THE CLOSING LEMMA AND THE PLANAR GENERAL DENSITY THEOREM FOR SOBOLEV MAPS

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Abstract. We prove that given a non-wandering point of a Sobolev-$(1,p)$ homeomorphism we can create closed trajectories by making arbitrarily small perturbations. As an application, in the planar case, we obtain that generically the closed trajectories are dense in the non-wandering set.

Keywords: *Periodic points, closing lemma, general density theorem, Sobolev-$(1,p)$ class*

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1. Introduction

A paramount result on dynamical systems, which goes back to Poincaré in the turn of the nineteenth century into the twentieth, is the well known *Closing Lemma*. In brief terms, we intend to *close*, in a sense that we transform into a periodic orbit, a given recurrent or non-wandering orbit by making a small perturbation on the original system. Poincaré believed that such a closing could be performed in quite general situations. However, until now, there are satisfactory answers to this problem if the perturbations are with respect to coarse topologies like e.g., C^0, Sobolev-$(1,p)$ and C^1. Even for the C^2 case the problem can get very difficult and a global approach, and no longer a local one, is needed (see e.g. [17]). Nevertheless, in the C^∞ topology we have Herman's C^∞-robust impressive example of Hamiltonians displaying an interval of energy levels without any closed orbit [11]. Furthermore, and as a consequence of a C^∞ closing lemma for Reeb flows on closed contact three-manifolds, recently Asaoka and Irie were able to obtain a surprising C^∞ closing lemma for Hamiltonian diffeomorphisms of closed surfaces [2].

Despite the fact that the C^0 closing lemma is a quite simple exercise except perhaps for the geodesic flows (see [19]), the C^1 statement reveals several difficulties. The C^1 closing lemma for non-conservative dynamics was first established by Pugh [15] in the late 1960's. For conservative dynamics the closing lemma was proved in the early 1980's by Pugh and Robinson [18]. Recently, in [8], was presented a non-conservative version of the closing lemma for the Sobolev-$(1,p)$ topology. These type of maps gained interest in recent times (see [3, 7, 8, 10, 13]). In this paper we present a simpler and different proof of the Sobolev closing lemma (see Theorem A) which also works in the conservative case.

The *general density theorem* is presumably the most important upshot coming out from the closing lemma. It asserts that, from the generic viewpoint, the closure of the set of periodic orbits is the set where the dynamics is truly relevant: in the non-conservative case this set is the non-wandering set and in the conservative case this set is the whole manifold. The general density theorem has been proved in several different contexts and there is now a vast literature on the subject (see [1, 2, 4, 5, 6, 9, 16, 21]).

Remarkably, the general density theorem turns out to be easier in the C^1 case (see [16]) when compared to the C^0 one ([6]). The main difficulty lies in the stability of the periodic points which in the differentiable case can be expressed through hyperbolicity but in the topological case is much more trickier. Within Sobolev homeomorphisms those issues concerning permanence of periodic points will have to be overcome using a well balanced twofold approach: in one hand we use *hyperbolicity* which is robust under C^1 perturbations and on the other hand we use *Brouwer index* which is robust under C^0 perturbations. In order to have hyperbolicity we had to demand for differentiability at least for a map arbitrarily close from the Sobolev point of view. This bypass through a differentiable map is very hard to obtain. Indeed, regularization of Sobolev-$(1,p)$ homeomorphisms is known only for planar domains (see [13, 10]). Moreover, some C^0 perturbation results which turn out to be quite simple for homeomorphisms, like e.g. the creation of periodic sinks from periodic points, are much more delicate for Sobolev-$(1,p)$ maps (see Proposition 4.3). In overall, we were able to obtain the planar general density theorem for Sobolev-$(1,p)$ maps (see Theorem B).

2. Preliminaries on Sobolev-$(1,p)$ maps

We denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$ and the Lebesgue measure on $\mathbb{R}^n$ by λ.

2.1. Sobolev maps.

Let U be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary (that is, the boundary is locally the graph of a Lipschitz map from an open set of $\mathbb{R}^{n-1}$ to $\mathbb{R}$) and let $1 \leq p, q \leq \infty$.

Recall that a measurable map $f = (f_1, \ldots, f_n)\colon U \to \mathbb{R}^n$ is in the Sobolev class $W^{1,p}(U, \mathbb{R}^n)$ if, for all $i = 1, \ldots, n$, f_i and all its distributional partial derivatives $\partial f_i / \partial x_j$ are in $L^p(U)$.

We endow $W^{1,p}(U, \mathbb{R}^n)$ with the norm defined by

$$\forall f \in W^{1,p}(U, \mathbb{R}^n), \quad \|f\|_{1,p} = \|f\|_p + \|Df\|_p,$$

where $\|f\|_p = \max_i \|f_i\|_p$ and $\|Df\|_p = \max_{i,j} \left\| \frac{\partial f_i}{\partial x_j} \right\|_p$.

