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On the Geometry of the Moduli Space of Certain Lattice Polarized K3 Surfaces and Their Picard-Fuchs Operators

Michael T. Schultz
Utah State University

ON THE GEOMETRY OF THE MODULI SPACE OF CERTAIN LATTICE POLARIZED K3 SURFACES AND THEIR PICARD-FUCHS OPERATORS

by

Michael T. Schultz

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Approved:

Andreas Malmendier, Ph.D. Major Professor	Ian Anderson, Ph.D. Committee Member
Nathan Geer, Ph.D. Committee Member	Zhaohu Nie, Ph.D. Committee Member
Oscar Varela, Ph.D. Committee Member	D. Richard Cutler, Ph.D. Interim Vice Provost for Graduate Studies

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ABSTRACT

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lattice polarized K3 surface and their
Picard-Fuchs operators

by

Michael T. Schultz, Doctor of Philosophy

Major Professor: Dr. Andreas Malmendier, PhD.
Department: Mathematics and Statistics

We study the moduli space of certain families of lattice polarized K3 surface via their period integrals and corresponding Picard-Fuchs system, in particular by utilizing explicit Jacobian elliptic fibrations that realize the lattice polarizations. Moreover, we study how other data that governs the complex structure of the elliptic fibres of certain generic fibrations determines global information about a Jacobian elliptic K3 surface in terms of string theoretic and index theoretic terms via holomorphic anomalies.

We demonstrate how the mixed-twist construction of Doran & Malmendier when applied to a certain family of rational elliptic surfaces yields the famous double sextic family, equipped with the canonical lattice polarization from the branching locus of the double cover. We show how restrictions of the moduli produce subvarieties on which the lattice polarization extends. Moreover, we show how the mixed-twist construction allows for the explicit computation of the monodromy of the mirror family of Calabi-Yau n-folds.

We show that for a certain family of lattice polarized K3 surfaces of Picard rank $\rho \geq 17$, the twisted Legendre pencil, that the moduli space carries an integrable holomorphic conformal structure. In the language of physics, this is known as a flat special geometry, whose existence has implications for a certain supersymmetric quantum field theory associated to the space. This construction is related to Seiberg-Witten theory via the mixed-twist construction. Implications and future directions are discussed at the end.

(251 pages)

PUBLIC ABSTRACT

On the geometry of the moduli space of certain

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Michael T. Schultz

K3 surfaces have a long and rich study in mathematics, and more recently in physics via string theory. Often, K3 surfaces come in multiparameter families - the parameters describing these surfaces fit together to form their own geometric space, a so-called *moduli space*. In particular, the moduli spaces of K3 surfaces equipped with a *lattice polarization* can sometimes be constructed explicitly, which subsequently reveals important information about the original K3 surface.

In this work, we construct such families explicitly from certain rational elliptic surfaces via the so-called *mixed-twist construction* of Doran & Malmendier, which in turn produces the moduli space. After identifying the lattice polarization by computing Jacobian elliptic fibrations, we find a rich differential geometric content imparted to the moduli space - an integrable holomorphic conformal structure - via quadratic relations satisfied by the period integrals of the K3 surface. This geometry allows one to compute crucial data about the K3 surface family, the *Picard-Fuchs operators*, by applying a general programme on uniformizing differential equations discovered by Sasaki & Yoshida. In physics, this differential geometric data is known as a flat special geometry, and has implications for a type of supersymmetric quantum field theory associated with the K3 surface. Via the mixed-twist construction, this is related to $N_f = 4$ Seiberg-Witten curves from $\mathcal{N} = 2$ SU(2) super Yang-Mills theory with various mass configurations.

We show as well how one can restrict the moduli, leading to subvarieties of the moduli space on which the lattice polarization extends. This can allow one to construct interesting families of Calabi-Yau manifolds, which are of crucial importance in string theory as well. Moreover, we study how other data that governs the complex structure of the elliptic fibres of certain generic fibrations determines global information about a Jacobian elliptic K3 surface in terms of string theoretic and index theoretic terms via holomorphic anomalies.

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I am especially grateful to my beautiful wife Cana Itchuaqiyaq, and our two dogs Reese and Pattycake, who have been with me through this entire process and giving me probably more grace than I deserve. I will always remember finishing writing my dissertation in our little Shasta camper with you. Thank you for helping me grow, growing with me, and helping me believe in myself.

I dedicate this work to my sister Megan and my friend Joe.

Michael Thomas Schultz

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CHAPTER 1

Introduction

The subject of K3 surfaces traces back to the time of Kummer, who in 1864 published *Über die Flächen vierten Grades mit sechzehn singulären Punkten* [82], which made systematic study of quartic hypersurfaces in $\mathbb{P}^3$ with a maximum number of sixteen nodal singularities. The study of the rich geometry of these surfaces, which can be traced to the configurations of such nodal singularities, or rational double points, continues to present day, and pervading algebraic and arithmetic geometry, as well as more recently mathematical physics.

One may show that the minimal resolution of Kummer's quartic surface $\mathbf{X}$ is simply connected and possesses a global nonvanishing holomorphic 2-form $\eta_{\mathbf{X}} \in H^{2,0}(\mathbf{X})$ that trivializes the canonical bundle $K_{\mathbf{X}} = \bigwedge^2 T^*_{\mathbb{C}}\mathbf{X}$, which makes $\mathbf{X}$ a K3 surface.

Definition 1.0.1. *A projective surface* $\mathbf{X} \subset \mathbb{P}^n$ *is a* K3 surface *if the canonical bundle* $K_{\mathbf{X}} \cong \mathcal{O}_{\mathbf{X}}$ *is trivial, and* $h^1(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = 0$.

A celebrated result in algebraic geometry - Chow's Theorem - asserts that smooth, closed, projective submanifolds of $\mathbb{P}^n$ are in fact *algebraic*, that is, realized as the zero locus of some collection of homogeneous polynomials. Thus, after resolving the potential nodal singularities, or rational double points of a K3 surface $\mathbf{X}$, we may restrict ourselves to studying *algebraic K3* surfaces. The algebraicity of the K3 surface is equivalent to choosing a *polarization*, that is, an quasi-ample line bundle $\mathcal{L} \to \mathbf{X}$ such that the image of $\mathbf{X}$ under the induced embedding has at worst rational double points.

Consider a homogeneous quartic polynomial

$$c_{ijkl}X_0^iX_1^jX_2^kX_3^l = 0\,, \tag{1.0.1}$$

with the summation convention implied on the homogeneous coordinates $[X_0 : X_1 : X_2 : X_3] \in \mathbb{P}^3$ and $c_{ijkl} \in \mathbb{C}$, $i, j, k, l = 0, 1, 2, 3, 4.$ with $i + j + k + l = 4$ representing an algebraic K3 surface $\mathbf{X} \subset \mathbb{P}^3$. A count of the parameters, accounting for the rescaling action of $\mathbb{C}^*$ on $\mathbb{P}^3$ and $\mathrm{PGL}(3, \mathbb{C})$ action reparameterizing the variables, including one parameter for the embedding $\mathbf{X} \hookrightarrow \mathbb{P}^3$, yields a total of 19 independent parameters. These are the *moduli* of an algebraic K3 surface, and together they form a 19-dimensional quasiprojective variety, the so-called *moduli space of complex structures* of a K3 surface.

The 19-dimensional moduli space is in general quite difficult to study directly, for one reason simply that the dimension is relatively large. A way to cut down on the dimension of the moduli space is to impose a *lattice polarization*, a primitive lattice embedding into the K3-lattice, $\mathrm{L} \hookrightarrow \Lambda_{\mathrm{K3}} \equiv H^2(\mathbf{X}, \mathbb{Z})$, the second integral cohomology lattice of the K3 surface $\mathbf{X}$, such that the lattice embedding contains a pseudo-ample class. This then specifies the Picard group of algebraic cycles on $\mathbf{X}$, i.e., how the Nerón-Severi group $\mathrm{NS}(\mathbf{X}) = H^{1,1}(\mathbf{X}) \cap H^2(\mathbf{X}, \mathbb{Z})$ sits inside $H^{1,1}(\mathbf{X})$.

One such lattice polarization comes from studying Kummer surfaces - after resolving the sixteen nodal singularities of a Kummer surface $\mathbf{X}$, we obtain a canonical lattice polarization on $\mathbf{X}$ of a rank $\rho \geq 16$ lattice $\mathrm{L} \hookrightarrow \Lambda_{\mathrm{K3}}$. This subsequently means that the moduli space of complex structures of such a surface has dimension $n \leq 4 = (19 + 1) - 16$. This makes the possibility of detailed study of such spaces much more feasible.

Another celebrated theorem - the Torelli theorem - asserts that the K3 surface

$\mathbf{X}$ can be recovered up to isomorphism from its period integrals (up to the action of a certain discrete arithmetic group), that is, integrating the holomorphic 2-form over suitable integral homology cycles. Stated differently, by studying how the period integrals

$$\int_{\Sigma} \eta_{\mathbf{X}} \tag{1.0.2}$$

vary as a function of the complex structure moduli c_{ijkl}, and $\Sigma \in H_2(\mathbf{X}, \mathbb{Z})$), we may study the geometry of the moduli space of complex structures itself. One such way of studying these period integrals is via systems of linear partial differential equations that they satisfy, the so-called *Picard-Fuchs system*.

1.1 Overview and statement of results

In the scope of this dissertation, we study the geometry of certain lattice polarized K3 surfaces via a multipronged approach.

1. Lattice theoretic analysis, in the form of explicit lattice polarizations

2. Geometry, in the form of simultaneous geometrization of the moduli space and K3 surface as a Jacobian elliptic fibration

3. Differential equations, in the form of the Picard-Fuchs system that annihilates the period integrals of the lattice polarized K3 surface

Our approach explicitly constructs the moduli space $\mathcal{M}_{\mathrm{L}}$ of L-polarized complex structures - or simply moduli space for short - of algebraic K3 surfaces equipped with a given lattice polarization $\mathrm{L} \hookrightarrow \Lambda_{\mathrm{K3}}$. The study is enhanced due to the existence of Jacobian elliptic fibrations on the K3 surfaces, or a surjective holomorphic map $\pi : \mathbf{X} \to \mathbb{P}^1$ such that the generic fibre of the map is an elliptic curve and the map π admits a section $\sigma : \mathbb{P}^1 \to \mathbf{X}$. At the level of lattice polarizations, the existence of Jacobian elliptic fibrations is traced to rank-2 sublattices of L isometric to the rank-2 hyperbolic lattice H. This allows for an explicit construction of the period integrals of such an L-polarized K3 surface via Weierstrass models compatible with the lattice polarization. We geometrize the moduli space $\mathcal{M}_{\mathrm{L}}$ via the program of Sasaki & Yoshida [123], which imparts an integrable holomorphich conformal structure (HCS) that decends from the *quadric condition* imposed by the lattice polarization: the period integrals satisfy a quadratic relation that is determined completely by the orthogonal complement of L in the ambient K3 lattice. We determined the HCS explicitly. Such analysis allows for the direct computation of the Picard-Fuchs system from the differential geometric data of the HCS. By doing so, we are able to completely understand the interaction of the moduli space for a special family of Picard rank $\rho \geq 17$ K3 surfaces, the twisted Legendre pencil, which had not previously been done.

Moreover, we show by a different, though explicit, analysis of a certain Jacobian elliptic surface - the universal bundle of elliptic curves - that the generic Jacobian elliptic K3 surface is associated to a type of holomorphic anomaly - one whose existence and resolution is understood through the lens of physics and string theory.

The novelty of this research is to show that the use of two types of functional invariants of Jacobian elliptic surfaces, Kodaira's classical functional invariant [77] and the generalized functional invariant [38, 40], allow for the study of the geometry of moduli spaces of families of elliptic curves and lattice polarized K3 surfaces, respectively. These functional invariants can roughly be characterized as those which "see" quadratic twists, and those that do not, respectively. In the latter case, the mixed-twist construction allows one to produce Jacobian elliptic K3 surfaces who are birational to the quadratic twist family of a family of rational elliptic surfaces. It is well known that the moduli space of lattice polarized K3 surfaces takes the form of a Hermitian symmetric domain of Type IV, but thus far the HCS of Sasaki & Yoshida has not been utilized to simultaneously geometrize both the moduli space and the

K3 surface itself. In this way, we are able to understand a rich interaction between the moduli space, K3 surface itself, and the period integrals, for example completely answering questions set forth by Hoyt in [69] about the twisted Legendre family of Jacobian elliptic K3 surfaces.

Our main results are as follows:

1. The computation of the Quillen anomaly of the $\bar{\partial}$ operator of the universal family of elliptic curves in Theorem 3.1.21, and for the generic Jacobian elliptic surface with only I_1 fibres in Corollary 3.2.25.

2. The cancellation of the local anomaly of the $\bar{\partial}$ operator by use of the construction of a rank-2 SU(2) bundle over the generic elliptic surface with the Poincaré line bundle in Theorem 3.2.28.

3. A chain of explicit lattice polarizations and associated moduli spaces to the family of Yoshida surfaces and restrictions thereof in Theorem 4.2.53. This identifies up to conjugacy and change of variables, the Picard-Fuchs system and associated monodromy groups.

4. The periods, Picard-Fuchs operators, and monodromy groups of the univariate mirror families of Calabi-Yau manifolds from string theory in §4.3.4. In Table 4.1, the monodromy of the families of mirror Calabi-Yau n-folds, $n = 1, 2, 3, 4$, are reproduced via our methods up to conjugacy, matching results known to Candelas et al. [16] and Chen et al. [17].

5. The calculation of the holomorphic conformal structure $\mathbf{g}$ in Equation (6.2.21) for the twisted Legenedre pencil in Theorem 6.2.85.

6. The computation of the Picard-Fuchs system for the twisted Legendre pencil in Corollary 6.2.86.

1.2 Summary of Chapters

In Chapter 2, we review all necessary material. This includes detailed discussion of the periods and Picard-Fuchs systems of algebraic surfaces, as well as how the existence of a Jacobian elliptic fibration and associated Weierstrass model may allow one to glean explicit information about the Picard-Fuchs system and moduli space. We also review all necessary differential topology and index theory related to Hirzebruch's signature to study holomorphic anomalies.

In Chapter 3, we study the vertical signature operator of a Jacobian elliptic surface. This provides an analytic measure - the analytic torsion - of how the complex structure varies on generic rational elliptic surface or Jacobian elliptic surface. We show that the analytic torsion of the universal bundle of elliptic curves lifts to the generic rational elliptic or elliptic K3 surface, and that this quantity manifest a holomorphic anomaly as the generalized first Chern class of the determinant line bundle of the vertical signature operator. We show furthermore that to "resolve" the anomaly completely, we need both string theory and the machinery of an algebro-geometric construction called the Poincaré line bundle, as well as the Riemann-Roch-Grothendieck-Quillen formula from index theory.

In Chapter 4, we study the mixed-twist construction of Doran & Malmendier of a certain two-parameter family of rational elliptic surfaces. We show that the mixed-twist construction yields the celebrated double sextic family of K3 surfaces of Picard rank $\rho \geq 16$. We study the explicit lattice polarizations and moduli spaces, as well as restrictions of the moduli that extend the lattice polarization, including for the twisted Legendre pencil. In §4.3, we show how the mixed-twist construction can be used to study the Picard-Fuchs system of the so-called mirror family of Calabi-Yau n-folds, and derive the explicit monodromy relations for the family.

In Chapter 5, we study in detail algebraic relationships for the twisted Legendre pencil that the periods satisfy - the quadratic period relations that come directly from the lattice polarization restricted to the transcendental lattice. This allows us to compute directly some of the Picard-Fuchs operators for the twisted Legendre pencil, and prepares us to study in detail the differential geometry of the moduli space in the next chapter.

In Chapter 6, we utilize the relationship between the twisted Legendre pencil and the double sextic family of K3 surfaces to explicitly study the geometry of the moduli space of the twisted Legendre pencil. In particular, we show that the moduli space admits and integrable holomorphic conformal structure. In the language of physics, such a structure is equivalent to a so-called *flat special geometry*. This rigid structure, which is a differential geometric manifestion of the lattice polarization, allows us to compute the full Picard-Fuchs system via the program of Sasaki & Yoshida. We also utilize the mixed-twist construction again to study the relationship between the flat special geometry of the moduli space, and the well known special geometry that comes from $\mathcal{N} = 2$ supersymmetric gauge theories in Seiberg-Witten theory. This is done via explicit analysis of the associated elliptic fibrations on the Seiberg-Witten curves and the K3 surfaces that come from the mixed-twist construction. We show that the Picard-Fuchs system of the Seiberg-Witten curves, when computed in the GKZ formalism, can be readily combined to produce the first order RG flow operators expected from physics for such gauge theories.

Finally, in Chapter 7, we provide an outlook of the future directions of this research project. This includes a more detailed look at the relationship between Seiberg-Witten theory and the twisted K3 surfaces and their periods, as well as discussion on how the mixed-twist construction can be used to build elliptically fibred Calabi-Yau threefolds that are simultaneously fibred by lattice polarized K3 surfaces. We show how our understanding of the moduli space of the twisted Legendre pencil puts one in the position to build such a threefold fibred by Picard rank $\rho = 17$ K3 surfaces.

1.3 Relations to published work

Parts of this dissertation research have come from the articles *From the Signature theorem to Anomaly cancellation* [88], published in the Rocky Mountain Journal of Mathematics in 2020, as well as the article *On the mixed-twist construction and monodromy of associated Picard-Fuchs systems* [89], at the time of writing under review at the Journal of Number Theory and Physics, both with Dr. Andreas Malmendier.

Parts of sections §2.1.2, 2.2 - 2.3.3, and Chapter 3 can be found in [88]. Section §2.1.5 and Chapter 4 can be found in [89].

CHAPTER 2

Preliminary Matter

In this chapter, we review all necessary material for the scope of this dissertation. This includes detailed discussions of the periods and Picard-Fuchs systems of algebraic surfaces, as well as how the existence of a Jacobian elliptic fibration and associated Weierstrass model may allow one to glean explicit information about the Picard-Fuchs system and moduli space. We also review all necessary differential topology and index theory related to Hirzebruch's signature to study holomorphic anomalies.

2.1 Algebraic Surfaces & their Periods

2.1.1 Abelian & K3 Surfaces

Recall that a K3 surface is a smooth, simply connected complex projective surface with trivial canonical bundle, $K_{\mathbf{X}} = \bigwedge^2 T^*_{\mathbb{C}}\mathbf{X} \cong \mathcal{O}_{\mathbf{X}}$. If $\mathbf{X}$ is a K3 surface, it is well known that the second integral cohomology with the intersection form is isometric to the lattice $H^2(\mathbf{X}, \mathbb{Z}) \cong \mathrm{H}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2}$, called the K3 lattice, Λ_{K3}. The lattice Λ_{K3} is the unique, even, integral lattice of rank 22 with signature $(3, 19)$ Here, H is the standard rank-two hyperbolic lattice, that is, $\mathbb{Z}^2$ together with the quadratic form $2xy$, and $\mathrm{E}_8(-1)$ is the negative definite lattice associated with the exceptional root system of E_8.

For a complex two-dimensional torus $\mathbf{Z} = \mathbb{C}^2/\Lambda$, where $\Lambda \subset \mathbb{C}^2$ is a rank-four lattice, it follows that $H^2(\mathbf{Z}, \mathbb{Z}) \cong \mathrm{H}^{\oplus 3}$. Moreover, the canonical bundle $K_{\mathbf{Z}} \cong \mathcal{O}_{\mathbf{Z}}$

is trivial, as the natural Euclidean coordinates $z_1, z_2 \in \mathbb{C}^2$ are periodic with respect to Λ. Hence, the nonvanishing holomorphic 2-form $dz_1 \wedge dz_2 \in \Omega^{2,0}(\mathbb{C}^2)$ descends to the quotient as a well defined holomorphic 2-form, thus providing a global trivializing section of $K_{\mathbf{Z}}$.

Most complex tori of dimension two are not algebraic. If $\mathbf{Z}$ is algebraic, we call $\mathbf{Z} = \mathbf{A}$ an *abelian surface*. For example, we may have $\mathbf{A} = \mathcal{E}_1 \times \mathcal{E}_2$ as the product of two elliptic curves, or $\mathbf{A} = \operatorname{Jac}(\mathbf{C})$ as the Jacobian variety of a curve $\mathbf{C}$ of genus two. Let $\mathbf{A}$ be an abelian surface. The minus identity map $-\mathbb{I} : \mathbf{A} \to \mathbf{A}$ has sixteen distinct fixed points - the two-torsion points of $\mathbf{A}$ - and hence, $\mathbf{A}/\{-\mathbb{I}\}$ is a singular surface with 16 rational double points,. Then minimal resolution of $\mathbf{A}/\{-\mathbb{I}\}$ is a special type of K3 surface called a *Kummer surface*, denoted $\operatorname{Kum}(\mathbf{A})$.

Let $\mathbf{X}$ be an abelian or K3 surface. The class $[\eta] \in H^2(\mathbf{X}, \mathbb{C})$ of the non-vanishing holomorphic two-form $\eta \in \Omega^{2,0}(\mathbf{X})$ is unique up to scale. The polarized Hodge structure of weight two will be denoted as follows:

$$
\begin{array}{ccccccc}
H^2(\mathbf{X}, \mathbb{C}) & = & H^{2,0}(\mathbf{X}) & \oplus & H^{1,1}(\mathbf{X}) & \oplus & H^{0,2}(\mathbf{X}) \\
 & & \| & & \| & & \| \\
 & & \langle \eta \rangle_{\mathbb{C}} & & \langle \eta, \overline{\eta} \rangle_{\mathbb{C}}^{\perp} & & \langle \overline{\eta} \rangle_{\mathbb{C}}
\end{array}
$$

A *polarization* is given by the intersection form, i.e., a non-degenerate, integral, symmetric bilinear form on $H^2(\mathbf{X}, \mathbb{Z})$ extended to $H^2(\mathbf{X}, \mathbb{C})$ by linearity. A *principally polarized* abelian surface is either the Jacobian variety $\mathbf{X} = \operatorname{Jac}(\mathbf{C})$ of a smooth projective curve $\mathbf{C}$ of genus two, where the polarization is the class of the theta divisor, or $\mathbf{X} = \mathcal{E}_1 \times \mathcal{E}_2$ is the product of two elliptic curves equipped with the product polarization.

Recall as well that an algebraic surface $\mathbf{Z}$ is said to be *rational* if it is birational to the projective plane $\mathbb{P}^2$, i.e., there is a birational map $\mathbf{Z} \dashrightarrow \mathbb{P}^2$. Any birational map can be realized as a sequence of blow-ups and blow-downs of exceptional divisors. For example, $\mathbf{Z} = \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ by blowing down an exceptional divisor from the product structure. Although the canonical bundle $K_{\mathbf{Z}}$ of a rational surface is never trivial (e.g., we have $K_{\mathbb{P}^2} \cong -3H$, a nonzero multiple of the hyperplane class), the periods and moduli of such surfaces play an important role in this research.

For any smooth algebraic surface $\mathbf{X}$, we have have the so-called *exponential* short exact sequence,

$$0 \to \underline{\mathbb{Z}} \hookrightarrow \mathcal{O}_{\mathbf{X}} \xrightarrow{\exp(2\pi i\,\cdot\,)} \mathcal{O}^*_{\mathbf{X}} \to 0.$$

The sequence remains exact on global sections, and hence we get an the induced sequence in cohomology, which begins as

$$0 \to H^1(\mathbf{X}, \mathbb{Z}) \to H^1(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to H^1(\mathbf{X}, \mathcal{O}^*_{\mathbf{X}}) \xrightarrow{c_1} H^2(\mathbf{X}, \mathbb{Z}) \to H^2(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \to \cdots .$$

The Picard group $\mathrm{Pic}(\mathbf{X})$ is the group of Cartier divisors modulo linear equivalence, and is thus naturally identified with the sheaf cohomology group $H^1(\mathbf{X}, \mathcal{O}^*_{\mathbf{X}})$. In turn, elements of $\mathrm{Pic}(\mathbf{X})$ are naturally identified with isomorphism classes of holomorphic line bundles $\mathcal{L} \to \mathbf{X}$. The kernel of the first Chern class map $c_1 : \mathrm{Pic}(\mathbf{X}) \to H^2(\mathbf{X}, \mathbb{Z})$ of degree zero line bundles is denoted by $\mathrm{Pic}^0(\mathbf{X})$ and the quotient $\mathrm{Pic}(\mathbf{X})/\mathrm{Pic}^0(\mathbf{X}) = \mathrm{NS}(\mathbf{X})$ is the Néron-Severi group. As usual, we identify $\mathrm{NS}(\mathbf{X})$ with its image in $H^2(\mathbf{X}, \mathbb{Z})$. Then the Néron-Severi group together with the intersection form is a Lorentzian lattice by the Hodge index theorem. The Picard number $\rho = \rho(\mathbf{X})$ is the rank of $\mathrm{NS}(\mathbf{X})$, and the Néron-Severi lattice is an even lattice of signature $(1, \rho - 1)$. For $\mathbf{X}$ an algebraic K3 surface, we have $h^{1,0} = 0$ as $\pi_1(\mathbf{X}) = 1$, and thus $\mathrm{NS}(\mathbf{X})$ is parameterized by the integral algebraic 2-cycles on $\mathbf{X}$ via Poincaré duality.

The first Chern class map restricts to an isomorphism $\mathrm{Pic}(\mathbf{X}) \to H^2(\mathbf{X}, \mathbb{Z}) \cap H^{1,1}(\mathbf{X})$ by the Lefschetz $(1, 1)$ theorem. Then $H^1(\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ maps onto $\mathrm{Pic}^0(\mathbf{X})$. If

$\mathbf{X}$ is an elliptic surface over $\mathbb{P}^1$, it follows that $H^1(\mathbf{X}, \mathcal{O}_{\mathbf{X}}) = 0$, since all 1-forms on $\mathbf{Z}$ are obtained by pullback from $\mathbb{P}^1$ by the elliptic structure. By adjunction, any rational elliptic or elliptic K3 surface must fibre over $\mathbb{P}^1$, and hence, for an elliptic K3 surface the natural map $\mathrm{Pic}(\mathbf{X}) \to \mathrm{NS}(\mathbf{X})$ is an isomorphism.[1] The orthogonal complement $\mathrm{T}(\mathbf{X}) = \mathrm{NS}(\mathbf{X})^{\perp} \subset H^2(\mathbf{X}, \mathbb{Z})$ is called the *transcendental lattice* and carries the induced Hodge structure. Then the lattice $\mathrm{T}(\mathbf{X})$ is of rank $n = 22 - \rho$ with signature $(2, n-2)$, and the plane $H^{2,0}(\mathbf{X}) \oplus H^{0,2}(\mathbf{X}) \subset \mathrm{T}(\mathbf{X}) \otimes \mathbb{C}$ is the positive definite subspace. Finally, we say that a mapping on cohomology between two of abelian or K3 surfaces is a *Hodge isometry* if it is an isometry of the intersection form that preserves the Hodge structure.

Both $\mathrm{NS}(\mathbf{X})$ and $\mathrm{T}(\mathbf{X})$ are independent of the choice of η, and are primitive sublattices[2] of $H^2(\mathbf{X}, \mathbb{Z})$. Conversely, for every integer $\rho = 0, \dots, 20$, given primitive sublattices of $\mathrm{H}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2}$ of signature $(1, \rho-1)$ and $(2, 20-\rho)$, respectively, there exists an algebraic K3 surface that realizes this lattice as Néron-Severi lattice and transcendental lattice, respectively. Hence, let $\mathrm{L} \subseteq \mathrm{H}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2}$ be a primitive, even sublattice of signature $(1, r)$, $0 \leq r \leq 19$. Then a *lattice polarization* on $\mathbf{X}$ is given by a primitive lattice embedding $\mathrm{L} \hookrightarrow H^2(\mathbf{X}, \mathbb{Z}) \cap H^{1,1}(\mathbf{X})$ whose image contains a pseudo-ample class. In this case, $\mathrm{L} = \mathrm{NS}(\mathbf{X})$ realizes the Néron-Severi lattice, and we say that $\mathbf{X}$ is L-*polarized.* Moreover, the Picard-rank of $\mathbf{X}$ is $\rho = r + 1$.

2.1.2 Jacobian elliptic surfaces

Let $\mathbf{Z}$ be a smooth, connected algebraic surface. An *elliptic fibration* over $\mathbb{P}^1$ on $\mathbf{Z}$ is a holomorphic map $\pi : \mathbf{Z} \to \mathbb{P}^1$ such that the general fiber of $\pi^{-1}(t)$ is a smooth curve of genus one with $t \in \mathbb{P}^1$. An *elliptic surface* is an algebraic surface with a

[1]We review elliptic fibrations in the following section.

[2]If Λ is a lattice, we say that a sublattice $\mathrm{L} \subseteq \Lambda$ is *primitive* if the quotient Λ/L is free. Moreover, we say that Λ is an *overlattice* of L.

given elliptic fibration. We require that the fibration is *relatively minimal*, meaning that there are no (-1)-curves in the fibres of π.[3] Since a curve of genus one, once a point has been chosen, is isomorphic to its Jacobian, i.e., an elliptic curve, we call an elliptic surface a *Jacobian elliptic surface* if it admits a section $\sigma : \mathbb{P}^1 \to \mathbf{Z}$ that equips each fiber with a smooth base point. In this way, each smooth fiber is an abelian group and the base point serves as the origin of the group law.

For elliptically fibered surfaces with a section, the two classes in $\mathrm{NS}(\mathbf{Z})$ associated with the generic elliptic fiber F and section σ span a sub-lattice $\mathcal{H} = \mathrm{span}_{\mathbb{Z}}\{\sigma, F\}$ isometric to the standard hyperbolic lattice H of rank two. The sublattice $\mathcal{H} \subset \mathrm{NS}(\mathbf{Z})$ completely determines the elliptic fibration with section on $\mathbf{Z}$. In fact, on a given K3 surface $\mathbf{X}$ there is a one-to-one correspondence between sub-lattices $\mathcal{H} \subset \mathrm{NS}(\mathbf{X})$ isometric to the standard hyperbolic lattice H that contain a pseudo-ample class, and elliptic structures with section on $\mathbf{X}$ which realize $\mathcal{H}$ [21, Thm. 2.3]. In this way, investigating elliptic fibrations on a given K3 surface $\mathbf{X}$, including whether or not $\mathbf{X}$ admits an elliptic fibrations at all, is purely cohomological in nature.

Again let $\mathbf{Z}$ be an arbitrary Jacobian elliptic surface. The distinct ways up to isometries to embed the standard rank-2 hyperbolic lattice H isometrically $\mathrm{NS}(\mathbf{Z})$ are distinguished by the isomorphism type of the orthogonal complement $\mathcal{W}$ of $\mathcal{H}$, such that the Néron-Severi lattice decomposes as a direct orthogonal sum

$$\mathrm{NS}(\mathbf{Z}) = \mathcal{H} \oplus \mathcal{W} \, .$$

A sub-lattice $\mathcal{W}^{\mathrm{root}} \subset \mathcal{W}$ is spanned by the roots, i.e., the algebraic classes of self-intersection -2 inside $\mathcal{W}$. The singular fibers of the elliptic fibration determine $\mathcal{W}^{\mathrm{root}}$ uniquely up to permutation. Moreover, there exists a canonical group isomorphism

[3] The definition of an elliptic surface over an arbitrary base curve $\mathbf{C}$ is defined analogously, though for higher genera the relative minimality may be impossible to impose.

[105]

$$\mathcal{W}/\mathcal{W}^{\text{root}} \xrightarrow{\cong} \text{MW}(\mathbf{Z}, \pi)\ , \tag{2.1.1}$$

where $\text{MW}(\mathbf{Z}, \pi)$ is Mordell-Weil group of sections on $\mathbf{Z}$ compatible with its elliptic structure.

To each Jacobian elliptic fibration $\pi : \mathbf{Z} \to \mathbb{P}^1$ there is an associated Weierstrass model obtained by contracting all components of fibers not meeting the section - this is the relative minimality condition. If we choose $t \in \mathbb{C}$ as a local affine coordinate on $\mathbb{P}^1$ and (x, y) as local coordinates of the elliptic fibers, we can write the Weierstrass model of an elliptic curve as

$$y^2 = 4x^3 - g_2(t)x - g_3(t), \tag{2.1.2}$$

where g_2 and g_3 are polynomials in the affine base coordinate t. When g_2, g_3 are of degree eight and twelve, then $\mathbf{Z}$ a $K3$ surface. Since the fibration is relatively minimal, the total space of Equation (2.1.2) is always singular with only rational double point singularities and irreducible fibers, and $\mathbf{Z}$ is the minimal desingularization. The discriminant $\Delta = g_2^3 - 27g_3^2$ vanishes where the fibers of Equation (2.1.2) are singular curves. It follows that if the degree of the discriminant Δ is a polynomial of degree or 24, considered as a homogeneous polynomial on $\mathbb{P}^1$, then the minimal desingularization of the total space of Equation (2.1.2) is a K3 surface).

In his seminal paper [77], Kodaira realized the importance of elliptic surfaces and proved a complete classification for the possible singular fibres of the Weierstrass models. Each possible singular fiber over a point t_0 with $\Delta(t_0) = 0$ is uniquely characterized in terms of the vanishing degrees of g_2, g_3, Δ as t approaches t_0. The classification encompasses two infinite families ($I_n, I_n^*, n \geq 0$) and six exceptional cases $(II, III, IV, II^*, III^*, IV^*)$.[4] Note that the vanishing degrees of g_2 and g_3 are always less or equal to three and five, respectively, as otherwise the singularity of Equation (2.1.2) is not a rational double point.

Closely related is the j-function, a holomorphic map $\mathsf{j} : \mathbb{H} \to J \equiv \mathbb{P}^1$ that can be computed form a Weierstrass model using the formula

$$\mathsf{j} = \frac{g_2^3}{\Delta}. \tag{2.1.3}$$

This map is called *Kodaira's functional invariant*, and was shown by Kodaira [77] to be a rational map. The codomain J is called the **j**-*line* and plays an important role in studying moduli of elliptic curves. Every smooth elliptic fiber $\mathcal{E}_t = \pi^{-1}(t)$ is a complex torus, and thus can be identified with a rank-two lattice Λ to obtain $\mathcal{E}_t \cong \mathbb{C}/\Lambda$. However, multiplying the lattice Λ by a complex number, which corresponds to rotating and scaling the lattice, preserves the isomorphism class of an elliptic curve, so we can always arrange for the lattice to be generated by 1 and some complex number $\tau \in \mathbb{H}$ in the upper half plane; we write $\Lambda_\tau = \langle 1, \tau \rangle$. Moreover, two τ-parameters τ_1 and τ_2 in $\mathbb{H}$ belong to isomorphic elliptic curves if and only if

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d} \qquad \text{for some} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) \,,$$

where the *modular group* $\mathrm{PSL}(2, \mathbb{Z})$ acts (projectively) on $\mathbb{H}$. It can be shown that the action of the modular group on the fundamental domain

$$D = \left\{ \tau \in \mathbb{H} \;\middle|\; \mathrm{Re}(\tau) \leq \frac{1}{2}, |\tau| \geq 1 \right\}$$

generates $\mathbb{H}$ [127] such that the moduli space of isomorphism classes of elliptic curves

[4]In fact, an I_0 fibre is just the generic smooth fibre, so all possible fibre types are accounted for.

is realized as $\mathbb{H}/\operatorname{PSL}(2,\mathbb{Z}) \cong D$. The one point compactifaction $D \cup \{\infty\} = \mathbb{P}^1$ is also called the *coarse moduli space* of elliptic curves. The "corners" of D are of fundamental importance: these are the numbers $\rho = e^{2\pi i/3}$, i, and $-\bar{\rho} = e^{\pi i/3}$.

It can be shown that under the identification $\mathcal{E}_t \cong \mathbb{C}/\Lambda_\tau$ the discriminant Δ becomes a modular form of weight twelve, and g_2 one of weight four, so that its third power is also of weight twelve. For example, we may express the discriminant as

$$\Delta_\tau = e^{2\pi i\tau} \prod_{r=1}^{\infty} \left(1 - e^{2\pi i\tau r}\right)^{24}. \tag{2.1.4}$$

Thus, the quotient in Equation (2.1.3) is a *modular function* of weight zero, in particular it defines a holomorphic function $j : \mathbb{H} \to \mathbb{P}^1$ invariant under the action of $\operatorname{PSL}(2,\mathbb{Z})$ such that for every smooth elliptic fiber $\mathcal{E}_t \cong \mathbb{C}/\Lambda_\tau$ we have $\mathsf{j}(t) = j(\tau)$ and $\Delta(t) = \Delta_\tau$. A more careful examination of the behavior at the corners yields $j(\rho) = 0$, $j(i) = 1$, and $j(-\bar{\rho}) = \infty$.

Notice that if one has a local affine coordinate t on a base curve B, and one replaces g_2 by g_2t^2 and g_3 by g_3t^3 in Equation (2.1.2), the j-function in Equation (2.1.3) is left invariant. This operation, called a *quadratic twist*, does change the nature of the singular fibers: it switches I_n and I_n^* fibers, as well as II and IV^*, IV and II^*, and III and III^*. Therefore, the j-function does not determine the elliptic surface, not even locally. However, the quadratic twist is the only way that two Jacobian elliptic surfaces can have the same j-function, and conversely, a Jacobian elliptic fibration is uniquely determined by the j-function up to quadratic twist. Moreover, the canonical holomorphic map $\mathsf{j} : B \to J$ in Equation (2.1.3) can be lifted to a (rational) map between the elliptic surfaces Z and S themselves. Thus, we have the following:

Corollary 2.1.2. *Let $\pi : Z \to B = \mathbb{P}^1$ be a Jacobian elliptic surface. There is a* *canonical holomorphic map* $\mathsf{j} : B \to J$ *that uniquely determines the Jacobian elliptic surface* Z *up to quadratic twist. Moreover, there is an induced rational map* $Z \dashrightarrow S$ *between the total spaces. The map* j *has degree* 1 *or* 2 *if* Z *is a rational or a K3 surface, respectively.*

The operation of the quadratic twist will appear again via the *mixed-twist construction* in §4.1.

Given a Jacobian elliptic surface $\pi : \mathbf{Z} \to \mathbb{P}^1$, we may use the adjunction formula to define the relative canonical bundle $K_{\mathbf{Z}|\mathbb{P}^1}$ of the elliptic surface in terms of the canonical bundles of $\mathbf{Z}$ and $\mathbb{P}^1$, respectively, by writing

$$K_{\mathbf{Z}|\mathbb{P}^1} = K_{\mathbf{Z}} \otimes \left(\pi^* K_{\mathbb{P}^1}\right)^{-1} . \tag{2.1.5}$$

The bundle $K_{\mathbf{Z}|\mathbb{P}^1}$ can be identified with the line bundle of vertical $(1,0)$-forms of the fibration $\pi : \mathbf{Z} \to \mathbb{P}^1$. Using the push-forward operation $\pi_* K_{\mathbf{Z}|\mathbb{P}^1}$, we obtain a bundle $\mathcal{K} = \pi_* K_{\mathbf{Z}|\mathbb{P}^1} \to \mathbb{P}^1$. In fact, we have the following result: on a Jacobian elliptic surface $\pi : \mathbf{Z} \to \mathbb{P}^1$ given by Equation (2.1.2) we have $\mathcal{K} = \pi_* K_{\mathbf{Z}|\mathbb{P}^1} \cong \mathcal{O}(n)$ with $n = 2$ if $\mathbf{Z}$ is a K3 surface. This viewpoint is crucial to our computations and analysis of the period integrals and Picard-Fuchs operators that annihilate them in §5.1.

We can rephrase the construction of the Weierstrass model in Equation (2.1.2) in terms of sections of the relative canonical bundle. We will use this point of view later. Let $\mathcal{L} \to \mathbb{P}^1$ be a holomorphic line bundle, and g_2 and g_3 sections of $\mathcal{L}^4$ and $\mathcal{L}^6$, respectively, such that the discriminant $\Delta = g_2^3 - 27\, g_3^2$ is a section of $\mathcal{L}^{12}$ not identically zero. Define $\mathbf{P} := \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ and let $p : \mathbf{P} \to \mathbb{P}^1$ be the natural projection and $\mathcal{O}_{\mathbf{P}}(1)$ the tautological line bundle. We denote by X, Y, and Z the sections of $\mathcal{O}_{\mathbf{P}}(1) \otimes \mathcal{L}^2$, $\mathcal{O}_{\mathbf{P}}(1) \otimes \mathcal{L}^3$, and $\mathcal{O}_{\mathbf{P}}(1)$, respectively, which correspond to the natural injections of $\mathcal{L}^2$, $\mathcal{L}^3$, and $\mathcal{O}$ into $p_* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$. We denote by W

the projective variety in $\mathbf{P}$ defined by the equation

$$Y^2 Z = 4X^3 - g_2(t)\, X Z^2 - g_3(t)\, Z^3 \,. \tag{2.1.6}$$

A canonical section $\sigma : \mathbb{P}^1 \to W$ is given by the point $[X : Y : Z] = [0 : 1 : 0]$ such that $\Sigma := \sigma(\mathbb{P}^1) \subset W$ is a divisor on W, and its *normal bundle* is isomorphic to the fundamental line bundle by $p_* \mathcal{O}_{\mathbf{P}}\big(-\Sigma\big) \cong \mathcal{L}$. In the affine chart $Z = 1$, and $X = x$, $Y = y$ the one-form dx/y is a section of the bundle $\mathcal{L}^{-1}$; hence, the dual of the normal bundle, also called the *conormal bundle*, is precisely the relative canonical bundle introduced above, i.e., $\mathcal{L}^{-1} \cong \mathcal{K}$.

2.1.3 The period map

Let $\mathbf{X}$ be an algebraic K3 surface of Picard rank $\rho < 20$ with given holomorphic two-form $\eta \in H^{2,0}(\mathbf{X})$, and let $\{c_1, \dots, c_n, \dots, c_{22}\}$ be a $\mathbb{Z}$-basis for $H_2(\mathbf{X}, \mathbb{Z})$, ordered so that the first $n = 22 - \rho \geq 3$ cycles lie in the transcendental lattice $\mathrm{T}(\mathbf{X})$. Then the *period point* of $\mathbf{X}$ is

$$\mathrm{per}(\mathbf{X}) := \Big[Z_1 : Z_2 : \cdots : Z_{22} \Big] \in \mathbb{P}^{21} \,, \tag{2.1.7}$$

where for $1 \leq j \leq 22$, we set $\vec{Z} = [Z_1 : Z_2 : \cdots : Z_{22}]$ with

$$Z_j = \int_{c_j} \eta \,,$$

the so-called *periods* or *period integrals* of $\mathbf{X}$. The projectification of the period vector $\vec{Z}$ reflects the fact that $\eta \in H^{2,0}(\mathbf{X})$ is unique up to scale.

For $i = n+1, \dots, 22$, we have $Z_i = 0$ as $c_i \in \mathrm{NS}(\mathbf{X})$. Therefore, the period point is can be taken to lie in $\mathbb{P}(\mathrm{T}(\mathbf{X}) \otimes \mathbb{C}) \subset \mathbb{P}^{n-1}$. In fact, there are two systems of transcendental cycles $c_1, \ldots, c_n \in H_2(\mathbf{X}, \mathbb{Z})$ and $c'_1, \ldots, c'_n \in H_2(\mathbf{X}, \mathbb{Z})$ such that $c'_1, \ldots, c'_n$ form a $\mathbb{Z}$-basis of $\mathrm{T}(\mathbf{X})$, and the cycles $c_1, \ldots, c_n$ are dual to $c'_1, \ldots, c'_n$, i.e., $(c_i \circ c'_j) = \delta_{ij}$ and its intersection matrix $Q_{ij} = (c'_i \circ c'_j)$ takes the fixed form of a symmetric, integral, bilinear form $Q = (Q_{ij})$ of signature $(2, n-2)$. We call a K3 surface $\mathbf{X}$ together with a basis of transcendental cycles $c_1, \ldots, c_n$ a *marked* K3 surface.

The periods of a marked K3 surface then satisfy the Riemann relation and Riemann inequality

$$Q(\vec{Z}, \vec{Z}) := \sum_{i,j} Q_{ij}, Z_i\, Z_j = 0\ , \qquad Q(\vec{Z}, \vec{Z}^*) := \sum_{i,j} Q_{ij}\, Z_i\, \overline{Z}_j > 0\ , \tag{2.1.8}$$

where $\vec{Z}^* = (\overline{Z}_1, \ldots, \overline{Z}_n)$. Thus, the period point $\vec{Z} \in \mathbb{P}^{n-1}$ lies on the hyperquadric $\mathbf{Q} \subset \mathbb{P}^{n-1}$ satisfying the first Riemann relation,

$$\mathbf{Q} = \left\{ [X_1 : \cdots : X_n] \in \mathbb{P}^{n-1} \,\middle|\, \sum_{i,j} Q_{ij}, X_i\, X_j = 0 \right\}. \tag{2.1.9}$$

Crucially, the entire discussion above carries through when the K3 surface $\mathbf{X}$ belongs to a suitably well behaved moduli space $\mathcal{M}$, such that the period vector $\vec{Z}$ varies holomorphically with respect to local coordinates in $\mathcal{M}$. For example, we shall be primarily concerned with the case where $\mathcal{M}$ is a complex orbifold. Such is the case when $\mathcal{M}$ is a (coarse) moduli space of lattice polarized K3 surfaces, for example, as established by Dolgachev [34].

If L is a primitive sublattice of $\mathrm{H}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2}$ of signature $(1, r)$, then the period map (2.1.7) is a holomorphic, multivalued map between the moduli space $\mathcal{M}_\mathrm{L}$ of isomorphism classes of marked, L-polarized K3 surfaces and the quasiprojective variety

$$\mathcal{D}_{\mathrm{L}} = \left\{ \vec{Z} = [Z_1 : \cdots : Z_{22-r}] \in \mathbb{P}^{21-r} \middle| \sum_{i,j=1}^{22-r} Q_{ij}\, Z_i\, Z_j = 0 \ , \ \sum_{i,j=1}^{22-r} Q_{ij}\, Z_i\, \overline{Z_j} > 0 \right\} . \tag{2.1.10}$$

The domain $\mathcal{D}_{\mathrm{L}}$ is isomorphic to two copies of what is called the symmetric homogeneous domain of type IV of the form

$$\mathrm{O}(2, 19-r)/\mathrm{SO}(2) \times \mathrm{O}(19-r) \ . \tag{2.1.11}$$

In this case, we define the *period map* $\psi : \mathcal{M}_{\mathrm{L}} \to \mathbb{P}^{n-1}$ by

$$\psi\left([\mathbf{X}]\right) = \mathrm{per}\left(\mathbf{X}\right), \tag{2.1.12}$$

where $[\mathbf{X}] \in \mathcal{M}_{\mathrm{L}}$ is the isomorphism class of the L-polarized K3 surface $\mathbf{X}$. It follows immediately that ψ is well defined, but ψ is multivalued by accruing nontrivial monodromy around the orbifold singularities in $\mathcal{M}_{\mathrm{L}}$. If $\pi : \ \mathcal{D}_{\mathrm{L}} \to \ \mathcal{M}_{\mathrm{L}}$ is the canonical projection map, then the period map ψ is the multivalued inverse map such that $\pi \circ \psi = \mathrm{id}_{\mathcal{M}_{\mathrm{L}}}$. Then we say that $\psi : \mathcal{M}_{\mathrm{L}} \to \mathcal{D}_{\mathrm{L}}$ is the *uniformizing map* of the orbifold $\mathcal{M}_{\mathrm{L}}$, and thus, by the general program of Wilczynski [143] established at the end of the 1800s, and brought into a more modern perspective by Sasaki & Yoshida, and co. [94, 96, 120, 121, 122, 123], there should be a system of linear differential equations of rank at most n annihilating ψ. Remarkably, this perspective has its roots founded in classical uniformization problems dating back to Riemann [118], Fricke, and Klein [75, 74] and their analysis of second order Fuchsian ODEs in the plane and the Gauss hypergeometric function. Our analysis of relevant elliptic fibrations reveals that this classical viewpoint is still very much embedded in the modern framework, at least as applied to the lattice polarized K3 surfaces in this research.

In fact, this system is the Picard-Fuchs system in the moduli variables of the family of K3 surfaces parameterized by $\mathcal{M}_{\mathrm{L}}$ that annihilates the periods. We will see in the sequel that considering explicit elliptic fibrations on this family of K3 surfaces provides a very simple way to obtain some of these equations, and to obtain the remaining equations, we must employ the differential geometric techniques relevant to uniformizing differential equations pioneered by Sasaki & Yoshida, thus demonstrating that the two methods of deriving the Picard-Fuchs equations provide linear systems that are projectively equivalent. Moreover, the projective gauge factor is obtained explicitly from the geometry of the elliptic fibration of our analysis.

2.1.4 Picard-Fuchs Equations

For $\mathbf{X}$ an L-polarized K3 surface, we want to study the behavior of the holomorphic 2-form $\eta = \eta_{\mathbf{X}} \in H^{2,0}(\mathbf{X})$ as we vary the complex structure moduli of $\mathbf{X}$. Since $\mathbf{X}$ is equipped with a lattice polarization, we can think of this measuring how the transcendental lattice $\mathrm{T}(\mathbf{X})$ rotates against the fixed cohomology lattice $H^2(\mathbf{X}, \mathbb{Z}) \cong \mathrm{H}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2} \subset \ H^2(\mathbf{X}, \mathbb{C})$ while varying the moduli for $\mathbf{X}$ in the moduli space $\mathcal{M}_{\mathrm{L}}$ discussed in §2.1.3. As $\mathrm{T}(\mathbf{X})$ has nonempty intersection with $H^{1,1}(\mathbf{X}) \cup H^{0,2}(\mathbf{X})$, it follows that $\mathrm{T}(\mathbf{X})$ does not vary holomorphically with complex structure moduli, so some additional machinery is needed to address the desired behavior of $\mathrm{T}(\mathbf{X})$; in the end, we remain within the complex analytic category. With this in mind, let us proceed with generalities.

We follow the exposition in [30, §5.1]. Let $\pi : \mathcal{X} \to S$ be a smooth morphism of relative dimension n, with the generic fibre $\mathbf{X}_t = \pi^{-1}(t)$ a complex projective variety, $t \in S$. Assume that S is quasismooth and quasiprojective. Then the cohomology groups $H^n(\mathbf{X}_t, \mathbb{C})$ patch together to form a locally free sheaf $\mathcal{F} = R^n \pi_* \underline{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_S$, where $\underline{\mathbb{C}} \to \mathcal{X}$ is the locally constant sheaf whose stalks are the abelian group $\mathbb{C}$. [5]

By construction, $\mathcal{F}$ contains the local system $R^n\pi_*\underline{\mathbb{C}}$. This uniquely determines a flat connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_S} \Omega^1(S)$ that annihilates precisely those (local) sections belonging to the local system. Then ∇ is called the *Gauss-Manin connection*. One can see by the definition of $\mathcal{F}$ that ∇ operates concretely on local sections as

$$\nabla(s \otimes f) = s \otimes df \in \mathcal{F} \otimes_{\mathcal{O}_S} \Omega^1(S),$$

where $s \in \Gamma(R^n\pi_*\underline{\mathbb{C}}, S)$ is a local section and $f \in \mathcal{O}_S$ is a regular function. Accordingly, if X is a vector field on S, we have $\nabla_X(s \otimes f) = s \otimes df(X) = X(f)\, s$. The Gauss-Manin connection ∇ determines a *variation of Hodge structure*, and was studied extensively in the seminal work by Griffiths [56, 57, 58, 59].

One must consider how the Hodge structure degenerates at singular points of S Indeed, this is where very interesting and rich behavior becomes manifest; for example, mirror symmetry, when $\mathcal{X}$ is a family of Calabi-Yau n-folds. Suppose that $\pi : \mathcal{X} \to S$ can be completed to a flat family $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \widetilde{S}$, where $\widetilde{S}$ is a quasismooth compactification of S with normal crossing boundary divisor $D = \cup_i D_i = \widetilde{S} - S$, meaning that in the compactification $\widetilde{S}$, each point of of D looks étale locally like the transverse intersection of coordinate hyperplanes. Hence, if $z_1, \ldots, z_m$ are local coordinates centered at the boundary divisor $D \subset \widetilde{S}$, where $m = \dim S$, D is described by the equation

$$z_1 \cdots z_k = 0$$

for some $1 \leq k \leq m$. Given $\pi : \mathcal{X} \to S$, the construction of such a compactification $\widetilde{S}$ is nontrivial and not guaranteed, especially finding one that is relevant to mirror

[5] Of course, the analogous construction can be made for any cohomology group $H^k(\mathbf{X}, \mathbb{C})$ - in this research, we are only concerned with the middle cohomology group $H^n(\mathbf{X}, \mathbb{C})$, and so focus exclusively on the construction for that.

symmetry.

The sheaf $\mathcal{F}$ has a canonical extension $\widetilde{\mathcal{F}}$ on $\widetilde{S}$, but the Gauss-Manin connection ∇ does not necessarily extend to a regular connection on $\widetilde{\mathcal{F}}$ because singularities may develop. However, these singularities are so-called *regular singular points*, which are quite mild. In fact, ∇ extends to a map $\widetilde{\nabla} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}} \otimes_{\mathbb{C}} \Omega^1_{\widetilde{S}}(\log D)$, where $\Omega^1_{\widetilde{S}}(\log D) \to \widetilde{S}$ is the sheaf generated by the differentials

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_m \,.$$

The presence of singularities induces monodromy around the boundary components D_j. If $\gamma_j : [0,1] \to \widetilde{S}$ is a small loop that winds around the boundary component D_j, with $\gamma_j(0) = \gamma_j(1) = t \in S$, then a cohomology class $\alpha \in H^n(\mathbf{X}_t, \mathbb{C})$ lifts uniquely to a ∇-flat section $\alpha(u) \in H^n(\mathbf{X}_{\gamma_j(u)}, \mathbb{C})$ such that $\alpha(0) = \alpha$. We obtain a monodromy transformation $T_j^\alpha \in \mathrm{GL}\,(H^n(\mathbf{X}_t, \mathbb{C}))$ by defining $T_j^\alpha(\alpha) = \alpha(1)$. Taking $T_1^\alpha, \ldots, T_k^\alpha \in \mathrm{GL}\,(H^n(\mathbf{X}_t, \mathbb{C}))$ as a generating set as α ranges over a basis of $H^n(\mathbf{X}_t, \mathbb{C})$, we obtain the *monodromy group* G of the family $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \widetilde{S}$, which is determined up to conjugation by the boundary components D_j and not the loops γ_j. In fact, the monodromy transformations T_j are quasi-unipotent, with index of unipotency at most $n+1$.

With this structure, we may describe the Picard-Fuchs equations of the family $\pi : \mathcal{X} \to S$. Fix a point $p \in S$, and let $z_1, \ldots, z_m$ be local coordinates centered at p. If

$$\mathcal{D} = \mathbb{C}_{\mathcal{O}_S}\left[\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_m}\right] \tag{2.1.13}$$

is the ring of linear differential operators whose coefficients are germs of the structure sheaf $\mathcal{O}_S$, then we obtain an $\mathcal{O}_S$-homomorphism $\phi : \mathcal{D} \to \mathcal{F}$ via the Gauss-Manin connection ∇ as follows. For local vector fields $X_1, \ldots, X_j$ on S, denote by concatenation

$X_1 \cdots X_j$ the j-fold composition of the first order differential operators represented by $X_1, \ldots, X_j$. Furthermore, fix a local section $\eta = \eta(t) \in \mathcal{F}$ at p. Then define ϕ by

$$\phi(X_1 \cdots X_j) = \nabla_{X_1} \cdots \nabla_{X_j} \eta \,,$$

and extending linearly over $\mathcal{O}_S$. It is immediate by the definition of ∇ the ϕ is an $\mathcal{O}_S$-homomorphism, giving $\mathcal{F}$ the structure of a $\mathcal{D}$-module.

From this data, the *Picard-Fuchs ideal* I_η is defined as $\ker(\phi)$, the collection of linear differential operators annihilating η. Then a differential operator $D \in I_\eta$ is seen to annihilate the a period integral

$$\omega(t) = \int_\Sigma \eta \tag{2.1.14}$$

where $\Sigma = \Sigma(t) \in H_n(\mathbf{X}_t, \mathbb{Q})$ is a rational homology n-cycle, as follows. Embed $H_n(\mathbf{X}_t, \mathbb{Q}) \hookrightarrow H_n(\mathbf{X}_t, \mathbb{C})$ with the inclusion map. Then we have the period sheaf $\Pi \to S$, whose stalks are generated by the local regular function

$$t \mapsto \omega(t) = \langle \Sigma(t), \eta(t) \rangle \,,$$

where $\langle \; , \; \rangle$ is the natural de Rham pairing given in Equation (2.1.14). Since the local system $R^n\pi_*\underline{\mathbb{C}}$ is parallel under ∇, it follows that the dual sheaf $(R^n\pi_*\underline{\mathbb{C}})^*$, whose stalks are canonically isomorphic to $H_n(\mathbf{X}_t, \mathbb{C})$, is also ∇-flat by insisting that the pairing $\langle \; , \; \rangle$ is compatible with ∇. In particular, this means we may "differentiate under the integral sign", and we have

$$\frac{\partial}{\partial z_i} \int_\Sigma \eta = \int_\Sigma \nabla_{\partial_{z_i}} \eta \,.$$

Hence, by linearity, we see that an operator $D \in I_\eta$ must annihilate the period integral $\omega(t)$ in Equation (2.1.14). Moreover, since $\Sigma \in H_n(\mathbf{X}_t, \mathbb{Q})$ was arbitrary, it follows that a differential operator D lies in the Picard-Fuchs ideal I_η if and only if D annihilates every period integral of η. Finally, we have the definition of Picard-Fuchs equations.

Definition 2.1.3. *Let $\pi : \mathcal{X} \to S$ be a smooth map of relative dimension n, with the generic fibre $\mathbf{X}_t = \pi^{-1}(t)$ a complex projective variety, $t \in S$. Suppose that the middle rational homology group $H_n(\mathbf{X}_t, \mathbb{Q})$ is of rank r. Given $\eta = \eta(t) \in H^n(\mathbf{X}_t, \mathbb{C})$, the* Picard-Fuchs equations *for η are the system of homogeneous linear PDEs corresponding to any minimal generating set of the Picard-Fuchs ideal $I_\eta \subseteq \mathcal{D}$ that annihilate the period integrals*

$$\omega_i(t) = \int_{\Sigma_i(t)} \eta(t) \,,$$

where $\Sigma_1, \ldots \Sigma_r \in H_n(\mathbf{X}_t, \mathbb{Q})$ are a basis.

In particular, based off the discussion above, we see that the Picard-Fuchs equations for $\eta \in H^n(\mathbf{X}, \mathbb{C})$ are Fuchsian, i.e., with at worst regular singular points. The rank and order of the system depends generically on the nature of the parameter space S and algebro-geometric data of the generic fibre $\mathbf{X}_t$. In practice, these both may be difficult to determine, though we will see shortly for our case of interest that geometric considerations allow us to quickly determine both. Moreover, the monodromy of the Picard-Fuchs system is naturally identified with a subgroup of the monodromy group generated by the monodromy transformations $T_1^\eta, \ldots, T_k^\eta$ for the full Gauss-Manin connection described above.

To end this subsection, we again discuss briefly the case of interest, that for the Picard-Fuchs equations of $\eta \in H^{2,0}(\mathbf{X})$ for an L-polarized K3 surface $\mathbf{X}$. It follows from the discussion of this section and of §2.1.3 that the Picard-Fuchs system of the family $\pi : \mathcal{X} \to \mathcal{M}_{\mathrm{L}}$ of L-polarized K3 surfaces is precisely the system of linear PDEs that annihilate the period map a la Wilczynski, Sasaki & Yoshida that uniformizes the coarse moduli space $\mathcal{M}_{\mathrm{L}}$. This is an example of so-called *Picard-Fuchs uniformization*, which was for example studied by Doran in similar contexts [36, 37]. As we shall discuss later in §6.1, the fact that $\mathcal{M}_{\mathrm{L}}$ is uniformized by the hyperquadric $\mathbf{Q}$ in Equation (2.1.9) has very strong differential geometric consequences for $\mathcal{M}_{\mathrm{L}}$; in particular, it admits an integrable holomorphic conformal structure [123]. This forces the Picard-Fuchs system for the holomorphic 2-form $\eta \in H^{2,0}(\mathbf{X})$ to be a second order linear system of rank $n = 22 - \rho = \mathrm{rank}(\mathrm{T}(\mathbf{X}))$ in $n - 2 = \dim \mathcal{M}_{\mathrm{L}}$ variables.

2.1.5 Generalities on Weierstrass models and their associated Picard-Fuchs Operators

In this section we generalize the discussion in §2.1.2 and §2.1.4 to higher dimensional elliptic fibrations and the associated Weierstrass models, prepare for the discussion in §4.3 and §7.2. Let X and S be normal complex algebraic varieties and $\pi : X \to S$ an elliptic fibration, that is, π is proper surjective morphism with connected fibers such that the general fiber is a nonsingular elliptic curve. Moreover, we assume that π is smooth over an open subset $S_0 \subset S$, whose complement in S is a divisor with at worst normal crossings. Thus, the local system $H^i_0 := R^i\pi_*\underline{\mathbb{Z}}_X|_{S_0}$ forms a variation of Hodge structure over S_0.

Elliptic fibrations possess the following canonical bundle formula: on S, the *fundamental line bundle* denoted $\mathcal{L} := (R^1\pi_*\mathcal{O}_X)^{-1}$ and the canonical bundles $\boldsymbol{\omega}_X := \wedge^{\mathrm{top}} T^{*(1,0)}X$, $\boldsymbol{\omega}_S := \wedge^{\mathrm{top}} T^{*(1,0)}S$ are related by

$$\boldsymbol{\omega}_X \cong \pi^*(\boldsymbol{\omega}_S \otimes \mathcal{L}) \otimes \mathcal{O}_X(D), \tag{2.1.15}$$

where D is a certain effective divisor on X depending only on divisors on S over which π has multiple fibers, and divisors on X giving (-1)-curves of π. When $\pi : X \to S$

is a *Jacobian elliptic fibration*, that is, when there is a section $\sigma : S \to X$, the case of multiple fibers is prevented. We may avoid the presence of (-1)-curves in the following way: For X an elliptic surface, we assume that the fibration is relatively minimal, meaning that there are no (-1)-curves in the fibers of π. When X is an elliptic threefold, we additionally assume that no contraction of a surface is compatible with the fibration.

Assuming these minimality constraints, we have $D = 0$, thus the canonical bundle formula (2.1.15) simplifies to $\boldsymbol{\omega}_X \cong \pi^*(\boldsymbol{\omega}_S \otimes \mathcal{L})$. In particular, for $\mathcal{L} \cong \boldsymbol{\omega}_S^{-1}$ we obtain $\boldsymbol{\omega}_X \cong \mathcal{O}_X$. Recall that X is a Calabi-Yau manifold if $\boldsymbol{\omega}_X \cong \mathcal{O}_X$ and $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < n = \dim(X)$. In this present context we will be concerned with Jacobian elliptic fibrations on Calabi-Yau manifolds. It is well known that for X an elliptic Calabi-Yau threefold, the base surface can have at worst log-terminal orbifold singularities. We will take the base surface S to be a Hirzebruch surface $\mathbb{F}_k$ (or its blowup).

It is well known that Jacobian elliptic fibrations admit *Weierstrass models*, i.e., given a Jacobian elliptic fibration $\pi : X \to S$ with section $\sigma : S \to X$, there is a complex algebraic variety W together with a proper, flat, surjective morphism $\hat{\pi} : W \to S$ with canonical section $\hat{\sigma} : S \to W$ whose fibers are irreducible cubic plane curves, together with a birational map $X \dashrightarrow W$ compatible with the sections σ and $\hat{\sigma}$; see [103]. The map from X to W blows down all components of the fibers that do not intersect the image $\sigma(S)$. If $\pi : X \to S$ is relatively minimal, the inverse map $W \dashrightarrow X$ is a resolution of the singularities of W.

