12-10-2021 10:30 AM

Equisingular Approximation of Analytic Germs

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics

Abstract

This thesis deals with the problem of approximating germs of real or complex analytic spaces by Nash or algebraic germs. In particular, we investigate the problem of approximating analytic germs in various ways while preserving the Hilbert-Samuel function, which is of importance in the resolution of singularities. We first show that analytic germs that are complete intersections can be arbitrarily closely approximated by algebraic germs which are complete intersections with the same Hilbert-Samuel function. We then show that analytic germs whose local rings are Cohen-Macaulay can be arbitrarily closely approximated by Nash germs whose local rings are Cohen-Macaulay and have the same Hilbert-Samuel function. Finally we prove that we may approximate arbitrary analytic germs by topologically equisingular Nash germs which have the same Hilbert-Samuel function.

Keywords:analytic space, Hilbert-Samuel function, approximation, Nash, Cohen-Macaulay, complete intersection

Summary for Lay Audience

Replacing geometric objects with simpler ones that share some properties with the original is an important operation in many branches of geometry. When the simpler replacement can be derived from the original by stopping a limiting process, it is called an approximation. In this thesis we deal with the local approximations of real or complex analytic spaces which preserve various properties. "Local" here indicates that we are interested in approximations near a point as opposed to global approximations that are approximations over some finite region of space. Analytic spaces near a point are defined by finite sets of power series. We look for approximations that preserve (i) the algebro-geometric class of the original, i.e., Complete Intersection or Cohen-Macaulay or (ii) the topological type of the original. In both cases we also impose the requirement that the approximants have the same Hilbert-Samuel function as the original. This additional constraint is motivated by the fact that the Hilbert-Samuel function is thought of as a measure of how singular an analytic space is near a point and plays an important role in Hironaka's seminal work on desingularization of analytic spaces. We approximate the original analytic space by approximating the power series that define it near a point. We show that it is possible to find approximations of the form (i) and (ii) whose defining power series belong to an algebraically simpler class than those of the original. Specifically, in the case of Complete Intersections we show that we can approximate the original set of defining power series by polynomials. In the other cases we show that the approximating power series can be chosen to be Nash, i.e., power series satisfying a polynomial equation.

Co-Authorship Statement

Some of the results in Chapter 2 for which full proofs are presented, are new, as well as all such results in Chapters 3 and 4 , and were obtained during the doctoral study of the author. The new results in Chapters 2 and 3 are joint work with Janusz Adamus, the author's doctoral thesis supervisor that have been published in a peer-reviewed journal [1].

Acknowledgements

I would like to thank my PhD advisor Prof. Janusz Adamus for being a truly excellent advisor. His timely and astute advice was invaluable in bringing my PhD research to fruition. His steadfast support and encouragement at times when my progress on my research was slow, were integral in enabling me to reach this point. Our many mathematical discussions during our meetings were a source of great enlightenment and intellectual fulfilment. I am also especially thankful to Prof. Adamus for his understanding and support during the times when I was suffering from bouts of depression and anxiety. He is my role-model for my future career as a mathematician.

I am grateful to the members of my supervisory committee Prof. Rasul Shafikov and Prof. Martin Pinsonnault for their scrutiny and advice in the context of this thesis and for very enlightening interactions in the context of courses that they have taught at Western University. I would like to also express my gratitude to all the professors who have taught me as a graduate student in the Department of Mathematics at Western University. In addition to those already mentioned I would like to particularly thank Prof. Masoud Khalkhali first for agreeing to be one of my thesis examiners, and also his excellent teaching in many courses I attended which included a great combination of mathematical anecdotes and historical commentary. I am also grateful to Prof. Guillaume Rond and Prof. Marc Moreno Maza for agreeing to be the External Examiner and University Examiner respectively for my thesis.

I am greatful to Prof. Richard Vuduc of the School of CSE, at Georgia Tech, who saw academic potential in me before I saw it in myself, and who helped me when I was making the challenging transition to Pure Mathematics from Computer Science. I am also thankful to Prof. Josephine Yu and Prof. Anton Leykin of the School of Mathematics at Georgia Tech, for their excellent advice and encouragement when I was making this change of fields. Their superbly taught graduate level courses on the subjects of Modern Algebra, Commutative Algebra and Computational Algebraic

Geometry, were also integral in helping me decide what type of work I wanted to do during before applying for admissions to PhD programs in Mathematics.

A statement of thanks would be incomplete without mentioning my beloved wife, Karuna DSouza, who entered my life in the middle of my PhD and without whose love, understanding and support I may never have finished this thesis. I would also like to thank my oldest friend Ramkumar Ramachandra, for the uncountable conversations that we've had and the experiences we shared as undergraduates that have served, at many times, to reinforce my faith in the many decisions that have ultimately contributed to bringing me to this point in my academic career.

Finally, and most importantly, I would like to thank my parents, Yusuf Patel and Shabnam Patel for a lifetime of love and support, and for a truly brilliant childhood. They have made great sacrifices to enable me to pursue my dreams as far as I have. I would never have been able to spend so many years studying abroad if it were not for their unwavering support. They shall always occupy a significant place in my heart.

Contents

Chapter 1

Introduction

For an analytic space $X \subset \mathbb{K}^n$ (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), its germ at a point $a \in X$, denoted by X_a, is defined by a finite set of convergent power series, and describes the local behaviour of X at the point a. The main question considered in this thesis is whether, given a germ of an analytic space X_a, we can find a germ $\hat{X}_a$ that is defined by polynomials or algebraic power series, such that $\hat{X}_a$ shares certain specified algebro-geometric properties with X_a. A standard approach to such problems in analytic geometry is to consider approximations $\hat{X}_a$ to X_a, defined by polynomials or algebraic power series that are μ-degree (for $\mu \in \mathbb{N}$) approximations to the defining convergent power series of X_a. Then by choosing μ to be sufficiently large one can expect to find germs $\hat{X}_a$ that have the same algebro-geometric properties as X_a. Indeed, the construction of such algebraic approximations to analytic objects is one of the central problems in analytic geometry.

Specifically, in this thesis we consider approximation of the germs of (real or complex) analytic spaces, X_a, by Nash germs, or even algebraic germs, which are equisingular with X_a in the sense of the Hilbert-Samuel function. Recall that, for an analytic germ X_a, the *Hilbert-Samuel function* $H_{X,a}$ is defined as

$$H_{X,a}(\eta) = \dim_{\mathbb{K}} \frac{\mathcal{O}_{X,a}}{\mathfrak{m}^{\eta+1}}\,, \qquad \text{for all } \eta \in \mathbb{N}\,,$$

where $\mathcal{O}_{X,a}$ is the local ring of X at a, with the maximal ideal $\mathfrak{m}$. The Hilbert-Samuel function encodes many important algebro-geometric properties of the germ and may be regarded as a measure of its singularity. It plays a central role in resolution of singularities (see [10]).

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $x = (x_1, \dots, x_n)$ and let $\mathbb{K}\{x\}$ denote the ring of convergent power series in variables x. If X_0 is an analytic germ at 0 in $\mathbb{K}^n$, its local ring $\mathcal{O}_{X,0}$ is of the form $\mathbb{K}\{x\}/I$ for some ideal I in $\mathbb{K}\{x\}$. Let $\mathbb{K}\langle x\rangle$ denote the ring of algebraic power series, that is, the convergent power series algebraic over the ring of polynomials $\mathbb{K}[x]$. One says that the germ X_0 is *Nash* if the ideal I can be generated by elements of $\mathbb{K}\langle x\rangle$. If I can be generated by polynomials then we call the germ X_0 *algebraic*.

In our first series of results, we deal with singularities of special types. Namely, those whose local ring is Cohen-Macaulay, or even better, a complete intersection. We prove that a complete intersection singularity can be arbitrarily closely approximated by algebraic germs which are also complete intersections and share the same Hilbert-Samuel function (Theorem 3.1.3). Polynomial approximation is not possible, in general, for Cohen-Macaulay singularities (see Example 3.2.2). The next best thing is approximation by Nash germs. In Theorem 3.2.1, we show that a Cohen-Macaulay singularity can be arbitrarily closely approximated by Nash germs which are also Cohen-Macaulay and share the same Hilbert-Samuel function.

We also consider the problem of the approximation of an analytic singularity X_0 by Nash germs which are homeomorphic with X_0. We give a variant of Mostowski's theorem [22], Theorem 3.4.1, showing that every analytic germ $X_0 \subset \mathbb{K}^n_0$ can be arbitrarily closely approximated by a Nash germ $\hat{X}_0 \subset \mathbb{K}^n_0$, such that the pairs $(\mathbb{K}^n, X)$

and $(\mathbb{K}^n, \hat{X})$ are topologically equivalent near zero, and the Hilbert-Samuel functions $H_{X,0}$ and $H_{\hat{X},0}$ coincide. This may be combined with the other results that we prove.

When dealing with questions about the Hilbert-Samuel function of the ring $\mathbb{K}\{x\}/I$, it is convenient to work with the so-called diagram of initial exponents of I, a combinatorial representation of the ideal I, denoted $\mathfrak{N}(I)$, which we recall in Section 2.4. Indeed, the Hilbert-Samuel function of $\mathbb{K}\{x\}/I$ may be read off from the sub-level sets of (the complement of) $\mathfrak{N}(I)$ (Lemma 2.8.3). The diagram itself is, in turn, uniquely determined by a standard basis of I, which is a special generating set of I (see Section 2.5). Our key tool in establishing Hilbert-Samuel equisingularity of a given germ and its approximants is a theorem of T. Becker [5], which gives a criterion for a collection $\{F_1, \dots, F_t\} \subset I$ to form a standard basis of I in terms of finitely many equations that depend polynomially on the F_i. It is therefore well suited for an application of the classical Algebraic Artin Approximation.

We call X_0 a Cohen-Macaulay (resp. complete intersection) singularity when the local ring $\mathcal{O}_{X,0}$ is Cohen-Macaulay (resp. a complete intersection); see Section 3.1 for definitions. The finite determinacy of the Hilbert-Samuel function of a complete intersection follows already from the work of Srinivas and Trivedi [31]. We give a new proof of this fact here, because it can also be applied in the Cohen-Macaulay case, which is new. Roughly speaking, we combine the equivalence of Cohen-Macaulayness and flatness (Remark 3.1.2) with a corollary to Hironaka's flatness criterion (Proposition 2.4.11), to show that with respect to a certain total ordering on $\mathbb{N}^n$ the diagrams of I and its suitable approximation I_μ coincide. We then show (Proposition 2.8.8) that this equality implies equality of the diagrams with respect to the standard ordering, and hence equality of the Hilbert-Samuel functions.

Finally, in the proof of Theorem 3.4.1, we combine the above Becker criterion with the original strategy of Mostowski, based on Płoski's parametrized Artin approximation [27] and a theorem of Varchenko stating that the algebraic equisingularity of

Zariski implies the topological equisingularity [32]. We use the modern exposition of Mostowski's theorem, due to Bilski, Parusiński and Rond [12], where the original Płoski theorem is replaced with a more powerful Theorem 2.6.1.

The structure of the thesis is as follows: Chapter 2 develops background material that establishes the setting in which we are working, and some essential propositions which we use as tools in the proofs of the main results. Chapter 3 contains the proofs of the main approximation results of this thesis. In Chapter 4 we explore consequences of these results and present two possible directions of future work. In Appendix A we present certain definitions and theorems from local algebra that we use at various points in the thesis.

Chapter 2

Background

In this chapter we present the basic theory of real and complex analytic sets and spaces, their germs and the germs of functions defined on them. Many well known results are presented without proofs. We also present other original results that we use in the proofs of the main approximation theorems in Chapter 3. For these we include complete proofs.

2.1 Basic Theory of Real and Complex Analytic sets and germs

In this section we present certain basic results in the theory of real and complex analytic sets. These are presented here without proof. There are various well known references on the theory of real and complex analytic sets and spaces, in particular we point the reader to [23], which was our primary source for the current section. This choice was made because of the fact that care is taken by the author in [23] to clearly separate the theory into parts that apply to both $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and parts that specifically only apply to $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$.

2.1.1 Analytic Sets - General Theory

We consider first results and definitions that are valid both in the case when the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Throughout this section $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. *A function defined in an open set $\Omega \subset \mathbb{K}^n$ is called* analytic *if for every point $p \in \Omega$ there is a neighborhood U of p and a power series that converges to f on U. That is,*

$$f(x) = \sum_{\alpha} c_\alpha (x-p)^\alpha \text{ for all } x \in U. \tag{2.1.1}$$

The power series on the right in the above equation is then called the Taylor series of f at p.

Remark 2.1.2. *In the case when $\mathbb{K} = \mathbb{C}$ analytic functions are called* holomorphic *functions.*

Definition 2.1.3. *A subset A of an open set $\Omega \subset \mathbb{K}^n$ is called an* analytic set *if for any $p \in \Omega$ there exists an open neighborhood U of p and a collection of functions $f_1, \dots, f_r$ analytic on U such that,*

$$A \cap U = \{x \in U | f_1(x) = \cdots = f_r(x) = 0\} \tag{2.1.2}$$

Remark 2.1.4. *It is easy to see that an analytic set $A \subset \Omega$ is closed in Ω.*

Proposition 2.1.5. *If $\Omega \subset \mathbb{K}^n$ is a connected open set and $A \subset \Omega$ is an analytic subset of Ω then $\Omega \setminus A$ is dense in Ω.*

Definition 2.1.6. *A point p in an analytic set A is called a* simple *(or* regular*) point of A if the functions $f_1, \dots, f_s$ defining A in some neighborhood U of p may be taken so that the differentials $df_1(p), \dots, df_s(p)$ are linearly independent. Points of A that are not regular are called* singular.

Remark 2.1.7. *It is a consequence of the above definition that an analytic set A is an analytic manifold in some neighborhood about every simple point. We define the dimension of A at a simple point to be the dimension of this analytic manifold.*

The dimension of an analytic set has two equivalent definitions:

Definition 2.1.8. *The dimension* $\dim_p A$*, of an analytic set A at a point p is defined as either:*

(1) $\dim_p A :=$ *the highest dimension of A as a manifold at simple points near p.*

(2) $\dim_p A :=$ *the highest codimension of a plane P through p such that p is an isolated point of $A \cap P$.*

There is also a global notion of the dimension of an analytic set which is defined in an intuitive way:

Definition 2.1.9. *The dimension of an analytic set $A \subset \Omega \subset \mathbb{K}^n$ is* $\dim A := \sup\{\dim_p A : p \in A\}$.

In this thesis, the set of singular points of an analytic set A is denoted by $\mathrm{Sing}(A)$.

2.1.2 Local Properties of Analytic sets

It is desirable to have tools to analyze the properties of an analytic set near a particular point. The mathematical object that formalizes the notion of a "set near a point" is a germ.

Definition 2.1.10. *Suppose X is a topological space, and $p \in X$. Consider the equivalence relation $\sim_p$ on the set of subsets of X, $\mathcal{P}(X)$,*

$$A \sim_p B \text{ iff there is an open set } U, \text{ with } p \in U \text{ such that } A \cap U = B \cap U. \tag{2.1.3}$$

The elements of $\mathcal{P}(X)/\sim_p$ are called set germs *at p. The equivalence class of $A \in \mathcal{P}(X)$ is denoted by A_p.*

We note the following regarding germs:

Remark 2.1.11. *(1) We write $A_p \subset B_p$ when $A \subset B$ for some representatives A and B of A_p and B_p respectively.*

(2) We set $A_p \cap B_p := (A \cap B)_p$ and $A_p \cup B_p := (A \cup B)_p$. Also, the cartesian product of germs can be defined $A_p \times B_q := (A \times B)_{(p,q)}$. These can be shown to be well-defined (i.e., independent of the representatives chosen for A_p and B_q)

(3) $A_p \neq \emptyset$ iff p is in the closure of A, i.e., $p \in \bar{A}$.

(4) Representatives of X_p are exactly those sets $A \subset X$ which satisfy $p \in \mathrm{int}(A)$.

We also have the notion of function germs which is defined as follows:

Definition 2.1.12. *Let X be a topological space, $p \in X$ and $\mathcal{F}(X,p)$ be the collection of all pairs (U,f) where U is an open neighborhood of p and $f : U \to \mathbb{K}$. Consider the equivalence relation $\sim_p$ on $\mathcal{F}(X,p)$,*

$$(U,f) \sim_p (W,g) \text{ iff there is an open neighborhood } V \text{ of } p \text{ such that } f|_V = g|_V. \tag{2.1.4}$$

The elements of $\mathcal{F}(X,p)/\sim_p$ are called function germs *at p. The equivalence class of (U,f) is denoted as f_p.*

Some useful facts about function germs are:

Remark 2.1.13. *(1) $f_p + g_p := (f+g)_p$, $f_p g_p := (fg)_p$, are well defined (i.e., independent of the choice of representatives). Also, $f_p/g_p := (f/g)_p$ is well defined provided g is non-zero in some neighborhood of p.*

(2) We say that the function germ f_p vanishes on A_p and write $f_p|_{A_p} = 0$ if some representative (U,f) of f_p vanishes on $A \cap U$ where A is a representative of A_p.

(3) The notion of the image $f_p(A_p)$ is in general not well defined.

2.1.3 Germs of Analytic Functions

Let $x = (x_1, \ldots, x_n)$. We denote by $\mathbb{K}\{x\}$ the ring of all power series $\sum_\alpha c_\alpha x^\alpha$ which converge for $|x| < \delta$, where $\delta > 0$ may depend on the power series in question. We write $\mathcal{O}_n$ for the ring of germs of analytic functions at $0 \in \mathbb{K}^n$. We may identify $\mathbb{K}\{x\}$ with $\mathcal{O}_n$ via the Taylor series expansion at 0.

Definition 2.1.14. *An* analytic $\mathbb{K}$-algebra R *is a* $\mathbb{K}$-*algebra isomorphic to a quotient of* $\mathbb{K}\{x\}$. *That is,*

$$R \cong \frac{\mathbb{K}\{x\}}{I} \tag{2.1.5}$$

where $I = (f_1, \ldots, f_s)$ *is a finitely generated ideal of* $\mathbb{K}\{x\}$.

In what follows we shall need the notion of a regular germ.

Definition 2.1.15. *Let* $f(z,w) \in \mathbb{K}\{z,w\} = \mathbb{K}\{z_1, \ldots, z_m, w\}$. *We say that* f *is* regular of order d *in* w *if* $f(0,w) = h(w)w^d$ *with* $h(w) \neq 0$. *We say that* f *is* regular in w *if it is regular of order* d *in* w *for some* d.

Two results of much utility in the theory of analytic function germs are the Weierstrass Division Theorem and the Weierstrass Preparation Theorem.

Theorem 2.1.16. *(Weierstrass Division Theorem) If* $f \in \mathbb{K}\{x\}$ *is regular of order* d *in* x_n, *then* $g(x_n) = f(0, \ldots, 0, x_n) = x_n^d h(x_n)$ *where* $h(x_n) \in \mathbb{K}\{x_n\}$ *is a unit, and, for any* $\phi \in \mathbb{K}\{x\}$ *there exist an* $a \in \mathbb{K}\{x\}$ *and* $b_1, \ldots, b_d \in \mathbb{K}\{x_1, \ldots, x_{n-1}\}$ *such that*

$$\phi = a \cdot f + \sum_{\nu=1}^{d} b_\nu x_n^{d-\nu}. \tag{2.1.6}$$

Furthermore, the a *and* b_ν *are uniquely determined.*

Theorem 2.1.17. *(Weierstrass Preparation Theorem) With* f *as in Theorem 2.1.16,* *there exist a unit $u \in \mathbb{K}\{x\}$ and $a_1, \ldots, a_d \in \mathbb{K}\{x_1, \ldots, x_{n-1}\}$ such that*

$$f = u \cdot \left(x_n^d + \sum_{\nu=1}^{d} a_\nu x_n^{d-\nu} \right) \tag{2.1.7}$$

Furthermore, the u and a_ν are uniquely determined.

Definition 2.1.18. *If $a_1, \ldots, a_n \in \mathbb{K}\{x_1, \ldots, x_{n-1}\}$ with $a_\nu(0) = 0$ for all ν and $P = x_n^d + \sum_{\nu=1}^d a_\nu x_n^{d-\nu}$, then P is called a* distinguished polynomial *with respect to x_n.*

Suppose $a \in \mathbb{K}^n$, we denote by $\mathcal{O}_{n,a}$ the ring of germs of analytic functions at a. From the remarks in the beginning of the section we have a canonical isomorphism $\mathbb{K}\{x_1 - a_1, \ldots, x_n - a_n\} \cong \mathcal{O}_{n,a}$ via the Taylor series expansion of germs $f \in \mathcal{O}_{n,a}$ about a. As a consequence of this isomorphism, $\mathcal{O}_{n,a}$ is a local ring. Some useful properties of $\mathcal{O}_{n,a}$ that can be obtained from Theorem 2.1.17 are:

Proposition 2.1.19. *(1) $\mathcal{O}_{n,a}$ is a Noetherian ring.*

(2) $\mathcal{O}_{n,a}$ is a regular local ring with $\dim \mathcal{O}_{n,a} = n$.

(3) $\mathcal{O}_{n,a}$ is an integral domain.

(4) $\mathcal{O}_{n,a}$ is a unique factorization domain.

(5) Every non-constant $f \in \mathbb{K}\{x\} \cong \mathcal{O}_n$ with $f(0) = 0$ is regular (after a linear change of coordinates at worst) with respect to some x_j.

2.1.4 Germs of Analytic Sets (General Theory)

In this section $\Omega \subset \mathbb{K}^n$ is an open set and $S \subset \Omega$ is an analytic set. Also, a is a point in Ω. All the results in this section are valid for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Definition 2.1.20. *The germ of an analytic set $S \subset \Omega$ at a point $a \in \Omega$ is called an* analytic set germ.

Further, in what follows we shall denote by $I = I(S_a) \subset \mathcal{O}_{n,a}$ the set of germs of analytic functions vanishing on S_a. I forms an ideal in $\mathcal{O}_{n,a}$.