We shall be interested only on Sobolev maps that are *continuous* up to the boundary. More precisely, we will consider the space

$$W^{1,p}(U,\mathbb{R}^n)\cap C^0(\overline{U},\mathbb{R}^n)\,.$$

The (natural) norm in this space is equivalent to the one defined by

$$\|f\|_{\infty,p} := \|f\|_\infty + \|Df\|_p\,,$$

since $C^0(\overline{U},\mathbb{R})$ is compactly included in $L^p(U)$.

Finally we define the Sobolev spaces we are going to work with.

Definition 2.1. (Sobolev homeomorphisms) We define $\mathbb{W}^{1,p}(U)$ as the set of all homeomorphisms $f\colon U\to U$ such that $f\in W^{1,p}(U,\mathbb{R}^n)\cap C^0(\overline{U},\mathbb{R}^n)$.

In this space we consider the natural metric defined by

$$d_{\infty,p}(f,g) = \|f-g\|_\infty + \|D(f-g)\|_p\,.$$

We also define $\mathbb{W}^{1,p}_\lambda(U)$ as the subspace of all volume preserving elements in $\mathbb{W}^{1,p}(U)$.

Definition 2.2. (Bi-Sobolev homeomorphisms) We define $\mathbb{W}^{1,p,q}(U)$ as the set of all elements in $\mathbb{W}^{1,p}(U)$ whose inverse is in $\mathbb{W}^{1,q}(U)$. In this space we consider the natural metric defined by $(f,g)\mapsto d_{\infty,p}(f,g)+d_{\infty,q}(f^{-1},g^{-1})$.

We also define $\mathbb{W}^{1,p,q}_\lambda(U)$ as the subspace of all volume preserving elements in $\mathbb{W}^{1,p,q}(U)$.

Since the space of homeomorphisms and the space of volume preserving homeomorphisms are topologically complete (see [14]) and the space $W^{1,p}(U,\mathbb{R}^n)\cap C^0(\overline{U},\mathbb{R}^n)$ is complete, we have the following.

Proposition 2.3. The spaces $\mathbb{W}^{1,p}(U)$, $\mathbb{W}^{1,p}_\lambda(U)$, $\mathbb{W}^{1,p,q}(U)$ and $\mathbb{W}^{1,p,q}_\lambda(U)$ satisfy the Baire property.

Finally, one can define similar spaces for smooth manifolds.

3. The Sobolev-$(1,p)$ closing lemma

Let $f\colon U\to U$ be a homeomorphism. A point $x\in U$ is said to be a *periodic point of period* n for f if $f^n(x)=x$ and $f^i(x)\neq x$ for all $i=1,\dots,n-1$. A point $x\in U$ is said to be a *non-wandering point* for f if for any neighbourhood V of x there exists $n\in\mathbb{N}$ such that $f^n(V)\cap V\neq\emptyset$. We denote by $\mathrm{Per}(f)$ the set of periodic points and by $\Omega(f)$ the set of non-wandering points for f.

Theorem A (Sobolev-$(1,p)$ closing lemma). Let $n \geq 2$. Consider X to be any of the spaces $\mathbb{W}^{1,p}(U)$, $\mathbb{W}^{1,p}_\lambda(U)$, $\mathbb{W}^{1,p,q}(U)$, $\mathbb{W}^{1,p,q}_\lambda(U)$, where $p,q \in [1,\infty[$. Let $f \in X$ and $z \in \Omega(f)$. Then, for all $\varepsilon > 0$ there exists $y_\varepsilon \in U$ and $h_\varepsilon \in X$ such that $\lim_{\varepsilon\to 0} |y_\varepsilon - z| = 0$, $\lim_{\varepsilon\to 0} \|h_\varepsilon - f\|_X = 0$ and $y_\varepsilon \in \mathrm{Per}(h_\varepsilon)$.

Remark 3.1. The Sobolev closing lemma also holds for the space $W^{1,p}(U,\mathbb{R}^n)\cap C^0(\overline{U},\mathbb{R}^n)$ with the additional hypothesis that $\lambda(f^{-1}(z)) = 0$.

In the proof of the Sobolev closing lemma we will use an auxiliary result proved in [3]. We include the proof of this result for completeness. For $a \geq b > 0$ we denote by $\Sigma_{a,b}$ the ellipsoid

$$\Sigma_{a,b} = \left\{ x \in \mathbb{R}^n : \left(\frac{x_1}{a}\right)^2 + \sum_{i=2}^{n} \left(\frac{x_i}{b}\right)^2 \leq 1 \right\}.$$

Lemma 3.2 ([3]). *Given $a \geq b > 0$ and $0 < \mu < 1$, there exists a C^∞ volume preserving diffeomorphism of $\mathbb{R}^n$, F, that is equal to the identity in $\mathbb{R}^n \setminus \Sigma_{a,b}$ and $F(x_1, x_2, \ldots, x_n) = (-x_1, -x_2, x_3, \ldots, x_n)$ in $\Sigma_{(1-\mu)a,(1-\mu)b}$.*

In addition, there exists $C > 0$ independent of a, b, μ, such that all partial derivatives of F and F^{-1} are bounded by $\frac{a}{b}\cdot\frac{C}{\mu}$.

