx \geq \frac{1}{2}\bar{x}^\top Q\bar{x} + \bar{x}^\top Q(x - \bar{x}).$$

By using this inequality and (A.6) then we deduce that

$$\begin{aligned} p(\bar{x}, \bar{y}) - p(x, y) &= -\frac{1}{2}\bar{x}^\top Q\bar{x} + b^\top \bar{y} + \frac{1}{2}x^\top Qx - b^\top y \\ &\geq x^\top Q(x - \bar{x}) + b^\top \bar{y} - b^\top y \\ &= \bar{x}^\top Qx + c^\top \bar{x} - b^\top y. \end{aligned}$$

Since (x, y) is feasible for (A.2), $Qx - A^\top y + c \in \mathcal{K}^*$ and $\bar{x}$ is feasible for (A.1), $\bar{x} \in \mathcal{K}$, we have

$$\bar{x}^\top Qx \geq \bar{x}^\top A^\top y - c^\top \bar{x} = b^\top y - c^\top \bar{x}$$

We then deduce that

$$p(\bar{x}, \bar{y}) - p(x, y) \geq 0,$$

which shows that $(\bar{x}, \bar{y})$ is optimal for (A.2). Finally by using (A.6) we have

$$p(\bar{x}, \bar{y}) = -\frac{1}{2}\bar{x}^\top Qx + b^\top \bar{y} = \frac{1}{2}\bar{x}^\top Qx + c^\top \bar{x} = q(\bar{x}),$$

which ends the proof of (ii).

To prove (iii), it suffices to reformulate the problem (A.2) under the standard form (A.1) and then apply (ii). To do that we introduce new variables $x \leftarrow x^+ - x^-$, with $x^+ \geq 0$ and $x^- \geq 0$, $y \leftarrow y^+ - y^-$, with $y^+ \geq 0$ and $y^- \geq 0$, and add the slack variable $z \in \mathcal{K}^*$, so that the problem (A.2) becomes

$$\begin{array}{ll} \min & \frac{1}{2}\begin{bmatrix} x^+ \\ x^- \end{bmatrix}^\top \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} - \begin{bmatrix} b \\ -b \end{bmatrix}^\top \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \\ \text{s.t.} & \begin{bmatrix} -Q & Q & A^\top & -A^\top & I \end{bmatrix} \xi = -c, \\ & \xi := \begin{bmatrix} x^+ \\ x^- \\ y^+ \\ y^- \\ z \end{bmatrix} \in L := \mathbb{R}_+^{2n+2m} \times \mathcal{K}^*. \end{array} \tag{A.7}$$

Suppose that $\bar{\xi}$ is an optimal solution of the problem (A.7). By assumption, the problem (A.2) is strictly feasible, therefore there exist (x, y) such that $Qx - A^\top y + c \in \text{int}(\mathcal{K}^*)$. Then set $\xi = (|x| + \epsilon, |-x| + \epsilon, |y| + \epsilon, |-y| + \epsilon, Qx - A^\top y + c)$ for some $\epsilon > 0$ and where the absolute value is applied component by component. We have $\xi \in \text{int}(L^*)$, therefore the problem (A.7) is also strictly feasible then by (ii) where exists $\bar{u} \in \mathbb{R}^n$ such that $(\bar{\xi}, \bar{u})$ is an optimal solution of

$$\begin{array}{ll} \max & -\frac{1}{2}(x^+ - x^-)^\top Q(x^+ - x^-) + c^\top u \\ \text{s.t.} & \begin{bmatrix} Qx^+ - Qx^- - Qu \\ -Qx^+ + Qx^- + Qu \\ -Au - b \\ Au + b \\ -u \end{bmatrix} \in L^*. \end{array} \tag{A.8}$$

It can be easily shown that $L^* = \mathbb{R}_+^{2n+2m} \times \mathcal{K}$. By making the substitution $x^+ - x^- \leftarrow x$ and $-u \leftarrow v$, changing max to min and simplifying the constraints, $\bar{x} = \bar{x}^+ - \bar{x}^-$, $\bar{v} = -\bar{u}$ is an optimal solution of

$$\begin{array}{ll} \min & \frac{1}{2}x^\top Qx + c^\top v \\ \text{s.t.} & Qv = Qx, \\ & Av = b, \\ & v \in \mathcal{K} \end{array} \tag{A.9}$$

It follows that $\bar{v}$ is an optimal solution of (A.1), because $\bar{v}$ satisfies the constraints of (A.1) and since $Q\bar{v} = Q\bar{x}$, we have $\bar{v}^\top Q\bar{v} = \bar{v}^\top Q\bar{x} = \bar{x}^\top Q\bar{x}$.

□

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