Remark 2.1.21. *$S_a \subset S'_a$ if and only if $I(S_a) \supset I(S'_a)$.*

Definition 2.1.22. *An analytic set germ S_a is called* irreducible *if whenever there are two analytic germs $S_{1,a}$ and $S_{2,a}$ such that $S_a = S_{1,a} \cup S_{2,a}$ then one of them must be S_a.*

Lemma 2.1.23. *S_a is irreducible if and only if $I(S_a) \subset \mathcal{O}_{n,a}$ is a prime ideal.*

Proposition 2.1.24. *Let S_a be an analytic set germ, then S_a can be written as a finite union $S_a = \cup_{\nu=1}^{k} S_{\nu,a}$ of irreducible analytic set germs such that $S_{\nu,a} \not\subset \cup_{\mu\neq\nu} S_{\mu,a}$. Further, this decomposition is uniquely determined up to order.*

Definition 2.1.25. *For S_a, an analytic set germ, the germs $S_{\nu,a}$ in Proposition 2.1.24, are called the* irreducible components *of S_a.*

Proposition 2.1.26. *Suppose I is a non-zero ideal of O_n, $p < n$, and R_p is the image of $\mathbb{K}\{x_1, \ldots, x_p\}$ via the canonical isomorphism $\mathbb{K}\{x\} \cong \mathcal{O}_n$. Let $\mathcal{O}_p = \mathcal{O}_n \cap R_p$. Let $A = \mathcal{O}_n/I$. Then we have a morphism $\eta : \mathcal{O}_p \to A$ induced by the canonical injection $\mathcal{O}_p \to \mathcal{O}_n$. Further, after a linear change of coordinates in $\mathbb{K}^n$ there is an integer $0 \leq p_0 < n$ such that $\eta : \mathcal{O}_{p_0} \to A$ makes A into a finite $\mathcal{O}_{p_0}$ module.*

Remark 2.1.27. *The necessary and sufficient condition that the coordinates satisfy the condition of Proposition 2.1.26 is that $I \cap \mathcal{O}_{p_0} = 0$ and for any $r > p_0$ there exists a $Q_r(x_1, \ldots, x_r) \in \mathcal{O}_{r-1}[x_r] \cap I$ which is distinguished in x_r.*

Suppose that $\Omega \subset \mathbb{K}^n$ is an open set and $S \subset \Omega$ is an analytic set with $0 \in S$. Further we suppose that S_0 is an irreducible germ, that is, $I = I(S_0) \subset \mathcal{O}_n$ is a prime ideal. Also, we assume that we choose the coordinates referred to in Prop. 2.1.26. Then we have,

Proposition 2.1.28. *There is a fundamental system of neighborhoods, $U = U' \times U''$ where $U' \subset \mathbb{K}^p$ and $U'' \subset \mathbb{K}^{n-p}$, of 0 such that if $\pi : S \cap U \to U'$ is the restriction of the projection of U onto U' then π is a proper map and every fibre of π, $\pi^{-1}(x')$ where $x' \in U'$ is a finite set.*

Proposition 2.1.29. *The integer p_0 of Proposition 2.1.26 for $I = I(S_a)$ where S_a is irreducible is* $\dim_a S$.

Definition 2.1.30. *Let S_a be an irreducible analytic set germ. The integer p_0 of Proposition 2.1.26, with respect to $I = I(S_a)$ is called the* dimension *of S_a and is denoted* $\dim S_a$.

We observe here that by definition $\dim S_a = \dim_a S$ for an irreducible analytic set germ S_a. We may now define the dimension of an arbitrary analytic set germ,

Definition 2.1.31. *Let S_a be an analytic set germ with irreducible components $S_{\nu,a}$ for $\nu = 1, \ldots, k$. The number $p = \max\{\dim S_{\nu,a} : \nu = 1, \ldots, k\}$ is called the dimension of S_a and is denoted* $\dim S_a$.

It turns out that this notion too coincides with the geometric definition of dimension at a point of an analytic set,

Theorem 2.1.32. *Let $S \subset \Omega \subset \mathbb{K}^n$ be an analytic set germ with Ω an open set. If $a \in S$ then $\dim S_a = \dim_a S$ (See Definition 2.1.8).*

2.1.5 Germs of analytic sets (Case $\mathbb{K} = \mathbb{C}$)

In this section we focus on the case when $\mathbb{K} = \mathbb{C}$. We shall see that complex analytic sets have some very nice properties not shared with their real analytic counterparts, including a very direct relationship between their geometry and algebra. In this entire section Ω will be an open set in $\mathbb{C}^n$.

Proposition 2.1.33. *If $S \subset \Omega \subset \mathbb{C}^n$ is a complex analytic set and $0 \in S$ with S_0 irreducible then the projection π of Proposition 2.1.28 satisfies $\pi(S \cap U) = U'$ (with U, U' as in Prop. 2.1.28).*

Theorem 2.1.34. *(Rückert's Nullstellensatz) Let I be any ideal in $\mathcal{O}_n$ and $S_0 = S(I)$ be the complex analytic germ defined by the set of common zeros of a finite set of generators of I. Then $I(S_0) = \mathrm{rad}(I)$.*

Definition 2.1.35. *Let $S \subset \Omega \subset \mathbb{C}^n$ be a complex analytic set. A function f on S is called* complex analytic *or* holomorphic *at $a \in S$ if there is a neighborhood U of a in Ω and a complex analytic function F defined on U such that $F|_{S\cap U} = f|_{S\cap U}$.*

We usually denote the ring of germs of complex analytic functions at $a \in S$ by $\mathcal{O}_{S,a}$.

Remark 2.1.36. *We have the following,*

$$\mathcal{O}_{S,a} \cong \mathcal{O}_a / I(S_a). \tag{2.1.8}$$

Hence $\mathcal{O}_{S,a}$ is an analytic $\mathbb{C}$-algebra.

2.1.6 Germs of Analytic Sets (Case $\mathbb{K} = \mathbb{R}$)

In this section we shall describe the concept of complexification of a real analytic set germ. This is an essential tool that has much utility in the study of the properties of real analytic set germs.

Suppose A_a is the germ of an analytic set at a point $a \in \mathbb{R}^n$. We may identify $\mathbb{R}^n$ with the subset of points in $\mathbb{C}^n$ whose coordinates have zero imaginary part. With this identification we can consider $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$. In order to distinguish between the $\mathbb{R}$ and $\mathbb{C}$ cases we shall denote the ring of germs of real analytic functions at $a \in \mathbb{R}^n \subset \mathbb{C}^n$ by $\mathcal{O}^{\mathbb{R}}_{n,a}$ and the ring of complex analytic functions vanishing at a by

$\mathcal{O}^{\mathbb{C}}_{n,a}$. Further we shall denote the ideal in $\mathcal{O}^{\mathbb{R}}_{n,a}$ of real analytic functions vanishing on A_a by $I^{\mathbb{R}}(A_a)$.

Proposition 2.1.37. *There is a unique germ $\tilde{A}_a$ of a complex analytic set at $a \in \mathbb{R}^n \subset \mathbb{C}^n$ such that $\tilde{A}_a \supset A_a$ and for which any germ $g \in \mathcal{O}^{\mathbb{C}}_{n,a}$ which vanishes on A_a also vanishes on $\tilde{A}_a$. Further we have $\tilde{A}_a \cap \mathbb{R}^n_a = A_a$ and any germ of a complex analytic set S_a which contains A_a also contains $\tilde{A}_a$. Also, if $I(\tilde{A}_a) = I^{\mathbb{C}}(A_a)$ is the ideal of $\mathcal{O}^{\mathbb{C}}_{n,a}$ of germs of holomorphic functions which vanish on $\tilde{A}_a$ then we have*

$$I(\tilde{A}_a) = I^{\mathbb{C}}(A_a) = I^{\mathbb{R}}(A_a) \otimes_{\mathbb{R}} \mathbb{C}. \tag{2.1.9}$$

Definition 2.1.38. *The germ $\tilde{A}_a$ in Proposition 2.1.37 is called the* complexification *of A_a.*

Proposition 2.1.39. *If A_a is irreducible as a real analytic set germ then $\tilde{A}_a$ is irreducible as a complex analytic germ. If $A_a = \cup_\nu A_{\nu,a}$ is the decomposition of A_a into its irreducible components and $\tilde{A}_{\nu,a}$ is the complexification of $A_{\nu,a}$ then $\tilde{A}_a = \cup_\nu \tilde{A}_{\nu,a}$ is the decomposition of $\tilde{A}_a$ into its irreducible components.*

We shall denote the dimension of the germ A_a by $\dim_{\mathbb{R}} A_a$ and that of $\tilde{A}_a$ as a complex analytic germ by $\dim_{\mathbb{C}} \tilde{A}_a$. With this notation we have,

Proposition 2.1.40. $\dim_{\mathbb{R}} A_a = \dim_{\mathbb{C}} \tilde{A}_a$.

As a direct consequence of this we have,

Corollary 2.1.41. *If A is a real analytic set in an open set $\Omega \subset \mathbb{R}^n$ then every point $a \in \Omega$ has a neighborhood U such that for each $b \in U$ we have* $\dim_{\mathbb{R}} A_b \leq \dim_{\mathbb{R}} A_a$.

It is important to note at this juncture that not all complex analytic set germs are complexifications of real analytic set germs. The following result tells us when they are in the irreducible case.

Proposition 2.1.42. *Let S_a be an irreducible complex analytic set germ at $a \in \mathbb{C}^n$. Then S_a is the complexification of a real analytic set germ A_a if and only if $S_a \cap \mathbb{R}^n$ contains a germ B_a with $\dim_{\mathbb{R}} B_a = \dim_{\mathbb{C}} S_a$.*

2.2 Analytic Spaces

In this section we generalize the notion of an analytic set so that we obtain a tighter correspondence between algebra and geometry. For the remainder of this section we will assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose that S_a is the germ of an analytic set at $a \in \mathbb{K}^n$ and that f_a is the germ of an analytic function on $\mathbb{K}^n$ such that $f_a^m|_{S_a} = 0$. Then we have $f_a|_{S_a} = 0$. That is, $f^m \in I(S_a)$, implies that $f \in I(S_a)$ for all positive integers m. Therefore we don't have a one to one correspondence with all local analytic $\mathbb{K}$-algebras and germs of analytic sets. What we do have is a correspondence between analytic set germs and local analytic $\mathbb{K}$-algebras without nilpotents (i.e., those which are *reduced*). This is the motivation for the definition of an *analytic space*.

Definition 2.2.1. *An* analytic space *(over $\mathbb{K}$) is a ringed space $X = (|X|, \mathcal{O}_X)$ (i.e., a Hausdorff topological space along with a sheaf of rings $\mathcal{O}_X$ on $|X|$) which is locally isomorphic to a ringed space Z defined as follows: Let U be an open set of $\mathbb{K}^n$, and let $f_1, \ldots, f_k$ be analytic functions on U (also denoted as $(f_1, \ldots, f_k) \in \mathcal{O}(U)$). Let $|Z|$ be the zero set of the f_i and $O_Z = O_U/I$ restricted to $|Z|$, where I is the ideal sheaf generated by the f_i.*

Definition 2.2.2. *A* morphism of analytic spaces *$\phi : X \to Y$ is a pair $\phi = (|\phi|, \phi^*)$, where $|\phi| : |X| \to |Y|$ is a continuous mapping, and $\phi^* : \mathcal{O}_Y \to |\phi|_* \mathcal{O}_X$ is a sheaf homomorphism.*

It is a consequence of the above definitions that the germs of analytic spaces are in one to one correspondence with the set of all local analytic $\mathbb{K}$-algebras $\mathbb{K}\{x\}/I$, with

$x = (x_1, \ldots, x_m)$, where I is an ideal in $\mathbb{K}\{x\}$, and morphisms of germs of analytic spaces correspond to homomorphisms of local analytic $\mathbb{K}$-algebras.

Definition 2.2.1 is from [7]. It should be noted at this point that this definition is slightly simpler than the definition of an analytic space that is used in some current literature (see, for example,[15]). This simplification is made to avoid a discussion of coherence in the case when $\mathbb{K} = \mathbb{R}$. Since we are only concerned with local results in this thesis, this simplified definition suffices, and allows us to avoid a lengthy digression focussed on issues that would have no involvement in the rest of the present work.

The following definitions establish the terminology that we use later when discussing our notion of approximation of germs of analytic spaces.

Definition 2.2.3. *We call the germ of a $\mathbb{K}$-analytic space $X \subseteq \mathbb{K}^n$ at a an* analytic germ *and denote it by X_a.*

Definition 2.2.4. *If the local ring $\mathcal{O}_{X,a}$ of an analytic germ X_a is isomorphic to $\mathbb{K}\{x\}/I$ where $I = (h_1, \ldots, h_s)$ and $h_i \in \mathbb{K}\langle x \rangle$ then we call X_a a* Nash germ.

Definition 2.2.5. *If the local ring $\mathcal{O}_{X,a}$ of an analytic germ X_a is isomorphic to $\mathbb{K}\{x\}/I$ where $I = (h_1, \ldots, h_s)$ and $h_i \in \mathbb{K}[x]$ then we call X_a an* algebraic germ.

2.3 Whitney's Tangent Cones

In [35] Whitney introduces the notion of a tangent cone at a point in a complex analytic set. This is a notion that generalizes the tangent space associated to a point in an analytic manifold. Tangent cones provide information about the nature of analytic sets at singular points and they have an algebraic structure associated with them that will be used in the sequel. In this section we will be specifically working with complex analytic sets in some open set in $\mathbb{C}^n$.

Definition 2.3.1. *A* cone K *in* $\mathbb{C}^n$ *is a set of vectors such that if* $v \in K$ *and* $a \in \mathbb{C}$ *then* $av \in K$. *A* cone from $p \in \mathbb{C}^n$*is a set of points* $p + v$ *where* v *is any vector of some cone.*

Definition 2.3.2. *Given a complex analytic set* $S \subset \Omega \subset \mathbb{C}^n$*, with* Ω *open and* $p \in S$ *the* tangent cone $C(S, p)$ *of* S *at* p *is the set of vectors* v *with the following property: There are sequences* $\{p_i\}$ *of points of* S *and scalars* $\{a_i\}$ *such that* $a_i(p - p_i) \to v$ *as* $i \to \infty$.

The above definition clearly implies that $C(S, p)$ is closed. Tangent cones have a remarkable algebraic realization that we shall have occasion to use in our later work.

If f is a holomorphic function in a neighborhood of a point $p \in \mathbb{C}^n$ then we may expand f in a power series about p as follows:

$$f(p + v) = f_p^{[0]} + f_p^{[1]} + f_p^{[2]} + \dots, \quad f_p^{[0]} = f(p), \tag{2.3.1}$$

where $f_p^{[m]}$ is a homogeneous polynomial of degree m in $v_1, \dots, v_m$ the components of v.

Definition 2.3.3. *The* initial polynomial *of* f *at* p*, denoted by* $f_p^{[*]}$ *is equal to* $f_p^{[m]}$ *where* m *is the smallest integer with* $f_p^{[m]} \not\equiv 0$.

Given the above we have the following result [35, Theorem 10.7],

Theorem 2.3.4. *Given a complex analytic set* $S \subset \Omega \subset \mathbb{C}^n$ *and* $p \in S$, $C(S, p)$ *is the set of solutions of* $f_p^{[*]}(x) = 0$*, for all* f *whose germs at* p *are in* $I(S_p)$.

We note that, in general, a set f_i generating $I(S_p)$ may be insufficient to generate $C(S, p)$. The following are some basic properties of tangent cones:

Lemma 2.3.5. *For complex analytic sets* $S, T \subset \Omega \subset \mathbb{C}^n$*:*

(1) $C(T, p) \subset C(S, p)$ *if* $p \in T \subset S$,

(2) $C(S \cup T, p) = C(S, p) \cup C(T, p)$ *if* $p \in S \cup T$,

(3) $C(S \cap T, p) \subset C(S, p) \cap C(T, p)$ *if* $p \in S \cap T$,

(4) [35, Lemma 8.11] $\dim C(S, p) = \dim_p S$ *if* $p \in S$ *and* $\dim_p S \geq 1$.

2.4 Hironaka's division algorithm and diagram of initial exponents

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let A denote the field $\mathbb{K}$ or the ring $\mathbb{K}\{y\}$ of convergent power series in variables $y = (y_1, \ldots, y_m)$. Let $A\{z\}$ denote the ring of convergent power series in variables $z = (z_1, \ldots, z_k)$ with coefficients in A (i.e., $\mathbb{K}\{z\}$ or $\mathbb{K}\{y, z\}$, depending on A). We will write z^α for $z_1^{\alpha_1} \ldots z_k^{\alpha_k}$, where $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$.

The mapping $A \ni F(y) \mapsto F(0) \in \mathbb{K} = A \otimes_A \mathbb{K}$ of evaluation of the y variables at 0 induces an evaluation mapping

$$A\{z\} \ni F = \sum_{\alpha \in \mathbb{N}^k} F_\alpha(y) z^\alpha \;\mapsto\; F(0) = \sum_{\alpha \in \mathbb{N}^k} F_\alpha(0) z^\alpha \in \mathbb{K}\{z\}\,.$$

(In case $A = \mathbb{K}$, this is just the identity mapping.)

Definition 2.4.1. *For an ideal* I *in* $A\{z\}$*, we call* $I(0) := \{F(0) : F \in I\}$ *in* $\mathbb{K}\{z\}$*, the* evaluated ideal.

Definition 2.4.2. *A function on* $\mathbb{K}^k$, $\Lambda(\alpha) = \sum_{j=1}^k \lambda_j \alpha_j$*, for some* $\lambda_j > 0$*, is called a* positive linear form *on* $\mathbb{K}^k$.

Given such Λ, we will regard $\mathbb{N}^k$ as endowed with the total ordering defined by the lexicographic ordering of the $(k+1)$-tuples $(\Lambda(\alpha), \alpha_k, \ldots, \alpha_1)$.

Definition 2.4.3. *For a non-zero* $F = \sum_{\alpha \in \mathbb{N}^k} F_\alpha z^\alpha \in A\{z\}$, $\mathrm{supp} F = \{\alpha \in \mathbb{N}^k : F_\alpha \neq 0\}$ *is called the* support *of* F.

Definition 2.4.4. *For $F \in A\{z\}$, the minimum (with respect to the above total ordering) over all $\alpha \in \mathrm{supp}F$ is called its* initial exponent *and denoted* $\exp_\Lambda F$.

Similarly,

$$\mathrm{supp}F(0) = \{\alpha \in \mathbb{N}^k : F_\alpha(0) \neq 0\} \quad \text{and} \quad \exp_\Lambda F(0) = \min_\Lambda\{\alpha \in \mathrm{supp}F(0)\}\,,$$

for the evaluated series. We have $\mathrm{supp}F(0) \subset \mathrm{supp}F$, and hence $\exp_\Lambda F \leq \exp_\Lambda F(0)$. We will write simply $\exp F$ and $\exp F(0)$ instead of $\exp_\Lambda F$ and $\exp_\Lambda F(0)$, when $\Lambda(\alpha) = |\alpha| = \alpha_1 + \cdots + \alpha_k$.

We now recall Hironaka's division algorithm.

Theorem 2.4.5 ([7, Thm. 3.1, 3.4]). *Let Λ be any positive linear form on $\mathbb{K}^k$. Let $G_1, \dots, G_t \in A\{z\}$, and let $\alpha^i := \exp_\Lambda G_i(0)$, $1 \leq i \leq t$. Then, for every $F \in A\{z\}$, there exist unique $Q_1, \dots, Q_t, R \in A\{z\}$ such that*

$$F = \sum_{i=1}^{t} Q_i G_i + R, \tag{2.4.1}$$

$$\alpha^i + \mathrm{supp}Q_i \subset \Delta_i,\ 1 \leq i \leq t, \quad \text{and} \quad \mathrm{supp}R \subset \Delta,$$

where

$$\Delta_1 := \alpha^1 + \mathbb{N}^k, \qquad \Delta_i := (\alpha^i + \mathbb{N}^k) \setminus \bigcup_{j=1}^{i-1} \Delta_j \quad \text{for } i \geq 2,$$

and $\Delta := \mathbb{N}^k \setminus \bigcup_{i=1}^{t} \Delta_i$.

Definition 2.4.6. *For an ideal I in $A\{z\}$,*

$$\mathfrak{N}_\Lambda(I) = \{\exp_\Lambda F : F \in I \setminus \{0\}\}.$$

is called the diagram of initial exponents *of I relative to Λ.*

Similarly, for the evaluated ideal $I(0)$, we set

$$\mathfrak{N}_\Lambda(I(0)) = \{\exp_\Lambda F(0) : F \in I, F(0) \neq 0\}.$$

We will write $\mathfrak{N}(I)$ and $\mathfrak{N}(I(0))$ instead of $\mathfrak{N}_\Lambda(I)$ and $\mathfrak{N}_\Lambda(I(0))$, when $\Lambda(\alpha) = |\alpha| = \alpha_1 + \cdots + \alpha_k$.

Note that every diagram $\mathfrak{N}_\Lambda(I)$ satisfies the equality $\mathfrak{N}_\Lambda(I) + \mathbb{N}^k = \mathfrak{N}_\Lambda(I)$. (Indeed, for $\alpha \in \mathfrak{N}_\Lambda(I)$ and $\gamma \in \mathbb{N}^k$, one can choose $F \in I$ with $\exp_\Lambda F = \alpha$; then $z^\gamma F \in I$, hence $\alpha + \gamma = \exp_\Lambda(z^\gamma F)$ is in $\mathfrak{N}_\Lambda(I)$.)

Remark 2.4.7. *It is easy to show that, for every ideal I in $A\{z\}$ and for every positive linear form Λ, there exists a unique smallest (finite) set $V_\Lambda(I) \subset \mathfrak{N}_\Lambda(I)$ such that $V_\Lambda(I) + \mathbb{N}^k = \mathfrak{N}_\Lambda(I)$ (see, e.g., [7, Lem. 3.8]). The elements of $V_\Lambda(I)$ are called the* vertices *of the diagram $\mathfrak{N}_\Lambda(I)$.*

Corollary 2.4.8 ([7, Cor. 3.9])**.** *Let Λ be any positive linear form on $\mathbb{K}^k$. Let I be an ideal in $\mathbb{K}\{z\}$, and let $\alpha^1, \dots, \alpha^t \in \mathbb{N}^k$ be the vertices of the diagram $\mathfrak{N}_\Lambda(I)$. Choose $G_i \in I$ such that $\exp_\Lambda G_i = \alpha^i$, $1 \leq i \leq t$, and let $\{\Delta_i, \Delta\}$ denote the partition of $\mathbb{N}^k$ determined by the α^i, as above. Then, $\mathfrak{N}_\Lambda(I) = \bigcup_{i=1}^t \Delta_i$ and the G_i generate the ideal I.*

Proof. The equality $\mathfrak{N}_\Lambda(I) = \bigcup_{i=1}^t \Delta_i$ follows immediately from Remark 2.4.7. According to Theorem 2.4.5, any $F \in \mathbb{K}\{z\}$ can be written as $F = \sum_{i=1}^t Q_i G_i + R_F$, where $\mathrm{supp} R_F \subset \Delta$. Therefore, $F \in I$ if and only if $R_F \in I$. But $\mathrm{supp} R_F \subset \Delta = \mathbb{N}^k \setminus \mathfrak{N}_\Lambda(I)$, hence $R_F \in I$ if and only if $R_F = 0$. $\square$

The remainder of this section will be concerned with the algebraic notion of flatness. First we recall its definition.

Definition 2.4.9. *A module M over a Noetherian ring A is called* flat *when, for* *every exact sequence*

$$0 \to N' \to N \to N'' \to 0$$

of A-modules, the sequence

$$0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$$

is also exact.

The following result of Hironaka expresses flatness in terms of his division algorithm.