Proof. Consider a C^∞ function $h_1\colon \mathbb{R} \to \mathbb{R}$ that is strictly decreasing in $]0,1[$, is constant equal to zero in $[1,+\infty[$ and constant equal to one in $]-\infty, 0]$. Let $h_\mu\colon \mathbb{R}\to\mathbb{R}$ be defined by $h_\mu(t) = h_1(\frac{1}{\mu}(t-(1-\mu)))$. Notice that $\|h'_\mu\|_\infty \leq \frac{\|h'_1\|_\infty}{\mu}$.

Consider now the function $F = (F_1, F_2, \ldots, F_n)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$F(x) = \left(x_1\cos(\alpha(x)) - \tfrac{a}{b}\, x_2 \sin(\alpha(x)), \tfrac{b}{a}\, x_1 \sin(\alpha(x)) + x_2\cos(\alpha(x)), \bar{x}\right)$$

where $\bar{x} = (x_3, \ldots, x_n)$ and
$$\begin{array}{rccc} \alpha: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \mapsto & h_\mu\left(\sqrt{\left(\frac{x_1}{a}\right)^2 + \sum_{i=2}^n \left(\frac{x_i}{b}\right)^2}\right) \end{array}.$$

Note that $\left|\frac{\partial\alpha}{\partial x_1}\right| \leq \frac{\|h'_1\|_\infty}{\mu\, a}$ and $\left|\frac{\partial\alpha}{\partial x_i}\right| \leq \frac{\|h'_1\|_\infty}{\mu\, b}$, if $i \geq 2$.

The function F preserves the volume because the determinant of its Jacobian matrix is equal to the determinant of the Jacobian matrix of (F_1, F_2), which can easily be seen to be equal to 1.

The conditions on the partial derivatives of $F_{a,b,\mu}$ are simply a consequence of the hypotheses on a, b, μ and of the inequalities $|\cos(\alpha(x))|, |\sin(\alpha(x))| \leq 1$, $|x_1| \leq a$, $|x_i| \leq b$, for $i \geq 2$.

To conclude just notice that F^{-1} is obtained by replacing $\alpha(x)$ by $-\alpha(x)$ in F. $\square$

Proof. (of Theorem A) Using Lemma (of Pugh) (see [8, Lemma 2.3.] and [15]) consider, for $\varepsilon > 0$, $y_\varepsilon \in B(z,\varepsilon)$ and $k \in \mathbb{N}$ such that $f^k(y_\varepsilon) \in B(z,\varepsilon)$ and

$$f^j(y_\varepsilon) \notin B(y_\varepsilon, \tfrac{3}{4}\rho) \cup B(f^k(y_\varepsilon), \tfrac{3}{4}\rho), \text{ for } j = 1, \ldots, k-1$$

where $\rho = |y_\varepsilon - f^k(y_\varepsilon)|$.

Without loss of generality we can suppose that $y_\varepsilon = -(\frac{\rho}{2}, 0, \ldots, 0)$ and $f^k(y_\varepsilon) = (\frac{\rho}{2}, 0, \ldots, 0)$. Let $F(= F_\varepsilon)$ be the function given by Lemma 3.2 for $a = \frac{5}{4}\rho$, $b = \frac{\sqrt{5}}{4}\rho$ and $\mu = \frac{1}{2}$.

Notice that $\Sigma_{a,b} \subseteq B(y_\varepsilon, \frac{3}{4}\rho) \cup B(f^k(y_\varepsilon), \frac{3}{4}\rho) \subseteq B(z, 4\varepsilon)$ (Figure 1). In fact, if $x = (x_1, \ldots, x_n) \in \Sigma_{a,b}$ with $x_1 \geq 0$ for example, then

$$(x_1 - \tfrac{\rho}{2})^2 + \sum_{i=2}^{n} x_i^2 < (x_1 - \tfrac{\rho}{2})^2 + \tfrac{5\rho^2}{16}\left(1 - \tfrac{16x_1^2}{25\rho^2}\right) = \tfrac{4}{5}x_1(x_1 - \tfrac{5\rho}{4}) + \left(\tfrac{3\rho}{4}\right)^2 \leq \left(\tfrac{3\rho}{4}\right)^2.$$

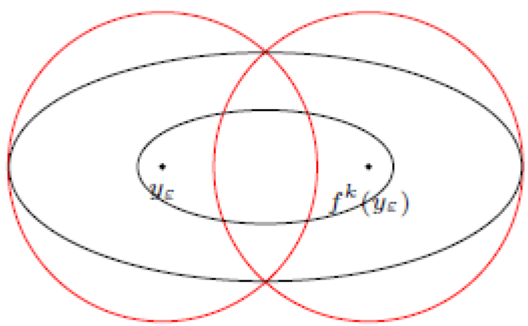

FIGURE 1. Illustration on the perturbative argument.