A Weierstrass model is constructed as follows: given a line bundle $\mathcal{L} \to S$, and sections g_2, g_3 of $\mathcal{L}^4$, $\mathcal{L}^6$ such that the discriminant $\Delta = g_2^3 - 27g_3^2$ as a section of $\mathcal{L}^{12}$ does not vanish, define a $\mathbb{P}^2$-bundle $p : \mathbf{P} \to S$ as $\mathbf{P} := \mathbb{P}\left(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3\right)$ with p the natural projection. Moreover, let $\mathcal{O}_{\mathbf{P}}(1)$ be the tautological line bundle. Denoting

x, y and z as the sections of $\mathcal{O}_{\mathbf{P}}(1) \otimes \mathcal{L}^2$, $\mathcal{O}_P(1) \otimes \mathcal{L}^3$ and $\mathcal{O}_{\mathbf{P}}(1)$ that correspond to the natural injections of $\mathcal{L}^2$, $\mathcal{L}^3$ and $\mathcal{O}_S$ into $\pi_* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$, the Weierstrass model W from above is given by the the sub-variety of $\mathbf{P}$ defined by the equation

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3. \tag{2.1.16}$$

The canonical section $\sigma : S \to W$ is given by the point $[x : y : z] = [0 : 1 : 0]$ in each fiber, such that $\Sigma := \sigma(S) \subset W$ is a Cartier divisor whose normal bundle is isomorphic to the fundamental line bundle $\mathcal{L}$ via $p_* \mathcal{O}_{\mathbf{P}}(-\Sigma) \cong \mathcal{L}$. It follows that W inherits the properties of normality and Gorenstein if S possesses these. Thus, the canonical bundle formula (2.1.15) reduces to

$$\boldsymbol{\omega}_W = \pi^* \left(\boldsymbol{\omega}_S \otimes \mathcal{L} \right). \tag{2.1.17}$$

The Jacobian elliptic fibration $p : W \to S$ then has a Calabi-Yau total space if $\mathcal{L} \cong \boldsymbol{\omega}_S^{-1} = \mathcal{O}_S(-K_S)$ (abusing notation slightly to denote the projection map p the same way as the projection from the ambient $\mathbb{P}^2$-bundle).

For a Jacobian elliptic fibration X the canonical bundle $\boldsymbol{\omega}_X$ is determined by the discriminant $\Delta = g_2^3 - 27 g_3^2$. For example, if $\pi : X \to S$ is a Jacobian elliptic fibration for a smooth algebraic surface X and $S = \mathbb{P}^1$ with homogeneous coordinates $[t : s]$, then X is a rational elliptic surface if the Δ is a homogeneous polynomial of degree 12 (meaning that $\mathcal{L} = \mathcal{O}(1)$), and X is a K3 surface when Δ is a homogeneous polynomial of degree 24 (meaning that $\mathcal{L} = \mathcal{O}(2)$); these results follow readily from adjunction and Noether's formula. The nature of the singular fibers and their effect on the canonical bundle was established by the seminal work of Kodaira [78, 79].

Of particular interest are multi-parameter families of elliptic Calabi-Yau n-folds over a base B, a quasi-projective variety of dimension r, denoted by $\pi : X \to B$.

Hence, each $X_p = \pi^{-1}(p)$ is a compact, complex n-fold with trivial canonical bundle. Moreover, each X_p is elliptically fibered with section over a fixed normal variety S. This means that we have a multi-parameter family of minimal Weierstrass models $p_b : W_b \to S$ representing a family of Jacobian elliptic fibrations $\pi_b : X_b \to S$. We denote the collective family of Weierstrass models as $p : W \to B$.

Working within affine coordinates for B and S we set $u = (u_1, \dots, u_{n-1}) \in \mathbb{C}^{n-1} \subset S$ and $b = (b_1, \dots, b_r) \in \mathbb{C}^r \subset B$. We then may write the Weierstrass model W_b in the form

$$y^2 = 4x^3 - g_2(u, b)x - g_3(u, b), \tag{2.1.18}$$

where for each fiber we have chosen the affine chart of W_b given by $z = 1$ in Equation (2.1.16).

Part of the utility of a Weierstrass model is the explicit construction of the holomorphic n-form on each X_b, up to fiberwise scale, allowing for the detailed study of the Picard-Fuchs operators underlying a variation of Hodge structure. In fact, consider the holomorphic sub-bundle $\mathcal{H} \to B$ of the vector bundle $V = R^n\pi_*\underline{\mathbb{C}}_X \to B$, whose fibers are given as the line $H^0(\omega_{X_b}) \subset H^n(X_b, \mathbb{C})$. Here, $\underline{\mathbb{C}} \to X$ is the constant sheaf whose stalks are $\mathbb{C}$. Griffiths showed [56, 57, 58, 59] that the vector bundle $\mathcal{V} = V \otimes_{\mathbb{C}} \mathcal{O}_B$ carries a canonical flat connection ∇, the Gauss-Manin connection. Again see §2.1.4 as well. A meromorphic section of $\mathcal{H} \subset \mathcal{V}$ is given fiberwise by the holomorphic n-form $\eta_b \in H^0(\boldsymbol{\omega}_{X_b}) \subset H^n(X_b, \mathbb{C})$

$$\eta_b = du_1 \wedge \cdots \wedge du_{n-1} \wedge \frac{dx}{y}, \tag{2.1.19}$$

where we denote the collective section as $\eta \in \Gamma(\mathcal{V}, B)$. It is natural to consider local parallel sections of the dual bundle $\mathcal{H}^*$; these are represented by transcendental cycles $\Sigma_b \in H_n(X_b, \mathbb{R})$ that vary continuously with $b \in B$, writing the collective section as $\Sigma \in \Gamma(\mathcal{V}^*, B)$. The sections are covariantly constant since the vector bundle $V = R^n\pi_*\underline{\mathbb{C}}_X$ is locally topologically trivial, and thus local sections of the dual V^* are as well. Utilizing the natural fiberwise de Rham pairing

$$\langle \Sigma_b, \eta_b \rangle = \oint_{\Sigma_b} \eta_b,$$

we obtain the period sheaf $\Pi \to B$, whose stalks are given by the local analytic function $b \mapsto \omega(b) = \langle \Sigma_b, \eta_b \rangle$. The function $\omega(b)$ is called a *period integral* (over Σ_b) and satisfies a system of coupled linear PDEs in the variables $b_1, \dots, b_r$ – the so called Picard-Fuchs system.

Given the affine local coordinates $(b_1, \dots, b_r) \in \mathbb{C}^r \subset B$, fix the meromorphic vector fields $\partial_j = \partial/\partial b_j$ for $j = 1, \dots, r$. Then each ∂_j induces a covariant derivative operator ∇_{∂_j} on $\mathcal{V}$. Since ∇ is flat, the curvature tensor $\Omega = \Omega_\nabla$ vanishes, and hence, for all meromorphic vector fields U, V on B we have

$$\Omega(U, V) = \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]} = 0.$$

Substituting in the commuting coordinate vector fields ∂_i, ∂_j, we conclude

$$\nabla_{\partial_i} \nabla_{\partial_j} = \nabla_{\partial_j} \nabla_{\partial_i}.$$

This integrability condition is crucial in obtaining a system of PDEs from the Gauss-Manin connection. Since $\mathcal{V}$ has rank $m = \dim H^n(X_b, \mathbb{C})$, each sequence of parallel sections $\nabla^i_{\partial_k} \nabla^l_{\partial_j} \eta$, for $i + l = 0, 1, 2$ and $1 \leq k, j \leq r$ form the linear dependence relations

$$\sum_{i+l=0}^{\hat{m}} \sum_{k,j=1}^{r} a^{kj}_{il}(b) \nabla^i_{\partial_k} \nabla^l_{\partial_j} \eta = 0$$

for some integer $0 < \hat{m} \leq m$, where $a^{kj}_{il}(b)$ are meromorphic. Here, it is understood that $\nabla^0 = \mathrm{id}$. As ∇ annihilates the transcendental cycle Σ and is compatible with the pairing $\langle \Sigma, \eta \rangle$, we may "differentiate under the integral sign" to obtain

$$\frac{\partial}{\partial b_j}\omega(b) = \frac{\partial}{\partial b_j}\oint_\Sigma \eta = \oint_\Sigma \nabla_{\partial_j}\eta.$$

It follows that the period integral $\omega(b)$ satisfies the system of linear PDEs of rank $\mathsf{r} \geq 1$, given by

$$\sum_{i+l=0}^{\hat{m}} \sum_{k,j=1}^{r} a_{il}^{kj}(b)\frac{\partial^{i+l}}{\partial^i b_k \partial^l b_j}\omega(b) = 0. \tag{2.1.20}$$

Equation (2.1.20) is the Picard-Fuchs system of the multi-parameter family $\pi : X \to B$ of Calabi-Yau n-folds,. By construction, it is a linear Fuchsian system, i.e., the system with at worst regular singularities.

The rank r and order $\hat{m}$ of the system depends on the parameter space B and algebro-geometric data of the generic fiber X_b. For example, let $\pi : X \to B$ be the family of Jacobian elliptic K3 surfaces which is polarized by a lattice L of rank $\rho \leq 18$ such that B realizes the coarse moduli space of pseudo-ample L-polarized K3 surfaces as defined by Dolgachev [34]. Then it follows that the Picard-Fuchs system (2.1.20) is a second order system of rank $\mathsf{r} = 22 - \rho$. Naturally, there are sub-loci of such moduli spaces where the lattice polarization extends to higher Picard-rank and the rank of the Picard-Fuchs system drops accordingly. This behavior was studied, for example, by Doran et al. in [39], and coined the *differential rank-jump property* therein. In the sequel, we will analyze it by studying corresponding Weierstrass model $p : W \to B$. Moreover, we will see that the Picard-Fuchs system can be explicitly computed from the geometry of the elliptic fibrations and the presentation of the associated period integrals as generalized Euler integrals using GKZ systems [51].

It is commonplace in the literature to study the Picard-Fuchs equations of one parameter families of Calabi-Yau n-folds; in this case, the base B is a punctured complex plane with local affine coordinate $t \in \mathbb{C} \subset B$, and an analogous construction leads to a regular Fuchsian ODE of order $\leq m$ with $m = \dim H^n(X_t, \mathbb{C})$ for the general fiber X_t. In the construction of Doran & Malmendier [40], this is the central focus, with $B = \mathbb{P}^1 - \{0, 1, \infty\}$ and $B = \mathbb{P}^1 - \{0, 1, p, \infty\}$. We will show that the restriction of the multi-parameter Picard-Fuchs system (2.1.20) above leads to the Picard-Fuchs ODE operators and families of lattice polarized K3 surfaces of Picard-rank $\rho = 19$, for example the mirror partners of the classic deformed Fermat quartic K3.

2.1.6 Branched double covers of $\mathbb{P}^2$ and double sextic K3 surfaces

Each Jacobian elliptic surface in the family of K3 surfaces constructed below is the minimal smooth model of the two-fold cover of $\mathbb{P}^2 = \mathbb{P}(t^1, t^2, t^3)$ branching along six lines in general position. Six lines in $\mathbb{P}^2$ are considered to be in general position if no three of them intersect in a point. Let us describe six lines l_j in $\mathbb{P}^2$ with $j = 1, \ldots, 6$ by setting

$$\ell_j = \left\{ [t^1 : t^2 : t^3] \in \mathbb{P}^2 \middle| \; t^1 v_{1j} + t^2 v_{2j} + t^3 v_{3j} = 0 \right\} . \tag{2.1.21}$$

Then we can define the so-called *double sextic* family of K3 surfaces in the weighted projective space $\mathbb{P}(1,1,1,3) = \mathbb{P}(t^1, t^2, t^3, z)$ as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along the configuration $\boldsymbol{\ell} = \{\ell_1, \ldots, \ell_6\}$,

$$z^2 = \prod_{j=1}^{6} \ell_j . \tag{2.1.22}$$

Let $\mathcal{X}$ be such a minimal resolution for a given configuration $\boldsymbol{\ell}$. As noted by Sasaki [119], $\mathcal{X}$ is a K3 surface[6]; to see this, consider the meromorphic 3-form on

[6]While Sasaki was certainly not the first to recognize that double sextics are K3 surfaces, the analysis provided in [119] of the relation of this family to the Aomoto-Gel'fand E(3,6) system is what is relevant to our analysis in this research.

$\mathbb{P}(1,1,1,3)$ given by $\mathbf{\Omega} = dt^1 \wedge dt^2 \wedge dt^3$, and let $\chi = t^1\partial_{t^1} + t^2\partial_{t^2} + t^3\partial_{t^3}$ be the Euler vector field. Then setting

$$\begin{aligned} \mathbf{dt} &= \mathbf{\Omega}(\chi,\,\cdot\,,\,\cdot\,) \\ &= t^1 dt^2 \wedge dt^3 - t^2 dt^1 \wedge dt^3 + t^3 dt^1 \wedge dt^2 \,, \end{aligned}$$

we have a global trivializing section of the canonical bundle $K_{\mathcal{X}}$ given by the holomorphic 2-form

$$\eta_{\mathbf{X}} = \frac{\mathbf{dt}}{z} \equiv \frac{\mathbf{dt}}{\sqrt{\prod_{i=1}^{6} \ell_j}} \,. \tag{2.1.23}$$

The period integrals of the double sextic family and explicit degenerations to higher Picard rank are the central focus of this research. With this in mind, let us analyze the Hodge theoretic data of the double sextics.

It follows from the discussion in the previous subsection §2.1.3 that one may know the Gram matrix of the restriction of the intersection form to the transcendental lattice $\mathrm{T}(\mathcal{X})$ by understanding the period domain and the period map. Let $\mathcal{X}(3,6)$ be the space of ordered systems $\boldsymbol{\ell}$ of six lines in general position in $\mathbb{P}^2$. Following [92] we can express elements in $\mathcal{X}(3,6)$ as equivalence classes of matrices $(v_{ij}) \in \mathrm{Mat}^{\circ}(3,6,\mathbb{C})$, i.e., 3×6 matrices whose 3×3 minors are all non-vanishing, modulo the action of $\mathrm{SL}(3,\mathbb{C})$ on the left and the action of $(\mathbb{C}^*)^6$ from the right. Then $\mathcal{X}(3,6)$ is the *moduli space* of six line configurations in $\mathbb{P}^2$.

Let us denote by $[(v_{ij})]$ the image of the matrix (v_{ij}) in the quotient. In fact, $\mathcal{X}(3,6)$ can be considered to be a Zariski open set of the affine 4-space if we choose coordinates (A, B, C, D) such that

$$(v_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & A & C \\ 0 & 0 & 1 & 1 & B & D \end{pmatrix} \in \mathrm{Mat}^\circ(3, 6, \mathbb{C}) \; . \tag{2.1.24}$$

Using the homogeneous coordinates $-T = t^2/t^1$ and $-x = t^3/t^1$, a two-fold cover of $\mathbb{P}^2$ branching along the six lines (v_{ij}) is then given by the following:

$$y^2 = T\,(T-1)\,x\,(x-1)\,(A\,T + B\,x - 1)\,(C\,T + D\,x - 1). \tag{2.1.25}$$

Let us denote this four parameter family of K3 surfaces by $\mathbf{X} = \mathbf{X}_{A,B,C,D}$, which was studied by Hoyt & Schwarz in [70]. Later, we will also use another set of moduli denoted as (a, b, c, d), defined in Equation (2.1.33) to describe the same family. When no confusion will arise, we will denote the four-parameter family as $\mathbf{X}$ with parameters supressed, and freely interchange with the full notation. As we will show in §2.1.7, the 4-parameter family $\mathbf{X}_{A,B,C,D}$ is a family of Jacobian elliptic K3 surfaces that are polarized by the even lattice $\mathrm{L} = \mathrm{H} \oplus \mathrm{E}_8(-1) \oplus \mathrm{A}_1^{\oplus 6}$ of rank sixteen that are birational to the double sextic $\mathcal{X}$. Surfaces in this class have Picard rank ρ taking the possible values 16, 17, 18, 19 or 20. For generic A, B, C, D, the Picard rank of $\mathbf{X}$ is $\rho = 16$, and certain values the parameters lead to higher Picard rank degenerations of the family. We will take up an earnest study of the family $\mathbf{X}$ in §4.1.

The authors in [92, 96] use the coordinates (x^1, x^2, x^3, x^4) such that

$$(\hat{v}_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x^1 & x^2 \\ 0 & 0 & 1 & 1 & x^3 & x^4 \end{pmatrix} \in \mathrm{Mat}^\circ(3, 6, \mathbb{C}) \; . \tag{2.1.26}$$

Notice that we have

$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot (\hat{v}_{ij}) \cdot \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} = (v_{ij}) \;, \tag{2.1.27}$$

i.e., we set $A = 1 - x^1, B = x^3, C = 1 - x^2, D = x^4$. Hence, we have $[(v_{ij})] = [(\hat{v}_{ij})]$ in $\mathcal{X}(3,6)$.

On $\mathbf{X}$ there are 16 linearly independent algebraic cycles; 15 exceptional curves come from blowing up the $\binom{6}{2} = 15$ rational double points of the branched cover, and one arises as section of the elliptic fibration. These algebraic curves span the Néron-Severi lattice $\mathrm{NS}(\mathbf{X})$. As discussed in 2.1.3, here are two systems of six transcendental cycles $c_1, \ldots, c_6 \in H_2(\mathbf{X}, \mathbb{Z})$ and $c'_1, \ldots, c'_6 \in H_2(\mathbf{X}, \mathbb{Z})$ such that $c'_1, \ldots, c'_6$ form a $\mathbb{Z}$-basis of the transcendental lattice $\mathrm{T}(\mathbf{X})$, and the cycles $c_1, \ldots, c_6$ are dual and its intersection matrix $Q_{ij} = (c'_i \circ c'_j)$ is the 6×6 matrix given by

$$\mathrm{H}(2)^{\oplus 2} \oplus \langle -2 \rangle \oplus \langle -2 \rangle \;, \tag{2.1.28}$$

where $\mathrm{H}(2)$ denotes the lattice whose Gram matrix is that of H scaled by 2. The correspondence sending the six lines (v_{ij}) in general position to the periods of the associated K3 surface $\mathbf{X}$ gives a multi-valued period map

$$\psi : \; [(v_{ij})] \in \mathcal{X}(3,6) \mapsto \vec{Z} = [Z_1 : \cdots : Z_6] \in \mathcal{D}^+ \;, \tag{2.1.29}$$

where $\mathcal{D}^+$ the positively oriented component

$$\mathcal{D}^+ = \left\{ \vec{Z} = [Z_1 : \cdots : Z_6] \in \mathbb{P}^5 \,\middle|\, Q(\vec{Z},\vec{Z}) = 0\,,\; Q(\vec{Z},\vec{Z}^*)\,,\; \mathrm{Im}\left(\frac{Z_3}{Z_1}\right) > 0 \right\}. \quad (2.1.30)$$

The monodromy group of this multivalued function is generated by the $\mathbb{Z}$-linear transformations of the marking of $\mathbf{X}$ caused by the move of the line l_j around the point $l_k \cap l_l$ for $1 \le j < k < l \le 6$ [92], and is the principal congruence subgroup

$$\Gamma = \left\{ G \in \mathrm{PGL}(6,\mathbb{Z}) \,\middle|\, G(\mathcal{D}^+) \subset \mathcal{D}^+\,,\; G^t Q G = Q\,,\; G \equiv \mathbb{I}(2) \right\}\,. \quad (2.1.31)$$

For special values of the moduli the Picard number $\rho(\mathbf{X})$ becomes $17, 18, 19,$ and the bilinear form will degenerate to a symmetric, integral, bilinear form of signature $(2, 20-\rho)$. The explicit form of Q in those cases is given in (2.1.45), (2.1.46), (2.1.47).

2.1.7 The Yoshida family of K3 surfaces with Picard rank $\rho \ge 16$

Let us examine the presentation of the double sextic surface from Equation (2.1.25). Consider the open subset of points $(A, B, C, D) \in \mathbb{C}^4$ such that the right hand side of the equation

$$Y^2 = T\,(T-1)\,X\,(X-1)\,(A\,T + B\,X - 1)\,(C\,T + D\,X - 1) \quad (2.1.32)$$

defines six lines in the complex projective plane, no three of which are concurrent. This space is biholomorphic to the moduli space $\mathcal{X}(3,6)$ of six lines in $\mathbb{P}^2$, and we will use the same character to refer to the moduli space of this suface, as well as any other space biholomorphic to it.

Setting $T = -B\,t/A + 1/A$ and $X = x$, we introduce the parameters

$$\begin{aligned} a = \frac{1}{B} - \frac{A}{B}\,, \quad b = \frac{1}{B}\,, \quad c = -\frac{A}{BC} + \frac{1}{B}\,, \quad d = \frac{AD}{BC}\,,\\ A = 1 - \frac{a}{b}\,, \quad B = \frac{1}{b}\,, \quad C = \frac{a-b}{c-b}\,, \quad D = \frac{d}{b-c}\,. \end{aligned} \tag{2.1.33}$$

Transforming $Y \mapsto \sqrt{-1}\,y/(a-b)$, Equation (2.1.25) becomes

$$y^2 = \frac{1}{b(b-c)}(t-a)\,(t-b)\,(t-c-d\,x)\,x\,(x-1)\,(x-t)\,. \tag{2.1.34}$$

We denote the minimal nonsingular model of (2.1.34) by $\mathbf{X}_{a,b,c,d}$. This surface is known as a *Yoshida surface*, following Hoyt & Schwartz. The open subset of $\mathbb{C}^4 = \mathbb{C}(a,b,c,d)$ such that Equation (2.1.34) defines a smooth surface is then biholomorphic to $\mathcal{X}(3,6)$. Then $\mathcal{X}(3,6)$ is realized as an affine subvariety of $\mathbb{C}^4$, realized as the $\mathbb{C}^4$ minus a union of hyperplanes

$$\{\,a=0,\ b=0,\ c=0,\ d=0,\ a-b=0,\dots,\ c-d-1=0\,\}\,.$$

Note the normalization factor in front the right hand side of (2.1.34) persists because the elements $b, b-c$ are not squares in the function field $\mathbb{C}(\mathcal{X}(3,6))$. Then the nonvanishing holomorphic 2-form $\eta \in H^{2,0}(\mathbf{X}_{a,b,c,d})$ is given by

$$\eta = \sqrt{b(b-c)}\,dt \wedge \frac{dx}{y}\,, \tag{2.1.35}$$

and the normalization factor $\sqrt{b(b-c)}$ is the generator of a projective gauge transformation of the Picard-Fuchs system annihilating η.

To transform Equation (2.1.34) into its Weierstrass normal form, we use the following transformation:

$$\tilde{X} = \frac{b\,(t-a)\,(t-b)\,(b-c)\,((-t^2+(c-d-1)\,t+c)\,x+3\,t\,(t-c))}{3x}\,,$$
$$\tilde{Y} = -\frac{2y\,t\,b^2\,(t-a)\,(t-b)\,(t-c)\,(b-c)^2}{x^2}\,.$$

Then Equation (2.1.34) becomes

$$\tilde{Y}^2 = 4\,\tilde{X}^3 - g_2\,\tilde{X} - g_3 \tag{2.1.36}$$

with

$$\begin{aligned}
g_2 &= \frac{4}{3}\,(t-a)^2\,(t-b)^2\,(b-c)^2\,b^2\\
&\times \Big(t^4+(-2\,c-d-1)\,t^3+\big(c^2+cd+d^2+2\,c-d+1\big)\,t^2\\
&\quad -c\,(c-d+2)\,t+c^2\Big)\,,\\
g_3 &= -\frac{4}{27}\,(-t+a)^3\,(-t+b)^3\,(b-c)^3\,b^3\\
&\times \left(-t^2+(c-d+2)\,t-2\,c\right)\left(-t^2+(c+2\,d-1)\,t+c\right)\,,\\
&\times \left(-2\,t^2+(2\,c+d+1)\,t-c\right)\,,\\
\Delta &= 256b^6(b-c)^6\,t^2\,(t-1)^2\,(t-c)^2\,(t-(c+d))^2\,((1-d)t-c)^2\\
&\times (t-a)^6\,(t-b)^6\,.
\end{aligned} \tag{2.1.37}$$

Notice that by letting $t \mapsto 1-t, x \mapsto 1-x$ in Equation (2.1.34) one establishes the symmetry

$$\mathbf{X}_{a,b,c,d} \cong \mathbf{X}_{1-a,1-b,1-c-d,d}\,.$$

One also obtains a symmetry by permuting the parameters a, b, i.e., $a \leftrightarrow b$, as well as a symmetry by permuting $(A, B) \leftrightarrow (C, D)$.

The transcendental lattice

For generic values of (a,b,c,d), it follows directly from analyzing the singular fibres of the Weierstrass model in Equation (2.1.36) that the configuration of singular fibres is $6\,I_2 + 2\,I_0^*$, with the six I_2 fibres located above $t = 0, 1, c+d,\ c/(1-d), \infty$, and the two I_0^* fibres above $t = a, b$. Moreover, we have $\mathrm{MW}(\pi_{\mathbf{X}}) \cong\ (\mathbb{Z}/2\mathbb{Z})^2$. We have $\mathcal{W}^{\mathrm{root}} = \mathrm{D}_4^{\oplus 2} \oplus \mathrm{A}_1^{\oplus 6}$. The Néron-Severi lattice $\mathrm{NS}(\mathbf{X})$ has signature $(1,15)$, and the determinant of the discriminant group is $(2^6)\,(2 \cdot 2)^2/4^2 =\ 2^6$. The computation of the discriminant group shows that it only contains factors of $\mathbb{Z}_2$, and it follows easily that

$$\mathrm{NS}(\mathbf{X}_{a,b,c,d}) = \mathrm{H} \oplus \mathrm{E}_8(-1) \oplus \mathrm{A}_1^{\oplus 6}\ . \tag{2.1.38}$$

It was proved in [70] that

$$\mathrm{T}(\mathbf{X}_{a,b,c,d}) = \mathrm{H}(2) \oplus \mathrm{H}(2) \oplus \langle -2 \rangle \oplus \langle -2 \rangle\ . \tag{2.1.39}$$

We have the following lemma:

Lemma 2.1.4. *The Jacobian elliptic surface $\mathbf{X}_{a,b,c,d} \to \mathbb{P}^1$ is a K3 surface with a $\mathrm{H} \oplus \mathrm{E}_8(-1) \oplus \mathrm{A}_1^6$-lattice polarization.*

Proof. The degree of the polynomials g_2 and g_3 make $\mathbf{X}_{a,b,c,d}$ a family of K3 surfaces. For generic values of a, b, c, the Néron-Severi lattice $\mathrm{NS}(\mathbf{X})$ has signature $(1,15)$ and discriminant 2^6. In fact, we will show that $\mathrm{NS}(\mathbf{X}) \cong \mathrm{H} \oplus \mathrm{E}_8(-1) \oplus \mathrm{A}_1^{\oplus 6}$ and $\mathrm{T}_{\mathbf{X}} = \mathrm{H}(2)^2 \oplus \langle -2 \rangle$. ∎

2.1.8 The twisted Legendre pencil of K3 surfaces with Picard rank $\rho \geq 17$

The Legendre family is the special case of the Yoshida surface (2.1.34) for $d = 0$. The family of Jacobian K3 surfaces $\mathbf{X}_{a,b,c} = \mathbf{X}_{a,b,c,d=0}$ is given by

$$\tilde{Y}^2 = 4\,\tilde{X}^3 - g_2(t)\,\tilde{X} - g_3(t) \tag{2.1.40}$$

with

$$\begin{aligned} g_2(t) &= h^2(t)\,G_2(t)\ , \\ g_3(t) &= h^3(t)\,G_3(t)\ , \\ h(t) &= (t-a)\,(t-b)\,(t-c)\ , \end{aligned} \tag{2.1.41}$$

where G_2 and G_3 are given by the modular elliptic surface for $\Gamma(2)$ with $u = 2t-1$ and rescaled such that

$$\begin{aligned} G_2(t) &= \left(\frac{3}{2}\right)^2 \left(\frac{1}{3}\,u^2 + 1\right) = 3\,t^2 - 3\,t + 3\ , \\ G_3(t) &= \left(\frac{3}{2}\right)^3 \left(\frac{1}{27}\,u\,(u-3)\,(u+3)\right) = t^3 - \frac{3}{2}\,t^2 - \frac{3}{2}\,t + 1\ , \end{aligned} \tag{2.1.42}$$

and $\Delta = \frac{729}{4}\,h^6(t)\,t^2\,(t-1)^2$ and $\mathcal{J} = \frac{4}{27}\frac{(t^2-t+1)^2}{t^2\,(t-1)^2}$. This elliptic fibration on $\mathbf{X} = \mathbf{X}_{a,b,c}$ has three I_0^* fibres located at $t = a, b, c$, and three I_2 fibres located at $t = 0, 1, \infty$, so long as a, b, c are distinct from each other, and not equal to $0, 1, \infty$. In this case, $\mathbf{X}$ is of Picard rank $\rho = 17$. Let $\mathfrak{T} \subset \mathbb{C}^3$ be the associated moduli space. The study of the geometry of $\mathfrak{T}$ is central to this research.

All this is to say, that in the variables of the Yoshida surface (2.1.34), we have the *twisted Legendre pencil* studied by Hoyt in [68, 69], given by

$$y^2 = (t-a)(t-b)(t-c)x(x-1)(t-t)\,. \tag{2.1.43}$$

This form of $\mathbf{X}$ will be instrumental to our calculations of Picard-Fuchs for the family in §5.3. In advance of that section, we have the following result.

Proposition 2.1.5. *On the K3 surface* $\mathbf{X}$*, a global trivializing section of the canonical bundle* $K_{\mathbf{X}} \to \mathbf{X}$ *is given by the holomorphic 2-form*

$$\eta_{\mathbf{X}} = dt \wedge \frac{dx}{y} \equiv \frac{dt}{\sqrt{h(t)}} \wedge \frac{dx}{\sqrt{x(x-1)(x-t)}}\,, \tag{2.1.44}$$

with $h(t) = (t-a)\,(t-b)\,(t-c)$ *as above.*

Notice that there are symmetries

$$\mathbf{X}_{a,b,c} \cong \mathbf{X}_{1-a,1-b,1-c} \cong \mathbf{X}_{1/a,1/b,1/c}\ ,$$

as well as the symmetry from permuting the parameters a, b, c.

The transcendental lattice

For $(a, b, c) \in \mathfrak{T}$, we have $\rho = 17$, and the Mordell-Weil group is $\mathrm{MW}(\mathbf{X}, \pi) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. By the Hodge index theorem, the Néron-Severi lattice $\mathrm{NS}(\mathbf{X})$ has signature $(1, 16)$, the configuration of singular fibres as $3I_0^* + 3I_2$ imply that the determinant of the discriminant group is $(2^3)\,(2\cdot 2)^3/4^2 = 2^5$. Moreover, the singular fibres determine the root lattice $\mathcal{W}^{\mathrm{root}} = \mathrm{D}_4^{\oplus 3} \oplus \mathrm{A}_1^{\oplus 3}$, and hence the discriminant group is

$$\Big(D(\mathcal{W}^{\mathrm{root}}), q_{\mathcal{W}^{\mathrm{root}}}\Big) = \left(\Big(\mathbb{Z}/2 \oplus \mathbb{Z}/2\Big) \oplus \mathbb{Z}/2, \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \oplus \Big[\frac{1}{2}\Big]\right)^{\oplus 3} .$$

We can pick generators are $\langle \bar{a}_1^* \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and $\langle \bar{d}_1^* \rangle \oplus \langle \bar{d}_4^* \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. To find isotopic subgroups, we need to involve at least two copies in the direct sum. This could lead to a discriminant group of an overlattice $(\mathbb{Z}/2\mathbb{Z})^{\oplus 5}$. The computation of the discriminant group shows that it only contains factors of $\mathbb{Z}/2\mathbb{Z}$, and it follows easily that the Néron-Severi group of the twisted Legendre pencil is

$$\mathrm{NS}(\mathbf{X}_{a,b,c}) = \mathrm{H} \oplus \mathrm{E}_8(-1) \oplus \mathrm{D}_4 \oplus \mathrm{A}_1^{\oplus 3} \ .$$

We will show by an explicit computation that the transcendental lattice is given by

$$\mathrm{T}(\mathbf{X}_{a,b,c}) = \mathrm{H}(2) \oplus \mathrm{H}(2) \oplus \langle -2 \rangle \ . \tag{2.1.45}$$

2.1.9 Further degeneration to Picard rank $\rho = 18, 19$

For $a = 0$ and b, c generic, the configuration of singular fibers is $2\,I_2 + 2\,I_0^* + I_2^*$, and we have the Mordell-Weil group $\mathrm{MW}(\mathbf{X}, \pi) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. Hence, we have Picard rank $\rho(\mathbf{X}_{0,b,c}) = 18$. Moreover, we have $\mathcal{W}^{\mathrm{root}} = \mathrm{D}_6 \oplus \mathrm{D}_4^{\oplus 2} \oplus \mathrm{A}_1^{\oplus 2}$. The Néron-Severi lattice $\mathrm{NS}(\mathbf{X})$ has signature $(1, 17)$ by the Hodge index theorem, and the determinant of the discriminant group is $(2^2)\,(2 \cdot 2)^3/4^2 = 2^4$. The computation of the discriminant group shows that

$$\mathrm{NS}(\mathbf{X}_{0,b,c}) = \mathrm{H} \oplus \mathrm{E}_8(-1) \oplus \mathrm{D}_6 \oplus \mathrm{A}_1^{\oplus 2} \ .$$

We will show by an explicit computation that the transcendental lattice is given by

$$\mathrm{T}(\mathbf{X}_{0,b,c}) = \langle 2 \rangle \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle \ . \tag{2.1.46}$$

For $a = 0$ and $b = 1$, the configuration of singular fibers is $I_2 + I_0^* + 2\,I_2^*$, and we have the Mordell-Weil group $\mathrm{MW}(\mathbf{X}, \pi) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. This means that the Picard rank is $\rho(\mathbf{X}_{0,1,c}) = 19$. We have $\mathcal{W}^{\mathrm{root}} = \mathrm{D}_6^{\oplus 2} \oplus \mathrm{D}_4 \oplus \mathrm{A}_1$. The Néron-Severi lattice $\mathrm{NS}(\mathbf{X})$ has signature $(1, 18)$ by the Hodge index theorem, and the determinant of the discriminant group is $(2)\,(2 \cdot 2)^3/4^2 = 2^3$. The computation of the discriminant group shows that

$$\mathrm{NS}(\mathbf{X}_{0,1,c}) = \mathrm{H} \oplus \mathrm{D}_{16} \oplus \mathrm{A}_1 \ .$$

By an explicit construction, Hoyt proved in [67, Sec. 2] that the transcendental lattice is given by

$$\mathrm{T}(\mathbf{X}_{0,1,c}) = \langle 2 \rangle \oplus \langle 2 \rangle \oplus \langle -2 \rangle \ . \tag{2.1.47}$$

This agrees with the result of van Geemen and Top [141]: by changing their variables according to $y \to y/\sqrt{bc}, x \to -x$ and $t \to c, z \to -t/c$ their hypersurface is transformed into the Legendre family investigated by Hoyt for $a = 0, b = \infty$. The transcendental lattice obtained in [141] agrees with Eq. (2.1.47).

2.1.10 Relations to Kummer surfaces from curves with full level-two structure

Let $\mathbf{C}$ be a genus two curve. Since $\mathbf{C}$ can be realized as a double cover $\mathbf{C} \to \mathbb{P}^1$, branched over the six points $\theta_1, \dots, \theta_6 \in \mathbb{P}^1$, using a fractional linear transformation we can move $\theta_4, \theta_5, \theta_6$ to $0, 1, \infty$. The resulting location of $\theta_1, \theta_2, \theta_3$, written as $\lambda_1, \lambda_2, \lambda$, yield the so-called *Rosenhain normal form* of $\mathbf{C}$, given by

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3) \, . \tag{2.1.48}$$

The values $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ are called the Rosenhain roots of $\mathbf{C}$. It is straightforward to establish that $\mathbf{C}$ is smooth if and only if $(\lambda_1, \lambda_2, \lambda_3)$ lie on the quasiprojective variety $\mathcal{M}[2] = \mathbb{P}^3 - \mathcal{P}$, where $\mathcal{P} \subset \mathbb{P}^3$ is the union of planes

$$\{ \, \lambda_i \neq 0, \, 1, \, \infty, \quad \lambda_i \neq \lambda_j, \quad i,j = 1,2,3 \, \}$$

Then $\mathcal{M}[2]$ is the moduli space of genus two curves with full level-two structure. This latter quality is determined because the Rosenhain normal form for $\mathbf{C}$ marks a 2-torsion point on the principally polarized Jacobian variety $\mathrm{Jac}(\mathbf{C})$.

Let $\mathcal{M}_2 = \widetilde{\mathcal{M}}[2]$ be the double cover, parameterized by l_1, l_2, l_3 that satisify $l_i^2 = \lambda_i'$ for $i = 1, 2, 3$, where λ_i' are the Rosenhain roots of a genus two curve $\mathbf{C}'$ whose Jacobian variety $\mathrm{Jac}(\mathbf{C}')$ is 2-isogenous to $\mathbf{C}$. See Clingher & Malmendier [26] for more details. Then it follows from results established by Braeger et al. [15], that there is a dominant rational map $\phi : \mathcal{M}_2 \dashrightarrow \mathfrak{T}$ given by

$$\phi : (l_1, l_2, l_3) \mapsto \left(a = \frac{4\, l_1 l_2 l_3}{(l_1 l_3 + l_2)^2}, \quad b = \frac{4\, l_1 l_2 l_3}{(l_1 l_2 + l_3)^2}, \quad c = \frac{4\, l_1 l_2 l_3}{(l_2 l_3 + l_1)^2} \right), \tag{2.1.49}$$

that is induced from a dominant, degree two rational map $\mathbf{Y}'_{l_1,l_2,l_3} \dashrightarrow \mathbf{X}_{a,b,c}$. Here $\mathbf{Y}'$ is the Kummer surface obtained the principally polarized abelian surface $\mathbf{A}'$ that is (2,2)-isogenous to $\mathbf{A}$. Such an explicit map is useful for not only relating results about the twisted Legendre pencil $\mathbf{X}_{a,b,c}$ and Kummer surfaces, but also because we are able to connect to known results for the periods and Picard-Fuchs equations that are of arithmetic interest as we degenerate to higher Picard rank. We take up this discussion in §6.2.3.

For example, as was shown in [15], we may degenerate the curve $\mathbf{C} = \mathbf{C}_{\lambda_1', \lambda_2', \lambda_3'}$ using a parabolic type $[I_{4-0-0}]$ degeneration from the Namikawa & Ueno classification [108] defined by

$$l_1 \mapsto k_1, \quad l_2 \mapsto \epsilon k_2, \quad l_3 \mapsto \epsilon \tag{2.1.50}$$

in the limit that $\epsilon \to 0$ to obtain the generically Picard rank $\rho = 18$ Kummer surface $\mathbf{Y}_{k_1^2, k_2^2, 0} = \mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$, where $\mathcal{E}_i$ is an elliptic curve in the Legendre normal form

$$\mathcal{E}_{k_i^2} : \quad \left\{ (Y, X) \mid Y^2 = X\,(X-1)\left(X - k_i^2\right) \right\} \tag{2.1.51}$$

whose elliptic modulus is given by k_i^2, $i = 1, 2$. In this way, we have restricted to the boundary component of $\mathcal{M}_2$ defined by the vanishing locus of the Siegel modular form χ_{10} [72]. Moreover, the degeneration of the $6I_2 + 2I_0^*$ Jacobian fibration on $\mathrm{Kum}(\mathrm{Jac}(\mathbf{C}))$ yields the $\mathfrak{J}_6$ Jacobian fibration on $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$. However, because the Legendre normal form in Equation (2.1.51) marks a 2-torsion point on $\mathcal{E}_i$, we naturally inherit a marked 2-torsion point on the abelian surface $\mathcal{E}_1 \times \mathcal{E}_2$.

On the twisted Legendre pencil, this degeneration corresponds to the limit $c \to 0$, yielding the two-parameter twisted Legendre pencil $\mathbf{X}_{a,b,0}$ equipped with the Jacobian fibration

$$\mathbf{X}_{a,b,0}: \quad \big\{(x, y, t) \mid y^2 = t(t-a)(t-b)\, x(x-1)(x-t)\big\} \tag{2.1.52}$$

with singular fibres $2I_0^* + 2I_2 + I_2^*$. The I_0^* fibres are located over $t = a, b$, the I_2 fibres are located over $t = 0, 1$, and the I_2^* fibre is located over $t = \infty$. In this way, assuming that $a, b \notin \{0, 1, \infty\}$, we recognize that the K3 surface $\mathbf{X}_{a,b,0}$ is the familiar quadratic twist family of the rational elliptic surface $\mathbf{Z}$ for the Legendre pencil, i.e., the modular elliptic surface for the rational curve $\mathbb{H}/\Gamma_0(2) \cong \mathbb{P}^1$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half plane, and $\Gamma_0(2) \subset \mathrm{SL}(2, \mathbb{Z})$ is the principal congruence subgroup of level-two. This is simply the total space of the family of elliptic curves

$$\mathbf{Z}: \quad \big\{(x, y, t) \mid y^2 = x(x-1)(x-t)\big\} \; . \tag{2.1.53}$$

It follows from fundamental results in [23] that the quadratic period relations can be expressed rather beautifully in terms of the factorization of hypergeometric functions that represent the period integrals on the K3 surface $\mathbf{X}_{a,b,0}$ and the Kummer surface $\mathbf{Y}'_{k_1^2, k_2^2, 0}$. We explain this further in §6.2.3.

We can further degenerate to Picard rank $\rho = 19$ by taking the limit in which

$b \to \infty$, yielding the K3 surface $\mathbf{X}_{a,\infty,0}$

$$\mathbf{X}_{a,\infty,0}: \quad \big\{(x,y,t) \mid y^2 = t(t-a)\,x(x-1)(x-t)\big\} \tag{2.1.54}$$

that was studied by Hoyt in [67]. This Jacobian fibration is of type $I_0^* + 2I_2^* + I_2$, with an I_0^* fibre over $t=a$, I_2^* fibres over $t=0,\infty$, and an I_2 fibre over $t=1$. On the side of the Kummer surface $\mathbf{Y}_{k_1,k_2,0}$, this degeneration corresponds to the limit in which $k_1 \mapsto \lambda, k_2 \mapsto -1/\lambda$. Then the abelian surface $\mathcal{E}_{k_1^2} \times \mathcal{E}_{k_2^2}$ degenerates to the product $\mathcal{E}_{\lambda^2} \times \mathcal{E}'_{\lambda^2}$, where $\mathcal{E}'_{\lambda^2}$ is 2-isogenous to $\mathcal{E}$. Again, the quadratic period relations for this family are encoded explicitly in terms of factorization of hypergeometric functions.

2.2 The Signature Theorem

In order to study a wider variety of analytical aspects of certain rational elliptic and elliptic K3 surfaces, we recall some fundamental notions of the differential topology of manifolds related the signature index.

2.2.1 The Cobordism Ring and Genera

One approach to investigating manifolds is through the construction of invariants. In this present context, we work to construct topological and geometric invariants in the category $\mathcal{S}$ of smooth, compact, orientable Riemannian manifolds. Given two such manifolds, we may generate new objects through the operations $+$ (disjoint union), $-$ (reversing the orientation), and $\times$ (Cartesian product). These operations turn $\mathcal{S}$ into a graded commutative monoid, graded naturally by dimension, which decomposes as the direct sum

$$\mathcal{S} = \bigoplus_{n=0}^{\infty} \mathcal{M}_n\,.$$

Here $\mathcal{M}_n \subseteq \mathcal{S}$ is the class of all smooth, compact, oriented Riemannian manifolds of dimension n. In $\mathcal{S}$, the additive operation is only well defined when restricted to elements in the same class $\mathcal{M}_n$. Furthermore, if $M \in \mathcal{M}_k$ and $N \in \mathcal{M}_j$, we have the graded commutativity relation $M \times N = (-1)^{kj} N \times M$. The positively oriented point $\{*\}$ serves as the unity element $1_{\mathcal{S}}$, but $\mathcal{S}$ fails to be a (graded) semi-ring because there is no neutral element $0_{\mathcal{S}}$ and there are no additive inverses. It is reasonable to attempt to define $0_{\mathcal{S}} = \emptyset$ and the additive inverse of M as $-M$ by reversing the orientation. However, $M \cup -M \neq \emptyset$, so the additive inverse is not well defined. If we insist that we want such as assignment of additive inverses and $0_{\mathcal{S}}$ to turn $\mathcal{S}$ or some quotient into a ring, we are led naturally to the notion of *cobordant* manifolds.

Definition 2.2.6. *An oriented differentiable manifold V^n bounds if there exists a compact oriented manifold X^{n+1} with oriented boundary $\partial X^{n+1} = V^n$. Two manifolds $V^n, W^n \in \mathcal{M}_n$ are cobordant if $V^n - W^n = \partial X^{n+1}$ for some smooth, compact $(n+1)$-manifold X with boundary, where by $V^n - W^n$ we mean $V^n \cup (-W^n)$ by reversing the orientation on W^n.*

The notion of "cobordant" is an equivalence relation $\sim$ on the class $\mathcal{M}_n$; hence we introduce the *oriented cobordism ring* Ω_* as the collection of all equivalence classes $[M] \in \Omega_n = \mathcal{M}_n/\sim$ for all $n \in \mathbb{N}$,

$$\Omega_* = \bigoplus_{n=0}^{\infty} \Omega_n.$$

Note first that for any boundary $V^n = \partial X^{n+1}$, V^n descends to the zero element $0 = [\emptyset] \in \Omega_*$. Since for any manifold M, we have that $M - M = \partial(M \times [0,1])$, the boundary of the oriented cylinder bounded by M, it follows that on Ω_*, $[M - M] = 0$. Thus $-[M] = [-M]$ for any manifold M. This in turn shows that the oriented cobordism ring Ω_* naturally inherits the structure of a graded commutative ring with respect to the operations on $\mathcal{S}$. For a thorough survey of cobordism (oriented and otherwise) we refer the reader to [135].

To study these structures, we look at nontrivial ring homomorphisms $\psi : \Omega_* \to \mathbb{C}$, i.e., for any two suitable $V, W \in \Omega_*$, the morphism ψ satisfies

$$\begin{aligned} \psi(V+W) &= \psi(V) + \psi(W)\,, \quad \psi(-V) \;=\; -\psi(V)\,, \\ \psi(V \times W) &= \psi(V) \cdot \psi(W)\,. \end{aligned}$$

Thus, we are interested in studying the dual space of Ω_*. This prompts the following:

Definition 2.2.7. *A genus is a ring homomorphism $\psi : \Omega_* \to \mathbb{Q}$. Hence, a genus $\psi \in \hom_{\mathbb{Q}}(\Omega_*; \mathbb{Q})$ is an element of the rational dual space to Ω_*.*

This implies that if $V^n = \partial X^{n+1}$ bounds, then necessarily $\psi(V^n) = 0$, as ψ must be compatible with the ring operations $(+, -, \times)$ and constant on the equivalence classes of manifolds that satisfy $V^n - W^n = \partial X^{n+1}$.

In the following section, we introduce the fundamental notion of the signature of a manifold of dimension $4k$. This quantity constitutes one of the most prominent examples of a ring homomorphism $\Omega_* \to \mathbb{Q}$. In fact, if the dimension of a compact manifold is divisible by four, the middle cohomology group is equipped with a real valued symmetric bilinear form ι, called the *intersection form*. The properties of this bilinear form allow us to define an important topological invariant, the *signature of a manifold*. The *Hirzebruch signature theorem* asserts that this topological index equals the analytic index of an elliptic operator on M, if M is equipped with a smooth Riemannian structure.

2.2.2 The definition of the signature

Let M be a compact manifold of dimension $n = 4k$. On the middle cohomology group $H^{2k}(M; \mathbb{R})$, a symmetric bilinear form ι called the *intersection form*, is obtained by evaluating the cup product of two $2k$-cocycles a, b on the fundamental homology class $[M]$ of M. Alternatively, we can obtain ι by integrating the wedge product of the corresponding differential $2k$-forms ω_a, ω_b over M via the deRham isomorphism. That is, we define ι and ι_{dR} as follows:

$$\begin{array}{llcllc}
\iota : & H^{2k}(M;\mathbb{R}) & \otimes & H^{2k}(M;\mathbb{R}) & \to & \mathbb{R} \\
 & & & (a,b) & \mapsto & \langle a \cup b,\ [M] \rangle \\
\iota_{dR} : & H^{2k}_{dR}(M;\mathbb{R}) & \otimes & H^{2k}_{dR}(M;\mathbb{R}) & \to & \mathbb{R} \\
 & & & (\omega_a, \omega_b) & \mapsto & \int_M \omega_a \wedge \omega_b \ .
\end{array}$$

Notice that ι is in fact symmetric since the dimension is a multiple of four.

Since M is compact, Poincaré duality asserts that for any cocycle $a \in H^{2k}(M;\mathbb{R})$ there is a corresponding cycle $\alpha \in H_{2k}(M;\mathbb{R})$. If we assume that the two cycles intersect transversally, then $\iota(a, b)$ can be expressed as the intersection number of the two cycles α and β, i.e., by computing

$$\iota(a,b) = \#(\alpha,\beta) = \sum_{p \in \alpha \cap \beta} i_p(\alpha,\beta)$$

where $i_p(\alpha, \beta)$ is either $+1$ or -1 depending on whether the orientation of the T_pM induced by the two cycles α, β agrees with the orientation of the manifold or not [139].

Sylvester's Theorem guarantees that any non-degenerate, real valued, symmetric bilinear form on a finite dimensional vector space can be diagonalized with only $+1$ or -1 entries on the diagonal. Thus, after a change of basis, we have $\iota \cong I_{u,v}$, where $I_{u,v}$ is the diagonal matrix with u entries $+1$ and v entries -1. This prompts the following definition.

Definition 2.2.8. *Let M be a compact $4k$-dimensional manifold with intersection form ι on the middle cohomology group $H^{2k}(M;\mathbb{R})$ such that $\iota \cong I_{u,v}$. Then the signature of M is given by* $\mathrm{sign}(M) = u - v$. *If the dimension of M is not a multiple of four, the signature is defined to be zero.*

It turns out that the signature is a fundamental topological *invariant* of M: if two $4k$ manifolds are homeomorphic, their signature will be equal; in fact, the signature is an invariant of the oriented homotopy class of M [73, 101]. Furthermore, the signature is constant on the oriented cobordism classes in Ω_*. We have the following theorem due to Hirzebruch [61].

Theorem 2.2.9. *The signature defines a ring homomorphism* $\mathrm{sign} : \Omega_* \to \mathbb{Z}$. *In particular, for all $k \in \mathbb{N}$ we have* $\mathrm{sign}(\mathbb{P}^{2k}) = 1$.

The theorem is proved by checking that the signature is compatible with the ring operations $(+, -, \times)$ on the classes $\mathcal{M}_{4k} \subseteq \mathcal{S}$, $k = 1, 2, \ldots$, by using the Künneth theorem and writing out the definition of the signature on a basis for the middle cohomology. Furthermore, one can show that $\mathrm{sign}(M) = 0$ for any manifold $M = \partial X$ that bounds. This is done by constructing a commutative exact ladder for $i : V^{4k} \hookrightarrow X^{4k+1}$, i.e., the embedding of V^{4k} as the boundary of X^{4k+1} using Lefschetz and Poincaré duality. The normalization for the even dimensional complex projective spaces $\mathbb{P}^{2k}$ can be checked as follows: the cohomology ring $H^*(\mathbb{P}^{2k};\mathbb{R})$ is a truncated polynomial ring in dimension $4k$, generated by the first Chern class of the hyperplane bundle $\mathrm{H} \in H^2(\mathbb{P}^{2k};\mathbb{R})$. It follows that $H^{2k}(\mathbb{P}^{2k};\mathbb{R})$ is generated by H^k, and

$$\iota(\mathrm{H}^k, \mathrm{H}^k) = \mathrm{H}^{2k}[\mathbb{P}^{2k}] = 1. \tag{2.2.55}$$

2.2.3 The signature as an analytical index

We will identify the signature of an n-dimensional smooth, compact oriented manifold M with the *index* of an elliptic operator acting on sections of a smooth vector bundle over M.[7]

The operator in question is the *Laplace* operator acting on smooth, complex valued differential p-forms on M. Over any oriented closed n-manifold M we have the exterior product bundles $\Lambda^p = \Lambda^p T^*_{\mathbb{C}} M$ of the complexified cotangent bundle $T^*_{\mathbb{C}} M = T^* M \otimes \mathbb{C}$. The complex-valued p-forms are the smooth sections $\omega : M \to \Lambda^p$; they form a vector space which we denote by $C^\infty(M, \Lambda^p)$. The vector spaces can be assembled into the so-called *de Rham complex* $C^\infty(M, \Lambda^*) = \bigoplus_{p=0}^n C^\infty(M, \Lambda^p)$. The summands are connected by the exterior derivative $d : C^\infty(M, \Lambda^p) \to C^\infty(M, \Lambda^{p+1})$, extended linearly over $T^*_{\mathbb{C}} M$, with the usual relation $d^2 = 0$.

Moreover, a Riemannian metric on M determines a Hermitian structure $(\cdot\,,\cdot)$ on $C^\infty(M, \Lambda^p)$ via the *Hodge-de Rham operator* $* : C^\infty(M, \Lambda^p) \to C^\infty(M, \Lambda^{n-p})$. The Hodge-de Rham operator is a bundle isomorphism $C^\infty(M, \Lambda^p) \to C^\infty(M, \Lambda^{n-p})$ which satisfies $*^2 = (-1)^{p(n-p)}\mathbb{I}$, where $\mathbb{I} : C^\infty(M, \Lambda^p) \to C^\infty(M, \Lambda^p)$ is the identity. Specifically, the Hodge dual of a p-form $\omega \in C^\infty(M, \Lambda^p)$ is the $(n-p)$-form denoted by $*\omega \in C^\infty(M, \Lambda^{n-p})$ determined by the property that for any p-form $\mu \in C^\infty(M, \Lambda^p)$ we have

$$\mu \wedge *\omega = \langle \mu, \omega \rangle \mathrm{vol}_M \,,$$

where the Riemannian structure on M induces both the inner product $\langle \cdot, \cdot \rangle$ on $C^\infty(M, \Lambda^p)$ and the volume form vol_M. A Hermitian structure on $C^\infty(M, \Lambda^p)$ is

[7] The results of this section can also be phrased in terms of spinors and spin bundles. We have chosen to leave this viewpoint out as to make the content more accessible.

then defined by setting

$$(\mu, \omega) = \int_M \mu \wedge *\bar{\omega}\,,$$

for any two p-forms μ, ω where $\bar{\omega}$ means complex conjugation. With respect to this inner product, we obtain an adjoint operator

$$\delta : C^\infty(M, \Lambda^{p+1}) \to C^\infty(M, \Lambda^p)$$

of the exterior derivative d defined by $(d\eta^p, \omega^{p+1}) = (\eta^p, \delta\omega^{p+1})$ and $\delta^2 = 0$. If $n = 2m$ the adjoint operator is $\delta = - * d*$.

The operator $D = d + \delta : C^\infty(M, \Lambda^*) \to C^\infty(M, \Lambda^*)$ is a first order differential operator that, by construction, is formally self-adjoint. We call D a *Dirac* operator, because it is, in a sense, the square root of the Laplace operator. In fact, the *Laplace operator*, also called the *Hodge Laplacian*, is given as its square by

$$\Delta_H = (d + \delta)^2 = d\delta + \delta d\,.$$

The Hodge Laplacian is homogeneous of degree zero, i.e., $\Delta_H : C^\infty(M, \Lambda^p) \to C^\infty(M, \Lambda^p)$, and is formally self-adjoint, i.e., $(\Delta_H\omega, \mu) = (\omega, \Delta_H\mu)$ for any p-forms ω, μ. A p-form ω is said to be *harmonic* if $\Delta_H\omega = 0$. One then shows that ω is harmonic if and only if ω is both closed and co-closed, i.e.,

$$\Delta_H\omega = 0 \;\Leftrightarrow\; d\omega = 0\,, \delta\omega = 0\,. \tag{2.2.56}$$

In fact, every $\eta \in H^p_{dR}(M; \mathbb{C})$ has a unique harmonic representative ω such that $\eta = \omega + d\phi$ for some $(p-1)$-form ϕ; this is the celebrated Hodge theorem [62]. This implies that there is an isomorphism $H^*_{dR}(M, \mathbb{C}) \cong \ker \Delta_H$.

Let us restrict to the case of an even-dimensional manifold M, i.e., $n = 2m$. Then, for $p = 0, \ldots, 2m$ we define the complex operators

$$\alpha_p = \sqrt{-1}^{p(p-1)+m} * : C^\infty(M, \Lambda^p) \to C^\infty(M, \Lambda^{2m-p}),$$

which for complex-valued differential forms are more natural than the Hodge-de Rham operator. We will denote the operators generically by $\alpha : C^\infty(M, \Lambda^*) \to C^\infty(M, \Lambda^*)$. One may check that the operators α_{2m-p} and α_p are inverses, $\alpha_{2m-p}\alpha_p = \alpha_p\alpha_{2m-p} = \mathbb{I}$. Hence, the eigenvalues of α_m are ± 1, so the de Rham complex splits into the two eigenspaces of α, given by $\alpha(\omega) = \pm\omega$, such that

$$C^\infty(M, \Lambda^*) \;=\; \underbrace{C_+^\infty(M, \Lambda^*)}_{\text{eigenvalue: } +1} \bigoplus \underbrace{C_-^\infty(M, \Lambda^*)}_{\text{eigenvalue:} -1} .$$

The projection operators onto the two eigenspaces given by $P_+ = (\mathbb{I} + \alpha)/2$ and $P_- = (\mathbb{I} - \alpha)/2$, respectively, and $P_+ + P_- = \mathbb{I}$, $P_\pm P_\mp = 0$. Since α anti-commutes with $d + \delta$, i.e., $\alpha(d + \delta) = -(d + \delta)\alpha$ and because of Equation (2.2.56), we get an orthogonal, direct-sum decomposition of the entire de Rham cohomology according to

$$H_{dR}^*(M; \mathbb{C}) \;=\; \bigoplus_{p=0}^{m} H_+^p(M; \mathbb{C}) \oplus H_-^p(M; \mathbb{C}) , \tag{2.2.57}$$

where $H_\pm^p(M; \mathbb{C}) = \{\omega \in H_{dR}^p(M; \mathbb{C}) \oplus H_{dR}^{2m-p}(M; \mathbb{C}) \mid \alpha(\omega) = \pm\omega\}$. Notice that for $p \neq m$, every element of $H_\pm^p(M; \mathbb{C})$ necessarily has the form $\omega \pm \alpha(\omega)$ for a non-trivial element $\omega \in H_{dR}^p(M; \mathbb{C})$, and we have for all $p \neq m$ the identity

$$\dim H_+^p(M, \mathbb{C}) = \dim H_-^p(M, \mathbb{C}) . \tag{2.2.58}$$

We now recall the notion of an *elliptic* differential operator, and the notion of its analytic index [63, 115]. Suppose we have two vector bundles $E, F \to M$ and and a q^{th}-order differential operator $\not{D} : C^\infty(M,E) \to C^\infty(M,F)$. Let $T'M$ be the cotangent bundle of M minus the zero section and $\pi : T'M \to M$ be the canonical projection. Then the *principal symbol* of the operator $\not{D}$ is a linear mapping $\sigma_{\not{D}} \in \mathrm{Hom}(\pi^* E, \pi^* F)$ such that $\sigma_{\not{D}}(x, \rho\xi) = \rho^q \sigma(x,\xi)$ for all $(x,\xi) \in T'M$ and for all scalars ρ. The operator $\not{D}$ is *elliptic* if the principal symbol $\sigma_{\not{D}}$ is a fiberwise isomorphism for all $x \in M$. It is a classical result that for an elliptic operator, the kernel and cokernel are finite dimensional [44]. The *analytic index* of $\not{D}$ is then defined as

$$\mathrm{ind}(\not{D}) = \dim(\ker \not{D}) - \dim(\mathrm{coker}\, \not{D}).$$

If we restrict the operator $D = d + \delta$ to the ± 1-eigenspaces of α, these restrictions become formal adjoint operators of one another. In physics, they are called *chiral* Dirac operators. We denote these operators by

$$\not{D} := (d+\delta)P_+ = P_-(d+\delta) : \ C^\infty_+(M,\Lambda^*) \ \to \ C^\infty_-(M,\Lambda^*)\,, \tag{2.2.59}$$

$$\not{D}^\dagger := (d+\delta)P_- = P_+(d+\delta) : \ C^\infty_-(M,\Lambda^*) \ \to \ C^\infty_+(M,\Lambda^*)\,, \tag{2.2.60}$$

where we again used the fact that α anti-commutes with $D = d + \delta$. We have that

$$\not{D}^\dagger \not{D} = \Delta_H P_+ = \Delta_H \Big|_{C^\infty_+(M,\Lambda^*)},$$

and that the operators $\not{D}$ and $\not{D}^\dagger$ are elliptic operators such that $\ker \not{D}^\dagger = \mathrm{coker}\, \not{D}$. To see this, we look to the principle symbols of the first order differential operators d and δ on the vector bundle $C^\infty(M,\Lambda^*)$ over M. One computes their principal symbols as $\sigma_d(x,\xi) = \mathrm{ext}(\xi)$ and $\sigma_\delta(x,\xi) = -\,\mathrm{int}(\xi)$, respectively, signifying the exterior algebra homomorphisms induced from the exterior product and interior product of forms, respectively. This implies that $\not{D} + \not{D}^\dagger = D$ is the Clifford multiplication on the exterior algebra of forms. Therefore, $\sigma_{\Delta_H}(x,\xi) = -|\xi|^2\mathbb{I}$. This mapping is invertible, and it follows that $\not{D}$, $\not{D}^\dagger$, and Δ_H are elliptic operators.

For the index of the elliptic operator $\not{D}$ on a compact even-dimensional manifold M^n with $n = 2m$ we obtain

$$\begin{aligned}\operatorname{ind}(\not{D}) &= \dim(\ker \not{D}) - \dim(\ker \not{D}^\dagger) \\ &= \sum_{p=0}^{m} \dim H_+^p(M;\mathbb{C}) - \dim H_-^p(M;\mathbb{C}) \\ &= \dim H_+^m(M;\mathbb{C}) - \dim H_-^m(M;\mathbb{C})\,,\end{aligned}$$

where we used $\ker \not{D}^\dagger = \operatorname{coker} \not{D}$ and Equation (2.2.58).

For $n = 2m$ with $m = 2k+1$ the operator α_m is an isomorphism between $H_+^m(M;\mathbb{C})$ and $H_-^m(M;\mathbb{C})$, thus $\operatorname{ind}(\not{D}) = 0$. In contrast, for $n = 4k$ we have

$$\operatorname{ind}(\not{D}) \;=\; \dim H_+^{2k}(M;\mathbb{C}) - \dim H_-^{2k}(M;\mathbb{C})\,,$$

and α maps $H_\pm^{2k}(M;\mathbb{C})$ to itself. Moreover, for $n = 4k$ we have $\alpha = *$ and the decomposition into ± 1-eigenspaces of α coincides with the orthogonal, direct-sum decomposition of $H_{dR}^{2k}(M;\mathbb{C})$ into self-dual and anti-self-dual, middle-dimensional, complex differential forms. Thus, we have

$$H_{dR}^{2k}(M;\mathbb{C}) \;=\; H_+^{2k}(M;\mathbb{C}) \oplus H_-^{2k}(M;\mathbb{C})\,.$$

On the other hand, using the de Rham isomorphism, we can evaluate the intersection form ι_{dR} on either two self-dual or anti-self-dual $2k$-forms ω, μ, to obtain

$$\begin{aligned} \iota_{dR}(\omega, \mu) &= (\omega, \mu)\,, \\ \Rightarrow \qquad \iota_{dR}(\omega, \omega)\big|_{\omega \in H^{2k}_{\pm}(M,\mathbb{C})} &= (\omega, \omega) = \pm \|\omega\|^2\,. \end{aligned}$$

This shows that $\iota_{dR} \cong I_{u,v}$ with $u = \dim H^{2k}_{+}(M;\mathbb{C})$ and $v = \dim H^{2k}_{-}(M;\mathbb{C})$, so that the index of the Dirac operator is equal to the difference of the number of linearly independent self-dual and anti-self-dual cohomology classes over $\mathbb{C}$. Hence, we have the following special case of the Atiyah-Singer index theorem [3, 61]:

Theorem 2.2.10 (Analytic index of signature operator)**.** *For all compact, oriented Riemannian manifolds M of dimension $4k$, the (analytic) index of the Dirac operator defined above equals the signature of M, i.e.,* $\mathrm{ind}(\not{D}) = \mathrm{sign}(M)$.