Theorem 2.4.10 ([7, Thm. 7.9])**.** *Let I be an ideal in $A\{z\}$. Let Λ be any positive linear form on $\mathbb{K}^k$, and let $\alpha^1, \ldots, \alpha^t$ be the vertices of $\mathfrak{N}_\Lambda(I(0))$. Let $G_1, \ldots, G_t \in I$ be such that $\exp_\Lambda G_i(0) = \alpha^i$, $1 \leq i \leq t$. Then, the following are equivalent:*

(i) $A\{z\}/I$ is flat as an A-module

(ii) For any $F \in I$, the remainder of F after division (2.4.1) *by $G_1, \ldots, G_t$ is zero.*

Let now $x = (x_1, \ldots, x_n)$, $n \geq 2$. Fix $k \in \{1, \ldots, n-1\}$. To simplify notation, let $x_{[k]}$ denote variables $(x_1, \ldots, x_k)$, and $\tilde{x}$ the remaining $(x_{k+1}, \ldots, x_n)$. In what follows, we will regard elements of $\mathbb{K}\{x\}$ either as power series in all the variables x with coefficients in $\mathbb{K}$, written $F = \sum_{\beta \in \mathbb{N}^n} f_\beta x^\beta$, $f_\beta \in \mathbb{K}$, or as power series in variables $x_{[k]}$ with coefficients in $\mathbb{K}\{\tilde{x}\}$, written $F = \sum_{\alpha \in \mathbb{N}^k} F_\alpha(\tilde{x}) x_{[k]}^\alpha$, $F_\alpha(\tilde{x}) \in \mathbb{K}\{\tilde{x}\}$.

For $F \in \mathbb{K}\{x\}$, we will denote by $F(0)$ the series with variables $\tilde{x}$ evaluated at 0. That is, if $F = \sum_{\alpha \in \mathbb{N}^k} F_\alpha(\tilde{x}) x_{[k]}^\alpha$ then $F(0) = \sum_{\alpha \in \mathbb{N}^k} F_\alpha(0) x_{[k]}^\alpha \in \mathbb{K}\{x_{[k]}\}$. Equivalently, if $F = \sum_{\beta \in \mathbb{N}^n} f_\beta x^\beta$ then

$$F(0) = \sum_{\beta \in \mathbb{N}^k \times \{0\}^{n-k}} f_\beta x^\beta \in \mathbb{K}\{x_{[k]}\}$$

(i.e., the sum is over those $\beta = (\beta_1, \ldots, \beta_n)$ for which $\beta_{k+1} = \cdots = \beta_n = 0$).

To avoid confusion, for $F \in \mathbb{K}\{x\}$, we will denote its support as an element of $\mathbb{K}\{\tilde{x}\}\{x_{[k]}\}$ by $\mathrm{supp}_{\mathbb{N}^k} F$, and the support as an element of $\mathbb{K}\{x\}$ as $\mathrm{supp}_{\mathbb{N}^n} F$. That is,

$$\mathrm{supp}_{\mathbb{N}^k} F = \{\alpha \in \mathbb{N}^k : F_\alpha(\tilde{x}) \neq 0\} \quad \text{and} \quad \mathrm{supp}_{\mathbb{N}^n} F = \{\beta \in \mathbb{N}^n : f_\beta \neq 0\}.$$

Note that a positive linear form $\Lambda(\beta) = \sum_{i=1}^n \lambda_i \beta_i$ on $\mathbb{K}^n$ gives rise to a positive form $\sum_{i=1}^k \lambda_i \beta_i$ on $\mathbb{K}^k$. By a slight abuse of notation, we will denote the latter also by Λ.

Proposition 2.4.11. *Let I be an ideal in $\mathbb{K}\{x\}$, let $1 \leq k < n$, and let $\tilde{x}$ denote the variables $(x_{k+1}, \ldots, x_n)$.*

(i) If there exist a positive linear form Λ on $\mathbb{K}^n$ and a set $D \subset \mathbb{N}^k$ such that $\mathfrak{N}_\Lambda(I) = D \times \mathbb{N}^{n-k}$, then $\mathbb{K}\{x\}/I$ is a flat $\mathbb{K}\{\tilde{x}\}$-module.

(ii) If $\mathbb{K}\{x\}/I$ is a flat $\mathbb{K}\{\tilde{x}\}$-module, then there exist $l_0 \in \mathbb{N}$ and a set $D \subset \mathbb{N}^k$ such that, for all $l \geq l_0$, the diagram $\mathfrak{N}_\Lambda(I)$ with respect to the linear form $\Lambda(\beta) = \displaystyle\sum_{i=1}^k \beta_i + \sum_{j=k+1}^n l\beta_j$ satisfies $\mathfrak{N}_\Lambda(I) = D \times \mathbb{N}^{n-k}$.

Remark 2.4.12. *Note that if Λ is such that $\mathfrak{N}_\Lambda(I) = D \times \mathbb{N}^{n-k}$ for some $D \subset \mathbb{N}^k$, then necessarily $D = \mathfrak{N}_\Lambda(I(0))$.*

Proof. For the proof of (i), we will need the following well known corollary to the classical "local criterion for flatness" (see, e.g., [7, Cor. 7.6]): *$\mathbb{K}\{x\}/I$ is flat as a $\mathbb{K}\{\tilde{x}\}$-module if and only if $I \cap (\tilde{x})\mathbb{K}\{x\} \subset (\tilde{x})\cdot I$.*

Suppose that $F \in I \cap (\tilde{x})\mathbb{K}\{x\}$. Then, $F(0) = 0$. Let $\beta^1, \ldots, \beta^t \in \mathbb{N}^k \times \{0\}^{n-k}$ be the vertices of $\mathfrak{N}_\Lambda(I)$, and let $G_1, \ldots, G_t \in I$ be such that $\exp_\Lambda G_i = \beta^i$, $1 \leq i \leq t$. By Theorem 2.4.5 and Corollary 2.4.8, there are $Q_1, \ldots, Q_t \in \mathbb{K}\{x\}$ such that $F = \sum_{i=1}^t Q_i G_i$ and the sets $\beta^i + \mathrm{supp}_{\mathbb{N}^n} Q_i$ are pairwise disjoint.

Write $\beta^i = (\alpha^i, 0)$, where $\alpha^i \in \mathbb{N}^k$, $1 \leq i \leq t$. It follows that the sets $\alpha^i + \mathrm{supp}_{\mathbb{N}^k} Q_i(0)$ in $\mathbb{N}^k$ are also pairwise disjoint, and hence the initial exponents

$\exp_\Lambda Q_i(0)G_i(0) = \exp_\Lambda Q_i(0) + \alpha^i$ are pairwise distinct. On the other hand, $\sum_{i=1}^t Q_i(0)G_i(0) = F(0) = 0$. This is only possible if $Q_i(0) = 0$ for all i. In other words, $Q_i \in (\tilde{x})\mathbb{K}\{x\}$. Hence $F = \sum_{i=1}^t Q_iG_i$ is in $(\tilde{x})\cdot I$, which proves (i).

Suppose now that $\mathbb{K}\{x\}/I$ is $\mathbb{K}\{\tilde{x}\}$-flat. Let $\lambda(\alpha) = |\alpha|$ for $\alpha \in \mathbb{N}^k$, and let $\alpha^1, \dots, \alpha^t$ be the vertices of $\mathfrak{N}(I(0)) = \mathfrak{N}_\lambda(I(0))$. Let $\{\Delta_i, \Delta\}$ be the partition of $\mathbb{N}^k$ determined by the α^i as in Theorem 2.4.5. Let $l_0 = 1 + \max\{|\alpha^i| : i = 1, \dots, t\}$, let $l \geq l_0$ be arbitrary, and set

$$\Lambda(\beta) := \sum_{i=1}^{k} \beta_i + \sum_{j=k+1}^{n} l\beta_j \,.$$

Set $\beta^i := (\alpha^i, 0) = (\alpha^i_1, \dots, \alpha^i_k, 0, \dots, 0) \in \mathbb{N}^n$, $1 \leq i \leq t$. We will show that the vertices of $\mathfrak{N}_\Lambda(I)$ are precisely $\{\beta^1, \dots, \beta^t\}$.

Let $G_1, \dots, G_t \in I$ be such that $\exp G_i(0) = \alpha^i$, $1 \leq i \leq t$. Write $G_i = \sum_{\alpha\in\mathbb{N}^k} G_{i,\alpha}x^\alpha_{[k]}$, where $G_{i,\alpha} = \sum_{\gamma\in\mathbb{N}^{n-k}} g_{i,\alpha,\gamma}\tilde{x}^\gamma$. For every i, there are at most finitely many $\alpha \in \mathrm{supp}_{\mathbb{N}^k} G_i$ with $\lambda(\alpha) < \lambda(\alpha^i)$. For each such α, by the choice of α^i, we have $\nu(G_{i,\alpha}) \geq 1$, where for a non-zero $F \in \mathbb{K}\{\tilde{x}\}$, $\nu(F) = \max\{r : F \in (\tilde{x})^r\}$. Therefore, for each such α and for every non-zero term $g_{i,\alpha,\gamma}\tilde{x}^\gamma$ of $G_{i,\alpha}$, we have $|\gamma| \geq 1$, and hence $\Lambda((\alpha,\gamma)) \geq l_0 > \Lambda(\beta^i)$. It follows that, with respect to the total ordering in $\mathbb{N}^n$ induced by Λ, we have

$$\exp_\Lambda(G_i) = \beta^i, \quad 1 \leq i \leq t.$$

Pick $F \in I$. By Theorem 2.4.10, there are $Q_1, \dots, Q_t \in \mathbb{K}\{x\}$ such that $F = \sum_{i=1}^t Q_iG_i$ and $\alpha^i + \mathrm{supp}_{\mathbb{N}^k} Q_i \subset \Delta_i$, $1 \leq i \leq t$. Then, $\beta^i + \mathrm{supp}_{\mathbb{N}^n} Q_i \subset \Delta_i \times \mathbb{N}^{n-k}$; in particular, $\beta^i + \exp_\Lambda Q_i \in \Delta_i \times \mathbb{N}^{n-k}$, $1 \leq i \leq t$. It thus follows that the $\exp_\Lambda(Q_iG_i) = \beta^i + \exp_\Lambda Q_i$ lie in pairwise disjoint regions, and hence are pairwise distinct. Consequently, $\exp_\Lambda F = \min_\Lambda\{\exp_\Lambda(Q_iG_i) : i = 1, \dots, t\}$ belongs to $\mathfrak{N}(I(0)) \times \mathbb{N}^{n-k}$, which completes the proof. $\square$

We shall need the following Lemma in the proof of Proposition 2.8.8.

Lemma 2.4.13. *Let I be an ideal in $\mathbb{K}\{x\}$, let $1 \leq k < n$ and let $\mathfrak{n}$ denote the ideal in $\mathbb{K}\{x\}$ generated by $\tilde{x} = (x_{k+1}, \ldots, x_n)$. Then $I \cap \mathfrak{n} = I \cdot \mathfrak{n}$ implies $I \cap \mathfrak{n}^m = I \cdot \mathfrak{n}^m$ for all $m \geq 1$.*

Proof. Suppose $I \cap \mathfrak{n} = I \cdot \mathfrak{n}$, and fix $m \geq 2$. As in the proof of Proposition 2.4.11, we then have that $\mathbb{K}\{x\}/I$ is flat as a $\mathbb{K}\{\tilde{x}\}$-module. Hence by Proposition 2.4.11 (ii), there exists a l_0 such that, for all $l \geq l_0$, the diagram $\mathfrak{N}_\Lambda(I)$ with respect to the linear form $\Lambda(\beta) = \sum_{i=1}^k \beta_i + \sum_{j=k+1}^n l\beta_j$ satisfies $\mathfrak{N}_\Lambda(I) = \mathfrak{N}(I(0)) \times \mathbb{N}^{n-k}$. Fix $G_1, \ldots, G_t \in I$ such that $\exp_\Lambda(G_i) = \beta^i$, where $\beta^i = (\alpha^i, 0) \in \mathbb{N}^k \times 0^{n-k}$, $i = 1, \ldots, t$ are the vertices of $\mathfrak{N}_\Lambda(I)$.

Pick $F \in I \cap \mathfrak{n}^m$, and let $F = \sum_{i=1}^t Q_i G_i$ be the unique Hironaka division of F (with respect to Λ). Set $\gamma^i := \exp_\Lambda(Q_i)$. Note that the Q_i depend only on the partition of $\mathbb{N}^n$ determined by the $\exp_\Lambda(G_i)$. In particular, they are independent of the choice of l, so long as $l \geq l_0$. Therefore by choosing l large enough, we may assume that

$$\text{If } \beta^{i_0} + \gamma^{i_0} = \min_\Lambda\{\beta^i + \gamma^i : 1 \leq i \leq t\} \text{ and } x^{\gamma^{i_0}} \in \mathfrak{n}^s \text{ then } Q_i \in \mathfrak{n}^s \text{ for all } i \quad (2.4.2)$$

Indeed, let s_0 be the least integer such that there exist $1 \leq i^* \leq t$ and $\gamma^* \in \operatorname{supp} Q_{i^*}$ with $x^{\gamma^*} \in \mathfrak{n}^{s_0} \setminus \mathfrak{n}^{s_0+1}$. Pick any i^* and γ^* with these properties. Write $\gamma^* = (\kappa^*, \lambda^*) \in \mathbb{N}^k \times \mathbb{N}^{n-k}$ and set

$$l^* := l_0 + |\kappa^*| + \max_{j=1,\ldots,t} |\alpha^j|, \text{ and } \Lambda^*(\beta) = \sum_{i=1}^k \beta_i + \sum_{j=k+1}^n l^*\beta_j$$

Then, for all $1 \leq i \leq t$, and $\gamma = (\kappa, \lambda) \in \operatorname{supp} Q_i$ with $x^\gamma \in \mathfrak{n}^{s_0+1}$, we have $|\lambda| \geq |\lambda^*| + 1$

and hence

$$\begin{aligned}\Lambda^*(\beta^i+\gamma) = |\alpha^i|+|\kappa|+l^*|\lambda| \geq |\alpha^i|+|\kappa|+l^*(|\lambda^*|+1)\\ = |\alpha^i|+|\kappa|+l_0+\max_j|\alpha^j|+|\kappa^*|+l^*|\lambda^*| > \Lambda(\beta^{i^*}+\gamma^*).\end{aligned}$$

This proves that the minimum $\min_{\Lambda^*}\{\beta^i+\gamma^i : 1\leq i\leq t\}$ is attained for some γ^i with $x^{\gamma^i}\in\mathfrak{n}^{s_0}$.

Now the fact that $\exp_\Lambda(F) = \min_\Lambda\{\exp_\Lambda G_iQ_i : 1\leq i\leq t\}$, together with $F\in\mathfrak{n}^m$ and (2.4.2), imply that $Q_i\in\mathfrak{n}^m$ for all i, and so $F\in I\cdot\mathfrak{n}^m$. □

Hironaka's division theorem can also be used to give a simple proof of the following classical result.

Theorem 2.4.14. *(Weierstrass Finiteness Theorem) Let* $\mathbb{K}=\mathbb{C}$ *and let* I *be an ideal in* $A\{x\}$*, where* $x=(x_1,\ldots,x_m)$*. Then* $A\{x\}/I$ *is a finitely generated* A*-module if and only if* $\dim_{\mathbb{C}}(\mathbb{C}\{x\}/I(0))<\infty$*.*

Proof. If $A\{x\}/I$ is finitely generated over A then $\dim_{\mathbb{C}}(\mathbb{C}\{x\}/I(0)) = \dim_{\mathbb{C}}(\frac{A\{x\}}{I}\otimes_A \frac{A}{\mathfrak{m}_A})<\infty$, by Nakayama's Lemma (Theorem A.0.7). Conversely, suppose that $\dim_{\mathbb{C}}(\mathbb{C}\{x\}/I(0))<\infty$. Let $G_1,\ldots,G_t$ be representatives of the vertices of the diagram $\mathfrak{N}(I(0))$; i.e. $G_1,\ldots,G_t\in I$ are such that $\exp(G_j(0))=\beta^j$, where $\beta^1,\ldots,\beta^t$ are the vertices of the diagram $\mathfrak{N}(I(0))$. Let $\{\Delta,\Delta_1,\ldots,\Delta_t\}$ be the decomposition of $\mathbb{N}^m$ determined by $\beta^1,\ldots,\beta^t$. Then, by Theorem 2.4.5, for every $F\in A\{x\}$, there are $Q_1,\ldots,Q_t,R\in A\{x\}$ such that $F=\sum_{j=1}^t Q_jG_j+R$ and $\operatorname{supp}(R)\subset\Delta$. On the other hand, the condition $\dim_{\mathbb{C}}(\mathbb{C}\{x\}/I(0))<\infty$ means that Δ consists of finitely many points, say $\gamma^1,\ldots,\gamma^s$. Thus every $R\in A\{x\}$ with $\operatorname{supp}(R)\subset\Delta$ is generated by the monomials $x^{\gamma^1},\ldots x^{\gamma^s}$. Hence, modulo I, every $F\in A\{x\}$ is generated over A by those finitely many monomials, which completes the proof. □

Remark 2.4.15. *Theorem 2.4.14 is also valid when* A *is a local analytic* $\mathbb{C}$*-algebra. That is when* A *is the quotient of* $\mathbb{C}\{y\}$ *by some ideal.*

2.5 Standard bases and Becker's s-series criterion

Let again $\Lambda(\beta) = \sum_{j=1}^{n} \lambda_j \beta_j$ be a positive linear form on $\mathbb{K}^n$, and let $\mathbb{N}^n$ be given the total ordering defined by the lexicographic ordering of the $(n{+}1)$-tuples $(\Lambda(\beta), \beta_n, \dots, \beta_1)$. For $F \in \mathbb{K}\{x\}$, let as before $\exp_\Lambda F = \min_\Lambda\{\beta \in \mathrm{supp} F\}$ denote the initial exponent of F relative to Λ.

The following definition is due to Becker [5].

Definition 2.5.1. *Let I be an ideal in $\mathbb{K}\{x\}$. A collection $S = \{G_1, \dots, G_t\} \subset I$ forms a* standard basis *of I (relative to Λ), when for every $F \in I$ there exists $i \in \{1, \dots, t\}$ such that $\exp_\Lambda F \in \exp_\Lambda G_i + \mathbb{N}^n$.*

Remark 2.5.2. *(1) It follows directly from definition that every standard basis S of I relative to Λ contains representatives of all the vertices of the diagram $\mathfrak{N}_\Lambda(I)$ (that is, for every vertex β^i of $\mathfrak{N}_\Lambda(I)$ there exists $G_i \in S$ with $\beta^i = \exp_\Lambda G_i$). Hence, by Corollary 2.4.8, every standard basis of I is a set of generators of I.*

(2) Note that the term "standard basis" in most of modern literature refers to a collection defined by more restrictive conditions than the one above (see, e.g., [7, Cor. 3.9] or [10, Cor. 3.19]). In particular, our standard basis is not unique and may contain elements which do not represent vertices of the diagram.

Becker in [5] also defines the following related to standard bases,

Definition 2.5.3. *For any $F = \sum_\beta f_\beta x^\beta$ and $G = \sum_\beta g_\beta x^\beta$ in $\mathbb{K}\{x\}$, one defines their* s-series *$S(F, G)$ with respect to Λ as follows: If $\beta_F = \exp_\Lambda F$, $\beta_G = \exp_\Lambda G$, and $x^\gamma = \mathrm{lcm}(x^{\beta_F}, x^{\beta_G})$, then*

$$S(F, G) \coloneqq g_{\beta_G} x^{\gamma - \beta_F} \cdot F - f_{\beta_F} x^{\gamma - \beta_G} \cdot G \,.$$

Definition 2.5.4. *Given $G_1, \dots, G_t, F \in \mathbb{K}\{x\}$, we say that F has a* standard representation *in terms of $\{G_1, \dots, G_t\}$ with respect to Λ, when there exist $Q_1, \dots, Q_t \in$*

$\mathbb{K}\{x\}$ *such that*

$$F = \sum_{i=1}^{t} Q_i G_i \qquad \text{and} \qquad \exp_\Lambda F \leq \min\{\exp_\Lambda(Q_i G_i) : i = 1, \dots, t\}\,.$$

Here, we adopt a convention that $\exp_\Lambda F < \exp_\Lambda 0$, for any Λ and any non-zero F.

The following s-series criterion of Becker will be our main tool in establishing Hilbert-Samuel equisingularity.

Theorem 2.5.5 ([5, Thm. 4.1])**.** *Let S be a finite subset of $\mathbb{K}\{x\}$. Then, S is a standard basis (relative to Λ) of the ideal it generates if and only if for any $G_1, G_2 \in S$ the s-series $S(G_1, G_2)$ has a standard representation in terms of S.*

Remark 2.5.6. *(1) The notion of s-series as defined in Definition 2.5.3 is analogous to the notion of* s-polynomials *in the context of Gröbner bases for ideals in polynomial rings [13, §6, Definition 4 (ii)].*

(2) The criterion in Theorem 2.5.5 is analogous to Buchberger's criterion for Gröbner bases [13, §6, Theorem 6].

2.6 Nested parametrized algebraic approximation

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$, and let $\mathbb{K}\langle x\rangle$ denote the ring of algebraic power series in x. Recall that a convergent power series $F \in \mathbb{K}\{x\}$ is called an *algebraic power series* when F is algebraic over the ring of polynomials $\mathbb{K}[x]$.

The following nested variant of Płoski's parametrized approximation theorem [27] is due to Bilski, Parusiński and Rond [12]. The result itself follows from Spivakovsky's nested approximation [30, Thm. 11.4], which in turn is a corollary of the Néron-Popescu Desingularization [28].

Theorem 2.6.1 ([12, Thm. 2.1])**.** *Let* $f(x,y) = (f_1(x,y), \ldots, f_p(x,y)) \in \mathbb{K}\langle x\rangle[y]^p$ *and let* $\bar{y}(x) = (\bar{y}_1(x), \ldots, \bar{y}_m(x)) \in \mathbb{K}\{x\}^m$ *be such that*

$$f(x, \bar{y}(x)) = 0.$$

Suppose that $\bar{y}_i(x)$ *depends only on variables* $(x_1, \ldots, x_{\sigma(i)})$*, where* $\{i \mapsto \sigma(i)\}$ *is an increasing function. Then, there exist a new set of variables* $z = (z_1, \ldots, z_s)$*, an increasing function* τ*, an* m*-tuple of algebraic power series* $\hat{y}(x,z) \in \mathbb{K}\left\langle x, z\right\rangle^m$ *such that*

$$f(x, \hat{y}(x,z)) = 0,$$

and for every i*,*

$$\hat{y}_i(x,z) \in \mathbb{K}\left\langle x_1, \ldots, x_{\sigma(i)}, z_1, \ldots, z_{\tau(i)}\right\rangle,$$

as well as convergent power series $\bar{z}_i(x) \in \mathbb{K}\{x\}$ *vanishing at* 0 *such that* $\bar{z}_1(x), \ldots,$ $\bar{z}_{\tau(i)}(x)$ *depend only on* $(x_1, \ldots, x_{\sigma(i)})$ *and*

$$\bar{y}(x) = \hat{y}(x, \bar{z}(x)).$$

The classical Algebraic Artin Approximation follows immediately from the above.