Let $h_\varepsilon = F_\varepsilon \circ f$ and notice that $(F_\varepsilon \circ f)^j(y_\varepsilon) = (F_\varepsilon \circ f^j)(y_\varepsilon)$, for $j = 1, \ldots, k$, as $f^j(y_\varepsilon) \notin \Sigma_{a,b}$. Therefore, $h_\varepsilon^k(y_\varepsilon) = \left(F_\varepsilon \circ f^k\right)(y_\varepsilon) = F_\varepsilon\left(f^k(y_\varepsilon)\right) = y_\varepsilon$, since y_ε and $f^k(y_\varepsilon)$ belong to $\Sigma_{\frac{a}{2},\frac{b}{2}}$.

As $\|h_\varepsilon - f\|_\infty \leq \operatorname{diam}(\Sigma_{a,b})$, we have that $\lim_{\varepsilon\to 0}\|h_\varepsilon - f\|_\infty = 0$.

On the other hand, by Lemma 3.2, we have that

$$\left|\frac{\partial (h_\varepsilon)_i}{\partial x_j}(x)\right| \leq 2\sqrt{5}\,C\sum_{k=1}^{n}\left|\frac{\partial f_k}{\partial x_j}(x)\right|, \quad i,j = 1, \ldots, n,$$

and then

$$\left\|\frac{\partial (h_\varepsilon - f)_i}{\partial x_j}\right\|_{L^p(U)} = \left\|\frac{\partial (h_\varepsilon - f)_i}{\partial x_j}\right\|_{L^p(f^{-1}(\Sigma_{a,b}))} \leq (2\sqrt{5}\,C + 1)\sum_{k=1}^{n}\left\|\frac{\partial f_k}{\partial x_j}\right\|_{L^p(f^{-1}(\Sigma_{a,b}))}.$$

As $\lim_{\varepsilon\to 0}\lambda\left(f^{-1}\left(B(z,4\,\varepsilon)\right)\right)=\lambda\left(\bigcap_{\varepsilon>0}f^{-1}\left(B(z,4\,\varepsilon)\right)\right)=\lambda(\{f^{-1}(z)\})=0$ then $\lim_{\varepsilon\to 0}\lambda(f^{-1}\left(\Sigma_{a,b}\right))=0$. Since $\frac{\partial f_k}{\partial x_j}\in L^p(U)$, then $\lim_{\varepsilon\to 0}\left\|\frac{\partial f_k}{\partial x_j}\right\|_{L^p(f^{-1}(\Sigma_{a,b}))}=0$. Hence, $\lim_{\varepsilon\to 0}\left\|\dfrac{\partial(h_\varepsilon-f)_i}{\partial x_j}\right\|_{L^p(U)}=0$.

For the bi-Sobolev case, first notice that

$$\left|\frac{\partial(h_\varepsilon^{-1})_i}{\partial x_j}(x)\right|\le 2\sqrt{5}\,C\sum_{k=1}^{n}\left|\frac{\partial(f^{-1})_i}{\partial x_k}(F_\varepsilon^{-1}(x))\right|\,,\quad i,j=1,\ldots,n\,.$$

Therefore, we just need to prove that

$$\lim_{\varepsilon\to 0}\int_{\Sigma_{a,b}}\left|\frac{\partial(f^{-1})_i}{\partial x_k}\left(F_\varepsilon^{-1}(x)\right)\right|^q\,dx=0\,.$$

This follows as before because $\lim_{\varepsilon\to 0}\lambda\left(\Sigma_{a,b}\right)=0$, $\frac{\partial(f^{-1})_k}{\partial x_j}\in L^q(U)$ and

$$\begin{aligned}\int_{\Sigma_{a,b}}\left|\frac{\partial(f^{-1})_i}{\partial x_k}\left(F_\varepsilon^{-1}(x)\right)\right|^q\,dx&=\int_{\Sigma_{a,b}}\left|\frac{\partial(f^{-1})_i}{\partial x_k}(y)\right|^q\left|\det\mathcal{J}F_\varepsilon(y)\right|\,dy\\&\le n!\,(2\sqrt{5}\,C)^n\int_{\Sigma_{a,b}}\left|\frac{\partial(f^{-1})_i}{\partial x_k}(y)\right|^q\,dy\,.\end{aligned}$$

On the other hand, we have that $\lim_{\varepsilon\to 0}\|h_\varepsilon^{-1}-f^{-1}\|_\infty=0$, since $h_\varepsilon^{-1}(x)=f^{-1}(x)$ outside $\Sigma_{a,b}$ and it is controlled by the continuity of f^{-1} in $\Sigma_{a,b}$.