2.3 The structure of the Cobordism ring and its genera

Having shown that the signature of a manifold M is a genus, we now investigate the structure of the oriented cobordism ring and its rational homomorphisms. In this spirit, it is beneficial to look at the *rational* oriented cobordism ring $\Omega_* \otimes \mathbb{Q}$. Tensoring with $\mathbb{Q}$ kills torsion subgroups; in fact, it is known that the cobordism groups Ω_n are finite and (graded) commutative when $n \neq 4k$ [139], implying that each such cobordism group is torsion. We shall see that $\Omega_* \otimes \mathbb{Q}$ has a generator of degree $4k$ for all $k \in \mathbb{N}$. This implies that the graded commutativity of Ω_* is erased by tensoring with $\mathbb{Q}$, leaving an honest commutative ring. Thus, the rational cobordism ring is of the form

$$\Omega_* \otimes \mathbb{Q} = \bigoplus_{k=0}^{\infty} \Omega_{4k} \otimes \mathbb{Q}$$

and forms a commutative ring with unity. We will soon be able to say much more about the structure of this ring and provide a generating set, along with a complete description of its dual $\hom_{\mathbb{Q}}\left(\Omega_* \otimes \mathbb{Q}, \mathbb{Q}\right)$. This will allow for a description of the signature of a $4k$-manifold as a topological index in terms of its Pontrjagin classes.

2.3.1 Pontrjagin numbers

We begin with the following motivation for the Pontrjagin classes [102]. Let ξ be a real vector bundle of rank r over a topological space B. Then the *complexification* $\xi \otimes \mathbb{C} = \xi \otimes_{\mathbb{R}} \mathbb{C}$ is a complex rank r vector bundle over B, with a typical fibre $F \otimes \mathbb{C}$ given by

$$F \otimes \mathbb{C} = F \oplus \sqrt{-1}F.$$

It follows that the underlying real vector bundle $(\xi \otimes \mathbb{C})_{\mathbb{R}}$ is canonically isomorphic to the Whitney sum $\xi \oplus \xi$. Furthermore, the complex structure on $\xi \otimes \mathbb{C}$ corresponds to the bundle endomorphism $J(x, y) = (-y, x)$ on $\xi \oplus \xi$.

In general, complex vector bundles η are not isomorphic to their conjugate bundles $\bar{\eta}$. However, complexifications of real vector bundles are *always* isomorphic to their conjugates: we have $\xi \otimes \mathbb{C} \cong \overline{\xi \otimes \mathbb{C}}$. Let us consider the total Chern class given as the formal sum

$$c(\xi \otimes \mathbb{C}) = 1 + c_1(\xi \otimes \mathbb{C}) + c_2(\xi \otimes \mathbb{C}) + \cdots + c_r(\xi \otimes \mathbb{C}).$$

We refer the reader to [18] for an introduction to Chern classes. From the above isomorphism, this expression is equal to the total Chern class of the conjugate bundle, that is

$$c(\overline{\xi \otimes \mathbb{C}}) = 1 - c_1(\overline{\xi \otimes \mathbb{C}}) + c_2(\overline{\xi \otimes \mathbb{C}}) - \cdots + (-1)^r c_r(\overline{\xi \otimes \mathbb{C}}).$$

Thus, the odd Chern classes $c_{2i+1}(\xi \otimes \mathbb{C})$ are all elements of order 2, so it is the even Chern classes of the complexification $\xi \otimes \mathbb{C}$ that encode relevant topological information about the real vector bundle ξ. This prompts the following definition.

Definition 2.3.11. *Let $\xi \to B$ be a real vector bundle of rank r. The i^{th} rational Pontrjagin class of ξ for $1 \leq i \leq r$ is defined as follows:*

$$p_i(\xi) \;\; = \;\; (-1)^i \, c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B; \mathbb{Q})\,.$$

where c_{2i} is the $2i^{th}$ rational Chern class of the complexified bundle $\xi \otimes \mathbb{C}$.The total rational Pontrjagin class is the formal sum

$$p(\xi) \;\; = \;\; \sum_i p_i(\xi) \; \in H^*(B; \mathbb{Q}).$$

The Pontrjagin classes of an oriented manifold M are the Pontrjagin classes of its tangent bundle TM.

As a practical way to compute the Pontrjagin classes of ξ, let ∇ be a bundle connection on $\xi \otimes \mathbb{C}$, and let Ω be the corresponding curvature tensor. Then the total Pontrjagin class $p(\xi) \in H^*_{dR}(B; \mathbb{R})$ is a polynomial in the curvature tensor given by the expansion of the right hand side in the formula

$$p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \cdots + p_r(\xi) = \det(I + \Omega/2\pi) \in H^*_{dR}(B; \mathbb{R}).$$

As the total Pontrjagin class is defined as a determinant, it is immediately invariant under the adjoint action of $\mathrm{GL}(r,\mathbb{R})$, and furthermore, that the cohomology classes of the resulting differential forms are independent of the connection ∇ [19, 20].

Suppose that M is an oriented $4k$-dimensional manifold. If $(i) = (i_1, \ldots, i_m)$ is a partition of k, i.e. $|(i)| = \sum_{a=1}^{m} i_a = k$, then we define the corresponding *rational Pontrjagin number* of M to be

$$p_{(i)}[M] = \big(p_{i_1}(TM) \cup \cdots \cup p_{i_m}(TM)\big)[M] \in \mathbb{Q}\,, \tag{2.3.61}$$

and, if the dimension is not a multiple of four, all Pontrjagin numbers of M are defined to be zero. When working with the Pontrjagin classes in the de Rham cohomology of M, the Pontrjagin numbers are given by integrating the top-dimensional form $p_{(i)} = p_{i_1} \wedge p_{i_2} \wedge \cdots \wedge p_{i_m}$ over the fundamental cycle $[M]$, i.e.,

$$p_{(i)}[M] = \int_M p_{i_1} \wedge p_{i_2} \wedge \cdots \wedge p_{i_m}. \tag{2.3.62}$$

Let us compute the Pontrjagin number for the class $p_1/3$ on $\mathbb{P}^2$. Equip the tangent bundle of $\mathbb{P}^2$ with the Fubini-Study metric [48, 136]. Then in the affine coordinate chart $[z_1 : z_2 : 1]$, a computation with the curvature tensor shows that

$$\frac{1}{3}p_1 = \frac{2}{\pi^2(|z_1|^2 + |z_2|^2 + 1)^3} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

Integrating this form over the fundamental cycle yields

$$\int_{\mathbb{P}^2} \frac{1}{3}p_1 = 1.$$

As we shall see in Section 2.3.3, this is consistent with $\mathrm{sign}(\mathbb{P}^2) = 1$ by Theorem 2.2.9.

The following explains the importance of the Pontrjagin classes [116]:

Proposition 2.3.12 (Theorem of Pontrjagin)**.** *The evaluation of the Pontrjagin classes are compatible with the operations* $(+,-,\times)$ *on* $\Omega_* \otimes \mathbb{Q}$*. Furthermore, all Pontrjagin numbers vanish if a manifold bounds. Thus, the Pontrjagin numbers define rational homomorphisms* $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$*.*

In Proposition 2.3.12 compatibility with $+$ is trivial and compatibility with $-$ follows from the fact that the Pontrjagin numbers are independent of the orientation [112, 113]. For multiplicativity, notice that the total Pontrjagin class of the Whitney sum satisfies $p(\xi_1 \oplus \xi_2) = p(\xi_1)p(\xi_2)$ up to two-torsion [116]. As we are working over $\mathbb{Q}$, this equality always holds.

2.3.2 The rational cohomology ring and genera

We have seen in Proposition 2.3.12 that the Pontrjagin numbers constitute prototypes of the rational homomorphisms $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$. It turns out that every genus can be constructed in this way [139]. This follows from a remarkable relationship between the rational oriented cobordism ring $\Omega_* \otimes \mathbb{Q}$ and the cohomology ring of the classifying space BSO for oriented vector bundles.

We assume that M is a smooth, compact n-manifold and recall the notion of a *classifying space* for oriented vector bundles.

Definition 2.3.13. *For every oriented rank* r *vector bundle* $\pi : E \to M$*, there is a topological space* $\mathrm{BSO}(r)$ *and a vector bundle* $\mathrm{ESO}(r) \to \mathrm{BSO}(r)$ *together with a classifying map* $f : M \to \mathrm{BSO}(r)$ *such that* $f^* \mathrm{ESO}(r) = E$*. The space* $\mathrm{BSO}(r)$ *is called a classifying space, and the vector bundle* $\mathrm{ESO}(r) \to \mathrm{BSO}(r)$ *is called the universal bundle of oriented* r*-planes over* $\mathrm{BSO}(r)$*.*

It is known that the isomorphism class of the vector bundle E determines the classifying map f up to homotopy. Moreover, the classifying space and the universal bundle can be constructed explicitly as the Grassmannian of oriented r-planes over $\mathbb{R}^\infty$, denoted by $\mathrm{BSO}(r) = \mathrm{Gr}_r(\mathbb{R}^\infty)$, and the tautological bundle of oriented r-planes over it, denoted by $\mathrm{ESO}(r) = \gamma^r \to \mathrm{BSO}(r)$ respectively [99, 100].

For any oriented rank r vector bundle $E \to M$ we can form the sphere bundle $\Sigma(E) \to M$ by taking the one-point compactification of each fiber E_p of the vector bundle over each point $p \in M$ and gluing them together to get the total space. We then construct the *Thom space* $T(E)$ [138] as the quotient by identifying all the new points to a single point $t_0 = \infty$, which we take as the base point of $T(E)$. Since M is assumed compact, $T(E)$ is the one-point compactification of E. Associated to the universal bundle $\mathrm{ESO}(r)$ is the *universal Thom space*, denoted by $\mathrm{MSO}(r)$.

Moreover, thinking of M as embedded into E as the zero section, there is a class $u \in \tilde{H}^r(T(E);\mathbb{Z})$, called the *Thom class*, such that for any fiber E_p the restriction of u is the (orientation) class induced by the given orientation of the fiber E_p. This class u is naturally an element of the reduced cohomology [138]

$$\tilde{H}^r(T(E);\mathbb{Z}) \cong H^r(\Sigma(E), M;\mathbb{Z}) \cong H^r(E, E - M;\mathbb{Z})\,.$$

It turns out that the map

$$H^i\big(E;\mathbb{Z}\big) \to \tilde{H}^{i+r}\big(T(E);\mathbb{Z}\big)\,, \qquad z \mapsto z \cup u\,,$$

is an isomorphism for every $i \geq 0$, called the *Thom isomorphism* [138]. Since the pullback map $\pi^* : H^*(M;\mathbb{Z}) \to H^*(E;\mathbb{Z})$ is a ring isomorphism as well, we obtain an isomorphism

$$H^i\big(M;\mathbb{Z}\big) \to \tilde{H}^{i+r}\big(T(E);\mathbb{Z}\big)\,, \qquad x \mapsto \pi^*(x) \cup u\,, \tag{2.3.63}$$

for every $i \geq 0$ which sends the identity element of $H^*(M;\mathbb{Z})$ to u. The following result of Thom is crucial [139]:

Theorem 2.3.14. *(Pontrjagin-Thom) For $r \geq n+2$, the homotopy group $\pi_{n+r}(\mathrm{MSO}(r),\infty)$ is isomorphic to the oriented cobordism group Ω_n.*

Proof. We offer only a sketch, and refer the reader to [138, 139, 102] for more details. Let M be an arbitrary smooth, compact, orientable n-manifold, and let $E \to M$ be a smooth, oriented vector bundle of rank r. The base manifold M is smoothly embedded in the total space E as the zero section, and hence in the Thom space $T = T(E)$. In particular, note that $M \subset T - \{t_0\}$. Note that T itself is not a smooth manifold – it is singular precisely at the base point t_0. Results in [102] show that every continuous map $f : S^{n+r} \to T$ is homotopic to a map $\widehat{f}$ that is smooth on $\widehat{f}^{-1}(T-\{\, t_0\})$. It follows that the oriented cobordism class of $\widehat{f}^{-1}(M)$ depends only on the homotopy class of $\widehat{f}$. Hence, the mapping $\widehat{f} \mapsto \widehat{f}^{-1}(M)$ induces a homomorphism $\pi_{n+r}(T, t_0) \to \Omega_n$. We will argue that this homomorphism is surjective. We refer the reader to [139] for the proof of injectivity.

A theorem of Whitney [142] shows that M can be smoothly embedded in $\mathbb{R}^{n+r}$. Identifying M with its image in Euclidean space, we choose a neighborhood U of M in $\mathbb{R}^{n+r}$, diffeomorphic to the total space $E(\nu^r)$ of the normal bundle ν^r to M. Let $m, q \geq n$ and γ_q^m be the tautological bundle of oriented m-planes over $\mathbb{R}^{m+q}$, and let $E(\gamma_a^m)$ be the total space of γ_a^m, $a = n, q$. Applying the Gauss map for Grassmannians, we obtain

$$U \cong E(\nu^r) \to E(\gamma_n^r) \subseteq E(\gamma_q^r).$$

We compose this mapping with the canonical mapping $E(\gamma_n^r) \to T(\gamma_q^r)$, and let $B \subseteq T(\gamma_q^r)$ be the smooth n-manifold identified with the zero section in the Thom space.

We have obtained a map $f_q : U \to T(\gamma_q^r)$ such that $f_q^{-1}(B) = M$. Hence, if t_q is the base point of the Thom space $T(\gamma_q^r)$, it follows from above that f_q is homotopic to a smooth map $\widehat{f}_q$ that is smooth on $T(\gamma_q^r) - \{t_q\}$. Thus, we obtain a surjective homomorphism $\pi_{n+r}(T(\gamma_q^r), t_q) \to \Omega_n$. Taking a direct limit on q the claim follows.

$\square$

Using the argument above, we can embed a representative manifold $M \in \Omega_n$ into $\mathrm{MSO}(r)$ for some $r \geq n+2$. By taking the direct limit, we make the construction independent of the embedding [139] and obtain a *canonical* isomorphism

$$\Omega_n \cong \varinjlim_{r\to\infty} \pi_{n+r}\big(\mathrm{MSO}(r), \infty\big)\,.$$

Another crucial result is a theorem by Serre [128] that asserts that in the range less or equal two times the connectivity of a space, rational homotopy is the same as rational cohomology. As shown in [138], the connectivity of the Thom space $\mathrm{MSO}(r)$ is $(r-1)$. Therefore, there are isomorphisms for all $r \geq n+2$ of the form

$$\pi_{n+r}\big(\mathrm{MSO}(r), \infty\big) \otimes \mathbb{Q} \xrightarrow{\cong} \tilde{H}^{n+r}\big(\mathrm{MSO}(r); \mathbb{Q}\big)\,.$$

From Equation (2.3.63) we also have the Thom isomorphisms

$$H^n\big(\mathrm{BSO}(r); \mathbb{Q}\big) \longrightarrow \tilde{H}^{n+r}\big(\mathrm{MSO}(r); \mathbb{Z}\big)\,.$$

Thus, for all $n \geq 0$ it follows that

$$H^n\big(\mathrm{BSO}(r); \mathbb{Q}\big) \cong \pi_{n+r}\big(\mathrm{MSO}(r), \infty\big) \otimes \mathbb{Q} \cong \Omega_n \otimes \mathbb{Q}. \tag{2.3.64}$$

Equation (2.3.64) together with $\mathrm{BSO} = \lim_{r\to\infty} \mathrm{BSO}(r)$ yields – after taking the appropriate limits on n – an isomorphism

$$\Omega_* \otimes \mathbb{Q} \xrightarrow{\cong} H^*\big(\mathrm{BSO}; \mathbb{Q}\big)\,. \tag{2.3.65}$$

Equation (2.3.65) shows that the study of genera is equivalent to studying rational homomorphisms $H^*(\mathrm{BSO};\mathbb{Q}) \to \mathbb{Q}$. However, the cohomology ring $H^*(\mathrm{BSO};\mathbb{Q}) = \mathbb{Q}[\mathbf{p}_1, \mathbf{p}_2, \dots]$ is easy to understand: it is a polynomial ring over $\mathbb{Q}$ generated by the Pontrjagin classes $\{\mathbf{p}_i\}$ of the universal classifying bundle ESO $\to$ BSO [139]. The ring isomorphism $\Omega_* \otimes \mathbb{Q} \to H^*(\mathrm{BSO};\mathbb{Q})$ identifies for all $1 \leq i \leq k$ the Pontrjagin classes of a $4k$-manifold M by pull-back $p_i(M) = f^*\mathbf{p}_i$ under the classifying map $f : M \to \mathrm{BSO}$ for the tangent bundle TM.

Thus, the structure of the ring $\Omega_* \otimes \mathbb{Q}$ and its dual space of rational homomorphisms $\hom_{\mathbb{Q}}(\Omega_* \otimes \mathbb{Q}; \mathbb{Q})$ is now easily understood: the latter consists of sequences of homogeneous polynomials in the Pontrjagin classes with coefficients in $\mathbb{Q}$. In fact, given a genus $\psi \in \hom_{\mathbb{Q}}(\Omega_* \otimes \mathbb{Q}; \mathbb{Q})$, there exists a homogeneous polynomial $L_k \in \mathbb{Q}[\mathbf{p}_1, \dots, \mathbf{p}_k]$ for every degree $4k$ such that for any manifold $M \in \Omega_{4k}$ we have

$$\psi([M]) = L_k(p_1, \dots, p_k)[M] \in \mathbb{Q}\,, \tag{2.3.66}$$

where the evaluation is carried out according to Equation (2.3.61) and Equation (2.3.62) [139]. Hence, the homomorphism ψ is associated to a sequence of homogeneous polynomials $\{L_1, L_2, \dots\} \subseteq \mathbb{Q}[\mathbf{p}_1, \mathbf{p}_2, \dots]$, where L_k is homogeneous of degree $4k$.

We summarize the results of this section in the following:

Theorem 2.3.15 (Rational cobordism ring)**.** *The rational oriented cobordism ring $\Omega_* \otimes \mathbb{Q}$ is isomorphic to the cohomology ring $H^*(\mathrm{BSO};\mathbb{Q})$. In particular, each element of $\hom_{\mathbb{Q}}(\Omega_* \otimes \mathbb{Q}; \mathbb{Q})$ is determined by a sequence of homogeneous polynomials in the Pontrjagin classes.*

The rich structure of $H^*(\mathrm{BSO};\mathbb{Q})$ also tells us how to find a suitable sequence of generators for $\Omega_* \otimes \mathbb{Q}$. Such a sequence is given by the cobordism classes of the even dimensional complex projective spaces (thought of as real $4k$-manifolds) [139]. Thus, each homomorphism is completely determined by its values on the even dimensional complex projective spaces, i.e., the generators of $\Omega_* \otimes \mathbb{Q}$.

2.3.3 The Hirzebruch L-genus

We now have the perspective to characterize the signature in terms of the Pontrjagin numbers. Theorem 2.2.9 asserts that the signature is a cobordism invariant $\mathrm{sign}(\cdot) \in \hom_{\mathbb{Q}}(\Omega_* \otimes \mathbb{Q};\mathbb{Q})$. Therefore, there must be a collection of polynomials $\{L_k(p_1,\ldots,p_k)\}_{k\in\mathbb{N}}$ such that for any smooth, compact, oriented manifold of dimension $4k$ the signature is

$$\mathrm{sign}(M) = L_k(p_1,p_2,\ldots,p_k)[M]\,,$$

where the evaluation is carried out according to Equation (2.3.61) and Equation (2.3.62). The right hand side is called the *Hirzebruch L-genus* [61]. By Theorem 2.3.15, the polynomials L_k have the general form

$$L_k = L_k(p_1,p_2,\ldots,p_k) = \sum_{|(i)|=k} \ell_k^{(i)}\, p_{i_1}\cdots p_{i_m}\,. \tag{2.3.67}$$

Let us explain how to compute a few of these L-polynomials: Theorem 2.2.9 asserts that the signature takes the value 1 on all even dimensional complex projective spaces. Thus, we have the sequence of equations

$$\begin{aligned} 1 &= \operatorname{sign}(\mathbb{P}^2) &= \ell_1^{(1)}\, p_1[\mathbb{P}^2] \\ 1 &= \operatorname{sign}(\mathbb{P}^4) &= \ell_2^{(1,1)}\, p_1^2[\mathbb{C}P^4] + \ell_2^{(2)}\, p_2[\mathbb{P}^4]\,, \\ & & \vdots \end{aligned}$$

and relations arising from the multiplicativity. The total Pontrjagin class of the even complex projective spaces are given by $p(\mathbb{P}^{2k}) = (1+\mathrm{H}^2)^{2k+1}$ with $\mathrm{H}^{2k+1} \equiv 0$. Thus, $p_1(\mathbb{P}^2) = 3\mathrm{H}^2$ and using Equation (2.2.55) we obtain $\ell_1^{(1)} = 1/3$. Compare this result with Example 2.3.1.

To find the polynomial L_2, we observe that generators of Ω_8 are given by $\mathbb{P}^4$ and $\mathbb{P}^2 \times \mathbb{P}^2$, with total Pontrjagin classes

$$p(\mathbb{P}^4) = 1 + 5\mathrm{H}^2 + 10\mathrm{H}^4 \qquad p(\mathbb{P}^2 \times \mathbb{P}^2) = (1+3\mathrm{H}_1^2)(1+3\mathrm{H}_2^2)\,,$$

where $\mathrm{H}_1^2, \mathrm{H}_2^2$ are the generators of the respective copies of $H^4(\mathbb{P}^2;\mathbb{Z})$. Then we have the following system of linear equations

$$\begin{aligned} 1 &= \ell_2^{(1,1)}\, p_1^2[\mathbb{P}^4] + \ell_2^{(2)}\, p_2[\mathbb{P}^4] \\ 1 &= \ell_2^{(1,1)}\, p_1^2[\mathbb{P}^2 \times \mathbb{P}^2] + \ell_2^{(2)}\, p_2[\mathbb{P}^2 \times \mathbb{P}^2]\,, \end{aligned}$$

which evaluates to the system

$$1 = 25\ell_2^{(1,1)} + 10\ell_2^{(2)}\,, \qquad 1 = 18\ell_2^{(1,1)} + 9\ell_2^{(2)}.$$

The solution is $\ell_2^{(1,1)} = -\frac{1}{45}, \ell_2^{(2)} = \frac{7}{45}$. Thus we can conclude

$$L_1 = \frac{1}{3}p_1\,, \qquad L_2 = \frac{1}{45}(7p_2 - p_1^2)\,.$$

In general, the homogeneous polynomials L_k are found by solving a system of linear equations on the generators of Ω_{4k}. These are the products $\mathbb{P}^{2i_1} \times \cdots \times \mathbb{P}^{22i_m}$ where $(i_1, \ldots, i_m)$ ranges over all partitions of k. The calculation can be formalized with the help of the so called *multiplicative sequences*. Results in [61] show that the coefficients of all polynomials $\{L_k\}_{k\in\mathbb{N}}$ can be efficiently stored in a formal power series $Q(z) = \sum_{i=0}^{\infty} b_i z^i$ with $b_0 = 1$ that satisfies $Q(zw) = Q(z)Q(w)$.

To find the multiplicative sequence from a formal power series $Q(z)$, first define the *Pontrjagin roots* by the formal factorization of the total Pontrjagin class $\sum_{i=1}^{k} p_i z^i = \prod_{i=1}^{k}(1 + t_i z)$, that is, consider the Pontrjagin classes the elementary symmetric functions in variables $t_1, \ldots, t_k$ of degree 2. Thus, given a $4k$-manifold M, we have the formal factorization of the total Pontrjagin class

$$p(M) = 1 + p_1 + p_2 + \ldots p_k = (1 + t_1)(1 + t_2) \cdots (1 + t_k),$$

whence

$$p_1 = t_1 + \cdots + t_k\,, \qquad p_2 = t_1 t_2 + t_1 t_3 + \cdots + t_{k-1} t_k\,, \quad \ldots$$

Then from the multiplicative property of Q we obtain

$$Q\big(p(M)\big) = Q(1 + t_1)\, Q(1 + t_2) \cdots Q(1 + t_k).$$

Putting the formal variable z back into the equation, it turns out that the equation

$$\sum_{i=0}^{k} L_i(p_1, \ldots, p_i)\, z^i + O(z^{k+1}) = \prod_{i=1}^{r} Q(1 + t_i z)$$

can be used to calculate the polynomial L_k recursively. The following result is crucial:

Lemma 2.3.16 ([61, 102])**.** *The coefficient of* $p_{i_1} \cdots p_{i_m}$ *in* L_k *in Equation (2.3.67) corresponding to the partition* $(i) = (i_1, \ldots, i_m)$ *with* $i_1 \geq \cdots \geq i_m$ *and* $\sum i_a = k$ *is calculated as follows: let* $s_{(i)}(p_1, \ldots, p_k)$ *be the unique polynomial such that*

$$s_{(i)}(p_1, \ldots, p_k) = \sum t_1^{i_1} \cdots t_m^{i_m}.$$

Then the coefficient of $p_{i_1} \cdots p_{i_m}$ *in* L_k *is* $s_{(i)}(b_1, \ldots, b_k)$*, where* $Q(z) = \sum_{i=0}^{\infty} b_i z^i$ *is the formal power series of the genus. Furthermore, the polynomial* L_k *is given by*

$$L_k(p_1, \ldots, p_k) = \sum_{(i)} s_{(i)}(b_1, \ldots, b_k)\, p_{(i)}\,,$$

where the sum is over all partitions (i) *of* k *and* $p_{(i)} = p_{i_1} \cdots p_{i_m}$*.*

Lemma 2.3.16 allows one to compute the multiplicative sequence of polynomials $\{L_k(p_1, \ldots, p_k)\}_{k\in\mathbb{N}}$ from a formal power series $Q(z)$, and conversely, from a given formal power series $Q(z)$, the multiplicative sequence of polynomials $\{L_k(p_1, \ldots, p_k)\}_{k\in\mathbb{N}}$. This leads to the following [61]:

Theorem 2.3.17 (Hirzebruch signature theorem)**.** *Let* $\{L_k(p_1, \ldots, p_k)\}_{k\in\mathbb{N}}$ *be the multiplicative sequence of polynomials corresponding to the formal power series*

$$Q(z) = \frac{\sqrt{z}}{\tanh\sqrt{z}} = 1 + \frac{1}{3}z - \frac{1}{45}z^2 + \frac{2}{945}z^3 - \frac{1}{4725}z^4 + \ldots.$$

For any smooth, compact, oriented manifold M *of dimension* $4k$ *the signature is*

$$\operatorname{sign}(M) = L_k(p_1, p_2, \ldots, p_k)[M]\,.$$

Let us illustrate the use of the theorem in the following: A computation with the first two symmetric polynomials gives

$$s_2(p_1, p_2) = p_1^2 - p_2 = t_1^2 + t_2^2\,, \qquad s_{1,1}(p_1, p_2) = p_2 = t_1 t_2\,.$$

Hence, we have

$$L_2 \;=\; s_2(1/3, -1/45)p_2 + s_{1,1}(1/3, -1/45)p_1^2 \;=\; \frac{1}{45}(7p_2 - p_1^2)\,,$$

which confirms the computation for L_2 above. Similarly, one checks that

$$\begin{aligned} s_3(p_1, p_2, p_3) &= p_1^3 - 3p_1p_2 + 3p_3 = t_1^3 + t_2^3 + t_3^3\,, \\ s_{2,1}(p_1, p_2, p_3) &= p_1p_2 - 3p_3 = t_1^2t_2 + t_1t_2^2 + \ldots\,, \\ s_{1,1,1}(p_1, p_2, p_3) &= p_3 = t_1t_2t_3\,, \end{aligned}$$

and one obtains

$$L_3 \;=\; \frac{1}{945}\big(62p_3 - 13p_2p_1 + 2p_1^3\big)\,.$$

For $\mathbb{P}^6$ we have $p_1 = 7\mathrm{H}^2$, $p_2 = 21\mathrm{H}^4$, $p_3 = 35\mathrm{H}^6$ so that $L_3[\mathbb{P}^6] = 1$ when using Equation (2.2.55).

We close this section with several remarks on the Hirzebruch signature theorem:

• The signature is an integer, so the Hirzebruch signature theorem imposes non-trivial integrality constraints on the rational combinations of the Pontrjagin numbers determined by the L-polynomials.

• The signature is an oriented homotopy invariant, whereas the L-polynomials are expressed in terms of Pontrjagin classes that rely heavily on the tangent bundle, and thus on the smooth structure, and orientation. This fact was used by Milnor to detect inequivalent differentiable structures on the 7-sphere [101]. According to a theorem of Kahn [73], the L-genus is (up to a rational multiple) the only rational linear combination of the Pontrjagin numbers that is an oriented homotopy invariant.

• Combining Theorem 2.2.10 and Theorem 2.3.17 implies that for a compact, oriented Riemannian manifold M of dimension $4k$, the index of the chiral Dirac operator $\not{D}$ is given by

$$\mathrm{ind}(\not{D}) = L_k(p_1, \ldots, p_k)[M] = \int_M L_k \,. \tag{2.3.68}$$

This statement is a special case of the celebrated Atiyah-Singer index theorem, which proves that for an elliptic differential operator on a compact manifold, the analytical index equals the topological index defined in terms of characteristic classes [3].

•The signature complex can be twisted by a complex vector bundle $\xi \to M$ of rank r [1, 41]. The coupling of the complexified signature complex to the vector bundle ξ yields a twisted chiral Dirac operator

$$\not{D}^\xi : C^\infty_+(M, \Lambda^* \otimes \xi) \to C^\infty_-(M, \Lambda^* \otimes \xi)\,.$$

For simplicity, we will assume $c_1(\xi) = 0$ and that the dimension of the manifold M is 4. Then the effect of the twisting on the index is as follows:

$$\mathrm{ind}(\not{D}^\xi) = r \cdot \mathrm{ind}(\not{D}) - \int_M c_2(\xi)\,, \tag{2.3.69}$$

where $c_2(\xi)$ is the second Chern class of the complex bundle $\xi \to M$ of rank r.

CHAPTER 3

The fibrewise signature operator on elliptic surfaces

The focus of this chapter is to study aspects of elliptic curves $\mathcal{E}$ that depend only on their complex structure

$$\tau = \frac{\int_\alpha \omega}{\int_\beta \omega} \in \mathbb{H}$$

by use of Kodaira's functional invariant in Equation (2.1.3). Here $\alpha, \beta \in H_1(\mathcal{E}, \mathbb{Z})$ is a symplectic basis.

As was discussed in §2.1.2, the functional invariant determines a Jacobian elliptic fibration up to quadratic twist of the Weierstrass model. The functional invariant is still quite powerful, as it provides reference to a *universal family* of elliptic curves, described by an explicit Weierstrass model in Equation (3.1.8). The total space of this family is a singular rational elliptic surface such that a map between two Jacobian elliptic surfaces factor through their functional invariants.

Using the explicit form of this universal family, we study how the signature of a smooth elliptic curve varies as a function of its complex structure. From more or less the definition of the signature in §2.2, the signature of $\mathcal{E}$ vanishes, $\operatorname{sign}(\mathcal{E}) = 0$. Thus, a more refined quantity is required to investigate this behavior.

The primary object of study is that of the determinant line bundle $\mathbf{Det}\,\bar{\partial} \to J^*$ associated to the Hodge de Rham Laplacian $\Delta_H = -4\partial\bar{\partial}$ from a choice of conformal class of flat Riemannian metric, introduced in §3.1.1. This line bundle, defined over the base J^* of the universal family of elliptic curves, is equipped with a certain smooth metric, the *Quillen* metric, that smooths out the potential jumps in dimension of kernel and cokernel of Δ_H. We compute the Quillen norm of a holomorphic trivializing section $s : J^* \to \mathbf{Det}\,\bar{\partial}$ in Proposition 3.1.19.

This computation allows us to compute the generalized first Chern class $c_1(\mathbf{Det}\,\bar{\partial})$, which is done in Theorem 3.1.21 from the Quillen norm computed in 3.1.19 via Quillen's anomaly theorem 3.1.20. This is a concrete instance of the Riemann-Roch-Grothendieck-Quillen (RRGQ) formula, and provides a measure of the so-called local and global anomaly associated to the $\bar{\partial}$ operator in physics. The computation uses known techniques for computing such anomalies from determinant line bundles, but to our knowledge this computation had not been previously carried out. Using the functional invariant, the anomaly is pulled back to a generic rational elliptic surface / elliptic K3 $\pi : Z \to B$ surface with only I_1 fibres, to which the $\bar{\partial}$ operator can be extended to the entire fibration.

To resolve this anomaly, we utilize the algebro-geometric construction of the Poincaré line bundle in §3.2.2, constructing a rank-2 SU(2) bundle that soaks up the zero modes of the $\bar{\partial}$ operator by use of the twisted RRGQ formula. This result, Theorem 3.2.28, is the main result of this chapter. In the case of the generic elliptic K3 surface with 24 I_1 fibres, the global anomaly is resolved via the insertion of D7 branes, as is required in generic 8D F theory compactifications. In this way, we see that our analysis connects with known examples of anomaly cancellations in physics, and can be in fact viewed as the *genesis* of D7 brane insertions.

3.1 Vertical signature operator on the j-line

In this section we consider the *family* of (complexified) signature operators $\{\not{D}_g\}$ – where we denote the operators as chiral Dirac operators defined in Section 2.2.3 – acting on sections of suitable bundles over a fixed even-dimensional manifold M, with the operators being parameterized by (conformal classes of) Riemannian metrics g. We will focus on the simplest case, when M is a flat two-torus, and g varies over the moduli space $\mathfrak{M}$ of flat metrics.

Given a two-torus M equipped with a complex structure $\tau \in \mathbb{H}$, we identify $M = \mathbb{C}/\langle 1, \tau \rangle$ where $\langle 1, \tau \rangle$ is a rank-two lattice in $\mathbb{C}$. This is done by identifying opposite edges of each parallelogram spanned by 1 and τ in the lattice to obtain $M = \mathbb{C}/\langle 1, \tau \rangle$. We endow M with a compatible flat torus metric g that descends from the flat metric on $\mathbb{C}$. Following Section 2.2.3 we define an operator $D = d + \delta$ on the even dimensional manifold M. The complexified signature operator, written as chiral Dirac operator $\not{D}_\tau$, is obtained from D in Equation (2.2.59). The subscript τ reminds us of the dependence on the metric in the definition of δ. We ask: as τ varies, what is the behavior of the family of Dirac operators (acting on complexified differential forms)

$$\not{D}_\tau : C^\infty_+(M, \Lambda^*) \to C^\infty_-(M, \Lambda^*)\,?$$

A more refined question is the following: consider a Jacobian elliptic surface, given as a holomorphic family of elliptic curves $\mathcal{E}_t$ (some of them singular) over the complex projective line $\mathbb{P}^1 \ni t$; the exact definition and the relation between τ and t will be discussed below. The numerical index of the chiral Dirac operator $\not{D}_t$ vanishes since the signature of each smooth fiber $\mathcal{E}_t$, is zero – a smooth torus forms the boundary of a smooth compact 3-manifold. Thus, the numerical index does not yield any interesting geometric information.

As a more refined invariant, we are interested in the *determinant line bundle* $\mathbf{Det}\,\not{D} \to \mathbb{P}^1$ associated with the family of Dirac operators $\not{D}_t$, and its first Chern class $c_1(\mathbf{Det}\,\not{D})$. As a generalized cohomology class, this class will measure the so-called *local and global anomaly*, revealing crucial information about the family of operators $\not{D}_t$, and it is this quantity that we will be studying for the remainder of this chapter.

3.1.1 The determinant line bundle

For an elliptic differential operator $\not{D}$ on a compact manifold M, the kernel and cokernel are finite dimensional vector spaces; thus, one can define the one-dimensional vector space

$$\mathbf{Det}\,\not{D} = (\Lambda^{\max} \ker \not{D})^* \otimes (\Lambda^{\max} \operatorname{coker} \not{D}). \tag{3.1.1}$$

The vector space $\mathbf{Det}\,\not{D}$ is the dual of the maximal exterior power of the index $\mathbf{Ind}\,\not{D}$ of D, that is, the formal difference of $\ker \not{D}$ and $\operatorname{coker} \not{D}$, given by

$$\mathbf{Ind}\,\not{D} = \ker \not{D} - \operatorname{coker} \not{D}.$$

Let $\pi : Z \to B$ be a smooth fibre bundle with compact fibers, and E and F be smooth vector bundles on Z, with a smooth family of elliptic operators $\not{D} = (\not{D}_t)_{t\in B}$ acting on the fibers $\pi^{-1}(t)$ as

$$\not{D}_t : C^\infty(\pi^{-1}(t), E) \to C^\infty(\pi^{-1}(t), F)\,.$$

Even though the dimensions of the kernel and cokernel of $\not{D}_t$ can jump, it turns out that that there is a canonical structure of a differentiable line bundle on the family of one-dimensional vector spaces $\{\mathbf{Det}\,\not{D}_t\}_{t\in B}$ [14]. Equivalently, we can say that the one-dimensional vector spaces $\{\mathbf{Det}\,\not{D}_t\}_{t\in B}$ patch together to form a line bundle by $\mathbf{Det}\,\not{D} \to B$. We remark that the formal difference used to define the index $\mathbf{Ind}\,\not{D}$ also makes sense in the context of K-theory, i.e., as a well-defined index bundle, an element of the K-theory group $K(B)$ [2].

We choose a smooth family of Riemannian metrics on the fibers $\pi^{-1}(t)$, and smooth Hermitian metrics on the bundles E and F. Then, for any $t \in B$, the adjoint operator $\not{D}_t^\dagger$ is well defined, and the vector spaces $\ker \not{D}_t$ and $\ker \not{D}_t^\dagger$ have natural L^2 metrics, which in turn define a metric $\|.\|_{L^2}$ on $\mathbf{Det}\,\not{D}_t$. However, because of the jumps in the dimensions of the kernel and cokernel, this does not define a smooth metric on the line bundle $\mathbf{Det}\,\not{D} \to B$. Instead, a way to assign a smooth metric is the *Quillen metric*, which uses the analytic torsion of the family $(\not{D}_t)_{t\in B}$ to smooth out these jumps [14]. The Quillen metric on $\mathbf{Det}\,\not{D} \to B$ is given by

$$\|.\|_Q = (\,\det{}' \not{D}_t^\dagger \not{D}_t)^{\frac{1}{2}}\, \|.\|_{L^2},$$

where $\det{}' \not{D}_t^\dagger \not{D}_t$ is the analytic torsion of the family of operators $(\not{D}_t)_{t\in B}$. It is crucial that the operators $\Delta_t = \not{D}_t^\dagger \not{D}_t$ form a family of positive, self-adjoint operators acting on sections of a vector bundle over compact manifolds.

The *analytic torsion*, or *regularized determinant* of a positive, self-adjoint operator Δ acting on sections over a compact manifold is defined as follows: by the hypotheses on Δ, it follows that the operator Δ has a pure point spectrum of eigenvalues, denoted by $\{\lambda_j\} \subset \mathbb{R}_{\geq 0}$. If there were only finitely many eigenvalues, then we could write down the identity

$$\frac{d}{ds}\left(\sum_{\lambda_j\neq 0} \lambda_j^{-s}\right)\Bigg|_{s=0} = -\sum_{\lambda_j\neq 0} \log \lambda_j,$$

and compute the product of eigenvalues, i.e., the determinant of the operator Δ, as

$$\prod_{\lambda_j\neq 0} \lambda_j = \exp\left(\sum_{\lambda_j\neq 0} \log \lambda_j\right).$$

Since there are infinitely many eigenvalues, we define a ζ-function instead, given by

$$\zeta_\Delta(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s} .$$

It turns that $\zeta_\Delta(s)$ is a well defined holomorphic function for $\mathrm{Re}(s) \gg 0$ with a meromorphic continuation to $\mathbb{C}$ such that $s = 0$ is not a pole of ζ_Δ [14, 117]. Then, the regularized determinant $\det' \Delta$ of Δ is defined as

$$\det{}' \Delta = \exp\big(- \zeta'_\Delta(0) \big) , \tag{3.1.2}$$

where the prime indicates that the zero eigenvalue has been dropped.

3.1.2 The analytic torsion for families of elliptic curves

Let us compute the analytic torsion for the Laplacian $\Delta_H = (d+\delta)^2$ from Section 2.2.3 in the special situation where the even dimensional manifold M is an elliptic curve $\mathcal{E}$, i.e., a flat two-torus equipped with a complex structure. We use the identification $\mathcal{E} \cong \mathbb{C}/\Lambda_\tau$, the local coordinate $z = x + iy$ on $\mathbb{C}$, and the notation $\partial = \partial_z$ and $\bar\partial = \partial_{\bar z}$. We have the following:

Lemma 3.1.18. *Let $\mathcal{E} \cong \mathbb{C}/\Lambda_\tau$ be a smooth elliptic curve endowed with the compatible flat torus metric g. The Laplacian when restricted to $C^\infty(\mathcal{E})$ equals $\Delta_H = -4\partial\bar\partial$, and its analytic torsion is given by*

$$\det{}' \Delta_H = \left(\frac{\mathrm{Im}(\tau)}{2\pi} \right)^2 |\Delta_\tau|^{\frac{1}{6}}, \tag{3.1.3}$$

where Δ_τ is the modular discriminant of the elliptic curve $\mathcal{E}$ in Equation (2.1.4). Moreover, the same answer holds for Δ_H restricted to $C^\infty(\mathcal{E}, T^*_{\mathbb{C}}\mathcal{E}^{(1,0)})$.

Proof. On $\mathcal{E}$ the operator $-\Delta_H$ is a positive, self adjoint operator. Endowing $\mathcal{E}_\tau$ with its canonical flat metric shows that for a local coordinate $z = x + iy$, we have the scalar Laplacian as

$$\Delta_H = -(\partial_x^2 + \partial_y^2) = -4\left(\frac{1}{2}(\partial_x - i\partial_y)\frac{1}{2}(\partial_x + i\partial_y)\right) = -4\partial_z\partial_{\bar{z}} = -4\partial\bar{\partial}.$$

The equality holds for the Laplacian acting on k-forms since one checks that

$$\begin{aligned}
&\Delta_H : f \mapsto -4\partial_z\partial_{\bar{z}} f\,, && f \in C^\infty(\mathcal{E})\,,\\
&\Delta_H : \phi = (f dz + g d\bar{z}) \mapsto -4(\partial_z\partial_{\bar{z}} f dz + \partial_z\partial_{\bar{z}} g d\bar{z})\,, && \phi \in C^\infty(\mathcal{E}, \Lambda^1)\,,\\
&\Delta_H : \omega = f dz \wedge d\bar{z} \mapsto -4\partial_z\partial_{\bar{z}} f dz \wedge d\bar{z}\,, && \omega \in C^\infty(\mathcal{E}, \Lambda^2)\,.
\end{aligned}$$

A function φ with a periodicity given by

$$\varphi(x+1, y) = \varphi(x, y)\ , \qquad \varphi(x + \mathrm{Re}\tau, y + \mathrm{Im}\tau) = \varphi(x, y)\ ,$$

descends to a well defined function on $\mathcal{E}$. For $n_1, n_2 \in \mathbb{N}$ such a function is given by

$$\varphi_{n_1,n_2}(x, y) = \exp 2\pi i\left(n_1 x + \frac{(n_2 - n_1\mathrm{Re}\tau)}{\mathrm{Im}\tau} y\right)\ .$$

In fact, the functions φ_{n_1,n_2} constitute a complete system of eigenfunctions for Δ_H with the eigenvalues

$$\lambda_{n_1,n_2} = \left(\frac{2\pi}{\mathrm{Im}\tau}\right)^2\ |n_1\tau - n_2|^2\ .$$

Notice that we have $\Delta_H\varphi_{n_1,n_2} = \lambda_{n_1,n_2}\varphi_{n_1,n_2} dz \wedge d\bar{z}$ and then use the Kähler form to identify $dz \wedge d\bar{z}$ with 1. We define a ζ-function $\zeta(s)$ by setting

$$\zeta(s) = \sum_{n_1,n_2}{}' \frac{1}{|n_1\tau - n_2|^{2s}}\,, \tag{3.1.4}$$

where the prime indicates that the summation does not include $n_1 = n_2 = 0$. One checks that $\zeta(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$, has a meromorphic extension to $\mathbb{C}$, and 0 is not a pole. It was shown in [117] that $\zeta(0) = -1$. The regularized determinant of Δ_H is then given by

$$\ln\det\Delta_H \;=\; -\left[\frac{1}{\left(\frac{2\pi}{\mathrm{Im}\tau}\right)^{2s}}\,\zeta(s)\right]' = -\zeta'(0) + \ln\left(\frac{2\pi}{\mathrm{Im}\tau}\right)^2\,\zeta(0)\,.$$

It was shown in [117] that $\exp\left[-\zeta'(0)\right] = |\eta(\tau)|^4$ using the Kronecker limit formula where the Dedekind η-function is given by

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}}\prod_{r=1}^{\infty}\left(1 - e^{2\pi i\tau r}\right)\,. \tag{3.1.5}$$

It follows from Equation (2.1.4) that

$$\det{}'\,\Delta_H = \left(\frac{\mathrm{Im}(\tau)}{2\pi}\right)^2 |\Delta_\tau|^{\frac{1}{6}}\,. \tag{3.1.6}$$

A similar argument can be repeated for the sections

$$\varphi_{n_1,n_2}(z)\,dz \in C^\infty(\mathcal{E}, T^*_{\mathbb{C}}\mathcal{E}^{(1,0)})\,. \tag{3.1.7}$$

Applying $\bar{\partial}$ we obtain

$$-\left(\frac{\pi}{\mathrm{Im}\tau}\right)\,(n_1\tau - n_2)\,\varphi_{n_1,n_2}(z)\,dz \wedge d\bar{z}\,.$$

Contracting with the Kähler form and applying ∂ shows that the sections in Equation (3.1.7) form a complete system of eigenfunctions for Δ_H restricted to the vector space $C^\infty(\mathcal{E}, T^*_{\mathbb{C}}\mathcal{E}^{(1,0)})$. $\square$

It follows from the definitions in Section 2.2.3 that

$$\not{D} = (d+\delta)P_+ = \bar{\partial}^\dagger + \bar{\partial}\Big|_{C^\infty_+(M,\Lambda^*)}, \quad \not{D}^\dagger = (d+\delta)P_- = \partial^\dagger + \partial\Big|_{C^\infty_-(M,\Lambda^*)}.$$

Recall that the kernel and cokernel of the chiral Dirac operator $\not{D}$ in Equation (2.2.59) are given by selfdual and anti-selfdual generators of the de Rham cohomology classes, respectively, i.e., by elements of $H^*_\pm(\mathcal{E};\mathbb{C}) = \{\omega \in H^*(\mathcal{E};\mathbb{C}) \mid \alpha(\omega) = \pm\omega\}$. On an elliptic curve $\mathcal{E}$, the forms $1-\frac{1}{2}dz\wedge d\bar{z}$ and dz are selfdual; similarly the forms $1+\frac{1}{2}dz\wedge d\bar{z}$ and $d\bar{z}$ are anti-selfdual where dz and $d\bar{z}$ are section of the holomorphic cotangent bundle $T^*_{\mathbb{C}}\mathcal{E}^{(1,0)} = K_{\mathcal{E}}$, also called the canonical bundle, and the bundle $T^*_{\mathbb{C}}\mathcal{E}^{(0,1)} = \overline{K}_{\mathcal{E}}$, respectively. These (anti-)selfdual differential forms descend to generators of $H^r_+(\mathcal{E};\mathbb{C})$ and $H^r_-(\mathcal{E};\mathbb{C})$, respectively, for $r = 0, 1$.

We now consider the Jacobian elliptic surface $\pi : S \to \mathbb{P}^1$ given by the Weierstrass model

$$y^2 = 4x^3 - 27t(t-1)^3x - 27t(t-1)^5\,, \tag{3.1.8}$$

where t is the affine coordinate on the base curve with $[t:1] \in \mathbb{P}^1$. This family was considered in [133], and it can easily be shown to satisfy

$$\Delta(t) = 27^3t^2(t-1)^9\,, \qquad \mathsf{j}(t) = \frac{g_2^3}{\Delta} = t\,. \tag{3.1.9}$$

The fibers over $t = 0, 1, \infty$ are singular; in the language of Kodaira's classification result, the singular fibers over $t = 0, 1, \infty$ correspond to fibers of type II, III^*, and

I_1, respectively. The total space $\overline{S}$ of Equation (3.1.8) is singular, and its minimal desingularization is the rational elliptic surface S. It is a rational Jacobian elliptic surface whose j-function is the coordinate on the base curve itself, it is also called the *universal family of elliptic curves*. We call the base curve the j-line and denote it by $J = \mathbb{P}^1$.

The elliptic fiber $\mathcal{E}_t = \pi^{-1}(t)$ given by Equation (3.1.8) is a smooth elliptic curve with discriminant $\Delta(t)$ for $t \in J^* = J - \{0, 1, \infty\}$. The chiral Dirac operator $\not{D}_t$ in Equation (2.2.59) on each smooth elliptic fiber $\mathcal{E}_t$ is the sum $\not{D}_t = \bar{\partial}_t \oplus \bar{\partial}_t^\dagger$ of the operators

$$\bar{\partial}_t : C^\infty\big(\mathcal{E}_t\big) \to C^\infty\big(\mathcal{E}_t, \overline{K}_{\mathcal{E}_t}\big)\,, \qquad \bar{\partial}_t^\dagger : C^\infty\big(\mathcal{E}_t, \overline{K}_{\mathcal{E}_t}\big) \to C^\infty\big(\mathcal{E}_t\big)\,,$$

where $\overline{K}_{\mathcal{E}_t}$ denotes the dual of the canonical bundle of $\mathcal{E}_t$. For our purposes, it is not necessary to investigate both components of the operator $\not{D}_t$; this follows from the factorization of the Laplacian in Lemma 3.1.18. Thus, we will focus on the operator $\bar{\partial}_t$ and its determinant line bundle $\mathbf{Det}\,\bar{\partial} \to J^*$.

It turns out that the vector spaces $H^r_+(\mathcal{E}_t;\mathbb{C})$ and $H^r_-(\mathcal{E}_t;\mathbb{C})$, respectively, for $r = 0, 1$, all patch together to form smooth line bundles over J^* with a smooth Hermitian metrics. The vector spaces $H^0_+(\mathcal{E}_t;\mathbb{C})$ and $H^0_-(\mathcal{E}_t;\mathbb{C})$ each form a trivial line bundle $\underline{\mathbb{C}} \to J^*$. Similarly, the vector spaces $K_{\mathcal{E}_t}$ generated by dz on each elliptic curve $\mathcal{E}_t$ patch together to generate the bundle of vertical $(1,0)$-forms $K_{S|J} \to J$ in § 2.1.2. Since we have described the kernel and cokernel of the chiral Dirac operator $\bar{\partial}_t$, it follows from Equation (3.1.1) that for each $t \in J^* = J - \{0, 1, \infty\}$ we have *fiberwise* an identification

$$\mathbf{Det}\,\bar{\partial}_t \cong \overline{K}_{S|J}\Big|_t\,.$$

Moreover, the Quillen metric induces a canonical holomorphic structure on the determinant line bundle $\mathbf{Det}\,\bar{\partial} \to J^*$; see [14]. We have the following:

Proposition 3.1.19. *Let s be the canonical holomorphic section of $\mathbf{Det}\,\bar{\partial} \to J^*$. For each $t \in J^* = J - \{0, 1, \infty\}$ the Quillen norm of the section s is given by*

$$\|s\|_Q^2 = \frac{\mathrm{Im}(\tau)^2}{4\pi^2} |\Delta(t)|^{\frac{1}{6}} . \tag{3.1.10}$$

Proof. For each $t \in J^* = J - \{0, 1, \infty\}$ we identify $\mathcal{E}_t = \mathbb{C}/\langle 1, \tau \rangle$ such that $j(\tau) = \mathsf{j}(t) = t$ and $\Delta_\tau = \Delta(t)$. We identify the one-form dx/y in §2.1.2 with dz in each smooth fiber $\mathcal{E}_t$ generating $\mathrm{coker}\,\bar{\partial}_t$. Similarly, the constant function 1 generates $\ker \bar{\partial}_t$. We have $\|1\|_{L^2}^2 = \|dz\|_{L^2}^2 = 2\mathrm{Im}(\tau)$, thus the canonical section s of $\ker \bar{\partial}^* \otimes \mathrm{coker}\,\bar{\partial}$ satisfies $\|s\|_{L^2} = 1$ and

$$\|s\|_Q^2 = \det{}' \, \bar{\partial}_t^\dagger \bar{\partial}_t \, \|s\|_{L^2}^2 = \frac{\mathrm{Im}(\tau)^2}{4\pi^2} |\Delta_\tau|^{\frac{1}{6}} ,$$

where we used Equation (3.1.3). □

3.1.3 The RRGQ formula

We observe that since $\Delta(t) = 0$ for $t = 0, 1, \infty$, we have that the Quillen norm vanishes at the punctures on J^*. The holomorphic determinant line bundle $\mathbf{Det}\,\bar{\partial} \to J^* = \mathbb{P}^1 - \{0, 1, \infty\}$ is locally trivial by means of the section s in Theorem 3.1.19 and in general does not extend to a bundle on the entire j-line $J \cong \mathbb{P}^1$. Using the section s we can compute the first Chern class of the bundle $\mathbf{Det}\,\bar{\partial} \to J^*$. We can then extend the curvature form representing the first Chern class to the entire j-line J by allowing so-called *currents*. These currents reflect the monodromy of s around the punctures of J^*.

In general, let $(\xi, \|.\|)$ be a holomorphic vector bundle over a complex algebraic variety, equipped with a smooth Hermitian metric. Then there exists a unique connection on ξ compatible with the holomorphic structure and the metric $\|.\|$. From Chern-Weil theory, there is a differential form that in the de Rham cohomology represents the first Chern class $c_1(\xi, \|.\|)$. We have the following result [12]:

Lemma 3.1.20. *The representative in the de Rham cohomology of the first Chern class $c_1(\xi, \|.\|)$ is given by*

$$c_1\big(\xi, \|.\|\big) = \frac{1}{2\pi i}\, \partial\bar{\partial} \log \|s\|^2 \tag{3.1.11}$$

where s is a nonzero holomorphic section of ξ.

Trying to extend the bundle from J^* to J, the points where s vanishes can cause problems, since for these points we might obtain *current contributions* to the curvature, which are distribution-valued differential forms. Currents are defined as distribution forms arising from the classical Cauchy integral formula for single variable complex analysis [55], such as

$$\bar{\partial}\left(\frac{1}{2\pi i}\frac{dt}{t}\right) = \delta_{(t=0)} \tag{3.1.12}$$

where $\delta_{(t=0)}$ is the Dirac delta function centered at $t = 0$. The current contributions in a generalized first Chern class encode the holonomy of the trivializing section σ of the holomorphic determinant line bundle around the punctures of J^*. The computation of the generalized first Chern class is achieved by the so-called *Riemann-Roch-Grothendieck-Quillen formula* or RRGQ formula for short; see [7, 8, 13, 83]. In the special situation of the Jacobian elliptic surface $\pi : S \to J \cong \mathbb{P}^1$ given by Equation (3.1.8) and the holomorphic determinant line bundle $\mathbf{Det}\,\bar{\partial} \to J^* = J - \{0, 1, \infty\}$ constructed above, we have the following:

Theorem 3.1.21 (RRGQ)**.** *In the situation described above, the generalized first Chern class of the determinant line bundle* $\mathbf{Det}\,\bar\partial \to J^*$ *is given by*

$$c_1\big(\,\mathbf{Det}\,\bar\partial, \|.\|_Q\big) = -\frac{1}{12}\big(2\,\delta_{(t=0)} + 9\,\delta_{(t=1)} + \delta_{(t=\infty)}\big) + c_1(\mathcal{K}) \qquad (3.1.13)$$

where $c_1(\mathcal{K})$ *is the first Chern class of the line bundle* $\mathcal{K} = \pi_* K_{S|J} \cong \mathcal{O}(1) \to J$.

Proof. Recall that the Quillen norm of the canonical holomorphic section σ in Proposition 3.1.19 is given by $\|s\|^2_Q = \frac{\mathrm{Im}(\tau)^2}{4\pi^2}|\Delta(t)|^{\frac{1}{6}}$. The discriminant $\Delta(t)$ of the Weierstrass equation defining the Jacobian elliptic surface $\pi : S \to J \cong \mathbb{P}^1$ was given in Equation (3.1.9) where we found $\Delta(t) = 27^3 t^2 (t-1)^9$ which vanishes at $t \in \{0, 1, \infty\}$.

Plugging the canonical section s of Theorem 3.1.19 into Equation (3.1.11) and applying the argument principle of Equation (3.1.12) yields

$$\frac{1}{2\pi i}\partial\bar\partial \log \|s\|^2_Q = -\frac{1}{12}\big(2\delta_{(t=0)} + 9\delta_{(t=1)} + \delta_{(t=\infty)}\big) + j^*\left(\frac{i}{4\pi \mathrm{Im}(\tau)^2}\, d\tau \wedge d\bar\tau\right) ,$$

where we used the j-function $j : \mathbb{H}/\operatorname{PSL}(2,\mathbb{Z}) \to J$ with $j(\tau) = t$. Since the Poincaré metric on the hyperbolic upper half plane is given by

$$\frac{i}{2\pi \mathrm{Im}(\tau)^2} d\tau \wedge d\bar\tau = \frac{dx \wedge dy}{\pi y^2} ,$$

and the j-function is a complex diffeomorphism, it follows that

$$j^*\left(\frac{i}{2\pi \mathrm{Im}(\tau)^2}\, d\tau \wedge d\bar\tau\right) = c_1\left(T_{\mathbb{C}} J^{(1,0)}\right) .$$

We have $J \cong \mathbb{P}^1$ and $T_{\mathbb{C}} J^{(1,0)} \cong \mathcal{O}(2)$. Since the first Chern class of products of line bundles on J is additive, i.e., $c_1(\xi_1 \otimes \xi_2) = c_1(\xi_1) + c_1(\xi_2)$ it follows that the continuous part of $c_1(\mathbf{Det}\,\bar\partial)$ is the first Chern class of $\mathcal{O}(1)$. Using the construction of the relative canonical bundle in §2.1.2, the claim follows. □

3.1.4 Extension as a meromorphic connection

The RRGQ formula is the key to studying the anomalies of the fiberwise Cauchy-Riemann operator $\bar{\partial}_t$ on a Jacobian elliptic surface. In particular, in the smooth category the *local anomaly* is the first Chern class of the determinant line bundle [14], and it is computed as the line bundle's curvature tensor. Hence if the curvature vanishes, then the bundle has no *local anomaly*. Conversely, Theorem 3.1.21 proves that the determinant line bundle of the fiberwise Cauchy-Riemann operators $\bar{\partial}_t$ on the Jacobian elliptic surface $\pi : S \to J \cong \mathbb{P}^1$ given by Equation (3.1.8) has a non-vanishing local anomaly.

Moreover, the current contributions encode the holonomy of the trivializing section σ for the bundle $\mathbf{Det}\,\bar{\partial} \to J^* = J - \{0, 1, \infty\}$ around the punctures over which the Weierstrass model has singular fibers. This represents the *global* anomaly of the bundle. To analyze the holonomy group we have the following:

Lemma 3.1.22. *There is a flat holomorphic line bundle $\mathcal{M}^* \to J^*$ with a $\mathbb{Z}_{12}$-holonomy such that $\mathbf{Det}\,\bar{\partial} \cong \mathcal{K} \otimes \mathcal{M}^*$.*

Proof. Since the first Chern class of products of line bundles on J^* is additive, i.e., $c_1(\xi_1 \otimes \xi_2) = c_1(\xi_1) + c_1(\xi_2)$, it follows from Equation (3.1.13) that $\mathbf{Det}\,\bar{\partial} \cong \mathcal{K} \otimes \mathcal{M}^*$ where $\mathcal{K}$ is (the restriction of) the bundle $\mathcal{K} \cong \mathcal{O}(1) \to J \cong \mathbb{P}^1$. The discriminant $\Delta(t) = 27^3 t^2 (t-1)^9$ vanishes at $t_0 \in \{0, 1, \infty\}$. At each point t_0, we compute the holonomy of $\Delta(t)^{1/12}$ (after fixing a base point of a smooth fiber) by encircling the point t_0 via the path $t = t_0 + \frac{1}{2} e^{2\pi i \epsilon}$ and calculating the result as ϵ goes from $0 \to 1$. At $t_0 = 0$, we obtain $\Delta^{1/12}\left(t_0 + \frac{1}{2}\right) \mapsto e^{\pi i/3} \Delta^{1/12}\left(t_0 + \frac{1}{2}\right)$. Similarly, at $t_0 = 1$, we obtain $\Delta^{1/12}\left(t_0 + \frac{1}{2}\right) \mapsto e^{\pi i/2} \Delta^{1/12}\left(t_0 + \frac{1}{2}\right)$, and at $t_0 = \infty$, we obtain $\Delta^{1/12}\left(t_0 + \frac{1}{2}\right) \mapsto e^{\pi i/6} \Delta^{1/12}\left(t_0 + \frac{1}{2}\right)$. The smallest subgroup of U(1) that contains these generators is $\mathbb{Z}_{12}$. Furthermore, this holonomy is not present in the holomorphic cotangent bundle on $\mathbb{P}^1$, we conclude that there is a flat line bundle $\mathcal{M}^*$ with $\mathbb{Z}_{12}$-holonomy. $\square$

We offer the following interpretation of the anomalies: when $t \neq 0, 1, \infty$, the automorphism group of the elliptic curve $\mathcal{E}_t$ is isomorphic to $\mathbb{Z}_2$. When $j = 0, 1, \infty$, the automorphism group is is isomorphic to $\mathbb{Z}_4, \mathbb{Z}_6$, and $\mathbb{Z}_4$, respectively [127]. These points give rise to the global anomaly. The jumping behavior in the symmetry of the elliptic fibre is also called the *holomorphic anomaly*; see [14].

The flat holomorphic line bundle $\mathcal{M}^* \to J^*$ can be extended to a line bundle $\mathcal{M} \to J$ with a flat meromorphic connection $\bar{\partial}_t$ that has only regular singular points. This follows from the *Riemann-Hilbert correspondence* which asserts that the restriction to J^* is an equivalence of categories between the category of flat meromorphic connections on J with only regular singular points and holomorphic on J^* and the category of flat holomorphic connections on J^* [32]. Here, it simply means that there exists a trivialization on a flat line bundle $\mathcal{M} \to J$ so that, when restricted to a punctured disc $D^*_{t_0}$ around any point $t_0 \in J - J^*$, the $\bar{\partial}_t$-operator on $\mathcal{M}$ is given by

$$\bar{\partial}_t\Big|_{D^*_{t_0}} = \bar{\partial} - \frac{a\,dt}{t - t_0} + \eta\,, \tag{3.1.14}$$

where $a \in \mathbb{C}$ and $\eta \in \Omega^1$ is a holomorphic one-form on D_{t_0}. We have the following:

Proposition 3.1.23. *The flat holomorphic line bundle $\mathcal{M}^* \to J^*$ extends to a line bundle $\mathcal{M} \to J$ with a flat meromorphic connection and regular singular points over $J - J^*$.*

Proof. Let $(\mathcal{M}^*, \bar{\partial})$ be the flat holomorphic flat connection on J^*. When restricted to a punctured disc $D^*_{t_0}$ around a point $t_0 \in J - J^*$, it is therefore determined by some monodromy matrix $A \in \mathrm{U}(1)$. Taking logarithms, there exists $a \in \mathfrak{u}(1)$ such that $A = \exp(2\pi i a)$. Then, the meromorphic connection on D_{t_0} given by $\bar{\partial} - \frac{a\,dt}{t-t_0}$ has flat sections of the form $t \mapsto v\, \exp(a \log(t - t_0))$ for any $v \in \mathbb{C}^*$. These sections have monodromy around t_0 given by $A = \exp(2\pi i a)$, so have their restriction to $D^*_{t_0}$. $\square$

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3.2 Twisting and anomaly cancellation on generic elliptic K3 surfaces

The Riemann-Roch-Grothendieck-Quillen (RRGQ) formula has a twisted analogue, similar to the twisted version of the signature theorem in Equation (2.3.69). However, to state this formula we will replace the Jacobian elliptic surface $S \to J$ given by Equation (3.1.8) with a Jacobian elliptic surface $\pi : Z \to B$ whose Weierstrass model has only nodes, i.e., fibers of Kodaira-type I_1; we will explain below how such a surface can be constructed using Corollary 2.1.2. This setup has the advantage that the total space of Equation (3.1.8) is smooth, and no additional blowups are needed to move from its total space to Z.