Theorem 2.6.2 ([3, Thm. 1.10])**.** *Let* $f(x,y) \in \mathbb{K}\langle x\rangle[y]^p$ *and let* $\bar{y}(x) \in \mathbb{K}\{x\}^m$ *be such that*

$$f(x, \bar{y}(x)) = 0.$$

Then, for any $c \in \mathbb{N}$*, there exists an* m*-tuple of algebraic power series* $\hat{y}(x) \in \mathbb{K}\left\langle x\right\rangle^m$ *such that*

$$f(x, \hat{y}(x)) = 0,$$

and $\hat{y}$ *coincides with* $\bar{y}$ *up to degree* c*, that is,* $\bar{y}(x) - \hat{y}(x) \in (x)^{c+1}$*.*

Proof. Let $\hat{y}(x,z) \in \mathbb{K}\left\langle x, z\right\rangle^m$ and the $\bar{z}_i(x) \in \mathbb{K}\{x\}$ be as in Theorem 2.6.1. Then, for a given $c \in \mathbb{N}$, the m-tuple $\hat{y}(x, j^c(\bar{z}(x))$ has the desired properties, where $j^c F$ denotes the c-jet of $F \in \mathbb{K}\{x\}$. $\square$

2.7 Reduction of the maximal ideal

Let, as before, $x = (x_1, \ldots, x_n)$, and for $k < n$, let $x_{[k]} = (x_1, \ldots, x_k)$ and $\tilde{x} = (x_{k+1}, \ldots, x_n)$. Let $\mathfrak{m}$ denote the maximal ideal of $\mathbb{K}\{x\}$. The purpose of this section is to find a suitable reduction (in the sense of Northcott-Rees [24]) of the maximal ideal $\mathfrak{m}/I$ in $\mathbb{K}\{x\}/I$, for a given ideal I in $\mathbb{K}\{x\}$.

The following is a consequence of Proposition 2.1.26.

Proposition 2.7.1. *Let I be a proper ideal in $\mathbb{K}\{x\}$. After a generic linear change of coordinates in $\mathbb{K}^n$, there exists $k \in \{0, \ldots, n-1\}$ such that the natural homomorphism $\mathbb{K}\{\tilde{x}\} \to \mathbb{K}\{x\}/I$ is injective and makes $\mathbb{K}\{x\}/I$ into a finite $\mathbb{K}\{\tilde{x}\}$-module.*

Lemma 2.7.2. *Let I be an ideal in $\mathbb{K}\{x\}$ with $\dim \mathbb{K}\{x\}/I = n - k$. Then, after a generic linear change of coordinates in $\mathbb{K}^n$, there is, for every $j = 1, \ldots, k$, a distinguished polynomial $P_j \in \mathbb{K}\{\tilde{x}\}[t]$ of degree d_j such that $P_j(x_j, \tilde{x}) \in I \cap \mathfrak{m}^{d_j}$, where $\tilde{x} = (x_{k+1}, \ldots, x_n)$.*

Proof. By Proposition 2.7.1, after a generic linear change of coordinates in $\mathbb{K}^n$, there exists $k' \leq n - 1$ and an injective homomorphism $\mathbb{K}\{\tilde{x}\} \to \mathbb{K}\{x\}/I$ such that $\mathbb{K}\{x\}/I$ is a finite $\mathbb{K}\{\tilde{x}\}$-module, where $\tilde{x} = (x_{k'+1}, \ldots, x_n)$. Since for a finite injective homomorphism of Noetherian rings $A \to R$ we have $\dim R = \dim A$, it follows that $k' = k$.

Suppose first that $\mathbb{K} = \mathbb{C}$. Let X_0 be the germ of an analytic space at 0 in $\mathbb{C}^n$ defined by $\mathcal{O}_{X,0} = \mathbb{C}\{x\}/I$. Further, let $C(X, 0)$ denote the tangent cone to X_0, in the sense of Whitney (Definition 2.3.2). Then, $\dim_0 C(X, 0) = \dim_0 X = n-k$, and after another generic linear change of coordinates if needed, we may assume that $C(X, 0)$

has a proper and surjective projection onto (an open neighborhood of 0 in) $\mathbb{C}^{n-k}$ spanned by the variables $\tilde{x}$. Finiteness of $\mathbb{C}\{x\}/I$ as a $\mathbb{C}\{\tilde{x}\}$-module implies that the images of $x_1, \ldots, x_k$ in $\mathbb{C}\{x\}/I$ are integral over $\mathbb{C}\{\tilde{x}\}$. Hence, for every $j = 1, \ldots, k$, there exist $d_j \in \mathbb{Z}_+$ and a distinguished polynomial $P_j \in \mathbb{C}\{\tilde{x}\}[t]$ of degree d_j, such that $P_j(x_j, \tilde{x}) \in I$. Write $P_j(x_j, \tilde{x}) = x_j^{d_j} + \sum_{r=1}^{d_j} a_r^j(\tilde{x}) x_j^{d_j - r}$, $j = 1, \ldots, k$.

Fix $j \in \{1, \ldots, k\}$. Let $LF(P_j)$ denote the leading form of P_j (i.e., the homogeneous polynomial consisting of the terms of P_j of lowest degree). By Theorem 2.3.4, $C(X, 0)$ is the set of common zeroes of leading forms $LF(F)$ for all F_0 vanishing on X_0. In particular, $C(X, 0) \subset LF(P_j)^{-1}(0)$.

To prove the lemma, it now suffices to show that $x_j^{d_j}$ is among the terms of $LF(P_j)$. We argue by induction on $n - k$, the number of variables $\tilde{x}$. If $n - k = 1$, then $\tilde{x}$ is a single variable x_n. If $x_j^{d_j}$ were not among the terms of $LF(P_j)$ then x_n would divide $LF(P_j)$, and so the image of $C(X, 0)$ under the projection to $\mathbb{C}^{n-k}$ would be $\{0\} = LF(P_j)^{-1}(0) \cap \{x_{[k]} = 0\}$, contradicting the surjectivity.

Suppose then that $n - k \geq 2$, and consider $\widetilde{P}_j := P_j(x_j, x_{k+1}, \ldots, x_{n-1}, 0)$. Then, $\widetilde{P}_j$ vanishes on $\widetilde{X} := X \cap \{x_n = 0\}$, and hence $LF(\widetilde{P}_j)$ vanishes on $C(\widetilde{X}, 0)$. Since $C(\widetilde{X}, 0)$ has a surjective projection onto (an open neighbourhood of 0 in) $\mathbb{C}^{n-k-1}$, then, by induction, $x_j^{d_j}$ is among the terms of $LF(\widetilde{P}_j)$. If $x_j^{d_j}$ were not among the terms of $LF(P_j)$, then we would have $\deg LF(P_j) < \deg LF(\widetilde{P}_j) = d_j$. Hence, x_n would divide $LF(P_j)$, and so the image of $C(X, 0)$ under projection to $\mathbb{C}^{n-k}$ would be contained in the hypersurface $\{x_n = 0\}$. This contradiction completes the proof in case $\mathbb{K} = \mathbb{C}$.

If $\mathbb{K} = \mathbb{R}$, the result follows by applying the above argument to the complexification $X^{\mathbb{C}}$ of X. Note that the linear changes of coordinates required at the beginning may be taken with integral coefficients, and hence the distinguished polynomials P_j will have real coefficients. $\square$

Let $P_j(x_j, \tilde{x}) = x_j^{d_j} + \sum_{r=1}^{d_j} a_r^j(\tilde{x}) x_j^{d_j - r}$, $j = 1, \dots, k$, be as above. Set

$$d := \sum_{j=1}^{k} (d_j - 1).$$

Corollary 2.7.3. *We have* $(x_{[k]})^{d+1} \subset I + (\tilde{x}) \cdot \mathfrak{m}^d$, *as ideals in* $\mathbb{K}\{x\}$.

Proof. Indeed, for any monomial $x_1^{\beta_1} \cdots x_k^{\beta_k} \in (x_{[k]})^{d+1}$, there exists j such that $\beta_j \geq d_j$. By Lemma 2.7.2, $x_j^{d_j} = P_j(x_j, \tilde{x}) - \sum_{r=1}^{d_j} a_r^j(\tilde{x}) x_j^{d_j - r}$ is an element of $I + (\tilde{x}) \cap \mathfrak{m}^{d_j} = I + (\tilde{x}) \cdot \mathfrak{m}^{d_j - 1}$. Consequently, $x_1^{\beta_1} \dots x_k^{\beta_k} \in I + (\tilde{x}) \cdot \mathfrak{m}^N$, where $N = \beta_1 + \dots + (\beta_j - 1) + \dots + \beta_k \geq d$. □

Remark 2.7.4. *The above corollary implies that* $I + (\tilde{x})/I$ *is a* reduction *(with exponent* d*) of the maximal ideal* $\mathfrak{m}/I$ *in* $\mathbb{K}\{x\}/I$, *in the sense of Northcott–Rees [24]. Indeed, one trivially has* $I + (\tilde{x}) \subset I + \mathfrak{m}$, *and by above,* $I + \mathfrak{m}^{d+1} \subset I + (\tilde{x}) \cdot \mathfrak{m}^d$. *It follows that* $I + \mathfrak{m}^{d+1} = I + (\tilde{x}) \cdot \mathfrak{m}^d$, *hence by induction*

$$I + \mathfrak{m}^{d+m} = I + (\tilde{x})^m \mathfrak{m}^d, \quad \text{for any } m \geq 1. \tag{2.7.1}$$

2.8 Approximation of ideals and diagrams

Let, as before, $\Lambda(\beta) = \sum_{j=1}^{n} \lambda_j \beta_j$ be a positive linear form on $\mathbb{K}^n$. For such Λ and $\mu \in \mathbb{N}$, define $\mathfrak{n}_{\Lambda,\mu}$ to be the ideal in $\mathbb{K}\{x\}$ generated by all the monomials $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$ with $\Lambda(\beta) \geq \mu$. (Note that, by positivity of the linear form Λ, the ideals $\mathfrak{n}_{\Lambda,\mu}$ are $\mathfrak{m}$-primary for every μ. Moreover, for $\Lambda(\beta) = |\beta|$ we have $\mathfrak{n}_{\Lambda,\mu} = \mathfrak{m}^\mu$.)

Definition 2.8.1. *For a natural number* $\mu \in \mathbb{N}$ *and a power series* $F \in \mathbb{K}\{x\}$, *the* μ-jet of F with respect to Λ, *denoted* $j^\mu_\Lambda(F)$, *is the image of* F *under the canonical epimorphism* $\mathbb{K}\{x\} \to \mathbb{K}\{x\}/\mathfrak{n}_{\Lambda,\mu+1}$.

In this thesis we will write $j^\mu(F)$ for μ-jets with respect to $\Lambda(\beta) = |\beta| = \beta_1 + \dots + \beta_n$.

Remark 2.8.2. *Given a power series* $F \in \mathbb{K}\{x\}$*, suppose that* $\mu \geq \Lambda(\exp_\Lambda(F))$*. Then,* $\exp_\Lambda(F) = \exp_\Lambda(G)$ *for every* $G \in \mathbb{K}\{x\}$ *with* $j^\mu_\Lambda(G) = j^\mu_\Lambda(F)$*.*

The following lemma expresses the Hilbert-Samuel function of an ideal in terms of its diagram of initial exponents.

Lemma 2.8.3. *Let* $\lambda_1, \ldots, \lambda_n > 0$ *be arbitrary, and let* $\Lambda(\beta) = \sum_{j=1}^n \lambda_j \beta_j$*. Then, for any ideal* I *in* $\mathbb{K}\{x\}$ *and for every* $\eta \geq 1$*,*

$$\#\{\beta \in \mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I) : \Lambda(\beta) \leq \eta\} \;=\; \dim_{\mathbb{K}} \frac{\mathbb{K}\{x\}}{I + \mathfrak{n}_{\Lambda,\eta+1}}\,,$$

where the dimension on the right side is in the sense of $\mathbb{K}$*-vector spaces. In particular, the Hilbert-Samuel function* H_I *of* $\mathbb{K}\{x\}/I$ *satisfies*

$$H_I(\eta) = \#\{\beta \in \mathbb{N}^n \setminus \mathfrak{N}(I) : |\beta| \leq \eta\}, \quad \text{for all } \eta \geq 1\,.$$

Proof. Fix $\eta \geq 1$. Suppose that $F \in \mathbb{K}\{x\}$ satisfies $\mathrm{supp}(F) \subset \{\beta \in \mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I) : \Lambda(\beta) \leq \eta\}$ and pick $G \in \mathfrak{n}_{\Lambda,\eta+1}$. Then, $\exp_\Lambda(F+G) = \exp_\Lambda(F)$, by Remark 2.8.2, and hence $\exp_\Lambda(F+G) \notin \mathfrak{N}_\Lambda(I)$. It follows that $F + G \notin I$, and so $F \notin I + \mathfrak{n}_{\Lambda,\eta+1}$. This proves that the set of monomials $\{x^\beta : \beta \in \mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I), \Lambda(\beta) \leq \eta\}$ is linearly independent in $\mathbb{K}\{x\}/(I + \mathfrak{n}_{\Lambda,\eta+1})$, whence

$$\dim_{\mathbb{K}} \frac{\mathbb{K}\{x\}}{I + \mathfrak{n}_{\Lambda,\eta+1}} \;\geq\; \#\{\beta \in \mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I) : \Lambda(\beta) \leq \eta\}\,.$$

Conversely, suppose that $F \notin I + \mathfrak{n}_{\Lambda,\eta+1}$. Let $G_1, \ldots, G_t \in I$ be representatives of the vertices of $\mathfrak{N}_\Lambda(I)$ and let $F = \sum_{i=1}^t Q_i G_i + R$ be the unique Hironaka division of F by the G_i in $\mathbb{K}\{x\}$, relative to Λ. Now, if $R \in \mathfrak{n}_{\Lambda,\eta+1}$ then $F \in I + \mathfrak{n}_{\Lambda,\eta+1}$; a contradiction. Therefore, we have $R = R_1 + R_2$, with $R_2 \in \mathfrak{n}_{\Lambda,\eta+1}$, $R_1 \neq 0$, and $\mathrm{supp}(R_1) \subset \{\beta \in \mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I) : \Lambda(\beta) \leq \eta\}$ (cf. Theorem 2.4.5). Then, $F - R_1 = \sum_{i=1}^t Q_i G_i + R_2$ is in $I + \mathfrak{n}_{\Lambda,\eta+1}$, which shows that F and R_1 represent the same element of $\mathbb{K}\{x\}/(I+\mathfrak{n}_{\Lambda,\eta+1})$. Thus,

$$\dim_{\mathbb{K}} \frac{\mathbb{K}\{x\}}{I+\mathfrak{n}_{\Lambda,\eta+1}} \ \le\ \#\{\beta\in\mathbb{N}^n\setminus\mathfrak{N}_\Lambda(I) : \Lambda(\beta)\le\eta\}\,.$$

The last claim of the lemma now follows from the definition of the Hilbert-Samuel function as $H_I(\eta)=\dim_{\mathbb{K}}\mathbb{K}\{x\}/(I+\mathfrak{m}^{\eta+1})$. □

Definition 2.8.4. *For an ideal $I=(F_1,\dots,F_s)\cdot\mathbb{K}\{x\}$, a positive linear form Λ and $\mu\ge 1$, we define the family of ideals $U^\mu_\Lambda(I)$ (or, more precisely, $U^\mu_\Lambda(F_1,\dots,F_s)$) as*

$$U^\mu_\Lambda(I)=\{(G_1,\dots,G_s)\cdot\mathbb{K}\{x\} : j^\mu_\Lambda(G_i)=j^\mu_\Lambda(F_i), 1\le i\le s\}.$$

We will write simply $U^\mu(I)$ for $U^\mu_\Lambda(I)$, when $\Lambda(\beta)=|\beta|$.

The following lemma shows that the reduction of the maximal ideal in $\mathbb{K}\{x\}/I$ is preserved by its sufficiently close Taylor approximations.

Lemma 2.8.5. *Let $I=(F_1,\dots,F_s)$ be an ideal in $\mathbb{K}\{x\}$ with* $\dim\mathbb{K}\{x\}/I=n-k$. *Then, after a generic linear change of coordinates in $\mathbb{K}^n$, there exists μ_0 such that, for every $\mu\ge\mu_0$ and $I_\mu\in U^\mu(I)$, we have*

$$I_\mu+\mathfrak{m}^{d+m} \ =\ I_\mu+(\tilde{x})^m\mathfrak{m}^d,\quad \text{for any } m\ge 1, \tag{2.8.1}$$

where d is the same as in (2.7.1).

Proof. After a generic linear change of coordinates from Lemma 2.7.2, we may assume that $(x_{[k]})^{d+1}\subset I+(\tilde{x})\cdot\mathfrak{m}^d$, where d is as in (2.7.1). Set $\mu_0:=d+1$. Pick $\mu\ge\mu_0$ and $I_\mu\in U^\mu(I)$. Then, $I\subset I_\mu+\mathfrak{m}^{d+2}$, and hence $(x_{[k]})^{d+1}\subset I_\mu+\mathfrak{m}^{d+2}+(\tilde{x})\mathfrak{m}^d$. It follows that

$$I_\mu+\mathfrak{m}^{d+1}\subset I_\mu+\mathfrak{m}^{d+2}+(\tilde{x})\mathfrak{m}^d\subset I_\mu+(\tilde{x})\mathfrak{m}^d+(I_\mu+\mathfrak{m}^{d+1})\mathfrak{m}\,,$$

hence $I_\mu + \mathfrak{m}^{d+1} \subset I_\mu + (\tilde{x})\mathfrak{m}^d$, by Nakayama's lemma. The claim now follows as in Remark 2.7.4. □

Let us recall now a results from [2] describing the connection between the diagram of initial exponents of I and those of its approximations I_μ. We include a short proof for the reader's convenience.

Lemma 2.8.6 (cf. [2, Lem. 3.2]). *Let I be an ideal in $\mathbb{K}\{x\}$ and let Λ be a positive linear form on $\mathbb{K}^n$. Let $l_0 = \max\{\Lambda(\beta^i) : 1 \le i \le t\}$, where $\beta^1, \dots, \beta^t$ are the vertices of the diagram $\mathfrak{N}_\Lambda(I)$. Then:*

(i) For every $\mu \ge l_0$ and $I_\mu \in U^\mu_\Lambda(I)$, we have $\mathfrak{N}_\Lambda(I_\mu) \supset \mathfrak{N}_\Lambda(I)$.

(ii) Given $l \ge l_0$, for every $\mu \ge l$ and $I_\mu \in U^\mu_\Lambda(I)$, we have

$$\mathfrak{N}_\Lambda(I_\mu) \cap \{\beta \in \mathbb{N}^n : \Lambda(\beta) \le l\} = \mathfrak{N}_\Lambda(I) \cap \{\beta \in \mathbb{N}^n : \Lambda(\beta) \le l\}\,.$$

Proof. Fix $\mu \ge l_0$ and let $G_1, \dots, G_s \in \mathbb{K}\{x\}$ be such that $I_\mu = (G_1, \dots, G_s)$ and $j^\mu_\Lambda(G_i) = j^\mu_\Lambda(F_i)$, $1 \le i \le s$. By Remark 2.4.7, for the proof of (i) it suffices to show that the vertices of $\mathfrak{N}_\Lambda(I)$ are contained in $\mathfrak{N}_\Lambda(I_\mu)$. Let then $F \in I$ be a representative of a vertex of $\mathfrak{N}_\Lambda(I)$ (i.e., $\exp_\Lambda(F) \in V_\Lambda(I)$). We can write $F = \sum_{i=1}^s H_i F_i$, for some $H_i \in \mathbb{K}\{x\}$. Then,

$$j^\mu_\Lambda(F) = j^\mu_\Lambda(\sum_{i=1}^s H_i F_i) = j^\mu_\Lambda(\sum_{i=1}^s H_i \cdot j^\mu_\Lambda F_i) = j^\mu_\Lambda(\sum_{i=1}^s H_i \cdot j^\mu_\Lambda G_i) = j^\mu_\Lambda(\sum_{i=1}^s H_i G_i)\,,$$

and hence, by Remark 2.8.2, we have $\exp_\Lambda(F) = \exp_\Lambda(\sum_{i=1}^s H_i G_i)$. It follows that $\exp_\Lambda(F) \in \mathfrak{N}_\Lambda(I_\mu)$, which proves (i).

For the proof of part (ii), fix $l \ge l_0$. Let $\mu \ge l$ and let $I_\mu = (G_1, \dots, G_s)$ with $j^\mu_\Lambda(G_i) = j^\mu_\Lambda(F_i)$, $1 \le i \le s$. By part (i), it now suffices to show that

$$\mathfrak{N}_\Lambda(I_\mu) \cap \{\beta \in \mathbb{N}^n : \Lambda(\beta) \le l\} \;\subset\; \mathfrak{N}_\Lambda(I) \cap \{\beta \in \mathbb{N}^n : \Lambda(\beta) \le l\}\,.$$

Pick $\beta^* \in \mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I)$ with $\Lambda(\beta^*) \leq l$. Suppose that $\beta^* \in \mathfrak{N}_\Lambda(I_\mu)$. Then, one can choose $G \in I_\mu$ with $\exp_\Lambda(G) = \beta^*$. Write $G = \sum_{i=1}^s H_i G_i$ for some $H_i \in \mathbb{K}\{x\}$. We have

$$j^\mu_\Lambda(G) = j^\mu_\Lambda(\sum_{i=1}^s H_i G_i) = j^\mu_\Lambda(\sum_{i=1}^s H_i \cdot j^\mu_\Lambda G_i) = j^\mu_\Lambda(\sum_{i=1}^s H_i \cdot j^\mu_\Lambda F_i) = j^\mu_\Lambda(\sum_{i=1}^s H_i F_i)\,,$$

and since $\mu \geq l \geq \Lambda(\exp_\Lambda(G))$, it follows that $\exp_\Lambda(G) = \exp_\Lambda(\sum_{i=1}^s H_i F_i)$, by Remark 2.8.2 again. Therefore $\beta^* \in \mathfrak{N}_\Lambda(I)$; a contradiction. □

Corollary 2.8.7. *Let I be an ideal in $\mathbb{K}\{x\}$ and let Λ be a positive linear form on $\mathbb{K}^n$. Suppose that the complement $\mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I)$ is finite. Then, there exists $\mu_0 \in \mathbb{N}$ such that, for every $\mu \geq \mu_0$ and $I_\mu \in U^\mu_\Lambda(I)$, we have $\mathfrak{N}_\Lambda(I_\mu) = \mathfrak{N}_\Lambda(I)$.*

Proof. Let $\beta^1, \dots, \beta^t$ be the vertices of $\mathfrak{N}_\Lambda(I)$. Since $\mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I)$ is finite, there exists $\mu_0 \geq \max_i \Lambda(\beta^i)$ such that

$$\mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I) \subset \{\beta \in \mathbb{N}^n : \Lambda(\beta) \leq \mu_0\}\,.$$

By Lemma 2.8.6 part (i), for every $\mu \geq \mu_0$ and $I_\mu \in U^\mu_\Lambda(I)$, we have

$$\mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I_\mu) \subset \mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I)$$

and by part (ii)

$$(\mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I_\mu)) \cap \{\beta \in \mathbb{N}^n : \Lambda(\beta) \leq \mu_0\} = (\mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I)) \cap \{\beta \in \mathbb{N}^n : \Lambda(\beta) \leq \mu_0\}\,.$$

Thus, $\mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I_\mu) = \mathbb{N}^n \setminus \mathfrak{N}_\Lambda(I)$, as required. □

The following proposition is a key tool in the proofs of Theorems 3.1.3 and 3.2.1. It shows that the equality of diagrams of an ideal I and its approximation I_μ with respect to some ordering on $\mathbb{N}^n$ implies the equality of diagrams with respect to the standard ordering.