Finally, we observe that when f is volume preserving, then h_ε is also volume preserving. □

Remark 3.3. We notice that the Sobolev closing lemma also holds in smooth manifolds. Indeed, since we only perform a single local perturbation, we can define an F_ε type function on the manifold.

4. The Sobolev-$(1,p)$ general density theorem on planar sets

In this section we prove the following result.

Theorem B (Sobolev-$(1,p)$ general density theorem). There exists a $\mathbb{W}^{1,p}$-residual subset $\mathcal{R}\subset\mathbb{W}^{1,p}(U)$ such that if $f\in\mathcal{R}$, then $\overline{\mathrm{Per}(f)}=\Omega(f)$.

Unless otherwise stated we assume that $U\subset\mathbb{R}^2$. First we give a synopsis of the strategy of the proof of Theorem B.

(i) Since the proof uses a topological fixed point index argument, in §4.1, we begin by presenting some definitions;
(ii) In §4.2 we obtain a result (Lemma 4.1) which assures the existence of a $\mathbb{W}^{1,p}$-residual $\mathscr{R}$ where the periodic orbits are permanent[1]. This guarantees that the map $\mathfrak{P}$ defined by $\mathfrak{P}(f) = \overline{\mathrm{Per}(f)}$ for $f \in \mathbb{W}^{1,p}(U)$ is lower semicontinuous in $\mathscr{R}$. In particular, the continuity points of $\mathfrak{P}|_{\mathscr{R}}$ form a residual subset of $\mathscr{R}$;
(iii) Lastly, a continuity argument using (ii), the Sobolev-$(1,p)$ closing lemma (Theorem A) and a key perturbation result on the creation of sinks (Proposition 4.3) will complete the proof of Theorem B.

4.1. **Brouwer fixed point index.** Given a continuous map f on $U \subset \mathbb{R}^2$, we recall the fixed point index used in [6]. Let B be an open ball on U. If f has no fixed points in ∂B then the *fixed point index* $\iota_f(B)$ is defined as follows: $\iota_f(B) = \deg(\gamma)$, otherwise, where $\deg(\gamma)$ denotes the Hopf degree of the map $\gamma : \partial B \simeq S^1 \to S^1$ which is defined (after a change of coordinates) by:

$$\gamma(x) = \frac{f(x) - x}{\|f(x) - x\|}\,.$$

It follows from the definition that $\iota_f(B) = 0$, if $f(\overline{B}) \cap \overline{B} = \emptyset$. This notion is constant in a small C^0 neighbourhood of f (see e.g. [12]).

4.2. **Permanence of periodic orbits.** We recall the notion of permanence from topological dynamics. We say that a periodic point x of a map $f \in \mathbb{W}^{1,p}(U)$ is *permanent* if for any $g \in \mathbb{W}^{1,p}(U)$, $\mathbb{W}^{1,p}$-arbitrarily close to f, the map g has a periodic point near x. Let $\mathscr{P}(f)$ denote the set of all permanent periodic points of f. Before proving Lemma 4.1 we recall two important facts that will play a crucial role along its proof.

- Every homeomorphism between planar open sets that belongs to $\mathbb{W}^{1,p}(U)$, $1 \leq p < \infty$, can be approximated by C^∞ smooth diffeomorphisms (see [13] for $p > 1$ and [10] for $p = 1$).
- Let $\mathrm{Diff}^1(U)$ be the space of C^1 diffeomorphisms $f \colon U \to U$ endowed with the C^1 topology. As a direct consequence of the Kupka-Smale theorem for diffeomorphisms [20] we have that there exists a C^1-residual subset $\tilde{\mathcal{R}} \subset \mathrm{Diff}^1(U)$ such that if $f \in \tilde{\mathcal{R}}$, then the periodic points of f are all hyperbolic[2]. Therefore, and since $\mathrm{Diff}^1(U)$ is a Baire space, $\tilde{\mathcal{R}}$ is C^1-dense in $\mathrm{Diff}^1(U)$.

[1] This is why we only prove our result in surfaces. We make use of an approximation result ([13, 10]) available only for planar domains.

[2] A periodic point p of period n for $f \in \mathrm{Diff}^1(U)$ is said to be *hyperbolic* if the eigenvalues of Df^n_p do not intersect S^1.

Lemma 4.1. *There exists a* $\mathbb{W}^{1,p}$*-residual subset* $\mathscr{R}$ *of* $\mathbb{W}^{1,p}(U)$ *such that* $\mathrm{Per}(f) = \mathscr{P}(f)$, *for any* $f \in \mathscr{R}$. *Moreover, the residual subset* $\mathscr{R}$ *contains* $\tilde{\mathcal{R}}$.