Let $\xi \to Z$ be a *holomorphic* vector bundle of rank r with a smooth Hermitian metric. Then, there is a unique unitary connection on ξ compatible with its holomorphic structure. Using this connection we can compute the Chern classes $c_j(\xi)$ for $j = 1, 2$. We also obtain a *twisted* Cauchy-Riemann operator $\bar{\partial}^{\xi}_t$ on any smooth elliptic curve $\mathcal{E}_t$ in the fibration $\pi : Z \to B$ coupled to the restriction of the holomorphic bundle ξ given by

$$\bar{\partial}^{\xi}_t : C^{\infty}\big(\mathcal{E}_t, \xi\Big|_{\mathcal{E}_t}\big) \to C^{\infty}\big(\mathcal{E}_t, \overline{K}_{\mathcal{E}_t} \otimes \xi\big|_{\mathcal{E}_t}\big)\,. \tag{3.2.15}$$

For the family of twisted operators $\{\bar{\partial}^{\xi}_t\}_{t \in B}$ a determinant line bundle $\mathbf{Det}\,\bar{\partial}^{\xi} \to B$ together with a Quillen metric $\|.\|_Q$ can be constructed as before; see [1, 14]. The analogue of Theorem 3.1.21 is the following:

Theorem 3.2.24 (tRRGQ)**.** *In the situation described above, the generalized first Chern class of the determinant line bundle* $\mathbf{Det}\,\bar{\partial}^{\xi}$ *is given by*

$$c_1\big(\mathbf{Det}\,\bar{\partial}^{\xi}, \|.\|_Q\big) = r \cdot c_1\big(\mathbf{Det}\,\bar{\partial}, \|.\|_Q\big) - \int_{X|B} c_2(\xi)\,, \tag{3.2.16}$$

where $c_2(\xi)$ *is the second Chern class of the holomorphic vector bundle* $\xi \to Z$ *of rank* r *assumed to satisfy* $c_1(\xi) = 0$*. Here,* $\int_{X|B} c_2(\xi)$ *is understood as integrating a the four-form* $c_2(\xi)$ *over the vertical fibers of* $\pi : Z \to B$*.*

Notice that Equation (3.2.16) is the generalization of Equation (2.3.69) for families. However, the last term on the right hand side of Equation (3.2.16) now yields upon integration a two-form on the base of the fibration. As we will show, Theorem 3.2.24 implies that we can choose the holomorphic vector bundle ξ in such a way that we can *cancel* the local anomaly of the determinant line bundle.

3.2.1 The generic elliptic surface

As an application of Corollary 2.1.2, we will consider the case of the most generic rational Jacobian elliptic surface $Z \to B \cong \mathbb{P}^1$. This is, a Jacobian elliptic fibration whose only singular fibers are twelve nodes, i.e., fibers of Kodaira-type I_1. This means that the discriminant $\Delta(t)$ has 12 distinct simple roots, and we have

$$\Delta(t) = \prod_{j=1}^{12} (t - t_j)\,, \tag{3.2.17}$$

for the distinct points $t_1, \ldots, t_{12} \in B$, and we set $B^* = B - \{t_1, \ldots, t_{12}\}$. It was shown in [114] that this Jacobian elliptic surface exists; it was denoted by #1 in the complete classification of rational Jacobian elliptic surfaces in [114]. It follows from general arguments in [77] that the total space of Equation (2.1.2) is rational and smooth. Similarly, there is the Jacobian elliptic surface where g_2 and g_3 are generic polynomials of degree 8 and 12, respectively, and the total space of Equation (2.1.2) is a smooth K3 surface. In this case, the singular fibers are 24 nodes, i.e., fibers of Kodaira-type I_1, and one has a discriminant $\Delta(t)$ with 24 distinct simple roots, i.e.,

$$\Delta(t) = \prod_{j=1}^{24} (t - t_j) \,, \tag{3.2.18}$$

for the distinct points $t_1, \dots, t_{24} \in B$, and we set $B^* = B - \{t_1, \dots, t_{24}\}$. We can adopt the construction of the holomorphic determinant line bundle from Section 3.1.3 to obtain $\mathbf{Det}\, \bar{\partial} \to B^*$ in both cases. The only difference between the case $n = 1$ (rational surface) and $n = 2$ (K3 surface) is that the degree of the holomorphic map $\mathsf{j}(t) = j(\tau)$ is 1 or 2, respectively. We adopt the proofs of Theorem 3.1.21, Lemma 3.1.22 and Proposition 3.1.23 to obtain the following:

Corollary 3.2.25. *Let $Z \to B \cong \mathbb{P}^1$ be the Jacobian elliptic surface whose Weierstrass model has $12n$ singular fibers of Kodaira-type I_1 for $n = 1$ (rational surface) or $n = 2$ (K3 surface). The generalized first Chern class of the determinant line bundle $\mathbf{Det}\, \bar{\partial} \to B^*$ is given by*

$$c_1\big(\mathbf{Det}\, \bar{\partial}, \|.\|_Q\big) = -\frac{1}{12} \left(\sum_{j=1}^{12n} \delta_{(t=t_j)} \right) + c_1(\mathcal{K}) \tag{3.2.19}$$

with $\mathcal{K} = \pi_* K_{Z|B} \cong \mathcal{O}(n) \to B$. Moreover, there is a flat holomorphic line bundle $\mathcal{M}^* \to B^*$ with a $\mathbb{Z}_{12}$-holonomy such that $\mathbf{Det}\, \bar{\partial} \cong \mathcal{K} \otimes \mathcal{M}^*$. In turn, the flat holomorphic line bundle $\mathcal{M}^* \to B^*$ extends to a line bundle $\mathcal{M} \to B$ with a flat meromorphic connection $\bar{\partial}_t$ for $t \in B^*$ given by

$$\bar{\partial}_t = \bar{\partial} - \frac{1}{12} \sum_{j=1}^{12n} \frac{dt}{t - t_j} \,. \tag{3.2.20}$$

Roughly speaking, pulling back the curvature from $\mathbf{Det}\, \bar{\partial} \to J^*$ has the effect of spreading out the current contributions over different points on the elliptic fibration, while the total flux of the current contributions is fixed by $\deg\Delta = 12n$. In fact, the canonical section s in Theorem 3.1.19 has a holonomy given by $\Delta^{1/12} \mapsto e^{\pi i/6}\Delta^{1/12}$. Hence, the nontrivial holonomy group is $\mathbb{Z}_{12} \subseteq \mathrm{U}(1)$. We also make the following remark.

It follows from results in [125, 85] that for a Jacobian elliptic surface $Z \to B \cong \mathbb{P}^1$ with only nodes in its Weierstrass model a suitable notion of a $\bar{\partial}$-operator and its regularized determinant can be established for all fibers, including the nodes, so that the meromorphic connection in Equation (3.2.20) arises naturally as the meromorphic connection of the extended determinant line bundle over B.

3.2.2 The Poincaré line bundle

Let $\pi : Z \to B \cong \mathbb{P}^1$ be the Jacobian elliptic surface with a zero-section denoted by $\sigma : B \to Z$ and a Weierstrass model with $12n$ singular fibers of Kodaira-type I_1 for $n = 1, 2$. Then, Z is the total space of Equation (2.1.2), is smooth, and a rational surface for $n = 1$ and a K3 surface for $n = 2$. This is important because it means that we can simply ignore all singularities when constructing the fiber product of Z. We also assume that the group of sections for the Jacobian elliptic surface Z admits no two-torsion. This is to avoid that the restriction of a holomorphic SU(2)-bundle $\xi \to Z$ of rank two to every fiber $\mathcal{E}_t = \pi^{-1}(t)$ can be an extension bundle.

We first build a rank-two SU(2)-bundle over a smooth elliptic curve $\mathcal{E}$. It follows from results in [4] that a rank-two vector bundle $V \to \mathcal{E}$ is a (semi-stable) holomorphic SU(2)-bundle if and only if $V \cong \mathcal{N}_1 \oplus \mathcal{N}_2$ for two holomorphic line bundles $\mathcal{N}_1, \mathcal{N}_2 \to \mathcal{E}$ with $\mathcal{N}_1 \otimes \mathcal{N}_2 \cong \mathcal{O}$. For simplicity, we assume that the line bundles $\mathcal{N}_1, \mathcal{N}_2$ have degree zero. Then, there are unique points $q_1, q_2 \in \mathcal{E}$ such that $\mathcal{N}_1, \mathcal{N}_2$ each have a holomorphic section vanishing at a point q_1 and q_2 respectively, and a simple pole at

$p = \infty$, i.e., the neutral point of the elliptic group law. Using the group law on $\mathcal{E}$, the condition $\mathcal{N}_1 \otimes \mathcal{N}_2 = \mathcal{O}$ implies $q_1 + q_2 = 0$. Hence, we write $q_1 = q$, $q_2 = -q$, and $V = \mathcal{O}(q-p) \oplus \mathcal{O}(-q-p)$. For each such a pair $(q, -q) \in \mathcal{E} \times \mathcal{E}$, there is a meromorphic function $w = a_0 - a_2 x$ on $\mathcal{E}$ given by Equation (2.1.2) with $a_0, a_2 \in \mathbb{C}$ that vanishes at q and $-q$ and has a simple pole at p. That is, we think of the points $\pm q \in \mathcal{E}$ as given by the coordinates

$$x = \frac{a_2}{a_0}\,, \quad y = \pm\sqrt{4\left(\frac{a_2}{a_0}\right)^3 - g_2\,\frac{a_2}{a_0} - g_3}\,.$$

We also introduce the *Poincaré line bundle*: for a smooth elliptic curve $\mathcal{E}$, the degree-zero holomorphic line bundles over a smooth elliptic curve $\mathcal{E}$ are parameterized by $\mathcal{E}$ itself since each point $q \in \mathcal{E}$ corresponds to the line bundle $\mathcal{O}(q-p)$. We denote by Δ the diagonal in $\mathcal{E} \times \mathcal{E}$. The Poincaré line bundle $\mathcal{P} \to \mathcal{E} \times \mathcal{E}$ is obtained from the divisor

$$D = \Delta - \mathcal{E} \times \{p\} - \{p\} \times \mathcal{E},$$

by setting $\mathcal{P} = \mathcal{O}_{\mathcal{E}\times\mathcal{E}}(D)$ so that $\mathcal{P}|_{\{q\}\times\mathcal{E}} \cong \mathcal{P}|_{\mathcal{E}\times\{q\}} \cong \mathcal{O}(q-p)$.

Now let the elliptic curve $\mathcal{E}$ vary over the elliptic fibers $\mathcal{E}_t$ of the Jacobian elliptic surface $Z \to B \cong \mathbb{P}^1$ with $t \in B$ such that the point at infinity in each fiber is given by the zero section $p = \sigma(t)$. Next, we consider a pair of points $\pm q$ which are the solutions of $w = a_0 - a_2 x = 0$ where the coefficients a_i are sections $a_i \in \Gamma(B, \mathcal{R} \otimes \mathcal{L}^{-i})$ for a non-trivial holomorphic line bundle $\mathcal{R} \to B$ and the normal bundle $\mathcal{L} \to B$. In this way, the vanishing locus of $w \in \Gamma(B, \mathcal{R})$ defines a ramified double covering $C_{\mathcal{R}} \subset Z$ of B, called a *spectral double cover*.

From the total space Z we form the *fibre product*, given by

$$Z \times_B Z = \{(z_1, z_2) \in Z \times Z \,|\, \pi(z_1) = \pi(z_2)\},$$

with a holomorphic projection map $\widetilde{\pi} : Z \times_B Z \to B$ given by $\widetilde{\pi}(z_1, z_2) = \pi(z_1)$; this is well defined by virtue of the definition of $Z \times_B Z$ and $\pi(z_1) = \pi(z_2)$. For $t \in B$, we have $\widetilde{\pi}^{-1}(t) = \mathcal{E}_t \times \mathcal{E}_t$ with $\mathcal{E}_t = \pi^{-1}(t)$. From the spectral cover $C_{\mathcal{R}} \subset Z$ we obtain, by using the fibre product, the topological subspace $C_{\mathcal{R}} \times_B Z \subset Z \times_B Z$ with $z_1 \in C$. The map $\pi_2 : C_{\mathcal{R}} \times_B Z \to Z$ obtained by forgetting z_1 is a two-fold covering.

The equation $z_1 = z_2$ forms a divisor $\Delta \subset Z \times_B Z$. The Poincaré line bundle $\mathcal{P} \to Z \times_B Z$ on the Jacobian elliptic surface $\pi : Z \to B$ with section σ is obtained from the divisor

$$D = \Delta - Z \times \sigma - \sigma \times Z\,,$$

by setting $\mathcal{P} = \mathcal{O}(D) \otimes \widetilde{\pi}^{*}\mathcal{L}$ where $\mathcal{L} \to B$ is the aforementioned normal bundle of the Jacobian elliptic fibration $\pi : Z \to B$. By restriction, we obtain the restricted Poincaré line bundle $\mathcal{P}_{\mathcal{R}} \to C \times_B Z$. Using results of [47], we have the following:

Proposition 3.2.26. *Given a spectral double cover $C_{\mathcal{R}} \subset Z \to B$ corresponding to a non-trivial holomorphic line bundle $\mathcal{R} \to B$, the bundle*

$$\xi = \pi_{2*}\big(\mathcal{P}_{\mathcal{R}}\big) \to Z \tag{3.2.21}$$

is a stable rank-two holomorphic SU(2) *bundle over Z.*

Proof. For any $z \in Z$ which is not in the branching locus of π_2 with $t = \pi(z) \in B$, we have $C_{\mathcal{R},t} = \{y_1, y_2\}$ and

$$\xi_z = \mathcal{P}_{(y_1,z)} \oplus \mathcal{P}_{(y_2,z)} \, .$$

Thus, the restriction of ξ to $\mathcal{E}_t = \pi^{-1}(t)$ is a sum of degree-zero line bundles given by

$$\xi\Big|_{\mathcal{E}_t} = \mathcal{O}\big(-q(t) + \sigma(t)\big) \oplus \mathcal{O}\big(q(t) + \sigma(t)\big) \, ,$$

where $\pm q(t)$ are obtained as the solutions with x-coordinate given by $w = a_0(t) - a_2(t)\, x = 0$. The restriction of ξ to any such fiber $\mathcal{E}_t = \pi^{-1}(t)$ carries a flat SU(2) connection. At the branching points of π_2 the preimage of $t \in B$ is a point of multiplicity two. Thus, the restriction $\xi|_{\mathcal{E}_t}$ is a non-trivial extension of a line bundle by a second isomorphic line bundle. This restriction bundle admits no flat SU(2) connection. To fit these two types of bundles together to form a holomorphic bundle on Z we replace some of the flat bundles by non-isomorphic, S-equivalent bundles. It follows from the results in [46] that after fitting these bundles together, we obtain a stable bundle with a Hermitian SU(2) connection. □

The following is a crucial computation in [47] which we cite without proof:

Lemma 3.2.27. *In the situation of Proposition 3.2.26 we have* $\pi_* c_2(\xi) = c_1(\mathcal{R})$.

3.2.3 Cancelling the local anomaly

We now prove our main theorem:

Theorem 3.2.28. *Let* $Z \to B \cong \mathbb{P}^1$ *be the Jacobian elliptic surface whose Weierstrass model has* $12n$ *singular fibers of Kodaira-type* I_1 *over* $\{t_j\}_{j=1}^{12n}$ *for* $n = 1$ *(rational surface) or* $n = 2$ *(K3 surface). Let* $C_{\mathcal{R}} \subset Z \to B$ *be the spectral double cover corresponding to the line bundle* $\mathcal{R} = \mathcal{O}(2n) \to B$ *that yields the stable rank-two* *holomorphic* $\mathrm{SU}(2)$ *bundle* $\xi = \pi_{2*}\big(\mathcal{P}_{\mathcal{R}}\big) \to Z$. *Then, the generalized first Chern class of the determinant line bundle* $\mathbf{Det}\,\bar{\partial}^{\,\xi} \to B^*$ *is given by*

$$c_1\big(\,\mathbf{Det}\,\bar{\partial}^{\,\xi}, \|.\|_Q\big) = -\frac{1}{6}\left(\sum_{j=1}^{12n} \delta_{(t=t_j)}\right)\,. \tag{3.2.22}$$

In particular, there is no local anomaly.

Proof. In Theorem 3.2.24 we use the rank-two ($r = 2$) bundle $\xi \to Z$ constructed in Proposition 3.2.26 with the contribution coming from the twist computed in Lemma 3.2.27 and Corollary 3.2.25. The continuous part of the first Chern class of the determinant line bundle $\mathbf{Det}\,\bar{\partial}^{\,\xi} \to B^*$ is given by

$$r \cdot c_1(\mathcal{K}) - \pi_* c_2(\xi) = r \cdot c_1(\mathcal{K}) - c_1(\mathcal{R})\,.$$

It follows from Corollary 3.2.25 that for $\mathcal{R} \cong \mathcal{O}(2n)$ the continuous part of the first Chern class vanishes. Notice that a global section $w \in \Gamma(B, \mathcal{R})$ (defining the rank-two holomorphic $\mathrm{SU}(2)$ bundle $\xi \to Z$) exists because the bundle $\mathcal{O}(2)$ is already very ample and defines an embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ by $[z_0 : z_1] \mapsto [z_0^2 : z_0 z_1 : z_1^2]$. □

The global anomaly, represented by the current contributions in Equation (3.2.22), is of critical importance in string theory. The traditional approach to producing low-dimensional physical models out of high-dimensional theories such as the string theories and M-theory has been to use a specific geometric compactification of the "extra" dimensions and derive an effective description of the lower-dimensional theory from the choice of geometric compactification. However, it has long been recognized that there are other possibilities: for example, one can couple perturbative string theory to an arbitrary superconformal two-dimensional theory (geometric or not) to obtain an effective perturbative string compactification in lower dimensions. One way of making an analogous construction in non-perturbative string theory is to exploit the nonperturbative duality transformations which relate various compactified string theories (and M-theory) to each other. This idea was the basis of the construction of F-theory [140].

In a standard compactification of the type IIB string, τ is a constant and D7-branes are absent. Vafa's idea in proposing F-theory [140] was to simultaneously allow a variable τ and the D7-brane sources, arriving at a new class of models in which the string coupling is never weak. Thus, one of the fundamental interpretations of F-theory is in terms of the type IIB string, where it depends on three ingredients: an $\mathrm{PSL}(2,\mathbb{Z})$ symmetry of the theory, a complex scalar field τ (the axio-dilaton) with positive imaginary part (in an appropriate normalization) on which $\mathrm{PSL}(2,\mathbb{Z})$ acts by fractional linear transformations, and D7-branes, which serve as a source for the multi-valuedness of τ if τ is allowed to vary. To do so, one needs to know what types of seven-branes have to be inserted. It turns out that there is a complete dictionary between the different types of seven-branes which must be inserted and the the possible singular limits in one-parameter families of elliptic curves given by the work of Kodaira [77] and Néron [110].

The total space of the Jacobian elliptic surface $\pi : Z \to B \cong \mathbb{P}^1$ in the case $n = 2$ (K3 surface) in Theorem 3.2.28 can now be interpreted as such an F-theory background with 24 disjoint D7-branes inserted into the physical theory. In the context of the physical description of the corresponding compactification of the type IIB string theory, Theorem 3.2.28 provides an explanation why and where these D7-branes have to be inserted: they had to be inserted at the points where $\Delta(t) = 0$ in order to cancel the current contributions of the generalized first Chern class of the determinant line bundle which plays a key role in the description of the path integral description of the physical theory.

CHAPTER 4

The Mixed-Twist Construction for Lattice Polarized K3 surfaces

In this chapter, we begin the study of Jacobian elliptic surfaces $\pi : X \to B$ via a more refined functional invariant than Kodaira's functional invariant $\mathsf{j} : B \to \mathbb{P}^1$, the *generalized functional invariant*. As j is determined by the ramification data over $0, 1, \infty \in J$, Kodaira's functional invariant does not "see" the quadratic twist of the fibration; the quadratic twist of X has the Weierstrass model

$$y^2 = 4x^3 - p(t)^2 g_2 x - p(t)^3 g_3\,,$$

and shares the same functional invariant j. The generalized functional invariant, introduced in 4.1.1, is a triple of numbers $\{i, j, \alpha\}$ that keeps track of the ramification data over other points, and is fine enough to detect the quadratic twist.

This results in the *mixed-twist construction*, and is fundamental for the remaining chapters in the dissertation. Starting with the Weierstrass model of a rational elliptic surface, the mixed-twist construction associated to the generalized functional invariant $\{i, j, \alpha\} = \{1, 1, 1\}$ returns the quadratic twist family after quotienting by a canonical Nikulin involution. The resulting Weierstrass model is that of an elliptic K3 surface, which is the primary object of study through the first half of this chapter.

In particular, we study the mixed-twist construction for a two parameter family of rational elliptic surfaces $S_{c,d} \to \mathbb{P}^1$ that results in the four parameter family $\mathbf{X}_{a,b,c,d}$ of Yoshida surfaces (Lemma 4.1.32). This allows us in Theorem 4.1.34 to explicitly identify the lattice polarization $\mathrm{L} \hookrightarrow \Lambda_{\mathrm{K3}}$ and construct the moduli space $\mathcal{M}_{\mathrm{L}}$. This family is birational to the famous double sextic family $\mathcal{X}_{x_1,x_2,x_3,x_4}$ introduced in §2.1.7 (Proposition 4.1.33), which extends to a biholomorphism of the respective moduli spaces (Corollary 4.1.35).

Subsequently, we produce convenient restrictions of the moduli a, b, c, d that result in extensions of the lattice polarization to Picard rank $\rho = 17, 18, 19$ in §4.2. The explicit lattice polarization $\mathrm{L}', \mathrm{L}'', \mathrm{L}'''$ is identified in each subsequent case, as well as the moduli space $\mathcal{M}_{\mathrm{L}'}, \mathcal{M}_{\mathrm{L}''}, \mathcal{M}_{\mathrm{L}'''}$. We identify as well the monodromy group in each subsequent case. The latter two cases, in Picard rank $\rho = 18, 19$, are identified with previous work of Clinger, Doran, and Malmendier [23] and Hoyt [67], respectively. The main result of this section is Theorem 4.2.53, which summarizes the relations described here.

Ultimately, our primary motivation is to study the Picard rank $\rho = 17$ case - that of the twisted Legendre pencil, introduced in §2.1.8 - which was partially analyzed by Hoyt in [69]. Our analysis in this chapter sets the stage for the remainder of the dissertation, where we in subsequent chapters complete the analysis initialized by Hoyt by geometrizing the moduli space $\mathcal{M}_{\mathrm{L}'}$.

The second half of the chapter begins in §4.3, where use the GKZ formalism to study the univariate families of mirror manifolds of the deformed Fermat family of hypersurfaces in string theory using the generalized functional invariant. In this case, while the initial GKZ data produced is resonant (and thus more difficult to study directly), we show how the mixed-twist construction can be used to produce a secondary set of *non-resonant* GKZ data that allows us to computed the Picard-Fuchs operators.

The utility of this non-resonant data is the explicit computation of the monodromy matrices for the mirror family, which is done in §4.3.4. This computation reproduces the monodromy computed by Candelas et al. [16] and Chen et al. [17] up to conjugacy.

4.1 Mixed-twist construction for multi parameter K3 surfaces

In this section, we use the mixed-twist construction to obtain a multi-parameter family of K3 surfaces of Picard-rank $\rho \geq 16$. Upon identifying a particular Jacobian elliptic fibration on its general member, we find the corresponding lattice polarization, the moduli space, and the Picard-Fuchs system for the family with its general monodromy group. We construct a sequence of restrictions that lead to extensions of the polarization keeping the polarizing lattice two-elementary. We show that the Picard-Fuchs operators under these restrictions coincide with well-known hypergeometric systems, the Aomoto-Gel'fand $E(3,6)$ system (for $\rho = 16, 17$), Appell's F_2 system (for $\rho = 18$), and Gauss' hypergeometric functions of type ${}_3F_2$ (for $\rho = 19$).

4.1.1 The generalized functional invariant

We first recall the *generalized functional invariant* of the mixed-twist construction studied by Doran & Malmendier [40], first introduced by Doran [38]. A generalized functional invariant is a triple (i, j, α) with $i, j \in \mathbb{N}$ and $\alpha \in \left\{\frac{1}{2}, 1\right\}$ such that $1 \leq i, j \leq 6$. To this end, the generalized functional invariant encodes a 1-parameter family of degree $i + j$ covering maps $\mathbb{P}^1 \to \mathbb{P}^1$, which is totally ramified over 0, ramified to degrees i and j over ∞, and simply ramified over another point $\tilde{t}$. For homogeneous coordinates $[v_0 : v_1] \in \mathbb{P}^1$, this family of maps (parameterized by $\tilde{t} \in \mathbb{P}^1 - \{0, 1, \infty\}$) is given by

$$[v_0, v_1] \mapsto [c_{ij} v_1^{i+j} \tilde{t} : v_0^i (v_0 + v_1)^j], \tag{4.1.1}$$

for some constant $c_{ij} \in \mathbb{C}^\times$. For a family $\pi : X \to B$ with Weierstrass models given by Equation (2.1.18) with complex n-dimensional fibers and a generalized functional invariant (i, j, α) such that

$$0 \leq \deg_t(g_2) \leq \min\left(\frac{4}{i}, \frac{4\alpha}{j}\right), \quad 0 \leq \deg_t(g_3) \leq \min\left(\frac{6}{i}, \frac{6\alpha}{j}\right), \tag{4.1.2}$$

Doran & Malmendier showed that a new family $\tilde{\pi} : \tilde{X} \to B$ can be constructed such that the general fiber $\tilde{X}_{\tilde{t}} = \tilde{\pi}^{-1}(\tilde{t})$ is a compact, complex $(n+1)$-manifold equipped with a Jacobian elliptic fibration over $\mathbb{P}^1 \times S$. In the coordinate chart $\{[v_0 : v_1], (u_1, \ldots, u_{n-1})\} \in \mathbb{P}^1 \times S$ the family of Weierstrass models $W_{\tilde{t}}$ is given by

$$\begin{aligned} \tilde{y}^2 = 4\tilde{x}^3 &- g_2\left(\frac{c_{ij}\tilde{t}v_1^{i+j}}{v_0^i(v_0+v_1)^j}, u\right) v_0^4 v_1^{4-4\alpha}(v_0+v_1)^{4\alpha}\tilde{x} \\ &- g_3\left(\frac{c_{ij}\tilde{t}v_1^{i+j}}{v_0^i(v_0+v_1)^j}, u\right) v_0^6 v_1^{6-6\alpha}(v_0+v_1)^{6\alpha} \end{aligned} \tag{4.1.3}$$

with $c_{ij} = (-1)^i i^i j^j/(i+j)^{i+j}$. The new family is called the *twisted family with generalized functional invariant* (i, j, α) *of* $\pi : X \to B$. It follows that conditions (4.1.2) guarantee that the twisted family is minimal and normal if the original family is. Moreover, they showed that if the Calabi-Yau condition is satisfied for the fibers of the twisted family if it is satisfied for the fibers of the original.

The twisting associated with the generalized functional invariant above is referred to as the *pure twist construction*; we may extend this notion to that of a *mixed twist construction*. This means that one combines a pure twist from above with a rational map $B \to B$, thus allowing one to change locations of the singular fibers and ramification data. This was studied in [40, Sec. 8] for linear and quadratic base changes. We may also perform a multi-parameter version of the mixed twist construction for a generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$. For us, it will be enough to consider the two-parameter family of ramified covering maps given by

$$[v_0 : v_1] \mapsto [4av_0(v_0+v_1) + (a-b)v_1^2 : 4v_0(v_0+v_1)], \tag{4.1.4}$$

such that for $a, b \in \mathbb{P}^1 - \{0, 1, \infty\}$ with $a \neq b$ the map in Equation (4.1.4) is totally ramified over a and b. We will apply the mixed twist construction to certain (families of) rational elliptic surfaces $X \to \mathbb{P}^1$. In [40, Sec. 5.5] the authors showed that the twisted family with generalized functional invariant $(1, 1, 1)$ in this case is birational to a quadratic twist family of $X \to \mathbb{P}^1$. We will explain the relationship in more detail and utilize it in the construction of the associated Picard-Fuchs operators in the next section.

4.1.2 Quadratic twists of a rational elliptic surface

A two-parameter family of rational elliptic surfaces $S_{c,d} \to \mathbb{P}^1$ is given by the affine Weierstrass model

$$y^2 = 4x^3 - g_2(t)x - g_3(t), \tag{4.1.5}$$

where $g_2(t)$ and $g_3(t)$ are the respective degree four and degree six polynomials

$$\begin{aligned} g_2 &= \frac{4}{3}\left(t^4 - (2c+d+1)t^3 + (c^2+cd+d^2+2c-d+1)t^2 - c(c-d+2)t + c^2\right), \\ g_3 &= \frac{4}{27}\left(t^2 - (c-d+2)t + 2c\right)\left(t^2 - (c+2d-1)t - c\right)\left(2t^2 - (2c+d+1)t + c\right). \end{aligned}$$

Assuming $c, d \neq 0, 1, \infty$, and $c \neq -d$, Equation (4.1.5) defines a rational elliptic surface with six singular fibers of Kodaira type I_2 occurring at $t = 0, 1, \infty, c, c+d$, and $c/(d-1)$. We have the following:

Lemma 4.1.29. *The rational elliptic surface $S = S_{c,d}$ in Equation (4.1.5) is bira-*

tionally equivalent to the twisted Legendre pencil given by

$$\tilde{y}^2 = \tilde{x}(\tilde{x}-1)(\tilde{x}-t)(t-c-d\tilde{x}). \tag{4.1.6}$$

Proof. The proof follows by direct computation using the transformation:

$$x = \frac{3t\,(t-c)}{3\tilde{x}+t^2+(d+1-c)\,t-c}, \quad y = \frac{3\tilde{y}t\,(t-c)}{2\,(3\tilde{x}+t^2+(d+1-c)\,t-c)^2}.$$

□

A standard quadratic twist applied to a rational elliptic surface can be identified with Doran & Malmendier's mixed-twist construction with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$. The two-parameter family of ramified covering maps in Equation (4.1.4) is totally ramified over $a, b \in \mathbb{P}^1 - \{0, 1, \infty\}$. We also require that $a, b \notin \{c, c+d, c/(d-1)\}$, $a \neq b$. We now apply the mixed-twist construction to the rational elliptic surface S:

Proposition 4.1.30. *The mixed-twist with generalized functional invariant* $(i, j, \alpha) = (1, 1, 1)$ *applied to the rational elliptic surface in Equation (4.1.5) yields the family of Jacobian elliptic K3 surfaces* $\mathbf{X}_{a,b,c,d} \to \mathbb{P}^1$ *given by*

$$\hat{y}^2 = 4\hat{x}^3 - (t-a)^2(t-b)^2 g_2(t)\hat{x} - (t-a)^3(t-b)^3 g_3(t). \tag{4.1.7}$$

Proof. In affine base coordinates $[v : 1] \in \mathbb{P}^1$, the map $f : \mathbb{P}^1 \to \mathbb{P}^1$ from the mixed-twist construction with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$ in Equation (4.1.4) is given by

$$f(v) = a + \frac{a-b}{4v(v+1)}.$$

The pullback of the Weierstrass model for the two-parameter family of the rational elliptic surfaces in Equation (4.1.5) along the map $t = f(v)$ is easily checked to yield a four-parameter family of Jacobian elliptic K3 surfaces $\tilde{\mathbf{X}}_{a,b,c,d} \to \mathbb{P}^1$. On $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}_{a,b,c,d}$, we have the anti-symplectic involution $\imath$ induced by the base transformation $v \mapsto -v-1$, and the fiberwise hyper-elliptic involution $-\mathrm{id}$. The composition map $\jmath = -\mathrm{id} \circ \imath$ leaves the holomorphic two-form $\eta_{\tilde{\mathbf{X}}} \in H^0(\boldsymbol{\omega}_{\tilde{\mathbf{X}}})$ invariant, $\jmath^* \eta_{\tilde{\mathbf{X}}} = \eta_{\tilde{\mathbf{X}}}$. Hence, $\jmath : \tilde{\mathbf{X}} \to \tilde{\mathbf{X}}$ is a Nikulin involution and the minimal resolution of the quotient $\tilde{\mathbf{X}}/\jmath$ is the four parameter family $\mathbf{X} = \mathbf{X}_{a,b,c,d} \to \mathbb{P}^1$ of Jacobian elliptic K3 surfaces given by the Weierstrass model in Equation (4.1.7). □

A direct computation yields the following:

Lemma 4.1.31. *Equation (4.1.7) defines a Jacobian elliptic fibration $\pi : \mathbf{X} \to \mathbb{P}^1$ on a general $\mathbf{X} = \mathbf{X}_{a,b,c,d}$ with two singular fibers of Kodaira type I_0^* over $t = a, b$, six singular fibers of Kodaira type I_2 over $t = 0, 1, \infty, c, c+d$, and $c/(d-1)$, and the Mordell Weil group $\mathrm{MW}(\mathbf{X}, \pi) = (\mathbb{Z}/2\mathbb{Z})^2$.*

The following echoes Lemma 4.1.29 and provides a convenient normal form for the family of K3 surfaces:

Lemma 4.1.32. *The family in Equation (4.1.7) is birationally equivalent to the family of Yoshida surfaces given by*

$$y^2 = x(x-1)(x-t)(t-a)(t-b)(t-c-dx). \tag{4.1.8}$$

Proof. By direct computation, with the following transformation:

$$\begin{aligned} \hat{x} &= \frac{3t\,(t-a)\,(t-b)\,(t-c)}{3x + (t-a)\,(t-b)\,(t^2 + (d+1-c)\,t - c)}, \\ \hat{y} &= \frac{3yt\,(t-a)\,(t-b)\,(t-c)}{2\,(3x + (t-a)\,(t-b)\,(t^2 + (d+1-c)\,t - c))^2}. \end{aligned}$$

□

The family of Yoshida surfaces was studied by Hoyt & Schwarz in [70], where the authors analyzed how certain restrictions and degenerations of the parameters increase the Picard-rank. There is a close connection between their full analysis and Dolgachev's mirror symmetry for K3 surfaces; this will be the subject of a forthcoming article.

Recall the family of double-sextic K3 surfaces from §2.1.7, i.e., the K3 surfaces obtained as the minimal resolution of the double cover of the projective plane $\mathbb{P}^2$ branched along a configuration of the six lines, denoted by $\boldsymbol{\ell} = \{\ell_1, \dots, \ell_6\}$ and given in weighted homogeneous coordinates $[t_1 : t_2 : t_3 : z] \in \mathbb{P}(1,1,1,3)$ by the equation

$$z^2 = \prod_{i=1}^{6} (a_{i1}t_1 + a_{i2}t_2 + a_{i3}t_3), \tag{4.1.9}$$

where $\ell_i = \{[t_1 : t_2 : t_3] \mid a_{i1}t_1 + a_{i2}t_2 + a_{i3}t_3 = 0\} \subset \mathbb{P}^2$ for parameters $a_{ij} \in \mathbb{C}$, $i = 1, \dots, 6$, $j = 1, 2, 3$ which are assumed to be general. The coordinates $x_1, x_2, x_3, x_4 \in \mathcal{X}(3,6)$ are moduli parameterizing the family of K3 surfaces for a configuration $\boldsymbol{\ell}$ of six lines in general position, and in the following we will denote this family by $\mathcal{X} = \mathcal{X}_{x_1,x_2,x_3,x_4}$. We have the following:

Proposition 4.1.33. *The family in Equation (4.1.8) is birationally equivalent to the family of double-sextic surfaces.*

Proof. In the affine coordinate system $\{t_1 = -1, t_2 = \tilde{t}, t_3 = u, z = z\} \subset \mathbb{P}(1,1,1,3)$, a birational transformation $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ is given by

$$\varphi \;:\quad \tilde{t} = \frac{x_3 t - 1}{x_3 t - x_1}, \qquad u = \frac{x(1 - x_1)}{x_3 t - x_1},$$

where we are also using the identification of the moduli given by

$$x_1 = \frac{a}{b}, \qquad x_2 = \frac{a-c}{b-c}, \qquad x_3 = \frac{1}{b}, \qquad x_4 = \frac{d}{b-c}, \tag{4.1.10}$$

Hence, the birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ induces a birational equivalence between the K3 surfaces $\mathbf{X}$ and $\mathcal{X}$ by the natural extension on the Weierstrass models, where $\mathbf{X}$ is identified with the (scaled) Yoshida surface

$$y^2 = \frac{1}{b(c-b)} x(x-1)(x-t)(t-a)(t-b)(t-c-dx)$$

of Lemma 4.1.32 and §2.1.7. $\square$

The double sextic family is a well studied, for example by Matsumoto [93], and Matsumoto et al. [94, 95, 96]. One takeaway from their work is that the family of double sextic K3 surfaces is in many ways analogous to the Legendre pencil of elliptic curves, realized as double covers of the line $\mathbb{P}^1$ branching over four points. More recently, the double sextic family $\mathcal{X}$ and closely related K3 surfaces have been studied further in the context of string dualities [87, 90, 28, 86, 24]. In Clingher et al. [28], the authors showed that four different elliptic fibrations on $\mathcal{X}$ have interpretations in F-theory/heterotic string duality. In [24] the authors classified all Jacobian elliptic fibrations on the Shioda-Inose surface associated with $\mathcal{X}$. Hosono et al. in [65, 65] constructed compactifications of $\mathcal{M}_6$ from GKZ data and toric geometry, suitable for the study of the Type IIA/Type IIB string duality.

In the following we will use the following standard notations for lattices: $L_1 \oplus L_2$ is orthogonal sum of the two lattices L_1 and L_2, $L(\lambda)$ is obtained from the lattice L by multiplication of its form by $\lambda \in \mathbb{Z}$, $\langle R \rangle$ is a lattice with the matrix R in some basis; A_n, D_m, and E_k are the positive definite root lattices for the corresponding root systems, H is the unique even unimodular hyperbolic rank-two lattice. A lattice L is two-elementary if its discriminant group A_L is a two-elementary abelian group, namely $A_L \cong (\mathbb{Z}/2\mathbb{Z})^\ell$ with ℓ being the minimal number of generators of the discriminant group A_L, also called the length of the lattice L. Even, indefinite, two-elementary lattices L are uniquely determined by the rank ρ, the length ℓ, and the parity δ – which equals 1 unless the discriminant form $q_L(x)$ takes values in $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}$ for all $x \in A_L$ in which case it is 0; this is a result by Nikulin [111, Thm. 4.3.2].

Dolgachev defined the notion of a lattice polarization for a K3 surface [34]. If L is an even lattice of signature $(1, \rho - 1)$ with $\rho \geq 1$, then an L-polarization on a K3 surface $\mathbf{X}$ is a primitive embeddings $\imath : L \hookrightarrow \mathrm{NS}(\mathbf{X})$ into the Néron-Severi lattice such that $\imath(L)$ contains a pseudo-ample class, i.e., a numerically effective class of positive self-intersection in the Néron-Severi lattice $\mathrm{NS}(\mathbf{X})$. If we assume that the lattice L has a primitive embeddings $\imath : L \hookrightarrow \Lambda_{K3}$ into the K3 lattice $\Lambda_{K3} \cong H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, then Dolgachev proves that there exists a coarse moduli space $\mathcal{M}_L$ of L-polarized K3 surfaces and an appropriate version of the global Torelli theorem holds; see [34]. We have the following:

Theorem 4.1.34. *The family in Equation (4.1.8) forms the 4-dimensional moduli space $\mathcal{M}_L$ of L-polarized K3 surfaces where L has the following isomorphic presentations:*

$$\begin{aligned} L \;&\cong H \oplus E_8(-1) \oplus A_1(-1)^{\oplus 6} \;\cong\; H \oplus E_7(-1) \oplus D_4(-1) \oplus A_1(-1)^{\oplus 3} \\ &\cong H \oplus D_6(-1) \oplus D_4(-1)^{\oplus 2} \;\cong\; H \oplus D_6(-1)^{\oplus 2} \oplus A_1(-1)^{\oplus 2} \\ &\cong H \oplus D_{10}(-1) \oplus A_1(-1)^{\oplus 4} \cong\; H \oplus D_8(-1) \oplus D_4(-1) \oplus A_1(-1)^{\oplus 2}. \end{aligned} \tag{4.1.11}$$

In particular, L is a primitive sub-lattice of the K3 lattice Λ_{K3}.

Proof. For a configuration $\boldsymbol{\ell}$ in general position the K3 surface $\mathbf{X}$ has the transcendental lattice $\mathrm{T}(\mathbf{X}) \cong H(2) \oplus H(2) \oplus \langle -2 \rangle^{\oplus 2}$; see [70]. Using Lemma 4.1.30 it follows that the family in Equation (4.1.5) forms a four-dimensional moduli space $\mathcal{M}_L$ of pseudo-ample L-polarized K3 surfaces where the lattice L has rank $\rho = 16$. From Lemma 4.1.31 we see that L is two-elementary such that $A_L \cong (\mathbb{Z}/2\mathbb{Z})^\ell$ with $\ell = 6$. It follows that in our case the lattice L is the unique two-elementary lattice with $\rho = 16$, $\ell = 6$, $\delta = 1$ (for $\rho = 16$ the two-elementary lattice must have $\delta = 1$; see [111]). We then use results in [76, Table 1] to read off the isomorphic presentations of L from the Jacobian elliptic fibrations on $\mathcal{X}$ with trivial Mordell Weil group. □

Combining Proposition 4.1.33 and Theorem 4.1.34 yields the following result:

Corollary 4.1.35. *The moduli spaces $\mathcal{M}_L$ and $\mathcal{X}(3,6)$ are isomorphic.*

It is well known that $\mathcal{X}(3,6)$ is a quasi-projective variety, with orbifold singularities arising from the quotient construction. This is part of the difficulty when attempting to study classical mirror symmetry for the family of double-sextic surfaces $\mathcal{X}$ or, equivalently, for the family $\mathbf{X}$ of Yoshida surfaces from the mixed-twist construction. Yet another pressing difficulty is that $\mathcal{M}_6$ does not carry, as a quasi-projective variety itself, domains known as *large scale structure limits* (LCSLs) where mirror symmetry is manifest. Although a compactification was constructed in the work of Matsumoto et al., the construction was insufficient for the purposes of mirror symmetry. This problem was solved completely by the Hosono et al. papers above, by constructing LCSLs as intersecting normal crossing divisors as the blowups of singular loci in a sequence of compactifications of $\mathcal{X}(3,6)$. We summarize the situation as follows:

Proposition 4.1.36. *Let $\Sigma \in \mathrm{T}(\mathbf{X})$ be a transcendental cycle on a general K3 surface $\mathbf{X}$, $\eta_{\mathbf{X}}$ the holomorphic two-form induced by $dt \wedge dx/y$ in Equation (4.1.8), and $\omega = \oint_\Sigma \eta_{\mathbf{X}}$ a period. The Picard-Fuchs system annihilating $\omega' = \omega/\sqrt{b(c-b)}$ is the rank-six Aomoto-Gel'fand system $E(3,6)$ of [94, 95, 96] in the variables*

$$x_1 = \frac{a}{b}, \quad x_2 = \frac{a-c}{b-c}, \quad x_3 = \frac{1}{b}, \quad x_4 = \frac{d}{b-c}.$$

Proof. In [119] Sasaki showed that the period integral

$$\omega' = \omega'(x_1, x_2, x_3, x_4) = \oint_{\Sigma'} \eta_{\mathcal{X}} = \oint_{\Sigma'} d\tilde{t} \wedge \frac{du}{z}$$

of the holomorphic two-form for Equation (4.1.12) over a transcendental cycle $\Sigma' \in \mathrm{T}(\mathcal{X})$ satisfies the rank-six Aomoto-Gel'fand system $E(3,6)$ in the variables x_1, x_2, x_3, x_4. Let $\Sigma = (\varphi^{-1})_* \Sigma'$ where φ was constructed in the proof of Proposition 4.1.33. In the affine coordinate system $\{t_1 = -1, t_2 = \tilde{t}, t_3 = u, z = z\} \subset \mathbb{P}(1,1,1,3)$ the non-vanishing holomorphic two-form $\eta_{\mathcal{X}} \in H^0(\boldsymbol{\omega}_{\mathcal{X}})$ is given by

$$\eta_{\mathcal{X}} = d\tilde{t} \wedge \frac{du}{z} = \frac{d\tilde{t} \wedge du}{\sqrt{\tilde{t}u(\tilde{t}+u-1)(x_1\tilde{t}+x_3u-1)(x_2\tilde{t}+x_4u-1)}}. \tag{4.1.12}$$

For the pullback of $\eta_{\mathcal{X}}$, a direct computation shows that

$$\varphi^* \eta_{\mathcal{X}} = \frac{\sqrt{b(c-b)}\; dt \wedge dx}{\sqrt{x(x-1)(x-t)(t-a)(t-b)(t-c-dx)}} \equiv \sqrt{b(c-b)}\; dt \wedge \frac{dx}{y}.$$

□

The moduli a, b, c, d parameterizing the Yoshida surfaces are different from the coordinate systems constructed near the LCSLs of Hosono et al. Nevertheless, the relation of the $E(3,6)$ system with the Picard-Fuchs system constructed near the LCSLs in the compactification still allows us to conclude the behavior of the monodromy group around the LCSLs for the double sextic family. Let $\overline{\mathcal{X}(3,6)}$ be the compactification of the moduli space $\mathcal{X}(3,6)$ constructed in [65], and let $p : \overline{\mathcal{X}(3,6)} \to \mathcal{P}$ be the period mapping, where $\mathcal{P} \subset \mathbb{P}^5$ is the period domain generated by the Hodge-Riemann relations of the six linearly independent period integrals. We have the following:

Corollary 4.1.37. *Let A be the Gram matrix of the lattice $H(2) \oplus H(2) \oplus \langle -2 \rangle^{\oplus 2}$.*

The global monodromy group G of the period map is, up to conjugacy, given by

$$G = \left\{ M \in \mathrm{GL}(6, \mathbb{Z}) \mid M^T A M = A,\ M \equiv \mathbb{I}_6 \mod 2 \right\} \ \subset\ \mathrm{O}(A, \mathbb{Z}).$$

Proof. Because of Proposition 4.1.36 the global monodromy group G of the period map coincides with the monodromy group of the Aomoto-Gel'fand $E(3,6)$ system. The statement follows from [65, Thm. 7.1], since the Picard-Fuchs system centered around the LCSLs constructed by Hosono et al. is a GKZ $\mathcal{A}$-hypergeometric system, to which the Aomoto-Gel'fand $E(3,6)$ system restricts near the LCSLs. Moreover, it follows from [64] that the solutions around different LCSLs in the compactified moduli space $\overline{\mathcal{X}(3,6)}$ are all analytic continuations of each other with trivial monodromy. □

4.2 Extending the lattice polarization and monodromy groups

Using the four-parameter family of Yoshida surfaces in Lemma 4.1.32 we can efficiently study certain extensions of the lattice polarization and identify the corresponding lattice polarizations, monodromy groups, and Picard-Fuchs operators. Moreover, it will follow from Corollary 4.1.37 above that the restricted monodromy groups extend to the LCSLs.

4.2.1 Picard-rank $\rho = 17$

We consider the extension of the lattice polarization for $d = 0$. In this case the Yoshida surface $\mathbf{X}'_{a,b,c} = \mathbf{X}_{a,b,c,0}$ becomes the *twisted Legendre Pencil*

$$y^2 = x(x-1)(x-t)(t-a)(t-b)(t-c). \tag{4.2.13}$$

The general member has Picard-rank 17, and was studied by Hoyt in [69]. We have the following:

Lemma 4.2.38. *Equation (4.2.13) defines a Jacobian elliptic fibration* $\pi : \mathbf{X}' \to \mathbb{P}^1$ *on a general* $\mathbf{X}' = \mathbf{X}'_{a,b,c}$ *with three singular fibers of Kodaira type* I_0^* *over* $t = a, b, c$*, three singular fibers of Kodaira type* I_2*, and the Mordell Weil group* $\mathrm{MW}(\mathbf{X}', \pi) = (\mathbb{Z}/2\mathbb{Z})^2$.

Proof. The proof is similar to the ones given in the preceding section. The statement about Picard-rank and the Mordell Weil group can be found in Hoyt [69]. □

In particular, $\mathbf{X}'$ is birational to the three-parameter quadratic twist family of the classical Legendre pencil of elliptic curves and hence, it is equivalently described by the mixed-twist construction with generalized functional invariant $(i, j, \alpha) = (1, 1, 1)$. We have the following:

Theorem 4.2.39. *The family in Equation (4.2.13) forms the 3-dimensional moduli space* $\mathcal{M}_{L'}$ *of* L'*-polarized K3 surfaces where* L' *has the following isomorphic presentations:*

$$\begin{aligned} L' &\cong H \oplus E_8(-1) \oplus D_4(-1) \oplus A_1(-1)^{\oplus 3} \;\cong\; H \oplus E_7(-1) \oplus D_4(-1)^{\oplus 2} \\ &\cong H \oplus D_{12}(-1) \oplus A_1(-1)^{\oplus 3} \;\cong\; H \oplus D_{10}(-1) \oplus D_4(-1) \oplus A_1(-1) \qquad (4.2.14) \\ &\cong H \oplus D_8(-1) \oplus D_6(-1) \oplus A_1(-1). \end{aligned}$$

In particular, L' *is a primitive sub-lattice of the K3 lattice* Λ_{K3}.

Proof. We use the same strategy as in the proof of Theorem 4.1.34. Using Lemma 4.2.38 it follows that the two-elementary lattice L' must have $\rho = 17$ and $\ell = 5$. Applying Nikulin's classification [111] it follows that there is only one such lattice admitting a primitive lattice embedding into Λ_{K3}, and it must have $\delta = 1$. We then go through the list in [129] to find the isomorphic presentations. □

In [27] it was shown that the configuration of six lines $\boldsymbol{\ell}$ in $\mathbf{X}'$ is specialized to one where three lines intersect in one point. The pencil of lines through the intersection point induces precisely the elliptic fibration of Proposition 4.2.38. Thus, the general K3 surface $\mathbf{X}'$ is not a Jacobian Kummer surface. Rather, it arises as the relative Jacobian of a Kummer surface associated with an abelian surface with a polarization of type $(1, 2)$; this was proved in [27, 29].

Setting $d = 0$ in Proposition 4.1.36 we obtain the following:

Proposition 4.2.40. *Let $\Sigma \in \mathrm{T}(\mathbf{X}')$ be a transcendental cycle on a general K3 surface $\mathbf{X}'$, $\eta_{\mathbf{X}'}$ the holomorphic two-form induced by $dt \wedge dx/y$ in Equation (4.2.13), and $\omega = \oint_\Sigma \eta_{\mathbf{X}'}$ a period. The Picard-Fuchs system annihilating $\omega' = \omega/\sqrt{b(b-c)}$ is the restricted rank-five Aomoto-Gel'fand system $E(3, 6)$ of [94, 95, 96] with $x_4 = 0$.*

It then follows:

Corollary 4.2.41. *The global monodromy group of the period map in Proposition 4.2.40 is, up to conjugacy, the Siegel congruence subgroup of level two $\Gamma_2(2) \subset \mathrm{Sp}(4, \mathbb{Z})$.*

Proof. Using Proposition 4.2.38 the statement follows from results of Hoyt [69], Matsumoto et al. [94, 95, 96], Hara et al. [60], Sasaki and Yoshida [124], and Braeger et al. [15]. □

One can ask what configurations of six lines $\boldsymbol{\ell}$ yield total spaces that are Kummer surfaces. In [11] the authors gave geometric characterizations of such six-line configurations. Here, we focus on the case of a Kummer surface associated with a principally polarized abelian surface. We have the following:

Proposition 4.2.42. *The general K3 surface in Equation (4.1.8) is a Jacobian Kummer surface, i.e., the Kummer surface associated with the Jacobian of a general genus-two curve if and only if a, b, c are generic and $d = \frac{(a-c)(b-c)}{ab-c}$.*

Proof. Using the methods of [28] we compute the square of the degree-two Dolgachev-Ortland R^2. It vanishes if and only if the six lines are tangent to a common conic.

It is well known that this is a necessary and sufficient criterion for the total space to be a Jacobian Kummer surface; see for example [26]. A direct computation of R^2 for the six lines in Equation (4.1.8) yields the result. $\square$

We also have the following:

Lemma 4.2.43. *Equation (4.1.8) with a, b, c generic and $d = \frac{(a-c)(b-c)}{ab-c}$ defines a Jacobian elliptic fibration $\pi : \widetilde{\mathbf{X}} \to \mathbb{P}^1$ with the singular fibers $2I_0^* + 6I_2$ and the Mordell Weil group $\mathrm{MW}(\widetilde{\mathbf{X}}, \pi) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus \langle 1 \rangle$.*

The connection between the parameters a, b, c and the moduli of genus-two curves was exploited in [91, 9]. We have the following:

Theorem 4.2.44. *The family in Equation (4.1.5) with $d = \frac{(a-c)(b-c)}{ab-c}$ forms the three-dimensional moduli space $\mathcal{M}_{\tilde{L}}$ of $\tilde{L}$-polarized K3 surfaces where $\tilde{L}$ has the following isomorphic presentations:*

$$\tilde{L} \;\cong\; H \oplus D_8(-1) \oplus D_4(-1) \oplus A_3(-1) \;\cong\; H \oplus D_7(-1) \oplus D_4(-1)^{\oplus 2}\,. \tag{4.2.15}$$

In particular, $\tilde{L}$ is a primitive sub-lattice of the K3 lattice Λ_{K3}.

Proof. We established in Proposition 4.2.42 that the K3 surface obtained from the Weierstrass model in Equation (4.1.8) is a Jacobian Kummer surface if and only if the parameters a, b, c, d satisfy a certain relation. In [81] Kumar classified all Jacobian elliptic fibrations on a generic Kummer surface. Among them are exactly two fibrations that have a trivial Mordell Weil group, called (15) and (17). The types of reducible fibers in the two fibrations then yield Equation (4.2.15). $\square$

It was shown in [27] that the general K3 surface $\widetilde{\mathbf{X}}_{a,b,c}$ in Theorem (4.2.44) arises as the rational double cover of a general K3 surface in Proposition (4.2.38). The double cover $\widetilde{\mathbf{X}}_{a,b,c} \dashrightarrow \mathbf{X}'_{a',b',c'}$ is branched along an even eight on $\mathbf{X}'$ consisting of the non-central components of two reducible fibers of type D_4.

4.2.2 Picard-rank $\rho = 18$

We consider the extension of the lattice polarization for $c = d = 0$. In this case the Yoshida surface $\mathbf{X}''_{a,b} = \mathbf{X}_{a,b,0,0}$ becomes the two-parameter twisted Legendre pencil given by

$$y^2 = x(x-1)(x-t)t(t-a)(t-b). \tag{4.2.16}$$

The general member of this family has Picard-rank 18. We have the following:

Lemma 4.2.45. *Equation (4.2.16) defines a Jacobian elliptic fibration $\pi : \mathbf{X}'' \to \mathbb{P}^1$ on a general $\mathbf{X}'' = \mathbf{X}''_{a,b}$ with the singular fibers $I_2^* + 2I_0^* + 2I_2$ and the Mordell Weil group* $\mathrm{MW}(\mathbf{X}'', \pi) = (\mathbb{Z}/2\mathbb{Z})^2$.

We then have the following:

Theorem 4.2.46. *The family in Equation (4.2.18) forms the 2-dimensional moduli space $\mathcal{M}_{L''}$ of L''-polarized K3 surfaces where L'' has the following isomorphic presentations:*

$$\begin{aligned} L'' &\cong H \oplus E_8(-1) \oplus D_6(-1) \oplus A_1(-1)^{\oplus 2} \;\cong\; H \oplus E_7(-1)^{\oplus 2} \oplus A_1(-1)^{\oplus 2} \\ &\cong H \oplus E_7(-1) \oplus D_8(-1) \oplus A_1(-1) \;\cong\; H \oplus D_{14}(-1) \oplus A_1(-1)^{\oplus 2} \\ &\cong H \oplus D_{10}(-1) \oplus D_6(-1). \end{aligned} \tag{4.2.17}$$

In particular, L'' is a primitive sub-lattice of the K3 lattice Λ_{K3}.

Proof. We use the same strategy as in the proof of Theorem 4.1.34. Using Lemma 4.2.45 it follows that the two-elementary lattice L'' must have $\rho = 18$ and $\ell = 4$. Applying Nikulin's classification [111] it follows that there are two such lattices admitting a primitive lattice embedding into Λ_{K3}, namely the ones with $\delta = 0, 1$. An extra computation shows that we have $\delta = 1$. We then go through the list in [129] to find the isomorphic presentations. □

From results in [23, 54] the Picard-Fuchs system can now be determined explicitly:

Proposition 4.2.47. *Let $\Sigma \in \mathrm{T}(\mathbf{X}'')$ be a transcendental cycle on a general K3 surface $\mathbf{X}''$, $\eta_{\mathbf{X}''}$ the holomorphic two-form induced by $dt \wedge dx/y$ in Equation (4.2.16), and $\omega = \oint_\Sigma \eta_{\mathbf{X}''}$ a period. The Picard-Fuchs system annihilating $\omega' = \omega/\sqrt{a(a-b)}$ is Appell's hypergeometric system $F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1|z_1, z_2)$ with $z_1 = 1/a, z_2 = 1 - b/a$.*

It then follows:

Corollary 4.2.48. *The global monodromy group of the period map in Proposition 4.2.47 is, up to conjugacy, the outer tensor product $\Gamma(2) \boxtimes \Gamma(2)$ where $\Gamma(2) \subset \mathrm{SL}(2, \mathbb{Z})$ is the principal congruence subgroup of level two.*

4.2.3 Picard-rank $\rho = 19$

We consider the extension of the lattice polarization for $c = d = 0$ and $b \to \infty$. In this case the Yoshida surface $\mathbf{X}'''_a = \mathbf{X}_{a,\infty,0,0}$ becomes the one-parameter twisted Legendre pencil given by

$$y^2 = x(x-1)(x-t)t(t-a). \tag{4.2.18}$$

This family was studied in detail by Hoyt [67]; the general member has Picard-rank $\rho = 19$. We have the following:

Lemma 4.2.49. *Equation (4.2.18) defines a Jacobian elliptic fibration $\pi : \mathbf{X}''' \to \mathbb{P}^1$ on a general $\mathbf{X}''' = \mathbf{X}'''_a$ with the singular fibers $2I_2^* + I_0^* + 2I_2$ and the Mordell Weil group $\mathrm{MW}(\mathbf{X}''', \pi) = (\mathbb{Z}/2\mathbb{Z})^2$.*

We then have the following:

Theorem 4.2.50. *The family in Equation (4.2.18) forms the 1-dimensional moduli space $\mathcal{M}_{L'''}$ of L'''-polarized K3 surfaces where L''' has the following isomorphic presentations:*

$$\begin{aligned} L''' \;&\cong H \oplus E_8(-1) \oplus E_7(-1) \oplus A_1(-1)^{\oplus 2} \;\cong\; H \oplus E_7(-1) \oplus D_{10}(-1) \\ &\cong H \oplus E_8(-1) \oplus D_8(-1) \oplus A_1(-1) \;\cong\; H \oplus D_{16}(-1) \oplus A_1(-1). \end{aligned} \tag{4.2.19}$$

In particular, L''' is a primitive sub-lattice of the K3 lattice Λ_{K3}.

Proof. We use the same strategy as in the proof of Theorem 4.1.34. Using Lemma 4.2.49 it follows that the two-elementary lattice L''' must have $\rho = 19$ and $\ell = 3$. Applying Nikulin's classification [111] it follows that there is only one such lattice admitting a primitive lattice embedding into Λ_{K3}, and it must have $\delta = 1$. We then go through the list in [129] to find the isomorphic presentations. □

We have the following:

Proposition 4.2.51. *Let $\Sigma \in \mathrm{T}(\mathbf{X}''')$ be a transcendental cycle on a general K3 surface $\mathbf{X}'''$, $\eta_{\mathbf{X}'''}$ the holomorphic two-form induced by $dt \wedge dx/y$ in Equation (4.2.18), and $\omega = \oint_\Sigma \eta_{\mathbf{X}'''}$ a period. The Picard-Fuchs operator is the operator annihilating the Gauss hypergeometric function ${}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1|\frac{1}{a})$, i.e., the univariate third-order linear differential operator given by*

$$8a^2\,(a-1)\,\frac{d^3\omega}{da^3} + 12a\,(3a-2)\,\frac{d^2\omega}{da^2} + (26a-8)\,\frac{d\omega}{da} + \omega = 0. \tag{4.2.20}$$

Proof. The period integral is given by the classical Gauss integral representation of the hypergeometric function ${}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1|\frac{1}{a}\right)$, that is

$$\omega(a) = \oint_\Sigma \eta_{\mathbf{X}'''} = \oint_\Sigma \frac{dt \wedge dx}{\sqrt{t(t-a)x(x-1)(x-t)}} = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1\Big|\frac{1}{a}\right). \tag{4.2.21}$$

The well known differential equation for ${}_3F_2$ yields Equation (4.2.20). □

It then follows:

Corollary 4.2.52. *The global monodromy group of the period map in Proposition 4.2.51 is, up to conjugacy,* $\Gamma(2)^* := \langle \Gamma(2), w \rangle$ *where* $w = \left(\begin{smallmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{smallmatrix}\right)$ *is the Fricke involution.*

Proof. Equation (4.2.21) proves that the monodromy group of the ODE annihilating ${}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1|, \cdot)$ can be obtained from the monodromy group of the ODE annihilating ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1|, \cdot)$ by adjoining the involution that is generated by the monodromy operator for loops around the singular fiber at $t = a$ or, equivalently, $t = 0$. One checks that in terms of the modular parameter the action is conjugate to w. □

In summary, we have the following main result:

Theorem 4.2.53. *Restricting (i)* $d = 0$, *(ii)* $c = d = 0$, *(iii)* $c = d = 0, b \to \infty$ *in the family of K3 surfaces in Equation (4.1.8), the lattice polarization* L *extends in a chain of even, indefinite, two-elementary lattices given by*

$$L \leqslant L' \leqslant L'' \leqslant L''' , \tag{4.2.22}$$

where the lattice are given by Equations (4.1.11), (4.2.14), (4.2.17), (4.2.19) and are uniquely determined by (rank, length, parity) with $(\rho, \ell, \delta) = (16 + k, 6 - k, 1)$ *for* $k = 0, 1, 2, 3$. *The corresponding moduli spaces form a chain of sub-varieties*

$$\mathcal{M}_{L'''} \subset \mathcal{M}_{L''} \subset \mathcal{M}_{L'} \subset \mathcal{M}_{L} ,$$

each of them admitting an appropriate version of the global Torelli theorem, with the Picard-Fuchs systems determined in Propositions 4.1.36, 4.2.40, 4.2.47, 4.2.51.

Proof. The theorem collect statements from Theorems 4.1.34, 4.2.39, 4.2.46, 4.2.50 and their respective proofs, as wells as from Propositions 4.1.36, 4.2.40, 4.2.47, 4.2.51. □

4.3 GKZ Description of the Univariate Mirror Families

In this section we will show that the generalized functional invariant of the mixed-twist construction captures all key features of the one-parameter mirror families for the Fermat pencils. In particular, we will show that the mixed-twist construction allows us to obtain a non-resonant GKZ system for which a basis of solutions in the form of absolutely convergent Mellin-Barnes integrals exists whose monodromy is computed explicitly.

4.3.1 The Mirror Families

Let us briefly review the construction of the mirror family for the deformed Fermat hypersurface. Let $\mathbb{P}^n(n+1)$ be the general family of hypersurfaces of degree $(n+1)$ in $\mathbb{P}^n$. The general member of $\mathbb{P}^n(n+1)$ is a smooth hypersurface Calabi-Yau $(n-1)$-fold. Let $[X_0 : \cdots : X_n]$ be the homogeneous coordinates on $\mathbb{P}^n$. The following family

$$X_0^{n+1} + \cdots + X_n^{n+1} + n\lambda X_0 X_1 \cdots X_n = 0 \tag{4.3.23}$$

determines a one-parameter single-monomial deformation $X_\lambda^{(n-1)}$ of the classical Fermat hypersurface in $\mathbb{P}^n(n+1)$. Cox and Katz determined [30] what deformations of Calabi-Yau hypersurfaces remain Calabi-Yau. For example, for $n = 5$ there are 101 parameters for the complex structure, which determine the coefficients of additional terms in the quintic polynomials. Starting with a Fermat-type hypersurfaces V in $\mathbb{P}^n$, Yui [145, 144, 137] and Goto [52] classified all discrete symmetries G such that the quotients V/G are singular Calabi-Yau varieties with at worst Abelian quotient singularities. A theorem by Greene, Roan, & Yau [53] guarantees that there are crepant resolutions of V/G. This is known as the *Greene-Plesser orbifolding* construction.