Proposition 2.8.8. *Let $I = (F_1, \ldots, F_s)$ be an ideal in $\mathbb{K}\{x\}$ with $\dim \mathbb{K}\{x\}/I = n - k$. Then, after a generic linear change of coordinates in $\mathbb{K}^n$, there exists μ_0 such that, for every $\mu \geq \mu_0$ the following holds: If $I_\mu \in U^\mu(I)$ is such that $I_\mu \cap (\tilde{x}) = I_\mu \cdot (\tilde{x})$ and $\dim_{\mathbb{K}} \mathbb{K}\{x\}/I + (\tilde{x})^m = \dim_{\mathbb{K}} \mathbb{K}\{x\}/I_\mu + (\tilde{x})^m$ for all $m \geq 1$, then $H_I = H_{I_\mu}$ (that is, the Hilbert-Samuel functions of I and I_μ coincide).*

Proof. To simplify notation, we shall write $\mathfrak{n}$ for the ideal $(\tilde{x})$ in $\mathbb{K}\{x\}$. By Lemma 2.7.2, Remark 2.7.4 and Lemma 2.8.5, after a generic linear change of coordinates in $\mathbb{K}^n$, we may assume that there exist positive integers d and μ_0 such that

$$I + \mathfrak{m}^{d+m} = I + \mathfrak{n}^m \mathfrak{m}^d, \text{ for all } m \geq 1\,, \tag{2.8.2}$$

and for every $\mu \geq \mu_0$ and $I_\mu \in U^\mu(I)$

$$I_\mu + \mathfrak{m}^{d+m} = I_\mu + \mathfrak{n}^m \mathfrak{m}^d, \text{ for all } m \geq 1\,. \tag{2.8.3}$$

By Lemmas 2.8.3 and 2.8.6, taking μ_0 sufficiently large, we always have $H_I(\eta) \geq H_{I_\mu}(\eta)$ for all $\eta \geq 1$ and $\mu \geq \mu_0$. Moreover by Lemma 2.8.6(ii), requiring further that $\mu_0 \geq d$ ensures that $H_I(\eta) = H_{I_\mu}(\eta)$ for all $\mu \geq \mu_0$ and $\eta \leq d$. Therefore, to prove the equality $H_I = H_{I_\mu}$ it suffices to show that $H_I(\eta) = H_{I_\mu}(\eta)$ for $\eta \geq d$, or equivalently that

$$\dim_{\mathbb{K}} \frac{\mathbb{K}\{x\}}{I + \mathfrak{n}^m \mathfrak{m}^d} = \dim_{\mathbb{K}} \frac{\mathbb{K}\{x\}}{I_\mu + \mathfrak{n}^m \mathfrak{m}^d} \qquad \text{for all } m \geq 1. \tag{2.8.4}$$

Fix $\mu \geq \mu_0$ and $I_\mu \in U^\mu(I)$. We have, for $m \geq 1$, the following exact sequences

$$0 \to \frac{I+\mathfrak{n}^m}{I+\mathfrak{n}^m\mathfrak{m}^d} \to \frac{\mathbb{K}\{x\}}{I+\mathfrak{n}^m\mathfrak{m}^d} \to \frac{\mathbb{K}\{x\}}{I+\mathfrak{n}^m} \to 0\,,$$
$$0 \to \frac{I_\mu+\mathfrak{n}^m}{I_\mu+\mathfrak{n}^m\mathfrak{m}^d} \to \frac{\mathbb{K}\{x\}}{I_\mu+\mathfrak{n}^m\mathfrak{m}^d} \to \frac{\mathbb{K}\{x\}}{I_\mu+\mathfrak{n}^m} \to 0\,.$$

By assumption, $\dim_{\mathbb{K}} \mathbb{K}\{x\}/(I+\mathfrak{n}^m) = \dim_{\mathbb{K}} \mathbb{K}\{x\}/(I_\mu+\mathfrak{n}^m)$, and hence to prove (2.8.4), it suffices to show that

$$\dim_{\mathbb{K}} \frac{I+\mathfrak{n}^m}{I+\mathfrak{n}^m\mathfrak{m}^d} = \dim_{\mathbb{K}} \frac{I_\mu+\mathfrak{n}^m}{I_\mu+\mathfrak{n}^m\mathfrak{m}^d} \qquad \text{for all } m \geq 1.$$

Note that

$$\frac{I+\mathfrak{n}^m}{I+\mathfrak{n}^m\mathfrak{m}^d} \cong \frac{\mathfrak{n}^m}{(I+\mathfrak{n}^m\mathfrak{m}^d)\cap\mathfrak{n}^m} = \frac{\mathfrak{n}^m}{(I\cap\mathfrak{n}^m)+\mathfrak{n}^m\mathfrak{m}^d} \tag{2.8.5}$$

and

$$\frac{I_\mu+\mathfrak{n}^m}{I_\mu+\mathfrak{n}^m\mathfrak{m}^d} \cong \frac{\mathfrak{n}^m}{(I_\mu+\mathfrak{n}^m\mathfrak{m}^d)\cap\mathfrak{n}^m} = \frac{\mathfrak{n}^m}{(I_\mu\cap\mathfrak{n}^m)+\mathfrak{n}^m\mathfrak{m}^d}\,. \tag{2.8.6}$$

Let λ be the Artin-Rees exponent of I relative to $\mathfrak{n}$. That is, we have $I\cap\mathfrak{n}^m = (I\cap\mathfrak{n}^\lambda)\mathfrak{n}^{m-\lambda}$ for all $m \geq \lambda$. For the remainder of the proof we are going to assume that $\mu_0 \geq d+\lambda$. Then, $I_\mu \subset I+\mathfrak{m}^{\mu+1} \subset I+\mathfrak{n}^\lambda\mathfrak{m}^d$, by (2.8.2), and conversely, $I \subset I_\mu+\mathfrak{m}^{\mu+1} \subset I_\mu+\mathfrak{n}^\lambda\mathfrak{m}^d$, by (2.8.3), whence

$$I+\mathfrak{n}^\lambda\mathfrak{m}^d = I_\mu+\mathfrak{n}^\lambda\mathfrak{m}^d, \quad \text{for any } \mu \geq \mu_0. \tag{2.8.7}$$

We now claim that $(I\cap\mathfrak{n}^m)+\mathfrak{n}^m\mathfrak{m}^d \subset (I_\mu\cap\mathfrak{n}^m)+\mathfrak{n}^m\mathfrak{m}^d$, for all $m \geq 1$. Indeed, for $m < \lambda$, the inclusion $I \subset I_\mu+\mathfrak{n}^\lambda\mathfrak{m}^d$ implies

$$I+\mathfrak{n}^m\mathfrak{m}^d \subset I_\mu+\mathfrak{n}^\lambda\mathfrak{m}^d+\mathfrak{n}^m\mathfrak{m}^d = I_\mu+\mathfrak{n}^m\mathfrak{m}^d\,,$$

and hence

$$(I \cap \mathfrak{n}^m) + \mathfrak{n}^m\mathfrak{m}^d = (I + \mathfrak{n}^m\mathfrak{m}^d) \cap \mathfrak{n}^m \subset (I_\mu + \mathfrak{n}^m\mathfrak{m}^d) \cap \mathfrak{n}^m = (I_\mu \cap \mathfrak{n}^m) + \mathfrak{n}^m\mathfrak{m}^d\,.$$

If, in turn, $m \geq \lambda$, then (2.8.7) yields

$$\begin{gathered}(I \cap \mathfrak{n}^m) + \mathfrak{n}^m\mathfrak{m}^d = (I \cap \mathfrak{n}^\lambda)\mathfrak{n}^{m-\lambda} + \mathfrak{n}^m\mathfrak{m}^d = ((I \cap \mathfrak{n}^\lambda) + \mathfrak{n}^\lambda\mathfrak{m}^d)\mathfrak{n}^{m-\lambda} \\ = ((I + \mathfrak{n}^\lambda\mathfrak{m}^d) \cap \mathfrak{n}^\lambda)\mathfrak{n}^{m-\lambda} = ((I_\mu + \mathfrak{n}^\lambda\mathfrak{m}^d) \cap \mathfrak{n}^\lambda)\mathfrak{n}^{m-\lambda} = ((I_\mu \cap \mathfrak{n}^\lambda) + \mathfrak{n}^\lambda\mathfrak{m}^d)\mathfrak{n}^{m-\lambda} \\ = (I_\mu \cap \mathfrak{n}^\lambda)\mathfrak{n}^{m-\lambda} + \mathfrak{n}^m\mathfrak{m}^d \subset (I_\mu \cap \mathfrak{n}^m) + \mathfrak{n}^m\mathfrak{m}^d\,.\end{gathered}$$

By (2.8.5) and (2.8.6), the above implies that there is, for every $m \geq 1$, a well-defined epimorphism

$$\frac{I + \mathfrak{n}^m}{I + \mathfrak{n}^m\mathfrak{m}^d} \cong \frac{\mathfrak{n}^m}{(I + \mathfrak{n}^m\mathfrak{m}^d) \cap \mathfrak{n}^m} \xrightarrow{\varphi_m} \frac{\mathfrak{n}^m}{(I_\mu \cap \mathfrak{n}^m) + \mathfrak{n}^m\mathfrak{m}^d} \cong \frac{I_\mu + \mathfrak{n}^m}{I_\mu + \mathfrak{n}^m\mathfrak{m}^d}\,.$$

To complete the proof, it thus suffices to show that $\ker \varphi_m = (0)$, or equivalently that $I_\mu \cap \mathfrak{n}^m \subset I + \mathfrak{n}^m\mathfrak{m}^d$, for all $m \geq 1$.

Recall that, by assumption, we have $I_\mu \cap \mathfrak{n} = I_\mu\mathfrak{n}$. By Lemma 2.4.13, we then have

$$I_\mu \cap \mathfrak{n}^m = I_\mu\mathfrak{n}^m,$$

and hence $I_\mu \cap (I_\mu + \mathfrak{n})^m \subset I_\mu(I_\mu + \mathfrak{n})^{m-1}$, for all $m \geq 1$. Moreover, by (2.8.2), $I_\mu \subset I + \mathfrak{m}^{\mu+1} \subset I + \mathfrak{n}$, and by (2.8.3), $I \subset I_\mu + \mathfrak{m}^{\mu+1} \subset I_\mu + \mathfrak{n}$, hence $I + \mathfrak{n} = I_\mu + \mathfrak{n}$. Finally, by (2.8.7), we also have $I_\mu \subset I + \mathfrak{n}\mathfrak{m}^d$, hence the sequence of inclusions

$$\begin{aligned}I_\mu \cap \mathfrak{n}^m \subset I_\mu \cap (I_\mu + \mathfrak{n})^m \subset I_\mu(I_\mu + \mathfrak{n})^{m-1} = I_\mu(I + \mathfrak{n})^{m-1} \\ \subset I + I_\mu\mathfrak{n}^{m-1} \subset I + (I + \mathfrak{n}\mathfrak{m}^d)\mathfrak{n}^{m-1} \subset I + \mathfrak{n}^m\mathfrak{m}^d\,.\end{aligned}$$

□

Chapter 3

Main Results

3.1 Approximation of complete intersections

The main result of this section, Theorem 3.1.3 below, asserts that a complete intersection singularity can be arbitrarily closely approximated by algebraic germs which are also complete intersections and share the same Hilbert-Samuel function.

We begin with a simple but useful observation.

Proposition 3.1.1. *For an ideal I in $\mathbb{K}\{x\}$, the following conditions are equivalent:*

(i) $\dim(\mathbb{K}\{x\}/I) \leq \dim \mathbb{K}\{x\} - k$.

(ii) After a generic linear change of coordinates in $\mathbb{K}^n$, the diagram $\mathfrak{N}(I)$ has a vertex on each of the first k coordinate axes in $\mathbb{N}^n$.

Proof. Let as before $x_{[k]} = (x_1, \dots, x_k)$ and $\tilde{x} = (x_{k+1}, \dots, x_n)$, and let $I(0)$ denote the ideal in $\mathbb{K}\{x_{[k]}\}$ obtained from I by evaluating the $\tilde{x}$ variables at zero.

Condition (ii) then implies that the diagram $\mathfrak{N}(I(0))$ has finite complement in $\mathbb{N}^k$, and hence $\dim_{\mathbb{K}} \mathbb{K}\{x_{[k]}\}/I(0) < \infty$. By the Weierstrass Finiteness Theorem (Theorem 2.4.14), it follows that $\dim(\mathbb{K}\{x\}/I) \leq \dim \mathbb{K}\{x\} - k$.

On the other hand, by Lemma 2.7.2, condition (i) implies that after a generic linear change of coordinates in $\mathbb{K}^n$, for every $j = 1, \dots, k$, I contains a distinguished polynomial $P_j(x_j, \tilde{x}) = x_j^{d_j} + \sum_{r=1}^{d_j} a_r^j(\tilde{x}) x_j^{d_j - r}$ such that $P_j(x_j, \tilde{x}) \in \mathfrak{m}^{d_j}$. Since the total ordering of $\mathbb{N}^n$ is induced by the lexicographic ordering of the $(n+1)$-tuples $(|\beta|, \beta_n, \dots, \beta_1)$, it follows that $\exp(P_j) = (0, \dots, d_j, 0 \dots, 0)$ with d_j in the j'th place. Hence (ii). □

We shall have occasion to use the following result which is a consequence of Theorem A.0.9 and Corollary A.0.10,

Remark 3.1.2. *Let I be a proper ideal in $\mathbb{K}\{x\}$ with $\dim \mathbb{K}\{x\}/I = n-k$, and suppose that $\mathbb{K}\{x\}/I$ is a finite $\mathbb{K}\{\tilde{x}\}$-module, where $\tilde{x} = (x_{k+1}, \dots, x_n)$. Then, $\mathbb{K}\{x\}/I$ is Cohen-Macaulay if and only if it is a flat $\mathbb{K}\{\tilde{x}\}$-module.*

Theorem 3.1.3. *Let $I = (F_1, \dots, F_k)$ be a complete intersection ideal in $\mathbb{K}\{x\}$ with $\dim \mathbb{K}\{x\}/I = n - k$. Then, there exists μ_0 such that for every $\mu \geq \mu_0$ and for any $G_1, \dots, G_k \in \mathbb{K}\{x\}$ satisfying $j^\mu G_i = j^\mu F_i$, $1 \leq i \leq k$, the ideal $I_\mu := (G_1, \dots, G_k)$ is a complete intersection ideal in $\mathbb{K}\{x\}$ and $H_{I_\mu} = H_I$.*

Proof. By Proposition 3.1.1, after a generic linear change of coordinates in $\mathbb{K}^n$, the diagram $\mathfrak{N}(I)$ has a vertex β^i on each of the first k coordinate axes in $\mathbb{N}^n$. Let $H_1, \dots, H_k \in I$ be representatives of these vertices, so that $\exp H_i = \beta^i$, $1 \leq i \leq k$. Let $Q_{i,j} \in \mathbb{K}\{x\}$ be such that $H_i = \sum_{j=1}^k Q_{i,j} F_j$. Set $\mu_1 := \max\{|\beta^1|, \dots, |\beta^k|\}$.

Since $\mathfrak{N}(I)$ has a vertex on each of the first k coordinate axes in $\mathbb{N}^n$, the complement $\mathbb{N}^k \setminus \mathfrak{N}(I(0))$ is finite. Hence, by Corollary 2.8.7, there exists $\mu_2 \geq 1$ such that, for every $\mu \geq \mu_2$ and $I_\mu \in U^\mu(I)$, $\mathfrak{N}(I(0)) = \mathfrak{N}(I_\mu(0))$. Let then $\mu_0 := \max\{\mu_1, \mu_2\}$.

Fix $\mu \geq \mu_0$ and $G_1, \dots, G_k \in \mathbb{K}\{x\}$ satisfying $j^\mu G_i = j^\mu F_i$, $1 \leq i \leq k$. Let $I_\mu = (G_1, \dots, G_k)$. Then, for every i,

$$j^\mu H_i = j^\mu(\sum_{j=1}^k Q_{i,j} F_j) = j^\mu(\sum_{j=1}^k Q_{i,j} j^\mu F_j) = j^\mu(\sum_{j=1}^k Q_{i,j} j^\mu G_j) = j^\mu(\sum_{j=1}^k Q_{i,j} G_j)\,,$$

hence, by Remark 2.8.2, $\exp(\sum_{j=1}^k Q_{i,j} G_j) = \beta^i$. It follows that $\beta^i \in \mathfrak{N}(I_\mu)$, $1 \leq$

$i \le k$, and thus $\mathfrak{N}(I_\mu)$ has a vertex on each of the first k coordinate axes in $\mathbb{N}^n$. By Proposition 3.1.1 again, we get $\dim \mathbb{K}\{x\}/I_\mu \le n-k$. Since I_μ is generated by k elements, it is thus a complete intersection ideal.

Since complete intersections are Cohen-Macaulay, then by Remark 3.1.2, both $\mathbb{K}\{x\}/I$ and $\mathbb{K}\{x\}/I_\mu$ are flat over $\mathbb{K}\{\tilde{x}\}$. Therefore, by Proposition 2.4.11 and Remark 2.4.12, there exists $l \ge 1$ such that for the linear form

$$\Lambda(\beta) = \sum_{i=1}^{k} \beta_i + \sum_{j=k+1}^{n} l\beta_j \,,$$

we have

$$\mathfrak{N}_\Lambda(I) = \mathfrak{N}(I(0)) \times \mathbb{N}^{n-k} \quad \text{and} \quad \mathfrak{N}_\Lambda(I_\mu) = \mathfrak{N}(I_\mu(0)) \times \mathbb{N}^{n-k} \,.$$

Thus, $\mathfrak{N}_\Lambda(I) = \mathfrak{N}_\Lambda(I_\mu)$, and hence

$$\dim_{\mathbb{K}} \frac{\mathbb{K}\{x\}}{I + \mathfrak{n}_{\Lambda,\eta+1}} \;=\; \dim_{\mathbb{K}} \frac{\mathbb{K}\{x\}}{I_\mu + \mathfrak{n}_{\Lambda,\eta+1}} \qquad \text{for all } \eta \in \mathbb{N} \,, \tag{3.1.1}$$

by Lemma 2.8.3. Note that $\mathfrak{n}_{\Lambda,l} = (x_{[k]})^l + (\tilde{x})$, and in general $\mathfrak{n}_{\Lambda,ml} = ((x_{[k]})^l + (\tilde{x}))^m$, for all $m \in \mathbb{N}$. Also, since $\mathbb{K}\{x\}/I$ is a finite $\mathbb{K}\{\tilde{x}\}$-module, then for l large enough one has $(x_{[k]})^l \subset I + (\tilde{x})$ (Corollary 2.7.3). It follows that $I + (\tilde{x}) = I + \mathfrak{n}_{\Lambda,l}$, and hence by induction

$$I + (\tilde{x})^m = I + \mathfrak{n}_{\Lambda,ml} \qquad \text{for all } m \in \mathbb{N} \,.$$

Therefore, by (3.1.1), we get $\dim_{\mathbb{K}} \dfrac{\mathbb{K}\{x\}}{I + (\tilde{x})^m} = \dim_{\mathbb{K}} \dfrac{\mathbb{K}\{x\}}{I_\mu + (\tilde{x})^m}$ for all $m \in \mathbb{N}$.

Note finally that $I_\mu \cap (\tilde{x}) = I_\mu \cdot (\tilde{x})$, by $\mathbb{K}\{\tilde{x}\}$-flatness of $\mathbb{K}\{x\}/I_\mu$ (see, e.g., [7, Cor. 7.6]). The theorem thus follows from Proposition 2.8.8. $\square$

Remark 3.1.4. *Observe that one can choose* polynomials $G_1, \dots, G_k \in \mathbb{K}[x]$ *satis-*

fying the hypothesis and hence the conclusion of Theorem 3.1.3. Therefore, Theorem 3.1.3 implies, in particular, that an arbitrary complete intersection singularity can be approximated arbitrarily well by algebraic complete intersection singularities that have the same Hilbert-Samuel function.

3.2 Approximation of Cohen-Macaulay singularities

At this point we recall that a module M over a Noetherian local ring $(A, \mathfrak{m})$ is Cohen-Macaulay if $\operatorname{depth}_A(M) = \dim M$, where $\operatorname{depth}_A(M)$ is the maximum length of an M-regular sequence in $\mathfrak{m}$. A local ring A is Cohen-Macaulay, when A is Cohen-Macaulay as an A-module (Definition A.0.4). In the context of the local rings of analytic germs, $\mathbb{K}\{x\}/I$, by Remark 3.1.2, this is equivalent to saying that if $\dim(\mathbb{K}\{x\}/I) = n - k$, then there exists a generic linear change of coordinates such that $\mathbb{K}\{x\}/I$ is flat as a $\mathbb{K}\{\tilde{x}\}$-module, where $\tilde{x} = (x_{k+1}, \ldots, x_n)$.

The polynomial approximation of analytic germs is, in general, not possible beyond the complete intersection case (see Example 3.2.2 below). The next best thing is an approximation by Nash germs. The following result shows that a Cohen-Macaulay singularity can be arbitrarily closely approximated by Nash germs which are also Cohen-Macaulay and share the same Hilbert-Samuel function.