Proof. Since $\mathrm{Diff}^1(U)$ is $\mathbb{W}^{1,p}$-dense in $\mathbb{W}^{1,p}(U)$ (see [10, 13]) then we conclude that $\tilde{\mathcal{R}}$ is $\mathbb{W}^{1,p}$-dense in $\mathbb{W}^{1,p}(U)$. In particular there exists a $\mathbb{W}^{1,p}$-dense set of maps in $\mathbb{W}^{1,p}(U)$ such that all periodic orbits are permanent.

We proceed to prove that there exists a $\mathbb{W}^{1,p}$-residual $\mathscr{R} \subset \mathbb{W}^{1,p}(U)$ such that any periodic orbit of a map in $\mathscr{R}$ is permanent. We begin by taking a countable base for the topology $\{\mathcal{B}_i\}_{i\in\mathbb{N}}$ of U consisting of open balls.

The index of periodic orbits will play a crucial role along the proof since, in rough terms, the existence of non-zero index on a set allows to conclude the existence of a periodic orbit that intersects that set and, moreover, displaying non-zero index persists under C^0 perturbations (thus $\mathbb{W}^{1,p}$ perturbations) of the map. We define, for every $i, n \in \mathbb{N}$, the following subsets $\mathscr{F}_{i,n}, \mathscr{I}_{i,n}$ of $\mathbb{W}^{1,p}(U)$ in the following way:

(i) $f \in \mathscr{F}_{i,n}$ if $f^n(x) \neq x$ for all $x \in \overline{\mathcal{B}_i}$;

(ii) $f \in \mathscr{I}_{i,n}$ if there exists $\mathcal{B}_j$ with $\mathrm{diam}(\mathcal{B}_j) < \mathrm{diam}(\mathcal{B}_i)$, $\mathcal{B}_i \cap \mathcal{B}_j \neq \emptyset$ and such that $f^n(x) \neq x$ for all $x \in \partial\mathcal{B}_j$ and $\iota_{f^n}(\mathcal{B}_j) \neq 0$.

The sets $\mathscr{F}_{i,n}$ and $\mathscr{I}_{i,n}$ are C^0-open subsets of $\mathbb{W}^{1,p}(U)$ (thus $\mathbb{W}^{1,p}$-open subsets of $\mathbb{W}^{1,p}(U)$).

We claim that $\tilde{\mathcal{R}} \subset \mathscr{F}_{i,n} \cup \mathscr{I}_{i,n}$ and, in particular, $\mathscr{F}_{i,n} \cup \mathscr{I}_{i,n}$ is a $\mathbb{W}^{1,p}$-open and dense subset of $\mathbb{W}^{1,p}(U)$ for all $i, n \geq 1$. Indeed, given $i, n \geq 1$ fixed and $f \in \tilde{\mathcal{R}}$ either there are no periodic points with period n in $\overline{\mathcal{B}_i}$ (in which case $f \in \mathscr{F}_{i,n}$) or there are periodic points with period n in $\overline{\mathcal{B}_i}$. In the later case, since the periodic orbits are hyperbolic, hence isolated, we have that $f \in \mathscr{I}_{i,n}$.

Now we claim that the $\mathbb{W}^{1,p}$-residual subset

$$\mathscr{R} := \bigcap_{i,n\geq 1} [\mathscr{F}_{i,n} \cup \mathscr{I}_{i,n}] \subset \mathbb{W}^{1,p}(U)$$

satisfies the requirements of the lemma. Take $f \in \mathscr{R}$ and let us show that $\mathrm{Per}(f) = \mathscr{P}(f)$. Let $p \in \mathrm{Per}(f)$ of period n and any $\mathcal{B}_i$ containing p. Since $f \in \mathscr{F}_{i,n} \cup \mathscr{I}_{i,n}$ and $p \in \mathcal{B}_i$ then $f \notin \mathscr{F}_{i,n}$. Hence, there exists $\mathcal{B}_j$ with $\mathrm{diam}(\mathcal{B}_j) < \mathrm{diam}(\mathcal{B}_i)$, such that $f^n(x) \neq x$ for all x in the boundary of $\mathcal{B}_j$, and the corresponding index $\iota_{f^n}(\mathcal{B}_j)$ is non-zero. We are left to see that $\iota_{g^n}(\mathcal{B}_j)$ is also non-zero for a small $\mathbb{W}^{1,p}$ perturbation g of the original map. This follows from the continuity of the composition, whenever defined, of the inclusion of $\mathbb{W}^{1,p}(U)$ in $C^0(\overline{U}, \mathbb{R}^2)$, the n-composition map in $C^0(\overline{U}, \mathbb{R}^2)$ and the function defined by the index relatively to $\mathcal{B}_j$. This clearly shows that p is permanent. □

4.3. **Finishing the proof of Theorem B.** Let $U^\star$ be the set of compact subsets of $\overline{U}$ endowed with the Hausdorff topology. We need the following semicontinuity result.