For the family (4.3.23), the discrete group of symmetries needed for the Greene-Plesser orbifolding is readily constructed: it is generated by the action $(X_0, X_j) \mapsto (\zeta_{n+1}^n X_0, \zeta_{n+1} X_j)$ for $1 \leq j \leq n$ and the root of unity $\zeta_{n+1} = \exp\left(\frac{2\pi i}{n+1}\right)$. Since the product of all generators multiplies the homogeneous coordinates by a common phase, the symmetry group is $G_{n-1} = (\mathbb{Z}/(n+1)\,\mathbb{Z})^{n-1}$. One checks that the affine variables

$$t = \frac{(-1)^{n+1}}{\lambda^{n+1}}, \quad x_1 = \frac{X_1^n}{(n+1)\,X_0 \cdot X_2 \cdots X_n\,\lambda}, \quad x_2 = \frac{X_2^n}{(n+1)\,X_0 \cdot X_1 \cdot X_3 \cdots X_n\,\lambda},$$

and similar equations hold for $x_2, \ldots, x_n$, are invariant under the action of G_{n-1}, hence coordinates on the quotient $X_\lambda^{(n-1)}/G_{n-1}$. A family of special hypersurfaces $Y_t^{(n-1)}$ is then defined by the remaining relation between $x_1, \ldots, x_n$, namely the equation

$$f_n(x_1, \ldots, x_n, t) = x_1 \cdots x_n \Big(x_1 + \cdots + x_n + 1\Big) + \frac{(-1)^{n+1}\,t}{(n+1)^{n+1}} = 0\,. \tag{4.3.24}$$

Moreover, it was proved by Batyrev & Borisov in [5] that the family of special Calabi-Yau hypersurfaces $Y_t^{(n-1)}$ of degree $(n+1)$ in $\mathbb{P}^n$ given by Equation (4.3.24) is in fact the *mirror family* of a general hypersurface $\mathbb{P}^n(n+1)$ of degree $(n+1)$ and co-dimension one in $\mathbb{P}^n$. For $n = 2, 3, 4$ the mirror family is a family of elliptic curves, K3 surfaces, and Calabi-Yau threefolds, respectively.

Each mirror family can be realized as a fibration of Calabi-Yau $(n-2)$-folds associated with a generalized functional invariant. The following was proved by Doran & Malmendier:

Proposition 4.3.54. *For $n \geq 2$ the family of hypersurfaces $Y_t^{(n-1)}$ in Equation (4.3.24) is a fibration over $\mathbb{P}^1$ by hypersurfaces $Y_{\tilde{t}}^{(n-2)}$ constructed as mixed-twist with the gen-*

eralized functional invariant $(1, n, 1)$.

Proof. For each $x_n \neq 0, -1$ substituting $\tilde{x}_i = x_i/(x_n+1)$ for $1 \leq i \leq n-1$ and $\tilde{t} = -n^n\, t/((n+1)^{n+1} x_n\, (x_n+1)^n)$ defines a fibration of the hypersurface (4.3.24) by $f_{n-1}(\tilde{x}_1, \ldots, \tilde{x}_{n-1}\tilde{t}) = 0$ since

$$f_n(x_1, \ldots, x_n, t) = x_n\, (x_n+1)^n\, f_{n-1}(\tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{t}\,) = 0\ . \tag{4.3.25}$$

This is the mixed-twist construction with generalized functional invariant $(1, n, 1)$. $\square$

4.3.2 GKZ data of the mirror family

In the GKZ formalism, the construction of the family $Y_t^{(n-1)}$ is described as follows: from the homogeneous degrees of the defining Equation (4.3.23) and the coordinates of the ambient projective space for the family $X_\lambda^{(n-1)}$ we obtain the lattice $\mathbb{L}' = \mathbb{Z}(-(n+1), 1, 1, \ldots, 1) \subset \mathbb{Z}^{n+2}$. We define a matrix $\mathsf{A}' \in \mathrm{Mat}(n+1, n+2; \mathbb{Z})$ as a matrix row equivalent to the $(n+1)\times(n+2)$ matrix with columns of the $(n+1)\times(n+1)$ identity matrix as the first $(n+1)$ columns, followed by the generator of $\mathbb{L}'$:

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & & (n+1) \\ 0 & 1 & 0 & \ldots & & -1 \\ 0 & \ddots & \ddots & \ddots & & -1 \\ \vdots & & & & & \vdots \\ 0 & 0 & \ldots & 0 & 1 & -1 \end{pmatrix} \sim \quad \mathsf{A}' = \begin{pmatrix} 1 & 1 & 1 & \ldots & & 1 \\ 0 & 1 & 0 & \ldots & & -1 \\ 0 & \ddots & \ddots & \ddots & & -1 \\ \vdots & & & & & \vdots \\ 0 & 0 & \ldots & 0 & 1 & -1 \end{pmatrix}, \tag{4.3.26}$$

and let $\mathcal{A}' = \{\vec{a}'_1, \ldots, \vec{a}'_{n+2}\}$ denote the columns of the right-handed matrix obtained by a basis transformation in $\mathbb{Z}^{n+1}$ from the matrix on the left hand side. The finite subset $\mathcal{A}' \subset \mathbb{Z}^{n+1}$ generates $\mathbb{Z}^{n+1}$ as an abelian group and can be equipped with a group homomorphism $h' : \mathbb{Z}^{n+1} \to \mathbb{Z}$, in this case the projection onto the first coordinate, such that $h'(\mathcal{A}') = 1$. This means that $\mathcal{A}'$ lies in an affine hyperplane in $\mathbb{Z}^{n+1}$. The lattice of linear relations between the vectors in $\mathcal{A}'$ is easily checked to be precisely $\mathbb{L}' = \mathbb{Z}(-(n+1), 1, 1, \ldots, 1) \subset \mathbb{Z}^{n+2}$. From A' we form the Laurent polynomial

$$
\begin{aligned}
P_{\mathsf{A}'}(z_1, \ldots, z_{n+1}) &= \sum_{\vec{a}' \in \mathcal{A}'} c_{\vec{a}}\, z_1^{a_1} \cdot z_2^{a_2} \cdots z_{n+1}^{a_{n+1}} \\
&= c_1\, z_1 + c_2\, z_1\, z_2 + c_3\, z_1\, z_3 + \cdots + c_{n+2} z_1\, z_2^{-1} \cdots z_{n+1}^{-1} \,,
\end{aligned}
$$

and observe that the dehomogenized Laurent polynomial yields

$$
\begin{aligned}
&\frac{x_1 \cdots x_n}{c_1}\; P_{\mathsf{A}'}\left(1, \frac{c_1 x_1}{c_2}, \frac{c_1 x_2}{c_3}, \ldots, \frac{c_1 x_n}{c_{n+1}}\right) \\
&\qquad\qquad = f_n\left(x_1, \ldots, x_n, t = (-1)^{n+1} \frac{(n+1)^{n+1}\, c_2 \cdots c_{n+2}}{c_1^{n+1}}\right) \,.
\end{aligned}
$$

In the context of toric geometry, this is interpreted as follows: a secondary fan is constructed from the data $(\mathcal{A}', \mathbb{L}')$. This secondary fan is a complete fan of strongly convex polyhedral cones in $\mathbb{L}'^{\vee}_{\mathbb{R}} = \mathrm{Hom}(\mathbb{L}', \mathbb{R})$ which are generated by vectors in the lattice $\mathbb{L}'^{\vee}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{L}', \mathbb{Z})$. As the coefficients $c_1, \ldots, c_{n+2}$ – or effectively t – vary, the zero locus of $P_{\mathcal{A}'}$ sweeps out the family of hypersurfaces $Y_t^{(n-1)}$ in $(\mathbb{C}^*)^{n+1}/\mathbb{C}^* = (\mathbb{C}^*)^n$. Both $(\mathbb{C}^*)^n$ and the hypersurfaces can then be compactified. The members of the family $Y_t^{(n-1)}$ are Calabi-Yau varieties since the original Calabi-Yau varieties had codimension one in the ambient space; see Batyrev & van Straten [6].

4.3.3 Recurrence relation between holomorphic periods

We now describe the construction of the period integrals. A result of Doran & Malmendier – referenced below as Lemma 4.3.55 – shows that the fibration on $Y_t^{(n-1)} \to \mathbb{P}^1$ by Calabi-Yau hypersurfaces $Y_{\tilde{t}}^{(n-2)}$ allows for a recursive construction of the period integrals for $Y_t^{(n-1)}$ by integrating a twisted period integral over a transcendental homology cycle. It turns out that the result can be obtained explicitly as the Hadamard product of certain generalized hypergeometric functions. Recall that the Hadamard of two analytic functions $f(t) = \sum_{k\geq 0} f_k t^k$, $g(t) = \sum_{k\geq 0} g_k t^k$ is the analytic function $f \star g$ given by

$$(f \star g)(t) = \sum_{k=0}^{\infty} f_k g_k t^k.$$

The unique holomorphic $(n-1)$-form on $Y_t^{(n-1)}$ is given by

$$\eta_t^{(n-1)} = \frac{dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n}{\partial_{x_1} f_n(x_1, \ldots, x_n, t)} \,. \tag{4.3.27}$$

The formula is obtained from the Griffiths-Dwork technique (see, for example, Morrison [106]). One then defines an $(n-1)$-cycle Σ_{n-1} on $Y_t^{(n-1)}$ by requiring that the period integral of $\eta_t^{(n-1)}$ over Σ_{n-1} corresponds by a residue computation in x_1 to the integral over the middle dimensional torus cycle $T_{n-1}(\vec{\mathbf{r}}) := S^1_{r_1} \times \cdots \times S^1_{r_{n-1}} \in H_{n-1}(Y_t^{n-1}, \mathbb{Q})$ with $S^1_{r_j} = \{|x| = r_j\} \subset \mathbb{C}$ and $\vec{\mathbf{r}}_{n-1} = (r_1, \ldots, r_{n-1}) \in \mathbb{R}_+^{n-1}$, i.e.,

$$\begin{aligned}
&\underbrace{\int \!\!\cdots\!\! \int}_{\Sigma_{n-1}} \frac{dx_2 \wedge \cdots \wedge dx_n}{\partial_{x_1} f_n(x_1, \ldots, x_n, t)} \\
&\qquad = \frac{c_1}{2\pi i} \underbrace{\int \!\!\cdots\!\! \int}_{T_{n-1}(r)} P_{\mathcal{A}} \left(1, \frac{c_1 x_1}{c_2}, \frac{c_1 x_2}{c_3}, \ldots, \frac{c_1 x_n}{c_{n+1}}\right)^{-1} \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_n}{x_n}.
\end{aligned} \tag{4.3.28}$$

The right hand side of Equation (4.3.28) is a resonant $\mathcal{A}$-hypergeometric integral in the sense of [51, Thm. 2.7] derived from the data $(\mathcal{A}', \mathbb{L}')$ and

$$\vec{\alpha}' = \langle \alpha'_1, -\beta'_1 - 1, \ldots, -\beta'_n - 1 \rangle^t = \langle -1, 0, \ldots, 0 \rangle^t = \sum_{i=1}^{n+2} \gamma'_i \, \vec{a}'_i \tag{4.3.29}$$

with $\boldsymbol{\gamma}'_0 = (\gamma'_1, \ldots, \gamma'_{n+2}) = (-1, 0, \ldots, 0)$. We will denote the period integral by $\omega_{n-1}(t) = \oint_{\Sigma_{n-1}} \eta_t^{(n-1)}$.

We recall the following result, which connects the GKZ data above to the iterative twist construction of Doran & Malmendier:

Proposition 4.3.55. *[40, Prop. 7.2] For $n \geq 1$ and $|t| \leq 1$, there is a family of transcendental $(n-1)$-cycles Σ_{n-1} on $Y_t^{(n-1)}$ such that*

$$\omega_{n-1}(t) = \oint_{\Sigma_{n-1}} \eta_t^{(n-1)} = (2\pi i)^{n-1} \, {}_nF_{n-1}\left(\left. \begin{array}{ccc} \frac{1}{n+1} & \cdots & \frac{n}{n+1} \\ 1 & \ldots & 1 \end{array} \right| t \right) . \tag{4.3.30}$$

The iterative structure in Proposition 4.3.54 induces the iterative period relation

$$\omega_{n-1}(t) = (2\pi i) \, {}_nF_{n-1}\left(\left. \begin{array}{ccc} \frac{1}{n+1} & \cdots & \frac{n}{n+1} \\ \frac{1}{n} & \ldots & \frac{n-1}{n} \end{array} \right| t \right) \star \omega_{n-2}(t) \quad \text{for } n \geq 2. \tag{4.3.31}$$

Here, the symbol $\star$ denotes the Hadamard product. The cycles Σ_{n-1} are determined by $\tilde{T}_{n-1}(\vec{\mathbf{r}}_{n-1}) := \frac{n}{n+1} \cdot \left(T_{n-2}(\vec{\mathbf{r}}_{n-2}) \times S^1_{r_{n-2}}\right)$ as in (4.3.28), with $r_j = 1 - \frac{j}{j+1}$, and $\frac{n}{n+1} \cdot \left(T_{n-2}(\vec{\mathbf{r}}_{n-2}) \times S^1_{r_{n-1}}\right)$ indicates that coordinates are scaled by a factor of $\frac{n}{n+1}$.

Hence, the iterative structure in Proposition 4.3.54, namely, the generalized functional invariant $(1, n, 1)$, determines the iterative period relations of the mirror family and the corresponding $\mathcal{A}$-hypergeometric data $(\mathcal{A}', \mathbb{L}', \boldsymbol{\gamma}_0')$ in the GKZ formalism.

The mirror family of K3 surfaces

Narumiya and Shiga [109] showed that the mirror family of K3 surfaces in Equation (4.3.24) with $n = 3$ is birationally equivalent to a family of Weierstrass model. In fact, if we set

$$\begin{aligned} x_1 &= -\frac{(4\,u^2\lambda^2 + 3\,X\lambda^2 + u^3 + u)\,(4\,u^2\lambda^2 + 3\,X\lambda^2 + u^3 - 2\,u)}{6\lambda^2 u\,(16\,u^3\lambda^2 - 3\,iY\lambda^2 + 12\,Xu\lambda^2 + 4\,u^4 + 4\,u^2)}\,, \\ x_2 &= -\frac{16\,u^3\lambda^2 - 3\,iY\lambda^2 + 12\,Xu\lambda^2 + 4\,u^4 + 4\,u^2}{8u\,(4\,u^2\lambda^2 + 3\,X\lambda^2 + u^3 - 2\,u)}\,, \\ x_3 &= \frac{u^2\,(4\,u^2\lambda^2 + 3\,X\lambda^2 + u^3 - 2\,u)}{2\lambda^2\,(16\,u^3\lambda^2 - 3\,iY\lambda^2 + 12\,Xu\lambda^2 + 4\,u^4 + 4\,u^2)}\,, \end{aligned} \tag{4.3.32}$$

in Equation (4.3.24), we obtain the Weierstrass equation

$$Y^2 = 4X^3 - g_2(u)\,X - g_3(u)\,, \tag{4.3.33}$$

with coefficients

$$\begin{aligned} g_2 &= \frac{4}{3\,\lambda^4}\,u^2\,\left(u^4 + 8\lambda^2 u^3 + (4\lambda^2-1)(4\lambda^2+1)u^2 + 8\lambda^2 u + 1\right)\,, \\ g_3 &= \frac{4}{27\,\lambda^6}\,u^3\,\left(u^2 + 4\lambda^2 u + 1\right)\left(2u^4 + 16\lambda^2 u^3 + (32\lambda^4 - 5)u^2 + 16\lambda^2 u + 2\right)\,. \end{aligned} \tag{4.3.34}$$

For generic parameter λ, Equation (4.3.33) defines a Jacobian elliptic fibration with the singular fibers $2I_4^* + 4I_1$ and the Mordell-Weil group $\mathbb{Z}/2\mathbb{Z} \oplus \langle 1 \rangle$, generated by a two-torsion section and an infinite-order section of height pairing one; see [109, 15]. Using the Jacobian elliptic fibration one has the following:

Proposition 4.3.56 ([109])**.** *The family in Equation (4.3.33) forms the moduli space $\mathcal{M}_{M_2}$ of M_2-polarized K3 surfaces with $M_2 \cong H \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -4 \rangle$.*

Proposition 4.3.56 shows why the family (4.3.33) can be called the mirror family of K3 surfaces. Dolgachev's mirror symmetry for K3 surfaces identifies marked deformations of K3 surfaces with given Picard lattice N with a complexified Kähler cone $K(M) = \{x + iy : \langle y, y \rangle > 0,\ x, y \in M_{\mathbb{R}}\}$ for some mirror lattice M; see [34]. In the case of the rank-one lattice $N_k = \langle 2k \rangle$, one can construct the mirror lattice explicitly by taking a copy of H out of the orthogonal complement $N_k^\perp$ in the K3 lattice Λ_{K3}. It turns out that the mirror lattice $M_k \cong H \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -2k \rangle$ is unique if k has no square divisor. In our situation, the general quartic hypersurfaces in Equation (4.3.23) with $n = 3$ have a Néron-Severi lattice isomorphic to $N_2 = \langle 4 \rangle$, and the mirror family in Equation (4.3.33) is polarized by the lattice M_2 such that $N_2^\perp \cong H \oplus M_2$.

It turns out that the holomorphic solution of the Picard-Fuchs equation governing the family of K3 surfaces in Equation (4.3.33) equals

$$\omega_2 = \left({}_2F_1\left(\left. \begin{array}{c} \frac{1}{8}, \frac{3}{8} \\ 1 \end{array} \right| \frac{1}{\lambda^4} \right) \right)^2 = {}_3F_2\left(\left. \begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{array} \right| t \right) . \tag{4.3.35}$$

The first equality was proved by Narumiya and Shiga, and the second equality is Clausen's formula, found by Thomas Clausen, expressing the square of a Gaussian hypergeometric series as a generalized hypergeometric series.

4.3.4 Monodromy of the mirror family

We will now show how the monodromy representations for the mirror families for general n are computed. The Picard-Fuchs operators of the periods given in Proposition 4.3.55 are the associated rank n-hypergeometric differential operators annihilating ${}_nF_{n-1}$. But yet more is afforded by pursuing the GKZ description of the period integrals. In fact, the Euler-integral formula for the hypergeometric functions ${}_nF_{n-1}$ generates a second set of *non-resonant* GKZ data $(\mathcal{A}, \mathbb{L}, \boldsymbol{\gamma}_0)$ from the *resonant* GKZ data $(\mathcal{A}', \mathbb{L}', \boldsymbol{\gamma}_0')$ by integration. The GKZ data $(\mathcal{A}, \mathbb{L}, \boldsymbol{\gamma}_0)$ determines local Frobenius bases of solutions around $t = 0$ and $t = \infty$. Their Mellin-Barnes integral representation determines the transition matrix between them by analytic continuation.

We will always assume that we have n rational parameters, namely $\rho_1, \dots, \rho_n \in (0,1) \cap \mathbb{Q}$, and consider the generalized hypergeometric function

$${}_nF_{n-1}\left(\left. \begin{array}{ccc} \rho_1 & \cdots & \rho_n \\ 1 & \dots & 1 \end{array} \right| t \right) ,$$

which include all periods from Propositions 4.3.55 and 4.2.51. The *Euler-integral* *formula* then specializes to the identity

$$\left[\prod_{i=1}^{n-1} \Gamma(\rho_i)\,\Gamma(1-\rho_i)\right] {}_nF_{n-1}\left(\begin{matrix}\rho_1 & \dots & \rho_n\\ 1 & \dots & 1\end{matrix}\,\middle|\, t\right)$$
$$= \left[\prod_{i=1}^{n-1} \int_0^1 \frac{dz_i}{z_i^{1-\rho_i}(1-z_i)^{\rho_i}}\right] (1-t\,z_1\cdots z_{n-1})^{-\rho_n}. \quad (4.3.36)$$

The rank-n hypergeometric differential equation satisfied by ${}_nF_{n-1}$ is given by

$$\Big[\theta^n - t\,(\theta+\rho_1)\cdots(\theta+\rho_n)\Big]\, F(t) = 0 \quad (4.3.37)$$

with $\theta = t\frac{d}{dt}$, and it has the Riemann symbol

$$\mathcal{P}\left(\begin{array}{ccc|c} 0 & 1 & \infty & \\ \hline 0 & 0 & \rho_1 & \\ 0 & 1 & \rho_2 & \\ \vdots & \vdots & \vdots & t \\ 0 & n-2 & \rho_{n-1} & \\ 0 & n-1-\sum_{j=1}^n \rho_j & \rho_n & \end{array}\right). \quad (4.3.38)$$

In particular, we read from the Riemann symbol that for each $n \geq 1$, the periods from Proposition 4.3.55 have a point of maximally unipotent monodromy at $t = 0$. This is well known to be consistent with basic considerations for mirror symmetry [107].

From the Euler-integral (4.3.36), using the GKZ formalism, we immediately read off the left hand side matrix, and convert to the A-matrix $\mathsf{A} \in \mathrm{Mat}(2n-1, 2n; \mathbb{Z})$

given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 1 & \dots & 0 & 1 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 1 \end{array}\right) \quad \sim \quad \mathsf{A} = \left(\begin{array}{cccc|c|cccc|c} 1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 1 & 0 \\ \hline 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ \hline 0 & 0 & \dots & 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & & 0 & 1 & 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 1 & 0 \end{array}\right), \qquad (4.3.39)$$

using elementary row operations, as in §4.3.2. Let $\mathcal{A} = \{\vec{a}_1, \dots, \vec{a}_{2n}\}$ denote the columns of the matrix A. The entries for the matrix on the left hand side of (4.3.39) are determined as follows: the first n entries in each column label which of the n terms $(1-z_i)^{\rho_i}$ or $(1-t\,z_1\cdots z_{n-1})^{-\rho_n}$ in the integrand of the Euler-integral (4.3.36) is specified. For each term, two column vectors are needed and the entries in rows $n+1,\dots,2n-1$ label the exponents of variables z_i appearing. For example, the last two columns determine the term $(1-t\,z_1\cdots z_{n-1})^{-\rho_n}$. The finite subset $\mathcal{A} \subset \mathbb{Z}^{2n-1}$ generates $\mathbb{Z}^{2n-1}$ as an abelian group and is equipped with a group homomorphism $h : \mathbb{Z}^{2n-1} \to \mathbb{Z}$, in this case the sum of the first n coordinates such that $h(\mathcal{A}) = 1$. The lattice of linear relations between the vectors in $\mathcal{A}$ is easily checked to be $\mathbb{L} = \mathbb{Z}(1,\dots,1,-1,\dots,-1) \subset \mathbb{Z}^{2n}$. The toric data $(\mathsf{A},\mathbb{L})$ has an associated GKZ system of differential equations which is equivalent to the differential equation (4.3.37). Equivalently, the right hand side of Equation (4.3.36) is the $\mathcal{A}$-hypergeometric integral in the sense of [51, Thm. 2.7] derived from the data $(\mathcal{A},\mathbb{L})$ and the additional vector

$$\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_{n-1}, -\beta_1 - 1, \ldots, -\beta_n - 1 \rangle^t$$
$$= \langle -\rho_1, \ldots, -\rho_n, -\rho_1, \ldots, -\rho_{n-1} \rangle^t = \sum_{i=1}^{2n} \gamma_i \, \vec{a}_i,$$

where we have set $\boldsymbol{\gamma}_0 = (\gamma_1, \ldots, \gamma_{2n}) = (0, \ldots, 0, -\rho_1, \ldots, -\rho_n) \subset \mathbb{Z}^{2n}$. We always have the freedom to shift $\boldsymbol{\gamma}_0$ by elements in $\mathbb{L} \otimes \mathbb{R}$ while leaving $\vec{\alpha}$ and any $\mathcal{A}$-hypergeometric integral unchanged. Thus we have the following:

Proposition 4.3.57. *The GKZ data* $(\mathcal{A}, \mathbb{L}, \boldsymbol{\gamma}_0)$ *is non-resonant.*

Proof. We observe that $\alpha_i, \beta_j \notin \mathbb{Z}$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$ and $\sum_i \alpha_i + \sum_j \beta_j \equiv -\rho_n \mod 1 \notin \mathbb{Z}$. It was proved in [51, Ex. 2.17] that this is equivalent to the non-resonance of the GKZ system. □

Construction of convergent period integrals

In this section, we show how from the toric data of the GKZ system convergent period integrals can be constructed. We are following the standard notation for GKZ systems; see, for example, Beukers [10].

Let us define the B-matrix of the lattice relations $\mathbb{L}$ for $\mathcal{A}$ as the matrix containing its integral generating set as the rows. Since the rank of $\mathbb{L}$ is 1, we simply have $\mathsf{B} = (1, \ldots, 1, -1, \ldots, -1) \in \mathrm{Mat}(1, 2n; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{2n}, \mathbb{Z})$. Of course, the B-matrix then satisfies $\mathsf{A} \cdot \mathsf{B}^t = 0$, as this is the defining property of the lattice $\mathbb{L}$. The space $\mathbb{L} \otimes \mathbb{R} \subset \mathbb{R}^{2n}$ is clearly a line, and is parameterized by the tuple $(s, \ldots, s, -s, \ldots, -s) \in \mathbb{R}^{2n}$ with $s \in \mathbb{R}$. To be used later in this subsection, the polytope $\Delta_{\mathcal{A}}$ defined as convex hull of the vectors contained in $\mathcal{A}$ is the primary polytope associated with $\mathcal{A}$. Also for later, we may also write $\mathsf{B} = \sum b_i \hat{e}_i$ in terms of the standard basis $\{\hat{e}_i\}_{i=1}^{2n} \subset \mathbb{Z}^{2n}$.

We can obtain a short exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{2n} \longrightarrow \mathbb{Z}^{2n-1} \to 0$$

by mapping each vector $\boldsymbol{\ell} = \sum l_i \hat{e}_i \in \mathbb{Z}^{2n}$ to the vector $\sum l_i \, \vec{a}_i \in \mathbb{Z}^{2n-1}$. As the linear relations between vectors in $\mathcal{A}$ are given by the lattice $\mathbb{L}$, this sequence is exact. The corresponding dual short exact sequence (over $\mathbb{R}$) is given by

$$0 \longrightarrow \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}^{2n} \xrightarrow{\pi} \mathbb{L}^{\vee}_{\mathbb{R}} \cong \mathbb{R} \longrightarrow 0,$$

with $\pi(u_1, \ldots, u_{2n}) = u_1 + \cdots + u_n - u_{n+1} - \cdots - u_{2n}$. Restricting π to the positive orthant in $\mathbb{R}^{2n}$ and calling it $\hat{\pi}$, we observe that for each $s \in \mathbb{R}$ the set $\hat{\pi}^{-1}(s)$ is a convex polyhedron. For $s \in \mathbb{L}^{\vee}_{\mathbb{R}}$, there are two maximal cones $\mathcal{C}_+$ and $\mathcal{C}_-$ in the secondary fan of $\mathcal{A}$ for positive and negative real value s, respectively. The lists of vanishing components for the vertex vectors in each $\hat{\pi}^{-1}(s)$ are given by

$$\begin{aligned} T_{\mathcal{C}_+} &= \bigcup_{k=1}^{n} \Big\{ \underbrace{\{1, \ldots, \widehat{k}, \ldots, n, n+1, \ldots 2n\}}_{=: I_k} \Big\}, \\ T_{\mathcal{C}_-} &= \bigcup_{k=1}^{n} \Big\{ \underbrace{\{1, \ldots, n, n+1, \widehat{k+n}, \ldots \ldots 2n\}}_{=: I_{k+n}} \Big\}. \end{aligned}$$

The symbol $\widehat{k}$ indicates that the entry k has been suppressed. For each member I of $T_{\mathcal{C}_\pm}$, we define $\boldsymbol{\gamma}^I = \boldsymbol{\gamma}_0 - \mu^I \mathsf{B}$ such that $\boldsymbol{\gamma}^I_i = 0$ for $i \notin I$. We then have

$$\boldsymbol{\gamma}^I = \begin{cases} \boldsymbol{\gamma}_0 & \text{for } I \in T_{\mathcal{C}_+}, \quad \mu^I = 0, \\ (-\rho_k, \ldots, -\rho_k, \rho_k - \rho_1, \ldots, 0, \ldots, \rho_k - \rho_n) & \text{for } I = I_{n+k} \in T_{\mathcal{C}_-}, \quad \mu^{I_{n+k}} = \rho_k. \end{cases}$$

Then for $I_k \in T_{\mathcal{C}_\pm}$ we denote the *convergence direction* by

$$\boldsymbol{\nu}^{I_k} = (\nu_1, \ldots, \nu_{2p}) = (\delta_i^k)_{i=1}^{2p} \in \mathbb{L} \otimes \mathbb{R}, \tag{4.3.40}$$

where δ_i^k is the Kronecker delta, such that $\hat{\pi}(\boldsymbol{\nu}^{I_k}) = \pm 1$.

Using the B-matrix, one defines the *zonotope*

$$\mathsf{Z}_\mathsf{B} = \left\{ \frac{1}{4} \sum_{i=1}^{2n} \mu_i \, b_i \,\middle|\, \mu_i \in (-1,1) \right\} = \left(-\frac{n}{2}, \frac{n}{2} \right) \subset \mathbb{L}_\mathbb{R}^\vee \cong \mathbb{R}.$$

The zonotope contains crucial data about the nature and form of the solutions to the GKZ system above. A crucial result of Beukers [10, Cor. 4.2] can then be phrased as follows:

Proposition 4.3.58. *[10, Cor. 4.2] Let $\boldsymbol{u}, \boldsymbol{\tau}$ be the vector with $\boldsymbol{u} = (u_1, \ldots, u_{2n})$, $u_j = |u_j| \exp(2\pi i \tau_j)$, and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{2n})$. For any $\boldsymbol{u}$ with $\boldsymbol{\tau}$ such that $\sum b_i \tau_i \in \mathsf{Z}_\mathcal{B}$ and any $\boldsymbol{\gamma}$ equivalent to $\boldsymbol{\gamma}_0$ up to elements in $\mathbb{L} \otimes \mathbb{R}$ with $\gamma_{n+i} < \sigma < -\gamma_i$ for all $i = 1, \ldots, n$, the Mellin-Barnes integral given by*

$$\mathsf{M}_{\boldsymbol{\tau}}(u_1, \ldots, u_{2n}) = \int_{\sigma + i\,\mathbb{R}} \left[\prod_{i=1}^{2n} \Gamma(-\gamma_i - b_i s) \; u_i^{\gamma_i + b_i s} \right] ds \,, \tag{4.3.41}$$

is absolutely convergent and satisfies the GKZ differential system for $(\mathsf{A}, \mathbb{L})$.

A toric variety $\mathcal{V}_\mathcal{A}$ can be associated with the secondary fan by gluing together certain affine schemes, one scheme for every maximal cone in the secondary fan. Details can be found in [132]. In the situation of the hypergeometric differential equation (4.3.37), the secondary fan has two maximal cones $\mathcal{C}_+$ and $\mathcal{C}_-$, and one can easily see that the toric variety $\mathcal{V}_\mathcal{A}$ is the projective line $\mathcal{V}_\mathcal{A} = \mathbb{P}^1$ which is the the domain of definition for the variable t in Equation (4.3.36). Each member in the list for a maximal cone contains $2n - 1$ integers and define a subdivision of the primary polytope $\Delta_{\mathcal{A}}$ by polytopes generated by the subdivision, called regular triangulations. In our case, these regular triangulations are unimodular, i.e.,

$$\text{for all } I_k \in T_{\mathcal{C}_\pm} : \quad \left| \det \left(\vec{a}_i\right)_{i \in I_k} \right| = \left| b_k \right| = 1 \; .$$

Given $\mathcal{A}$ and its secondary fan, we define a ring $\mathcal{R}_{\mathcal{A}}$ by dividing the free polynomial ring in $2n$ variables by the ideal $\mathcal{I}_{\mathcal{A}}$ generated by the linear relations of $\mathcal{A}$ and the ideal $\mathcal{I}_{\mathcal{C}_\pm}$ generated by the regular triangulations. In our situation, we obtain $\mathcal{R}_{\mathcal{A}}$ from the list of generators given by

$$\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_{2n}) = \epsilon \,(1, \ldots, 1, -1, \ldots, -1) \in \mathcal{R}_{\mathcal{A}}$$

with relation $\epsilon^n = 0$, i.e., $\mathcal{R}_{\mathcal{A}} = \mathbb{Z}[\epsilon]/(\epsilon^n)$ is a free $\mathbb{Z}$-module of rank n. Thus, we have the following:

Corollary 4.3.59. *A solution for the hypergeometric differential equation (4.3.37) is given by restricting to $u_2 = \cdots = u_{2n} = 1$ and $u_1 = (-1)^n t$ in Equation (4.3.41).*

In the case of the hypergeometric differential equation (4.3.37), it follows crucially from Beukers [10, Prop. 4.6] that there is a basis of Mellin-Barnes integrals since the zonotope $\mathsf{Z}_{\mathcal{B}}$ contains n distinct points $\{-\frac{n-1}{2} + k\}_{k=0}^{n-1}$ whose coordinates differ by integers.

A basis of solutions around zero

Using the toric data, we may now derive a local basis of solutions of the differential equation (4.3.37) around the point $t = 0$ [132]. For the convergence direction $\boldsymbol{\nu}^{I_1}$ in

$T_{\mathcal{C}_+}$, the Γ-series is a series solutions of the GKZ system for $(\mathbb{L}, \boldsymbol{\gamma}_0)$ and given by

$$\Phi_{\mathbb{L},\boldsymbol{\gamma}_0}(u_1, \ldots, u_{2n}) = \sum_{\boldsymbol{\ell} \in \mathbb{L}} \frac{u_1^{\gamma_1+\ell_1} \cdots u_{2n}^{\gamma_{2n}+\ell_{2n}}}{\Gamma(\gamma_1 + \ell_1 + 1) \cdots \Gamma(\gamma_{2n} + \ell_{2n} + 1)} \,. \tag{4.3.42}$$

We have the following:

Lemma 4.3.60. *For the convergence direction $\boldsymbol{\nu}^{I_1}$ in $T_{\mathcal{C}_+}$, the Γ-series for $(\mathbb{L}, \boldsymbol{\gamma}_0)$ equals*

$$\Phi_{\mathbb{L},\boldsymbol{\gamma}_0}(u_1, \ldots, u_{2n}) = \left[\prod_{i=1}^{n} \frac{1}{\Gamma(1-\rho_i)\, u_{n+i}^{\rho_i}} \right] \; {}_nF_{n-1}\left(\begin{matrix} \rho_1 \; \ldots \; \rho_n \\ 1 \; \ldots \; 1 \end{matrix} \middle| \, t \right) \tag{4.3.43}$$

for $t = (-1)^n u_1 \cdots u_n / (u_{n+1} \cdots u_{2n}) > 0$. Moreover, convergence of Equation (4.3.43) in the convergence direction $\boldsymbol{\nu}^{I_1} = (\nu_1, \ldots, \nu_{2p})$ is guaranteed for all $u_1, \ldots, u_{2n}$ with $|u_i| = t^{\nu_i}$ and $0 \leq t < 1$.

Proof. We observe that

$$\begin{aligned} \Phi_{\mathbb{L},\boldsymbol{\gamma}_0}(u_1, \ldots, u_{2n}) & \sum_{k \geq 0} \frac{u_1^k \cdots u_n^k \cdot u_{n+1}^{-\rho_1 - k} \cdots u_{2n}^{-\rho_n - k}}{(k!)^n \, \Gamma(-\rho_1 - k + 1) \cdots \Gamma(-\rho_n - k + 1)} \\ = \left[\prod_{i=1}^{n} \frac{1}{\Gamma(1-\rho_i)\, u_{n+i}^{\rho_i}} \right] & \sum_{k \geq 0} \frac{(\rho_1)_k \cdots (\rho_n)_k}{(k!)^n} \, t^k \,. \end{aligned} \tag{4.3.44}$$

The summation over $\mathbb{L}$ reduces to non-negative integers as the other terms vanish when $1/\Gamma(k+1) = 0$ for $k < 0$. Using the identities

$$(\rho)_k = (-1)^k \frac{\Gamma(1-\rho)}{\Gamma(1-k-\rho)}, \quad \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin{(\pi z)}} \,, \tag{4.3.45}$$

we obtain Equation (4.3.43). Equation (4.3.42) shows that restricting the variables $u_2 = \cdots = u_{2n} = 1$ to a base point, the convergence of the Γ-series $\Phi_{\mathbb{L},\boldsymbol{\gamma}_0}((-1)^n t, 1 \ldots, 1)$ is guaranteed for $|u_1| = t$ with t sufficiently small. $\square$

We obtain the same Γ-series for all convergence directions $\boldsymbol{\nu}^{I_r}$ with $1 \leq r \leq n$ in $T_{\mathcal{C}_+}$. This is due to the fact that in the Riemann symbol (4.3.38) at $t = 0$ the critical exponent 0 has multiplicity n.

However, from the maximal cone $\mathcal{C}_+$ of the secondary fan of $\mathcal{A}$, we can still construct a local basis of solutions of the GKZ system around $t = 0$ by expanding the twisted power series $\Phi_{\mathbb{L},\boldsymbol{\gamma}_0+\boldsymbol{\epsilon}}(u_1, \ldots, u_{2n})$ over $\mathcal{R}_{\mathcal{A}}$; see [132]. Similarly, a twisted hypergeometric series can be introduced, for example, by defining the following renormalized generating function:

$$f(\epsilon, t) = t^{\epsilon}\, {}_nF^{(\epsilon)}_{n-1}\left(\begin{matrix} \rho_1 & \cdots & \rho_n \\ 1 & \ldots & 1 \end{matrix} \,\middle|\, t \right) = \sum_{k \geq 0} \frac{(\rho_1 + \epsilon)_k \cdots (\rho_n + \epsilon)_k}{(1 + \epsilon)^n_k}\, t^{k+\epsilon}. \tag{4.3.46}$$

We have the following:

Lemma 4.3.61. *For $|t| < 1$, choosing the principal branch of $t^{\epsilon} = \exp(\epsilon \ln t)$ the twisted power series over $\mathcal{R}_{\mathcal{A}}$ is given by*

$$\Phi_{\mathbb{L},\boldsymbol{\gamma}_0+\boldsymbol{\epsilon}}(u_1, \ldots, u_{2n}) = \frac{e^{2\pi i \epsilon}}{\Gamma(1+\epsilon)^n} \left[\prod_{i=1}^{n} \frac{1}{\Gamma(1 - \rho_i - \epsilon)\, u_{n+i}^{\rho_i}} \right] \; t^{\epsilon}\, {}_nF^{(\epsilon)}_{n-1}\left(\begin{matrix} \rho_1 & \cdots & \rho_n \\ 1 & \ldots & 1 \end{matrix} \,\middle|\, t \right). \tag{4.3.47}$$

Proof. The proof uses $1/(1+\epsilon)^n_k = O(\epsilon^n) = 0$ for $k < 0$, where $(a)_k$ is the Pochammer symbol, because for $k \in \mathbb{Z}$ we have

$$\frac{1}{(1+\epsilon)_k} = \frac{\Gamma(1+\epsilon)}{\Gamma(k+1+\epsilon)} = \begin{cases} \epsilon(\epsilon - 1) \cdots (\epsilon + k + 1) & \text{if } k < 0, \\ 1 & \text{if } k = 0, \\ \dfrac{1}{(1+\epsilon)(2+\epsilon) \cdots (k+\epsilon)} & \text{if } k > 0. \end{cases}$$

$\square$

For $r = 0, \dots, n-1$, we also introduce the functions

$$y_r(t) = \frac{1}{r!}\,\frac{\partial^r}{\partial \epsilon^r}\bigg|_{\epsilon=0} {}_nF^{(\epsilon)}_{n-1}\left(\begin{matrix} \rho_1 & \dots & \rho_n \\ 1 & \dots & 1 \end{matrix}\bigg|\, t\right),\; y_0(t) = f(0,t) = {}_nF_{n-1}\left(\begin{matrix} \rho_1 & \dots & \rho_n \\ 1 & \dots & 1 \end{matrix}\bigg|\, t\right).$$

We have the following:

Lemma 4.3.62. *For $|t| < 1$, the following identity holds*

$$f(\epsilon, t) = \sum_{m=0}^{n-1} \Big(2\pi i \epsilon\Big)^m f_m(t) = \sum_{m=0}^{n-1} \Big(2\pi i \epsilon\Big)^m \sum_{r=0}^{m} \frac{1}{r!} \left(\frac{\ln t}{2\pi i}\right)^r \frac{y_{m-r}(t)}{(2\pi i)^{m-r}}, \tag{4.3.48}$$

where $f_m(t) = \frac{1}{(2\pi i)^m m!} \frac{\partial^m}{\partial \epsilon^m}|_{\epsilon=0} f(\epsilon, t)$ for $m = 0, \dots, n-1$.

As proved in [132], the functions $\{f_r\}_{r=0}^{n-1}$ form a local basis of solutions around $t = 0$, and the functions $y_r(t)$ with $r = 0, \dots n-1$ are holomorphic in a neighborhood of $t = 0$. The local monodromy group is generated by the cycle $(u_1, \dots, u_{2n}) = (R_1 \exp{(i\varphi)}, R_2, \dots, R_{2n})$ based at the point $(R_1, \dots, R_{2n})$ for $\varphi \in [0, 2\pi]$. Equivalently, we consider the local monodromy of the hypergeometric differential equation generated by $t = t_0 \exp{(i\varphi)}$ for $0 < t_0 < 1$ and $\varphi \in [0, 2\pi]$ (by setting $|u_2| = \dots = |u_{2n}| = 1$ and $|u_1| = t$). The monodromy of the functions $\{f_r\}_{r=0}^{n-1}$ can be read off Equation (4.3.48) immediately. We have the following:

Proposition 4.3.63. *The local monodromy of the basis $\boldsymbol{f}^t = \langle f_{n-1}, \dots, f_0\rangle^t$ of solutions to the differential equation (4.3.37) at $t = 0$ is given by*

$$\mathsf{m}_0 = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \dots & \frac{1}{(n-2)!} \\ 0 & 1 & 1 & \dots & \frac{1}{(n-3)!} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}. \tag{4.3.49}$$

Proof. Lemma 4.3.62 proves that

$$f_m(t) = \sum_{r=0}^{m} \frac{1}{r!} \left(\frac{\ln t}{2\pi i}\right)^r \frac{y_{m-r}(t)}{(2\pi i)^{m-r}}\,.$$

The functions $y_k(t)$ are invariant for $t = t_0 \exp{(i\varphi)}$ for $0 < t_0 < 1$ and $\varphi \to 2\pi$. The result then follows. $\square$

Corollary 4.3.64. *The monodromy matrix* m_0 *is maximally unipotent.*

A basis of solutions around infinity

We assume $0 < \rho_1 < \cdots < \rho_n < 1$. Using the toric data we can derive a local basis of solutions of the differential equation (4.3.37) around the point $t = \infty$. For the convergence direction $\boldsymbol{\nu}^{I_{n+r}}$ in $T_{\mathcal{C}_-}$, the Γ-series is a series solutions of the GKZ system for $(\mathbb{L}, \boldsymbol{\gamma}^{I_{n+r}})$ and given by

$$\Phi_{\mathbb{L},\boldsymbol{\gamma}^{I_{n+r}}}(u_1,\dots,u_{2n}) = \sum_{\boldsymbol{\ell}\in\mathbb{L}} \frac{u_1^{\gamma_1-\mu^{I_{r+n}}+\ell_1}\cdots u_{2n}^{\gamma_{2n}+\mu^{I_{r+n}}+\ell_{2n}}}{\Gamma(\gamma_1-\mu^{I_{r+n}}+\ell_1+1)\cdots\Gamma(\gamma_{2n}+\mu^{I_{r+n}}+\ell_{2n}+1)}\,. \tag{4.3.50}$$

We have the following:

Lemma 4.3.65. *For the convergence direction* $\boldsymbol{\nu}^{I_{n+r}}$ *in* $T_{\mathcal{C}_-}$ *Equation (4.3.50) is a series solution for* $(\mathbb{L}, \boldsymbol{\gamma}^{I_{n+r}})$*. The following identity holds*

$$\begin{aligned}\Phi_{\mathbb{L},\boldsymbol{\gamma}^{I_{n+r}}}(u_1,\dots,u_{2n}) &= \frac{e^{\pi i n\rho_r}}{\Gamma(1-\rho_r)^n}\left[\prod_{i=1}^{n}\frac{1}{\Gamma(1+\rho_r-\rho_i)\,u_{n+i}^{\rho_i}}\right]\\ &\times\, t^{-\rho_r}\;{}_nF_{n-1}\left(\begin{matrix}\rho_r & \cdots & \cdots & \rho_r\\ 1+\rho_r-\rho_1 & \dots\widehat{1}\dots & & 1+\rho_r-\rho_n\end{matrix}\,\middle|\,\frac{1}{t}\right)\end{aligned} \tag{4.3.51}$$

for $t = (-1)^n u_1\cdots u_n/(u_{n+1}\cdots u_{2n}) > 0$*. The symbol* $\widehat{1}$ *indicates that the entry* $1+\rho_r-\rho_i$ *for* $i = r$ *has been suppressed. In particular, restricting variables* $u_1 = \cdots =$

$\widehat{u_{n+r}} = \cdots = u_{2n} = 1$ *the convergence of the* Γ*-series* $\Phi_{\mathbb{L},\boldsymbol{\gamma}^{I_{n+r}}}(1,\ldots,(-1)^n/t,\ldots,1)$ *is guaranteed for* $t > 1$.

Proof. A direct computation shows that the Γ-series satisfies

$$\Phi_{\mathbb{L},\boldsymbol{\gamma}^{I_{n+r}}}(u_1,\ldots,u_{2n}) = \frac{e^{\pi i n \rho_r}}{\Gamma(1-\rho_r)^n}\left[\prod_{i=1}^{n}\frac{1}{\Gamma(1+\rho_r-\rho_i)\,u_{n+i}^{\rho_i}}\right]\left(\frac{u_{n+1}\cdots u_{2n}}{(-1)^n u_1\cdots u_n}\right)^{\rho_r}$$
$$\times\sum_{k\geq 0}\frac{(\rho_r)_k^n}{(1+\rho_r-\rho_1)_k\cdots(1+\rho_r-\rho_n+1)_k}\left(\frac{u_{n+1}\cdots u_{2n}}{(-1)^n u_1\cdots u_n}\right)^k.$$

The result follows. □

Based on the assumption that $0 < \rho_1 < \cdots < \rho_n < 1$, we have the following:

Lemma 4.3.66. *There are* n *different* Γ*-series for the convergence directions* $\boldsymbol{\nu}^{I_{n+r}}$ *with* $1 \leq r \leq n$ *in* $T_{\mathcal{C}_-}$.

The local monodromy group is generated by the cycle based at $(R_1,\ldots,R_{2n})$ given by $(u_1,\ldots,u_{n+r},\ldots,u_{2n}) = (R_1,\ldots,R_{n+r}\exp(-i\varphi),\ldots,R_{2n})$ for $\varphi\in[0,2\pi]$ Equivalently, we consider the local monodromy generated by $t = t_0\exp(i\varphi)$ for $t_0 \gg 1$ and $\varphi\in[0,2\pi]$ (by setting $|u_1| = \cdots = |u_{2n}| = 1$ and $|u_{n+r}| = 1/t$). We have the following:

Proposition 4.3.67. *The local monodromy of the basis* $\boldsymbol{F}^t = \langle F_n,\ldots,F_1\rangle^t$ *of solutions to the differential equation (4.3.37) at* $t=\infty$ *is given by*

$$\mathsf{M}_\infty = \begin{pmatrix} e^{-2\pi i\rho_n} & & \\ & \ddots & \\ & & e^{-2\pi i\rho_1}\end{pmatrix}. \tag{4.3.52}$$

Proof. From the Riemann symbol (4.3.38), we observe that the functions

$$F_r(t) = A_r\, t^{-\rho_r}\;{}_nF_{n-1}\left(\begin{matrix}\rho_r & \cdots & \cdots & \rho_r\\ 1+\rho_r-\rho_1 & \ldots\widehat{1}\ldots & 1+\rho_r-\rho_n\end{matrix}\,\middle|\,\frac{1}{t}\right) \tag{4.3.53}$$

for $r = 1, \ldots, n$ and any non-zero constants A_r, form a Frobenius basis of solutions to the differential equation (4.3.37) at $t = \infty$. The claim follows. $\square$

The transition matrix

The solution (4.3.46) has an integral representation of Mellin-Barnes type [10] given by

$$f(\epsilon, t) = \frac{t^\epsilon}{2\pi i} \frac{\Gamma(1+\epsilon)^n}{\Gamma(\rho_1+\epsilon)\cdots\Gamma(\rho_n+\epsilon)} \int_{\sigma+i\mathbb{R}} ds \, \frac{\Gamma(s+\rho_1+\epsilon)\cdots\Gamma(s+\rho_n+\epsilon)}{\Gamma(s+1+\epsilon)^n} \cdot \frac{\pi\,(-t)^s}{\sin(\pi s)}\,, \tag{4.3.54}$$

where $\sigma \in (-\rho_1, 0)$. For $|t| < 1$ the contour integral can be closed to the right. We have the following:

Lemma 4.3.68. *For $|t| < 1$, Equation (4.3.54) coincides with Equation (4.3.46).*

Proof. For $|t| < 1$ the contour integral can be closed to the right, and the Γ-series in Equation (4.3.46) is recovered as a sum over the enclosed residua at $r \in \mathbb{N}_0$ where we have used

$$\text{for all } r \in \mathbb{N}_0: \ \operatorname{Res}_{s=r}\left(\frac{\pi\,(-t)^s}{\sin(\pi s)}\right) = t^r.$$

$\square$

For $|t| > 1$ the contour integral must be closed to the left. The relation to the local basis of solutions at $t = \infty$ can be explicitly computed:

Proposition 4.3.69. *For $|t| > 1$, we obtain for $f(\epsilon, t)$ in Equation (4.3.54)*

$$f(\epsilon, t) = \sum_{r=1}^{n} B_r(\epsilon)\, F_r(t) \tag{4.3.55}$$

where $F_r(t)$ is given by

$$F_r(t) = A_r\, t^{-\rho_r}\, {}_nF_{n-1}\left(\begin{array}{cccc} \rho_r & \dots & \dots & \rho_r \\ 1+\rho_r-\rho_1 & \dots \widehat{1} \dots & & 1+\rho_r-\rho_n \end{array} \middle| \frac{1}{t} \right) \tag{4.3.56}$$

and

$$A_r = -e^{-\pi i \rho_r} \prod_{\substack{i=1 \\ i\neq r}}^{n} \frac{\Gamma(\rho_i-\rho_r)}{\Gamma(\rho_i)\,\Gamma(1-\rho_r)},\ B_r(\epsilon) = e^{-\pi i \epsilon} \left[\prod_{i=1}^{n} \frac{\Gamma(\rho_i)\,\Gamma(1+\epsilon)}{\Gamma(\rho_i+\epsilon)} \right] \frac{\sin(\pi\rho_r)}{\sin(\pi\rho_r+\pi\epsilon)}, \tag{4.3.57}$$

such that $B_r(0) = 1$ for $r = 1, \dots, n$.

Proof. For $|t| > 1$ the contour integral in Equation (4.3.54) must be closed to the left. Using $1/(1+\epsilon)_k^n = O(\epsilon^n) = 0$ for $k < 0$, we observe that the poles are located at $s = -\epsilon - \rho_i - k$ for $i = 1, \dots, n$ and $k \in \mathbb{N}_0$. Using

$$\forall r \in \mathbb{N}_0 :\ \mathrm{Res}_{s=-r} \Big(\Gamma(s)\,(-t)^s \Big) = \frac{t^{-r}}{r!}.$$

and Equations (4.3.45) the result follows. □

Equation (4.3.55) allows to compute the transition matrix between the Frobenius basis $\langle f_{n-1}, \dots, f_0\rangle^t$ of solutions for the differential equation (4.3.37) at $t = 0$ with local monodromy given by the matrix (4.3.49) and the Frobenius basis $\langle F_n, \dots, F_1\rangle^t$ of solutions at $t = \infty$ with local monodromy given by the matrix (4.3.52). We obtain:

Corollary 4.3.70. *The transition matrix P between the analytic continuations of the* *bases* $\boldsymbol{f}^t = \langle f_{n-1}, \ldots, f_0 \rangle^t$ *at* $t = 0$ *and* $\boldsymbol{F}^t = \langle F_n, \ldots, F_1 \rangle^t$ *at* $t = \infty$ *is given by*

$$\begin{pmatrix} f_{n-1} \\ \vdots \\ f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} \frac{B_n^{(n-1)}(0)}{(2\pi i)^{n-1}(n-1)!} & \cdots & \frac{B_1^{(n-1)}(0)}{(2\pi i)^{n-1}(n-1)!} \\ \vdots & \ddots & \vdots \\ \frac{B_n'(0)}{2\pi i} & \cdots & \frac{B_1'(0)}{2\pi i} \\ 1 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} F_n \\ \vdots \\ F_2 \\ F_1 \end{pmatrix} \tag{4.3.58}$$

with $B_r(\epsilon)$ *given in Equation (4.3.57).*

Proof. The transition matrix P between the analytically continued Frobenius basis of solutions $\boldsymbol{f}^t = \langle f_{n-1}, \ldots, f_0 \rangle^t$ at $t = 0$ and the analytic continuation of the Frobenius basis $\boldsymbol{F}^t = \langle F_n, \ldots, F_1 \rangle^t$ at $t = \infty$ is obtained by first comparing the expression of $f(\epsilon, t)$ from Equation (4.3.46) as a linear combination of the solutions $\boldsymbol{F}$ at $t = \infty$ from Equation (4.3.55), and subsequently applying Lemma 4.3.62 to find the explicit linear relations between $\boldsymbol{f}$ and $\boldsymbol{F}$. By differentiation of the functions $B_r(\epsilon)$ in Equation (4.3.57) and evaluating at $\epsilon = 0$, we recover the matrix (4.3.58). □

We can now compute the monodromy of the analytic continuation of $\boldsymbol{f}$ around any singular point:

Corollary 4.3.71. *The monodromy of the analytic continuation of* $\boldsymbol{f}$ *around* $t = 0$, $t = \infty$, *and* $t = 1$ *is given by* m_0 *in (4.3.49),* $\mathsf{m}_\infty = \mathsf{P} \cdot \mathsf{M}_\infty \cdot \mathsf{P}^{-1}$ *for* M_∞ *in (4.3.52), and* $\mathsf{m}_1 = \mathsf{m}_\infty \cdot \mathsf{m}_0^{-1}$, *respectively.*

Monodromy after rescaling

For $C > 0$ the rescaled hypergeometric differential equation satisfied by $\tilde{F}(t) = {}_nF_{n-1}(Ct)$ is given by

$$\Big[\theta^n - C\, t\, (\theta + \rho_1) \cdots (\theta + \rho_n)\Big]\, \tilde{F}(t) = 0\ . \tag{4.3.59}$$

For $|t| < 1/C$ we introduce $\tilde{f}(\epsilon, t) = C^{-\epsilon} f(\epsilon, Ct)$ such that

$$\tilde{f}(\epsilon, t) = \sum_{m=0}^{n-1} \Big(2\pi i \epsilon\Big)^m \tilde{f}_m(t) \quad \text{with} \quad \tilde{f}_m(t) = \frac{1}{(2\pi i)^m m!} \frac{\partial^m}{\partial \epsilon^m}\bigg|_{\epsilon=0} f(\epsilon, Ct) \tag{4.3.60}$$

for $j = 0, \ldots, n-1$. The local monodromy around $t = 0$ with respect to the Frobenius basis $\langle \tilde{f}_{n-1}, \ldots, \tilde{f}_0 \rangle^t$ is still given by the matrix m_0 in (4.3.49). Similarly, for $|t| > 1/C$ we introduce $\tilde{F}_k(t) = F_k(Ct)$ for $k = 1, \ldots, n$. The local monodromy (around $t = \infty$) with respect to the Frobenius basis $\langle \tilde{F}_n, \ldots, \tilde{F}_1 \rangle^t$ is given by the matrix M_∞ in (4.3.52). We obtain:

Proposition 4.3.72. *The transition matrix $\tilde{\mathsf{P}}$ between the analytic continuation of $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{F}}$ such that $\tilde{\boldsymbol{f}} = \tilde{\mathsf{P}} \cdot \tilde{\boldsymbol{F}}$ is given by*

$$\tilde{\mathsf{P}} = \Big(\tilde{\mathsf{P}}_{n-j,n+1-k}\Big)_{j=0,k=1}^{n-1,n} \quad \textit{with} \quad \tilde{\mathsf{P}}_{n-j,n+1-k} = \frac{1}{(2\pi i)^j j!} \frac{\partial^j}{\partial \epsilon^j}\bigg|_{\epsilon=0} \Big[C^{-\epsilon} B_k(\epsilon)\Big] \,. \tag{4.3.61}$$

The monodromy of the analytic continuation of $\tilde{\boldsymbol{f}}$ around $t = \infty$ and $t = 1/C$ is given by $\mathsf{m}_\infty = \tilde{\mathsf{P}} \cdot \mathsf{M}_\infty \cdot \tilde{\mathsf{P}}^{-1}$ and $\mathsf{m}_{1/C} = \mathsf{m}_\infty \cdot \mathsf{m}_0^{-1}$, respectively.

Proof. One emulates the proof of Corollaries 4.3.70 and 4.3.71 directly with new analytic continuations $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{F}}$ around $t = 0$ and $t = \infty$, respectively. In this case, one finds that the functions $B_r(\epsilon)$ appearing in Equation (4.3.55) acquire a factor of $C^{-\epsilon}$. The result then follows suit as claimed. □

In summary, we obtained the monodromy matrices m_0 in (4.3.49), $\mathsf{m}_\infty = \tilde{\mathsf{P}} \cdot \mathsf{M}_\infty \cdot \tilde{\mathsf{P}}^{-1}$ for M_∞ in (4.3.52) and $\tilde{\mathsf{P}}$ in Equation (4.3.61), and $\mathsf{m}_{1/C} = \mathsf{m}_\infty \cdot \mathsf{m}_0^{-1}$ for the hypergeometric differential equation (4.3.59). Thus, we have the following main result:

Theorem 4.3.73. *For the family of hypersurfaces $Y_t^{(n-1)}$ in Equation (4.3.24) with $n \geq 2$ the mixed-twist construction defines a non-resonant GKZ system. Then a* *basis of solutions exists given as absolutely convergent Mellin-Barnes integrals whose monodromy around* $t = 0, 1/C, \infty$ *is, up to conjugation,* $\mathsf{m}_0, \mathsf{m}_{1/C}, \mathsf{m}_\infty$*, respectively, for* $\rho_k = k/(n+1)$ *with* $k = 1, \ldots, n$ *and* $C = (n+1)^{n+1}$*.*

Proof. The theorem combines the statements of Propositions 4.3.57, 4.3.58, 4.3.63, 4.3.67, 4.3.72 that were proven above. □

We have the following:

Corollary 4.3.74. *Set* $\kappa_4 = -200\frac{\zeta(3)}{(2\pi i)^3}$*, and* $\kappa_5 = 420\frac{\zeta(3)}{(2\pi i)^3}$*. The monodromy matrices of Theorem 4.3.73 for* $2 \leq n \leq 5$ *are given by Table 4.1.*

Proof. We obtain from the multiplication formula for the Γ-function, i.e.,

$$\prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right) = (2\pi)^{\frac{1}{2}(m-1)}\, m^{\frac{1}{2}-mz}\, \Gamma(mz),$$

the identity

$$C^{-\epsilon} B_k(\epsilon) = \frac{\Gamma(1+\epsilon)^{n+1}}{\Gamma\big(1+(n+1)\epsilon\big)} \frac{\sin(\pi\rho_k)}{\sin(\pi\rho_k + \pi\epsilon)} e^{-\pi i\epsilon} \; .$$

We then compute the monodromy of the analytic continuation of $\tilde{\boldsymbol{f}}$ around $t = 0, 1/C, \infty$ where we have set $\kappa_4 = -200\frac{\zeta(3)}{(2\pi i)^3}$ and $\kappa_5 = 420\frac{\zeta(3)}{(2\pi i)^3}$. We obtain the results listed in Table 4.1. □

The case $n = 4$, reproduces up to conjugation the monodromy matrices for the quintic threefold case by Candelas et al. [16] and Chen et al. [17].

n	$Y_t^{(n-1)}$	m_0	$\mathsf{m}_{1/C}$	m_∞
2	EC	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$
3	K3	$\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 1 \\ -4 & -4 & -2 \end{pmatrix}$
4	CY3	$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1+\kappa_4 & 0 & \frac{5\kappa_4}{12} & \frac{\kappa_4^2}{5} \\ -\frac{25}{12} & 1 & -\frac{125}{144} & -\frac{5\kappa_4}{12} \\ 0 & 0 & 1 & 0 \\ -5 & 0 & -\frac{25}{12} & 1-\kappa_4 \end{pmatrix}$	$\begin{pmatrix} 1+\kappa_4 & 1+\kappa_4 & \frac{1}{2}+\frac{11\kappa_4}{12} & \frac{1}{6}+\frac{7\kappa_4}{12}+\frac{\kappa_4^2}{5} \\ -\frac{25}{12} & -\frac{13}{12} & -\frac{131}{144} & -\frac{103}{144}-\frac{5\kappa_4}{12} \\ 0 & 0 & 1 & 1 \\ -5 & -5 & -\frac{55}{12} & -\frac{23}{12}-\kappa_4 \end{pmatrix}$
5	CY4	$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{75}{64} & 0 & \frac{55}{512} & -\frac{11\kappa_5}{384} & -\frac{121}{24576} \\ -\kappa_5 & 1 & -\frac{5\kappa_5}{8} & \frac{\kappa_5^2}{6} & \frac{11\kappa_5}{384} \\ -\frac{15}{4} & 0 & -\frac{43}{32} & \frac{5\kappa_5}{8} & \frac{55}{512} \\ 0 & 0 & 0 & 1 & 0 \\ -6 & 0 & -\frac{15}{4} & \kappa_5 & \frac{75}{64} \end{pmatrix}$	$\begin{pmatrix} \frac{75}{64} & \frac{75}{64} & \frac{355}{512} & -\frac{11\kappa_5}{384}+\frac{155}{512} & -\frac{11\kappa_5}{384}+\frac{2399}{24576} \\ -\kappa_5 & -\kappa_5+1 & -\frac{9\kappa_5}{8}+1 & \frac{(4\kappa_5-3)(\kappa_5-4)}{24} & \frac{\kappa_5^2}{6}-\frac{125\kappa_5}{384}+\frac{1}{6} \\ -\frac{15}{4} & -\frac{15}{4} & -\frac{103}{32} & \frac{5\kappa_5}{8}-\frac{63}{32} & \frac{5\kappa_5}{8}-\frac{369}{512} \\ 0 & 0 & 0 & 1 & 1 \\ -6 & -6 & -\frac{27}{4} & \kappa_5-\frac{19}{4} & \kappa_5-\frac{61}{64} \end{pmatrix}$

Table 4.1: Monodromy matrices for the mirror families with $2 \le n \le 5$

CHAPTER 5

Quadratic period relations for K3 surfaces of high Picard-rank

We study in this chapter aspects of the quadratic period relations for the twisted Legenedre pencil $\pi : \mathbf{X}_{a,b,c} \to \mathbb{P}^1$,

$$\mathbf{X}_{a,b,c} : \, y^2 = x(x-1)(x-t)(t-a)(t-b)(t-c)\,.$$

The existence of a quadratic relation amoung the period integrals follows by virtue of the lattice polarization; equipping a K3 surface $\mathbf{X}$ with a lattice polarization $\mathrm{L} \hookrightarrow \Lambda_{\mathrm{K3}}$ determines the transcendental lattice $\mathrm{T}(\mathbf{X}) := \mathrm{L}^{\perp} \subset \Lambda_{\mathrm{K3}}$, upon which the Riemann relations hold, see §2.1.3. The investigation of these quadratic period relations is crucial for the geometrization of the moduli space, which we undertake in Chapter 6.

This analysis is undertaken in §5.1, where we construct the parabolic cohomology from the homological marking defined on the punctured base of the fibration. The explicit quadratic relations given in §5.1.4 comes from parabolic cusp forms realizing such cohomology classes, based off Endo's analysis of generalized Eichler type [43]. Such a relation implies that the period domain is uniformized by a hyperquadric in $\mathbb{P}^4$ defined by the quadratic form.