Theorem 3.2.1. *Let $I = (F_1, \ldots, F_s)$ be an ideal in $\mathbb{K}\{x\}$ such that $\mathbb{K}\{x\}/I$ is Cohen-Macaulay with $\dim \mathbb{K}\{x\}/I = n - k$. Then, there exists $\mu_0 \in \mathbb{N}$, such that for any $\mu \geq \mu_0$ there are algebraic power series $G_1, \ldots, G_s \in \mathbb{K}\langle x \rangle$ with $j^\mu G_i = j^\mu F_i$, $1 \leq i \leq s$, the ideal $I_\mu = (G_1, \ldots, G_s)$ satisfies $H_{I_\mu} = H_I$, and $\mathbb{K}\{x\}/I_\mu$ is Cohen-Macaulay with $\dim \mathbb{K}\{x\}/I_\mu = n - k$.*

Proof. By Proposition 3.1.1, after a generic linear change of coordinates in $\mathbb{K}^n$, the diagram $\mathfrak{N}(I)$ has a vertex on each of the first k coordinate axes in $\mathbb{N}^n$. It follows that $\mathbb{K}\{x\}/I$ is $\mathbb{K}\{\tilde{x}\}$-finite, and hence $\mathbb{K}\{\tilde{x}\}$-flat (Remark 3.1.2). Therefore, by Proposition 2.4.11 and Remark 2.4.12, there exists $l \geq 1$ such that for the linear form

$$\Lambda(\beta) = \sum_{i=1}^{k} \beta_i + \sum_{j=k+1}^{n} l\beta_j \,,$$

we have

$$\mathfrak{N}_\Lambda(I) = \mathfrak{N}(I(0)) \times \mathbb{N}^{n-k} \,.$$

We can extend the given set of generators $\{F_1, \dots, F_s\}$ by power series $F_{s+1}, \dots, F_r \in I$ such that the collection $\{F_1, \dots, F_r\}$ contains representatives of all the vertices of $\mathfrak{N}_\Lambda(I)$. Since I is generated by $\{F_1, \dots, F_s\}$, there are $H_p^q \in \mathbb{K}\{x\}$ such that

$$F_{s+p} = \sum_{q=1}^{s} H_p^q F_q \,, \qquad p = 1, \dots, r-s.$$

Then, $\{F_1, \dots, F_r\}$ is a set of generators of I and a standard basis of I relative to Λ (Corollary 2.4.8). For $i, j \in \{1, \dots, r\}$, $i < j$, let $S_{i,j} = S(F_i, F_j)$ denote the s-series of the pair (F_i, F_j). By Theorem 2.5.5, there exist $Q_m^{i,j} \in \mathbb{K}\{x\}$, $i, j, m \in \{1, \dots, r\}$, such that

$$S_{i,j} = \sum_{m=1}^{r} Q_m^{i,j} F_m \qquad \text{and} \qquad \exp_\Lambda S_{i,j} \leq \min\{\exp_\Lambda(Q_m^{i,j} F_m) : m = 1, \dots, r\} \,.$$

Recall that, for all $1 \leq i < j \leq r$, there are monomials $P_{i,j}, P_{j,i} \in \mathbb{K}[x]$, which depend only on the initial terms of F_i, F_j, such that $S_{i,j} = P_{i,j}F_i - P_{j,i}F_j$. Consider a system

$$\begin{cases} P_{i,j}(x)y_i - P_{j,i}(x)y_j - \displaystyle\sum_{m=1}^{r} z_m^{i,j} y_m = 0 \\ y_{s+p} - \displaystyle\sum_{q=1}^{s} w_p^q y_q = 0 \end{cases} \tag{3.2.1}$$

of $\binom{r}{2} + r - s$ polynomial equations in variables $y = (y_1, \dots, y_r)$, $z = (z_1^{1,2}, \dots, z_r^{r-1,r})$

and $w = (w_1^1, \ldots, w_{r-s}^s)$. The system has a convergent solution $\{F_i, Q_m^{i,j}, H_p^q\}$, and hence by Theorem 2.6.2, for every $\mu \in \mathbb{N}$, an algebraic power series solution $\{G_i, R_m^{i,j}, K_p^q\}$ with $j^\mu G_i = j^\mu F_i$, $j^\mu R_m^{i,j} = j^\mu Q_m^{i,j}$, and $j^\mu K_p^q = j^\mu H_p^q$ for all i, j, m, p, q.

Let now $\mu_0 \coloneqq \max\{\Lambda(\exp_\Lambda Q_m^{i,j}) + \Lambda(\exp_\Lambda F_m)) : 1 \le i < j \le r, 1 \le m \le r\}$, and fix $\mu \ge \mu_0$. Then, for any algebraic solution $\{G_i, R_m^{i,j}, K_p^q\}$ to (3.2.1) which coincides with $\{F_i, Q_m^{i,j}, H_p^q\}$ up to degree μ, we have $S(G_i, G_j) = P_{i,j}G_i - P_{j,i}G_j$ and

$$S(G_i, G_j) = \sum_{m=1}^{r} R_m^{i,j} G_m,$$

$$\text{with } \exp_\Lambda S(G_i, G_j) \le \min\{\exp_\Lambda(R_m^{i,j} G_m) : m = 1, \ldots, r\}.$$

Hence the G_i form a standard basis for the ideal $I_\mu = (G_1, \ldots, G_r)$, by Theorem 2.5.5 again. In particular, the set $\{G_1, \ldots, G_r\}$ contains representatives of all the vertices of $\mathfrak{N}_\Lambda(I_\mu)$ (see Remark 2.5.2(1)). Since, by construction, $\exp_\Lambda G_i = \exp_\Lambda F_i$ for all i, it follows that $\mathfrak{N}_\Lambda(I_\mu) = \mathfrak{N}_\Lambda(I)$. Thus, $\mathfrak{N}_\Lambda(I_\mu) = \mathfrak{N}(I(0)) \times \mathbb{N}^{n-k}$ and so $\mathbb{K}\{x\}/I_\mu$ is $\mathbb{K}\{\tilde{x}\}$-flat, by Proposition 2.4.11. Note also that I_μ is, in fact, generated by $\{G_1, \ldots, G_s\}$, since the remaining generators $G_{s+1}, \ldots, G_r$ are combinations of the former, by (3.2.1).

The equality of diagrams $\mathfrak{N}_\Lambda(I_\mu) = \mathfrak{N}_\Lambda(I)$ implies, as in the proof of Theorem 3.1.3, that we have $\dim_{\mathbb{K}} \dfrac{\mathbb{K}\{x\}}{I + (\tilde{x})^m} = \dim_{\mathbb{K}} \dfrac{\mathbb{K}\{x\}}{I_\mu + (\tilde{x})^m}$, for all $m \in \mathbb{N}$. Moreover, $I_\mu \cap (\tilde{x}) = I_\mu \cdot (\tilde{x})$, by $\mathbb{K}\{\tilde{x}\}$-flatness of $\mathbb{K}\{x\}/I_\mu$. The theorem thus follows from Proposition 2.8.8. $\square$

In contrast with complete intersections, the Cohen-Macaulay singularities are not, in general, finitely determined. This can be shown using Becker's s-series criterion, as follows.

Example 3.2.2. *Let I be an ideal in $\mathbb{K}\{x,y,z\}$ generated by*

$$F_1 = x^8, \quad F_2 = y^5 + y^2z^4e^z, \quad F_3 = x^2y^3 + x^2z^4e^z\,.$$

Let $\mathbb{N}^3$ be equipped with the standard ordering induced by lexicographic ordering of the 4-tuples $(|\beta|, \beta_3, \beta_2, \beta_1)$.

We claim that $\{F_1, F_2, F_3\}$ are a standard basis of I. Indeed, the s-series of pairs (F_1, F_3) and (F_2, F_3) are as follows:

$$S_{1,3} = y^3F_1 - x^6F_3 = (-z^4e^z)F_1, \qquad S_{2,3} = x^2F_2 - y^2F_3 = 0\,,$$

which are standard representations in terms of $\{F_1, F_2, F_3\}$. The $S_{1,2}$, in turn, has a standard representation in terms of F_1 and F_2, because their initial exponents are relatively prime (see [6, Thm. 3.1]). The claim thus follows from Theorem 2.5.5.

The diagram $\mathfrak{N}(I)$ contains vertices on the first two coordinate axes in $\mathbb{N}^3$, namely $\mathrm{exp}F_1$ and $\mathrm{exp}F_2$, hence $\mathbb{K}\{x,y,z\}/I$ is a finite $\mathbb{K}\{z\}$-module. On the other hand, by Remark 2.5.2(1), the only vertices of $\mathfrak{N}(I)$ are the $\mathrm{exp}F_1$, $\mathrm{exp}F_2$, and $\mathrm{exp}F_3$, which all lie in $\mathbb{N}^2 \times \{0\}$. Thus, by Proposition 2.4.11, $\mathbb{K}\{x,y,z\}/I$ is $\mathbb{K}\{z\}$-flat, and hence Cohen-Macaulay (Remark 3.1.2).

Let now $\mu \geq 8$ be arbitrary, and let

$$G_1 = x^8, \quad G_2 = y^5 + y^2z^4(e^z + z^{\mu-6}h(z)), \quad G_3 = x^2y^3 + x^2z^4e^z\,,$$

where $h(z) \in \mathbb{K}\{z\}$ is an arbitrary non-zero series with $h(0) = 0$. Then, $j^\mu G_i = j^\mu F_i$ for all i, but for the ideal $I_\mu = (G_1, G_2, G_3)$, the ring $\mathbb{K}\{x,y,z\}/I_\mu$ is not Cohen-Macaulay. Indeed, consider the s-series $S(G_2, G_3)$. We have

$$S(G_2, G_3) = x^2G_2 - y^2G_3 = x^2y^2z^{\mu-2}h(z)\,,$$

and hence x^2y^2 is a zero-divisor in $\mathbb{K}\{x,y,z\}/I_\mu$ regarded as a $\mathbb{K}\{z\}$-module. Thus, $\mathbb{K}\{x,y,z\}/I_\mu$ is not $\mathbb{K}\{z\}$-flat, and hence not a Cohen-Macaulay ring, by Remark 3.1.2 again.

3.3 Zariski equisingularity and Varchenko theorem

In this section we recall a result of Varchenko on topological equisingularity of algebraically equisingular families. This is a central tool in the proof of Mostowski's theorem.

Let V be a complex analytic hypersurface in a neighbourhood U of the origin in $\mathbb{C}^l \times \mathbb{C}^n$, and let $T = V \cap (\mathbb{C}^l \times \{0\})$. Suppose there is, for every $0 \le i \le n$, a distinguished polynomial

$$F_i(t, x_{[i]}) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(t, x_{[i-1]}) x_i^{p_i-j} ,$$

where $t \in \mathbb{C}^l$, $x_{[i]} = (x_1, \dots, x_i) \in \mathbb{C}^i$, $a_{i-1,j} \in \mathbb{C}\{t, x_{[i-1]}\}$, all such that the following hold:

(1) $V = F_n^{-1}(0)$.

(2) $a_{i,j}(t, 0) \equiv 0$, for all i, j.

(3) $F_{i-1}(t, x_{[i-1]}) = 0$ if and only if $F_i(t, x_{[i-1]}, x_i)$, regarded as a polynomial in x_i with $(t, x_{[i-1]})$ fixed, has fewer roots than for generic $(t, x_{[i-1]})$.

(4) Either $F_i(t, 0) \equiv 0$ or $F_i \equiv 1$, and in the latter case $F_k \equiv 1$ for all $k \le i$ by convention.

(5) $F_0 \equiv 1$.

A system of distinguished polynomials $\{F_i(t, x_{[i]})\}$ satisfying the above conditions is called *algebraically equisingular.* Answering a question posed by Zariski [38], Varchenko showed that algebraic equisingularity of a system $\{F_i(t, x_{[i]})\}$ implies topological equisingularity of V along T. More precisely, we have the following.

Theorem 3.3.1 ([32, 33], cf. [12, Thms. 3.1, 3.2])**.** *Under the above hypotheses, let* $V_t = V \cap (\{t\} \times \mathbb{C}^n)$ *and* $U_t = U \cap (\{t\} \times \mathbb{C}^n)$*, for* $t \in T$*. Then, for every* $t \in T$*, there exists a homeomorphism* $h_t : U_0 \to U_t$ *such that* $h_t(V_0) = V_t$ *and* $h_t(0) = 0$*. Moreover, if* $F_n = G_1 \dots G_r$ *is a product of distinguished polynomials in* x_n*, then*

$$h_t(G_j^{-1}(0) \cap (\{0\} \times \mathbb{C}^n)) = G_j^{-1}(0) \cap (\{t\} \times \mathbb{C}^n) \qquad \text{for all } 1 \leq j \leq r\,.$$

3.4 Mostowski theorem with Hilbert-Samuel equisingularity

The goal of this section is to prove a strong variant of Mostowski's theorem [22], showing that every analytic germ $X_0 \subset \mathbb{K}^n_0$ can be arbitrarily closely approximated by a Nash germ $\hat{X}_0 \subset \mathbb{K}^n_0$ with the same Hilbert-Samuel function, and such that the pairs $(\mathbb{K}^n, X)$ and $(\mathbb{K}^n, \hat{X})$ are topologically equivalent near zero.

Theorem 3.4.1. *Let* $g_1, \dots, g_s \in \mathbb{K}\{x\}$ *and let* $X_0 \subseteq \mathbb{K}^n_0$ *be an analytic germ defined by* $g_1 = \dots = g_s = 0$*. Then, there exists* μ_0 *such that for all* $\mu \geq \mu_0$ *there are algebraic power series* $\hat{g}_1, \dots, \hat{g}_s \in \mathbb{K}\langle x\rangle$ *and a homeomorphism germ* $h : \mathbb{K}^n_0 \to \mathbb{K}^n_0$ *such that:*

(i) $j^\mu \hat{g}_k = j^\mu g_k$ *for* $k = 1, \dots, s$

(ii) If $\hat{X}_0$ *is the Nash germ defined by* $\hat{g}_1 = \dots = \hat{g}_s = 0$*, then* $H_{\hat{X},0} = H_{X,0}$

(iii) $h(X_0) = \hat{X}_0$*.*

Our proof of Theorem 3.4.1 combines the exposition of [12] with the Becker s-series criterion (Theorem 2.5.5). We include the details of the argument for the reader's convenience.

3.4.1 Generalized discriminants

Let $T = (T_1, \ldots, T_p)$ be variables. For $j \geq 1$, consider

$$\Delta_j = \sum_{r_1,\ldots,r_{j-1}} \prod_{\substack{k \neq l \\ k,l \neq r_1,\ldots,r_{j-1}}} (T_k - T_l).$$

The Δ_j are symmetric in variables T, and hence each $\Delta_j = \Delta_j(A_0, \ldots, A_{p-1})$ is a polynomial in the elementary symmetric functions $A_0 = T_1 \cdots T_p, \ldots, A_{p-1} = T_1 + \cdots + T_p$. We have: A polynomial $X^p + a_{p-1}X^{p-1} + \cdots + a_1 X + a_0$ has precisely $p - j$ distinct roots if and only if $\Delta_1(a_0, \ldots, a_{p-1}) = \cdots = \Delta_j(a_0, \ldots, a_{p-1}) = 0$ and $\Delta_{j+1}(a_0, \ldots, a_{p-1}) \neq 0$.

3.4.2 Construction of a normal system of equations

Let $g_1, \ldots, g_s \in \mathbb{K}\{x\}$ and let $I \coloneqq (g_1, \ldots, g_s) \cdot \mathbb{K}\{x\}$. After a generic linear change of coordinates, if needed, all the g_k become regular in variable x_n. We may thus, without loss of generality, assume that each g_k is a distinguished polynomial in x_n. That is,

$$g_k(x) = x_n^{r_k} + \sum_{j=1}^{r_k} a_{n-1,k,j}(x_{[n-1]}) x_n^{r_k - j}\,, \tag{3.4.1}$$

where $a_{n-1,k,j} \in \mathbb{K}\{x_{[n-1]}\}$ and $a_{n-1,k,j}(0) = 0$.

The coefficients $a_{n-1,k,j}$ can be arranged in a row vector $a_{n-1} \in \mathbb{K}\{x_{[n-1]}\}^{p_n}$, where $p_n = \sum_k r_k$. Set $f_n \coloneqq g_1 \cdots g_s$. Then, the generalized discriminants $\Delta_{n,i}$ of f_n are polynomials in a_{n-1}. Let j_n be such that

$$\Delta_{n,i}(a_{n-1}) \equiv 0 \quad \text{for } i < j_n\,,$$

and $\Delta_{n,j_n}(a_{n-1}) \not\equiv 0$. Then, after a linear change of coordinates $x_{[n-1]}$, we may write

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x_{[n-1]})(x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x_{[n-2]})x_{n-1}^{p_{n-1}-j})\,,$$

where $u_{n-1}(0) \neq 0$, and for all j, $a_{n-2,j}(0) = 0$. Set

$$f_{n-1} \coloneqq x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x_{[n-2]})x_{n-1}^{p_{n-1}-j},$$

and denote the vector of its coefficients $a_{n-2,j}$ by $a_{n-2} \in \mathbb{K}\{x_{[n-2]}\}^{p_{n-1}}$. Let j_{n-1} be such that the first $j_{n-1}-1$ generalized discriminants $\Delta_{n-1,i}$ of f_{n-1} are identically zero and $\Delta_{n-1,j_{n-1}}$ is not. Then, again, we define $f_{n-2}(x_{[n-2]})$ as the distinguished polynomial associated to $\Delta_{n-1,j_{n-1}}$, and so on.

By induction, we define a system of distinguished polynomials $f_i \in \mathbb{K}\{x_{[i-1]}\}[x_i]$, $i = 1, \ldots, n-1$, such that

$$f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x_{[i-1]})x_i^{p_i-j}$$

is the distinguished polynomial associated to the first non identically zero generalized discriminant $\Delta_{i+1,j_{i+1}}(a_i)$ of f_{i+1}:

$$\Delta_{i+1,j_{i+1}}(a_i) = u_i(x_{[i]})(x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x_{[i-1]})x_i^{p_i-j})\,, \quad i = 0, ..., n-1, \tag{3.4.2}$$

where, in general, $a_i = (a_{i,1}, \ldots, a_{i,p_{i+1}})$. Thus, the vector of functions a_i satisfies

$$\Delta_{i+1,k}(a_i) \equiv 0 \qquad \text{for} \quad k < j_{i+1}, \quad i = 0, ..., n-1. \tag{3.4.3}$$

This means in particular that

$$\Delta_{1,k}(a_0) \equiv 0 \quad \text{for } k < j_1 \qquad \text{and} \quad \Delta_{1,j_1}(a_0) \equiv u_0,$$

where u_0 is a non-zero constant.

3.4.3 Incorporating a standard basis

Consider now the diagram of initial exponents $\mathfrak{N}(I)$ of the ideal I in $\mathbb{K}\{x\}$ (with respect to the linear form $\Lambda(\beta) = |\beta|$ on $\mathbb{N}^n$). We can extend the given set of generators $\{g_1, \dots, g_s\}$ by power series $g_{s+1}, \dots, g_r \in I$ such that the collection $\{g_1, \dots, g_r\}$ contains representatives of all the vertices of $\mathfrak{N}(I)$. Since I is generated by $\{g_1, \dots, g_s\}$, there are $h_p^q \in \mathbb{K}\{x\}$ such that

$$g_{s+p}(x) = \sum_{q=1}^{s} h_p^q(x) \cdot \left(x_n^{r_q} + \sum_{j=1}^{r_q} a_{n-1,q,j}(x_{[n-1]}) x_n^{r_q - j} \right) , \tag{3.4.4}$$

for $p = 1, \dots, r-s$, by (3.4.1).

Now, $\{g_1, \dots, g_r\}$ is a set of generators of I and a standard basis of I (Corollary 2.4.8). For $i, j \in \{1, \dots, r\}$, $i < j$, let $S_{i,j} = S(g_i, g_j)$ denote the s-series of the pair (g_i, g_j). By Theorem 2.5.5, there exist $v_m^{i,j} \in \mathbb{K}\{x\}$, $i, j, m \in \{1, \dots, r\}$, such that

$$S_{i,j} = \sum_{m=1}^{r} v_m^{i,j} g_m \qquad \text{and} \qquad \exp S_{i,j} \leq \min\{\exp(v_m^{i,j} g_m) : m = 1, \dots, t\} . \tag{3.4.5}$$

Recall that, for all $1 \leq i < j \leq r$, there are monomials $P_{i,j}, P_{j,i} \in \mathbb{K}[x]$, which depend only on the initial terms of g_i, g_j, such that $S_{i,j} = P_{i,j} g_i - P_{j,i} g_j$. Therefore, the $v_m^{i,j}$, h_p^q, and $a_{n-1,q,j}$ satisfy the following system of $\binom{r}{2}$ polynomial equations

$$P_{i,j}(x) g_i - P_{j,i}(x) g_j - \sum_{m=1}^{r} v_m^{i,j} g_m = 0 , \qquad 1 \leq i < j \leq r , \tag{3.4.6}$$

in which the h_p^q and $a_{n-1,q,j}$ are present via (3.4.1) and (3.4.4). We will denote the vector of functions $v_m^{i,j}$ by $v \in \mathbb{K}\{x\}^{r\binom{r}{2}}$, and the vector of h_p^q by $h \in \mathbb{K}\{x\}^{s(r-s)}$, to simplify notation.

3.4.4 Approximation by Nash functions

Consider (3.4.2), (3.4.3), and (3.4.6) as a system of polynomial equations in $a_i(x_{[i]})$, $u_i(x_{[i]})$, $v(x)$, and $h(x)$. By construction, this system admits a convergent solution. Therefore, by Theorem 2.6.1, there exist a new set of variables $z = (z_1, \dots, z_k)$, an increasing function $\tau : \mathbb{N} \to \mathbb{N}$, convergent power series $z_i(x) \in \mathbb{K}\{x\}$ vanishing at zero, algebraic power series $\hat{u}_i(x_{[i]}, z) \in \mathbb{K}\big\langle x_{[i]}, z_1, \dots, z_{\tau(i)}\big\rangle$, and vectors of algebraic power series $\hat{a}_i(x_{[i]}, z) \in \mathbb{K}\big\langle x_{[i]}, z_1, \dots, z_{\tau(i)}\big\rangle^{p_i}$, $\hat{v}(x, z) \in \mathbb{K}\langle x, z\rangle^{r\binom{r}{2}}$, and $\hat{h}(x, z) \in \mathbb{K}\langle x, z\rangle^{s(r-s)}$ all such that the following hold:

(a) $z_1(x), \dots, z_{\tau(i)}(x)$ depend only on variables $x_{[i]} = (x_1, \dots, x_i)$

(b) $\hat{u}_i(x_{[i]}, z), \hat{a}_i(x_{[i]}, z), \hat{v}(x, z), \hat{h}(x, z)$ are solutions of (3.4.2), (3.4.3), and (3.4.6)

(c) The convergent solutions satisfy:
$u_i(x_{[i]}) = \hat{u}_i(x_{[i]}, z(x_{[i]}))$, $a_i(x_{[i]}) = \hat{a}_i(x_{[i]}, z(x_{[i]}))$, $v(x) = \hat{v}(x, z(x))$, and $h(x) = \hat{h}(x, z(x))$.