Lemma 4.2. *Let $\mathbb{W}^{1,p}(U)$ be endowed with the $\mathbb{W}^{1,p}$ topology. Then the map*

$$\begin{array}{rccc} \mathfrak{P}: & \mathbb{W}^{1,p}(U) & \to & U^\star \\ & f & \mapsto & \overline{\mathrm{Per}(f)} \end{array}$$

is lower semicontinuous on the residual $\mathscr{R}$ given by Lemma 4.1.

Proof. We must prove that for any $f \in \mathscr{R}$, and any $\varepsilon > 0$ there exists a neighbourhood $\mathcal{V} \subset \mathbb{W}^{1,p}(U)$ of f such that $\mathfrak{P}(f) \subseteq \mathcal{B}_\varepsilon(\mathfrak{P}(g))$ for all $g \in \mathcal{V}$, or in other words there are no implosions of the set of periodic points when we $\mathbb{W}^{1,p}$ perturb f. But Lemma 4.1 says that $\mathrm{Per}(f) = \mathscr{P}(f)$ and the proof follows immediately from the definition of permanent periodic points. $\square$

We will also need the following perturbation result. We recall that a periodic point p of period k is a *sink* of f if there exists a small disk D containing p such that $\bigcap_{j\geq 0} f^{jk}(D) = \{p\}$.

Proposition 4.3. Let $f \in \mathbb{W}^{1,p}(U)$, p a periodic point of f and $\varepsilon > 0$. There exists $g \in \mathbb{W}^{1,p}(U)$, ε-$\mathbb{W}^{1,p}$-close to f such that p is a sink of g.

The proof of this proposition follows immediately from the following two lemmas. We remark that Lemma 4.5 holds for U in $\mathbb{R}^n$, $n \geq 2$.

Lemma 4.4. *(Regularization lemma) Let $f \in \mathbb{W}^{1,p}(U)$, p a periodic point of f with period k and $\varepsilon > 0$. There exists a C^∞ map $g \in \mathbb{W}^{1,p}(U)$ such that $\|g - f\|_{\infty,p} \leq \varepsilon$ and p is a periodic point of g with period k.*

Proof. Let $\delta > 0$ such that the balls $B(f^i(p), \delta)$, for $i = 0, \ldots, k-1$, are disjoint and contained in U.

For $\eta \leq \frac{\delta}{4}$ let h_η be a C^∞ homeomorphism such that $\|f - h_\eta\|_{\infty,p} \leq \eta$ (see [13, 10]). We can also assume that $\|f^i - h_\eta^i\|_\infty \leq \eta$ for all $i = 1, \ldots, k$. Let q_η be the middle point of p and $h_\eta^k(p)$. Without lost of generality we can suppose that $q_\eta = 0$. Consider the function F_η given by Lemma 3.2 for $a = b = 2\eta$ and $\mu = \frac{1}{2}$. Notice that the partial derivatives of F_η are bounded independently of η.

Take the function $F_\eta \circ h_\eta$. We have that p is a periodic point of this function with period k and

$$\begin{aligned} \|F_\eta \circ h_\eta - f\|_{\infty,p} &\leq \|F_\eta \circ h_\eta - h_\eta\|_{\infty,p} + \|h_\eta - f\|_{\infty,p} \\ &\leq \|F_\eta \circ h_\eta - h_\eta\|_\infty + \|D(F_\eta \circ h_\eta - h_\eta)\|_p + \|h_\eta - f\|_{\infty,p} \\ &\leq 4\eta + \|D(F_\eta \circ h_\eta - h_\eta)\|_{L^p(h_\eta^{-1}(B(q_\eta,2\eta)))} + \eta \\ &\leq 5\eta + \|D(F_\eta \circ h_\eta - h_\eta)\|_{L^p(f^{-1}(B(p,4\eta)))} \,. \end{aligned}$$

As in the proof of Theorem A, we have that

$$\begin{aligned} \|D(F_\eta \circ h_\eta - h_\eta)\|_{L^p(f^{-1}(B(p,4\eta)))} &\leq C\,\|Dh_\eta\|_{L^p(f^{-1}(B(p,4\eta)))} \\ &\leq C\,\left(\|Df\|_{L^p(f^{-1}(B(p,4\eta)))} + \eta\right)\,, \end{aligned}$$

and the conclusion follows making η goes to 0. □

Lemma 4.5. *Let f be a C^1 map in $\mathbb{W}^{1,p}(U)$, p a periodic point of f and $\varepsilon > 0$. There exists $g \in \mathbb{W}^{1,p}(U)$, ε-$\mathbb{W}^{1,p}$-close to f such that p is a sink of g.*