Our understanding of the parabolic cohomology of the fibration allows us to compute some of the Picard-Fuchs equations for the twisted Legendre pencil in §5.3. Since we now have an understanding of the transcendental cycles, this is done by a relatively elementary method: Fubini's theorem for multiple integrals, and integration by parts.

Roughly speaking, we are "twisting" the second order Picard-Fuchs operator for the Legendre pencil of elliptic curves, the second order Fuchsian operator annihilating ${}_2F_1(1/2, 1/2; 1|t)$, by the twist factor $1/\sqrt{h(t)} = 1/\sqrt{(t-a)(t-b)(t-c)}$.

Except for restrictions of the twisted Legendre pencil to Picard rank $rho \geq 19$ (§5.3.3 and §5.3.4) method does not allow us to produce all of the Picard-Fuchs equations, but only some. However, the method is sufficient for generating higher order differential relations, and we use this method in §7.2 to determine a generic fifth order Picard-Fuchs ODE operator for the twisted Legendre pencil.

5.1 Quadratic period relations for $\rho \geq 17$

5.1.1 The homological invariant

The elliptic fibration on $\pi : \mathbf{X}_{a,b,c} \to \mathbb{P}^1$ has singular fibers of Kodaira type I_2 over the points $t = 0, 1, \infty$, and singular fibers of Kodaira type I_0^* over $t = a, b, c$. With the singular locus of the elliptic fibration being $\Sigma = \{0, 1, a, b, c, \infty\} \subset \mathbb{P}^1$, we set $C = \mathbb{P}^1 \backslash \Sigma$. Let t_0 be a fixed point in C, and denote the fundamental group of C bases at t_0 by $\Gamma = \pi_1(C, t_0)$. Generators for Γ are suitable simple loops α_v around the base points of the singular fibers with $v \in \Sigma$ such that the following relation holds

$$[\alpha_c] * [\alpha_b] * [\alpha_a] * [\alpha_1] * [\alpha_0] * [\alpha_\infty] = 1 \, , \tag{5.1.1}$$

where $[\,\cdot\,]$ denotes a homotopy equivalence class and $*$ the group multiplication of loops up to homotopy. Denote by $t : \tilde{C} \to C$ the universal cover of C with meromorphic functions on C regarded as quotients of polynomials in t. We also let $\tilde{C}^*$ denote the union of $\tilde{C}$ and the set of cusps for Γ on $\tilde{C}$. Let D the polygonal fundamental domain for Γ on $\tilde{C}^*$ with pair of edges A_v and $\alpha_v A_v$ lying over the paths $t(A_v)$ that extend from cusps ∞^* to 0^* to $\alpha_0\infty^*$ to 1^* to $\alpha_1\alpha_0\infty^*$ to a^* to $\alpha_a\alpha_1\alpha_0\infty^*$ to b^* to

$\alpha_b\alpha_a\alpha_1\alpha_0\infty^*$ to c^* to ∞^* and with covering transformations α_v which generate Γ, stabilize the vertices v^*, and satisfy Equation (5.1.1). The boundary of D is

$$\partial D = A_0 - \alpha_0 A_0 + A_1 - \alpha_1 A_1 + \cdots + A_c - \alpha_c A_c \,. \tag{5.1.2}$$

We will now fix a so-called *homological marking* on $\mathbf{X}_{a,b,c}$. For an elliptic surface $\pi : \mathbf{X} \to C$, a homological invariant $\mathcal{G}$ is obtained by defining a locally constant sheaf over C whose generic stalk is isomorphic to $H_1(\mathcal{E}_t, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, where $\mathcal{E}_t = \pi^{-1}(t)$ is a generic smooth fibre. A monodromy representation

$$M : \Gamma \to \mathrm{SL}(2, \mathbb{Z}) \tag{5.1.3}$$

defines the transition functions for this sheaf. Then $\mathcal{G} = M(\Gamma) \subseteq \mathrm{SL}(2, \mathbb{Z})$ is the homological invariant of the fibration $(\mathbf{X}, \pi)$. From Kodaira's classification [78, 79], it follows that $M(\alpha_0)$, $M(\alpha_1)$, $M(\alpha_\infty)$ are conjugate to T^2, and $M(\alpha_a)$, $M(\alpha_b)$, $M(\alpha_c)$ are conjugate to $-\mathbb{I}$ where we used the following $\mathrm{SL}(2, \mathbb{Z})$-generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \,, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,. \tag{5.1.4}$$

We fix the homological invariant of the Jacobian elliptic K3 surface $\pi : \mathbf{X}_{a,b,c} \to \mathbb{P}^1$ by fixing the boundary ∂D in Eq. (5.1.2) and setting

$$M_\infty = TST^2\,(TS)^{-1}\,,\; M_0 = ST^2\,S^{-1}\,, \quad M_1 = T^2\,,\; M_a = M_b = M_c = -\mathbb{I}\,, \tag{5.1.5}$$

where $M(\alpha_v) = M_v$ for $v \in \Sigma$. Notice that the matrices (5.1.5) are conjugate to the elements of $\mathrm{SL}(2, \mathbb{Z})$ required by the Kodaira-type of the singular fibers and satisfy the condition

$$M_c \cdot M_b \cdot M_a \cdot M_1 \cdot M_0 \cdot M_\infty = \mathbb{I}\ . \tag{5.1.6}$$

5.1.2 The Picard-Fuchs equation and differentials of the second kind

For the twisted Legendre pencil $\mathbf{X} = \mathbf{X}_{a,b,c}$, let

$$Y^2 = 4X^3 - g_2 X - g_3$$

denote the Weierstrass form of the elliptic fibres. Let $\Delta = g_2^3 - 27 g_3^2$ be the modular discriminant. We will use $dX/Y \in H^{1,0}(\mathcal{E}_t)$ as the holomorphic one form on each regular fiber $\mathcal{E}_t = \pi^{-1}(t)$. This is called an *analytic marking* of the elliptic surface $\mathbf{X}$. The Picard-Fuchs equation for the Weierstrass elliptic surface is given by the Fuchsian system (see for example [37])

$$\frac{d}{dt}\begin{pmatrix} \omega \\ \mathfrak{a} \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\ln\Delta}{dt} & \frac{3\,\delta}{2\,\Delta} \\ -\frac{g_2\,\delta}{8\,\Delta} & \frac{1}{12}\frac{d\ln\Delta}{dt} \end{pmatrix} \cdot \begin{pmatrix} \omega \\ \mathfrak{a} \end{pmatrix}\ , \tag{5.1.7}$$

where

$$\omega = \int_\gamma \frac{dX}{Y}\ , \qquad \mathfrak{a} = \int_\gamma \frac{X\,dX}{Y}\ , \tag{5.1.8}$$

for a one-cycle γ and $\delta = 3\,g_3\,g_2' - 2\,g_2\,g_3'$. The Picard-Fuchs equations are equivalent to the following second-order ordinary differential equation of hypergeometric type

$$\frac{1}{\sqrt{h(t)}}\left(t(t-1)\,\frac{d^2}{dt^2} + (2t-1)\,\frac{d}{dt} + \frac{1}{4}\right)\left(\sqrt{h(t)}\,\omega(t)\right) = 0\ . \tag{5.1.9}$$

Kummer found all six solutions to the underlying hypergeometric differential equation which account for all possible behaviors at the three regular singular points $0, 1, \infty$.

Using the family of an ordered basis γ_1 and γ_2 for $H_1(\mathcal{E}_t, \mathbb{Z})$ that changes analytically in t, we obtain the two solutions near $t = 0$ for the periods

$$\begin{aligned} \omega_1 &= \int_{\gamma_1} \frac{dX}{Y} = \frac{r}{\sqrt{h(t)}} \; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) , \\ \omega_2 &= \int_{\gamma_2} \frac{dX}{Y} = -\frac{\pi\, r}{\sqrt{h(t)}} \; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-t\right) , \end{aligned} \tag{5.1.10}$$

where $r = 2\pi/\sqrt{6}$. We have the following asymptotic expansions of the hypergeometric functions

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) &= 1 + \frac{1}{4} t + O(t^2) , \\ -\pi \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-t\right) &= \ln(t) \; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) \\ & \qquad + \text{(terms holomorphic in } t) . \end{aligned} \tag{5.1.11}$$

We set $\tau = \frac{1}{2\pi i} \frac{\omega_2}{\omega_1} : C \to \mathbb{H}$ and $q = \exp(2\pi i \tau)$ such that

$$\mathcal{J} = J \circ \tau : C \to \mathbb{H} \to \mathbb{P}^1 \cong \mathrm{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}^* , \tag{5.1.12}$$

where $J = E_4^3(\tau)/[E_4^3(\tau) - E_6^2(\tau)]$ and $J = j/1728$ where $j = q^{-1} + \dots$ is the classical modular function. Moreover, the comparison with the the modular elliptic surface for $\Gamma(2)$ implies that $t = -t_{\Gamma(2)}/16$ where $t_{\Gamma(2)}$ is the canonical Hauptmodul for $\Gamma(2)$

$$t_{\Gamma(2)} = 2^8 \, \frac{\eta^8(2\tau)}{\eta^8(\tau/2)} = 2^8 \, \sqrt{q} \, \left(1 + O(\sqrt{q})\right) . \tag{5.1.13}$$

It follows that

$$\frac{\sqrt{h(t)}}{c} \, \omega_1 = \frac{\eta^4(\tau/2)}{\eta^2(\tau)} = 1 - 4\sqrt{q} + O(q)$$

and

$$\Delta = \frac{729}{4}\, h^6(t)\, t^2\, (t-1)^2 = \frac{(2\pi)^{12}\, \eta^{24}(\tau)}{\omega_1^{12}} = \frac{64\, \pi^{12}\, [E_4^4(\tau) - E_6^2(\tau)]}{27\, \omega_1^{12}}\ , \tag{5.1.14}$$

in accordance with the expansions in Equations (5.1.11) and (5.1.13).

There are covering maps, a period map $\tilde{\tau} : \tilde{C} \to \mathbb{H}$ and a non-vanishing holomorphic $\tilde{\omega}_1$ on $\tilde{C}$ such that

$$\mathcal{J} \circ t = J \circ \tilde{\tau}\ , \tag{5.1.15a}$$

$$\tilde{\tau} \circ \alpha = M(\alpha) \cdot \tilde{\tau} \quad \text{for all } \alpha \in \Gamma\ , \tag{5.1.15b}$$

and

$$\tilde{\omega}_1 \circ \alpha = (c\, \tilde{\tau} + d)\, \tilde{\omega}_1 \quad \text{for all } \alpha \in \Gamma \text{ and } M(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ , \tag{5.1.16}$$

and

$$\begin{aligned} g_2 \circ t &= \frac{4\, \pi^4\, E_4(\tilde{\tau})}{3\, \tilde{\omega}_1^4}\ , \qquad g_3 \circ t = \frac{8\, \pi^6\, E_6(\tilde{\tau})}{27\, \tilde{\omega}_1^6}\ , \\ \Delta \circ t &= \frac{64\, \pi^{12}\, [E_4^4(\tilde{\tau}) - E_6^2(\tilde{\tau})]}{27\, \tilde{\omega}_1^{12}}\ , \quad \mathcal{J} \circ t = \frac{E_4^3(\tilde{\tau})}{[E_4^3(\tilde{\tau}) - E_6^2(\tilde{\tau})]}\ . \end{aligned} \tag{5.1.17}$$

We also set $\tilde{\omega}_2 = \tilde{\tau}\, \tilde{\omega}_1$. Letting $\tilde{C}^*$ denote the union of $\tilde{C}$ and the set of cusps for Γ on $\tilde{C}$ ($\mathbb{H}$ resp.), we can extend $\tilde{\tau}$ and $\mathcal{J}$ to surjective maps $\tilde{\tau}^* : \tilde{C}^* \to \mathbb{H}^*$ such that Equations (5.1.15)-(5.1.17) remain valid.

The period map pd maps from the space of vector-valued meromorphic one-form on $\tilde{C}^*$ to the space of vector-valued meromorphic functions on $\tilde{C}^*$ by setting

$$\mathrm{pd} : \vec{\xi} \mapsto \vec{\Xi}(u) = \int_{c^*}^{u} \vec{\xi}\,, \tag{5.1.18}$$

for any vector-valued meromorphic differential $\vec{\xi}$ on $\tilde{C}^*$ and $u \in \tilde{C}^*$. Since $\eta = dt \wedge \frac{dX}{Y}$ is a two-form on $\mathbf{X}_{a,b,c}$ we obtain a vector-valued one-form by integrating over every non-singular fiber

$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} dt \;, \qquad \text{with} \quad \omega_i \, dt = \int_{\gamma_i} \eta \;. \tag{5.1.19}$$

Using the lift to the universal cover described above, we obtain a meromorphic vector-valued one-form of on $\tilde{C}^*$ as follows

$$\vec{w} = \begin{pmatrix} \tilde{\omega}_2 \\ \tilde{\omega}_1 \end{pmatrix} dt \;. \tag{5.1.20}$$

If we set $\vec{W}(u) := \mathrm{pd}(\vec{w})(u)$ where $\vec{W}(u) = (w_u, z_u)^t$, it follows that $\vec{W}(u)$ converges as u approaches any cusp in D. The convergence of the integrals at cusps can be inferred from expansions of the integrand. From now on, we will not distinguish between $\tilde{\omega}_1$ and ω_1 any further to simplify notation. Hence, we have

$$\begin{aligned} w_u &= \int_{c^*}^{u} dt \; \omega_2 \;, \\ z_u &= \int_{c^*}^{u} dt \; \omega_1 \;. \end{aligned} \tag{5.1.21}$$

5.1.3 Generalized cusp forms

The construction of the Hodge structure on the parabolic cohomology is based on the results by Hoyt [66] and its extension by Endo [43]. The results concerns families of Weierstrass equations $Y^2 = 4X^3 - g_2 \, X - g_3$ with transcendental invariant $\mathcal{J}$ with g_2, g_3 in an arbitrary finite algebraic extension $\mathbb{K}$ of $\mathbb{C}(\mathcal{J})$. In the situation of Equations (5.1.17) we use the period ω_1 to define the space of generalized modular forms of weight three relative to τ to be the one-dimensional $\mathbb{K}$-module $\mathbb{K}\,\omega_1^3 = \{(f \circ t)\, \omega_1^3 \mid f \in \mathbb{K}\}$ generated by ω_1^3. One can ask whether the multi-valued modular form ω_1^3 of weight three is a generalized cusp form of the second kind. This means that ω_1^3 is a cusp form at the parabolic cusps, the cusps where $\mathcal{J} \circ t = \infty$, and satisfies a second kind condition at the cusps which are not parabolic. We denote the space of generalized cusp forms of the second kind of weight three by $\mathcal{T}$. Hoyt proves in [69] that ω_1^3 in Equation (5.1.10) is a generalized cusp form of the second kind with parabolic cusps at $t = 0, 1, \infty$ and non-parabolic cusps at $t = a, b, c$.

In fact, every element of $\mathcal{T}$ is a two-form on $\mathbf{X}_{a,b,c}$ that is of the second kind and holomorphic on singular fibers and has the form $\vec{w}_f = f\, dt \wedge dX/Y$ for a suitable $f \in \mathbb{C}(t)$. On $\mathcal{T}$, a quadratic form Q is defined by

$$Q(\vec{w}_f) = \int_{\partial D} \vec{W}_f^t \cdot S \cdot \vec{w}_f \;, \tag{5.1.22}$$

where S is defined in Eq. (5.1.4) and the boundary ∂D is given in Equation (5.1.2). A theorem by Endo [43] then proves that the quadratic form is well defined provided one modifies $\vec{w}_f$ by adding a suitable exact $\vec{w}_g$ with $\mathrm{pd}(\vec{w}_g) = 0$ if necessary to ensure that the integrals converge. Endo proves that for each $f \in \mathcal{T}$ there is a generalized Eichler integral F of f such that $f = \frac{d^2}{d\tau^2} F$. On $\mathcal{T}$, there is a well-defined quadratic form for $f \in \mathcal{T}$

$$Q(f) = 2\pi i \sum_{v \in C^*} \widetilde{\mathrm{Res}}_v \Big(F \cdot f \Big)\, d\tilde{\tau} \;. \tag{5.1.23}$$

Here, $\widetilde{\mathrm{Res}}$ is an extended residue, and the quadratic form (5.1.23) is the pull-back of the quadratic form in (5.1.22). In particular, Endo's theorem implies that the following equations hold

$$0 = \int_{\partial D} \vec{W}^t \cdot S \cdot \vec{w} \;, \tag{5.1.24a}$$

$$0 < \int_{\partial D} \mathrm{Re}\Big(\vec{W}^t\Big) \cdot S \cdot \mathrm{Re}\Big(\vec{w}\Big) \;. \tag{5.1.24b}$$

These equations are special cases of relations in Shimura [131] and are called the Hodge-Riemann relations

$$Q(\vec{w}) = 0 \;, \qquad Q(\mathrm{Re}(\vec{w})) > 0 \;. \tag{5.1.25}$$

5.1.4 The construction of the parabolic cocycle

For the marked elliptic surface $\mathbf{X}_{a,b,c}$ one defines the parabolic cohomology group $H^1_{\text{par}}(\Gamma, M)$ associated to the monodromy representation $M : \Gamma \to \mathrm{SL}(2,\mathbb{Z})$ of the elliptic surface $\pi : \mathbf{X}_{a,b,c} \to \mathbb{P}^1$. We refer to [134, 71] for its definition and for the relation to the parabolic cohomology group as originally defined by Shimura [131] and Eichler [42]. Cox and Zucker [31] showed that the natural Hodge structures on $H^2(\mathbb{P}^1, R^1\pi_*\mathbb{Z})$ and the parabolic cohomology $H^1_{\text{par}}(\Gamma, M)$ are isomorphic. The isomorphism between the parabolic cohomology group and the space $\mathcal{T}$ is given by the period map (5.1.18). We describe the natural Hodge structure on $H^2(\mathbb{P}^1, R^1\pi_*\mathbb{Z})$ following [66, 43]. The vector-valued function $\vec{W}$ defines a parabolic cocyle $\vec{Y}$ by

$$\vec{Y}(\alpha) = \vec{W} \circ \alpha - M(\alpha) \cdot \vec{W} \tag{5.1.26}$$

for $\alpha \in \Gamma$. Immediate consequences of the definition are

$$\vec{Y}(\alpha\beta) = \vec{Y}(\alpha) + M(\alpha) \cdot \vec{Y}(\beta) \;, \tag{5.1.27a}$$

$$\vec{0} = \vec{Y}\Big(\alpha_c \alpha_b \alpha_a \alpha_1 \alpha_0 \alpha_\infty\Big) \;. \tag{5.1.27b}$$

We can then compute quadratic relations between the periods of the two-form η. The following is based on a similar computation in [69]. For $v \in \Sigma$ we set

$$\vec{Y}_v := \vec{Y}(\alpha_v) = \Big(\mathbb{I} - M(\alpha_v)\Big) \cdot \vec{W}(v^*) \;. \tag{5.1.28}$$

It follows from Equations (5.1.28) and (5.1.5) that

$$\vec{Y}_\infty = \begin{pmatrix} 2\,w_\infty - 2\,z_\infty \\ 2\,w_\infty - 2\,z_\infty \end{pmatrix} , \quad \vec{Y}_0 = \begin{pmatrix} 0 \\ 2\,w_0 \end{pmatrix} , \quad \vec{Y}_1 = \begin{pmatrix} -2\,z_1 \\ 0 \end{pmatrix} ,$$
$$\vec{Y}_v = \begin{pmatrix} 2\,w_v \\ 2\,z_v \end{pmatrix} \text{ for } v = a, b, c \, .$$

The linear relation (5.1.27b) now becomes

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\,w_c - 2\,w_b + 2\,w_a + 2\,z_1 - 4\,w_0 + 2\,w_\infty - 2\,z_\infty \\ 2\,z_c - 2\,z_b + 2\,z_a - 2\,w_0 + 2\,w_\infty - 2\,z_\infty \end{pmatrix} . \tag{5.1.29}$$

The period integrals $\int_{A_v} \vec{w}$ can be be simplified as follows

$$\begin{aligned}
&\int_{A_0} \vec{w} = \vec{W}(0^*) - \vec{W}(\infty^*) \, , \\
&\int_{A_1} \vec{w} = \vec{W}(1^*) - \left[\vec{Y}_0 + M(\alpha_0)\,\vec{W}(\infty^*)\right] \, , \\
&\int_{A_a} \vec{w} = \vec{W}(a^*) - \left\{\vec{Y}_1 + M(\alpha_1)\left[\vec{Y}_0 + M(\alpha_0)\,\vec{W}(\infty^*)\right]\right\} \, ,
\end{aligned}$$

and similar relations hold for $\int_{A_v} \vec{w}$ with $v = b, c$. We introduce the following linear combinations of the periods

$$\begin{aligned}
Z_1 &= z_a + z_\infty - w_\infty \, , \\
Z_2 &= z_1 - 2\,z_a + w_a \, , \\
Z_3 &= -w_\infty + z_\infty + w_0 - z_1 + 2\,z_a - w_a \, , \\
Z_4 &= w_\infty - z_\infty \, , \\
Z_5 &= z_a - z_1 \, ,
\end{aligned} \tag{5.1.30}$$

where $z_v = \int_{c^*}^{v^*} \int_{\gamma_1} dt \wedge dX/Y = \int_{c^*}^{v^*} dt\, \omega_1$ for $v \in \{0, 1, a, \infty\}$. Using the variables in (5.1.38) and the quadratic form

$$Q\Big(Z_1, Z_2, Z_3, Z_4, Z_5\Big) = 2\, Z_1^2 + 2\, Z_2^2 - 2\, Z_3^2 - 2\, Z_4^2 - 2\, Z_5^2 \ , \tag{5.1.31}$$

the Hodge-Riemann relations (5.1.24a) and (5.1.24b) on $\mathbf{X}_{a,b,c}$ become the following quadratic period relations:

$$Q\Big(Z_1, Z_2, Z_3, Z_4, Z_5\Big) = 0 \ , \tag{5.1.32a}$$

$$Q\Big(\mathrm{Re}(Z_1), \mathrm{Re}(Z_2), \mathrm{Re}(Z_3), \mathrm{Re}(Z_4), \mathrm{Re}(Z_5)\Big) > 0 \ . \tag{5.1.32b}$$

Here, we have used that $z_c = 0$ and $w_c = 0$ and the linear relation (5.1.29) to express all w_b, z_b in terms of the other variables.

Alternate choice of variables

If we use the following linear combinations of the periods

$$\begin{aligned} Z_1' &= z_a \ , \\ Z_2' &= -z_\infty + z_1 + w_\infty - 2\, w_0 \ , \\ Z_3' &= w_a \ , \\ Z_4' &= z_\infty - w_\infty + w_0 \ , \\ Z_5' &= z_\infty - z_1 - w_\infty + w_0 \ , \end{aligned} \tag{5.1.33}$$

and the quadratic form

$$Q'\Big(Z_1', Z_2', Z_3', Z_4', Z_5'\Big) = 4\, Z_1'\, Z_2' + 4\, Z_3'\, Z_4' - 2\, (Z_5')^2 \ , \tag{5.1.34}$$

the Hodge-Riemann relations (5.1.24a) and (5.1.24b) on $\mathbf{X}_{a,b,c}$ become the following quadratic period relations:

$$Q'\Big(Z_1', Z_2', Z_3', Z_4', Z_5'\Big) = 0 \ , \tag{5.1.35a}$$

$$Q'\Big(\mathrm{Re}(Z_1'), \mathrm{Re}(Z_2'), \mathrm{Re}(Z_3'), \mathrm{Re}(Z_4'), \mathrm{Re}(Z_5')\Big) > 0 \ . \tag{5.1.35b}$$

Degeneration of Hodge-structures

For $a \to 1$ the singular fiber of Kodaira type I_0^* will coalesce with the singular fiber of Kodaira type I_2 at 1 to form a fiber of Kodaira type I_2^*. The elliptic fibration then has four singular fibers of Kodaira type I_1 over the points $t_1, \dots, t_4$; For $a \to 1$ it follows $z_a \to z_1$ and $Z_5 \to 0$. In terms of the period relation if follows that

$$Q\Big(Z_1, Z_2, Z_3, Z_4, Z_5\Big) \to 2\, Z_1^2 + 2\, Z_2^2 - 2\, Z_3^2 - 2\, Z_4^2 \ . \tag{5.1.36}$$

For $a \to 1$ and $c \to \infty$ the singular fibers of Kodaira type I_0^* at $t = a$ and $t = c$ will coalesce with the singular fiber of Kodaira type I_2 at $t = 1$ and $t = \infty$, respectively, to form fibers of Kodaira type I_2^*. It follows that $z_a \to z_1, w_a \to w_1$ and $z_\infty, w_\infty \to 0$, hence $Z_4, Z_5 \to 0$. In terms of the period relation, it follows that

$$Q\Big(Z_1, Z_2, Z_3, Z_4, Z_5\Big) \to 2\, Z_1^2 + 2\, Z_2^2 - 2\, Z_3^2 \ . \tag{5.1.37}$$

So, in this case we have

$$\begin{aligned} Z_1 &= z_1 \ , \\ Z_2 &= -z_1 + w_1 \ , \\ Z_3 &= w_0 - w_1 + z_1 \ , \end{aligned} \tag{5.1.38}$$

where $z_v = \int_{\infty^*}^{v^*} \int_{\gamma_1} dt \wedge dX/Y = \int_{\infty^*}^{v^*} dt \;\; \omega_1$ for $v \in \{0, 1\}$. Notice that in this normalization the boundary does not depend on the remaining parameter b.

5.2 The period maps for Picard rank $\rho \geq 17$

For $\mathbf{X}_{a,b,c}$ the transcendental lattice is $\mathrm{T}(\mathbf{X}_{a,b,c}) = \mathrm{H}(2)^{\oplus 2} \oplus \langle -2 \rangle$. We look at the period domain

$$\mathcal{D} = \left\{ \vec{Z} \in \mathbb{P}(\mathrm{T} \otimes \mathbb{C}) \mid Q(\vec{Z}, \vec{Z}) = 0 \, , \; Q(\vec{Z}, \vec{Z}^*) > 0 \right\} . \tag{5.2.39}$$

Based on the exposition in [21] we give a description of $\mathcal{D}$ in Narain coordinates. One starts with a fixed choice of a sublattice V of $\mathrm{T}(\mathbf{X}_{a,b,c})$ which has rank 2 and is primitive and isotropic. For any $\vec{Z}$ representing a class in $\mathcal{D}$, the homomorphism

$$Q(\vec{Z}, \, \cdot \,) : \; \mathrm{V} \otimes \mathbb{R} \to \mathbb{C} \tag{5.2.40}$$

is an isomorphism of real vector spaces. If an orientation is chosen on $\mathrm{V} \otimes \mathbb{R}$, then the map (5.2.40) is either orientation preserving or orientation reversing depending on the component in which $[\vec{Z}]$ lies. We pick an orientation on $\mathcal{D}$ by choosing a connected component $\mathcal{D}^+$ such that for all $[\vec{Z}] \in \mathcal{D}^+$ the map (5.2.40) is orientation reversing.

We select linearly independent isotropic elements $\{x_1, x_2, y_1, y_2, u\}$ such that

$$\mathrm{T}(\mathbf{X}_{a,b,c}) = \mathbb{Z}\, y_1 \oplus \mathbb{Z}\, y_2 \oplus \mathbb{Z}\, x_1 \oplus \mathbb{Z}\, x_2 \oplus \mathbb{Z}\, u \; ,$$

with and $Q(x_i, y_i) = 4$ for $i = 1, 2$, $Q(u, u) = -2$. $\{y_1, y_2\}$ forms an oriented basis of V. The choice of $\{x_1, x_2, y_1, y_2, u\}$ is the same as defining an embedding of $\mathrm{H}(2)^{\oplus 2}$ into $\mathrm{T}(\mathbf{X}_{a,b,c})$ such that the image contains a sublattice V of rank 2 which is primitive and isotropic. With respect to the basis $\langle y_1, y_2, x_1, x_2, u\rangle$ the intersection form is

$$Q = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} . \tag{5.2.41}$$

For any class in $\mathcal{D}^+$ there is a normalized representative $\vec{Z}$ such that $Q(\vec{Z}, y_2) = 1$, hence $\vec{Z} = (\tau_1, 1)(\tau_2, v)(\mu)$ for some $\tau_1, \tau_2, v, \mu \in \mathbb{C}^*$. The first Hodge-Riemann relation $Q(\vec{Z}, \vec{Z}) = 0$ becomes

$$4(\tau_1\, \tau_2 + v) - 2\,\mu^2 = 0 \;\Rightarrow v = -\tau_1\, \tau_2 + \frac{1}{2}\mu^2 \; .$$

The second Hodge-Riemann relation $Q(\vec{Z}, \vec{Z}^*) > 0$ becomes

$$4\,\mathrm{Im}\tau_1\;\mathrm{Im}\tau_2 - 4\,(\mathrm{Im}\,\mu)^2 > 0 \; .$$

In the above convention $[\vec{Z}] \in \mathcal{D}^+$ means $\mathrm{Im}\tau_1 > 0$. Hence we have

$$2\,\mathrm{Im}\tau_1\;\mathrm{Im}\tau_2 > (\mathrm{Im}\,\mu)^2 > 0 \; .$$

We then have the following lemma:

Lemma 5.2.75. *There is a C^∞-isomorphism $\mathcal{D}^+ \to \mathfrak{H} \times \mathfrak{H} \times \mathbb{C}^*$ which associates to* *a period line $[\vec{Z}]$ the triple (τ_1, τ_2, μ). The isomorphism is given by*

$$\begin{aligned} \tau_1 &= \frac{z_1 - z_\infty - 2\,w_0 + w_\infty}{z_a} \,, \\ \tau_2 &= \frac{z_\infty - w_\infty + w_0}{z_a} \,, \\ \mu &= \frac{-z_1 + z_\infty + w_0 - w_\infty}{z_a} \,, \end{aligned} \tag{5.2.42}$$

where for $v \in \{0, 1, a, \infty\}$ we have used the K3 periods of $\mathbf{X}_{a,b,c}$

$$\begin{aligned} z_v &= \int_{c^*}^{v^*} \int_{\gamma_1} dt \wedge dX/Y = \int_{c^*}^{v^*} dt\ \omega_1 \,, \\ w_v &= \int_{c^*}^{v^*} \int_{\gamma_2} dt \wedge dX/Y = \int_{c^*}^{v^*} dt\ \omega_2 \,. \end{aligned} \tag{5.2.43}$$

Proof. Comparison of $\eta = (\tau_1, 1)(\tau_2, v)(\mu)$ with Equation (5.1.34) leads to

$$\tau_1 = \frac{Z_2'}{Z_1'} \,, \quad \tau_2 = \frac{Z_4'}{Z_1'} \,, \quad \mu = \frac{Z_5'}{Z_1'} \,, \tag{5.2.44}$$

where Z_1', Z_2', Z_4', Z_5' where defined in Equations (5.1.33). □

5.3 Some Picard-Fuchs equations for the twisted Legendre pencil

The quadratic period relations for the twisted Legendre pencil $\mathbf{X} = \mathbf{X}_{a,b,c}$ in §5.1 are manifest in the structure of the Picard-Fuchs operators annihilating their periods. In this section, we will see how some of the quadratic relations can be discovered by applying a very simple method: Fubini's theorem for multiple integrals, and integration by parts. In order to obtain the full Picard-Fuchs system, we appeal in §6.1 to a differential geometric structure on the period domain that is inherited by the existence of the quadratic period relations.

5.3.1 The Picard-Fuchs equations for $\rho \geq 17$

Recall that the Picard-Fuchs equation for the periods of the elliptic fibers on **X** was given in Equation (5.1.9) by the twisted second order hypergeometric operator

$$\frac{1}{\sqrt{h(t)}}\left(t(t-1)\,\frac{d^2}{dt^2}+(2t-1)\,\frac{d}{dt}+\frac{1}{4}\right)\left(\sqrt{h(t)}\,\omega(t)\right)=0\;,$$

where $h(t) = (t-a)(t-b)(t-c)$ is the twist factor for **X**. A basis of solutions is given by

$$\begin{aligned}\omega_1 &= \int_{\gamma_1}\frac{dX}{Y} = \frac{r}{\sqrt{h(t)}}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;t\right)\;,\\ \omega_2 &= \int_{\gamma_2}\frac{dX}{Y} = -\frac{\pi\,r}{\sqrt{h(t)}}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-t\right)\;,\end{aligned} \tag{5.3.45}$$

where $r = 2\pi/\sqrt{6}$. It follows from the explicit form Eqn (5.3.45) of the solution that

$$\frac{\partial^m}{\partial c^m}\,\omega(t) = \left(\frac{1}{2}\right)_m\,\frac{1}{(t-c)^m}\,\omega(t)\;, \tag{5.3.46}$$

where $(1/2)_m = 1/2\cdot(1/2+1)\cdots(1/2+m)$. We also have similar equations for derivatives with respect to a, b. Thus, for the period $Z = \int_0^1 dt\,\omega(t)$ we obtain

$$\frac{\partial^2 Z}{\partial b\partial c} = \frac{1}{2(b-c)}\left(\frac{\partial Z}{\partial b}-\frac{\partial Z}{\partial c}\right)\;, \tag{5.3.47}$$

and similar equations in volving (a, b) and (a, c). In other words, multiplying the period integral by rational functions in t of the type $1/(t-a)^m, 1/(t-b)^m, 1/(t-c)^m$, where $m \in \mathbb{N}$ is the same as differentiating the period integral m times with respect to that variable. This simple observation allows us to quickly compute differential relations, and hence, Picard-Fuchs operators, for the twisted Legendre pencil. We have the following easy computational result.

Proposition 5.3.76. *Let* $Z = \int_0^1 dt\,\omega(t)$ *be a period integral on the twisted Legendre pencil* $\mathbf{X}$*, and let* $Z_a = \partial Z/\partial a$*,* $Z_{ab} = \partial^2 Z/\partial a \partial b$*, etc. Then we have the following relations:*

$$\begin{aligned} \frac{1}{t-a} &= \frac{2Z_a}{Z}, \quad \frac{1}{t-b} = \frac{2Z_b}{Z}, \quad \frac{1}{t-c} = \frac{2Z_c}{Z}, \\ \frac{1}{(t-a)^2} &= \frac{4}{3}\frac{Z_{a,a}}{Z}, \quad \frac{1}{(t-b)^2} = \frac{4}{3}\frac{Z_{b,b}}{Z}, \quad \frac{1}{(t-c)^2} = \frac{4}{3}\frac{Z_{c,c}}{Z} . \end{aligned} \tag{5.3.48}$$

Similar relations hold for higher powers of $1/(t-a), 1/(t-b), 1/(t-c)$*.*

We invite the reader to verify these easy computations themselves.

The second order equation in (5.3.47) is a differential equation of Euler-Poisson-Darboux type for $N = 1/2$. The relationship of these equations, especially as related to the hypergeometric differential operator, was studied for example, by Miller in [98]. In addition to these three equations, we can obtain one more differential relation from the twisted hypergeometric opertator in Equation (5.1.9) as follows. We rewrite Equation (5.1.9)

$$\Big(R_2(t)\,\frac{d^2}{dt^2} + R_1(t)\,\frac{d}{dt} + R_0(t)\Big)\,\omega(t) = 0\,, \tag{5.3.49}$$

where by allowing the hypergeometric differential operator to interact with the twist factor $\sqrt{h(t)}$. By integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_0^1 dt\,\Big(R_2(t)\,\frac{d^2}{dt^2} + R_1(t)\,\frac{d}{dt} + R_0(t)\Big)\,\omega(t) \\ &= \int_0^1 dt\,\Big(R_2''(t) - R_1'(t) + R_0(t)\Big)\,\omega(t) \\ &+ \Big[R_2(t)\,\omega'(t) + \Big(-R_2'(t) + R_1(t)\Big)\,\omega(t)\Big]_0^1 . \end{aligned} \tag{5.3.50}$$

A careful expansion around the boundary points yields

$$\Big[R_2(t)\,\omega'(t) + \Big(-R_2'(t) + R_1(t)\Big)\,\omega(t)\Big]_0^1 = 0\,. \tag{5.3.51}$$

We conclude then that $R_2''(t) - R_1'(t) + R_0(t) = 0$. A partial fraction decomposition yields

$$\begin{aligned}
&R_2''(t) - R_1'(t) + R_0(t) \\
&= 1 + \frac{\alpha_2}{(t-a)^2} + \frac{\alpha_1}{t-a} + \frac{\beta_2}{(t-b)^2} + \frac{\beta_1}{t-b} \\
&\quad + \frac{\gamma_2}{(t-c)^2} + \frac{\gamma_1}{t-c} \,,
\end{aligned} \tag{5.3.52}$$

where the coefficients $\gamma_1, \gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2$ depend only on the parameters a, b, c. Upon substitution of the relations in Proposition 5.3.76, we obtain the following second order equation for the periods:

$$\begin{aligned}
0 = &\,(a-1)a\,\frac{\partial^2 Z}{\partial a^2} + b(b-1)\,\frac{\partial^2 Z}{\partial b^2} + c(c-1)\,\frac{\partial^2 Z}{\partial c^2} \\
&+ \frac{(6a^3 - 4a^2b - 4a^2c + 2abc - 5a^2 + 3ab + 3ac - bc)}{2(a-b)(a-c)}\,\frac{\partial Z}{\partial a} \\
&+ \frac{(4ab^2 - 2abc - 6b^3 + 4b^2c - 3ab + ac + 5b^2 - 3bc)}{2(b-c)(a-b)}\,\frac{\partial Z}{\partial b} \\
&+ \frac{(2abc - 4ac^2 - 4bc^2 + 6c^3 - ab + 3ac + 3bc - 5c^2)}{2(b-c)(a-c)}\,\frac{\partial Z}{\partial c} + Z
\end{aligned} \tag{5.3.53}$$

We are allowed to multiply the differential operator in Equation (5.1.9) by algebraic functions $f(t) \in \mathcal{O}_C^*$ - choosing $f(t)$ carefully allows us to obtain higher order equations using Proposition 5.3.76. For example, consider the following, where we have chosen $f(t) = t(t-1)/(t-c)^2$:

$$\frac{t(t-1)}{(t-c)^2\sqrt{h(t)}}\left(t(t-1)\,\frac{d^2}{dt^2} + (2t-1)\,\frac{d}{dt} + \frac{1}{4}\right)\left(\sqrt{h(t)}\,\omega(t)\right) = 0\,. \tag{5.3.54}$$

Following the steps above, a partial fraction decomposition yields

$$\begin{aligned} &R_2''(t) - R_1'(t) + R_0(t) \\ &= 1 + \frac{\alpha_2}{(t-a)^2} + \frac{\alpha_1}{t-a} + \frac{\beta_2}{(t-b)^2} + \frac{\beta_1}{t-b} \\ &\quad + \frac{\gamma_4}{(t-c)^4} + \frac{\gamma_3}{(t-c)^3} + \frac{\gamma_2}{(t-c)^2} + \frac{\gamma_1}{t-c} \ , \end{aligned} \tag{5.3.55}$$

where the coefficients $\gamma_1, \ldots, \gamma_4, \beta_1, \beta_2, \alpha_1, \alpha_2$ depend only on the parameters a, b, c. We then obtain the partial differential equation

$$\begin{aligned} 0 = &\frac{4\,\gamma_4}{35}\, Z_{cccc} + \frac{2\,\gamma_3}{5}\, Z_{ccc} + \gamma_2\, Z_{cc} + \frac{3\,\gamma_1}{2}\, Z_c \\ &+ \alpha_2\, Z_{aa} + \frac{3\,\alpha_1}{2}\, Z_a + \beta_2\, Z_{bb} + \frac{3\,\beta_1}{2}\, Z + \frac{3}{4}\, Z \ , \end{aligned} \tag{5.3.56}$$

with

$$
\begin{aligned}
\frac{4\,\gamma_4}{35} =& c^2\,(c-1)^2\ , \\
\frac{2\,\gamma_3}{5} =& \frac{c(c-1)\left(8abc-9ac^2-9bc^2+10c^3-4ab+5ac+5bc-6c^2\right)}{(a-c)(b-c)}\ , \\
\gamma_2 =& \frac{55a^2c^2-138ac^3+76c^4-55a^2c+152ac^2-83c^3+9a^2-32ac+16c^2}{4(c-a)^2} \\
& -\frac{7}{4}\frac{c^2\left(c^2-2c+1\right)}{(c-b)^2}-\frac{c\left(14ac^2-15c^3-21ac+23c^2+7a-8c\right)}{2(b-c)(a-c)}\ , \\
\frac{3\,\gamma_1}{2} =& \frac{3}{8}\frac{10a^3c-54a^2c^2+62ac^3-22c^4-5a^3+39a^2c-39ac^2+13c^3-4a^2}{(a-c)^3} \\
& -\frac{3}{2}\frac{c^2\left(c^2-2c+1\right)}{(b-c)^3}-\frac{3c\left(8ac^2-9c^3-12ac+14c^2+4a-5c\right)}{4(c-b)^2(a-c)} \\
& +\frac{-36a^2c^2+84ac^3-45c^4+36a^2c-90ac^2+48c^3-6a^2+18ac-9c^2}{4(b-c)(a-c)^2}\ , \\
\alpha_2 =& \frac{3}{4}\,\frac{a^2\,(a-1)^2}{(a-c)^2}\ , \\
\frac{3\,\alpha_1}{2} =& \frac{3}{4}\,\frac{a^2\,(a-1)^2\,(3a-2b-c)}{(a-c)^3(a-b)}\ , \\
\beta_2 =& \frac{3}{4}\,\frac{b^2\,(b-1)^2}{(b-c)^2}\ , \\
\frac{3\,\beta_1}{2} =& \frac{3}{4}\,\frac{b^2\,(b-1)^2\,(3b-2a-c)}{(b-c)^3(b-a)}\ .
\end{aligned}
\tag{5.3.57}
$$

The utility of these complicated expressions will become evident through the rest of this section.

5.3.2 The Picard-Fuchs equation for $\rho \geq 18$

Setting $a = 1$, we start with the Picard-Fuchs equation for the periods of the elliptic fibers

$$\begin{aligned}
\frac{(t-1)(t-b)}{(t-c)\sqrt{h(t)}}\Big(t(t-1)\,\frac{d^2}{dt^2} + (2t-1)\,\frac{d}{dt} + \frac{1}{4}\Big)\left(\sqrt{h(t)}\,\omega(t)\right) &= 0\;,\\
\frac{(t-1)(t-c)}{(t-b)\sqrt{h(t)}}\Big(t(t-1)\,\frac{d^2}{dt^2} + (2t-1)\,\frac{d}{dt} + \frac{1}{4}\Big)\left(\sqrt{h(t)}\,\omega(t)\right) &= 0\;,\\
\frac{(t-1)}{\sqrt{h(t)}}\Big(t(t-1)\,\frac{d^2}{dt^2} + (2t-1)\,\frac{d}{dt} + \frac{1}{4}\Big)\left(\sqrt{h(t)}\,\omega(t)\right) &= 0\;.
\end{aligned} \tag{5.3.58}$$

We have chosen a factor in front which make the boundary contribution disappear. We can then employ the same strategy of integrating by part and evaluating the boundary contributions as before. From those we only need the last one. The two equation we get are then the following:

$$Z_{bc} = \frac{1}{2(b-c)}\Big(Z_b - Z_c\Big)\;, \tag{5.3.59}$$

and

$$\begin{aligned}
0 = &\frac{b(b-1)^2}{2}Z_{bb} + \frac{c(c-1)^2}{2}Z_{cc}\\
&+ \frac{(b-1)(5b^2-3bc-3b+c)}{4(b-c)}Z_b + \frac{(c-1)(5c^2-3bc-3c+b)}{4(c-b)}Z_c\\
&+ \frac{2b+2c-3}{8}\,Z\;.
\end{aligned} \tag{5.3.60}$$

5.3.3 The Picard-Fuchs equation for $\rho \geq 19$

For $a = 1$, $b = 0$ and Picard rank $\rho \geq 19$ the differential equation (5.3.56) becomes

$$\begin{aligned}
0 = &\,c^2\,(c-1)^2 Z_{cccc} + 5c(2c-1)(c-1)\,Z_{ccc}\\
&+ \left(\frac{99}{4}c^2 - \frac{99}{4}c + 4\right)\,Z_{cc} + \frac{57}{8}(2c-1)\,Z_c + \frac{3}{4}\,Z\;.
\end{aligned} \tag{5.3.61}$$

This can be simplified to

$$
\begin{aligned}
0 = \frac{d}{dc}\bigg(& c^2\,(c-1)^2 Z_{ccc} + 3c(2c-1)(c-1)\,Z_{cc} \\
& + \frac{1}{4}\left(27c^2 - 27c + 4\right)\,Z_c + \frac{3}{8}(2c-1)\,Z\bigg)\ .
\end{aligned}
\tag{5.3.62}
$$

We could have obtained the differential equation

$$
0 = c^2\,(c-1)^2 Z_{ccc} + 3c(2c-1)(c-1)\,Z_{cc} \quad + \frac{1}{4}\left(27c^2 - 27c + 4\right)\,Z_c + \frac{3}{8}(2c-1)\,Z\ ,
\tag{5.3.63}
$$

by starting from the differential equation for the periods of the elliptic fiber

$$
\frac{t(t-1)}{(t-c)\,\sqrt{h(t)}}\left(t(t-1)\,\frac{d^2}{dt^2} + (2t-1)\,\frac{d}{dt} + \frac{1}{4}\right)\left(\sqrt{h(t)}\,\omega(t)\right) = 0\ ,
\tag{5.3.64}
$$

integrating by parts and evaluating the boundary contributions. In turn, the differential operator

$$
\widehat{\mathcal{O}}_2 := c^2\,(c-1)^2 \frac{d^3}{dc^3} + 3c(2c-1)(c-1)\,\frac{d^2}{dc^2} \quad + \frac{1}{4}\left(27c^2 - 27c + 4\right)\,\frac{d}{dc} + \frac{3}{8}(2c-1)
\tag{5.3.65}
$$

is the symmetric square $\widehat{\mathcal{O}}_2 = \widehat{\mathcal{O}}_1^{\otimes 2}$ of the differential operator

$$
\widehat{\mathcal{O}}_1 := c(c-1)\frac{d^2}{dc^2} + (2c-1)\,\frac{d}{dc} + \frac{3}{16}\ .
\tag{5.3.66}
$$

The periods of the K3 surface $\mathbf{X}_{0,1,c}$ satisfy the differential equation $\widehat{\mathcal{O}}_2 Z = 0$ and its projective solution is the period map. The K3 surface $\mathbf{X}_{0,1,c}$ is related to the K3 surface $\mathbf{Y}_{0,1,c}$ by a Shioda-Inose structure. The K3 surface $\mathbf{Y}_{0,1,c}$ is the Kummer surface $\mathrm{Kum}(\mathcal{E}_c \times \mathcal{E}_c)$ where the elliptic curve $\mathcal{E}_c$ is given by

$$\mathcal{E}_c : \quad \left\{ (y,x) \mid y^2 = 4\,x^3 - 3\left(1 - \frac{3\,c}{4}\right)\,x + \left(1 - \frac{9\,c}{8}\right) \right\} \,. \tag{5.3.67}$$

The periods of the elliptic curve $\mathcal{E}_c$ satisfy die Picard-Fuchs equation $\widehat{\mathcal{O}}_2\,\omega(c) = 0$.

5.3.4 The case of $\rho \geq 19$ with full level-two structure

Lastly in this section we consider the Picard rank $\rho = 19$ K3 surface $\mathbf{X}_{a,\infty,0}$. By following the methods above, we arrive at the Picard-Fuchs equation

$$0 = 2(a-1)a^2 Z_{aaa} + 3a(3a-2)Z_{aa} + \left(\frac{13}{2}a - 2\right) Z_a + \frac{1}{4} Z \,. \tag{5.3.68}$$

This operator is, in fact, the Fuchsian operator that annihilates the hypergeometric function ${}_3F_2(1/2, 1/2, 1/2; 1, 1 \mid a)$. Similar to above, the operator

$$\widetilde{\mathcal{O}}_2 := 2(a-1)a^2 \frac{d^3}{da^3} + 3a(3a-2)\frac{d^2}{da^2} + \left(\frac{13}{2}a - 2\right) \frac{d}{da} + \frac{1}{4} \tag{5.3.69}$$

is the symmetric square $\widetilde{\mathcal{O}}_2 = \widetilde{\mathcal{O}}_1^{\otimes 2}$ of the second order Fuchsian operator

$$\widetilde{\mathcal{O}}_1 := 2a(a-1)\frac{d^2}{da^2} + (3a-2)\frac{d}{da} + \frac{1}{8} \,. \tag{5.3.70}$$

This is the hypergeometric operator for ${}_2F_1(1/4, 1/4; 1/2 \mid 1-a)$. For reasons that will be explained in §6.2.3, this operator is naturally associated to the elliptic curve

$$\mathcal{E}_{\lambda^2} : \quad \left\{ (y,x) \mid y^2 = \frac{1}{(\lambda^2 - 1)} x(x-1)\left(x - \lambda^2\right) \right\} \,, \tag{5.3.71}$$

where $a = -4\lambda^2/(\lambda^2-1)^2$. The K3 surface $\mathbf{X}_{a,\infty,0}$ is related to the Kummer surface $\mathbf{Y}'_{\lambda^2,\lambda'^2,0} = \mathrm{Kum}(\mathcal{E}_{\lambda^2} \times \mathcal{E}'_{\lambda^2})$, where $\mathcal{E}'_{\lambda^2}$ is 2-isogenous to $\mathcal{E}_{\lambda^2}$ and λ'^2 is the elliptic modulus of $\mathcal{E}'_{\lambda^2}$, by Shioda-Inose structure and underlying $(2,2)$-isogeny of Kummer surfaces [26].

CHAPTER 6

Special geometry of the moduli space

In the previous sections, we have seen that the presentation of the $\mathbf{X}_{a,b,c}$ as the twisted Legendre pencil

$$\mathbf{X}_{a,b,c} \;:\; y^2 = (t-a)(t-b)(t-c)x(x-1)(x-t)$$

allows one to compute differential relations satisfied by the period integral by simply applying Fubini's theorem for multivariable integrals, and integration by parts. Due to rank considerations of the Picard-Fuchs system for $\rho = 19$, this method is completely sufficient to capture the entire differential system annihilating the period integrals, and also for finding higher order differential relations for univariate restrictions of the twisted Legendre pencil. However, the case of primary interest is that of Picard rank $\rho = 17$, for which there is only an incomplete story known in current literature to the best of our knowledge. The method employed above only captures part of the full rank 5 system. However, the work expended by computing the quadratic period relations in §5.1 pays off greatly, since we know that the solutions of the Picard-Fuchs system are quadratically related, or satisfies the *quadric condition*.

This imposes very strong geometric consequences on the moduli space $\mathfrak{T}$ for $\mathbf{X}_{a,b,c}$. By employing the differential geometric techniques of Sasaki & Yoshida [123], combined with their work with Matsumoto for the double sextic family [96], we will compute the full Picard-Fuchs system of the family $\mathbf{X}_{a,b,c}$. Moreover, by utilizing the dominant rational map $\phi : \mathcal{M}_2 \dashrightarrow \mathfrak{T}$ from Equation (2.1.49), where $\mathcal{M}_2$ is the double cover of the moduli space $\mathcal{M}[2]$ of genus 2 curves $\mathbf{C}$ with level-2 structure parameterized by the Rosenhain roots of a 2-isogenous curve $\mathbf{C}'$, we recover the uniformizing Picard-Fuchs equations for the Kummer surface $\mathrm{Kum}(\mathrm{Jac}(\mathbf{C}))$, which were computed in [123]. Here $\mathrm{Jac}(\mathbf{C})$ is equipped with full level-two structure. This differential system, being the Picard-Fuchs system of a family of lattice polarized K3 surfaces, also satisfies the quadric condition - in this case, the quadric condition is explicitly manifest, as the Picard-Fuchs system was computed in [60] to be the exterior product of the Lauricella F_D system that annihilates the hyperelliptic period integrals of the genus 2 curve $\mathbf{C}$. In this way, the work in this dissertation completely answers the questions posed by Hoyt in [69]. Moreover, we show how restricting this system in the limit $c \to 0$ recovers exactly previous work of Clingher, Doran, and Malmendier [26] in the Picard rank $\rho = 18$ case.

6.1 Geometry of certain uniformizing differential equations

In this section, we recall the theory of orbifold uniformizing differential equations of Sasaki & Yoshida [123], and the connection to holomorphic conformal geometry. We show that such an integrable holomorphic conformal structure can be realized as a "flat" special geometry on the moduli space, as in physics.

6.1.1 Linear differential equations associated to projective hypersurfaces

Let $\mathbf{M}$ be a connected, complex n-dimensional orbifold.[1] We recall the fundamental construction of linear differential equations of rank $n+2$ in n variables that are associated to immersed hypersurfaces $\psi : \mathbf{M} \to \mathbb{P}^{n+1}$. The case of $n = 2$ for surfaces

[1]This construction carries over in its entirety to the real category, where perhaps the geometric content is more directly recognizable. We remark that the geometric content of this section is *not* reflective of the hermitian geometry of $\mathbf{M}$; rather, it describes the algebro-geometric nature of $\mathbf{M}$, realized as an immersed hypersurface of $\mathbb{P}^{n+1}$.

was classically treated by Wilczynski [143], and $n \geq 3$ by Sasaki [120] and Sasaki & Yoshida [123]. We are exclusively concerned with the latter case, in fact solely for $n = 3$, but present the general story. Let $z^1, \ldots, z^n \in \mathbf{M}$ be local coordinates, and $\mathbb{P}^{n+1} = \mathbb{P}(t^0, \ldots, t^{n+1})$ be homogeneous coordinates. Assume that for some fixed $1 \leq \alpha, \beta \leq n$, the $n+2$ vectors $\{\psi, \psi_1, \ldots, \psi_n, \psi_{\alpha\beta}\} \subset J^2(\underline{\mathbb{C}}^{n+2}, \mathbb{P}^{n+1})$ form a rank $n+2$ linear subbundle $\mathbf{S} \subseteq J^2(\underline{\mathbb{C}}^{n+2}, \mathbb{P}^{n+1})$, where $\psi_\mu = \partial\psi/\partial z^\mu$, $\psi_{\mu\nu} = \partial^2\psi/\partial z^\mu \partial z^\nu$, etc, and $J^2(\underline{\mathbb{C}}^{n+2}, \mathbb{P}^{n+1})$ is the second jet bundle of the trivial $(n+2)$-plane bundle $\underline{\mathbb{C}}^{n+2} \to \mathbb{P}^{n+1}$. Then after projecting into $\mathbf{S}$, the remaining second order derivatives $\psi_{\mu\nu}$ satisfy linear dependence relations, which we express in the form

$$\frac{\partial^2\psi}{\partial z^\mu \partial z^\nu} = G_{\mu\nu}\frac{\partial^2\psi}{\partial z^\alpha \partial z^\beta} + A^\gamma_{\mu\nu}\frac{\partial\psi}{\partial z^\gamma} + A^0_{\mu\nu}\psi \tag{6.1.1}$$

where we utilize the summation convention for repeated indices, for some meromorphic functions $G_{\mu\nu}, A^k_{\mu\nu}, A^0_{\mu\nu}$. Moreover, we assume the indices μ, ν are symmetric, and we set $G_{\alpha\beta} = 1, A^k_{\alpha\beta} = 0 = A^0_{\alpha\beta}$.

A crucial behaviour of the system above is the how the components behave under projective rescaling $\psi \mapsto f\psi$, where $f \in \mathcal{O}^*_{\mathbf{M}}$ is a suitable regular function. We call such a transformation on ψ a *projective gauge transformation*, because of how the connection matrix representing the linear system in Equation (6.1.1) transforms under this transformation. One may verify directly that under any projective gauge transformation, the coefficients $G_{\mu\nu}$ of $\psi_{\alpha\beta}$ are invariant. This suggests heavily that they are actually the components of a symmetric tensor $\mathbf{G} = G_{\mu\nu}\, dz^\mu \otimes dz^\nu \in S^2(\mathbf{M})$ that determines a conformal structure on $\mathbf{M}$. We will see shortly that this conformal structure $\mathbf{G}$ is actually the driving force behind the theory of these differential equations, and is indispensable in our study of Picard-Fuchs equations for the the twisted Legendre pencil. Because of their prominent role, the components $G_{\mu\nu}$ are called the *principal part* of the Equation (6.1.1).

Consider the following rank four system in $n = 2$ variables, with $(\alpha, \beta) = (1, 2)$, given by

$$\begin{aligned}
\frac{\partial^2\psi}{\partial z_1^2} &= \frac{z_2}{1-z_1}\frac{\partial^2\psi}{\partial z_2\partial z_1} - \frac{(1-2z_1)}{z_1(1-z_1)}\frac{\partial\psi}{\partial z_1} + \frac{z_2}{2z_1(1-z_1)}\frac{\partial\psi}{\partial z_2} + \frac{1}{4z_1(1-z_1)}\psi\,,\\
\frac{\partial^2\psi}{\partial z_2^2} &= \frac{z_1}{1-z_2}\frac{\partial^2\psi}{\partial z_1\partial z_2} - \frac{(1-2z_2)}{z_2(1-z_2)}\frac{\partial\psi}{\partial z_2} + \frac{z_1}{2z_2(1-z_2)}\frac{\partial\psi}{\partial z_1} + \frac{1}{4z_2(1-z_2)}\psi\,.
\end{aligned}$$

This Fuchsian system is that which annihilates Appell's bivariate hypergeometric function $F_2(1/2, 1/2, 1/2\,; 1, 1\,|z_1, z_2)$, and is defined on the quasiprojective variety $\mathbf{M} = \mathbb{P}^2 - \mathcal{L}$, where $\mathcal{L} \subset \mathbb{C}^2$ is the union of the lines

$$\{\, z_1 = 0, \quad z_1 = 1, \quad z_1 = \infty, \quad z_2 = 0, \quad z_2 = 1, \quad z_2 = \infty, \quad z_1 + z_2 = 1\,\}\ .$$

In this case, the principal part of the equation is given by $G_{1,1} = z_2/(1-z_1)$ and $G_{2,2} = z_1/(1-z_2)$. Let $f \in \mathcal{O}^*_{\mathbf{M}}$ be any function. Then the projective gauge transformation $\psi \mapsto f\psi$ yields the new system

$$\begin{aligned}
\frac{\partial^2\psi}{\partial z_1^2} &= \frac{z_2}{1-z_1}\frac{\partial^2\psi}{\partial z_2\partial z_1} + \frac{z^2\left(2\left(\frac{\partial f}{\partial z^1}\right)z^1 + f\right)}{2fz^1(1-z^1)}\frac{\partial\psi}{\partial z^2} + \cdots\,,\\
\frac{\partial^2\psi}{\partial z_2^2} &= \frac{z_1}{1-z_2}\frac{\partial^2\psi}{\partial z_1\partial z_2} + \frac{z^1\left(2\left(\frac{\partial f}{\partial z^2}\right)z^2 + f\right)}{2fz^2(1-z^2)}\frac{\partial\psi}{\partial z^1} + \cdots
\end{aligned}$$

which shows that the principal part $G_{1,1}, G_{2,2}$ of the equations remains unchanged.

When $\psi : \mathbf{M} \to \mathbb{P}^{n+1}$ takes $\mathbf{M}$ to its universal cover $\mathbb{U} \subset \mathbb{P}^{n+1}$, then the map ψ is called the *developing map* of the orbifold $\mathbf{M}$, and the system (6.1.1) is called the system of *uniformizing differential equations* for $\mathbf{M}$. When the system (6.1.1) is the Picard-Fuchs system for a family of complex projective varieties parameterized by $\mathbf{M}$, we have *Picard-Fuchs uniformization.* This phenomena was studied for example by Doran in [36, 37].

For the cases we are interested in, we will see in Theorem 6.1.78 below that the universal cover $\mathbb{U}$ is contained in a hyperquadric $\mathbf{Q} \subset \mathbb{P}^{n+1}$. This is, at the very least, feasible, since all hyperquadrics are simply connected over $\mathbb{C}$. In this case, we say that the map ψ - and equivalently, the system (6.1.1) - satisfies the *quadric condition.* We have mentioned an algebraic incarnation of the quadric condition above - the one described here is geometric. They are equivalent.

Definition 6.1.77. *A system (6.1.1) of rank $n+2$ satisfies the* quadric condition *if its solutions are quadratically related. This is equivalent to projectivized vector of solutions $\psi : \mathbf{M} \to \mathbb{P}^{n+1}$ lying on a hyperquadric $\mathbf{Q} \subset \mathbb{P}^{n+1}$. In this case, the universal cover $\pi : \mathbb{U} \to \mathbf{M}$ of the orbifold $\mathbf{M}$ is the image of the multivalued developing map ψ, $\psi(\mathbf{M}) = \mathbb{U} \subseteq \mathbf{Q}$, and the linear system (6.1.1) is the system of uniformizing differential equations for $\mathbf{M}$. Moreover, the solution ψ of (6.1.1) is the multivalued inverse of π, $\pi \circ \psi = \mathrm{id}_{\mathbf{M}}$. We may say equivalently that $\psi, \mathbf{M}$ or Equation (6.1.1) satisfies the quadric condition.*

From this point forward, we assume that $\mathbf{M}$ satisfies the quadric condition. There is a strong analogy between the intrinsic conformal geometry of $\mathbf{M}$, induced by pull-back of the canonical (holomorphic) conformal metric $\boldsymbol{\gamma} = \delta_{\mu\nu}\, dt^{\mu} \otimes dt^{\nu} \in S^2(\mathbb{P}^{n+1})$ by the developing map, and the classical hypersurface geometry of a Riemannian n-manifold immersed in $\mathbb{R}^{n+1}$. Here, $\delta_{\mu\nu}$ is the Kronecker delta. Let $\mathbf{g} = \psi^* \boldsymbol{\gamma} \in S^2(\mathbf{M})$, which is a nondegenerate, symmetric tensor uniquely determined up to conformal transformation. This allows one to realize the coefficients of the linear equations in Equation (6.1.1) purely in terms of the holomorphic conformal geometry of $\mathbf{M}$. In particular, there are analogues of the classical Gauss and Codazzi equations that express the compatibility of the intrinsic holomorphic conformal geometry of $\mathbf{M}$ with the natural holomorphic conformal geometry induced by the immersion ψ. This is fortunate, because we can detect the quadric condition purely from the intrinsic geometry of $\mathbf{M}$. The primary notion is that of *conformally flat*.