3.4.5 Proof of Theorem 3.4.1

Let $g_1, \dots, g_s \in \mathbb{K}\{x\}$ and let $X_0 \subseteq \mathbb{K}_0^n$ be an analytic germ defined by $g_1 = \dots = g_s = 0$. Suppose first that $\mathbb{K} = \mathbb{C}$.

Let $g_{s+1}(x), \dots, g_r(x)$, $u_i(x_{[i]})$, $a_i(x_{[i]})$, $v(x)$, and $h(x)$ be as in Sections 3.4.2 and 3.4.3. Set

$$\mu_0 := \max\{|\exp(v_m^{i,j})| + |\exp(g_m)| : 1 \le i < j \le r, 1 \le m \le r\},$$

and fix $\mu \geq \mu_0$.

Let $z_i(x) \in \mathbb{C}\{x\}$ be the convergent power series, and let $\hat{u}_i(x_{[i]}, z)$, $\hat{a}_i(x_{[i]}, z)$, $\hat{v}(x, z)$, and $\hat{h}(x, z)$ be the (vectors of) algebraic power series constructed above. To simplify notation, we will write $\bar{z}_i^\mu(x)$ for $z_i(x) - j^\mu z_i(x)$, where as before $j^\mu z_i(x)$ denotes the μ-jet of z_i as a power series in variables x.

For $t \in \mathbb{C}$, we define

$$F_n(t, x) := \prod_{k=1}^{s} G_k(t, x)\,,$$

where

$$G_k(t, x) := x_n^{r_k} + \sum_{j=1}^{r_k} \hat{a}_{n-1,k,j}(x_{[n-1]}, j^\mu z(x_{[n-1]}) + t\bar{z}^\mu(x_{[n-1]}))\cdot x_n^{r_k - j}\,, \qquad (3.4.7)$$

and

$$F_i(t, x) := x_i^{p_i} + \sum_{j=1}^{p_i} \hat{a}_{i-1,j}(x_{[i-1]}, j^\mu z(x_{[i-1]}) + t\bar{z}^\mu(x_{[i-1]}))\cdot x_i^{p_i - j}\,, \quad i = 1, \dots, n-1.$$

Finally, we set $F_0(t) \equiv 1$. Now, since $u_i(0, 0) = \hat{u}_i(0, z(0)) \neq 0$, $i = 1, \dots, n-1$, it follows that the family $\{F_i(t, x_{[i]})\}$ is algebraically equisingular (with $|t| < R$, for any $R < \infty$).

Set $\hat{g}_k(x) := G_k(0, x)$, and let $\hat{X}_0$ be the Nash germ in $\mathbb{C}^n_0$ defined by $\hat{g}_1 = \cdots = \hat{g}_s = 0$. Note that $g_k = G_k(1, x)$, $k = 1, \dots, s$. Therefore, by Theorem 3.3.1, there is a homeomorphism germ $h : \mathbb{C}^n_0 \to \mathbb{C}^n_0$ such that $h(X_0) = \hat{X}_0$.

By construction, we have $j^\mu \hat{g}_k = j^\mu g_k$ for $k = 1, \dots, s$. Finally, as in the proof of Theorem 3.2.1, observe that the collection $\{\hat{g}_1, \dots, \hat{g}_s, \dots, \hat{g}_r\}$ forms a standard basis for the ideal I_μ that it generates (by (3.4.5)). In particular, the set $\{\hat{g}_1, \dots, \hat{g}_r\}$ contains representatives of all the vertices of the diagram $\mathfrak{N}(I_\mu)$ (see Remark 2.5.2(1)). Since, by construction and the choice of μ_0, we have $\exp(\hat{g}_k) = \exp(g_k)$ for all k, it follows that $\mathfrak{N}(I_\mu) = \mathfrak{N}(I)$. Hence, $H_{\hat{X},0} = H_{X,0}$, by Lemma 2.8.3. Note also that I_μ

is, in fact, generated by $\{\hat{g}_1, \ldots, \hat{g}_s\}$, since the remaining generators $\hat{g}_{s+1}, \ldots, \hat{g}_r$ are combinations of the former, by (3.4.4). This completes the proof in the complex case.

The real case follows from the complex one, since by [32, §6], if the distinguished polynomials F_i of Section 3.3 have real coefficients, then the homeomorphism h constructed in Varchenko's Theorem 3.3.1 is conjugation invariant. □

Remark 3.4.2. *We note the following:*

(1) A parameterization such as $F_n(t, x)$ of a power series is referred to as an unfolding *(see [15, Section II.1.2]).*

(2) By [15, Corollary II.1.6, Proposition II.1.7] the unfolding of f_n in the proof of Theorem 3.4.1, given by $F_n(t, x)$, defines an analytic deformation of the hypersurface defined by $f_n = 0$. Also, in the case when the germ X_0 is complex analytic and a complete intersection and the g_i are a minimal set of defining functions, the unfoldings $G_i(t, x)$ define a deformation as well, and in this case one can easily see that the deformation is equisingular both topologically and in the sense of the Hilbert-Samuel function.

(3) In general, any unfolding of the defining functions of a complex analytic germ which is not a complete intersection does not define a deformation (see [15, Example II.1.7.1]).

(4) It is an open question whether the particular unfoldings $G_i(t, x)$ arising in the proof of Theorem 3.4.1 always define deformations of the germ X_0 even when it is not a complete intersection.

Chapter 4

A look ahead

This section explores work that is a continuation of the main results of this thesis and points to possible future directions of research. There are two main threads that can be followed. One is the approximation of germs of flat analytic mappings. This has potential applications to deformation theory. The second one concerns exploration into the problem of equiresolution.

4.1 Approximation of flat maps from Cohen-Macaulay germs

This section begins with a result that builds upon previous work by Adamus and Seyedinejad [2, Theorem 4.9]. In [2] the authors prove that an analytic flat map from a complete intersection is finitely determined, that is, by taking sufficiently large jets of the functions defining the map we can obtain approximations to it that are flat as well. Subsequently, they asked whether this could be extended to flat maps from germs of analytic spaces whose local rings are Cohen-Macaulay. The Cohen-Macaulay case is interesting because in the context of complex analytic spaces the Cohen-Macaulay case is well understood geometrically - for analytic maps from germs of complex analytic spaces to euclidean germs, flatness is equivalent to openness [14, Proposition 3.20]. The work in this section appears in the preprint [26].

In what follows we show first that finite determinacy can be extended to the case of flat maps from germs of analytic spaces whose local rings are Cohen-Macaulay.

Theorem 4.1.1. *Let X be a $\mathbb{K}$-analytic subspace of $\mathbb{K}^n$. Suppose that $0 \in X$ and the local ring $\mathcal{O}_{X,0}$ is Cohen-Macaulay. Also, let $\phi = (\phi_1, \ldots, \phi_m) : X \to \mathbb{K}^m$ with $\phi(0) = 0$ be a $\mathbb{K}$-analytic mapping which is flat at 0. Then there exists $\mu_0 \in \mathbb{N}$, such that for every $\mu \geq \mu_0$ every mapping $\psi = (\psi_1, \ldots, \psi_m) : X \to \mathbb{K}^m$ satisfying $j^\mu \phi = j^\mu \psi$ is flat at zero.*

Proof. Let $\dim \mathcal{O}_{X,0} = n - k$. Suppose that $\mathcal{O}_{X,0} = \mathbb{K}\{x\}/I$ and $I = (F_1, \ldots, F_s)$. Further, let J be the ideal in $\mathbb{K}\{x\}$ generated by $\phi_1, \ldots, \phi_m$. Then by Theorem A.0.6, $\dim \mathbb{K}\{x\}/(I + J) = n - (k + m)$. Up to a generic linear change of coordinates this is equivalent, by Proposition 3.1.1, to the condition that $\mathfrak{N}(I + J)$ has a vertex on each of the first $k + m$ coordinate axes. Suppose that $G_1, \ldots, G_{k+m} \in I + J$ are representatives of these vertices. Then we have $Q^i_p \in \mathbb{K}\{x\}$ such that,

$$G_i = \sum_{p=1}^{s} Q^i_p F_p + \sum_{q=1}^{m} Q^i_{s+q} \phi_q.$$

For all sufficiently large μ we have,

$$\mathrm{exp} j^\mu G_i = \mathrm{exp} G_i \tag{4.1.1}$$

and,

$$j^\mu G_i = j^\mu (\sum_{p=1}^{s} Q^i_p F_p + \sum_{q=1}^{m} Q^i_{s+q} j^\mu \phi_q).$$

Now suppose that $\psi = (\psi_1, \ldots, \psi_m) : X \to \mathbb{K}^m$ is a mapping such that $j^\mu \psi_q = j^\mu \phi_q$

for $q = 1, \dots, m$. We then have,

$$j^\mu G_i = j^\mu(\sum_{p=1}^{s} Q_p^i F_p + \sum_{q=1}^{m} Q_{s+q}^i j^\mu \psi_q).$$

Therefore, by (4.1.1), we may conclude that for all sufficiently large μ, $\mathfrak{N}(I + \hat{J})$, where $\hat{J}$ is the ideal in $\mathbb{K}\{x\}$ generated by $\psi_1, \dots, \psi_m$, has a vertex each of the first $k + m$ coordinate axes. By Proposition 3.1.1 this implies that,

$$\dim \mathbb{K}\{x\}/(I + \hat{J}) \le (n - k) - m.$$

But we have $\dim \mathbb{K}\{x\}/(I + \hat{J}) \ge (n - k) - m$ because $\hat{J}$ has m generators, therefore, $\dim \mathbb{K}\{x\}/(I + \hat{J}) = n - (k + m)$. Theorem A.0.9 then allows us to conclude that $\psi = (\psi_1, \dots, \psi_m)$ is flat at zero. $\square$

Flat maps from germs of real or complex analytic spaces are central to deformation theory. In this context, it is of interest to know if analytic flat maps can be approximated by maps that are Nash in such a way that algebro-geometric properties of the special fibre are preserved. This is of interest because if we are approximating deformations of analytic spaces we would like the approximant to belong to the same class, which is usually determined based on properties of the special fibre. Using Theorem 3.2.1, we can prove that there exist Nash flat approximations to flat maps whose domains are germs of analytic spaces with Cohen-Macaulay local rings, which preserve the Hilbert-Samuel function of the special fibre.

Theorem 4.1.2. *Let X be a $\mathbb{K}$-analytic subspace of $\mathbb{K}^n$. Suppose that $0 \in X$ and the local ring $\mathcal{O}_{X,0}$ is Cohen-Macaulay and that $\mathcal{O}_{X,0} = \mathbb{K}\{x\}/I$ where $I = (F_1, \dots, F_s)$. Also, let $\phi = (\phi_1, \dots, \phi_m) : X \to \mathbb{K}^m$ with $\phi(0) = 0$ be a $\mathbb{K}$-analytic mapping which is flat at 0. Then there exists $\mu_0 \in \mathbb{N}$, such that for each $\mu \ge \mu_0$ there exist:*

(a) A Nash, Cohen-Macaulay germ $\hat{X}_0 \subseteq \mathbb{K}^n$ with $\mathcal{O}_{\hat{X},0} = \mathbb{K}\{x\}/\hat{I}$, where $\hat{I} =$

$(G_1, \dots, G_s)$, for some $G_i \in \mathbb{K}\langle x\rangle$ that satisfy $j^\mu G_i = j^\mu F_i$ for all i, such that $H_{\hat{I}} = H_I$.

(b) A Nash map $\psi = (\psi_1, \dots \psi_n) : \hat{X} \to \mathbb{K}^m$ which is flat at 0 with $j^\mu\phi_i = j^\mu\psi_i$ for all i, such that if $J = (\phi_1, \dots, \phi_m) \subseteq \mathbb{K}\{x\}$ and $\hat{J} = (\psi_1, \dots, \psi_m) \subseteq \mathbb{K}\{x\}$, then $H_{I+J} = H_{\hat{I}+\hat{J}}$.

Proof. Suppose that $\mathcal{O}_{X,0} = \mathbb{K}\{x\}/I$ and $I = (F_1, \dots, F_s)$. We proceed exactly as in the proof of Theorem 3.2.1 to establish the following:

(1) After a generic linear change of coordinates in $\mathbb{K}^n$ there exists $l_1 \geq 1$ such that for the linear form
$$\Lambda_1(\beta) = \sum_{i=1}^{k} \beta_i + \sum_{j=k+1}^{n} l_1\beta_j \,,$$
we have
$$\mathfrak{N}_{\Lambda_1}(I) = D_1 \times \mathbb{N}^{n-k}, \text{ for some } D_1 \subset \mathbb{N}^k. \tag{4.1.2}$$

(2) We may extend $\{F_1, \dots, F_s\}$ by power series $F_{s+1}, \dots,\ F_r \in I$ such that the collection $\{F_1, \dots, F_r\}$ contains representatives of all the vertices $\mathfrak{N}_{\Lambda_1}(I)$. Since I is generated by $\{F_1, \dots, F_s\}$, there are $H_p^q \in \mathbb{K}\{x\}$ such that
$$F_{s+p} = \sum_{q=1}^{s} H_p^q F_q \,, \qquad p = 1, \dots, r-s.$$

(3) By Theorem 2.5.5, there exist $Q_m^{i,j} \in \mathbb{K}\{x\}$, $i, j, m \in \{1, \dots, r\}$, such that
$$S_{i,j} = \sum_{m=1}^{r} Q_m^{i,j} F_m \qquad \text{and} \qquad \exp_{\Lambda_1} S_{i,j} \leq \min\{\exp_{\Lambda_1}(Q_m^{i,j} F_m) : m = 1, \dots, r\}\,.$$

Let $J = (\phi_1, \dots, \phi_m) \subseteq \mathbb{K}\{x\}$. The flatness of ϕ implies that the representatives of $\phi_1, \dots, \phi_m$ form a regular sequence in $\mathbb{K}\{x\}/I$. By Theorem A.0.6 the ring $\mathbb{K}\{x\}/(I+J)$ is Cohen-Macaulay with dimension $\dim(\mathbb{K}\{x\}/I) - m = n-k-m$. Therefore, as in (1), after a generic linear change of coordinates in $\mathbb{K}^n$ there exists $l_2 \geq 1$ such that for the linear form

$$\Lambda_2(\beta) = \sum_{i=1}^{k+m} \beta_i + \sum_{j=k+m+1}^{n} l_2 \beta_j \,,$$

we have

$$\mathfrak{N}_{\Lambda_2}(I+J) = D_2 \times \mathbb{N}^{n-k-m}, \text{ for some } D_2 \subset \mathbb{N}^{k+m}. \tag{4.1.3}$$

Since *generic* linear changes of coordinates form an open dense subset of the space of all linear changes of coordinates, and both the coordinate changes made above are generic, we can, in fact, chose one change of coordinates for which both (4.1.2) and (4.1.3) hold.

Now, we observe that $I + J = (F_1, \ldots, F_s, \phi_1, \ldots, \phi_m)$. We may extend the set of generators of this ideal by $\phi_{m+1}, \ldots, \phi_l$, such that $F_1, \ldots, F_s, \phi_1, \ldots, \phi_l$ form a standard basis for $I + J$ with respect to Λ_2. We then have $\bar{H}_p^q \in \mathbb{K}\{x\}$ such that

$$\phi_{m+p} = \sum_{q=1}^{s} \bar{H}_p^q F_q + \sum_{q=1}^{m} \bar{H}_p^{q+s} \phi_q \,, \qquad p = 1, \ldots, l-m.$$

Let $S_{i,j}^{(1)} = S(F_i, F_j)$ for $1 \leq i < j \leq s$, $S_{i,j}^{(2)} = S(F_i, \phi_j)$ for $i \in \{1, \ldots, s\}$, $j \in \{1, \ldots, l\}$, and $S_{i,j}^{(3)} = S(\phi_i, \phi_j)$ for $1 \leq i < j \leq l$ be the s-series with respect to the ordering induced by Λ_2. We may now apply Theorem 2.5.5 to conclude that there exist: $\tilde{Q}_m^{i,j}, \bar{Q}_m^{i,j}, \hat{Q}_m^{i,j} \in \mathbb{K}\{x\}$ with index ranges $1 \leq i < j \leq s$, $i \in \{1, \ldots, s\}$, $j \in \{1, \ldots, l\}$, and $1 \leq i < j \leq l$ respectively, such that

$$S_{i,j}^{(1)} = \sum_{m=1}^{s} \tilde{Q}_m^{i,j} F_m + \sum_{m=1}^{l} \tilde{Q}_{m+s}^{i,j} \phi_m$$

$$S_{i,j}^{(2)} = \sum_{m=1}^{s} \bar{Q}_m^{i,j} F_m + \sum_{m=1}^{l} \bar{Q}_{m+s}^{i,j} \phi_m$$

and

$$S_{i,j}^{(3)} = \sum_{m=1}^{s} \hat{Q}_m^{i,j} F_m + \sum_{m=1}^{l} \hat{Q}_{m+s}^{i,j} \phi_m$$

In the above equations we have $\exp_{\Lambda_2} S_{i,j}^{(1)} \leq \min\{e_1, e_2\}$, $\exp_{\Lambda_2} S_{i,j}^{(2)} \leq \min\{f_1, f_2\}$, and $\exp_{\Lambda_2} S_{i,j}^{(3)} \leq \min\{g_1, g_2\}$ where,

$$e_1 = \min\{\exp_{\Lambda_2}(\tilde{Q}_m^{i,j} F_m) : m = 1, \ldots, s\},$$

$$e_2 = \min\{\exp_{\Lambda_2}(\tilde{Q}_{m+r}^{i,j} \phi_m) : m = 1, \ldots, l\},$$

$$f_1 = \min\{\exp_{\Lambda_2}(\bar{Q}_m^{i,j} F_m) : m = 1, \ldots, s\},$$

$$f_2 = \min\{\exp_{\Lambda_2}(\bar{Q}_{m+r}^{i,j} \phi_m) : m = 1, \ldots, l\},$$

$$g_1 = \min\{\exp_{\Lambda_2}(\hat{Q}_m^{i,j} F_m) : m = 1, \ldots, s\},$$

$$g_2 = \min\{\exp_{\Lambda_2}(\hat{Q}_{m+r}^{i,j} \phi_m) : m = 1, \ldots, l\}.$$

Now, there are monomials $P_{i,j}, \bar{P}_{j,i}, P_{i,j}^{(1)}, \bar{P}_{j,i}^{(1)}, P_{i,j}^{(2)}, \bar{P}_{j,i}^{(2)}, P_{i,j}^{(3)}, \bar{P}_{i,j}^{(3)}$ such that

$$S_{i,j} = P_{i,j} F_i - \bar{P}_{i,j} F_j$$

$$S_{i,j}^{(1)} = P_{i,j}^{(1)} F_i - \bar{P}_{i,j}^{(1)} F_j$$

$$S_{i,j}^{(2)} = P_{i,j}^{(2)} F_i - \bar{P}_{i,j}^{(2)} \phi_j$$

$$S_{i,j}^{(3)} = P_{i,j}^{(3)} \phi_i - \bar{P}_{i,j}^{(3)} \phi_j$$

Further, these monomials only depend on the initial terms of the F_k, ϕ_k involved on the right hand sides of the above equations taken with respect to the appropriate ordering (i.e., the one induced by Λ_1, or Λ_2). We now consider the following system of equations in variables $y = (y_1, \ldots, y_r)$, $z = (z_1^{1,2}, \ldots, z_r^{r-1,r})$, $w = (w_1^1, \ldots, w_{r-s}^s)$, $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_l)$, $\tilde{z} = (\tilde{z}_1^{1,1}, \ldots, \tilde{z}_{s+l}^{s,l})$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_l)$, $\bar{z} = (\bar{z}_1^{1,1}, \ldots, \bar{z}_{s+l}^{s,l})$, $\hat{z} =$

$(\hat{z}_1^{1,2}, \ldots, \hat{z}_{s+l}^{l-1,l})$, $\bar{w} = (\bar{w}_1^1, \ldots, \bar{w}_{l-m}^{s+l})$.

$$\begin{cases} P_{i,j}(x)y_i - \bar{P}_{j,i}(x)y_j - \sum_{m=1}^{r} z_m^{i,j} y_m = 0 \\ P_{i,j}^{(1)}(x)y_i - \bar{P}_{j,i}^{(1)}(x)\tilde{y}_j - \sum_{m=1}^{s} \tilde{z}_m^{i,j} y_m - \sum_{m=1}^{l} \tilde{z}_{m+s}^{i,j} \tilde{y}_m = 0 \\ P_{i,j}^{(2)}(x)y_i - \bar{P}_{j,i}^{(2)}(x)\bar{y}_j - \sum_{m=1}^{s} \bar{z}_m^{i,j} y_m - \sum_{m=1}^{l} \bar{z}_{m+s}^{i,j} \bar{y}_m = 0 \\ P_{i,j}^{(3)}(x)\bar{y}_i - \bar{P}_{j,i}^{(3)}(x)\bar{y}_j - \sum_{m=1}^{s} \hat{z}_m^{i,j} y_m - \sum_{m=1}^{l} \hat{z}_{m+s}^{i,j} \bar{y}_m = 0 \\ y_{s+p} - \sum_{q=1}^{s} w_p^q y_q = 0 \\ \bar{y}_{s+p} - \sum_{q=1}^{s} \bar{w}_p^q y_q - \sum_{q=1}^{m} \bar{w}_p^{q+s} = 0 \end{cases} \tag{4.1.4}$$

This system has a solution in convergent power series $\{F_i, Q_m^{i,j}, \tilde{Q}_m^{i,j}, \bar{Q}_m^{i,j}, H_p^q, \phi_i, \hat{Q}_m^{i,j}, \bar{H}_p^q\}$, and hence by Theorem 2.6.2, for every $\mu \in \mathbb{N}$ an algebraic power series solution $\{G_i, R_m^{i,j}, \tilde{R}_m^{i,j}, \bar{R}_m^{i,j}, K_p^q, \psi_i, \hat{R}_m^{i,j}, \bar{K}_p^q\}$ such that

$$\begin{aligned} j^\mu G_i &= j^\mu F_i \\ j^\mu R_m^{i,j} &= j^\mu Q_m^{i,j} \\ j^\mu \tilde{R}_m^{i,j} &= j^\mu \tilde{Q}_m^{i,j} \\ j^\mu \bar{R}_m^{i,j} &= j^\mu \bar{Q}_m^{i,j} \\ j^\mu \hat{R}_m^{i,j} &= j^\mu \hat{Q}_m^{i,j} \\ j^\mu \psi_i &= j^\mu \phi_i \end{aligned}$$

for all allowable values of the indices.

Taking μ_0 sufficiently large, so as to satisfy all the inequalities on the initial exponents with respect to both orderings (i.e., the ordering corresponding to Λ_1 and the one corresponding to Λ_2) we can conclude, as in the proof of Theorem 3.2.1, the following:

(1) The analytic germ $\hat{X}_0$ with local ring $\mathbb{K}\{x\}/\hat{I}$ where $\hat{I} = (G_1, \ldots, G_s)$ is Cohen-Macaulay and has the same Hilbert-Samuel function as X_0. This is by an argument in terms of the ordering corresponding to Λ_1.