Proof. Let $\delta_0 > 0$ be such that $f(p), \ldots, f^{k-1}(p) \notin B(p, 2\delta_0)$. Let $\alpha < 1$ to be chosen later. We define, for $\delta < \delta_0$, the function $\varphi = \varphi^\delta \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\varphi(x) = \begin{cases} \alpha(x-p)+p & \text{if } |x-p| \leq \delta \\ [(2-\alpha)(|x-p|-\delta)+\alpha\delta]\,\frac{x-p}{|x-p|} + p & \text{if } \delta \leq |x-p| \leq 2\delta \\ x & \text{if } |x-p| \geq 2\delta\,. \end{cases}$$

Notice that

- φ is an homeomorphism which is C^∞ in $\{x \in \mathbb{R}^2 : |x-p| \neq \delta, 2\delta\}$;
- all partial derivatives of φ are bounded by α in $\{x \in \mathbb{R}^2 : |x-p| < \delta\}$ and by 6 in $\{x \in \mathbb{R}^2 : |x-p| > \delta\}$. Consider the function $g = \varphi \circ f$.

We have that $g \in \mathbb{W}^{1,p}(U)$. Moreover, in a neighbourhood of p, $g^k = \varphi \circ f^k$ and g^k is C^1. Consequently,

(1) p is a periodic point of g of period k;
(2) p is a sink of g as

$$\begin{aligned} \left|\frac{\partial (g^k)_i}{\partial x_j}(p)\right| &= \left|\frac{\partial \varphi_i}{\partial x_1}(p)\frac{\partial (f^k)_1}{\partial x_j}(p) + \frac{\partial \varphi_i}{\partial x_2}(p)\frac{\partial (f^k)_2}{\partial x_j}(p)\right| \\ &\leq \alpha\left(\left|\frac{\partial (f^k)_1}{\partial x_j}(p)\right| + \left|\frac{\partial (f^k)_2}{\partial x_j}(p)\right|\right)\,, \end{aligned}$$

which is less than 1 for a convenient choice of α;

(3) for δ small enough, g is ε-$\mathbb{W}^{1,p}$-close to f, as

$$\begin{aligned} \|g-f\|_{\infty,p} &= \|g-f\|_\infty + \|D(g-f)\|_p \\ &\leq 2\delta + \|D(g-f)\|_{L^p(f^{-1}(B(p,2\delta))} \\ &\leq 2\delta + \|Dg\|_{L^p(f^{-1}(B(p,2\delta))} + \|Df\|_{L^p(f^{-1}(B(p,2\delta))}. \end{aligned}$$

□

Remark 4.6. We observe that, in Lemma 4.5, the approximating function g, which is a contraction, can be chosen with arbitrarily small contraction rate.

We are now in a position to prove Theorem B. Since the map $\mathfrak{P}\colon\ \mathbb{W}^{1,p}(U) \to U^\star$ defined by $\mathfrak{P}(f) = \overline{\mathrm{Per}(f)}$ is lower semicontinuous on $\mathscr{R}$ (by Lemma 4.2) then the continuity points of $\mathfrak{P}|_{\mathscr{R}}$ form a residual subset $\mathscr{R}_1 \subset \mathscr{R}$. Thus, to prove Theorem B it is enough to show that $\Omega(f) = \overline{\mathrm{Per}(f)}$ for every map $f \in \mathscr{R}_1$. Assume, by contradiction, that there exists a map $f \in \mathscr{R}_1$ such that $\Omega(f) \setminus \overline{\mathrm{Per}(f)} \neq \emptyset$ and take $p \in \Omega(f) \setminus \overline{\mathrm{Per}(f)}$. Theorem A guarantees that f can be $\mathbb{W}^{1,p}$ approximated by a map $g_1 \in \mathbb{W}^{1,p}(U)$ with a periodic point $\tilde{p}$ arbitrarily close to p.

Due to Proposition 4.3, we can perform an extra $\mathbb{W}^{1,p}$ small perturbation of g_1 so that the resulting map $g_2 \in \mathbb{W}^{1,p}(U)$ has $\tilde{p}$ as a periodic sink.

As $\mathbb{W}^{1,p}(U)$ is a Baire space we get that $\mathscr{R}$ is $\mathbb{W}^{1,p}$-dense in $\mathbb{W}^{1,p}(U)$. Therefore, g_2 can be arbitrarily $\mathbb{W}^{1,p}$ approximated by some map $g_3 \in \mathscr{R}$ with a periodic point $\overline{p}$ arbitrarily close to $\tilde{p}$. This is in contradiction to the fact that f is a continuity point of $\mathfrak{P}\,|_{\mathscr{R}}$.

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