For the purpose of computations, recall that any symmetric, nondegenerate tensor $\mathbf{g} \in S^2(\mathbf{M})$ defines a linear, $\mathbf{g}$-compatible torsion free connection ∇ on the tangent bundle $T\mathbf{M}$ by $\nabla_\nu \partial_\mu = \Gamma^\rho_{\mu\nu} \partial_\rho$, where $\partial_\mu = \partial/\partial z^\mu$ and $\nabla_\mu = \nabla_{\partial_\mu}$. The coefficients - the *Christoffel symbols* $\Gamma^\rho_{\mu\nu}$ - are computed explicitly from $\mathbf{g} = g_{\mu\nu}\, dz^\mu \otimes dz^\nu$ as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\, g^{\rho\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial z^\nu} + \frac{\partial g_{\nu\lambda}}{\partial z^\mu} - \frac{\partial g_{\mu\nu}}{\partial z^\lambda} \right), \tag{6.1.2}$$

where $g^{\mu\nu}$ are the components of the inverse tensor to $\mathbf{g}$. The curvature tensor $\mathrm{R}(\mathbf{g}) = R^\mu_{\nu\kappa\lambda} \partial_\mu \otimes dz^\nu \otimes dz^\kappa \otimes dz^\lambda \in T^1_3(\mathbf{M})$ is defined in terms of ∇ as

$$\mathrm{R}(\mathbf{g})(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\,.$$

for (local) vector fields X, Y on $\mathbf{M}$. In this way, $\mathrm{R}(\mathbf{g})$ can be thought of as a section of the bundle $\Lambda^2(\mathrm{End}(T\mathbf{M}))$ of $T\mathbf{M}$-endomorphism valued 2-forms. Then the components of the curvature tensor can be expressed in terms of the Christoffel symbols as

$$R^\mu_{\nu\kappa\lambda} = \partial_\kappa \Gamma^\mu_{\lambda\nu} - \partial_\lambda \Gamma^\mu_{\kappa\nu} + \Gamma^\mu_{\kappa\alpha} \Gamma^\alpha_{\lambda\nu} - \Gamma^\mu_{\lambda\alpha} \Gamma^\alpha_{\kappa\nu}\,, \tag{6.1.3}$$

and the *Ricci curvature* tensor $\mathrm{Ric}(\mathbf{g}) = R_{\mu\nu}\, dz^\mu \otimes dz^\nu \in S^2(\mathbf{M})$ is the contraction

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}\,. \tag{6.1.4}$$

Contracting $\mathrm{Ric}(\mathbf{g})$ with $\mathbf{g}$ yields the *scalar curvature* $\mathrm{s}(\mathbf{g})$

$$\mathrm{s}(\mathbf{g}) = g^{\mu\nu} R_{\mu\nu}\,. \tag{6.1.5}$$

The curvature tensor $\mathrm{R}(\mathbf{g})$ can be decomposed into a conformally invariant piece, the *conformal curvature tensor* or *Weyl tensor* $\mathrm{W}(\mathbf{g})$ such that

$$W^{\mu}_{\nu\kappa\lambda} = R^{\mu}_{\nu\kappa\lambda} + S_{\nu\kappa}\delta^{\mu}_{\lambda} - S_{\nu\lambda}\delta^{\mu}_{\kappa} + g_{\nu\kappa}g^{\mu\alpha}S_{\alpha\lambda} - g_{\nu\lambda}g^{\mu\alpha}S_{\alpha\kappa}\,, \tag{6.1.6}$$

where $S_{\mu\nu}$ are the components of the *Schouten tensor* $\mathrm{S}(\mathbf{g}) = S_{\mu\nu}\,dz^{\mu}\otimes dz^{\nu} \in S^2(\mathbf{M})$, defined as

$$S_{\mu\nu} = \frac{1}{n-2}\left(R_{\mu\nu} - \frac{\mathrm{s}(\mathbf{g})}{2(n-1)}g_{\mu\nu}\right)\,. \tag{6.1.7}$$

The conformal class $[\mathbf{g}]$ of the tensor $\mathbf{g}$ is a linear subbundle of $S^2(\mathbf{M})\times\mathbb{C}^*$ generated by ray through $\mathbf{g}$. Hence, $[\mathbf{g}]$ is given by the collection of all $\exp(\theta)\mathbf{g}$ such that $\exp(\theta) \in \mathcal{O}^*_{\mathbf{M}}$ is a global nonvanishing regular function. We say $(\mathbf{M}, \mathbf{g})$ is *conformally flat* if there is a $\exp(\theta) \in \mathcal{O}^*_{\mathbf{M}}$ such that the tensor $\widehat{\mathbf{g}} = \exp(\theta)\mathbf{g}$ is flat, $\mathrm{R}(\widehat{\mathbf{g}}) = 0$. If $n \geq 4$, M is conformally flat if and only if the conformal curvature tensor $\mathrm{W}(\mathbf{g}) = 0$. If $n = 3$, the failure of $\mathbf{M}$ to be conformally flat is measured by the Cotton tensor $\mathrm{C}(\mathbf{g}) = C_{\mu\nu\lambda}\,dz^{\mu}\otimes dz^{\nu}\otimes dz^{\lambda} \in T^0_3(\mathbf{M})$, which is defined in terms of the Schouten tensor $\mathrm{S}(\mathbf{g})$ as

$$C_{\mu\nu\lambda} = \left(\nabla_{\mu}\,\mathrm{S}(\mathbf{g})\right)_{\nu\lambda} - \left(\nabla_{\nu}\,\mathrm{S}(\mathbf{g})\right)_{\mu\lambda}\,. \tag{6.1.8}$$

We have that $(\mathbf{M}, \mathbf{g})$ is conformally flat in $n = 3$ if and only if $\mathrm{C}(\mathbf{g}) = 0$.

We have the following primordial example. [The case that $\mathbf{M} = \mathbf{Q}$, $\psi : \mathbf{M} = \mathbf{Q} \hookrightarrow \mathbb{P}^{n+1}$ is the inclusion map] Let $\mathbf{Q} \subset \mathbb{P}^{n+1}$ by a hyperquadric as in Equation (2.1.9). Hence $\mathbf{Q}$ is the locus of the equation $Q_{\mu\nu}\,t^{\mu}t^{\nu} = 0$, where $(Q_{\mu\nu}) \in \mathrm{Mat}(n+2, \mathbb{C})$ is a square, nondegenerate, symmetric matrix. After a change of basis by an automorphism $X \in \mathrm{GL}(n+2, \mathbb{C})$ of $\mathbf{Q}$, we can always write the equation as $-t^0t^{n+1} + \widetilde{Q}_{\mu\nu}\,t^{\mu}t^{\nu} = 0$. Then we obtain a canonical conformal on $\mathbf{Q}$ by utilizing local coordinates. Take, for example, a local coordinate chart $f : \mathbb{C}^n \hookrightarrow \mathbf{Q} \subset \mathbb{P}^{n+1}$ is given by

$$f : (z^1, \dots, z^n) \mapsto \Big[t^0 = 1 \,:\, t^1 = z^1 \,:\, \cdots \,:\, t^{n+1} = \widetilde{Q}_{\mu\nu}\, t^\mu t^\nu \Big] \in \mathbb{P}^{n+1} .$$

Then pulling back the holomorphic tensor $\boldsymbol{\gamma} = \delta_{\mu\nu}\, dt^\mu \otimes dt^\nu$ by the chart map f gives the nondegenerate, symmetric tensor

$$\boldsymbol{\varphi} = f^* \boldsymbol{\gamma} = \widetilde{Q}_{\mu\nu}\, dt^\mu \otimes dt^\nu \in S^2(\mathbf{M}) . \tag{6.1.9}$$

Note that $\boldsymbol{\varphi}$ flat, and hence, conformally flat. Moreover, this construction is clearly induced by the inclusion map $\mathbf{Q} \hookrightarrow \mathbb{P}^{n+1}$. We call $\boldsymbol{\varphi}$ the *canonical conformal structure* of the hyperquadric $\mathbf{Q}$.

We utilize an old result of Kuiper, phrased it in slightly more modern language.

Theorem 6.1.78 (Kuiper [80], Theorem 4′, Theorem 4″)**.** *Suppose that* $(\mathbf{M}, \mathbf{g})$ *is a connected, conformally flat orbifold of dimension* n*,* $\pi : \mathbb{U} \to \mathbf{M}$ *be the universal cover, and* $\psi : \mathbf{M} \to \mathbb{U}$ *be the developing map. Then there is a hyperquadric* $(\mathbf{Q}, \boldsymbol{\varphi}) \subset \mathbb{P}^{n+1}$ *equipped with the canonical conformal structure* $\boldsymbol{\varphi}$ *and an embedding* $\imath : \mathbb{U} \hookrightarrow \mathbf{Q}$*, unique up to ambient conformal transformations, such that* $\mathbf{g} = (\imath \circ \psi)^* \varphi$*. In particular,* $\mathbf{M}$ *is conformally immersed in a hyperquadric* $\mathbf{Q}$*. Conversely, if* $\psi : \mathbf{M} \to (\mathbf{Q}, \boldsymbol{\varphi})$ *is an immersion, then* $(\mathbf{M}, \psi^* \boldsymbol{\varphi})$ *is conformally flat.*

We will be slightly sloppy from this point forward and not distinguish between the developing map $\psi : \mathbf{M} \to \mathbb{U}$ and the map $\imath \circ \psi : \mathbf{M} \to \mathbf{Q}$, and simply say "the developing map $\psi : \mathbf{M} \to \mathbf{Q}$".

We have established that the period map $\psi : \mathfrak{T} \to \mathbb{P}^4$ for the twisted Legendre family $\mathbf{X}_{a,b,c}$ lies on the hyperquadric $\mathbf{Q} \subset \mathbb{P}^4$ from the matrix in Equation (5.2.41), given in the variables of this section by the locus of the equation

$$2t^1t^3 + 2t^2t^4 - (t^5)^2 = 0 \tag{6.1.10}$$

If we knew explicitly the conformal metric $\mathbf{g} = \psi^*\boldsymbol{\varphi}$, we could directly appeal to the following result that allows us to compute the Picard-Fuchs system directly.

Theorem 6.1.79 (Sasaki & Yoshida [123], Theorem 2.5)**.** *Assume $n \geq 3$, and let $\mathbf{M}$ be an n-manifold with local coordinates $z^1, \dots, z^n \in \mathbf{M}$. Let $\mathbf{g} = g_{\mu\nu}\, dz^\mu \otimes dz^\nu \in S^2(\mathbf{M})$ be a symmetric, nondegenerate tensor that is conformally flat. Then the linear system*

$$g_{\mu\nu}\left(\psi_{\kappa\lambda} - \Gamma^\rho_{\kappa\lambda}\psi_\rho + \frac{1}{n-2}R_{\kappa\lambda}\psi\right) = g_{\kappa\lambda}\left(\psi_{\mu\nu} - \Gamma^\rho_{\mu\nu}\psi_\rho + \frac{1}{n-2}R_{\mu\nu}\psi\right) \tag{6.1.11}$$

is of rank $n+2$ and satisfies the quadric condition for some hyperquadric $\mathbf{Q}$. The vector of solutions $\psi : \mathbf{M} \to \mathbf{Q}$ is the developing map.

While the analysis of Endo & Hoyt utilized in §5.1 allows one to know the explicit quadratic period relations of the family $\mathbf{X}$, the period map as described therein does not provide a very useful way to know the explicit relationship between the moduli (a, b, c) the hyperquadric $\mathbf{Q}$ in Equation (6.1.10). Using the GKZ formalism (under appropriate restriction) for the period integral of the twisted Legendre pencil [50], one may compute an analytic, and thus, transcendental, expression for the period map, but we would like a suitable algebro-geometric description. Thus, we seek out another method to find the explicit conformal structure $\mathbf{g}$ on $\mathfrak{T}$. Fortunately, as might be guessed from the explicit geometric content of the linear equations in Equation (6.1.11), it turns out that any rank $n+2$ linear system in n variables of the form given in Equation (6.1.1) carries differential geometric content. We have the following result.

Theorem 6.1.80 (Sasaki & Yoshida [123], Theorem 2.4)**.** *Assume $n \geq 3$. Fix indices $1 \leq \alpha, \beta \leq n$. Let $\mathbf{G} = G_{\mu\nu}\, dz^{\mu} \otimes dz^{\nu} \in S^2(\mathbf{M})$ be a symmetric, nondegenerate tensor that is conformally flat, scaled such that $G_{\alpha\beta} = 1$. Define θ so that $\det \widehat{\mathbf{G}} = 1$, where $\widehat{\mathbf{G}} = \exp(\theta)\mathbf{G}$. Define functions $A^{\lambda}_{\mu\nu}, A^{0}_{\mu\nu}$ by*

$$A^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - G_{\mu\nu}\Gamma^{\lambda}_{\alpha\beta},$$
$$A^{0}_{\mu\nu} = -S_{\mu\nu} + G_{\mu\nu}S_{\alpha\beta}$$

with the Christoffel symbols and Schouten tensor computed according to the tensor $\widehat{\mathbf{G}}$. Then the linear system

$$\frac{\partial^2 \psi}{\partial z^{\mu} \partial z^{\nu}} = G_{\mu\nu} \frac{\partial^2 \psi}{\partial z^{\alpha} \partial z^{\beta}} + A^{\gamma}_{\mu\nu} \frac{\partial \psi}{\partial z^{\gamma}} + A^{0}_{\mu\nu}\psi$$

from Equation (6.1.1) is of rank $n+2$ and satisfies the quadric condition for some hyperquadric $\mathbf{Q} \subset \mathbb{P}^{n+1}$. The converse is true, in the sense the following sense: given a linear system of the form (6.1.1) of rank $n+2$ in n variables that satisfies the quadric condition for some hyperquadric $\mathbf{Q} \subset \mathbb{P}^{n+1}$, then $\mathbf{G} = G_{\mu\nu}\, dz^{\mu} \otimes dz^{\nu} \in S^2(\mathbf{M})$ is conformally flat, and the coefficients $A^{\lambda}_{\mu\nu}, A^{0}_{\mu\nu}$ of the system are realized in terms of the normalized tensor $\widehat{\mathbf{G}}$, its Christoffel symbols, and Schouten tensor as above. The vector of solutions $\psi : \mathbf{M} \to \mathbf{Q}$ is the developing map.

One should think of Theorem 6.1.80 as the differential equations version of the geometric incarnation in Theorem 6.1.78 of Kuiper above.

6.1.2 The quadric condition and special geometry

There is an additional piece of geometric data that allows one to conclude that $\mathbf{M}$ satisfies the quadric condition, the vanishing of the so-called *Wilczynski-Fubini-Pick form* $\mathbf{\Phi} = \Phi_{ijk}\, dz^i \otimes dz^j \otimes dz^k \in S^3(M)$. The components Φ_{ijk} are obtained from the conformal structure $G_{\mu\nu}$ on $\mathbf{M}$ and the connection matrix $\boldsymbol{\omega}$ from writing the differential system (6.1.1) as a Pfaffian system, $d\psi = \boldsymbol{\omega}\psi$, where $\boldsymbol{\omega} = (\omega^i_j)$ is the $(n+2) \times (n+2)$ matrix of 1-forms obtained from the linear system. See Sasaki, Sasaki & Yoshida [120, 123]. The following result is crucial.

Theorem 6.1.81 (Sasaki [120])**.** *Let* $\mathbf{M}$ *be a connected piece of a hypersurface in* $\mathbb{P}^{n+1}$. *Assume the quadratic form* $\mathbf{G}$ *is nondegenerate and the Wilczynski-Fubini-Pick form* $\boldsymbol{\Phi}$ *vanishes identically. Then* $\mathbf{M}$ *is contained in a hyperquadric.*

Sasaki & Yoshida utilized the holomorphic cubic form $\boldsymbol{\Phi}$ in order to connect with the classical work of Wilczynski, as well as to demonstrate that the quadric condition can be detected from the intrinsic algebro-geometric data of $\mathbf{M}$, i.e., an ample line bundle $\mathcal{L} \to \mathbf{M}$ yielding the embedding $\psi : \mathbf{M} \hookrightarrow \mathbb{P}^{n+1}$, and an intrinsic rank-$(n+2)$ holomorphic $\mathrm{SL}(n+2, \mathbb{C})$-bundle $E \to \mathbf{M}$ on which $\boldsymbol{\omega}$ is an integrable connection. This perspective is equivalent to the discussion at the beginning of this section. In particular, one may think of $\boldsymbol{\Phi}$ as measuring the failure of the linear system (6.1.1) to satisfy the quadric condition.

Since the orbifolds that we are interested in satisfy the quadric condition, we do not need to compute $\boldsymbol{\Phi}$ directly to check that the conclusion of Theorem 6.1.81 holds. We conclude this indirectly by showing that $(\mathbf{M}, \mathbf{g})$ is conformally flat directly by using either the Weyl tensor $\mathrm{W}(\mathbf{g})$ or the Cotton tensor $\mathrm{C}(\mathbf{g})$ depending on the dimension of $\mathbf{M}$. In this way, we conclude that $\boldsymbol{\Phi} = 0$ on $\mathbf{M}$. This allows us to characterize $\mathbf{M}$ in terms of so-called *special geometry*, which is well known to physicists in the context of string theory and Seiberg-Witten theory.

Special geometry specifies a stringent differential geometric condition on $\mathbf{M}$ - a certain flat symplectic connection ∇ on the underlying real tangent sheaf of $\mathbf{M}$ - that is governed by a symmetric, holomorphic cubic form $\boldsymbol{\Xi} \in S^3(\mathbf{M})$ that measures the failure of ∇ to preserve a given fixed complex structure on $\mathbf{M}$.[2] See Freed [45]. As was discussed by Donagi & Witten [35], there are crucial connections of special geometry, their physical interpretations in $\mathcal{N}=2$ supersymmetric Yang-Mills theory via Seiberg-Witten theory, algebraic integrable systems, and families of polarized abelian varieties that is quite pertinent to the scope of this research. We take up some of that analysis in §6.3.

In particular, we identify the holomorphic cubic form $\boldsymbol{\Xi}$ arising in special geometry and the Wilczynski-Fubini-Pick form $\boldsymbol{\Phi}$. This is natural to do, as we will show through the course of this chapter that such algebraic integrable systems are a geometric incarnation of the Picard-Fuchs equations we study, via a generalization of the Shioda-Inose structure provided by Mehran [97]. As explained in Freed [45], it can be shown that there is a local holomorphic function $\mathfrak{F} \in \mathcal{O}(\mathbf{M})$ such that $\boldsymbol{\Xi}$ can be expressed as

$$\boldsymbol{\Xi} = \frac{\partial^3 \mathfrak{F}}{\partial z^i \partial z^j \partial z^k}\, dz^i \otimes dz^j \otimes dz^k \tag{6.1.12}$$

in certain local coordinates $\{z^i\} \subset \mathbf{M}$ related to the flat symplectic structure specified by ∇. The function $\mathfrak{F}$ is called the *holomorphic prepotential*, which actually determines the local hermitian and Riemannian geometry of $\mathbf{M}$. It can be shown that $\mathrm{Im}(\partial_i \partial_j \mathfrak{F})$ is a potential function for the canonical (1,1)-class obtained from the Fubini-Study metric, and moreover determines the associated Riemannian metric $\mathbf{g}$. In fact, $\boldsymbol{\Xi}$ can be used, together with a given symplectic (1,1)-class, associated Riemannian metric $\mathbf{g}$, and Levi-Civita connection $\tilde{\nabla}$ to generate the flat symplectic connection ∇ [45].

Let us end this introductory section with a statement about the physical significance of special geometry. Although one can obtain a classical lagrangian from a special coordinate system, the special geometry of $\mathbf{M}$ does not yield a classical field

[2]Since $\mathbf{M}$ is quasiprojective, the choice of complex structure is naturally inherited from $\mathbb{P}^n$

theory. The local change of special coordinates must be accompanied by a duality transformation on the gauge field in the vector multiplet, and this has only a quantum interpretation. In fact, the holonomy group of ∇ lies in some *integral* symplectic group $\mathrm{Sp}(2n, \mathbb{Z})$ - we find for all examples in this research that the monodromy groups of the Picard-Fuchs equations can be embedded in an integral symplectic group. Hence, the special geometry on $\mathbf{M}$ is naturally associated with symplectic automorphisms of an integral lattice Λ that determines a quantum field theory, which locally has a semiclassical description in terms of $\mathcal{N} = 2$ vector multiplets. Then $\mathbf{M}$ acts as the moduli space of quantum vacua. Such descriptions are specified by an algebraic integrable system, whose fibres consist of polarized abelian varieties.

Physically, this determines an abelian description of the low energy behavior of the Coulomb branch of nonabelian $\mathcal{N} = 2$ supersymmetric gauge theories, that may or may not have matter multiplets. In the simplest case, initially studied by Seiberg & Witten [126], the gauge group is $\mathrm{SU}(2)$ and no matter content is specified. Then M is the rational elliptic surface $\mathbf{Z} \to B$, with $B = \Gamma_0(2)\backslash\mathbb{H} \cong \mathbb{P}^1$ as the modular curve for the principle congruence subgroup $\Gamma(2) \subset \mathrm{PSL}(2, \mathbb{Z})$ of level-two structure. We refer to the fibration $Z \to B$ as a *Seiberg-Witten curve*. In classical algebraic geometry, this is simply the Legendre pencil of elliptic curves, given by the equation

$$y^2 = (x^2 - \Lambda^4)(x - u)\,, \tag{6.1.13}$$

with $u \in \mathbb{P}^1$. Here $\Lambda \in \mathbb{P}^1$ is the dynamically generated scale of the quantum field theory, and the location $u = 0, \pm\Lambda^2$ of the singular fibres of the elliptic fibration are where the hypermultiplets become massless. The description of $\mathbf{Z}$ as a Jacobian elliptic fibration is the algebraic integrable system defining the model. In fact, we shall see that this Seiberg-Witten curve plays a cruicial role in the analysis we undertake in §6.3.

6.2 Holomorphic conformal geometry from the twisted Legendre pencil

Our strategy for computing the Picard-Fuchs system for the twisted Legendre family $\mathbf{X}$ has been reduced, in the section preceding, to computing the holomorphic conformal structure on the moduli space $\mathfrak{T}$ induced from the quadratic period relations. We shall obtain the explicit conformal structure by utilizing the previous work of Matsumoto, Sasaki, & Yoshida [96] for the double sextic family, and then pulling back to the twisted Legendre pencil.

6.2.1 Holomorphic Conformal geometry of the moduli space $\mathcal{X}(3,6)$

In [96], the authors computed the uniformizing differential equations of the four dimensional moduli space $\mathcal{X}(3,6)$ of six lines in $\mathbb{P}^2$ in general position using the GKZ formalism [50, 49] for functions on Grassmannians, putting the equations in the form of Equation (6.1.1). Thus, they have explicitly obtained the conformal structure on $\mathcal{X}(3,6)$. In the variables x^1, x^2, x^3, x^4 of Equation (2.1.26), this conformal structure is given explicitly by the tensor $\mathbf{G} \in S^2(\mathcal{X}(3,6))$ given by

$$\begin{aligned}
\mathbf{G} = & \left(\frac{x^2x^3 - x^4}{x^1\left(-x^1+1\right)} - \frac{x^3\left(-x^2+x^4\right)}{x^1\left(x^1-x^3\right)} - \frac{x^2\left(-x^3+x^4\right)}{x^1\left(x^1-x^2\right)} \right) dx^1 \otimes dx^1 \\
& - \left(\frac{x^3 - x^4}{x^1 - x^2} \right) \left(dx^1 \otimes dx^2 + dx^2 \otimes dx^1\right) - \left(\frac{x^2 - x^4}{x^1 - x^3} \right) \left(dx^1 \otimes dx^3 + dx^3 \otimes dx^1\right) \\
& + \left(\frac{x^1x^4 - x^3}{x^2\left(-x^2+1\right)} - \frac{x^1\left(x^3-x^4\right)}{x^2\left(-x^1+x^2\right)} - \frac{x^4\left(-x^1+x^3\right)}{x^2\left(x^2-x^4\right)} \right) dx^2 \otimes dx^2 \\
& - \left(\frac{x^1 - x^3}{x^2 - x^4} \right) \left(dx^2 \otimes dx^4 + dx^4 \otimes dx^2\right) + \left(dx^2 \otimes dx^3 + dx^3 \otimes dx^2\right) \\
& + \left(\frac{x^1x^4 - x^2}{x^3\left(-x^3+1\right)} - \frac{x^1\left(x^2-x^4\right)}{x^3\left(-x^1+x^3\right)} - \frac{x^4\left(-x^1+x^2\right)}{x^3\left(x^3-x^4\right)} \right) dx^3 \otimes dx^3 \\
& + \left(\frac{x^1 - x^2}{x^3 - x^4} \right) \left(dx^3 \otimes dx^4 + dx^4 \otimes dx^3\right) + \left(dx^1 \otimes dx^4 + dx^4 \otimes dx^1\right) \\
& + \left(\frac{x^2x^3 - x^1}{x^4\left(-x^4+1\right)} - \frac{x^3\left(x^1-x^2\right)}{x^4\left(-x^3+x^4\right)} - \frac{x^2\left(x^1-x^3\right)}{x^4\left(-x^2+x^4\right)} \right) dx^4 \otimes dx^4
\end{aligned} \tag{6.2.14}$$

We now may analyze this tensor in the context of Picard-Fuchs uniformization for the double sextic family.

Lemma 6.2.82. *The orbifold $(\mathcal{X}(3,6), \mathbf{G})$ is conformally flat. In particular, the components $G_{\mu\nu}$ are the principal part of the rank six system of uniformizing differential equations (6.1.1) which satisfies the quadric condition.*

Proof. By direct computation the Weyl tensor vanishes, $\mathrm{W}(\mathbf{G}) = 0$. □

Lemma 6.2.83. *The rank six differential system in Lemma 6.2.82 is the Picard-Fuchs system for the double sextic family of K3 surfaces $\mathcal{X}_{x^1,x^2,x^3,x^4}$ in Equation (2.1.22).*

Proof. On $\mathcal{X}$, we have the nonvanishing holomorphic 2-form $\eta_{\mathbf{X}}$ from Equation (2.1.23), given in the affine coordinate chart $[\, t^1 = 1 \,:\, t^2 = T : \, t^3 = X \,:\, z = z\,] \subset \mathbb{P}(1,1,1,3)$ by

$$\eta_{\mathcal{X}} = \frac{1}{\sqrt{TX(1+T+X)\,(x_3T + x_1X + 1)\,(x_4T + x_2X + 1)}} dT \wedge dX\,. \tag{6.2.15}$$

In [119], Sasaki showed that the period integral

$$\omega(x^1, x^2, x^3, x^4) = \int_\Sigma \eta_{\mathcal{X}} ,$$

satisfies the differential system in Lemma 6.2.82, where $\Sigma \in \mathrm{T}(\mathbf{X})$ is a transcendental cycle. Since the Picard-Fuchs system for $\mathbf{X}$ is also of rank six in four variables, the claim follows. □

6.2.2 Embedding the moduli space $\mathfrak{T}$ as a boundary component of $\mathcal{X}(3,6)$

One may compute the explicit Picard-Fuchs system for the double sextic family from the conformal structure $\mathbf{G}$ using Theorem 6.1.80, but the system not particularly inspiring, and more importantly is not needed for this dissertation. Since the double sextic family is birational to the family $\mathbf{X}_{a,b,c,d}$ of Yoshida surfaces, we can find a restriction of the double sextic family that yields the twisted Legendre pencil (2.1.43); after pulling back the conformal structure $\mathbf{G}$ to the moduli space $\mathfrak{T}$ of the twisted Legendre pencil, we will implement Theorem 6.1.79 and compute the system explicitly. However, we will see that one *must* use the elliptic fibration in Equation (2.1.43) and the induced projective gauge transformation on the holomorphic 2-form (2.1.44) to obtain the equations that we computed in §5.3.

Lemma 6.2.84. *Let us write $\mathcal{X} = \mathcal{X}_{x^1,x^2,x^3,x^4}$ for the double sextic family, and $\mathbf{X} = \mathbf{X}_{a,b,c}$ the twisted Legendre pencil. Then the birational map $\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ defined by*

$$\Phi : ([t,1],[x,1]) \mapsto \left[1 : T = -\frac{x^1 t - 1}{x^1 t - x^3} : X = \frac{x\,(x^3-1)}{x^1 t - x^3} \right] \tag{6.2.16}$$

together with the rational map $\sigma : \mathfrak{T} \dashrightarrow \mathcal{X}(3,6)$ defined by

$$\sigma : (a,b,c) \mapsto \left(x^1 = \frac{1}{a} \,,\; x^2 = 0 \,,\; x^3 = \frac{c}{a} \,,\; x^4 = \frac{b-c}{b-a} \right) \tag{6.2.17}$$

takes the restricted Yoshida surface $\mathcal{X}_{x^1,x^2=0,x^3,x^4}$ to the (scaled) twisted Legendre pencil $\mathbf{X}_{a,b,c}$

$$y^2 = \frac{1}{a(a-b)}\,(t-a)(t-b)(t-c)x(x-1)(x-t)\,. \tag{6.2.18}$$

At the level of holomorphic 2-forms, we have

$$\Phi^*\,\eta_{\mathcal{X}} = \sqrt{a(a-b)}\,dt\wedge\frac{dx}{y} \equiv \sqrt{a(a-b)}\,\eta_{\mathbf{X}}\,. \tag{6.2.19}$$

Proof. By direct computation using the maps Φ and σ. The scaling factor $1/a(a-b)$ persists because $a, a-b$ are not squares in the function field $\mathbb{C}(\mathfrak{T})$. □

Using a natural variation of the map $\sigma : \mathfrak{T} \dashrightarrow \mathcal{X}(3,6)$, we may pullback the conformal structure $(\mathcal{X}(3,6), \mathbf{G})$ to obtain the conformal structure $(\mathfrak{T}, \mathbf{g})$. In this way, we realize $\mathfrak{T}$ as birational to the three-dimensional boundary component $\mathcal{P}_2$ of $\mathcal{X}(3,6)$ determined by the hyperplane $x^2 = 0$.

Theorem 6.2.85. *Define a one-parameter family of rational maps $\sigma_\epsilon : \mathfrak{T} \dashrightarrow \mathcal{X}(3,6)$ by*

$$\sigma_\epsilon : (a,b,c) \mapsto \left(x^1 = \frac{1}{a}\,,\, x^2 = \epsilon\,,\, x^3 = \frac{c}{a}\,,\, x^4 = \frac{b-c}{b-a}\right). \tag{6.2.20}$$

Then the symmetric tensor $\mathbf{g} = \lim_{\epsilon\to 0}\,\sigma_\epsilon^*\,\mathbf{G} \in S^2(\mathfrak{T})$ is nondegenerate, and is given up to conformal scaling by

$$\mathbf{g} = \frac{(1-b)(b-c)}{a^2\,(a-b)^2(a-1)}da\otimes da + \frac{a-c}{ab\,(a-b)^2}db\otimes db + \frac{1-b}{ac\,(c-1)(a-b)}dc\otimes dc \tag{6.2.21}$$

Then $\mathbf{g}$ determines a conformal structure on $\mathfrak{T}$ that is conformally flat. Moreover, the family of maps σ_ϵ factors through the inclusion map $\jmath : \mathcal{P}_2 \hookrightarrow \mathcal{X}(3,6)$ that restricts the fourfold down to the boundary hyperplane $\mathcal{P}_2$ defined by the equation $x_2 = 0$ and the birational map $\widehat{\sigma} : \mathfrak{T} \dashrightarrow \mathcal{P}_2$ assigning the values to x^1, x^3, x^4 appearing in Equation

6.2.20 above:

$$\begin{tikzcd} \mathcal{P}_2 \arrow[r, "\jmath"] & \mathcal{X}(3,6) \\ \mathfrak{T} \arrow[u, "\phi"] \arrow[ur, "\sigma_\epsilon"] & \end{tikzcd}$$

The conformal structure on the boundary hyperplane $\mathcal{P}_2$ induced by the inclusion map is given by the symmetric tensor

$$\begin{aligned}\mathbf{h} = &-\frac{x^4(x^3-1)}{(x^1-1)(x^1-x^3)}\,dx^1\otimes dx^1 + \frac{x^4}{x^1-x^3}\,(dx^1\otimes dx^3 + dx^3\otimes dx^1)\\ &+\frac{x^4x^1\,(x^1x^4-(x^3)^2-x^1+2x^3-x^4)}{x^3(x^3-1)(x^3-x^4)(x^1-x^3)}\,dx^3\otimes dx^3 - \frac{x^1}{x^3-x^4}(dx^3\otimes dx^4+dx^4\otimes dx^3)\\ &+(dx^1\otimes dx^4+dx^4\otimes dx^1)+\frac{x^1(x^3-1)}{(x^4-1)(x^3-x^4)}dx^4\otimes dx^4\,,\end{aligned} \tag{6.2.22}$$

which is conformally flat.

Proof. By direct computation of $\mathbf{g} = \lim_{\epsilon\to 0}\sigma_\epsilon^*\,\mathbf{G}$, one finds the tensor presented in Equation (6.2.21). This shows that the diagram above commutes. A subsequent computation reveals that the Cotton tensor vanishes, $\mathrm{C}(\mathbf{g}) = 0$. Thus $(\mathfrak{T},\mathbf{g})$ is conformally flat. To compute the conformal structure on $\mathcal{P}_2$ induced by the inclusion map, as given, the conformal structure $\mathbf{G}$ on $\mathcal{X}(3,6)$ blows up along $\mathcal{P}_2$, and this can not be scaled away without sacrificing nondegeneracy. One then obtains the conformal structure $\mathbf{h}$ on $\mathcal{P}_2$ induced by the inclusion map by computing $\mathbf{h} = (\sigma_\epsilon\circ\phi^{-1})^*\mathbf{G}$, yielding the tensor in Equation 6.2.22. Similarly, one computes that $\mathbf{h}$ has vanishing Cotton tensor $\mathrm{C}(\mathbf{h}) = 0$, so $(\mathcal{P}_2,\mathbf{h})$ is conformally flat. □

Now with the explict conformal structures $(\mathfrak{T},\mathbf{g})$ and $(\mathcal{P}_2,\mathbf{h})$ in hand, we may compute the full Picard-Fuchs system that annihilates the period map $\psi:\mathfrak{T}\to\mathbf{Q}$ for the twisted Legendre family by appealing to Theorem 6.1.79.

Corollary 6.2.86. *The Picard-Fuchs system for the twisted Legendre pencil the rank five system in the variables a, b, c is given by the following linear equations. Three equations are hyperbolic equations of so-called Euler-Poisson-Darboux type for $N = 1/2$ [98]*

$$\frac{\partial^2\psi}{\partial a\partial b} = \frac{1}{2(a-b)}\left(\frac{\partial\psi}{\partial a} - \frac{\partial\psi}{\partial b}\right), \tag{6.2.23}$$

$$\frac{\partial^2\psi}{\partial a\partial c} = \frac{1}{2(a-c)}\left(\frac{\partial\psi}{\partial a} - \frac{\partial\psi}{\partial c}\right), \tag{6.2.24}$$

$$\frac{\partial^2\psi}{\partial b\partial c} = \frac{1}{2(b-c)}\left(\frac{\partial\psi}{\partial b} - \frac{\partial\psi}{\partial c}\right), \tag{6.2.25}$$

and the remaining two equations are given by

$$\begin{aligned}\frac{\partial^2\psi}{\partial b^2} = &\frac{a\,(1-a)(a-c)}{b\,(b-c)(b-1)}\,\frac{\partial^2\psi}{\partial a^2} - \frac{(5a^3-3a^2b-4a^2c+2abc-4a^2+2ab+3ac-bc)}{2b\,(b-c)(b-1)(a-b)}\,\frac{\partial\psi}{\partial a}\\ &- \frac{(3ab^2-2abc-5b^3+4b^2c-2ab+ac+4b^2-3bc)}{2b\,(b-c)(b-1)(a-b)}\,\frac{\partial\psi}{\partial b} + \frac{c(c-1)}{2b\,(b-c)(b-1)}\,\frac{\partial\psi}{\partial c}\\ &- \frac{(2a-2c-1+2b)}{4b\,(b-c)(b-1)}\,\psi\,,\end{aligned} \tag{6.2.26}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial c^2} = &\frac{a\,(a-b)(a-1)}{c(c-1)(b-c)}\,\frac{\partial^2\psi}{\partial a^2} + \frac{(5a^3-4a^2b-3a^2c+2abc-4a^2+3ab+2ac-bc)}{2c\,(c-1)(b-c)(a-c)}\,\frac{\partial\psi}{\partial a}\\ &- \frac{b(b-1)}{2c\,(c-1)(b-c)}\,\frac{\partial\psi}{\partial b} - \frac{(2abc-3ac^2-4bc^2+5c^3-ab+2ac+3bc-4c^2)}{2c\,(c-1)(b-c)(a-c)}\,\frac{\partial\psi}{\partial c}\\ &+ \frac{(2a-2b+2c-1)}{4c\,(c-1)(b-c)}\psi\,.\end{aligned} \tag{6.2.27}$$

Proof. Using the conformal structure $(\mathcal{P}_2, \mathbf{h})$ from Theorem 6.2.85 with $(\alpha, \beta) = (1, 4)$ and

$$\theta = \frac{1}{3}\log\left(\frac{(x^3-1)\,(x^4-1)\,(x^4-x^3)\,x^3\,(x^1-1)\,(x^1-x^3)}{x^4\left((x^4-1)\,x^1+x^3-x^4\right)^2 x^1}\right),$$

for an appropriate branch of the logarithm, one writes out the five equations from

Theorem 6.1.80 after computing the Christoffel symbols and Schouten tensor with respect to the $\widehat{\mathbf{h}} = \exp(\theta)\mathbf{h}$. These equations are projectively equivalent to the rank 5 system given in [96, Equation 0.15.1], using the projective gauge factor

$$f(x^1, x^3, x^4) = \left(x^3\left(x^3-1\right)\right)^{1/6}\left(-x^3+x^4\right)^{1/6}\left(x^4-1\right)^{1/6}\left(x^1x^4\right)^{(1/3)}\left(x^1-x^3\right)^{1/6}$$
$$\left(x^1-1\right)^{1/6}\left(\left(x^4-1\right)x^1+x^3-x^4\right)^{1/6} \in \mathcal{O}^*_{\mathcal{P}_2}\,.$$

After changing variables defined by the map ϕ in Theorem 6.2.85, one solves for the second order derivatives in terms of $\partial^2\psi/\partial a^2, \partial\psi/\partial a, \partial\psi/\partial b, \partial\psi/\partial c$, and ψ. That the equations must be expressed in terms of $\partial^2\psi/\partial a^2$ was anticipated by Hoyt [69]. Afterwards, one makes the projective gauge transformation $\psi \mapsto \sqrt{a(a-b)}\,\psi$ to yield the equations above. Alternatively, one may use Theorem 6.1.79 on the conformal structure $(\mathfrak{T}, \mathbf{g})$, and use the projective gauge factor

$$\tilde{f}(a,b,c) = \sqrt{\frac{(b-1)(a-c)(b-c)}{a(a-b)}} \in \mathcal{O}^*_{\mathfrak{T}}$$

to yield the same equations. It follows then from Lemma 6.2.83 and Lemma 6.2.84 that the system is the Picard-Fuchs system the annihilates the period integral of the twisted Legendre pencil

$$\omega(a,b,c) = \int_\Sigma \eta_{\mathbf{X}} = \int_\Sigma dt \wedge \frac{dx}{y}\,,$$

where $\Sigma \in \mathrm{T}(\mathbf{X})$ is a transcendental cycle. $\square$

Note that the Euler-Poisson-Darboux equations in Equation (6.2.23) are precisely what was computed by directly differentiating the period integral in Equation (5.3.47). We verify the harmony of these different perspectives for the remaining two equations (6.2.26,6.2.27) by showing that a simple linear combination of the Picard-Fuchs operators for these equations yields the second order Picard-Fuchs operator in Equation (5.3.53).

Corollary 6.2.87. *The Picard-Fuchs equations for the twisted Legendre pencil obtained from the differential geometric method of Sasaki & Yoshida matches the differential relations obtained by differentiating the period integral directly.*

Proof. For each sequential Picard-Fuchs equation in Corollary 6.2.86, let $\mathcal{L}_i$ denote the corresponding second order differential operator, $i = 1 \dots 5$. As noted above, the operators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ from Equation (6.2.23) match directly to those computed in Equation (5.3.47). Similarly let $\mathcal{D}$ denote the second order operator from Equation (5.3.53), given by

$$
\begin{aligned}
\mathcal{D} = {} & (a-1)a\,\frac{\partial^2}{\partial a^2} + b(b-1)\,\frac{\partial^2}{\partial b^2} + c(c-1)\,\frac{\partial^2}{\partial c^2} \\
& + \frac{(6a^3 - 4a^2b - 4a^2c + 2abc - 5a^2 + 3ab + 3ac - bc)}{2(a-b)(a-c)}\,\frac{\partial}{\partial a} \\
& + \frac{(4ab^2 - 2abc - 6b^3 + 4b^2c - 3ab + ac + 5b^2 - 3bc)}{2(b-c)(a-b)}\,\frac{\partial}{\partial b} \\
& + \frac{(2abc - 4ac^2 - 4bc^2 + 6c^3 - ab + 3ac + 3bc - 5c^2)}{2(b-c)(a-c)}\,\frac{\partial}{\partial c} + 1
\end{aligned}
$$

It is straightforward then to see that the operators $\mathcal{L}_4$ from 6.2.26 and $\mathcal{L}_5$ from 6.2.27 satisfy

$$b(b-1)\,\mathcal{L}_4 + c(c-1)\,\mathcal{L}_5 = \mathcal{D}\,.$$

□

6.2.3 Picard-Fuchs operators for Kummer surfaces with level-two structure

In [96], Matsumoto, Sasaki, & Yoshida found a rational map $\imath : \mathcal{M}[2] \dashrightarrow \mathcal{XQ} \subset \mathcal{X}(3,6)$ given by

$$\imath : (\lambda_1, \lambda_2, \lambda_3) \mapsto \left(x^1 = \frac{1-\lambda_1}{1-\lambda_2}, \quad x^2 = \frac{1-\lambda_1}{1-\lambda_3}, \quad x^3 = \frac{\lambda_1}{\lambda_2}, \quad x^4 = \frac{\lambda_1}{\lambda_3} \right) , \tag{6.2.28}$$

where $\mathcal{XQ} \subset \mathcal{X}(3,6)$ is the subvariety parameterizing configurations of six lines in $\mathbb{P}^2$ that are tangent to a smooth conic. Then the pullback $\mathbf{g}_{\mathcal{Q}} = \imath^* \mathbf{G}$ of the conformal structure $(\mathcal{X}(3,6), \mathbf{G})$ is easily checked to be conformal to the following tensor:

$$\begin{aligned} \mathbf{g}_{\mathcal{Q}} = & (\lambda_1 - \lambda_2)\, \lambda_3\, (\lambda_3 - 1)\, (d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) \\ & + (\lambda_2 - \lambda_3)\, \lambda_1\, (\lambda_1 - 1)\, (d\lambda_2 \otimes d\lambda_3 + d\lambda_3 \otimes d\lambda_2) \\ & + (\lambda_3 - \lambda_1)\, \lambda_2\, (\lambda_2 - 1)\, (d\lambda_3 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_3) \; . \end{aligned} \tag{6.2.29}$$

Recall that the Siegel upper half space of degree two $\mathbb{H}_2$ is given by

$$\left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mid (\operatorname{Im} \tau_1)\, (\operatorname{Im} \tau_3) - (\operatorname{Im} \tau_2)^2 > 0, \operatorname{Im} \tau_1 > 0 \right\} .$$

Let $\Gamma(2) \subset \operatorname{Sp}(4, \mathbb{Z})$ be the Siegel modular group of level-two, i.e., the principal congruence subgroup of level-two. Then as in Example 6.1.1, $\mathbb{H}_2$ is a quasiprojective variety, realized as a part of a nondegenerate hyperquadric $\mathbf{Q} : t^0 t^4 = t^1 t^3 - (t^2)^2$ in $\mathbb{P}^4$ by

$$\imath : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \left[1 : \tau_1 : \tau_2 : \tau_3 : \tau_1 \tau_3 - (\tau_2)^2 \right] \in \mathbf{Q} \subset \mathbb{P}^4 .$$

Hence, $\mathbb{H}_2$ carries a canonical conformally flat structure given by

$$\boldsymbol{\varphi}_{\mathbb{H}_2} = d\tau_1 \otimes d\tau_3 + d\tau_3 \otimes d\tau_1 - 2d\tau_2 \otimes d\tau_2 \,.$$

Moreover, the quotient $\mathbb{H}_2/\Gamma(2)$ is known to be the space $\mathcal{M}[2]$. Sasaki & Yoshida proved the following result.

Theorem 6.2.88 (Sasaki & Yoshida [123], Theorem 3.1)**.** *Let $\left(\mathbb{H}_2, \boldsymbol{\varphi}_{\mathbb{H}_2}\right)$ be the Siegel upper half space equipped with the canonical conformal structure. Let $\Gamma(2) \subset \mathrm{Sp}(4, \mathbb{Z})$ be the Siegel modular group of level-two. Then tensor $\mathbf{g}_{\mathcal{Q}} \in S^2(\mathcal{M}[2])$ is conformal to $\pi_* \boldsymbol{\varphi}_{\mathbb{H}_2}$, where $\pi : \mathbb{H}_2 \to \mathbb{H}_2/\Gamma(2) \cong \mathcal{M}[2]$ is the canonical projection. Then $(\mathcal{M}[2], \mathbf{g}_{\mathcal{Q}})$ is conformally flat, and hence the rank 5 $E_{\mathcal{Q}}$ system obtained from applying Theorem 6.1.79 to $\mathbf{g}_{\mathcal{Q}}$ is the system of uniformizing differential equations for $\mathcal{M}[2]$.*

In this way, we recognize that the differential system $E_{\mathcal{Q}}$ built from $(\mathcal{M}[2], \mathbf{g}_{\mathcal{Q}})$ is the Picard-Fuchs system for the family of Picard rank $\rho = 17$ Kummer surfaces $\mathrm{Kum}(\mathrm{Jac}(\mathbf{C}))$ equipped with level-two structure. Since $(\mathcal{M}[2], \mathbf{g}_{\mathcal{Q}})$ is conformally flat, this rank 5 system satisfies the quadric condition. In this case, it was explicitly computed by Hara, Sasaki, & Yoshida that $E_{\mathcal{Q}}$ is gauge equivalent to the exterior product $E_D \wedge E_D$, where E_D is the rank 4 system annihilating the Lauricella function $F_D(1/2; 1/2, 1/2, 1/2; 1 \mid \lambda_1, \lambda_2, \lambda_3)$ [60]. This is a rather beautiful result, as the system E_D is the Picard-Fuchs system for the genus two curve $\mathbf{C}_{\lambda_1,\lambda_2,\lambda_3}$, and the construction of the Kummer surface $\mathrm{Kum}(\mathrm{Jac}(\mathbf{C}))$ involves the explicit exterior product of holomorphic differentials in $H^{1,0}(\mathbf{C})$ to trivialize the canonical bundle of the Kummer surface.

We may connect the conformal structure $(\mathfrak{T}, \mathbf{g})$ of the twisted Legendre pencil with that of the double cover $\mathcal{M}_2$ of $\mathcal{M}[2]$, as in section §2.1.10, using the dominant rational map $\phi : \mathcal{M}_2 \dashrightarrow \mathfrak{T}$ from Equation (2.1.49). Recall that we are parameterizing

$\mathcal{M}_2$ with the squares of Rosenhain roots λ_i' from a curve $\mathbf{C}'$ that is 2-isogenous to $\mathbf{C}$. This is simply for ease of presentation, as the Picard-Fuchs operators of $\mathrm{Kum}(\mathrm{Jac}(\mathbf{C}))$ and $\mathrm{Kum}(\mathrm{Jac}(\mathbf{C}'))$ are identical. This relationship completely answers the questions set forth by Hoyt [69].

Theorem 6.2.89. *Let $\pi : \mathcal{M}_2 \to \mathcal{M}[2]$ be the projection map from the double cover, and $\phi : \mathcal{M}_2 \dashrightarrow \mathfrak{T}$ be the dominant rational map from Equation (2.1.49). Let $\mathbf{g}_{\mathcal{Q}}$ be the conformally flat structure on $\mathcal{M}[2]$ from Equation (6.2.29) and $\mathbf{g}$ be the conformally flat structure on $\mathfrak{T}$ from Equation (6.2.21). Then the orbifolds $(\mathcal{M}_2, \pi^*\mathbf{g}_{\mathcal{Q}})$ and $(\mathfrak{T}, \mathbf{g})$ are conformally isomorphic.*

Proof. It is straightforward to show that $\phi^*\mathbf{g}$ is conformal to $\pi^*\mathbf{g}_{\mathcal{Q}}$. □

Hence, we have the following result, whose proof is immediate.

Corollary 6.2.90. *The uniformizing Picard-Fuchs equations for the twisted Legendre pencil in Corollary 6.2.86 are equivalent to the uniformizing Picard-Fuchs equations for $\mathrm{Kum}(\mathrm{Jac}(\mathbf{C}'))$ with full level-two structure, in the variables*

$$a = \frac{4\, l_1 l_2 l_3}{(l_1 l_3 + l_2)^2}, \quad b = \frac{4\, l_1 l_2 l_3}{(l_1 l_2 + l_3)^2}, \quad c = \frac{4\, l_1 l_2 l_3}{(l_2 l_3 + l_1)^2}.$$

Degeneration to Picard rank $\rho \geq 18$

In this section, we explain quadratic period relations for the higher Picard rank restrictions discussed in §2.1.10. Fundamental to this discussion the integral representation of the classical Gauss hypergeometric function ${}_2F_1(1/2, 1/2; 1)$, which we state here for convenience:

$$ {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right| t \right) = \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-tx)}} . \tag{6.2.30}$$

The periods of the two-parameter twisted Legendre pencil $\mathbf{X}_{a,b,0}$ was studied by Clingher, Doran, & Malmendier in [23], where they showed the following result (we have specialized their more general result for the situation at hand).

Proposition 6.2.91 (Clingher, Doran, & Malmendier [23], Corollary 2.2)**.** *The following integral relation between Gauss' hypergeometric function ${}_2F_1$ and Appell's hypergeometric function F_2 :*

$$\frac{1}{\sqrt{a}}\,F_2\left(\begin{array}{c}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{array}\Big|\,\frac{1}{a},1-\frac{b}{a}\right)=-\frac{1}{\pi}\int_a^b\frac{dt}{\sqrt{t(a-t)(t-b)}}\;{}_2F_1\left(\begin{array}{c}\frac{1}{2},\frac{1}{2}\\ 1\end{array}\Big|\,\frac{1}{t}\right).$$

Hence, we infer that for the twisted Legendre pencil $\mathbf{X}_{a,b,0}$, there exists a transcendental cycle $\Sigma\in\mathrm{T}(\mathbf{X}_{a,b,0})$, homologous to $[a,b]\times[0,1]$, such that

$$\omega(a,b)=\int_\Sigma dt\wedge\frac{dx}{y}=\frac{-\pi}{\sqrt{a}}\,F_2\left(\begin{array}{c}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{array}\Big|\,\frac{1}{a},1-\frac{b}{a}\right) \tag{6.2.31}$$

This means that the Picard-Fuchs system of $\mathbf{X}_{a,b,0}$ is the rank 4 system annihiliating Appell's hypergeometric function F_2 given in Example 6.1.1, in the variables

$$(z_1,z_2)=\left(\frac{1}{a},1-\frac{b}{a}\right).$$

These equations are easily then computed to be the following system:

Proposition 6.2.92. *The Picard-Fuchs operators of the twisted Legendre pencil $\mathbf{X}_{a,b,0}$ is given by*

$$\frac{\partial^2\omega}{\partial b\partial a}=\frac{1}{2(a-b)}\left(\frac{\partial\omega}{\partial a}-\frac{\partial\omega}{\partial b}\right), \tag{6.2.32}$$

$$\frac{\partial^2\omega}{\partial b^2} = -\frac{(a-1)a}{b(b-1)}\frac{\partial^2\omega}{\partial a^2} - \frac{(4a^2-2ab-3a+b)}{2b(b-1)(a-b)}\frac{\partial\omega}{\partial a} - \frac{(2ab-4b^2-a+3b)}{2b(b-1)(a-b)}\frac{\partial\omega}{\partial b} - \frac{1}{4b(b-1)}\omega\,. \tag{6.2.33}$$

This system of equations satisfies the quadric condition, as originally noted by Sasaki & Yoshida in [121]. However, it was not until the work of Clingher, Doran, & Malmendier [23] that the geometric underpinnings of the quadric condition was realized. Therein, the authors showed the following, where again, we have specialized to the situation at hand:

Theorem 6.2.93 (Clingher, Doran, & Malmendier [23], Theorem 2.5 (Multivariate Clausen Identity))**.** *For $|z_1| + |z_2| < 1$, $|k_1^2| < 1$, and $|1-k_2^2| < 1$, Appell's hypergeometric series factors into two hypergeometric functions according to*

$$F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array}\middle| z_1, z_2\right) = (k_1+k_2)\,{}_2F_1\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array}\middle| k_1^2\right){}_2F_1\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array}\middle| 1-k_2^2\right) \tag{6.2.34}$$

with

$$(z_1, z_2) = \left(\frac{4k_1k_2}{(k_1+k_2)^2}\,, -\frac{(k_1^2-1)(k_2^2-1)}{(k_1+k_2)^2}\right)\,. \tag{6.2.35}$$

This is the explicit quadratic period relation for the twisted Legendre pencil $\mathbf{X}_{a,b,0}$, which demonstrates the relationship between this K3 surface and Shioda-Inose partner, the Kummer surface $\mathbf{Y}'_{k_1^2,k_2^2,0} = \mathrm{Kum}(\mathcal{E}_{k_1^2} \times \mathcal{E}_{k_2^2})$ from section §2.1.10. Moreover, by explicit computation, one shows that that Appell's hypergeometric system from Example 6.1.1 in the variables from Equation (6.2.38), or eqivalently, the differential system from Propositon 6.2.92 in the variables

$$a = \frac{4k_1k_2}{(k_1+k_2)^2}\,, \quad b = \frac{4k_1k_2}{(k_1k_2+1)^2} \tag{6.2.36}$$

decomposes into the tensor product ${}_2F_1 \boxtimes {}_2F_1$, where the hypergeometric operators annihilating the system are given by the rank 2 ordinary differential operators

$$k_i(k_i^2-1)\frac{d^2}{dk_i^2}+\left(3k_i^2-1\right)\frac{d}{dk_i}+k_i\,, \tag{6.2.37}$$

and hence are associated to ${}_2F_1(1/2,1/2,1/2;1\mid k_i^2)$, $i=1,2$. Although the argument of the second factor of ${}_2F_1$ in Theorem 6.2.93 contains $1-k_2^2$, one can verify directy that ${}_2F_1(1/2,1/2,1/2;1\mid k_2^2)$ and ${}_2F_1(1/2,1/2,1/2;1\mid 1-k_2^2)$ satisfy the same differential equation.

Degeneration to Picard rank $\rho\geq 19$

As in §2.1.10, we may further degenerate the twisted Legendre pencil in the limit $b\to\infty$ to obtain the Picard rank $\rho=19$ K3 surface $\mathbf{X}_{a,\infty,0}$ studied by Hoyt in [67]. We can see from Equation (6.2.36) that this corresponds to the limit in which

$$k_1\mapsto\lambda\,,\quad k_2\mapsto -1/\lambda\,,$$

which moreover yields

$$a=\frac{-4\,\lambda^2}{(\lambda^2-1)^2}\,. \tag{6.2.38}$$

We showed in §5.3.4 that the Picard-Fuchs operator for $\mathbf{X}_{a,\infty,0}$ is the third order hypergeometric operator $\widetilde{\mathcal{O}}_2$ for ${}_3F_2(1/2,1/2,1/2;1,1\mid a)$, given in Equation (5.3.69). Moreover, $\widetilde{\mathcal{O}}_2=\widetilde{\mathcal{O}}_1^{\otimes 2}$ is the symmetric square of the second order hypergeometric operator $\widetilde{\mathcal{O}}_1$ for ${}_2F_1(1/2,1/2;1\mid 1-a)$. Transforming the operator $\widetilde{\mathcal{O}}_1$ according to Equation (6.2.38), we arrive at the operator

$$\underline{\mathcal{O}}_1:=\frac{d^2}{d\lambda^2}+\frac{1}{\lambda}\frac{d}{d\lambda}+\frac{1}{(\lambda^2-1)^2} \tag{6.2.39}$$

which is checked directly to annihilate the periods of the elliptic curve

$$\mathcal{E}_{\lambda^2}: \quad \left\{ (y,x) \mid y^2 = \frac{1}{(\lambda^2-1)} x(x-1)\left(x-\lambda^2\right) \right\} \ ,$$

from Equation (5.3.71). This verifies the claims that the Picard rank $\rho = 19$ Kummer surface $\mathbf{Y}'_{\lambda^2,\lambda'^2,0} = \mathrm{Kum}(\mathcal{E}_{\lambda^2} \times \mathcal{E}'_{\lambda^2})$ is the Shioda-Inose partner of $\mathbf{X}_{a,\infty,0}$.

Similarly to the Picard rank $\rho = 18$ case studied in §6.2.3, we may obtain explict analytic expressions for the periods in terms of hypergeometric functions. In this case, we have the classical Gauss representation of ${}_3F_2(1/2,1/2,1/2;1,1 \mid a)$ given by

$$ {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1,1 \end{array} \Big| \, a \right) = \frac{1}{\pi} \int_0^1 \frac{dt}{\sqrt{t(1-t)}} \, {}_2F_1\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \Big| \, at \right) . \tag{6.2.40}$$

As before, we infer that there exists a transcendental cycle $\Sigma \in \mathrm{T}(\mathbf{X}_{a,\infty,0})$, homologous to $[0,1] \times [0,1]$, such that

$$\omega(a) = \int_\Sigma dt \wedge \frac{dx}{y} = \pi \, {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1,1 \end{array} \Big| \, a \right) .$$

In this case, there is also a factorization of the hypergeometric function ${}_3F_2$, given by a specialization of the Multivariate Clausen Identity in Theorem 6.2.93:

$$ {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1,1 \end{array} \Big| \, a \right) = \left(1-\lambda^2\right) \, {}_2F_1\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \Big| \, \lambda^2 \right)^2 ,$$

where again $a = -4\lambda^2/(\lambda^2-1)^2$. This exact analytic expression shows the connection between the periods of $\mathbf{X}_{a,\infty,0}$ and the generalized Shioda-Inose partner $\mathbf{Y}'_{\lambda^2,\lambda'^2,0}$ from section §2.1.10.

6.3 Relations to $\mathcal{N} = 2$ supersymmetric gauge theories

In this section, we connect the analysis of the mixed-twist construction of section §4.1 and the special geometry of the moduli space $\mathfrak{T}$ with a celebrated example in physics: the four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories of Seiberg and Witten [126]. Therein, the authors constructed multi-parameter families of rational elliptic surfaces $\pi : \mathbf{Z} \to \mathbb{P}^1$ with certain configurations of singular fibres of type I_n^*, I_k for $n = 0, \ldots, 4$ and $k = 1, 2, 3, 4$ based off of electric-magnetic duality considerations for BPS states in the 4-dimensional supersymmetric quantum field theory. Here, the gauge group of the is SU(2) and we have $n = 4 - N_f$, where $N_f = 0, \ldots, 4$ is the number of SU(2) hypermultiplets, or flavors of quarks. The classification of possible configurations of singular fibres is given in [85], whose locations encode locations in the moduli space $\mathbb{P}^1 - \{\text{singular fibers}\}$ of quantum vacua where the hypermultiplets become massless. The base coordinate $u \in \mathbb{P}^1$ is given by $u = \langle \operatorname{Tr} \phi^2 \rangle$, where ϕ is the complex scalar of the adjoint representation of SU(2); accordingly, u is invariant under the action of the Weyl group $\mathbb{Z}/2\mathbb{Z}$ of SU(2).

The most familiar such rational elliptic surface is given by the Legendre like pencil $\pi : \mathbf{Z} \to \mathbb{P}^1$ given in Equation (6.1.13)

$$y^2 = (x^2 - \Lambda^4)(x - u)\,,$$

where Λ is the dynamically generated mass scale of the quantum theory. The fibration $\mathbf{Z}$ describes a quantum SU(2) super Yang-Mills theory with $N_f = 2$ massless hypermultiplets. This fibration is the natural geometric manifestation of the the τ-parameter of the theory, which is defined as [84]

$$\tau = \tau(a) = \frac{\theta(a)}{\pi} + \frac{8\pi i}{g^2(a)} \in \mathbb{H}\,, \tag{6.3.41}$$

where a is the Higgs field and $\theta = \theta(a)$ is the θ angle of the low energy effective Lagrangian, as in quantum chromodynamics, and $g = g(a)$ is the effective, renormalized gauge coupling. In terms of the geometry of the fibration, τ is defined as usual as the ratio of the periods of the elliptic fibre,

$$\tau = \frac{\int_\alpha \frac{dx}{y}}{\int_\beta \frac{dx}{y}}$$

where $\alpha, \beta \in H_1(\mathcal{E}_u, \mathbb{Z})$ is a symplectic basis and $dx/y \in H^{1,0}(\mathcal{E}_u)$ is the analytic marking of the surface $\mathbf{Z}$.

Hence, the elliptic fibration has singular fibres of type I_2 over $u = \pm\Lambda^2$, and an I_2^* fibre over $u = \infty$, the latter of which is the so-called *semi-classical* limit of the theory. The realm in which u lies close to $\pm\Lambda^2$ is where nonperturbative quantum effects dominate.

It is straightforward to see that $\mathbf{Z}$ is birational to the honest Legendre pencil

$$y^2 = x(x-1)(x-t),$$

by applying a Möbius transformation to the singular fibres and simultaneously rescaling x, y, u to remove all factors of Λ. In this way, we conclude that the Mordell-Weil group $\mathrm{MW}(\mathbf{Z}, \pi) = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{P}^1 \cong \Gamma_0(2)\backslash\mathbb{H} = X_0(2)$, so that $\mathbf{Z}$ is the modular elliptic surface over the modular curve $X_0(2)$.

While this is useful from an arithmetic or algebraic perspective, it does little to explain the physical significance played by the fibration in equation (6.1.13). Indeed, we have eliminated the only relevant piece of data, Λ from the fibration, by utilizing the honest geometry of the elliptic surface. This is further evidenced by the fact that $\mathrm{MW}(\mathbf{Z}, \pi) = \mathbb{Z}/2\mathbb{Z}$; the scale Λ does not contribute to the algebro-geometric data of the elliptic fibration. It does not parameterize any *moduli* of the surface, since the

Legendre pencil has no moduli.

However, the scale Λ plays a crucial role in the quantum field theory determined by (6.1.13). In the limit $\Lambda \to 0$, we obtain the *classical* limit, in which both I_2 fibers coalesce and the moduli space of vacua becomes singular. Moreover, Λ is a fundamental independent parameter in the so-called *renomalization group flow equation* (RG) of the quantum theory, which describes how the gauge coupling runs with the scale of the theory. It is therefore in our interest to keep Λ in the picture, even though it does not describe in this case the intrinsic moduli of the fibration.

As is well known, Seiberg & Witten showed that there is a natural special geometry on the base curve $\mathbb{P}^1 - \{\text{singular fibres}\}$ of the fibration $\mathbf{Z}$, defined in terms of a meromorphic differential 1-form λ (defined in the following section) on the fibres of the π, such that the holomorphic prepotential $\mathfrak{F}_{SW}$ can be expressed as [84]

$$\mathfrak{F}_{SW}(A) = \frac{1}{2}\tau_0 A^2 + \frac{i}{\pi}A^2 \log\left[\frac{A^2}{\Lambda^2}\right] + \frac{1}{2\pi i}A^2 \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\Lambda}{A}\right)^{4\ell} , \tag{6.3.42}$$

where A is a super SU(2) gauge field. Physically, τ_0 is the bare coupling constant, the coefficients c_ℓ are instanton corrections to the theory, and the multivaluedness of $\mathfrak{F}_{SW}$ is determined by charges of the BPS states near the semi-classical limit. The general form of the prepotential had been previously surmised by the considerations of supersymmetry in the low energy effective lagrangian, but the power of Seiberg & Witten's analysis was to show how to explicitly compute all of the instanton terms in terms of the geometry of the elliptic fibration. From this expression, upon knowing the dependence of A on the quantum vacua u, one may compute the holomorphic cubic form Ξ_{SW} in Equation (6.1.12). Then in general $\Xi_{SW} \neq 0$ unless the quantum theory is free, i.e., noninteracting.

One may twist the massless $N_f = 2$ Seiberg-Witten curve in Equation (6.1.13) to the twisted Legendre pencil via the mixed-twist construction, and attempt to relate the special geometry of the Seiberg-Witten curve to the flat special geometry of the K3 moduli space $\mathfrak{T}$. We thus see that any attempt to connect these two special geometries *must* retain the physical parameters of the Seiberg-Witten data, even if they do not determine moduli of the fibration, since the holomorphic prepotential $\mathfrak{F}_{SW}$ depends nontrivially on Λ.

In the following subsections, we will review the connection between the Seiberg-Witten data and period integrals & Picard-Fuchs equations. Then it will be shown that the GKZ method discussed in §4.3.1 allows one to compute the Picard-Fuchs operators of certain Seiberg-Witten curves, and that some of these differential operators can be combined in a straightforward way to yield the homogeneous components of the expected first order RG flow operators for $\mathcal{N}=2$ supersymmetric gauge theories from physics [33].

From this point, we will show how to twist fibrations on rational elliptic surfaces corresponding to certain mass configurations of the $N_f = 4$ Seiberg-Witten curve to obtain pencils of Picard rank $\rho = 17$ and $\rho = 16$ K3 surfaces, the familiar twisted Legendre pencil and double sextic K3. We will then make a comment on the mathematical and physical interpretations of these constructions.