(2) If $\hat{J} = (\psi_1, \ldots, \psi_m)$ then $\mathbb{K}\{x\}/(\hat{I} + \hat{J})$ is Cohen-Macaulay and $(\hat{I} + \hat{J})$ has the same Hilbert-Samuel function as $(I+J)$. This implies that $\dim \mathbb{K}\{x\}/(I+J) = \dim \mathbb{K}\{x\}/(\hat{I}+\hat{J})$. This is by an argument in terms of the ordering corresponding to Λ_2.

Point (2) above allows us to apply Theorem A.0.9 to conclude that the map $\psi : \hat{X}_0 \to \mathbb{K}^n_0$ defined by $\psi = (\psi_1, \ldots, \psi_m)$ is flat at zero. $\square$

In the case when the germ X is already Nash, we have the following corollary,

Corollary 4.1.3. *Let X be a $\mathbb{K}$-analytic subspace of $\mathbb{K}^n$. Suppose that $0 \in X$, X_0 is a Nash germ, the local ring $\mathcal{O}_{X,0}$ is Cohen-Macaulay, and that $\mathcal{O}_{X,0} = \mathbb{K}\{x\}/I$ where $I = (F_1, \ldots, F_s)$. Also, let $\phi = (\phi_1, \ldots, \phi_m) : X \to \mathbb{K}^m$ with $\phi(0) = 0$ be a $\mathbb{K}$-analytic mapping which is flat at 0. Then for some $\mu_0 \in \mathbb{N}$, and each $\mu \geq \mu_0$ there is a Nash map $\psi = (\psi_1, \ldots \psi_m) : X \to \mathbb{K}^m$ which is flat at 0 and such that $j^\mu \phi_i = j^\mu \psi_i$ for all i. Further, if $J = (\phi_1, \ldots, \phi_m) \subseteq \mathbb{K}\{x\}$, and $\hat{J} = (\psi_1, \ldots, \psi_m) \subseteq \mathbb{K}\langle x\rangle \subseteq \mathbb{K}\{x\}$, then we have $H_{I+J} = H_{I+\hat{J}}$.*

In this case there is no need to approximate the generators of I because they are already algebraic power series. They appear as coefficients in the system of equations for approximating ϕ. As in the proof of Theorem 4.1.2, Theorem 2.6.2 will yield the required approximating map ψ.

Remark 4.1.4. *In the proof of Theorem 4.1.2 the system of equations that we are applying Theorem 2.6.2 to has coefficients that are polynomials in x so, in a sense, we are not using the full strength of the theorem. This is not true for Corollary 4.1.3.*

4.2 Equisingular deformations

As stated in Remark 3.4.2, it is an open question whether the unfoldings $G_i(t,x)$ of the defining functions g_i of X_0 in the proof of Theorem 3.4.1 in Section 3.4.5 define a deformation in the case when X_0 is not a complete intersection. If this is indeed the case, then this would be a deformation that would be both topologically equisingular and equisingular in the sense of the Hilbert-Samuel function. Further, the proof of Theorem 3.4.1 would imply that there is a collection of such deformations, one associated to each $\mu > \mu_0$ and preserving the μ-jets of the defining functions of X_0. This remains an avenue for future work.

4.3 Equiresolution

In this section we point to a potential approximation result that is closely connected to the resolution of singularities of an analytic space.

Hironaka's Desingularization theorem [16] is an extremely important result in singularity theory. This was subsequently extended to analytic spaces in [17, 18]. Subsequently, several alternative proofs and formulations of the result have been published [10, 37, 34] with the aim to simplify the method of proof used by Hironaka, underscoring the fact that it remains a pivotal result in the field. In this section we follow the notation and language used in [10], specifically [10, Theorem 1.6], which is equivalent to [16, Main Theorem I].

Let $X \subseteq \mathbb{K}^n$ be a compact reduced analytic space. We are making these assumptions to simplify the following exposition (see Remark 4.3.1). Hironaka's Desingularization Theorem states that we can find a finite sequence of blowups σ_i of the ambient space, for $i = 1, \ldots, p$, with non-singular centers C_{i-1} that resolve the singularities of X,

$$X_p \xrightarrow{\sigma_p|_{X_p}} X_{p-1} \xrightarrow{\sigma_{p-1}|_{X_{p-1}}} \ldots \xrightarrow{\sigma_2|_{X_2}} X_1 \xrightarrow{\sigma_1|_{X_1}} X_0 = X \qquad (4.3.1)$$

where the X_j denotes the strict transform of X_{j-1} by σ_j, and X_p is smooth. We denote by E_j the set of exceptional hypersurfaces, that is, E_j is the set of the strict transforms of $H \in E_{j-1}$ together with $\sigma_j^{-1}(C_{j-1})$. It is important to emphasize here that unlike in the rest of this thesis X_0 *does not* represent the germ of X at 0, instead it represents an integer subscript. Further, [10, Theorem 1.6] states that $C_j \subseteq \mathrm{Sing} X_j$ if X_j is singular or $X_j \cap E_j$ if X_j is smooth and that the X_p and E_p have only simple normal crossings.

Remark 4.3.1. *(i) For non-compact analytic spaces, the sequences of blow-ups for resolving singularities by Hironaka's Theorem will be locally finite instead of finite. Since all our considerations in this thesis are local in nature, little is lost by making this assumption.*

(ii) The statement of Hironaka's Theorem becomes slightly more complicated in the non-reduced case. A development of the concepts required to incorporate this more complicated statement would represent a significant excursion into concepts that would have limited relevance to the main exposition in this section. We refer the reader to [10, Theorem 13.4] for a very general version of Hironaka's Theorem.

A detailed development of the general notion of blowups, and strict transforms that are central to Hironaka's resolution of singularities is beyond the scope of this thesis, and we refer the reader to [10] for this. That said, it is instructive at this juncture to explicitly present the equations defining the blowup of $\mathbb{K}^n$ with center given by a coordinate subspace, say, $C := \{x \in \mathbb{K}^n | x_{k+1} = \cdots = x_n = 0\}$.

Definition 4.3.2. *Let $\mathbb{P}^{n-k-1}$ be the projective space of dimension $n-k-1$ over $\mathbb{K}$, and let $(u_{k+1} : \cdots : u_n)$ be projective coordinates on $\mathbb{P}^{n-k-1}$. The blowup $\tilde{K}^n$ of $\mathbb{K}^n$ with center given by $\{(x_1, \dots, x_n) \in \mathbb{K}^n | x_{k+1} = \cdots = x_n = 0\}$ is the subset of*

$\mathbb{K}^n \times \mathbb{P}^{n-k-1}$ *defined by the following equations:*

$$u_i x_j = u_j x_i \quad \text{for } i, j = k+1, \ldots, n.$$

with the blowup map $\sigma : \tilde{K}^n \to \mathbb{K}^n$ *given by the restriction of the natural projection* $\pi : \mathbb{K}^n \times \mathbb{P}^{n-k-1} \to \mathbb{K}^n$ *to* $\tilde{K}^n$*.*

We can use local charts to cover $\mathbb{P}^{n-k-1}$ and arrive at local description of $\tilde{K}^n$ and $\sigma : \tilde{K}^n \to \mathbb{K}^n$. For $j \in \{k+1, \ldots, n\}$ we set $U_j := \mathbb{K}^n$ and define a glueing map between any two such *charts* U_j and U_l as follows,

$$\begin{aligned}
x_i &\mapsto x_i/x_j \quad \text{for } i \in \{k+1, \ldots, n\} \setminus \{j, l\}, \\
x_j &\mapsto 1/x_l, \\
x_l &\mapsto x_j x_l, \\
x_i &\mapsto x_i \quad \text{for } i \in \{1, \ldots, k\}
\end{aligned}$$

This defines the variety $\tilde{K}^n$. We may now define the morphism $\sigma : \tilde{K}^n \to \mathbb{K}^n$ by specifying it explicitly in the charts U_j. That is, by maps, $\sigma^j : \mathbb{K}^n \to \mathbb{K}^n$ for $j = k+1, \ldots, n$, defined as follows,

$$\begin{aligned}
x_i &\mapsto x_i, \quad \text{for } i \in \{1, \ldots, k\} \cup j, \\
x_i &\mapsto x_i x_j \quad \text{for } i \in \{k+1, \ldots, n\} \setminus j.
\end{aligned}$$

We see from this that locally a blowup is a quadratic map.

Now returning to Hironaka's resolution of singularities, the smoothness of the centers of the blowups in (4.3.1) implies that locally there exists a system of coordinates in which each C_j is a coordinate subspace. That is, C_j is locally analytically isomorphic to $\{x \in U \subseteq \mathbb{K}^n | x_{k+1} = \cdots = x_n = 0\}$ for some neighborhood U of 0, and some k. If we make these coordinate changes explicit, then from (4.3.1) we have $U_0, U_1, \dots, U_p$ such that,

$$X_p \cap U_p \xrightarrow{\sigma_p|_{X_p \cap U_p}} X_{p-1} \cap U_{p-1} \xrightarrow{\sigma_{p-1}|_{X_{p-1} \cap U_{p-1}}} \cdots\cdots \xrightarrow{\sigma_2|_{X_2 \cap U_2}} X_1 \cap U_1 \xrightarrow{\sigma_1|_{X_1 \cap U_1}} X_0 \cap U_0 \qquad (4.3.2)$$

where U_0 is a sufficiently small neighborhood of 0 in $\mathbb{K}^n$, $U_i = \sigma_i^{-1}(U_{i-1})$ for $i = 1, \dots, p$, and $\sigma_i|_{U_i} = \tilde{\sigma}_i \circ \phi_i|_{U_i}$ where $\phi_i : \mathbb{K}^n \to \mathbb{K}^n$ is an analytic map representing the change of coordinates and $\tilde{\sigma}_i$ is a blowup of the ambient space with center given by a coordinate subspace.

Now, by Theorem 3.4.1 there exists a homeomorphism h defined on a sufficiently small neighborhood of zero in $\mathbb{K}^n$ such that, restricting U_0 if necessary, $h_0(X_0 \cap U_0) = \hat{X}_0 \cap h(U_0)$, where $\hat{X}_0$ is an analytic space whose germ at 0 is Nash. A question that is of interest is whether we can find $\bar{\sigma}_i = \tilde{\sigma} \circ \hat{\phi}_i$, where $\hat{\phi}_i$ are Nash morphisms that approximate the ϕ_i, and homeomorphisms h_i such that the following diagram commutes,

$$\begin{array}{ccccccccc}
X_p \cap U_p & \xrightarrow{\sigma_p|_{X_p \cap U_p}} & X_{p-1} \cap U_{p-1} & \xrightarrow{\sigma_{p-1}|_{X_{p-1} \cap U_{p-1}}} \cdots & \xrightarrow{\sigma_2|_{X_2 \cap U_2}} & X_1 \cap U_1 & \xrightarrow{\sigma_1|_{X_1 \cap U_1}} & X_0 \cap U_0 \\
\downarrow{\scriptstyle h_p|_{X_p \cap U_p}} & & \downarrow{\scriptstyle h_{p-1}|_{X_{p-1} \cap U_{p-1}}} & & & \downarrow{\scriptstyle h_1|_{X_1 \cap U_1}} & & \downarrow{\scriptstyle h|_{X_0 \cap U_0}} \\
\hat{X}_p \cap V_p & \xrightarrow{\bar{\sigma}_p|_{\hat{X}_p \cap V_p}} & \hat{X}_{p-1} \cap V_{p-1} & \xrightarrow{\bar{\sigma}_{p-1}|_{\hat{X}_{p-1} \cap V_{p-1}}} \cdots & \xrightarrow{\bar{\sigma}_2|_{\hat{X}_2 \cap V_2}} & \hat{X}_1 \cap V_1 & \xrightarrow{\bar{\sigma}_1|_{\hat{X}_1 \cap V_1}} & \hat{X}_0 \cap V_0
\end{array} \qquad (4.3.3)$$

where $V_i = h_i(U_i)$. An affirmative answer to this question would tell us that, up to homeomorphism, in the sense of Hironaka's resolution, analytic singularities are locally Nash.

We expect that obtaining an answer to this question will be a challenging endeavor. As a first step, it is logical to consider a simpler problem. We assume that we have a homeomorphism h between $X \cap U$ and $\hat{X} \cap V$, where U is a small neighborhood of the origin in $\mathbb{K}^n$ and $V = h(U)$, such as the one we get from Theorem 3.4.1, and that the coordinate changes required to align the first blow-up in the sequence required for desingularization of X and $\hat{X}$, with a coordinate subspace have already been made. Then we may ask whether we can find a map H that is a homeomorphism near the origin in $\mathbb{K}^n$ such that the following diagram commutes,

$$\begin{array}{ccc} X_1 \cap U_1 & \xrightarrow{\tilde{\sigma}|_{X_1 \cap U_1}} & X \cap U \\ \Big\downarrow H|_{X_p \cap U_p} & & \Big\downarrow h|_{X \cap U} \\ \hat{X}_1 \cap V_1 & \xrightarrow{\tilde{\sigma}|_{\hat{X}_1 \cap V_1}} & \hat{X} \cap V \end{array} \tag{4.3.4}$$

Obviously a negative answer to this question would imply a negative answer to the more challenging question we posed before.

The most promising line of attack for obtaining H from (4.3.4) is based on a recent paper [25], in which the authors prove that the homeomorphism h in Theorem 3.4.1 can be chosen to be arc-analytic and subanalytic. These classes of maps have many nice properties and are of much importance in real analytic and algebraic geometry (see [20], [8], [9]). It is possible that these much stronger properties can be used to show that h lifts through the blowups in (4.3.4).

4.4 Simultaneous topological equisingularity of resolutions

In this section we shall outline another possible avenue of future research. In order to keep the exposition simple we shall describe our objective for the case of a single blowup as in the last part of the previous section. Once again, for simplicity of exposition, we assume that X is a reduced and compact analytic space in $\mathbb{K}^n$. Suppose the germ of X at zero defined by equations $\{f_1 = \cdots = f_k = 0\}$ where $f_i \in \mathbb{K}\{x\}$. We see from Definition 4.3.2, that the blowup X_1 of X with center given by $C = \{x \in \mathbb{K}^n | x_{r+1} = \cdots = x_n = 0\}$ is an analytic space in $\mathbb{K}^n \times \mathbb{P}^{n-r-1}$ defined by the following equations,

$$f_i = 0 \quad \text{for } i = 1, \dots, k \tag{4.4.1}$$

$$u_i x_j = u_j x_i \quad \text{for } i, j \in \{r+1, \dots n\}. \tag{4.4.2}$$

Since we are only interested in local considerations, by choosing a chart for $\mathbb{P}^{n-r-1}$ we can consider this to be an analytic space embedded in $\mathbb{K}^n \times \mathbb{K}^{n-r-1}$. Now we may ask if we can find a topological map $H : \mathbb{K}^n \times \mathbb{K}^{n-r-1} \to \mathbb{K}^n \times \mathbb{K}^{n-r-1}$ that is a homeomorphism in some neighborhood of the origin U such that $H(X_1 \cap U_1) = \hat{X}_1 \cap V_1$ where V_1 is a neighborhood of the origin in $\mathbb{K}^n \times \mathbb{K}^{n-r-1}$ and $\hat{X}_1$ is the blowup of an analytic space $\hat{X}$ in $\mathbb{K}^n$ whose germ at the origin is Nash and which approximates X. Further, if we set $S := \{(x, u) \in \mathbb{K}^n \times \mathbb{K}^{n-r-1} | u = 0\}$ and identify $(X \times \mathbb{K}^{n-r-1}) \cap S$ with X, and $(\hat{X} \times \mathbb{K}^{n-r-1}) \cap S$ with $\hat{X}$, we may also impose the additional condition on H that $H|_S(X \cap U) = \hat{X} \cap V$ where U and V are open neighborhoods of the origin in $\mathbb{K}^n \times \{0\}^{n-r-1}$.

This would give us something that can best be described as simultaneous (local) topological equisingularity of X and $\hat{X}$ and their blowups. It is important to note at this point that H and $H|_S$ may not necessarily commute as H and h do in the diagram (4.3.4). Suppose we have $(\tilde{x}, u) \in X_1$ and that we are working in the chart corresponding to $u_n \neq 0$. Further, let $H := (H_1, \dots, H_n, H'_{r+1}, \dots, H'_{n-1})$. Then, working around one leg of the commutative diagram (4.3.4) we have,

$$(\tilde{x}, u) = (x_1, \dots, x_r, u_{r+1}x_n, \dots, u_{n-1}x_n, x_n, u_{r+1}, \dots, u_{n-1}) \quad \text{for some } x_i, u_j \in \mathbb{K}. \tag{4.4.3}$$

$$\sigma(x, u) = (x_1, \dots, x_r, u_{r+1}x_n, \dots, u_{n-1}x_n, x_n) \tag{4.4.4}$$

This point is then identified with $(\tilde{x}, 0) = (x_1, \dots, x_r, u_{r+1}x_n, \dots, u_{n-1}x_n, x_n, 0, \dots, 0) \in$

$\mathbb{K}^n \times \mathbb{K}^{n-r-1}$. Then we have,

$$H((\tilde{x},0) = (H_1(\tilde{x},0),\ldots,H_n(\tilde{x},0),0,\ldots,0) \tag{4.4.5}$$

Now, going around the other leg of the commutative diagram we observe,

$$H(\tilde{x},u) = (H_1(\tilde{x},u),\ldots,H_n(\tilde{x},u),H'_{r+1}(\tilde{x},u),\ldots,H'_{n-1}(\tilde{x},u)) \tag{4.4.6}$$

$$\sigma(H_1(\tilde{x},u),\ldots,H_n(\tilde{x},u)) = (H_1(\tilde{x},u),\ldots,H_n(\tilde{x},u)). \tag{4.4.7}$$

This point is then identified with $(H_1(\tilde{x},u),\ldots,H_n(\tilde{x},u),0,\ldots,0)$ which need not be the same as $(H_1(\tilde{x},0),\ldots,H_n(\tilde{x},0),0\ldots,0)$.

One possible way of approaching this problem is to follow an approach that is similar to the one used in Section 3.4. The most direct approach would be to construct a normal system of equations starting with an initial product f consisting of the generators of X (i.e. f_i) along with additional factors required to define the blowup (4.4.2). The hope is that, we can proceed as in Section 3.4 and apply Theorem 3.3.1 to get the required homeomorphism. There is, however, an obstacle to using a simple modification of the technique in Section 3.4. This is the fact that in the step in Section 3.4.4 where the approximation theorem (Theorem 2.6.1) is applied, the statement that we are pursuing would require the approximations to the f_i to be independent of the variables $u = (u_{r+1},\ldots,u_n)$, and this, in turn, would violate the nestedness requirement on the dependencies that is present in Theorem 2.6.1.

The problem of Artin's approximation with restrictions on the variable dependencies is referred to in some literature as *Artin's approximation with constraints* (see [29]). Due to an example by J. Becker [4], which shows that a similar statement where the dependencies are *disjoint* is false, there is a reason to suspect that obtaining a version of Artin's approximation theorem with the constraints required for the approach outlined above to succeed may be impossible. That does not preclude, however, the existence of algebraic power series solutions to the *particular* system of equations that is relevant to our considerations, and the proof of this existence is likely to be a fundamental component in a successful endeavor to obtain the weaker result that we describe in this section.

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Appendix A

Miscellaneous concepts from local algebra

In this appendix we present certain definitions and theorems from local algebra. This material may be found in any standard textbook on commutative algebra such as [21]. In the remainder of this section $(A, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated module over A.

Definition A.0.1. *An ideal I of A is called a* complete intersection ideal *if I can be generated by* $\dim A - \dim A/I$ *elements of A.*

Definition A.0.2. *An analytic germ X_0 in $\mathbb{K}^n_0$ is called a* complete intersection singularity *if its local ring $\mathcal{O}_{X,0}$ is isomorphic to $\mathbb{K}\{x\}/I$ where I is a complete intersection ideal.*

Definition A.0.3. *A sequence $a_1, \ldots, a_l \in \mathfrak{m}$ is called M-*regular *if a_1 is not a zero-divisor in M and a_{i+1} is not a zero-divisor in $M/(a_1, \ldots, a_i)M$ for $i = 1, \ldots, l-1$.*

Definition A.0.4. *M is called* Cohen-Macaulay *when* $\operatorname{depth}_A(M) = \dim M$*, where* $\operatorname{depth}_A(M)$ *is the maximum length of an M-regular sequence in $\mathfrak{m}$. A local ring A is Cohen-Macaulay, when A is Cohen-Macaulay as an A-module.*

Definition A.0.5. *An analytic germ X_0 in $\mathbb{K}^n_0$ is called* Cohen-Macaulay *(or a* Cohen-Macaulay singularity*) if its local ring $\mathcal{O}_{X,0}$ is Cohen-Macaulay.*

Theorem A.0.6. *Let $x_1, \ldots, x_n \in \mathfrak{m}_A$, and M be a finitely generated A-module.*

(1) If $x_1, \ldots, x_n$ is M-regular then M is Cohen-Macaulay if and only if $M/(x_1, \ldots, x_n)$ is Cohen-Macaulay.

(2) Let M be Cohen-Macaulay, then the sequence $x_1, \ldots, x_n$ is M-regular if and only if $\dim(M/(x_1, \ldots, x_n)) = \dim(M) - n$.

Theorem A.0.7. *(Nakayama's Lemma) Let I be an ideal in A, and M a finitely generated module over A. If $IM = M$ then there exists an $r \equiv 1(\mathrm{mod} I)$ such that $rM = 0$.*

Remark A.0.8. *Theorem A.0.7 is also valid when the ring A is not a local ring.*

We shall also have occasion to use the following result regarding flatness ([15, Theorem B.8.12]).

Theorem A.0.9. *Let $\phi : A \to B$ be a morphism of local rings with A regular and M a finitely generated B-module. Let $x_1, \ldots, x_d$ be a minimal set of generators of the maximal ideal of A, $\mathfrak{m}_A$, and $f_i = \phi(x_i)$ for $i = 1, \ldots, d$. Then the following are equivalent:*

(1) M is A-flat.

(2) $\mathrm{depth}_A(M) = d$, or equivalently $\mathrm{depth}_B(\mathfrak{m}_A B, M) = d$.

(3) $f_1, \ldots, f_d$ is an M-regular sequence.

In particular, if B is Cohen-Macaulay, the ϕ is flat if and only if,

$$\dim B = \dim A + \dim B/\mathfrak{m}_A B. \tag{A.0.1}$$

The following result is a consequence of the equivalence of (1) and (2) in the above and the fact that flatness is equivalent to freeness for finitely generated modules.

Corollary A.0.10. *Let M be a finitely generated module over a regular local ring A. Then M is Cohen-Macaulay if and only if it is free.*