6.3.1 The Seiberg-Witten differential and periods of rational elliptic surfaces

The primary reference for this section is Shimizu [130]. In Seiberg & Witten's analysis of the exact solution, the meromorphic differential 1-form obtained from Equation (6.1.13) given by

$$\lambda = \frac{y\,dx}{1-x^2} = (u-x)\frac{dx}{y} \tag{6.3.43}$$

plays a crucial role in the following way. Let $\pi : \mathbf{Z} \to \mathbb{P}^1$ be a Seiberg-Witten curve, and α, β be a symplectic basis for the first integral homology group $H_1(\mathcal{E}_u, \mathbb{Z})$ of the generic fibre $\mathcal{E}_u = \pi^{-1}(u)$. Then define quantities a, a_D as

$$a = a(u) = \int_\alpha \lambda \quad , \quad a_D = a_D(u) = \int_\beta \lambda \, . \tag{6.3.44}$$

Then the pair (a_D, a) determines a holomorphic section of an $\mathrm{SL}(2,\mathbb{Z})$-bundle over the punctured line $\mathbb{P}^1 - \{\text{singular fibres}\}$. We recognize such quantities as a twisted fibre-wise period integral on the rational elliptic surface $\mathbf{Z}$; in physics, the $\mathcal{N} = 2$ supersymmetry algebra acts on (a_D, a) and determines a $\mathrm{U}(1)$ gauge multiplet where a is related to the semiclassical photon and a_D its dual, the magnetic photon. As such, both a, a_D are gauge fields of the associated $\mathrm{SU}(2)$ super Yang-Mills theory, and determine holomorphic local expressions of the prepotential function $\mathfrak{F}_{SW}$ from Equation (6.3.42) that are related by $\mathrm{SL}(2,\mathbb{Z})$ on the quantum moduli space $\mathbb{P}^1 - \{\text{singular fibres}\}$ [84]. Moreover, the pair (a_D, a) determine the holomorphic prepotential $\mathfrak{F}$ as

$$a_D = \frac{\partial}{\partial a} \mathfrak{F}_{SW}(a)$$

Additionally, the Seiberg-Witten differential λ is required to have residues in terms of the masses of the hypermultiplets [126, 130]. The masses form flat sections of the variation of Hodge structure naturally determined by the elliptic fibration. We have the following definition.

Definition 6.3.94 (Shimizu [130], 3.1.4)**.** *Let ∇ be the Gauss-Manin connection on the elliptic fibration $\pi : \mathbf{Z} \to \mathbb{P}^1$ from §2.1.4. Let $\Omega = -du \wedge dx/y \in H^0(\mathbf{Z}, \Omega^2_{\mathbf{Z}}(\log \mathcal{E}_u))$ be the rational 2-form on $\mathbf{Z}$ corresponding to the analytic marking dx/y on the generic* *fibre $\mathcal{E}_u$. Then the* Seiberg-Witten equation *is the following equality of 2-forms on* $\mathbf{Z}$*:*

$$\nabla(\lambda) = \Omega\,. \tag{6.3.45}$$

According to Shimizu, the masses of the hypermultiplets are flat sections of ∇, and so may be included on the left hand side without altering the solutions of the equation. Hence, we have the following result, whose proof follows from this discussion.

Lemma 6.3.95. *Let $\pi : \mathbf{Z} \to \mathbb{P}^1$ be a Seiberg-Witten curve, with or without matter. Then the period integrals of $\mathbf{Z}$ can be expressed in terms of the Seiberg-Witten differential as*

$$-\int_\Sigma \Omega = \int_\Sigma du \wedge \frac{dx}{y} = -\int_\Sigma d\lambda\,. \tag{6.3.46}$$

Moreover, the prepotential $\mathfrak{F}_{SW}$ satisfies an inhomogeneous differential equation determined by the Picard-Fuchs operators of $\mathbf{Z}$.

With this in mind, let us compute the Picard-Fuchs systems of some particular Seiberg-Witten curves relevant to the our goals in this section.

6.3.2 The $N_f = 4$ curve with two massive hypermultiplets

Our primary Seiberg-Witten curve of study will be the $N_f = 4$ curve, which is determined by a configuration of (m_1, m_2, m_3, m_4) of four masses. A generic configuration results in a rational elliptic surface $\pi : \mathbf{Z} \to \mathbb{P}^1$ with an I_0^* fibre at $u = \infty$ and six I_1 fibres elsewhere. In addition to the four mass parameters, $\mathbf{Z}$ is also parameterized by three other parameters e_1, e_2, e_3 whose significance are as follows.

The $N_f = 4$ SU(2)-super Yang-Mills theory is very closely related to the four dimensional $\mathcal{N} = 4$ super Yang-Mills theory, since the latter can be regarded as $\mathcal{N} = 2$ with an additional matter field hypermultiplet that transforms in the adjoint representation of SU(2) [126]. Both theories are scale invariant, and in fact form superconformal field theories in the absence of bare masses. Once a bare mass is given to the additional hypermultiplet in the $\mathcal{N} = 4$ theory, the supersymmetry is explicitly broken to $\mathcal{N} = 2$.

Hence, regarding the $N_f = 4$ theory as a mass deformed $\mathcal{N} = 4$ theory, the parameters e_1, e_2, e_3 are artifacts from the pure $\mathcal{N} = 4$ theory that can be used to label a choice of spin structure on the generic fibre $\mathcal{E}_u = \pi^{-1}(u)$ of $\mathbf{Z}$; in fact, due to the exact $\mathrm{SL}(2, \mathbb{Z})$ symmetry of the $\mathcal{N} = 4$ theory, the parameters e_1, e_2, e_3 are related naturally to Jacobi theta functions. The $N_f = 4$ theory can be seen to possess $\mathrm{SL}(2, \mathbb{Z})$ symmetry as follows: the modular symmetry that permutes the e_i parameters can be combined with the Spin(8) triality symmetry acting on the extended Dynkin diagram $\widetilde{D}_4$ of the I_0^* fibre at $u = \infty$ to yield full modular invariance of the $N_f = 4$ theory. This symmetry is broken for $N_f < 4$.

Bearing these considerations in mind, the Seiberg-Witten curve $\mathbf{Z}$ with mass configuration $(m_1, m_2, 0, 0)$ is described by the elliptic fibration

$$y^2 = W_1 W_2 W_3 + A \left(W_1 t_1 \left(e_2 - e_3\right) + W_2 t_2 \left(e_3 - e_1\right) + W_3 t_3 \left(e_1 - e_2\right)\right) - A^2 N \,, \tag{6.3.47}$$

where

$$\begin{aligned}
W_i &= x - e_i u - e_i^2 R \\
A &= \left(e_1 - e_2\right)\left(e_2 - e_3\right)\left(e_3 - e_1\right) \\
R &= \frac{1}{2}\left(m_1^2 + m_2^2\right) \\
t_1 &= \frac{1}{12} m_2^2 m_1^2 - \frac{1}{24} m_1^4 - \frac{1}{24} m_2^4 \\
t_2 &= -\frac{1}{24} m_2^2 m_1^2 + \frac{1}{48} m_1^4 + \frac{1}{48} m_2^4 \\
t_3 &= -\frac{1}{24} m_2^2 m_1^2 + \frac{1}{48} m_1^4 + \frac{1}{48} m_2^4 \\
N &= \frac{1}{96} m_1^6 - \frac{1}{96} m_2^2 m_1^4 - \frac{1}{96} m_1^2 m_2^4 + \frac{1}{96} m_2^6 \,.
\end{aligned}$$

Then the fibration has generically a singular fibre of type I_0^* at $u = \infty$, two singular fibres of type I_2, and two singular fibres of type I_1 at various finite values of u, as the effect of two hypermultiplets becoming massless coalesces two pairs of I_1 fibres together, forming the I_2 fibres. Furthermore, following Oguiso & Shioda [114], Table Entry 34, we see that the Mordell-Weil lattice is of rank two and is given by $\mathrm{MW}(\mathbf{Z}, \pi) = (A_1^*)^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$, where A_1^* is the dual root lattice of the Dynkin diagram A_1.

Notice that this curve has only bare masses, and no scaling parameter Λ, as was seen in the massless $N_f = 2$ curve in Equation (6.1.13). In particular, m_1, m_2 determine honest moduli of the surface $\mathbf{Z}$, while e_1, e_2, e_3 do not. To simplify the analysis, we choose to set these parameters as $e_1 = 0, e_2 = 1, e_3 = -1$, which corresponds to a some weak coupling limit in the $N_f = 4$ theory [126].

In this case, it is straightforward to write the resulting Weierstrass model from Equation (6.3.47) in the form

$$Y^2 = X(X^2 + bX + ac)\,, \tag{6.3.48}$$

where $b = -\left(m_1^2 + m_2^2\right)$, $a = u + m_1 m_2$, $c = u - m_1 m_2$. As indicated by Lemma 6.3.95, we can study the Seiberg-Witten equation for this mass configuration by finding the Picard-Fuchs system. We write the fibrewise period integral as

$$\hat{\omega} = \hat{\omega}(u, m_1, m_2) = \sqrt{u - m_1 m_2} \int_\sigma \frac{dX}{Y}, \tag{6.3.49}$$

where $\sigma \in H_1(\mathcal{E}_u, \mathbb{Z})$ is some integral cycle, and the normalization factor has been chosen for convenience that will be shown shortly. In the GKZ formalism for generalized

Euler integrals [50], we write ω as

$$\omega = \int_\sigma P(X)^\alpha X^\beta dX\,, \tag{6.3.50}$$

where $P(X) = u_3X^2 + u_2X + u_1$, $\alpha = \beta = -1/2$. This corresponds to the $\mathcal{A}$-hypergeometric system with

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \subset \mathbb{Z}^2 \tag{6.3.51}$$

whose lattice relations are given by $\mathbb{L} = \mathrm{span}_{\mathbb{Z}} \{[1,-2,1]\} \subset \mathbb{Z}^{3*}$. the resulting GKZ equations are given by

$$\begin{aligned} \frac{\partial^2}{\partial u_1 \partial u_3}\Phi - \frac{\partial^2}{\partial u_2^2}\Phi &= 0 \\ u_1\left(\frac{\partial}{\partial u_1}\Phi\right) + u_2\left(\frac{\partial}{\partial u_2}\Phi\right) + u_3\left(\frac{\partial}{\partial u_3}\Phi\right) + \frac{\Phi}{2} &= 0 \\ u_2\left(\frac{\partial}{\partial u_2}\Phi\right) + 2u_3\left(\frac{\partial}{\partial u_3}\Phi\right) + \frac{\Phi}{2} &= 0\,. \end{aligned} \tag{6.3.52}$$

Then upon transforming the system (6.3.52) by the transformation $\left\{u_1 = a, u_2 = \frac{b}{c}, u_3 = \frac{1}{c}\right\}$, we find the following system annihilates the honest period integral

$$\omega = \omega(u, m_1, m_2) = \int_\Sigma du \wedge \frac{dX}{Y} \tag{6.3.53}$$

with $\Sigma \in H_2(\mathbf{Z}, \mathbb{Z})$ a transcendental cycle:

$$0 = m_2 m_1 \left(\frac{\partial}{\partial u}\omega\right) - \frac{u m_2 \left(\frac{\partial}{\partial m_1}\omega\right)}{(m_1 - m_2)(m_1 + m_2)} + \frac{m_1 u \left(\frac{\partial}{\partial m_2}\omega\right)}{(m_1 - m_2)(m_1 + m_2)} + \frac{\omega}{2} \tag{6.3.54}$$

$$0 = (m_2 m_1 - u)\left(\frac{\partial}{\partial u}\omega\right) - \frac{\left(m_1^3 - m_2^2 m_1 + 2u m_2\right)\left(\frac{\partial}{\partial m_1}\omega\right)}{2\left(m_1 - m_2\right)\left(m_1 + m_2\right)} + \frac{\left(-m_2 m_1^2 + m_2^3 + 2m_1 u\right)\left(\frac{\partial}{\partial m_2}\omega\right)}{2\left(m_1 - m_2\right)\left(m_1 + m_2\right)} + \frac{\omega}{2} \tag{6.3.55}$$

$$\begin{aligned}
0 = {} & \frac{\left(-m_2 m_1 + u\right)^2\left(\frac{\partial^2}{\partial u^2}\omega\right)}{4} + \frac{m_1\left(m_1^2 + m_2^2\right)\left(-m_2 m_1 + u\right)\left(\frac{\partial^2}{\partial u \partial m_1}\omega\right)}{4\left(m_1 - m_2\right)\left(m_1 + m_2\right)} \\
& - \frac{m_2\left(m_1^2 + m_2^2\right)\left(-m_2 m_1 + u\right)\left(\frac{\partial^2}{\partial u \partial m_2}\omega\right)}{4\left(m_1 - m_2\right)\left(m_1 + m_2\right)} - \frac{\left(-m_2 m_1 + u\right) u\left(m_1^2 + m_2^2\right)\left(\frac{\partial^2}{\partial m_1^2}\omega\right)}{4\left(m_1 - m_2\right)^2\left(m_1 + m_2\right)^2} \\
& + \frac{\left(-m_2 m_1 + u\right)\left(m_1^4 - 2m_2^2 m_1^2 + m_2^4 + 4u m_1 m_2\right)\left(\frac{\partial^2}{\partial m_1 \partial m_2}\omega\right)}{4\left(m_1 - m_2\right)^2\left(m_1 + m_2\right)^2} \\
& - \frac{\left(-m_2 m_1 + u\right) u\left(m_1^2 + m_2^2\right)\left(\frac{\partial^2}{\partial m_2^2}\omega\right)}{4\left(m_1 - m_2\right)^2\left(m_1 + m_2\right)^2} \\
& + \frac{\left(-m_2 m_1 + u\right)\left(m_1^4 m_2 - 2m_1^2 m_2^3 + m_2^5 + 2u m_1^3 + 6u m_1 m_2^2\right)\left(\frac{\partial}{\partial m_1}\omega\right)}{4\left(m_1 - m_2\right)^3\left(m_1 + m_2\right)^3} \\
& - \frac{\left(-m_2 m_1 + u\right)\left(m_1^5 - 2m_1^3 m_2^2 + m_1 m_2^4 + 6u m_1^2 m_2 + 2u m_2^3\right)\left(\frac{\partial}{\partial m_2}\omega\right)}{4\left(m_1 - m_2\right)^3\left(m_1 + m_2\right)^3}
\end{aligned} \tag{6.3.56}$$

Combining together (6.3.54),(6.3.55), we arrive at the differential equation

$$0 = u\left(\frac{\partial}{\partial u}\omega\right) + \frac{m_1\left(\frac{\partial}{\partial m_1}\omega\right)}{2} + \frac{m_2\left(\frac{\partial}{\partial m_2}\omega\right)}{2} \tag{6.3.57}$$

which is the homogeneous part of the RG flow operator for the Seiberg-Witten curve expected from physics [33].

6.3.3 The isomassive $N_f = 4$ curve with two massless hypermultiplets

We deem the $N_f = 4$ curve with mass configuration $(m, m, 0, 0)$ the isomassive curve with two massless hypermultiplets. In the same weak coupling limit as in §6.3.2, the Seiberg-Witten curve in Equation (6.3.47) can be written in the following convenient form:

$$y^2 = (x-1)\left(x-1-\frac{m}{2}-\frac{t}{2}\right)\left(x-1-\frac{m}{2}+\frac{t}{2}\right). \tag{6.3.58}$$

This Weierstrass model determines an elliptic fibration $\widetilde{\pi} : \widetilde{\mathbf{Z}} \to \mathbb{P}^1$ three singular fibres of type I_2 at $t = 0, \pm m$ and an I_0^* fibre at $t = \infty$. From Oguiso & Shioda [114], Table Entry 57, the Mordell-Weil lattice is given by $\mathrm{MW}(\widetilde{\mathbf{Z}}, \widetilde{\pi}) = A_1^* \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

Then fibrewise period integral of $(\widetilde{\mathbf{Z}}, \widetilde{\pi})$ is naturally expressed in terms of Appell's F_1 function as

$$\begin{aligned}
\widetilde{\omega} = \widetilde{\omega}(t,m) &= \int_0^1 \frac{dx}{y} \\
&= \int_0^1 \frac{dx}{\sqrt{(x-1)(x-1-m/2-t/2)(x-1-m/2+t/2)}} \\
&= F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; 1 \middle| \, 1+m/2+t/2, 1+m/2-t/2\right).
\end{aligned} \tag{6.3.59}$$

Accordingly, using the well known differential system for F_1, the rank three Picard-Fuchs system annihilating $\widetilde{\omega}$ is given by

$$\begin{aligned}
\frac{\partial^2}{\partial m^2}\widetilde{\omega} = &- \frac{t\left(\frac{\partial^2}{\partial m \partial t}\widetilde{\omega}\right)}{2+m} - \frac{(3m^2+2mt-3t^2+6m+4t)\left(\frac{\partial}{\partial t}\widetilde{\omega}\right)}{2(2+m)(m-t)(m+t)} \\
&- \frac{(m^2-3t)\left(\frac{\partial}{\partial m}\widetilde{\omega}\right)}{(2+m)(m-t)(m+t)} - \frac{m\widetilde{\omega}}{2(2+m)(m-t)(m+t)},
\end{aligned} \tag{6.3.60}$$

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\widetilde{\omega} = &- \frac{(2+m)\left(\frac{\partial^2}{\partial m \partial t}\widetilde{\omega}\right)}{t} + \frac{(t^2+2m+3t)\left(\frac{\partial}{\partial t}\widetilde{\omega}\right)}{(m+t)t(m-t)} \\
&- \frac{(3m^2-2mt-3t^2+6m)\left(\frac{\partial}{\partial m}\widetilde{\omega}\right)}{2(m+t)t(m-t)} + \frac{\widetilde{\omega}}{2(m+t)(m-t)}.
\end{aligned} \tag{6.3.61}$$

6.3.4 Flowing to the isomassive $N_f = 2$ curve

Seiberg & Witten showed originally in their analysis that in certain weak coupling limits - meaning, taking particular values for the constants e_1, e_2, e_3, and taking various degenerative limits on the mass configurations of the $N_f = 4$ curve, one recovers the $N_f < 4$ curves.

To illustrate the robustness of the analysis in the previous section §6.3.2, we show how the GKZ computation for the Picard-Fuchs system of the $N_f = 4$ curve with two massive hypermultiplets exactly reproduces the homogeneous component of the RG flow operator for the isomassive $N_f = 2$ curve, i.e., with mass configuration (m, m).

Indeed, it was shown in [126] that taking the $N_f = 4$ curve with generic mass configurations can be flowed to the massive $N_f = 2$ curve in the limit $\tau \to i\infty$, $m_3, m_4 \to \infty$ while keeping $\Lambda^2 := 64q^{1/2}m_3m_4$ fixed, where $q^{1/2} = \exp(\pi i\tau)$ is the single instanton contribution to the $N_f = 4$ curve.

The resulting elliptic fibration $\hat{\pi} : \hat{\mathbf{Z}} \to \mathbb{P}^1$ can be described by the Weierstrass model

$$Y^2 = X(X^2 + \hat{b}X + \hat{a}\hat{c})\,, \tag{6.3.62}$$

with $\hat{b} = 3\Lambda^2/8 - u$, $\hat{a} = \Lambda^2/32$, and $\hat{c} = \Lambda^2 + 8m^2 - 8u$, which describes an elliptic fibration with an I_2^* fibre at $u = \infty$, an I_2 fibre at $u = m^2 + \Lambda^2/8$, and I_1 fibres at $u = -\Lambda^2/8 - m\Lambda, -\Lambda(\Lambda - 8m)/8$. The Mordell-Weil lattice is given by $\mathrm{MW}(\hat{\mathbf{Z}}, \hat{\pi}) = A_1^* \oplus \mathbb{Z}/2\mathbb{Z}$ [114], Table Entry 48. In this way, the exact GKZ system given in Equation (6.3.52) can be seen to annihilate the period integral

$$\hat{\omega} = \hat{\omega}(u, m, \Lambda) = \frac{\Lambda}{4\sqrt{2}} \int_\Sigma du \wedge \frac{dX}{Y}\,. \tag{6.3.63}$$

In the coordinates u, m, Λ, the Picard-Fuchs system becomes

$$0 = \frac{\Lambda\left(\frac{\partial}{\partial\Lambda}\hat{\omega}\right)}{2} + \frac{(\Lambda^2 - 8m^2 + 8u)\left(\frac{\partial}{\partial m}\hat{\omega}\right)}{16m} + \frac{3\Lambda^2\left(\frac{\partial}{\partial u}\hat{\omega}\right)}{8} + \frac{\hat{\omega}}{2} \tag{6.3.64}$$

$$0 = \frac{(\Lambda^2 - 16m^2 + 8u)\left(\frac{\partial}{\partial m}\hat{\omega}\right)}{16m} + \left(\frac{3\Lambda^2}{8} - u\right)\left(\frac{\partial}{\partial u}\hat{\omega}\right) + \frac{\hat{\omega}}{2} \tag{6.3.65}$$

$$
\begin{aligned}
0 =& -\frac{\left(\Lambda^2+8m^2-8u\right)\left(\Lambda^2-16m^2+8u\right)\left(\frac{\partial}{\partial m}\hat{\omega}\right)}{4096m^3} \\
&+\frac{\left(\Lambda^2+8m^2-8u\right)(\Lambda-2m)(\Lambda+2m)\left(\frac{\partial^2}{\partial\Lambda\partial m}\hat{\omega}\right)}{512m\Lambda} \\
&+\frac{\left(3\Lambda^2-8u\right)\left(\Lambda^2+8m^2-8u\right)\left(\frac{\partial^2}{\partial\Lambda\partial u}\hat{\omega}\right)}{512\Lambda} \\
&+\frac{\left(\Lambda^2+8m^2-8u\right)\left(\Lambda^2-16m^2+8u\right)\left(\frac{\partial^2}{\partial m^2}\hat{\omega}\right)}{4096m^2} \\
&+\frac{\left(\Lambda^2+8m^2-8u\right)\left(\Lambda^2-7m^2+2u\right)\left(\frac{\partial^2}{\partial m\partial u}\hat{\omega}\right)}{512m} \\
&+\frac{\left(\Lambda^2+8m^2-8u\right)\left(7\Lambda^2-16m^2-8u\right)\left(\frac{\partial^2}{\partial u^2}\hat{\omega}\right)}{2048}\,.
\end{aligned}
\tag{6.3.66}
$$

Once again, combining the first two Picard-Fuchs operators, we arrive at the homogeneous component of the RG flow operator expected from physics [33],

$$
0=\frac{\Lambda\left(\frac{\partial}{\partial\Lambda}\hat{\omega}\right)}{2}+\frac{m\left(\frac{\partial}{\partial m}\hat{\omega}\right)}{2}+u\left(\frac{\partial}{\partial u}\hat{\omega}\right) \tag{6.3.67}
$$

6.3.5 The mixed-twist construction for the isomassive $N_f = 4$ curve

We return now to the isomassive $N_f = 4$ curve with two massless hypermultiplets, i.e., mass configuration given by $(m, m, 0, 0)$. Instead of the weak coupling limit considered in §6.3.3, we impose the restriction $e_1 = 0, e_2 = \alpha/m^2, e_3 = \beta/m^2$ with $\alpha, \beta \in \mathbb{C}^*$ and $\alpha \neq \beta$. After simultaneously rescaling X, Y, we arrive at the Weierstrass model

$$
\widetilde{y}^2 = 4\widetilde{x}^3 - g_2(u,\alpha,\beta)\widetilde{x} - g_3(u,\alpha,\beta)\,, \tag{6.3.68}
$$

where g_2, g_3 are given by

$$
g_2 = \frac{4}{3}\left(\left(\alpha^2-\alpha\beta+\beta^2\right)u^2-(\alpha+\beta)\left(2\alpha^2-3\alpha\beta+2\beta^2\right)u+\alpha^4-\alpha^2\beta^2+\beta^4\right)\,,
$$

$$g_3 = \frac{4}{27}\left(\alpha^2 - \alpha u - 2\beta^2 + 2\beta u\right)\left(2\alpha^2 - 2\alpha u - \beta^2 + \beta u\right)\left(\alpha^2 - \alpha u + \beta^2 - \beta u\right)\,.$$

The Weierstrass model determines an elliptic fibration $\widetilde{\pi} : \widetilde{\mathbf{Z}} \to \mathbb{P}^1$ with a singular fibre of type I_0^* at $u = \infty$, and three singular fibres of type I_2 at $u = \alpha, \beta, \alpha+\beta$. Using a Möbius transformation that moves the I_0^* fibre from $u = \infty$, we write the elliptic fibration as a twisted Legendre pencil over the quadratic field extension $\mathbb{C}\left(\sqrt{\alpha - \beta}\,\right)$

$$y^2 = (t - c)x(x - 1)(x - t) \tag{6.3.69}$$

with $c = \alpha/(\alpha - \beta)$. This defines a rational elliptic surface birational to $(\widetilde{\mathbf{Z}}, \widetilde{\pi})$ that has three singular fibres of type I_2 at $t = 0, 1, \infty$, and a singular fibre of type I_0^* at $t = c$.

Applying the mixed-twist construction to this fibration, branched over $a, b \neq 0, 1, \infty, c$ yields the full twisted Legendre of Picard rank $\rho \geq 17$,

$$y^2 = (t - a)(t - b)(t - c)x(x - 1)(x - t)\,.$$

We thus surmise that the mixed-twist construction applied to this particular configuration of the isomassive $N_f = 4$ curve lifts the special geometry of the Seiberg-Witten curve to the special geometry of the moduli space $\mathfrak{T}$ of the twisted Legendre pencil, the latter of which is flat by the analysis of Theorem 6.2.85. Physically, this is consistent with the general notion that 10D string compactifications on K3 $\times\, T^2$ have $\mathcal{N} = 4$ supersymmetry, which is free from instanton corrections.

One could say roughly that "the mixed-twist construction twists away the instanton contributions", leaving a free super Yang-Mills theory. Mathematically, this corresponds to an explicit immersion of the ball quotient corresponding to the moduli of rational elliptic surfaces to the Type IV symmetric domain of the lattice polarized

K3 moduli space, where the extra parameters of the Seiberg-Witten data manifest in certain configurations as isomonodromic deformation parameters. Proof of a more general statement is given in Theorem 6.3.96.

6.3.6 The mixed-twist construction for the $N_f = 4$ curve with two massive hypermultiplets

Based off of the observations at the end of the last section, we would like to construct a relationship between an elliptic fibration representing an $N_f = 4$ curve and the double sextic family of K3 surfaces with Picard rank $\rho \geq 16$. A similar statement would then hold about the special geometry on the Seiberg-Witten curve and the flat special geometry on the moduli space $\mathcal{M}_6$ of six line configurations in $\mathbb{P}^2$.

In this case, we begin with the $N_f = 4$ curve with two massless hypermultiplets, so the generic mass configuration is $(m_1, m_2, 0, 0)$. Again, this is an elliptic fibration with one I_0^* fibre, two singular fibres of type I_2, and two I_1 fibres. Since the analogous conclusion will hold in terms of lifting the special geometry, at least over some finite degree field extension for the accessory parameters from the Seiberg-Witten data, for the purposes of this construction, we care only about starting with some isomorphic model of the Seiberg-Witten curve.

So our starting point will be the isomorphic model, abusing notation slightly by denoting it as $\pi : \mathbf{Z} \to \mathbb{P}^1$

$$y^2 = x\left(x^2 + 4s\left(\left(K_2 - L_2\right)t - \left(K_1 - L_1\right)s\right)x + 4s^2\left(K_1^2 - 4K_2\right)\left(s^2 - L_1 st + L_2 t^2\right)\right) \tag{6.3.70}$$

with $K_1 = \gamma_1 + \gamma_2$, $K_2 = \gamma_1\gamma_2$, $L_1 = \delta_1 + \delta_2$, and $L_2 = \delta_1\delta_2$, and $[s : t] \in \mathbb{P}^1$ are homogeneous coordinates. The fibration has the required singular fibre structure: there is an I_0^* fibre at $s = 0$, two singular fibres of type I_2 at $s = \delta_1, \delta_2$, and two singular fibres of type I_1 at $s = \delta_3, \delta_4$ with

$$\delta_3 = \frac{\delta_1\delta_2 - \gamma_1^2}{\delta_1 + \delta_2 - 2\gamma_1}, \qquad \delta_4 = \frac{\delta_1\delta_2 - \gamma_2^2}{\delta_1 + \delta_2 - 2\gamma_2}\,. \tag{6.3.71}$$

In this case, since the Mordell-Weil lattice is given by $\mathrm{MW}(\mathbf{Z}, \pi) = (A_1^*)^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$, there are two honest moduli of the fibration. The only pair of parameters that do *not* parameterize moduli of $(\mathbf{Z}, \pi)$ are (δ_1, δ_2). This can be verified in the following way.

The fibration in (6.3.71) is of the form

$$y^2 = x(x^2 + bx + ac)$$

with $b = 4s\left(\left(K_2 - L_2\right)t - \left(K_1 - L_1\right)s\right)$, $a = 4s^2\left(K_1^2 - 4K_2\right)$, $c = (s^2 - L_1 st + L_2 t^2)$. Hence, this is of the form we have previously considered in (6.3.48), and we can compute the Picard-Fuchs system of the fibration using the GKZ system (6.3.52) by selecting u and two other parameters - choosing (u, δ_1, δ_2) does not yield an invertible transformation, and hence δ_1, δ_2 do not parameterize moduli. This shows that we have a two parameter family of Picard-Fuchs systems that annihilate the period integrals of (6.3.70).

We connect $(\mathbf{Z}, \pi)$ with the six-line configuration as follows. First, applying the mixed-twist construction to (6.3.70), branched over $s = 0, \infty$ yields the four-parameter family of K3 surfaces $\widehat{\pi} : \mathbf{X} \to \mathbb{P}^1$, given by the elliptic fibration (again in homogeneous base coordinates $[s : t]$)

$$\begin{aligned} y^2 =& x\left(x^2 - 4s(s-t)t\left(\left(K_1 - L_1\right)s - \left(K_2 - L_2\right)t\right)x\right. \\ &\left.+4s^2(s-t)^2t^2\left(K_1^2 - 4K_2\right)\left(s^2 - L_1 st + L_2 t^2\right)\right) \end{aligned} \tag{6.3.72}$$

The fibration is then has three I_0^* fibres, at $s = 0, 1, \infty$, and the same configuration of I_2 and I_1 fibres as the rational elliptic surface $(\mathbf{Z}, \pi)$.

The K3 surface $\mathbf{X}$ admits a special Nikulin involution $\imath : \mathbf{X} \to \mathbf{X}$, a Van Geemen-Sarti involution, which results in translating the elliptic fibration (6.3.72) by the two-torsion section $(x = 0, y = 0)$. Crucially, Nikulin involutions are *symplectic* involutions, which preserve the holomorphic 2-form. It is a standard result (see, for example [25]) that the resulting elliptic fibration $\widetilde{\pi} : \mathbf{Y} \to \mathbb{P}^1$ given by

$$Y^2 = X(X^2 - 2bX + b^2 - 4ac) \tag{6.3.73}$$

yields a K3 surface after resolving the eight isolated fixed points of of $\imath$, $\mathbf{Y} = \mathbf{X}/\langle \imath \rangle$.

In this case, the the K3 surface $\mathbf{Y}$ is given by the fibration

$$\begin{aligned} Y^2 =& X \left(X^2 + 2s(s-t)t \left(\left(K_1 - L_1\right) s - \left(K_2 - L_2\right) t \right) X \right. \\ & + s^2(s-t)^2 t^2 \left(\left(\delta_1 + \delta_2 - 2\gamma_1\right) s - \left(\delta_1\delta_2 - \gamma_2^2\right) t \right) \\ & \left. \left(\left(\delta_1 + \delta_2 - 2\gamma_1\right) s - \left(\delta_1\delta_2 - \gamma_1^2\right) t \right) \right) \end{aligned} \tag{6.3.74}$$

One checks that the fibration structure is the same, except the location of the I_1 and I_2 fibres have swapped. Then $\mathbf{Y}$ can be realized as a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a divisor of bidegree $(4,4)$ as follows. Let the branching divisor be written as $3D_{1,0} + 2D_{0,1} + D_{1,2}$, where $D_{i,j}$ indicates the bidegree of the divisor. In homogeneous coordinates $([s:t],[\tilde{x}:\tilde{y}])$ on $\mathbb{P}^1 \times \mathbb{P}^1$, the double cover is given by

$$\tilde{z}^2 = s(s-t)t \left(\tilde{x}^2 - K_1\tilde{x}\tilde{y} + K_2\tilde{y}^2 \right) \left(\left(\tilde{x}^2 - L_2\tilde{y}^2 \right) t - \left(2\tilde{x} - L_1\tilde{y} \right) \tilde{y}s \right) \tag{6.3.75}$$

Regarded as an elliptic fibration over $[s:t] \in \mathbb{P}^1$, Equation (6.3.75) is isomorphic to (6.3.74), since the functional invariants of each fibration agree.

Introduce parameters r_1, r_2, r_3, r_4 such that

$$L_2 = \frac{(r_1+r_2)^2}{(r_2-1)^2}, L_1 = \frac{2\left(r_1 r_2 + r_2^2 + r_1 - r_2\right)}{(r_2-1)^2}, \gamma_1 = \frac{r_2+r_1-2r_3}{r_2-1}, \gamma_2 = \frac{r_2+r_1-2r_4}{r_2-1} \tag{6.3.76}$$

After a linear shift in the fibre variables, one arrives at the model

$$z^2 = s(s-t)t\,(x-r_3y)\,(x-r_4y)\,((x-(r_1+r_2)\,y)\,xt - ((1-r_2)\,x - r_1y)\,ys)\ , \tag{6.3.77}$$

which allows one to blow down the exceptional divisor $-sy+tx$, yielding a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ and the arrive at the K3 surface $\widehat{\pi} : \widehat{\mathbf{X}} \to \mathbb{P}^1$, elliptically fibred in homogeneous coordinates as

$$y^2 = (\tilde{t}-x)(z-x)x\left(\tilde{t}-r_3z\right)\left(\hat{t}-r_4z\right)\left(\tilde{t}-r_1x-r_2z\right)\ . \tag{6.3.78}$$

This is clearly the four parameter family of Yoshida surfaces obtained in Lemma 4.1.8, by setting $r_3 = a, r_4 = b, r_2 = c, r_1 = d$. As such, the fibration (6.3.78) has two singular fibres of type I_0^* over $\hat{t} = r_3, r_4$ and six fibres of type I_2 over $\hat{t} = 0, 1, \infty, r_2, r_1+r_2, r_2/(r_1-1)$, with Mordell-Weil given by $\mathrm{MW}(\widehat{\mathbf{X}}, \widehat{\pi}) = (\mathbb{Z}/2\mathbb{Z})^2$.

We have thus demonstrated that $\hat{\mathbf{X}}$ can be realized as the quadratic twist of the rational elliptic surface $\mathbf{S}_{r_1,r_2}$ from Lemma 4.1.29 with six singular fibres of type I_2 at the same values as $\hat{X}$. Hence, the Picard-Fuchs system and monodromy group of $\hat{X}$ are the same as that of the family of Yoshida surfaces; since all of the operations described here preserve the holomorphic 2-form up to scale, the same holds for the quadratic twist $\mathbf{X}$ of the $N_f = 4$ Seiberg-Witten curve with two massless hypermultiplets. Moreover, since the Yoshida surface $\mathbf{X}$ is birational to the double sextic surface $\mathcal{X}$ from Equation (2.1.22), we conclude that the quadratic twist operation lifts the nontrivial special geometry of the Seiberg-Witten curve to the flat special geometry of the double sextic.

We have arrived at the following result.

Theorem 6.3.96. *The quadratic twist* $\mathbf{X}$ *of the rational elliptic surface* $\mathbf{Z}$ *corresponding to the* $N_f = 4$ *Seiberg-Witten curve with two massless hypermultiplets is birational to the double sextic family* $\mathcal{X}$ *of Picard rank* $\rho \geq 16$. *The Picard-Fuchs system of* $\mathbf{X}$ *and subsequent monodromy group is identified with the Aomoto-Gel'fand* $E(3,6)$ *hypergeometric system, with monodromy group* G *from Corollary 4.1.37, in the variables*

$$\begin{aligned}
&\gamma_1 = \frac{2x_1x_2 - x_4x_1 - x_1 - x_2 + x_4}{x_3x_2 + x_1 - x_2 - x_3}, \gamma_2 = -\frac{x_4x_1 - x_1 - x_2 - x_4 + 2}{x_3x_2 + x_1 - x_2 - x_3},\\
&\delta_1\delta_2 = \frac{(x_4x_1 - x_1 + x_2 - x_4)^2}{(x_3x_2 + x_1 - x_2 - x_3)^2},\\
&\delta_1 + \delta_2 = -2\left(-x_1x_2x_3x_4 + x_1^2x_4 - x_1x_2x_3 - x_1x_2x_4 + x_1x_3x_4 + x_2^2x_3 + x_2x_3x_4\right.\\
&\left.-x_1^2 + 2x_1x_2 + x_1x_3 - x_4x_1 - x_2^2 - x_3x_2 + x_2x_4 - x_4x_3\right)/(x_3x_2 + x_1 - x_2 - x_3)^2
\end{aligned} \tag{6.3.79}$$

Here x_1, x_2, x_3, x_4 *are the moduli of the double sextic surface* $\mathcal{X}$. *As such, over some quadratic field extension of* $\mathbb{C}(\gamma_1, \gamma_2, \delta_1, \delta_2)$ *the nontrivial special geometry of the* $N_f = 4$ *Seiberg-Witten curve lifts via the mixed-twist construction to the flat special geometry of the moduli space* $\mathcal{X}(3,6)$ *of six lines in* $\mathbb{P}^2$.

Proof. The proof follows from the fact that all maps

$$\mathbf{X} \dashrightarrow \mathbf{Y} \dashrightarrow \widehat{\mathbf{X}} \dashrightarrow \mathcal{X}$$

preserve the holomorphic 2-form up to scale. One obtains the expressions in (6.3.79) by subsituting the moduli for the Yoshida surface in terms of the double sextic moduli and combining with the expressions in (6.3.76). $\square$

CHAPTER 7

Outlook

In this final chapter, we describe the outlook and future directions of the research of this dissertation.

7.1 RG flow operators and the Picard-Fuchs system twisted K3 surfaces

It is interesting, though perhaps not a surprise that the GKZ method pursued in §6.3.2, §6.3.4 yielded the homogeneous component of the first order RG flow operator expected from physics [33]. As was demonstrated in §6.3.5 , §6.3.6 , the isomassive curve and double massless hypermultiplet $N_f = 4$ curve can be twisted via the mixed-twist construction in such a way that the new twist parameters are independent of the mass configuration Seiberg-Witten data, though potentially dependent on the weak coupling limit chosen (up to a quadratic field extension).

In this way, the corresponding Picard-Fuchs operators determined by the mass configuration should lift as well to the K3 periods. The role that these operators play, especially the first order RG flow operators computed in Equations (6.3.57,6.3.67) should play some role in generating the Picard-Fuchs ideal described in §2.1.4 of the twisted K3 surface.

However, the role does seem to depend on what choice of weak coupling limit and which quadratic field extension of the moduli / accessory parameters from the Seiberg-Witten data is chosen, if any. Consider the following computation.

In the case of the isomassive $N_f = 4$ curve in §6.3.3, we demonstrated that after choosing $e_1 = 0, e_2 = \alpha/m^2, e_3 = \beta/m^2$, and the quadratic field extension of the Seiberg-Witten data $\mathbb{C}(\sqrt{\alpha-\beta})$, the curve could be written as a twisted Legendre pencil

$$y^2 = (t-c)x(x-1)(x-t)\,.$$

Using the direct computational methods in §5.3.1, one computes that the period integral $\hat{Z} = \int_0^1 dt\omega(t)$ satisfies the second order ODE

$$c(c-1)Z_{cc} + \left(c - \frac{1}{2}\right) Z_c = 0\,. \tag{7.1.1}$$

This ODE has the general solution with logarithmic singularities given by

$$Z = c_1 + c_2 \log\left(c - \frac{1}{2} + \sqrt{c(c-1)}\right) \tag{7.1.2}$$

with $c_1, c_2 \in \mathbb{C}$ arbitrary, after choosing some appropriate branch of log. This reveals some sort of integral transform of the hypergeometric function ${}_2F_1$ with the kernel function ${}_1F_0\left(\frac{1}{2}\middle|\frac{t}{c}\right)$, though such an identity is assuredly already known.

Moreover, the logarthmic singularities are only apparent, as the argument $c - \frac{1}{2} + \sqrt{c(c-1)}$ is everywhere nonzero. One may then check that the operator

$$\mathcal{O}_1 := c(c-1)\frac{d^2}{dc^2} + \left(c - \frac{1}{2}\right)\frac{d}{dc} \tag{7.1.3}$$

is *not* the derivative of some first order operator, nor is the square of a first order operator. We deem this second order operator "RG-like", in that it is not an honest RG flow operator, but annihilates the periods, and has a connection to the special geometry of the associated Seiberg-Witten curve. This operator should then lift to the second order operator

$$\mathcal{O}_1 := c(c-1)\frac{\partial^2}{\partial c^2} + \left(c - \frac{1}{2}\right)\frac{\partial}{\partial c} \tag{7.1.4}$$

in the Picard-Fuchs ideal for the K3 periods, though it is unclear what role this operator plays for the full twisted Legendre pencil.

A similar statement holds for the periods of the double massless hypermultiplet $N_f = 4$ curve. By virtue of Theorem 6.3.96, the RG flow equation derived in Equation (6.3.57) should lift to the Picard-Fuchs ideal of the double sextic family, the Aomoto-Gel'fand $E(3,6)$ hypergeometric system. Determining the solution to this problem will be further illuminating on how the mixed-twist construction interacts with the special geometry of the Seiberg-Witten curves. Moreover, one should determine how the massive $N_f = 2$ curve considered in §6.3.4 lifts to the twisted Legendre pencil / double sextic family.

Another connection between the special geometries can be seen in the following general result.

Theorem 7.1.97. *Let $\pi : \mathbf{Z} \to \mathbb{P}^1$ be a Seiberg-Witten curve described by the Weierstrass model*

$$y^2 = 4x^3 - g_2^{\mathrm{SW}}(t)x - g_2^{\mathrm{SW}}(t) \tag{7.1.5}$$

and consider the K3 $\mathbf{X}$ surface obtained from the mixed-twist construction applied to $\mathbf{Z}$, branched over $a, b \notin D_{\mathbf{Z}}$, the discriminant locus of $\mathbf{Z}$. Then the period integral of $\mathbf{X}$ can be written as

$$\omega_{\mathrm{K3}} = \int_\Sigma dx \wedge du \left(\lambda^{\mathrm{SW}} d\left(\frac{1}{\sqrt{h}}\right)\right) \tag{7.1.6}$$

where $h = (t-a)(t-b)$, λ^{SW} is the Seiberg-Witten differential of $\mathbf{Z}$ and $\Sigma \in \mathrm{T}(\mathbf{X})$ is a transcendental avoiding the branching locus of the mixed-twist construction.

Proof. The holomorphic 2-form $\eta_{\mathbf{X}} \in H^{2,0}(\mathbf{X})$ can be written as

$$\eta_{\mathbf{X}} = \frac{dt}{\sqrt{h}} \wedge \frac{dx}{y} = -\left(\frac{1}{\sqrt{h}}\right) \Omega^{\text{SW}} \equiv -\left(\frac{1}{\sqrt{h}}\right) d\lambda^{\text{SW}}\,.$$

After carefully choosing the transcendental cycle $\Sigma \in \mathrm{T}(\mathbf{X})$ to avoid the branching locus of the mixed-twist construction, the result follows from integration by parts. $\square$

7.2 Constructing Calabi-Yau threefolds

The main point of interest in continuing this research trajectory is to iteratively build families of Calabi-Yau threefolds via the mixed-twist construction of Doran & Malmendier [40]. More specifically, given an elliptically fibred K3 surface $\pi : \mathbf{X} \to \mathbb{P}^1$, the mixed-twist construction allows one to explicitly construct an elliptically fibred Calabi-Yau threefold $\widehat{\pi} : \widehat{X} \to B$, where B is some rational surface, such that $\widehat{X}$ is simultaneously fibred over $\mathbb{P}^1$ by Jacobian elliptic K3 surfaces. In fact, their iterative construction allows one to build, starting from a rational elliptic surface, a chain of Jacobian elliptic Calabi-Yau n-folds that fibres simultaneously over $\mathbb{P}^1$ by the Calabi-Yau $(n-1)$-folds.

For the purposes of this section, we recall the definition of an elliptic threefold.

Definition 7.2.98 (Miranda, [104]). *An elliptic threefold is a threefold X together with a map $\pi : X \to B$ from X to a surface B, whose generic fiber is a smooth elliptic curve. We say that the fibration $\pi : X \to B$ is a Jacobian elliptic threefold if a section for the map π is given.*

Miranda also outlines the following criteria, to study whether or not the fibration $\pi : X \to B$ has a smooth minimal model:

- X and B are both smooth.

• The map π is $flat$, i.e., all fibers are one-dimensional.

• The map π is minimal, in the sense that there is no generically contractible surface Y whose contractible fibers lie in the fibers of π. Hence no contraction or generic contraction of a surface in X will be compatible with the map π.

• The discriminant locus $D \subset B$, over which the fibers of π are singular, is a curve with at worst ordinary double points as singularities.

• At a smooth point $\mathrm{p} \in \mathrm{D}$, the singular fiber $\pi^{-1}(\mathrm{p})$ is a singular elliptic curve on Kodaira's list of singular fibers of elliptic surfaces [**?**]. Moreover, this fiber type is locally constant near p, and so is constant on irreducible components of $\mathrm{D} - \mathrm{D}_{\mathrm{sing}}$.

• At a singular point $\mathrm{p} \in \mathrm{D}$, the singular fiber $\pi^{-1}(\mathrm{p})$ is determined by the singular fiber types over the two branches of D at p.

Indeed, the possibility of codimension-2 singularities complicates the analysis of the existence of a smooth minimal model; moreover, even if such a smooth model does exist, the resolution of singular fibres may not be *crepant*, i.e., preserving the canonical bundle.

We have the following example:

- Start with the extremal rational elliptic surface with singular fibres give by $2I_1 + II^*$, with $g_2 = 3$ and $g_3(t) = -1 + 2t$ and $\Delta(t) = -108\, t\,(t-1)$ such that

  $$y^2 = 4x^3 - 3\,x + 1 - 2t.$$

  The fiberwise periods of dx/y satisfy the ODE of ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1|t)$.

- Carry out a base transformation and a twist to obtain a convenient model for a one-parameter subfamily of the family of M-polarized K3 surfaces. By setting

$t \mapsto t\left(2 + \frac{1}{s(s+1)}\right)$ we obtain

$$y^2 = 4x^3 - 3\left(s(s+1)\right)^4 x - g_3\left(t\left(2 + \frac{1}{s(s+1)}\right)\right)\left(s(s+1)\right)^6.$$

This family has the singular fibres given by $2II^* + 4I_1$; a simple base transformation shows that this family can be identified with the M-polarized sub-family for $\alpha = 1, \beta = 1, \gamma = 0, \delta = (4t)^2$. The K3 periods of $ds \wedge dx/y$ satisfy the ODE of

$${}_4F_3\left(\begin{array}{c} \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12} \\ 1, 1, \frac{1}{2} \end{array} \middle| (2t)^2\right).$$

- Setting $t \mapsto t\,\frac{(1+u^2)}{u}$ and carry out another twist to obtain a family of Calabi-Yau threefolds over $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$y^2 = 4x^3 - 3\left(s(s+1)\,u\right)^4 x - g_3\left(t\frac{(1+u^2)}{u}\left(2 + \frac{1}{s(s+1)}\right)\right)\left(s(s+1)\,u\right)^6.$$

  Notice that in generic u- and s-direction we have elliptic fibrations with $2II^* + 4I_1$. In u-direction, the II^*-fibers are located at $u = 0, \infty$, in s-direction the II^*-fibers are at $s = 0, -1$. A residue computation shows that the periods of $du \wedge ds \wedge dx/y$ satisfy the ODE of

$${}_4F_3\left(\begin{array}{c} \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12} \\ 1, 1, 1 \end{array} \middle| (2t)^2\right).$$

The analysis of this fibration via the framework of Miranda reveals, after a relatively long sequence of blow-ups, that the threefold X produced here by the mixed-twist construction is *singular*, i.e., the smooth-minimal model is *not* Calabi-Yau. Part of what makes this interesting, besides being the so-called 14th and final case of hypergeometric pencils of Calabi-Yau threefolds [22], is that this threefold can also be constructed by means of a different rational elliptic surface - and the fibering K3 surfaces have *different* Picard rank.

- Start with the extremal rational elliptic surface with singular fibres $I_2+I_1+III^*$, with $g_2 = \frac{16}{3} - 4\,t$ and $g_3(t) = -\frac{64}{27} + \frac{8}{3}t$ and $\Delta = -64\,t^2\,(t-1)$ such that

  $$y^2 = 4x^3 - \left(\frac{16}{3} - 4\,t\right)\,x + \frac{64}{27} - \frac{8}{3}t.$$

  The fiberwise periods of dx/y satisfy the ODE of ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1|t)$.

- Carry out a base transformation and a twist to obtain a convenient model for a one-parameter family of K3 surfaces with M_2-polarization - so Picard-rank 19.

- By setting $t \mapsto -t\frac{1}{4s(s+1)}$ we obtain

  $$y^2 = 4x^3 - g_2\left(\frac{-t}{4s(s+1)}\right)\,\big(s(s+1)\big)^4\,x - g_3\left(\frac{-t}{4s(s+1)}\right)\,\big(s(s+1)\big)^6.$$

  This family has singular fiber content $2III^* + 2I_1 + I_4$. The K3 periods of $ds \wedge dx/y$ satisfy the ODE of

  $${}_3F_2\left(\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{array}\middle|\, t\right).$$

- Set $t \mapsto \frac{16t}{729u^4(u+1)^2}$ and carry out another twist to obtain a family of Calabi-Yau threefolds over $\mathbb{P}^1 \times \mathbb{P}^1$ given by

  $$y^2 = 4x^3 - g_2(T)\,(s(s+1))^4\,(u^2(u+1))^2\,x - g_3\,(T)\;(s(s+1))^6\,(u^2(u+1))^3$$

  with $T = \frac{-2^2t}{3^6s\,(s+1)\,(u^2(u+1))^2}$. In generic s-direction we have an elliptic fibration with $2III^* + 2I_1 + I_4$ with the III^*-fibers located at $s = 0, -1$. In u-direction, we have an elliptic fibration with $I_{12}^* + 6I_1$ with the I_{12}^*-fiber located at $u = \infty$ - and thus, are M-polarized. A residue computation shows that the periods of $du \wedge ds \wedge dx/y$ satisfy the ODE of

$$
{}_4F_3\left(\begin{array}{c} \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12} \\ 1, 1, 1 \end{array} \middle| \, t \right).
$$

Importantly, the 14th case has been reconstructed in such a way that in various directions of the base surface, the fibering K3 surfaces have different Picard rank! This could be a manifestation of the singular structure of the total space, which as we remarked before, does not possess a crepant resolution.

Beyond illustrating the utility of the mixed-twist construction in analyzing interesting Calabi-Yau threefolds, which may or may not possess smooth minimal models, one may provide explicit geometric constructions of Calabi-Yau threefolds that are known to exist solely from a Hodge theoretic argument [40]. In order to do so, one must sample larger families of rational elliptic surfaces - in particular, those rational elliptic surfaces whose quadratic twists yield Jacobian elliptic K3 surfaces of at least Picard rank 17 or 16. This has of course been the focus of this dissertation.

Moreover, knowing the Picard-Fuchs systems of such K3 surfaces allows one to take 1-parameter restrictions by adequately sampling the moduli space. A generic, 1-parameter restriction of the moduli space $\mathfrak{T}$ of the twisted Legendre pencil over the line $\{a = \alpha_0 z + \beta_0, b = \alpha_1 z + \beta_1, c = z\} \subset \mathfrak{T}$ has been produced during the course of this research, and the corresponding rank-5 Picard-Fuchs ODE operator is given as follows:

$$a_5(z)\frac{d^5\omega}{dz^5}+a_4(z)\frac{d^4\omega}{dz^4}+a_3(z)\frac{d^3\omega}{dz^3}+a_2(z)\frac{d^2\omega}{dz^2}+a_1(z)\frac{d\omega}{dz}+a_0(z)\omega=0\,,$$

with coefficients given by

$$a_5(z)=\frac{8}{15}z(z-1)\left(z\beta_1-z+\beta_0\right)^2\left(z\alpha_1-z+\alpha_0\right)^2,$$

$$\begin{aligned}a_4(z)=&\frac{4}{15}\left(\beta_1 z-z+\beta_0\right)\left(\alpha_1 z-z+\alpha_0\right)\big(14z^3\alpha_1\beta_1-26\alpha_1 z^3-26z^3\beta_1+14z^2\alpha_0\beta_1\\&+14z^2\alpha_1\beta_0-7z^2\alpha_1\beta_1+38z^3-26z^2\alpha_0+19z^2\alpha_1-26z^2\beta_0+19z^2\beta_1+\\&14z\alpha_0\beta_0-7z\alpha_0\beta_1-7z\alpha_1\beta_0-31z^2+19z\alpha_0+19z\beta_0-7\alpha_0\,,\beta_0\big)\end{aligned}$$

$$
\begin{aligned}
a_3(z) =& \frac{124}{15}\alpha_0^2\beta_0 + \frac{124}{15}\alpha_0\beta_0^2 - \frac{172}{15}z^3\alpha_1^2 - \frac{172}{15}z^3\beta_1^2 + \frac{224}{5}\alpha_1^3 - \frac{172}{15}z\alpha_0^2 - \frac{172}{15}z\beta_0^2 \\
&+ \frac{224}{5}z^2\alpha_0 + \frac{224}{5}z^2\beta_0 + \frac{224}{5}z^3\beta_1 + \frac{24}{5}\alpha_0^2\beta_0^2 \\
&+ \frac{368}{15}z^4\alpha_1^2 + \frac{368}{15}z^4\beta_1^2 + \frac{368}{15}z^2\alpha_0^2 \\
&+ \frac{368}{15}z^2\beta_0^2 - \frac{396}{5}z^3\alpha_0 - \frac{336}{5}z^3\beta_0 - \frac{396}{5}z^4\alpha_1 - \frac{396}{5}z^4\beta_1 + \frac{96}{5}z^2\alpha_0\alpha_1\beta_0\beta_1 \\
&- \frac{548}{15}z^3 + \frac{868}{15}z^4 + \frac{248}{15}z\alpha_0\beta_0\beta_1 + \frac{248}{15}z^2\alpha_0\alpha_1\beta_1 \\
&+ \frac{248}{15}z^2\alpha_1\beta_0\beta_1 + \frac{248}{15}\alpha_0\alpha_1\beta_0 \\
&+ \frac{48}{5}z^3\alpha_0\alpha_1\beta_1^2 + \frac{48}{5}z^3\alpha_1^2\beta_0\beta_1 - \frac{784}{15}z^3\alpha_0\alpha_1\beta_1 \\
&- \frac{784}{15}z^3\alpha_1\beta_0\beta_1 - \frac{784}{15}z^2\alpha_0\alpha_1\beta_0 \\
&- \frac{784}{15}z^2\alpha_0\beta_0\beta_1 + \frac{48}{5}z\alpha_0^2\beta_0\beta_1 + \frac{48}{5}z\alpha_0\alpha_1\beta_0^2 \\
&+ \frac{124}{15}z\alpha_1\beta_0^2 - \frac{344}{15}z^2\beta_0\beta_1 \\
&+ \frac{124}{15}z^2\alpha_0\beta_1^2 - \frac{140}{3}z^2\alpha_1\beta_0 + \frac{124}{15}z^2\alpha_1^2\beta_0 \\
&- \frac{140}{3}z\alpha_0\beta_0 - \frac{140}{3}z^2\alpha_0\beta_1 - \frac{344}{15}z^2\alpha_0\alpha_1 \\
&+ \frac{124}{15}z\alpha_0^2\beta_1 + \frac{124}{15}z^3\alpha_1\beta_1^2 + \frac{124}{15}z^3\alpha_1^2\beta_1 - \frac{140}{3}z^3\alpha_1\beta_1 - \frac{392}{15}z^3\alpha_1^2\beta_0 \\
&- \frac{392}{15}z^4\alpha_1^2\beta_1 + \frac{1484}{15}z^3\alpha_0\beta_1 - \frac{392}{15}z^2\alpha_1\beta_0^2 - \frac{392}{15}z^2\alpha_0^2\beta_1 + \frac{736}{15}z^3\beta_0\beta_1 \\
&+ \frac{24}{5}z^4\alpha_1^2\beta_1^2\beta_1^2 + \frac{736}{15}z^3\alpha_0\alpha_1 + \frac{24}{5}z^2\alpha_0^2\beta_1^2 - \frac{392}{15}z\alpha_0^2\beta_0 + \frac{1484}{15}z^2\alpha_0\beta_0 \\
&- \frac{392}{15}z^3\alpha_0\beta_1^2 + \frac{24}{5}z^2\alpha_1^2\beta_0^2 + \frac{1484}{15}z^3\alpha_1\beta_0 \\
&+ \frac{1484}{15}z^4\alpha_1\beta_1 - \frac{392}{15}z\alpha_0\beta_0^2 - \frac{332}{15}z^4\alpha_1\beta_1^2
\end{aligned}
$$

$$\begin{aligned}
a_2(z) =& -\frac{16}{3}z^2\alpha_1^2 - \frac{16}{3}z^2\beta_1^2 + 42z^2\alpha_1 + 42z^2\beta_1 + 42z\alpha_0 + 42z\beta_0 - \frac{70}{3}\alpha_0\beta_0 \\
& - 14\alpha_0^2\beta_0 - 14\alpha_0\beta_0^2 + \frac{74}{3}z^3\alpha_1^2 + \frac{74}{3}z^3\beta_1^2 - 118\alpha_1 z^3 \\
& + \frac{74}{3}z\alpha_0^2 + \frac{74}{3}z\beta_0^2 - 118z^2\alpha_0 - 118z^2\beta_0 \\
& - 118z^3\beta_1 - 50z^2 - \frac{16}{3}\alpha_0^2 - \frac{16}{3}\beta_0^2 + 112z^3 \\
& - 28z\alpha_0\beta_0\beta_1 - 28z^2\alpha_0\alpha_1\beta_1 - 28z^2\alpha_1\beta_0\beta \\
& - 28z\alpha_0\alpha_1\beta_0 - \frac{70}{3}z^2\alpha_1\beta_1 - \frac{32}{3}z\alpha_0\alpha_1 - \frac{70}{3}z\alpha_0\beta_1 \\
& - \frac{70}{3}z\alpha_1\beta_0 - \frac{32}{3}z\beta_0\beta_1 - 14z\alpha_1\beta_0^2 + \frac{148}{3}z^2\beta_0\beta_1 \\
& - 14z^2\alpha_0\beta_1^2 + \frac{308}{3}z^2\alpha_1\beta_0 - 14z^2\alpha_0^2\beta_0 + \frac{308}{3}z\alpha_0\beta_0 \\
& + \frac{308}{3}z^2\alpha_0\beta_1 + \frac{148}{3}z^2\alpha_0\alpha_1 - 14z\alpha_0^2\beta_1 - 14z^3\alpha_1\beta_1^2 \\
& - 14z^3\alpha_1^2\beta_1 + \frac{308}{3}z^3\alpha_0\beta_1\,,
\end{aligned}$$

$$\begin{aligned}
a_1(z) =& \frac{9}{2}z^2\alpha_1^2 + 21z^2\alpha_1\beta_1 + \frac{9}{2}z^2\beta_1^2 - 44z^2\alpha_1 \\
& - 44z^2\beta_1 + 9z\alpha_0\alpha_1 + 21z\alpha_0\beta_1 + 21z\alpha_1\beta_0 \\
& + 9z\beta_0\beta_1 + \frac{119}{2}z^2 - 44z\alpha_0 + 7\alpha_1 z - 44z\beta_0 + 7\beta_1 z + \frac{9}{2}\alpha_0^2 \\
& + 21\alpha_0\beta_0 + \frac{9}{2}\beta_0^2 - \frac{31}{2}z + 7\alpha_0 + 7\beta_0\,,
\end{aligned}$$

$$a_0(z) = -\frac{1}{4}\left(6z\alpha_1 + 6z\beta_1 - 14z + 6\alpha_0 + 6\beta_0 + 1\right)$$

Studying how Picard-Fuchs operators such as these will allow for the construction, upon taking careful values of $\alpha_0, \beta_0, \alpha_1, \beta_1$, will allow for the explicit geometric construction of Calabi-Yau threefolds that have thus far only been realized by abstract Hodge-theoretic data.

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VITA

Personal Data and Bio

NAME: Michael Thomas Schultz

EMAIL: michael.schultz@usu.edu

Michael Thomas Schultz is a fifth year PhD candidate in mathematics at Utah State University, in the DGCAMP group (Differential Geometry, Computer Algebra, and Mathematical Physics) supervised by Dr. Andreas Malmendier. He is studying the geometry of lattice polarized K3 surfaces of high Picard rank, and the geometry of their period domains via Picard-Fuchs equations. His other interests include variational principles on period domains for Calabi-Yau manifolds, F-theory / heterotic string duality, and black hole geometries in general relativity.

Education

Current	PhD in Mathematics, **Utah State University**, Logan, UT Dissertation: *On the geometry of families of lattice polarized K3 surfaces of high Picard rank and their period domains* Supervisor: Dr. Andreas Malmendier Expected Date of Completion: December 2021 Gpa: 3.91/4.0
August 2016	Master of Science in Mathematics, **Idaho State University**, Pocatello, ID Emphasis of Study: Differential Geometry, Mathematical Physics, Calculus of Variations Supervisor: Dr. Bennett Palmer Gpa: 3.6/4.0
December 2013	Honors Bachelor of Science in Mathematics, **Idaho State University**, Pocatello, ID Thesis: Characterizing integrality in circulant graphs Supervisor: Dr. Catherine Kriloff Gpa: 3.8/4.0

Academic Appointments

January 2022 - 2025	Visiting Assistant Professor Virginia Tech, Blacksburg, VA
January 2017 - Present	Graduate Teaching Assistant Utah State University, Logan, UT Main instructor for sections of Differential Equations, Linear Algebra, and Calculus I. Recitation leader for a variety of undergraduate mathematics classes. Developed curriculum, facilitated lecture, prepared lessons, reported grades, conducted office hours.
August 2014 - May 2016	Graduate Teaching Assistant Idaho State University, Pocatello, ID Main instructor for a variety of undergraduate mathematics classes. Designed syllabi, wrote and administered exams, reported final grades and conducted office hours.

Research Appointments

JANUARY 2020 - MAY 2020	Excellence in Teaching Research Assistantship (Awarded) Dept. Mathematics and Statistics UTAH STATE UNIVERSITY, Logan, UT
JANUARY 2018 - MAY 2018	Research Assistantship for study of Abelian and Calabi-Yau varieties Funded by Dr. Andreas MALMENDIER, NSA Conference Grant (H98230-18-1-0285)

Publications

Rocky Mountain Journal of Mathematics Volume 50 (2020) No. 1, 181-212	*From the Signature Theorem* *to Anomaly Cancellation*, With Andreas MALMENDIER
SUBMITTED Journal of Number Theory & Physics AUGUST 2021	*On the Mixed-Twist Construction* *and Monodromy of Associated* *Picard-Fuchs Systems* With Andreas MALMENDIER
IN PREPARATION	*On Special Geometry associated* *to certain lattice polarized* *K3 surfaces and Seiberg-Witten curves* With Andreas MALMENDIER

Awards

USU Dept. of Math and Stats May 3, 2020	2020 Doctoral Researcher of the Year
USU Dept. of Math and Stats April 28, 2019	Graduate Student Departmental Service Award
Utah State University April 12, 2019	Outstanding Oral Presentation at Student Research Symposium
690. WE-Heraeus-Seminar Bonn, DEU February 15, 2019	“Best Poster” Teaching Resources

Conference Presentations

Joint Mathematics Meeting 2021 Virtual January 8, 2021	Title: *Differential Geometry of the Seigel Modular Threefold & algebro-arithmetic data of certain K3 surfaces*
Algebraic Geometry and Mathematical Physics: Explicit Methods for Abelian and Calabi-Yau varieties Utah State University, Logan, UT July 5 - 7, 2019	Title: *Resolving Holomorphic Anomalies on Elliptic Surfaces*
USU Student Research Symposium April 10, 2019 Utah State University, Logan, UT	Title: *Gravitational Anomalies and Elliptic Curves* Notes: Awarded "Outstanding Graduate Oral Presentation" in Physical Sciences
Jim Isenberg Pacific Coast Gravity Meeting 35 March 29, 2019 Utah State University, Logan, UT	Title: *Gravitational Anomalies and the Universal Bundle of Elliptic Curves*